IOP Publishing Classical and Quantum Gravity
Class. Quantum Grav. **41** (2024) 157001 (6pp) https://doi.org/10.1088/1361-6382/ad563b

Note

Can a radiation gauge be horizon-locking?

Leo C Stein

Department of Physics and Astronomy, University of Mississippi, University, MS 38677, United States of America

E-mail: lcstein@olemiss.edu

Received 15 April 2024; revised 20 May 2024
Accepted for publication 10 June 2024
Published 8 July 2024

Abstract
In this short Note, I answer the titular question: yes, a radiation gauge can be horizon-locking. Radiation gauges are very common in black hole perturbation theory. It's also very convenient if a gauge choice is horizon-locking, i.e. the location of the horizon is not moved by a linear metric perturbation. Therefore it is doubly convenient that a radiation gauge can be horizon-locking, when some simple criteria are satisfied. Though the calculation is straightforward, it seemed useful enough to warrant writing this Note. Finally I show an example: the ℓ vector of the Hartle–Hawking tetrad in Kerr satisfies all the conditions for ingoing radiation gauge to keep the future horizon fixed.

Keywords: black hole, perturbation theory, radiation gauge, horizon-locking

The context of this Note is black hole perturbation theory (see [1] for a review). Suppose we have a Lorentzian spacetime $(M, \mathring{g})$ where $\mathring{g}$ is the background metric, e.g. the Kerr metric (see [2] for a review), which has a future horizon $\mathcal{H}^+$ (see [3] for a pedagogical introduction). We work to first order in perturbation theory, with a metric

$$g_{ab} = \mathring{g}_{ab} + \varepsilon h_{ab} + \mathcal{O}\left(\varepsilon^2\right) , \tag{1}$$

where ε is a formal order-counting parameter.

Chrzanowski introduced two 'radiation gauges' for perturbations in [4]. These radiation gauges are adapted for algebraically special [5] spacetimes. If ℓ^a is an *outgoing* principal null vector field, then *ingoing* radiation gauge (IRG) is specified by

$$\ell^a h_{ab} = 0 , \quad h \equiv \mathring{g}^{ab} h_{ab} = 0 \qquad \text{(IRG)} . \tag{2}$$

Similarly, if n^a is an ingoing principal null vector field, then outgoing radiation gauge (ORG) is the same but with n replacing ℓ. These gauges at first seem over-specified, with 5 algebraic conditions. Price *et al* [6] showed that one of IRG or ORG is admissible in a Petrov type II metric, whereas in type D, both are admissible.

The event horizon is the defining feature of a black hole [7], and thus it is of great physical interest to locate the horizon, e.g. to study thermodynamics [8] or tides [9], or to compute fluxes down the horizon [10]. In general, locating a horizon is challenging since it is teleological, requiring global knowledge of the entire future development of the spacetime [7]. This challenge is lessened in perturbation theory, but replaced with the new challenge that we are free to make $\mathcal{O}(\varepsilon)$ coordinate transformations. These generate the gauge transformations $h_{ab} \to h'_{ab} = h_{ab} + \mathcal{L}_\xi \mathring{g}_{ab} = h_{ab} + \mathring{\nabla}_{(a}\xi_{b)}$ where ξ^a generates the infinitesimal diffeomorphism. We are describing the same physical spacetime, but the horizon moves by $\mathcal{O}(\varepsilon)$ in coordinates.

On the other hand, we can exploit this freedom to make coordinates of the horizon of g_{ab} coincide with the analytically-known horizon of $\mathring{g}_{ab}$. A gauge choice achieving this is called 'horizon-locking,' possibly introduced by [11], though the idea is surely older. There is still considerable freedom in achieving a horizon-locking gauge: only components of ξ^a transverse to the horizon are relevant [12]. We can now pose the question asked in the title of this Note: Can a radiation gauge be horizon-locking? Yes.

Theorem . *Let $(M, \mathring{g})$ be a stationary, Ricci-flat, Lorentzian spacetime with future horizon $\mathcal{H}^+$. Let ℓ^a: (i) be null, (ii) be geodesic, and (iii) generate $\mathcal{H}^+$. Let h_{ab} be the perturbation as in equation (1), and let h_{ab} vanish either in the distant past or future. Further let $R_{ab} = \mathcal{O}(\varepsilon^2)$ with R_{ab} the Ricci tensor of g_{ab}. Then the gauge in equation (2) is horizon-locking.*

Proof . First, we follow [12] to see that the event horizon and apparent horizon agree to first order in ε. Consider the Raychaudhuri equation for a geodesic null congruence k^a that generates the horizon, with affine parameter v,

$$\frac{\mathrm{d}\theta}{\mathrm{d}v} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}k^a k^b \, . \tag{3}$$

Here θ is the expansion scalar, σ_{ab} is the shear, and ω_{ab} is the twist. By assumption, the Ricci term vanishes at zeroth and first order. Since the horizon generator is hypersurface orthogonal, $\mathring{\omega}_{ab}|_{\mathcal{H}^+} = 0 = \omega_{ab}|_{\mathcal{H}^+}$.

Expand all quantities as a series in ε, e.g. $\sigma_{ab} = \mathring{\sigma}_{ab} + \varepsilon\sigma^{(1)}_{ab} + \mathcal{O}(\varepsilon^2)$. Stationarity of the background then tells us that $\mathrm{d}\mathring{\theta}/\mathrm{d}v|_{\mathcal{H}^+} = 0$, and thus $\mathring{\theta}|_{\mathcal{H}^+} = \mathring{\sigma}_{ab}|_{\mathcal{H}^+} = 0$. Now study the $\mathcal{O}(\varepsilon^1)$ equation, which says

$$\frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}v} = -\mathring{\theta}\theta^{(1)} - 2\mathring{\sigma}^{ab}\sigma^{(1)}_{ab} + 2\mathring{\omega}^{ab}\omega^{(1)}_{ab} \, . \tag{4}$$

Evaluating at the background horizon, all terms on the right-hand side vanish, so $\theta^{(1)}|_{\mathcal{H}^+}$ is constant. Since h_{ab} vanishes in the distant past or future, this constant must be $\theta^{(1)}|_{\mathcal{H}^+} = 0$. Therefore the perturbed event horizon is an apparent horizon to $\mathcal{O}(\varepsilon^1)$, and our job has reduced to locating the apparent horizon at first order.

Now for locating the apparent horizon. First note that in the gauge (2), the vector field ℓ^a is automatically null up to our desired order,

$$g(\ell, \ell) = \mathring{g}_{ab}\ell^a\ell^b + \varepsilon h_{ab}\ell^a\ell^b + \mathcal{O}\left(\varepsilon^2\right) = 0 + \varepsilon 0 + \mathcal{O}\left(\varepsilon^2\right) \, . \tag{5}$$

Correspondingly, lowering ℓ^a with either metric gives the same one-form, $\ell_a \equiv g_{ab}\ell^b = \mathring{g}_{ab}\ell^b + O(\varepsilon^2)$. Therefore we find no need to expand ℓ^a in a series in ε. Below we need an identity arising from a gradient of the gauge conditions (2),

$$\mathring{\nabla}_a(\ell^c h_{cd}) = 0 \qquad \Rightarrow \qquad \ell^c \mathring{\nabla}_a h_{cd} = -h_{cd}\mathring{\nabla}_a \ell^c \,. \tag{6}$$

Here $\mathring{\nabla}$ is the Levi-Civita connection of $\mathring{g}$.

Now let us check that ℓ^a is geodesic with respect to the perturbed metric, not just the background metric. To do this we need to express the Levi-Civita connection of g, which we call ∇, in terms of $\mathring{\nabla}$. The two connections are related by

$$\nabla_b v^a - \mathring{\nabla}_b v^a = \varepsilon \dot{C}^a{}_{bc} v^c + \mathcal{O}\left(\varepsilon^2\right) , \tag{7}$$

where the linearized difference of connections tensor is [7]

$$\dot{C}^a{}_{bc} = \frac{1}{2}\mathring{g}^{ad}\left[\mathring{\nabla}_b h_{cd} + \mathring{\nabla}_c h_{bd} - \mathring{\nabla}_d h_{bc}\right] . \tag{8}$$

By assumption, with the background connection we have a geodesic congruence, not affinely parameterized,

$$\ell^a \mathring{\nabla}_a \ell^b = \mathring{\kappa}\ell^b \,. \tag{9}$$

Evaluate $\nabla_\ell \ell$ to see if it is geodesic:

$$\begin{aligned}
\ell^a \nabla_a \ell^b &= \ell^a \mathring{\nabla}_a \ell^b + \varepsilon \ell^a \dot{C}^b{}_{ac}\ell^c + \mathcal{O}\left(\varepsilon^2\right) , && (10)\\
&= \mathring{\kappa}\ell^b + \varepsilon\frac{1}{2}\ell^a\ell^c\mathring{g}^{bd}\left[\mathring{\nabla}_a h_{cd} + \mathring{\nabla}_c h_{ad} - \mathring{\nabla}_d h_{ac}\right] + \mathcal{O}\left(\varepsilon^2\right) , && (11)\\
&= \mathring{\kappa}\ell^b + \varepsilon\mathring{g}^{bd}\left[\ell^a\ell^c\mathring{\nabla}_a h_{cd} - \frac{1}{2}\ell^a\ell^c\mathring{\nabla}_d h_{ac}\right] + \mathcal{O}\left(\varepsilon^2\right) , && (12)\\
&= \mathring{\kappa}\ell^b + \varepsilon\mathring{g}^{bd}\left[-\ell^a h_{cd}\mathring{\nabla}_a\ell^c + \frac{1}{2}\ell^a h_{ac}\mathring{\nabla}_d\ell^c\right] + \mathcal{O}\left(\varepsilon^2\right) , && (13)\\
&= \mathring{\kappa}\ell^b + \varepsilon\mathring{g}^{bd}\left[-h_{cd}\mathring{\kappa}\ell^c + 0\right] + \mathcal{O}\left(\varepsilon^2\right) , && (14)\\
\ell^a \nabla_a \ell^b &= \mathring{\kappa}\ell^b + \mathcal{O}\left(\varepsilon^2\right) . && (15)
\end{aligned}$$

Therefore ℓ is also still a null geodesic congruence with respect to g, not just $\mathring{g}$. Furthermore, the inaffinity has not changed, $\kappa = \mathring{\kappa} + \mathcal{O}(\varepsilon^2)$, a result we need below.

Next we need to check that ℓ^a is still hypersurface-orthogonal. From the Frobenius theorem, the one-form ℓ_a is hypersurface-orthogonal when $\ell \wedge d\ell = 0$. This has implicit dependence on the metric, lowering the vector ℓ^a into the one-form. As we saw above, the gauge condition makes $g_{ab}\ell^b = \mathring{g}_{ab}\ell^b + \mathcal{O}(\varepsilon^2)$. Therefore whenever $\ell \wedge d\ell$ vanishes according to the background metric, it also vanishes according to the perturbed metric, up to $\mathcal{O}(\varepsilon^2)$. Thus ℓ_a is hypersurface-orthogonal at $\mathcal{H}^+$ with respect to both metrics.

Finally we want to check that the congruence ℓ^a has vanishing expansion—as measured with g_{ab}—at the unperturbed horizon. To find the expansion, we proceed as usual [3] by studying $B_{ab} \equiv \nabla_b \ell_a$. Specifically we will need to take an orthogonal projection with the aid of an auxiliary null vector n_a, satisfying $n_a\ell^a = -1$ (we work in signature $-{+}{+}{+}$). Next construct the orthogonal projector $\gamma_{ab} = g_{ab} + \ell_a n_b + n_a \ell_b$, and use it to project out $\hat{B}_{ab} = \gamma_a{}^c\gamma_b{}^d B_{cd}$. The expansion scalar is the trace,

$$\theta = \gamma^{ab}\hat{B}_{ab} = \gamma^{ab}B_{ab} = g^{ab}\nabla_b\ell_a + \ell^a n^b \nabla_b \ell_a + n^a \ell^b \nabla_b \ell_a \,, \tag{16}$$

$$\theta = \nabla_a \ell^a + n^b \nabla_b \left(\tfrac{1}{2}\ell^a \ell_a\right) + n^a \kappa \ell_a = \nabla_a \ell^a - \kappa\,. \tag{17}$$

In this final expression we see that all references to B_{ab} and the auxiliary n^a have disappeared, so we do not have to worry about their perturbations; we just need this last expression along with $\mathring{\theta} = \mathring{\nabla}_a \ell^a - \mathring{\kappa}$. The perturbed expansion is

$$\theta = \nabla_a \ell^a - \kappa = \mathring{\nabla}_a \ell^a + \varepsilon \dot{C}^a{}_{ab} \ell^b - \mathring{\kappa} + \mathcal{O}\left(\varepsilon^2\right) \tag{18}$$

$$= \mathring{\theta} + \varepsilon \ell^b \frac{1}{2} \mathring{g}^{ad} \left[\mathring{\nabla}_a h_{bd} + \mathring{\nabla}_b h_{ad} - \mathring{\nabla}_d h_{ab}\right] + \mathcal{O}\left(\varepsilon^2\right), \tag{19}$$

$$\theta = \mathring{\theta} + \varepsilon \ell^b \frac{1}{2} \left[\mathring{\nabla}_a h_b{}^a + \mathring{\nabla}_b h - \mathring{\nabla}_d h^d{}_b\right] + \mathcal{O}\left(\varepsilon^2\right) = \mathring{\theta} + \mathcal{O}\left(\varepsilon^2\right). \tag{20}$$

The first and third term in parentheses cancel, and the middle term vanishes from the gauge condition for vanishing trace $h = 0$. Thus we have shown that the perturbed expansion is the same as the background expansion up to $\mathcal{O}(\varepsilon^2)$. In particular, θ (as measured by ∇_a) vanishes at the unperturbed horizon $\mathcal{H}^+$, thus locating the perturbed apparent horizon; which we saw above is the same as the perturbed event horizon. □

Remark 1. Notice that the conditions for the theorem are weaker than what is usually done in black hole perturbation theory: ℓ^a does not need to be a principal null direction.

Remark 2. The condition $R_{ab} = \mathcal{O}(\varepsilon^2)$ is satisfied if h_{ab} solves the linearized Einstein equations with vanishing first-order source T_{ab}. For example, in the EMRI problem we have a point-particle source, so $R_{ab} = \mathcal{O}(\varepsilon^2)$ everywhere except the location of the particle. Horizon-locking can be achieved at all times except when the particle passes through the horizon.

Remark 3. The condition $R_{ab} = \mathcal{O}(\varepsilon^2)$ can be generalized to the weaker condition $R_{ab}\ell^a\ell^b = \mathcal{O}(\varepsilon^2)$.

Remark 4. Throughout the derivation, we only needed the gauge condition (2) and its first derivative evaluated along $\mathcal{H}^+$. Therefore, the theorem still holds replacing the global gauge condition with just the horizon boundary condition

$$\ell^a h_{ab}\Big|_{\mathcal{H}^+} = 0, \quad h\Big|_{\mathcal{H}^+} = 0, \quad \mathring{\nabla}_c\left(\ell^a h_{ab}\right)\Big|_{\mathcal{H}^+} = 0, \quad \mathring{\nabla}_a h\Big|_{\mathcal{H}^+} = 0. \tag{21}$$

Remark 5. Using n^a and its ingoing expansion in place of ℓ^a and its outgoing expansion, and using $\mathcal{H}^-$ in place of $\mathcal{H}^+$, the theorem also applies to ORG being compatible with fixing the past horizon.

Example: Hartle–Hawking tetrad for the Kerr metric

Here we show that the ℓ vector of the Hartle–Hawking tetrad for the Kerr metric satisfies the conditions for the above theorem. Our metric is compactly represented by specifying our tetrad. We use ingoing coordinates $(v, r, \theta, \tilde{\phi})$ to give the Hartle–Hawking tetrad components [10, 13, 14],

$$\ell^a = \left(1, \frac{1}{2}\frac{\Delta}{r^2 + a^2}, 0, \frac{a}{r^2 + a^2}\right), \tag{22}$$

$$n^a = \left(0, -\frac{r^2 + a^2}{\Sigma}, 0, 0\right), \tag{23}$$

$$m^a = \frac{1}{2(r + ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right), \tag{24}$$

where as is typical in Kerr, $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$, and $\Sigma = r^2 + a^2\cos^2\theta$. The roots $r_\pm = M \pm \sqrt{M^2 - a^2}$ are the locations of the outer and inner horizons. This tetrad is clearly regular at the future horizon, where ℓ^a coincides with the horizon generator, which in terms of the Killing vectors ∂_v and $\partial_{\tilde{\phi}}$ and angular velocity of the horizon Ω_H is

$$\ell^a\Big|_{\mathcal{H}^+} = \frac{\partial}{\partial v} + \Omega_H \frac{\partial}{\partial\tilde{\phi}}, \qquad \Omega_H = \frac{a}{2Mr_+}. \tag{25}$$

The coordinate v here should not be confused with the affine parameter in equation (3). This tetrad is related to the very common Kinnersley tetrad [15], which is not regular at $\mathcal{H}^+$, by the boost $\ell_{\mathrm{HH}} = \lambda\bar{\lambda}\ell_K$ and $n_{\mathrm{HH}} = \lambda^{-1}\bar{\lambda}^{-1}n_K$ where $\lambda^{-2} = 2(r^2 + a^2)/\Delta$. Therefore ℓ and n are both geodesic principal null congruences. From the tetrad we can assemble the inverse metric

$$g^{ab} = -\ell^a n^b - n^a \ell^b + m^a \bar{m}^b + \bar{m}^a m^b, \tag{26}$$

or invert for the more common form [2],

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)\left(\mathrm{d}v - a\sin^2\theta\,\mathrm{d}\tilde{\phi}\right)^2 \\ &+ 2\left(\mathrm{d}v - a\sin^2\theta\,\mathrm{d}\tilde{\phi}\right)\left(\mathrm{d}r - a\sin^2\theta\,\mathrm{d}\tilde{\phi}\right) + \Sigma\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\tilde{\phi}^2\right). \end{aligned} \tag{27}$$

It's interesting to inspect a few of the Newman–Penrose (NP) spin coefficients [16] $\rho, \sigma_{\mathrm{NP}}, \epsilon_{\mathrm{NP}}$. We can find ρ and σ_{NP} from the boost transformations $\rho_{\mathrm{HH}} = \lambda^1\bar{\lambda}^1\rho_K$ and $\sigma_{\mathrm{HH}} = \lambda^3\bar{\lambda}^{-1}\sigma_K$ (where $\{1,1\}$ and $\{3,-1\}$ are the GHP weights [17] for the spin coefficients ρ and σ_{NP}),

$$\rho = -m^b\bar{m}^c\nabla_c\ell_b = \frac{-1}{r - ia\cos\theta}\frac{\Delta}{2(r^2 + a^2)}, \tag{28}$$

$$\sigma_{\mathrm{NP}} = -m^b m^c\nabla_c\ell_b = 0, \tag{29}$$

$$\epsilon_{\mathrm{NP}} = -\frac{1}{2}\left(n^b\ell^c\nabla_c\ell_b - \bar{m}^b\ell^c\nabla_c m_b\right) = \frac{M\left(r^2 - a^2\right)}{2\left(r^2 + a^2\right)^2}. \tag{30}$$

These are not the only non-vanishing spin coefficients, but the only ones needed below. The inaffinity κ (not to be confused with the NP coefficient κ_{NP}) and the tensor $\hat{B}_{ab}$ associated to the congruence ℓ^a can be expressed in terms of the above spin coefficients. Notice that we can get the inaffinity from

$$\epsilon_{\mathrm{NP}} + \bar{\epsilon}_{\mathrm{NP}} = -n^b\ell^c\nabla_c\ell_b = \kappa. \tag{31}$$

Evaluating at the horizon, we get the surface gravity of the Kerr black hole,

$$\kappa_+ = \frac{r_+ - M}{2Mr_+}. \tag{32}$$

When constructing $\hat{B}_{ab}$, the projector in NP language becomes $\gamma_{ab} = m_a\bar{m}_b + \bar{m}_a m_b$. Assembling the expansion, shear, and twist, we get the NP translations

$$\theta = \gamma^{ab}B_{ab} = -(\rho + \bar{\rho}), \tag{33}$$

$$\omega_{ab} = \hat{B}_{[ab]} = \frac{1}{2}\left(\rho - \bar{\rho}\right)\left(m_a \bar{m}_b - \bar{m}_a m_b\right), \tag{34}$$

$$\sigma_{ab} = \hat{B}_{\langle ab \rangle} = -m_a m_b \bar{\sigma}_{\rm NP} - \bar{m}_a \bar{m}_b \sigma_{\rm NP}. \tag{35}$$

Since $\sigma_{\rm NP} = 0$, indeed ℓ^a is shear-free. Noting that $\rho \propto \Delta$, we see that $\rho|_{\mathcal{H}^+} = 0$, so both the expansion and twist vanish at the future horizon, as they must for ℓ to be the horizon generator. [Contrast this with the Kinnersley tetrad, where $\rho_K = -(r - ia\cos\theta)^{-1}$, $\rho_K|_{\mathcal{H}^+} \neq 0$; and so ℓ_K has non-zero expansion and twist at the horizon.] Since $\ell_{\rm HH}$ is a null geodesic that generates $\mathcal{H}^+$, we can satisfy the conditions of the theorem with a perturbation h_{ab} which vanishes in the distant past or future, and which is Ricci-flat to first order, $R_{ab} = \mathcal{O}(\varepsilon^2)$. Then the ingoing radiation gauge $\ell^a_{\rm HH} h_{ab} = 0 = h$ will be horizon-locking.

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

I would like to thank Alex Lupsasca and Eric Poisson for valuable discussions. This work was supported by NSF CAREER Award PHY–2047382 and a Sloan Foundation Research Fellowship.

ORCID iD

Leo C Stein https://orcid.org/0000-0001-7559-9597

References

[1] Pound A and Wardell B 2021 arXiv:2101.04592
[2] Teukolsky S A 2015 *Class. Quantum Grav.* **32** 124006
[3] Poisson E 2004 *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press)
[4] Chrzanowski P L 1975 *Phys. Rev.* D **11** 2042–62
[5] Stewart J M 1991 *Advanced General Relativity* (*Cambridge Monographs on Mathematical Physics*) (Cambridge University Press)
[6] Price L R, Shankar K and Whiting B F 2007 *Class. Quantum Grav.* **24** 2367–88
[7] Wald R M 1984 *General Relativity* (Chicago University Press)
[8] Hollands S, Wald R M and Zhang V G 2024 arXiv:2402.00818
[9] O'Sullivan S and Hughes S A 2016 *Phys. Rev.* D **94** 044057
[10] Hawking S W and Hartle J B 1972 *Commun. Math. Phys.* **27** 283–90
[11] Seidel E and Suen W-M 1992 *Phys. Rev. Lett.* **69** 1845–8
[12] Poisson E 2004 *Phys. Rev.* D **70** 084044
[13] Teukolsky S A 1973 *Astrophys. J.* **185** 635–47
[14] Teukolsky S A and Press W H 1974 *Astrophys. J.* **193** 443–61
[15] Kinnersley W 1969 *J. Math. Phys.* **10** 1195–203
[16] Newman E and Penrose R 1962 *J. Math. Phys.* **3** 566–78
[17] Geroch R P, Held A and Penrose R 1973 *J. Math. Phys.* **14** 874–81