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BLOW-UP OF SOLUTIONS OF CRITICAL ELLIPTIC EQUATIONS IN THREE DIMENSIONS

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We describe the asymptotic behavior of positive solutions u_ε of the equation $-\Delta u + au = 3u^{5-\varepsilon}$ in $\Omega \subset \mathbb{R}^3$ with a homogeneous Dirichlet boundary condition. The function a is assumed to be critical in the sense of Hebey and Vaugon, and the functions u_ε are assumed to be an optimizing sequence for the Sobolev inequality. Under a natural nondegeneracy assumption we derive the exact rate of the blow-up and the location of the concentration point, thereby proving a conjecture of Brezis and Peletier (1989). Similar results are also obtained for solutions of the equation $-\Delta u + (a + \varepsilon V)u = 3u^5$ in Ω.

1. Introduction and main results

We are interested in the behavior of solutions to certain semilinear elliptic equations that are perturbations of the critical equation

$$-\Delta U = 3U^5 \quad \text{in } \mathbb{R}^3.$$

It is well known that all positive solutions to the latter equation are given by

$$U_{x,\lambda}(y) := \frac{\lambda^{1/2}}{(1+\lambda^2|y-x|^2)^{1/2}} \tag{1-1}$$

with parameters $x \in \mathbb{R}^3$ and $\lambda > 0$. This equation arises as the Euler–Lagrange equation of the optimization problem related to the Sobolev inequality

$$\int_{\mathbb{R}^3} |\nabla z|^2 \geq S \left(\int_{\mathbb{R}^3} z^6 \right)^{\frac{1}{3}}$$

with sharp constant [Aubin 1976; Rodemich 1966; Rosen 1971; Talenti 1976]

$$S := 3\left(\frac{\pi}{2}\right)^{\frac{4}{3}}.$$

The perturbed equations that we are interested in are posed in a bounded open set $\Omega \subset \mathbb{R}^3$ and involve a function a on Ω such that the operator $-\Delta + a$ with Dirichlet boundary conditions is coercive. (Later, we will be more precise concerning regularity assumptions on Ω and a.) One of the two families of equations also involves another rather arbitrary function V on Ω. The case where a and V are constants is also of interest.

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We consider solutions $u = u_\varepsilon$, parametrized by $\varepsilon > 0$, to the following two families of equations:

$$\begin{cases} -\Delta u + au = 3u^{5-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-2}$$

and

$$\begin{cases} -\Delta u + (a+\varepsilon V)u = 3u^{5} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-3}$$

While there are certain differences between the problems (1-2) and (1-3), the methods used to study them are similar, and we will treat both in this paper. We are interested in the behavior of the solutions u_ε as $\varepsilon \to 0$, and we assume that in this limit the solutions form a minimizing sequence for the Sobolev inequality. More precisely, for (1-3) we assume

$$\lim_{\varepsilon\to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\left(\int_\Omega u_\varepsilon^6\right)^{1/3}} = S, \tag{1-4}$$

and for (1-2) we assume

$$\lim_{\varepsilon\to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\left(\int_\Omega u_\varepsilon^{6-\varepsilon}\right)^{2/(6-\varepsilon)}} = S. \tag{1-5}$$

For example, when Ω is the unit ball, $a = -\frac{1}{4}\pi^2$, and $V = -1$, then (1-3) has a solution if and only if $0 < \varepsilon < \frac{3}{4}\pi^2$; see [Brezis and Nirenberg 1983, Section 1.2]. Note that in this case π^2 is the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions on Ω.

Returning to the general situation, the existence of solutions to (1-2) and (1-3) satisfying (1-4) and (1-5) can be proved via minimization under certain assumptions on a and V; see, e.g., [Frank et al. 2021] for (1-3). Moreover, it is not hard to prove, based on the characterization of optimizers in Sobolev's inequality, that these functions converge weakly to zero in $H_0^1(\Omega)$ and that u_ε^6 converges weakly in the sense of measures to a multiple of a delta function; see Proposition 2.2. In this sense, the functions u_ε blow up.

The problem of interest is to describe this blow-up behavior more precisely. This question was advertised in an influential paper by Brezis and Peletier [1989], who presented a detailed study of the case where Ω is a ball and a and V are constants. For earlier results on (1-2) with $a \equiv 0$, see [Atkinson and Peletier 1987; Budd 1987]. Concerning the case of general open sets $\Omega \subset \mathbb{R}^3$, the Brezis–Peletier paper contains three conjectures, the first two of which concern the blow-up behavior of solutions to the analogues of (1-2) and (1-3) in dimensions $N \geq 3$ ($N \geq 4$ for (1-3)) with $a \equiv 0$. These conjectures were proved independently in seminal works of Han [1991] and Rey [1989; 1990].

In the present paper, under a natural nondegeneracy condition, we prove the third Brezis–Peletier conjecture, which has remained open so far. It concerns the blow-up behavior of solutions of (1-2) for certain nonzero a in the three-dimensional case. We also prove the corresponding result for (1-3). This latter result is not stated explicitly as a conjecture in [Brezis and Peletier 1989], but it is contained there in spirit and could have been formulated using the same heuristics. Indeed, it is the version with $a \not\equiv 0$ of the second Brezis–Peletier conjecture in the same way as, concerning (1-2), the third conjecture is the $a \not\equiv 0$ version of the first one.

A characteristic feature of the three-dimensional case is the notion of criticality for the function a. To motivate this concept, let

$$S(a) := \inf_{0 \not\equiv z \in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla z|^2 + a z^2)}{\left(\int_\Omega z^6\right)^{1/3}}.$$

One of the findings of [Brezis and Nirenberg 1983] is that if a is small (for instance, in $L^\infty(\Omega)$) but possibly nonzero, then $S(a) = S$. This is in stark contrast to the case of dimensions $N \geq 4$, where the corresponding analogue of $S(a)$ (with the exponent 6 replaced by $2N/(N-2)$) is always strictly below the corresponding Sobolev constant, whenever a is negative somewhere.

This phenomenon leads naturally to the following definition due to [Hebey and Vaugon 2001]. A continuous function a on $\overline{\Omega}$ is said to be *critical* in Ω if $S(a) = S$ and if for any continuous function $\tilde{a}$ on $\overline{\Omega}$ with $\tilde{a} \leq a$ and $\tilde{a} \not\equiv a$ one has $S(\tilde{a}) < S(a)$. Throughout this paper we assume that a is critical in Ω.

A key role in our analysis is played by the regular part of the Green's function and its zero set. To introduce these, we follow the sign and normalization convention of [Rey 1990]. Since the operator $-\Delta + a$ in Ω with Dirichlet boundary conditions is assumed to be coercive, it has a Green's function G_a satisfying, for each fixed $y \in \Omega$,

$$\begin{cases} -\Delta_x G_a(x,y) + a(x) G_a(x,y) = 4\pi \delta_y & \text{in } \Omega, \\ G_a(\cdot, y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-6}$$

The regular part H_a of G_a is defined by

$$H_a(x,y) := \frac{1}{|x-y|} - G_a(x,y). \tag{1-7}$$

It is well known that for each $y \in \Omega$ the function $H_a(\cdot, y)$, which is originally defined in $\Omega \setminus \{y\}$, extends to a continuous function in Ω, and we abbreviate

$$\phi_a(y) := H_a(y,y).$$

It was proved by Brezis [1986] that $\inf_{y \in \Omega} \phi_a(y) < 0$ implies $S(a) < S$. The reverse implication, which was stated in [Brezis 1986] as an open problem, was proved by Druet [2002]. Hence, as a consequence of criticality we have

$$\inf_{y \in \Omega} \phi_a(y) = 0; \tag{1-8}$$

see also [Esposito 2004] and [Frank et al. 2021, Proposition 5.1] for alternative proofs. Note that (1-8) implies, in particular, that each point x with $\phi_a(x) = 0$ is a critical point of ϕ_a.

Let us summarize the setting in this paper. In the sequel we set

$$\mathcal{N}_a := \{x \in \Omega : \phi_a(x) = 0\}.$$

Assumptions 1.1. (a) $\Omega \subset \mathbb{R}^3$ is a bounded, open set with C^2 boundary.

(b) $a \in C^{0,1}(\overline{\Omega}) \cap C^{2,\sigma}_{\mathrm{loc}}(\Omega)$ for some $\sigma > 0$.

(c) a is critical in Ω.

(d) Any point in $\mathcal{N}_a$ is a nondegenerate critical point of ϕ_a, that is, for any $x_0 \in \mathcal{N}_a$, the Hessian $D^2\phi_a(x_0)$ does not have a zero eigenvalue.

Let us briefly comment on these items. Assumptions (a) and (b) are modest regularity assumptions, which can probably be further relaxed with more effort. Concerning assumption (d) we first note that $\phi_a \in C^2(\Omega)$ by Lemma 4.1, and therefore any point in $\mathcal{N}_a$ is a critical point of ϕ_a; see (1-8). We believe that assumption (d) is "generically" true. (For results in this spirit, but in the noncritical case $a \equiv 0$, see [Micheletti and Pistoia 2014].) The corresponding assumption for $a \equiv 0$ appears frequently in the literature, for instance, in [Rey 1990; del Pino et al. 2004]. Assumption (d) holds, in particular, if Ω is a ball and a is a constant, as can be verified by explicit computation.

To leading order, the blow-up behavior of solutions of (1-3) will be given by the projection of a solution (1-1) of the unperturbed whole space equation to $H_0^1(\Omega)$. For parameters $x \in \mathbb{R}^3$ and $\lambda > 0$ we introduce $PU_{x,\lambda} \in H_0^1(\Omega)$ as the unique function satisfying

$$\Delta PU_{x,\lambda} = \Delta U_{x,\lambda} \quad \text{in } \Omega, \qquad PU_{x,\lambda} = 0 \quad \text{on } \partial\Omega. \tag{1-9}$$

Moreover, let

$$T_{x,\lambda} := \operatorname{span}\{PU_{x,\lambda},\ \partial_\lambda PU_{x,\lambda},\ \partial_{x_1} PU_{x,\lambda} \partial_{x_2} PU_{x,\lambda} \partial_{x_3} PU_{x,\lambda}\},$$

and let $T_{x,\lambda}^\perp$ be the orthogonal complement of $T_{x,\lambda}$ in $H_0^1(\Omega)$ with respect to the inner product $\int_\Omega \nabla u \cdot \nabla v$. By $\Pi_{x,\lambda}$ and $\Pi_{x,\lambda}^\perp$ we denote the orthogonal projections in $H_0^1(\Omega)$ onto $T_{x,\lambda}$ and $T_{x,\lambda}^\perp$, respectively.

Here are our main results. We begin with those pertaining to (1-2), and we first provide an asymptotic expansion of u_ε with a remainder in $H_0^1(\Omega)$.

Theorem 1.2 (asymptotic expansion of u_ε). *Let (u_ε) be a family of solutions to* (1-2) *satisfying* (1-5). *Then there are sequences $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$ and $(r_\varepsilon) \subset T_{x_\varepsilon,\lambda_\varepsilon}^\perp$ such that*

$$u_\varepsilon = \alpha_\varepsilon \left(PU_{x_\varepsilon,\lambda_\varepsilon} - \lambda_\varepsilon^{-1/2} \Pi_{x_\varepsilon,\lambda_\varepsilon}^\perp (H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)) + r_\varepsilon\right) \tag{1-10}$$

and a point $x_0 \in \Omega$ with $\nabla\phi_a(x_0) = 0$ such that, along a subsequence,

$$|x_\varepsilon - x_0| = o(1), \tag{1-11}$$

$$\lim_{\varepsilon\to 0} \varepsilon\lambda_\varepsilon = \frac{32}{\pi}\phi_a(x_0), \tag{1-12}$$

$$\alpha_\varepsilon^{4-\varepsilon} = 1 + \frac{\varepsilon}{2}\log\lambda_\varepsilon + \begin{cases} \mathcal{O}(\lambda_\varepsilon^{-1}) & \text{if } \phi_a(x_0) \neq 0, \\ \frac{64}{3\pi}\phi_0(x_0)\lambda_\varepsilon^{-1} + o(\lambda_\varepsilon^{-1}) & \text{if } \phi_a(x_0) = 0, \end{cases} \tag{1-13}$$

$$\|\nabla r_\varepsilon\|_2 = \begin{cases} \mathcal{O}(\lambda_\varepsilon^{-1}) & \text{if } \phi_a(x_0) \neq 0, \\ \mathcal{O}(\lambda_\varepsilon^{-3/2}) & \text{if } \phi_a(x_0) = 0. \end{cases} \tag{1-14}$$

Moreover, if $\phi_a(x_0) = 0$, then

$$\lim_{\varepsilon\to 0} \varepsilon\lambda_\varepsilon^2 = -32a(x_0). \tag{1-15}$$

Our second main result concerns the pointwise blow-up behavior, both at the blow-up point and away from it, and, in the special case of constant a, verifies the conjecture from [Brezis and Peletier 1989] under the natural nondegeneracy assumption (d).

Theorem 1.3 (Brezis–Peletier conjecture). *Let* (u_ε) *be a family of solutions to* (1-2) *satisfying* (1-5).

(a) *The asymptotics close to the concentration point* x_0 *are given by*

$$\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_\infty^2 = \lim_{\varepsilon\to 0} \varepsilon|u_\varepsilon(x_\varepsilon)|^2 = \frac{32}{\pi}\phi_a(x_0).$$

If $\phi_a(x_0) = 0$, *then*

$$\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_\infty^4 = \lim_{\varepsilon\to 0} \varepsilon|u_\varepsilon(x_\varepsilon)|^4 = -32a(x_0). \tag{1-16}$$

(b) *The asymptotics away from the concentration point* x_0 *are given by*

$$u_\varepsilon(x) = \lambda_\varepsilon^{-1/2} G_a(x, x_0) + o(\lambda_\varepsilon^{-1/2})$$

for every fixed $x \in \Omega \setminus \{x_0\}$. *The convergence is uniform for* x *away from* x_0.

Strictly speaking, the Brezis–Peletier conjecture [1989] is stated without the criticality assumption (c) on a, but rather under the assumption $\phi_a \geq 0$ on Ω. (Note that [Brezis and Peletier 1989] uses the opposite sign convention for the regular part of the Green's function. Also, their Green's function is normalized to be $\frac{1}{4\pi}$ times ours.) The remaining case, however, is much simpler and can be proved with existing methods. Indeed, by Druet's theorem [2002], the inequality $\phi_a \geq 0$ on Ω is equivalent to $S(a) = S$, and the assumption that a is critical is equivalent to $\min \phi_a = 0$. Thus, the case of the Brezis–Peletier conjecture that is not covered by our Theorem 1.3 is when $\min \phi_a > 0$. This case can be treated in the same way as the case $a \equiv 0$ in [Han 1991; Rey 1989] (or as we treat the case $\phi_a(x_0) > 0$). Note that in this case the nondegeneracy assumption (d) is not needed. Whether this assumption can be removed in the case where $\phi_a(x_0) = 0$ is an open problem.

We note that Theorems 1.2 and 1.3 and, in particular, the asymptotics (1-15) and (1-16) hold independently of whether $a(x_0) = 0$ or not. We note that $a(x_0) \leq 0$ if $\phi_a(x_0) = 0$, as shown in [Frank et al. 2021, Corollary 2.2]. We are grateful to H. Brezis (personal communication) for raising the question of whether $a(x_0) = 0$ can happen and what the asymptotics of λ_ε and $\|u_\varepsilon\|_\infty$ would be in this case, or whether one can show that $\phi_a(x_0) = 0$ implies $a(x_0) < 0$. Deciding which alternative holds does not appear to be easy, in particular due to the nonlocal nature of $\phi_a(x_0)$. Here is a simple observation that may illustrate the expected level of difficulty: In the spirit of [Frank et al. 2021, Theorem 2.1 and Corollary 2.2], $a(x_0) < 0$ would follow if one could exhibit a family of very refined test functions $\eta_{x_0,\lambda}$ such that when $\inf_\Omega \phi_a = \phi_a(x_0) = 0$, the Sobolev quotient defining $S(a)$ satisfies $\mathcal{S}_a[\eta_{x_0,\lambda}] = S - c_1 a(x_0)\lambda^{-2} - c_2\lambda^{-\tau} + o(\lambda^{-\tau})$ for some $c_1, c_2 > 0$ and $\tau > 2$, say. However, extracting such an explicit term $c_2\lambda^{-\tau}$ is beyond the precision of both [Frank et al. 2021] and the present paper.

We also point out that the conjecture in [Brezis and Peletier 1989] is formulated with assumption (1-4) rather than (1-5). However, the latter assumption is typically used in the posterior literature dealing with problem (1-2), see, e.g., [Grossi and Pacella 2005; Han 1991], and we follow this convention.

We now turn our attention to the results for the second family of equations, namely (1-3). Whenever we deal with that problem, we impose the following additional assumptions:

Assumptions 1.4. (e) $a < 0$ in $\mathcal{N}_a$.

(f) $V \in C^{0,1}(\overline{\Omega})$.

Again, assumption (f) is a modest regularity assumption, which can probably be further relaxed with more effort. Assumption (e) is not severe, as we know from [Frank et al. 2021, Corollary 2.2] that any critical a satisfies $a \le 0$ on $\mathcal{N}_a$; see also the above discussion of the question by Brezis of whether or not this assumption is automatically satisfied. In particular, it is fulfilled if a is a negative constant.

Let

$$Q_V(x) := \int_\Omega V(y) G_a(x,y)^2, \quad x \in \Omega. \tag{1-17}$$

Again, we first provide an asymptotic expansion of u_ε with a remainder in $H_0^1(\Omega)$.

Theorem 1.5 (asymptotic expansion of u_ε). *Let (u_ε) be a family of solutions to* (1-3) *satisfying* (1-4). *Then there are sequences $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$ and $(r_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ such that*

$$u_\varepsilon = \alpha_\varepsilon (PU_{x_\varepsilon,\lambda_\varepsilon} - \lambda_\varepsilon^{-1/2} \Pi^\perp_{x_\varepsilon,\lambda_\varepsilon}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)) + r_\varepsilon) \tag{1-18}$$

and a point $x_0 \in \mathcal{N}_a$ with $Q_V(x_0) \le 0$ such that, along a subsequence,

$$|x_\varepsilon - x_0| = o(\varepsilon^{1/2}), \tag{1-19}$$

$$\phi_a(x_\varepsilon) = o(\varepsilon), \tag{1-20}$$

$$\lim_{\varepsilon\to 0} \varepsilon\lambda_\varepsilon = 4\pi^2 \frac{|a(x_0)|}{|Q_V(x_0)|}, \tag{1-21}$$

$$\alpha_\varepsilon = 1 + \frac{4}{3\pi^3} \frac{\phi_0(x_0)|Q_V(x_0)|}{|a(x_0)|}\varepsilon + o(\varepsilon), \tag{1-22}$$

$$\|\nabla r_\varepsilon\|_2 = \mathcal{O}(\varepsilon^{3/2}). \tag{1-23}$$

If $Q_V(x_0) = 0$, the right side of (1-21) *is to be interpreted as ∞.*

The following result concerns the pointwise blow-up behavior.

Theorem 1.6. *Let (u_ε) be a family of solutions to* (1-3) *satisfying* (1-4).

(a) *The asymptotics close to the concentration point x_0 are given by*

$$\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_\infty^2 = \lim_{\varepsilon\to 0} \varepsilon|u_\varepsilon(x_\varepsilon)|^2 = 4\pi^2 \frac{|a(x_0)|}{|Q_V(x_0)|}.$$

If $Q_V(x_0) = 0$, the right side is to be interpreted as ∞.

(b) *The asymptotics away from the concentration point x_0 are given by*

$$u_\varepsilon(x) = \lambda_\varepsilon^{-1/2} G_a(x,x_0) + o(\lambda_\varepsilon^{-1/2})$$

for every fixed $x \in \Omega \setminus \{x_0\}$. The convergence is uniform for x away from x_0.

Theorems 1.2 and 1.5 state that, to leading order, the solution is given by a projected bubble $PU_{x_\varepsilon,\lambda_\varepsilon}$. One of the main points of these theorems, which enters crucially in the proof of Theorems 1.3 and 1.6, is the identification of the localization length λ_ε^{-1} of the projected bubble as an explicit constant times ε (for (1-2) if $\phi_a(x_0) \neq 0$ and for (1-3) if $Q_V(x_0) < 0$) or $\varepsilon^{1/2}$ (for (1-2) if $\phi_a(x_0) = 0$ and $a(x_0) \neq 0$).

The fact that the solutions are given to leading order by a projected bubble is a rather general phenomenon, which is shared, for instance, also by the higher-dimensional generalizations of (1-2) and (1-3). In contrast to the higher-dimensional case, however, in order to compute the asymptotics of the localization length λ_ε^{-1}, we need to extract the leading order correction to the bubble. Remarkably, for both problems (1-2) and (1-3) this correction is given by $\lambda_\varepsilon^{-1/2}\Pi^\perp_{x_\varepsilon,\lambda_\varepsilon}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot))$.

In this relation it is natural to wonder whether the above projected bubble $PU_{x,\varepsilon}$ can be replaced by a different projected bubble $\widetilde{PU}_{x,\lambda}$, namely where the projection is defined with respect to the scalar product coming from the operator $-\Delta + a$, leading to

$$(-\Delta + a)\widetilde{PU}_{x,\lambda} = (-\Delta + a)U_{x,\lambda}, \quad \widetilde{PU}_{x,\lambda}|_{\partial\Omega} = 0.$$

Such a choice is probably possible and would even simplify some computations, but it would lead to additional difficulties elsewhere (for instance, in the proofs of Propositions 2.2 and 5.1 our choice allows us to apply the classical results by Bahri and Coron).

Moreover, for both problems the concentration point x_0 is shown to satisfy $\nabla\phi_a(x_0) = 0$. Here, however, we see an interesting difference between the two problems. Namely, for (1-3) we also know that $\phi_a(x_0) = 0$, whereas we know from [del Pino et al. 2004, Theorem 2(b)] that there are solutions of (1-2) concentrating at any critical point of ϕ_a which is not necessarily in $\mathcal{N}_a$. (These solutions also satisfy (1-4).)

An asymptotic expansion very similar to that in Theorem 1.5 is proved in [Frank et al. 2021] for energy-minimizing solutions of (1-3); see also [Frank et al. 2020] for the simpler higher-dimensional case. There, we did not assume the nondegeneracy of $D^2\phi_a(x_0)$, but we did assume that $Q_V < 0$ in $\mathcal{N}_a$. Moreover, in the energy minimizing setting we showed that x_0 satisfies

$$\frac{Q_V(x_0)^2}{|a(x_0)|} = \sup_{x\in\mathcal{N}_a,\, Q_V(x)<0} \frac{Q_V(x)^2}{|a(x)|},$$

but this cannot be expected in the more general setting of the present paper.

Before describing the technical challenges that we overcome in our proofs, let us put our work into perspective. In the past three decades there has been an enormous literature on blow-up phenomena of solutions to semilinear equations with critical exponent, which is impossible to summarize. We mention here only a few recent works from which, we hope, a more complete bibliography can be reconstructed. In some sense, the situation in the present paper is the simplest blow-up situation, as it concerns single bubble blow-up of positive solutions in the interior. Much more refined blow-up scenarios have been studied, including, for instance, multibubbling, sign-changing solutions or concentration on the boundary under Neumann boundary conditions. For an introduction we refer to [Druet et al. 2004; Hebey 2014]. In this paper we are interested in the description of the behavior of a given family of solutions. For the converse problem of constructing blow-up solutions in our setting, see [Musso and Salazar 2018; del Pino et al. 2004], and for a survey of related results, see [Pistoia 2013] and references therein. Obstructions to the existence of solutions in three dimensions were studied in [Druet and Laurain 2010]. The spectrum near zero of the linearization of solutions was studied in [Choi et al. 2016; Grossi and Pacella 2005]. There are also connections to the question of compactness of solutions; see [Brendle and Marques 2009; Khuri et al. 2009].

What makes the critical case in three dimensions significantly harder than the higher-dimensional analogues solved by Han [1991] and Rey [1989; 1990] is a certain cancellation, which is related to the fact that $\inf \phi_a = 0$. Thus, the term that in higher dimensions completely determines the blow-up vanishes in our case. Our way around this impasse is to iteratively improve our knowledge about the functions u_ε. The mechanism behind this iteration is a certain coercivity inequality, due to Esposito [2004], which we state in Lemma 2.3, and a crucial feature of our proof is to apply this inequality repeatedly, at different orders of precision. To arrive at the level of precision stated in Theorems 1.2 and 1.5, two iterations are necessary (plus a zeroth one, hidden in the proof of Proposition 2.2).

The first iteration, contained in Sections 2 and 5, is relatively standard and follows Rey's ideas [1990] with some adaptions due to Esposito [2004] to the critical case in three dimensions. The two main outcomes of the first iteration are that concentration occurs in the interior, and an order-sharp bound in H_0^1 on the remainder $\alpha_\varepsilon^{-1} u_\varepsilon - PU_{x_\varepsilon,\lambda_\varepsilon}$.

The second iteration, contained in Sections 3 and 6, is more specific to the problem at hand. Its main outcome is the extraction of the subleading correction

$$\lambda_\varepsilon^{-1/2} \Pi^{\perp}_{x_\varepsilon,\lambda_\varepsilon}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)).$$

Using the nondegeneracy of $D^2\phi_a(x_0)$ we will be able to show in the proofs of Theorems 1.2 and 1.5 that λ_ε is proportional to ε^{-1} (for (1-2) if $\phi_a(x_0) \neq 0$ and for (1-3) if $Q_V(x_0) < 0$) or $\varepsilon^{-1/2}$ (for (1-2) if $\phi_a(x_0) = 0$ and $a(x_0) \neq 0$).

The arguments described so far are, for the most part, carried out in H_0^1 norm. Once one has completed the two iterations, we apply in Sections 4C and 7B a Moser iteration argument in order to show that the remainder $\alpha_\varepsilon^{-1} u_\varepsilon - PU_{x_\varepsilon,\lambda_\varepsilon}$ is negligible also in L^∞ norm. This will then allow us to deduce Theorems 1.3 and 1.6.

As we mentioned before, Theorem 1.5 is the generalization of the corresponding theorem in [Frank et al. 2021] for energy-minimizing solutions. In that previous paper, we also used a similar iteration technique. Within each iteration step, however, minimality played an important role, and we used the iterative knowledge to further expand the energy functional evaluated at a minimizer. There is no analogue of this procedure in the current paper. Instead, as in most other works in this area, starting with [Brezis and Peletier 1989], Pohozaev identities now play an important role. These identities were not used in [Frank et al. 2021]. In fact, in that paper we did not use (1-3) at all and our results there are valid as well for a certain class of "almost minimizers".

There are five types of Pohozaev-type identities corresponding, in some sense, to the five linearly independent functions in the kernel of the Hessian at an optimizer of the Sobolev inequality on $\mathbb{R}^3$ (resulting from its invariance under multiplication by constants, by dilations and by translations). All five identities will be used to control the five parameters α_ε, λ_ε and x_ε in (1-10) and (1-18), which precisely correspond to the five asymptotic invariances. In fact, all five of these identities are used in the first iteration and then again in the second iteration. (To be more precise, in the first iteration in the proof of Theorem 1.5 it is more economical to only use four identities, since the information from the fifth identity is not particularly useful at this stage, due to the above mentioned cancellation $\phi_a(x_0) = 0$.)

Thinking of the five Pohozaev-type identities as coming from the asymptotic invariances is useful, but it is an oversimplification. Indeed, there are several possible choices for the multipliers in each category, for instance, u, $PU_{x,\lambda}$, $\psi_{x,\lambda}$ corresponding to multiplication by constants, $y \cdot \nabla u$, $\partial_\lambda PU_{x,\lambda}$, $\partial_\lambda \psi_{x,\lambda}$ corresponding to dilations, and $\partial_{x_j} u$, $\nabla_{x_j} PU_{x,\lambda}$, $\nabla_{x_j} PU_{x,\lambda}$ corresponding to translations. (Here $\psi_{x,\lambda}$ is a modified bubble defined below in (3-1).) The choice of the multiplier is subtle and depends on the available knowledge at the moment of applying the identity and the desired precision of the outcome. In any case, the upshot is that these identities can be brought together in such a way that they give the final result of Theorems 1.2 and 1.5 concerning the expansion in $H^1_0(\Omega)$. As mentioned before, the desired pointwise bounds in Theorems 1.3 and 1.6 then follow in a relatively straightforward way using a Moser iteration.

We believe that our techniques are robust enough to derive blow-up asymptotics for (1-2) and (1-3) in more general situations containing a nonzero weak limit and/or multiple concentration points. Since our main motivation was to solve the Brezis–Peletier conjecture stated for single blow-up [1989] and to limit the amount of calculations needed, we do not attempt to pursue this further here.

Let us also mention that a problem similar to, but different from, (1-2) has been studied in the recent article [Malchiodi and Mayer 2021] using a similar approach. While the analysis there, carried out on a Riemannian manifold M of dimension $n \geq 5$, is rather comprehensive and also treats the case of multiple blow-up points, it does not seem to contain an analogue of the vanishing phenomenon for $\phi_a(x_0)$ nor, as a consequence, of our refined iteration step described above.

The structure of this paper is as follows. The first part of the paper, consisting of Sections 2, 3 and 4, is devoted to problem (1-3), while the second part, consisting of Sections 5, 6 and 7, is devoted to (1-2). The two parts are presented in a parallel manner, but the emphasis in the second part is on the necessary changes compared to the first part. The preliminary Sections 2 and 5 contain an initial expansion, the subsequent Sections 3 and 6 contain its refinement and, finally, in Sections 4 and 7 the main theorems presented in this introduction are proved. Some technical results are deferred to two appendices.

2. Additive case: a first expansion

In this and the following section we will prepare for the proof of Theorems 1.5 and 1.6.

The main result from this section is the following preliminary asymptotic expansion of the family of solutions (u_ε).

Proposition 2.1. *Let* (u_ε) *be a family of solutions to* (1-3) *satisfying* (1-4). *Then, up to extraction of a subsequence, there are sequences* $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$ *and* $(w_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ *such that*

$$u_\varepsilon = \alpha_\varepsilon (PU_{x_\varepsilon,\lambda_\varepsilon} + w_\varepsilon) \tag{2-1}$$

and a point $x_0 \in \Omega$ *such that*

$$|x_\varepsilon - x_0| = o(1), \quad \alpha_\varepsilon = 1 + o(1), \quad \lambda_\varepsilon \to \infty, \quad \|\nabla w_\varepsilon\|_2 = \mathcal{O}(\lambda^{-1/2}). \tag{2-2}$$

This proposition follows to a large extent by an adaptation of existing results in the literature. We include the proof since we have not found the precise statement and since related arguments will appear in the following section in a more complicated setting.

An initial qualitative expansion follows from works of Struwe [1984] and Bahri and Coron [1988]. In order to obtain the statement of Proposition 2.1, we then need to show two things, namely, the bound on $\|\nabla w\|$ and the fact that $x_0 \in \Omega$. The proof of the bound on $\|\nabla w\|$ that we give is rather close to that of Esposito [2004]. The setting in [Esposito 2004] is slightly different (there, V is equal to a negative constant and, more importantly, the solutions are assumed to be energy minimizing), but this part of the proof extends to our setting. On the other hand, the proof in [Esposito 2004] of the fact that $x_0 \in \Omega$ relies on the energy-minimizing property and does not work for us. Instead, we adapt some ideas from Rey [1990]. The proof in [Rey 1990] is only carried out in dimensions ≥ 4 and without the background a, but, as we will see, it extends with some effort to our situation.

We subdivide the proof of Proposition 2.1 into a sequence of subsections. The main result of each subsection is stated as a proposition at the beginning and summarizes the content of the corresponding subsection.

2A. ***A qualitative initial expansion.*** As a first important step, we derive the following expansion, which is already of the form of that in Proposition 2.1 except that all remainder bounds are nonquantitative and the limit point x_0 may a priori be on the boundary $\partial\Omega$.

Proposition 2.2. *Let* (u_ε) *be a family of solutions to* (1-3) *satisfying* (1-4)*. Then, up to extraction of a subsequence, there are sequences* $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$ *and* $(w_\varepsilon) \subset T^{\perp}_{x_\varepsilon,\lambda_\varepsilon}$ *such that* (2-1) *holds and a point* $x_0 \in \overline{\Omega}$ *such that*

$$|x_\varepsilon - x_0| = o(1), \quad \alpha_\varepsilon = 1 + o(1), \quad d_\varepsilon \lambda_\varepsilon \to \infty, \quad \|\nabla w_\varepsilon\|_2 = o(1), \tag{2-3}$$

where we write $d_\varepsilon := d(x_\varepsilon, \partial\Omega)$.

Proof. We shall only prove that $u_\varepsilon \rightharpoonup 0$ in $H^1_0(\Omega)$. Once this is shown, we can use standard arguments, due to Lions [1985], Struwe [1984] and Bahri and Coron [1988], to complete the proof of the proposition; see, for instance, [Rey 1990, Proof of Proposition 2].

Step 1: We begin by showing that (u_ε) is bounded in $H^1_0(\Omega)$ and that $\|u_\varepsilon\|_6 \gtrsim 1$. Integrating the equation for u_ε against u_ε, we obtain

$$\int_\Omega (|\nabla u_\varepsilon|^2 + (a - \varepsilon V) u_\varepsilon^2) = 3 \int_\Omega u_\varepsilon^6, \tag{2-4}$$

and therefore

$$3\left(\int_\Omega u_\varepsilon^6\right)^{\frac{2}{3}} = \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\left(\int_\Omega u_\varepsilon^6\right)^{1/3}} + \frac{\int_\Omega (a + \varepsilon V) u_\varepsilon^2}{\left(\int_\Omega u_\varepsilon^6\right)^{1/3}}.$$

On the right side, the first quotient converges by (1-4) and the second quotient is bounded by Hölder's inequality. Thus, (u_ε) is bounded in $L^6(\Omega)$. By (1-4) we obtain boundedness in $H^1_0(\Omega)$. By coercivity of $-\Delta + a$ in $H_0^1(\Omega)$ and Sobolev's inequality, for all sufficiently small $\varepsilon > 0$, the left side in (2-4) is bounded from below by a constant times $\|u_\varepsilon\|_6^2$. This yields the lower bound on $\|u_\varepsilon\|_6 \gtrsim 1$.

Step 2: According to Step 1, (u_ε) has a weak limit point in $H_0^1(\Omega)$ and we denote by u_0 one of those. Our goal is to show that $u_0 \equiv 0$. Throughout this step, we restrict ourselves to a subsequence of ε's along which $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(\Omega)$. By Rellich's lemma, after passing to a subsequence, we may also assume that $u_\varepsilon \to u_0$ almost everywhere. Moreover, passing to a further subsequence, we may also assume that $\|\nabla u_\varepsilon\|$ has a limit. Then, by (1-4), $\|u_\varepsilon\|_6$ has a limit as well and, by Step 1, none of these limits is zero.

We now argue as in the proof of [Frank et al. 2021, Proposition 3.1] and note that, by weak convergence,

$$\mathcal{T} = \lim_{\varepsilon \to 0} \int_\Omega |\nabla(u_\varepsilon - u_0)|^2 \text{ exists and satisfies } \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega |\nabla u_0|^2 + \mathcal{T}$$

and, by the Brezis–Lieb lemma [Brezis and Lieb 1983],

$$\mathcal{M} = \lim_{\varepsilon \to 0} \int_\Omega (u_\varepsilon - u_0)^6 \text{ exists and satisfies } \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon^6 = \int_\Omega u_0^6 + \mathcal{M}.$$

Thus, (1-4) gives

$$S\left(\int_\Omega u_0^6 + \mathcal{M}\right)^{\frac{1}{3}} = \int_\Omega |\nabla u_0|^2 + \mathcal{T}.$$

We bound the left side from above with the help of the elementary inequality

$$\left(\int_\Omega u_0^6 + \mathcal{M}\right)^{\frac{1}{3}} \le \left(\int_\Omega u_0^6\right)^{\frac{1}{3}} + \mathcal{M}^{1/3},$$

and, by the Sobolev inequality for $u_\varepsilon - u_0$, we bound the right side from below using

$$\mathcal{T} \ge S\mathcal{M}^{1/3}.$$

Thus,

$$S\left(\int_\Omega u_0^6\right)^{\frac{1}{3}} \ge \int_\Omega |\nabla u_0|^2.$$

Thus, either $u_0 \equiv 0$ or u_0 is an optimizer for the Sobolev inequality. Since u_0 has support in $\Omega \subsetneq \mathbb{R}^3$, the latter is impossible and we conclude that $u_0 \equiv 0$, as claimed. □

Convention. Throughout the rest of the paper, we assume that the sequence (u_ε) satisfies the assumptions and conclusions from Proposition 2.2. We will make no explicit mention of subsequences. Moreover, we typically drop the index ε from u_ε, α_ε, x_ε, λ_ε, d_ε and w_ε.

2B. *Coercivity.*

The following coercivity inequality from [Esposito 2004, Lemma 2.2] is a crucial tool for us in subsequently refining the expansion of u_ε. It states, roughly speaking, that the subleading error terms coming from the expansion of u_ε can be absorbed into the leading term, at least under some orthogonality condition.

Lemma 2.3. *There are constants $T_* < \infty$ and $\rho > 0$ such that, for all $x \in \Omega$, all $\lambda > 0$ with $d\lambda \geq T_*$ and all $v \in T_{x,\lambda}^{\perp}$,*

$$\int_\Omega (|\nabla v|^2 + av^2 - 15U_{x,\lambda}^4 v^2) \geq \rho \int_\Omega |\nabla v|^2. \tag{2-5}$$

The proof proceeds by compactness, using the inequality [Rey 1990, (D.1)]

$$\int_\Omega (|\nabla v|^2 - 15U_{x,\lambda}^4 v^2) \geq \frac{4}{7} \int_\Omega |\nabla v|^2 \quad \text{for all } v \in T_{x,\lambda}^{\perp}.$$

For details of the proof, we refer to [Esposito 2004].

In the following subsection, we use Lemma 2.3 to deduce a refined bound on $\|\nabla w\|_2$. We will use it again in Section 3B below to obtain improved bounds on the refined error term $\|\nabla r\|_2$, with $r \in T_{x,\lambda}^{\perp}$ defined in (3-4).

2C. ***The bound on $\|\nabla w\|_2$.*** The goal of this subsection is to prove:

Proposition 2.4. *As $\varepsilon \to 0$,*

$$\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2}) + \mathcal{O}((\lambda d)^{-1}). \tag{2-6}$$

Using this bound, in Section 2D we prove that $d^{-1} = \mathcal{O}(1)$ and therefore the bound in Proposition 2.4 becomes $\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2})$, as claimed in Proposition 2.1.

Proof. The starting point is the equation satisfied by w. Since

$$-\Delta P U_{x,\lambda} = -\Delta U_{x,\lambda} = 3U_{x,\lambda}^5,$$

from (2-1) and (1-3) we obtain

$$(-\Delta + a)w = -3U_{x,\lambda}^5 + 3\alpha^4 (PU_{x,\lambda} + w)^5 - (a + \varepsilon V) PU_{x,\lambda} - \varepsilon V w. \tag{2-7}$$

Integrating this equation against w and using

$$\int_\Omega U_{x,\lambda}^5 w = \frac{1}{3} \int_\Omega \nabla PU_{x,\lambda} \cdot \nabla w = 0,$$

we get

$$\int_\Omega (|\nabla w|^2 + aw^2) = 3\alpha^4 \int_\Omega (PU_{x,\lambda} + w)^5 w - \int_\Omega (a + \varepsilon V) PU_{x,\lambda} w - \int_\Omega \varepsilon V w^2. \tag{2-8}$$

We estimate the three terms on the right-hand side separately.

The second and third terms are easy: We have by Lemma A.1

$$\left| \int_\Omega (a + \varepsilon V) PU_{x,\lambda} w \right| \lesssim \|w\|_6 \|U_{x,\lambda}\|_{6/5} \lesssim \lambda^{-1/2} \|\nabla w\|_2.$$

Moreover,

$$\left| \int_\Omega \varepsilon V w^2 \right| \lesssim \varepsilon \|w\|_6^2 = o(\|\nabla w\|_2^2).$$

The first term on the right side of (2-8) needs a bit more care. We write $PU_{x,\lambda} = U_{x,\lambda} - \varphi_{x,\lambda}$ as in Lemma A.2 and expand

$$\begin{aligned}\int_\Omega (PU_{x,\lambda}+w)^5 w &= \int_\Omega U_{x,\lambda}^5 w + 5\int_\Omega U_{x,\lambda}^4 w^2 + \mathcal{O}\Big(\int_\Omega (U_{x,\lambda}^4\varphi_{x,\lambda}|w| + U_{x,\lambda}^3(|w|^3+|w|\varphi_{x,\lambda}^2) + \varphi_{x,\lambda}^5|w| + w^6)\Big)\\ &= 5\int_\Omega U_{x,\lambda}^4 w^2 + \mathcal{O}\Big(\int_\Omega U_{x,\lambda}^4\varphi_{x,\lambda}|w| + \|\nabla w\|_2\|\varphi_{x,\lambda}\|_6^2 + \|\nabla w\|_2^3\Big),\end{aligned}$$

where we again used $\int_\Omega U_{x,\lambda}^5 w = 0$. By Lemmas A.1 and A.2, we have $\|\varphi_{x,\lambda}\|_6^2 \lesssim (d\lambda)^{-1}$ and

$$\int_\Omega U_{x,\lambda}^4\varphi_{x,\lambda}|w| \lesssim \|w\|_6\|\varphi_{x,\lambda}\|_\infty\|U_{x,\lambda}\|_{24/5}^4 \lesssim \|\nabla w\|_2(d\lambda)^{-1}.$$

Putting all the estimates together, we deduce from (2-8) that

$$\int_\Omega (|\nabla w|^2 + aw^2 - 15\alpha^4U^4w^2) = \mathcal{O}((d\lambda)^{-1}\|\nabla w\|_2 + \lambda^{-1/2}\|\nabla w\|_2) + o(\|\nabla w\|_2^2).$$

Due to the coercivity inequality from Lemma 2.3, the left side is bounded from below by a positive constant times $\|\nabla w\|_2^2$. Thus, (2-6) follows. ☐

2D. *Excluding boundary concentration.* The goal of this subsection is to prove:

Proposition 2.5. $$d^{-1} = \mathcal{O}(1).$$

By integrating the equation for u against ∇u, one obtains the Pohozaev-type identity

$$-\int_\Omega (\nabla(a+\varepsilon V))u^2 = \int_{\partial\Omega} n\Big(\frac{\partial u}{\partial n}\Big)^2. \tag{2-9}$$

Inserting the decomposition $u = \alpha(PU + w)$, we get

$$\int_{\partial\Omega} n\Big(\frac{\partial PU_{x,\lambda}}{\partial n}\Big)^2 = -\int_{\partial\Omega} n\Big(2\frac{\partial PU_{x,\lambda}}{\partial n}\frac{\partial w}{\partial n} + \Big(\frac{\partial w}{\partial n}\Big)^2\Big) - \int_\Omega (\nabla(a+\varepsilon V))(PU_{x,\lambda}+w)^2. \tag{2-10}$$

Since $a, V \in C^1(\overline{\Omega})$, the volume integral is bounded by

$$\Big|\int_\Omega (\nabla(a+\varepsilon V))(PU_{x,\lambda}+w)^2\Big| \lesssim \|PU_{x,\lambda}\|_2^2 + \|w\|_2^2 \lesssim \lambda^{-1} + (\lambda d)^{-2}, \tag{2-11}$$

where we used (2-6) and Lemmas A.1 and A.2.

The function $\partial PU_{x,\lambda}/\partial n$ on the boundary is discussed in Lemma A.3. We now control the function $\partial w/\partial n$ on the boundary.

Lemma 2.6. $$\int_{\partial\Omega}\Big(\frac{\partial w}{\partial n}\Big)^2 = \mathcal{O}(\lambda^{-1}d^{-1}) + o(\lambda^{-1}d^{-2}).$$

Proof. The following proof is analogous to [Rey 1990, Appendix C]. It relies on the inequality

$$\Big\|\frac{\partial z}{\partial n}\Big\|_{L^2(\partial\Omega)}^2 \lesssim \|\Delta z\|_{L^{3/2}(\Omega)}^2 \quad \text{for all } z \in H^2(\Omega)\cap H_0^1(\Omega). \tag{2-12}$$

This inequality is well known and is contained in [Rey 1990, Appendix C]. A proof can be found, for instance, in [Hang et al. 2009].

We write (2-7) for w as $-\Delta w = F$ with

$$F := 3\alpha^4 (PU_{x,\lambda} + w)^5 - 3U_{x,\lambda}^5 - (a + \varepsilon V)(PU_{x,\lambda} + w). \tag{2-13}$$

We fix a smooth $0 \le \chi \le 1$ with $\chi \equiv 0$ on $\{|y| \le \frac{1}{2}\}$ and $\chi \equiv 1$ on $\{|y| \ge 1\}$ and define the cut-off function

$$\zeta(y) := \chi\left(\frac{y-x}{d}\right). \tag{2-14}$$

Then $\zeta w \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$-\Delta(\zeta w) = \zeta F - 2\nabla\zeta \cdot \nabla w - (\Delta\zeta) w.$$

The function F satisfies the simple pointwise bound

$$|F| \lesssim U_{x,\lambda}^5 + |w|^5 + U_{x,\lambda} + |w|, \tag{2-15}$$

which, when combined with inequality (2-12), yields

$$\begin{aligned} \left\|\frac{\partial w}{\partial n}\right\|_{L^2(\partial\Omega)}^2 = \left\|\frac{\partial(\zeta w)}{\partial n}\right\|_{L^2(\partial\Omega)}^2 &\lesssim \|\zeta F - 2\nabla\zeta\cdot\nabla w - (\Delta\zeta)w\|_{3/2}^2 \\ &\lesssim \|\zeta(U_{x,\lambda}^5 + |w|^5 + U_{x,\lambda} + |w|)\|_{3/2}^2 + \||\nabla\zeta|\,|\nabla w|\|_{3/2}^2 + \|(\Delta\zeta)w\|_{3/2}^2. \end{aligned}$$

It remains to bound the norms on the right side. The term most difficult to estimate is $\|\zeta w^5\|_{3/2}$, because $5 \cdot \frac{3}{2} = \frac{15}{2} > 6$, and we shall come back to it later. The other terms can all be estimated using bounds on $\|U\|_{L^p(\Omega\setminus B_{d/2}(x))}$ from Lemma A.1, as well as the bound $\|w\|_6 \lesssim \lambda^{-1/2} + \lambda^{-1}d^{-1}$ from Proposition 2.4. Indeed, we have

$$\begin{aligned} \|\zeta U_{x,\lambda}^5\|_{3/2}^2 &\lesssim \|U_{x,\lambda}\|_{L^{15/2}(\Omega\setminus B_{d/2}(x))}^{10} \lesssim \lambda^{-5}d^{-6} = o(\lambda^{-1}d^{-2}), \\ \|\zeta U_{x,\lambda}\|_{3/2}^2 &\lesssim \|U_{x,\lambda}\|_{L^{3/2}(\Omega\setminus B_d)}^2 \lesssim \lambda^{-1} = \mathcal{O}(\lambda^{-1}d^{-1}), \\ \|\zeta w\|_{3/2}^2 &\lesssim \|w\|_6^2 \lesssim \lambda^{-1} + \lambda^{-2}d^{-2} = \mathcal{O}(\lambda^{-1}d^{-1}) + o(\lambda^{-1}d^{-2}), \\ \||\nabla\zeta|\,|\nabla w|\|_{3/2}^2 &\lesssim \|\nabla w\|_2^2 \|\nabla\zeta\|_6^2 \lesssim (\lambda^{-1} + \lambda^{-2}d^{-2})d^{-1} = \mathcal{O}(\lambda^{-1}d^{-1}) + o(\lambda^{-1}d^{-2}) \end{aligned}$$

and

$$\|(\Delta\zeta)w\|_{3/2}^2 \lesssim \|w\|_6^2 \|\Delta\zeta\|_2^2 \lesssim (\lambda^{-1} + \lambda^{-2}d^{-2})d^{-1} = \mathcal{O}(\lambda^{-1}d^{-1}) + o(\lambda^{-1}d^{-2}).$$

In order to estimate the difficult term $\|\zeta w^5\|_{3/2}$, we multiply the equation $-\Delta w = F$ by $\zeta^{1/2}|w|^{1/2}w$ and integrate over Ω to obtain

$$\int_\Omega \nabla(\zeta^{1/2}|w|^{1/2}w) \cdot \nabla w \le \int_\Omega |F|\zeta^{1/2}|w|^{3/2}. \tag{2-16}$$

We now note that there are universal constants $c > 0$ and $C < \infty$ such that, pointwise a.e.,

$$\nabla(\zeta^{1/2}|w|^{1/2}w) \cdot \nabla w \ge c|\nabla(\zeta^{1/4}|w|^{1/4}w)|^2 - C|w|^{5/2}|\nabla(\zeta^{1/4})|^2. \tag{2-17}$$

Indeed, by repeated use of the product rule and chain rule for Sobolev functions, one finds

$$\nabla(\zeta^{1/2}|w|^{1/2}w)\cdot\nabla w = \tfrac{3}{2}\big(\tfrac{4}{5}\big)^2|\nabla(\zeta^{1/4}|w|^{1/4}w)|^2 + \big(\tfrac{3}{2}\big(\tfrac{4}{5}\big)^2 - \tfrac{4}{5}\cdot 2\big)|w|^{5/2}|\nabla(\zeta^{1/4})|^2 - \big(\tfrac{3}{2}\big(\tfrac{4}{5}\big)^2\cdot 2 - \tfrac{4}{5}\cdot 2\big)|w|^{1/4}w\nabla(\zeta^{1/4})\cdot\nabla(\zeta^{1/4}|w|^{1/4}w).$$

The claimed inequality (2-17) follows by applying Schwarz's inequality $v_1\cdot v_2 \geq -\varepsilon|v_1|^2 - |v_2|^2/(4\varepsilon)$ to the cross term on the right side with $\varepsilon > 0$ small enough.

As a consequence of (2-17), we can bound the left side in (2-16) from below by

$$\int_\Omega \nabla(\zeta^{1/2}|w|^{1/2}w)\cdot\nabla w \geq c\int_\Omega |\nabla(\zeta^{1/4}|w|^{1/4}w)|^2 - C\int_\Omega |w|^{5/2}|\nabla(\zeta^{1/4})|^2.$$

Thus, by the Sobolev inequality for the function $\zeta^{1/4}|w|^{1/4}w$ and (2-16), we get

$$\|\zeta w^5\|_{3/2}^2 = \bigg(\int_\Omega \big|\zeta^{1/4}|w|^{1/4}w\big|^6\bigg)^{\frac{4}{3}} \lesssim \bigg(\int_\Omega |\nabla(\zeta^{1/4}|w|^{1/4}w)|^2\bigg)^4 \lesssim \bigg(\int_\Omega |w|^{5/2}|\nabla(\zeta^{1/4})|^2\bigg)^4 + \bigg(\int_\Omega |F|\zeta^{1/2}|w|^{3/2}\bigg)^4. \tag{2-18}$$

For the first term on the right side, we have

$$\bigg(\int_\Omega |w|^{5/2}|\nabla(\zeta^{1/4})|^2\bigg)^4 \leq \|w\|_6^{10}\bigg(\int_\Omega |\nabla(\zeta^{1/4})|^{24/7}\bigg)^{\frac{7}{3}} \lesssim (\lambda^{-5}+\lambda^{-10}d^{-10})d^{-1} = \mathcal{O}(\lambda^{-1}d^{-1}) + o(\lambda^{-1}d^{-2}).$$

To control the second term on the right side of (2-18), we use again the pointwise estimate (2-15). The contribution of the $|w|^5$ term to the second term on the right side of (2-18) is

$$\bigg(\int_\Omega |w|^{5+3/2}\zeta^{1/2}\bigg)^4 = \bigg(\int_\Omega (\zeta^{1/2}w^{5/2})w^4\bigg)^4 \leq \|\zeta w^5\|_{3/2}^2\|w\|_6^{16} = o(\|\zeta w^5\|_{3/2}^2),$$

which can be absorbed into the left side of (2-18).

For the remaining terms, we have

$$\bigg(\int_\Omega |w|^{3/2}U_{x,\lambda}^5\zeta^{1/2}\bigg)^4 \lesssim \|w\|_6^6\|U_{x,\lambda}\|_{L^{20/3}(\Omega\setminus B_{d/2}(x))}^{20} = (\lambda^{-3}+(d\lambda)^{-6})(\lambda^{-10}d^{-11}),$$

$$\bigg(\int_\Omega |w|^{3/2}U_{x,\lambda}\zeta^{1/2}\bigg)^4 \lesssim \|w\|_6^6\|U_{x,\lambda}\|_{L^{4/3}(\Omega)}^4 = (\lambda^{-3}+(d\lambda)^{-6})\lambda^{-2},$$

$$\bigg(\int_\Omega |w|^{5/2}\zeta^{1/2}\bigg)^4 \lesssim \|w\|_6^{10} = \lambda^{-5}+(d\lambda)^{-10},$$

all of which is $\mathcal{O}(\lambda^{-1}d^{-1}) + o(\lambda^{-1}d^{-2})$. This concludes the proof of the bound

$$\|\zeta w^5\|_{3/2}^2 = \mathcal{O}(\lambda^{-1}d^{-1}) + o(\lambda^{-1}d^{-2})$$

and thus of Lemma 2.6. □

It is now easy to complete the proof of the main result of this section.

Proof of Proposition 2.5. The identity (2-10), together with the bound (2-11) and Lemma A.3(a), yields

$$C\lambda^{-1}\nabla\phi_0(x) = \mathcal{O}(\lambda^{-1}) + o(\lambda^{-1}d^{-2}) + \mathcal{O}\left(\left\|\frac{\partial PU_{x,\lambda}}{\partial n}\right\|_{L^2(\partial\Omega)}\left\|\frac{\partial w}{\partial n}\right\|_{L^2(\partial\Omega)} + \left\|\frac{\partial w}{\partial n}\right\|^2_{L^2(\partial\Omega)}\right)$$

for some $C > 0$. By Lemmas A.3(c) and 2.6, the last term on the right-hand side is bounded by $\lambda^{-1}d^{-3/2} + o(\lambda^{-1}d^{-2})$, so we get

$$\nabla\phi_0(x) = \mathcal{O}(d^{-3/2}) + o(d^{-2}).$$

On the other hand, according to [Rey 1990, (2.9)], we have $|\nabla\phi_0(x)| \gtrsim d^{-2}$. Hence

$$d^{-2} = \mathcal{O}(d^{-3/2}) + o(d^{-2}),$$

which yields $d^{-1} = \mathcal{O}(1)$, as claimed. □

2E. *Proof of Proposition 2.1.* Existence of the expansion follows from Proposition 2.2. Proposition 2.5 implies that $d^{-1} = \mathcal{O}(1)$, which implies that $x_0 \in \Omega$. Moreover, inserting the bound $d^{-1} = \mathcal{O}(1)$ into Proposition 2.4, we obtain $\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2})$, as claimed in Proposition 2.1. This completes the proof of the proposition. □

3. Additive case: refining the expansion

Our goal in this section is to improve the decomposition given in Proposition 2.1. As in [Frank et al. 2021], our goal is to discover that a better approximation to u_ε is given by the function

$$\psi_{x,\lambda} := PU_{x,\lambda} - \lambda^{-1/2}(H_a(x,\cdot) - H_0(x,\cdot)). \tag{3-1}$$

Let us set

$$q_\varepsilon := w_\varepsilon + \lambda_\varepsilon^{-1/2}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)), \tag{3-2}$$

so that

$$u_\varepsilon = \alpha_\varepsilon(\psi_{x_\varepsilon,\lambda_\varepsilon} + q_\varepsilon).$$

As in [Frank et al. 2021], we further decompose

$$q_\varepsilon = s_\varepsilon + r_\varepsilon, \tag{3-3}$$

with $s_\varepsilon \in T_{x_\varepsilon,\lambda_\varepsilon}$ and $r_\varepsilon \in T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ given by

$$r_\varepsilon := \Pi^\perp_{x_\varepsilon,\lambda_\varepsilon} q \quad \text{and} \quad s_\varepsilon := \Pi_{x_\varepsilon,\lambda_\varepsilon} q. \tag{3-4}$$

We note that the notation r_ε is consistent with that used in Theorem 1.5 since, using $w_\varepsilon \in T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ where we write $w_\varepsilon = q_\varepsilon + \lambda_\varepsilon^{-1/2}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot))$, we have

$$s_\varepsilon = \lambda_\varepsilon^{-1/2}\Pi_{x_\varepsilon,\lambda_\varepsilon}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)). \tag{3-5}$$

The following proposition summarizes the results of this section.

Proposition 3.1. *Let (u_ε) be a family of solutions to* (1-3) *satisfying* (1-4). *Then, up to extraction of a subsequence, there are sequences $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0, \infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$, $(s_\varepsilon) \subset T_{x_\varepsilon,\lambda_\varepsilon}$ and $(r_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ such that*

$$u_\varepsilon = \alpha_\varepsilon(\psi_{x_\varepsilon,\lambda_\varepsilon} + s_\varepsilon + r_\varepsilon) \tag{3-6}$$

and a point $x_0 \in \Omega$ such that, in addition to Proposition 2.1,

$$\begin{aligned} \|\nabla r_\varepsilon\|_2 &= \mathcal{O}(\varepsilon\lambda_\varepsilon^{-1/2}), \\ \phi_a(x_\varepsilon) &= a(x_\varepsilon)\pi\lambda_\varepsilon^{-1} - \frac{\varepsilon}{4\pi} Q_V(x_\varepsilon) + o(\lambda_\varepsilon^{-1}) + o(\varepsilon), \\ \nabla\phi_a(x_\varepsilon) &= \mathcal{O}(\varepsilon^\mu) \quad \text{for any } \mu < 1, \\ \lambda_\varepsilon^{-1} &= \mathcal{O}(\varepsilon), \\ \alpha_\varepsilon^4 &= 1 + \frac{64}{3\pi}\phi_0(x_\varepsilon)\lambda_\varepsilon^{-1} + \mathcal{O}(\varepsilon\lambda_\varepsilon^{-1}). \end{aligned} \tag{3-7}$$

The expansion of $\phi_a(x)$ will be of great importance also in the final step of the proof of Theorem 1.5. Indeed, by using the bound on $|\nabla\phi_a(x)|$ we will show that in fact $\phi_a(x) = o(\lambda^{-1}) + o(\varepsilon)$. This allows us to determine $\lim_{\varepsilon\to 0} \varepsilon\lambda_\varepsilon$.

We prove Proposition 3.1 in the following subsections. Again the strategy is to expand suitable energy functionals.

3A. *Bounds on s.*

In this section we record bounds on the function s introduced in (3-4) and on the coefficients β, γ and δ_j defined by the decomposition

$$s = \Pi_{x,\lambda} q =: \lambda^{-1}\beta PU_{x,\lambda} + \gamma\partial_\lambda PU_{x,\lambda} + \lambda^{-3}\sum_{i=1}^{3} \delta_i \partial_{x_i} PU_{x,\lambda}. \tag{3-8}$$

Since $PU_{x,\lambda}$, $\partial_\lambda PU_{x,\lambda}$ and $\partial_{x_i} PU_{x,\lambda}$, $i = 1, 2, 3$, are linearly independent for sufficiently small ε, the numbers β, γ and δ_i, $i = 1, 2, 3$, (depending on ε, of course) are uniquely determined. The choice of the different powers of λ multiplying these coefficients is motivated by the following proposition.

Proposition 3.2. *The coefficients appearing in* (3-8) *satisfy*

$$\beta,\ \gamma,\ \delta_i = \mathcal{O}(1). \tag{3-9}$$

Moreover, we have the bounds

$$\|s\|_\infty = \mathcal{O}(\lambda^{-1/2}), \quad \|\nabla s\|_2 = \mathcal{O}(\lambda^{-1}) \quad \text{and} \quad \|s\|_2 = \mathcal{O}(\lambda^{-3/2}), \tag{3-10}$$

as well as

$$\|\nabla s\|_{L^2(\Omega\setminus B_{d/2}(x))} = \mathcal{O}(\lambda^{-3/2}). \tag{3-11}$$

Proof. Because of (3-5), s_ε depends on u_ε only through the parameters λ and x. Since these parameters satisfy the same properties $\lambda \to \infty$ and $d^{-1} = \mathcal{O}(1)$ as in [Frank et al. 2021], the results on s_ε there are applicable. In particular, the bound (3-9) follows from [Frank et al. 2021, Lemma 6.1].

The bounds stated in (3-10) follow readily from (3-8) and (3-9), together with the corresponding bounds on the basis functions $PU_{x,\lambda}$, $\partial_\lambda PU_{x,\lambda}$ and $\partial_{x_i} PU_{x,\lambda}$, $i = 1, 2, 3$, which come from

$$\|U_{x,\lambda}\|_\infty \lesssim \lambda^{1/2}, \quad \|\nabla U_{x,\lambda}\|_2 \lesssim 1, \quad \|U_{x,\lambda}\|_2 \lesssim \lambda^{-1/2},$$

and similar bounds on $\partial_\lambda U_{x,\lambda}$ and $\partial_{x_i} U_{x,\lambda}$, compare Lemma A.1, as well as

$$\|H_0(x,\cdot)\|_2 + \|\nabla_x H_0(x,\cdot)\|_2 + \|\nabla_x \nabla_y H_0(x,y)\|_2 \lesssim 1.$$

It remains to prove (3-11). Again by (3-8) and (3-9), it suffices to show that

$$\lambda^{-1}\|\nabla PU_{x,\lambda}\|_{L^2(\Omega\setminus B_{d/2}(x))} + \|\nabla \partial_\lambda PU_{x,\lambda}\|_{L^2(\Omega\setminus B_{d/2}(x))} + \lambda^{-3}\|\nabla \partial_{x_i} PU_{x,\lambda}\|_{L^2(\Omega\setminus B_{d/2}(x))} \lesssim \lambda^{-3/2}. \quad (3\text{-}12)$$

(In fact, there is a better bound on $\nabla \partial_{x_i} PU_{x,\lambda}$, but we do not need this.) Since the three bounds in (3-12) are all proved similarly, we only prove the second one.

By integration by parts, we have

$$\int_{\Omega\setminus B_{d/2}(x)} |\nabla \partial_\lambda PU_{x,\lambda}|^2 = 15\int_{\Omega\setminus B_{d/2}(x)} U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} \partial_\lambda PU_{x,\lambda} + \int_{\partial B_{d/2}(x)} \frac{\partial(\partial_\lambda PU_{x,\lambda})}{\partial n} \partial_\lambda PU_{x,\lambda}.$$

By the bounds from Lemmas A.1 and A.2, the volume integral is estimated by

$$\int_{\Omega\setminus B_{d/2}(x)} U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} \partial_\lambda PU_{x,\lambda} \le \int_{\mathbb{R}^3\setminus B_{d/2}(x)} U_{x,\lambda}^4 (\partial_\lambda U_{x,\lambda})^2 + \|\partial_\lambda \varphi_{x,\lambda}\|_\infty \int_{\mathbb{R}^3\setminus B_{d/2}(x)} U_{x,\lambda}^4 |\partial_\lambda U_{x,\lambda}| \lesssim \lambda^{-5}.$$

Since

$$\nabla \partial_\lambda U_{x,\lambda}(y) = \frac{\lambda^{3/2}}{2} \frac{(-5 + 3\lambda^2|y-x|^2)(y-x)}{(1+\lambda^2|y-x|^2)^{5/2}},$$

we find $|\nabla \partial_\lambda U_{x,\lambda}| \lesssim \lambda^{-3/2}$ on $\partial B_{d/2}(x)$. By the mean value formula for the harmonic function $\partial_\lambda \varphi_{x,\lambda}$ and the bound from Lemma A.2,

$$|\nabla \partial_\lambda \varphi_{x,\lambda}(y)| = \|\partial_\lambda \varphi_{x,\lambda}\|_\infty \lesssim \lambda^{-3/2} \quad \text{for all } y \in \partial B_{d/2}(x).$$

This implies that $|\nabla(\partial_\lambda PU_{x,\lambda})| \lesssim \lambda^{-3/2}$ on $\partial B_{d/2}(x)$. Thus, the boundary integral is estimated by

$$\int_{\partial B_{d/2}(x)} \frac{\partial(\partial_\lambda PU_{x,\lambda})}{\partial n} \partial_\lambda PU_{x,\lambda} = \|\nabla(\partial_\lambda PU_{x,\lambda})\|_{L^\infty(\partial B_{d/2}(x))} (\|\partial_\lambda U_{x,\lambda}\|_{L^\infty(\Omega\setminus B_{d/2}(x))} + \|\partial_\lambda \varphi_{x,\lambda}\|_\infty) \lesssim \lambda^{-3},$$

since $\|\partial_\lambda U_{x,\lambda}\|_{L^\infty(\Omega\setminus B_{d/2}(x))} \lesssim \lambda^{-3/2}$ by Lemma A.1. Collecting these estimates, we find that

$$\|\nabla \partial_\lambda PU_{x,\lambda}\|_{L^2(\Omega\setminus B_{d/2}(x))} \lesssim \lambda^{-3/2},$$

which is the second bound in (3-12). □

Later we will also need the leading order behavior of the zero-mode coefficients β and γ in (3-8).

Proposition 3.3. *As* $\varepsilon \to 0$,

$$\beta = \frac{16}{3\pi}(\phi_a(x) - \phi_0(x)) + \mathcal{O}(\lambda^{-1}), \quad \gamma = -\frac{8}{5}\beta + \mathcal{O}(\lambda^{-1}). \tag{3-13}$$

Proof. According to (3-5), we have

$$\int_\Omega \nabla s \cdot \nabla PU_{x,\lambda} = \lambda^{-1/2} \int_\Omega \nabla(H_a(x,\cdot) - H_0(x,\cdot)) \cdot \nabla PU_{x,\lambda}, \tag{3-14}$$

$$\int_\Omega \nabla s \cdot \nabla \partial_\lambda PU_{x,\lambda} = \lambda^{-1/2} \int_\Omega \nabla(H_a(x,\cdot) - H_0(x,\cdot)) \nabla \partial_\lambda PU_{x,\lambda}. \tag{3-15}$$

By (3-8), the left side of (3-14) is

$$\beta\lambda^{-1} \int_\Omega |\nabla PU_{x,\lambda}|^2 + \gamma \int_\Omega \nabla \partial_\lambda PU_{x,\lambda} \cdot \nabla PU_{x,\lambda} + \lambda^{-3} \sum_{i=1}^3 \delta_i \int_\Omega \nabla \partial_{x_i} PU_{x,\lambda} \cdot \nabla PU_{x,\lambda} = 3\beta\lambda^{-1}\frac{\pi^2}{4} + \mathcal{O}(\lambda^{-2}),$$

where we used the facts that, by [Rey 1990, Appendix B],

$$\int_\Omega |\nabla PU_{x,\lambda}|^2 = \frac{3\pi^2}{4} + \mathcal{O}(\lambda^{-1}), \quad \int_\Omega \nabla \partial_\lambda PU_{x,\lambda} \cdot \nabla PU_{x,\lambda} = \mathcal{O}(\lambda^{-2}),$$
$$\int_\Omega \nabla \partial_{x_i} PU_{x,\lambda} \cdot \nabla PU_{x,\lambda} = \mathcal{O}(\lambda^{-1}). \tag{3-16}$$

On the other hand, the right side of (3-14) is

$$\begin{aligned}\lambda^{-1/2} \int_\Omega \nabla(H_a(x,\cdot) - H_0(x,\cdot)) \cdot \nabla PU &= 3\lambda^{-1/2} \int_\Omega (H_a(x,\cdot) - H_0(x,\cdot)) U_{x,\lambda}^5 \\ &= 4\pi(\phi_a(x) - \phi_0(x))\lambda^{-1} + \mathcal{O}(\lambda^{-2})\end{aligned} \tag{3-17}$$

by Lemma B.3. Comparing both sides yields the expansion of β stated in (3-13).

Similarly, by (3-8), the left side of (3-15) is

$$\frac{\beta}{\lambda^2} \int_\Omega \nabla PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} + \gamma \int_\Omega |\nabla \partial_\lambda PU_{x,\lambda}|^2 + \lambda^{-3} \sum_{i=1}^3 \delta_i \int_\Omega \nabla \partial_{x_i} PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} = \frac{15\pi^2\gamma}{64\lambda^2} + \mathcal{O}(\lambda^{-3}),$$

where, besides (3-16), we used $\int_\Omega \nabla \partial_{x_i} PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} = \mathcal{O}(\lambda^{-2})$ by [Rey 1990, Appendix B] and

$$\int_\Omega |\nabla \partial_\lambda PU_{x,\lambda}|^2 = \int_\Omega |\nabla \partial_\lambda U_{x,\lambda}|^2 + \mathcal{O}(\lambda^{-3}) = \frac{15\pi^2}{64}\lambda^{-2} + \mathcal{O}(\lambda^{-3}).$$

(The numerical value comes from an explicit evaluation of the integral in terms of beta functions, which we omit.) On the other hand, the right side of (3-15) is

$$\begin{aligned}\lambda^{-1/2} \int_\Omega \nabla(H_a(x,\cdot) - H_0(x,\cdot)) \cdot \nabla \partial_\lambda PU_{x,\lambda} &= 15\lambda^{-1/2} \int_\Omega (H_a(x,\cdot) - H_0(x,\cdot)) U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} \\ &= -2\pi(\phi_a(x) - \phi_0(x))\lambda^{-2} + \mathcal{O}(\lambda^{-3})\end{aligned}$$

by Lemma B.3. Comparing both sides yields the expansion of γ stated in (3-13). ☐

3B. ***The bound on*** $\|\nabla r\|_2$**.** The goal of this subsection is to prove:

Proposition 3.4. *As* $\varepsilon \to 0$,

$$\|\nabla r\|_2 = \mathcal{O}(\phi_a(x)\lambda^{-1}) + \mathcal{O}(\lambda^{-3/2}) + \mathcal{O}(\varepsilon\lambda^{-1/2}). \tag{3-18}$$

Using $\Delta(H_a(x,\cdot) - H_0(x,\cdot)) = -aG_a(x,\cdot)$ and introducing the function $g_{x,\lambda}$ from (A-4), we see that (2-7) for w implies

$$(-\Delta + a)r = -3U_{x,\lambda}^5 + 3\alpha^4(\psi_{x,\lambda} + s + r)^5 + a(f_{x,\lambda} + g_{x,\lambda}) - as - \varepsilon V(\psi_{x,\lambda} + s + r) + \Delta s. \tag{3-19}$$

Integrating against r and using the orthogonality conditions

$$\int_\Omega (\Delta s)r = -\int_\Omega \nabla s \cdot \nabla r = 0 \quad \text{and} \quad 3\int_\Omega U_{x,\lambda}^5 r = \int_\Omega \nabla PU_{x,\lambda} \cdot \nabla r = 0,$$

we obtain

$$\int_\Omega (|\nabla r|^2 + ar^2) = 3\alpha^4 \int_\Omega (\psi_{x,\lambda} + s + r)^5 r - \int_\Omega a(s - f_{x,\lambda} - g_{x,\lambda})r - \int_\Omega \varepsilon V(\psi_{x,\lambda} + s + r)r. \tag{3-20}$$

The terms appearing in (3-20) satisfy the following bounds.

Lemma 3.5. *As* $\varepsilon \to 0$, *the following hold*:

(a) $\left|3\alpha^4 \int_\Omega (\psi_{x,\lambda} + s + r)^5 r - 15\alpha^4 \int_\Omega U_{x,\lambda}^4 r^2\right| \lesssim (\lambda^{-3/2} + \lambda^{-1}\phi_a(x) + \|r\|_6^2)\|r\|_6$.

(b) $\left|\int_\Omega (a(s - f_{x,\lambda} - g_{x,\lambda}) + \varepsilon V(\psi_{x,\lambda} + s + r))r\right| \lesssim (\lambda^{-3/2} + \varepsilon\lambda^{-1/2})\|r\|_6$.

Proof. (a) We write $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) - f_{x,\lambda}$ and bound pointwise

$$(\psi_{x,\lambda} + s + r)^5 = U_{x,\lambda}^5 + 5U_{x,\lambda}^4(s + r) + \mathcal{O}(U_{x,\lambda}^4(\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|) + U_{x,\lambda}^3(r^2 + s^2)) \\ + \mathcal{O}(\lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5 + |r|^5 + |s|^5). \tag{3-21}$$

When integrated against r, the first term vanishes by orthogonality. Let us bound the contribution coming from the second term, that is, from $5U_{x,\lambda}^4 s$. We write

$$s = \lambda^{-1}\beta U_{x,\lambda} + \gamma\partial_\lambda U_{x,\lambda} + \tilde{s},$$

so $\tilde{s}$ consists of the zero-mode contributions involving the δ_i, plus contributions from the difference between $PU_{x,\lambda}$ and $U_{x,\lambda}$ in the terms involving β and γ. By orthogonality, we have

$$\int_\Omega U_{x,\lambda}^4 sr = \int_\Omega U_{x,\lambda}^4 \tilde{s}r = \mathcal{O}(\|U_{x,\lambda}\|_6^4 \|\tilde{s}\|_6 \|r\|_6),$$

and, by Lemmas A.1 and A.2 as well as Proposition 3.2,

$$\|\tilde{s}\|_6 \le (|\beta| + |\gamma|)(\lambda^{-1}\|\varphi_{x,\lambda}\|_6 + \|\partial_\lambda \varphi_{x,\lambda}\|_6) + \lambda^{-3}\sum_{i=1}^3 |\delta_i| \|\partial_{x_i} PU_{x,\lambda}\|_6 \lesssim \lambda^{-3/2}.$$

This proves

$$\int_\Omega U_{x,\lambda}^4 sr = \mathcal{O}(\lambda^{-3/2}\|r\|_6). \tag{3-22}$$

It remains to bound the remainder terms in (3-21). We write $H_a(x,y) = \phi_a(x) + \mathcal{O}(|x-y|)$ and bound

$$\int_\Omega U_{x,\lambda}^{24/5} |H_a(x,\cdot)|^{6/5} \lesssim \phi_a(x)^{6/5} \int_\Omega U_{x,\lambda}^{24/5} + \int_\Omega U_{x,\lambda}^{24/5} |x-y|^{6/5} \lesssim \lambda^{-3/5} \phi_a(x)^{6/5} + \lambda^{-9/5}.$$

Hence

$$\begin{aligned} \left| \int_\Omega U_{x,\lambda}^4 (\lambda^{-1/2} |H_a(x,\cdot)| + |f_{x,\lambda}|) |r| \right| &\le (\lambda^{-1/2} \| U_{x,\lambda}^4 H_a(x,\cdot) \|_{6/5} + \| U_{x,\lambda}^4 \|_{6/5} \| f_{x,\lambda} \|_\infty) \|r\|_6 \\ &\lesssim (\lambda^{-1} \phi_a(x) + \lambda^{-2}) \|r\|_6. \end{aligned} \tag{3-23}$$

Finally, using Proposition 3.2,

$$\begin{aligned} &\int_\Omega U_{x,\lambda}^3 (r^2 + s^2)|r| + \int_\Omega (\lambda^{-5/2} |H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5 + |r|^5 + |s|^5)|r| \\ &\qquad \lesssim (\|r\|_6^2 + \|s\|_6^2 + \lambda^{-5/2} + \|f_{x,\lambda}\|_\infty^5 + \|r\|_6^5 + \|s\|_6^5) \|r\|_6 \lesssim (\|r\|_6^2 + \lambda^{-2}) \|r\|_6. \end{aligned}$$

(b) We have

$$\begin{aligned} &\left| \int_\Omega (a(s - f_{x,\lambda} - g_{x,\lambda}) + \varepsilon V(\psi_{x,\lambda} + s + r)) r \right| \\ &\qquad \lesssim (\|s\|_{6/5} + \|f_{x,\lambda}\|_{6/5} + \|g_{x,\lambda}\|_{6/5} + \varepsilon \|\psi_{x,\lambda}\|_{6/5} + \varepsilon \|r\|_{6/5}) \|r\|_6. \end{aligned}$$

By Proposition 3.2, $\|s\|_{6/5} \lesssim \|s\|_2 \lesssim \lambda^{-3/2}$. By Lemma A.2, $\|f_{x,\lambda}\|_{6/5} \lesssim \|f_{x,\lambda}\|_\infty \lesssim \lambda^{-5/2}$. By Lemma A.4, $\|g_{x,\lambda}\|_{6/5} \lesssim \lambda^{-2}$. By Lemmas A.1 and A.2, $\|\psi_{x,\lambda}\|_{6/5} \lesssim \lambda^{-1/2}$. Finally, $\|r\|_{6/5} \lesssim \|r\|_6$. This proves the claimed bound. □

Proof of Proposition 3.4. We deduce from identity (3-20) together with Lemma 3.5 that

$$\int_\Omega (|\nabla r|^2 + ar^2 - 15\alpha^4 U_{x,\lambda}^4 r^2) \lesssim (\lambda^{-1} \phi_a(x) + \lambda^{-3/2} + \varepsilon \lambda^{-1/2} + \|\nabla r\|_2^2 + \varepsilon \|\nabla r\|_2) \|\nabla r\|_2.$$

Since $\alpha^4 \to 1$ and $r \in T_{x,\lambda}^\perp$, the coercivity inequality (2-5) implies that for all sufficiently small $\varepsilon > 0$ the left side is bounded from below by $c\|\nabla r\|_2^2$ with a universal constant $c > 0$. Thus,

$$\|\nabla r\|_2 \lesssim \lambda^{-1} \phi_a(x) + \lambda^{-3/2} + \varepsilon \lambda^{-1/2} + \|\nabla r\|_2^2 + \varepsilon \|\nabla r\|_2.$$

For all sufficiently small $\varepsilon > 0$, the last two terms on the right side can be absorbed into the left side and we obtain the claimed inequality (3-18). □

Proposition 3.4 is a first step to prove the bound (3-7) in Proposition 3.1. In Section 3D we will show that $\phi_a(x) = \mathcal{O}(\lambda^{-1} + \varepsilon)$ and $\lambda^{-1} = \mathcal{O}(\varepsilon)$. Combining these with Proposition 3.4 we will obtain (3-7).

3C. *Expanding α^4*. In this subsection, we will prove:

Proposition 3.6. *As $\varepsilon \to 0$,*

$$\alpha^4 = 1 - 4\beta\lambda^{-1} + \mathcal{O}(\phi_a(x)\lambda^{-1} + \lambda^{-2} + \varepsilon\lambda^{-1}), \tag{3-24}$$

where β is the zero-mode coefficient from (3-8).

To prove (3-24), we expand the energy identity obtained by integrating the equation for u against u. Writing $u = \alpha(\psi_{x,\lambda} + q)$, this yields

$$\int_\Omega |\nabla(\psi_{x,\lambda} + q)|^2 + \int_\Omega (a + \varepsilon V)(\psi_{x,\lambda} + q)^2 = 3\alpha^4 \int_\Omega (\psi_{x,\lambda} + q)^6,$$

which we write as

$$\int_\Omega (|\nabla\psi_{x,\lambda}|^2 + (a+\varepsilon V)\psi_{x,\lambda}^2 - 3\alpha^4\psi_{x,\lambda}^6) + 2\int_\Omega (\nabla q\cdot\nabla\psi_{x,\lambda} + (a+\varepsilon V)q\psi_{x,\lambda} - 9\alpha^4 q\psi_{x,\lambda}^5) = \mathcal{R}_0, \quad (3\text{-}25)$$

with

$$\mathcal{R}_0 := -\int_\Omega (|\nabla q|^2 + (a + \varepsilon V)q^2) + 3\alpha^4 \sum_{k=2}^{6} \binom{6}{k} \int_\Omega \psi_{x,\lambda}^{6-k} q^k.$$

The following lemma provides the expansions of the terms in (3-25).

Lemma 3.7. *As* $\varepsilon \to 0$, *the following hold*:

(a) $\displaystyle\int_\Omega (|\nabla\psi_{x,\lambda}|^2 + (a + \varepsilon V)\psi_{x,\lambda}^2 - 3\alpha^4\psi_{x,\lambda}^6) = (1 - \alpha^4)\frac{3\pi^2}{4} + \mathcal{O}(\phi_a(x)\lambda^{-1} + \lambda^{-2} + \varepsilon\lambda^{-1}).$

(b) $\displaystyle\int_\Omega (\nabla q \cdot \nabla\psi_{x,\lambda} + (a + \varepsilon V)q\psi_{x,\lambda} - 9\alpha^4 q\psi_{x,\lambda}^5) = (1 - 3\alpha^4)\frac{3\pi^2}{4}\beta\lambda^{-1} + \mathcal{O}(\lambda^{-2} + \varepsilon^2\lambda^{-1}).$

(c) $\mathcal{R}_0 = \mathcal{O}(\lambda^{-2} + \varepsilon^2\lambda^{-1}).$

Proof. (a) In [Frank et al. 2021, Theorem 2.1], we have shown the expansions

$$\int_\Omega (|\nabla\psi_{x,\lambda}|^2 + (a + \varepsilon V)\psi_{x,\lambda}^2) = \frac{3\pi^2}{4} + \mathcal{O}(\phi_a(x)\lambda^{-1} + \lambda^{-2} + \varepsilon\lambda^{-1}),$$

$$3\int_\Omega \psi_{x,\lambda}^6 = \frac{3\pi^2}{4} + \mathcal{O}(\phi_a(x)\lambda^{-1} + \lambda^{-2}),$$

which immediately imply the bound in (a).

(b) Since $\Delta(H_a(x,\cdot) - H_0(x,\cdot)) = -aG_a(x,\cdot)$, we have $-\Delta\psi_{x,\lambda} = 3U_{x,\lambda}^5 - \lambda^{-1/2}aG_a(x,\cdot)$. Since $\psi_{x,\lambda} = \lambda^{-1/2}G_a(x,\cdot) - f_{x,\lambda} - g_{x,\lambda}$ with $g_{x,\lambda}$ from (A-4), we can rewrite this as

$$-\Delta\psi_{x,\lambda} + a\psi_{x,\lambda} = 3U_{x,\lambda}^5 - a(f_{x,\lambda} + g_{x,\lambda}). \quad (3\text{-}26)$$

Thus,

$$\int_\Omega (\nabla q \cdot \nabla\psi_{x,\lambda} + (a + \varepsilon V)q\psi_{x,\lambda} - 9\alpha^4 q\psi_{x,\lambda}^5)$$
$$= 3(1 - 3\alpha^4)\int_\Omega qU_{x,\lambda}^5 - \int_\Omega q(9\alpha^4(\psi_{x,\lambda}^5 - U_{x,\lambda}^5) + a(f_{x,\lambda} + g_{x,\lambda}) + \varepsilon V\psi_{x,\lambda}).$$

By orthogonality and the computations in the proof of Proposition 3.3,

$$3\int_\Omega qU_{x,\lambda}^5 = \int_\Omega \nabla s \cdot \nabla PU_{x,\lambda} = \frac{3\pi^2}{4}\beta\lambda^{-1} + \mathcal{O}(\lambda^{-2}).$$

Moreover,

$$\left|\int_\Omega q(9\alpha^4(\psi_{x,\lambda}^5 - U_{x,\lambda}^5) + a(f_{x,\lambda} + g_{x,\lambda}) + \varepsilon V\psi_{x,\lambda})\right| \\ \lesssim \|q\|_6(\|\psi_{x,\lambda}^5 - U_{x,\lambda}^5\|_{6/5} + \|f_{x,\lambda}\|_{6/5} + \|g_{x,\lambda}\|_{6/5} + \varepsilon\|\psi_{x,\lambda}\|_{6/5}).$$

By Propositions 3.2 and 3.4 we have

$$\|q\|_6 \lesssim \|\nabla q\|_2 \lesssim \lambda^{-1} + \varepsilon\lambda^{-1/2}, \tag{3-27}$$

by Lemma A.2 we have $\|f_{x,\lambda}\|_\infty \lesssim \lambda^{-5/2}$ and, by Lemma A.4 we have $\|g_{x,\lambda}\|_{6/5} \lesssim \lambda^{-2}$. Moreover, writing $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2} H_a(x,\cdot) - f_{x,\lambda}$ and using Lemmas A.1 and A.2 and (B-1), we get $\|\psi_{x,\lambda}\|_{6/5} \lesssim \lambda^{-1/2}$. Also, bounding

$$|\psi_{x,\lambda}^5 - U_{x,\lambda}^5| \lesssim \psi_{x,\lambda}^4(\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|) + \lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5,$$

we obtain from Lemmas A.1 and A.2 and (B-1)

$$\|\psi_{x,\lambda}^5 - U_{x,\lambda}^5\|_{6/5} \lesssim \lambda^{-1/2}\|\psi_{x,\lambda}\|_{24/5}^4 + \lambda^{-5/2} \lesssim \lambda^{-1}.$$

Collecting all the terms, we obtain the claimed bound.

(c) Because of the second inequality in (3-27), the first integral in the definition of $\mathcal{R}_0$ is $\mathcal{O}(\lambda^{-2} + \varepsilon^2\lambda^{-1})$. The second integral is bounded, in absolute value, by a constant times

$$\int_\Omega (\psi_{x,\lambda}^4 q^2 + q^6) \le \|\psi_{x,\lambda}\|_6^4 \|q\|_6^2 + \|q\|_6^6 \lesssim \lambda^{-2} + \varepsilon^2\lambda^{-1}.$$

This completes the proof. □

Proof of Proposition 3.6. The claim follows from (3-25) and Lemma 3.7. □

3D. *Expanding $\phi_a(x)$.*

In this subsection we prove the following important expansion.

Proposition 3.8. *As $\varepsilon \to 0$,*

$$\phi_a(x) = \pi a(x)\lambda^{-1} - \frac{\varepsilon}{4\pi} Q_V(x) + o(\lambda^{-1}) + o(\varepsilon) \tag{3-28}$$

Before proving it, let us note the following consequence.

Corollary 3.9. *We have $\phi_a(x_0) = 0$, $Q_V(x_0) \le 0$ and*

$$\lambda^{-1} = \mathcal{O}(\varepsilon), \tag{3-29}$$

as $\varepsilon \to 0$. Moreover, $\|\nabla r\|_2 = \mathcal{O}(\varepsilon\lambda^{-1/2})$ and $\alpha^4 = 1 + \frac{64}{3\pi}\phi_0(x)\lambda^{-1} + \mathcal{O}(\varepsilon\lambda^{-1})$.

Proof. The fact that $\phi_a(x_0) = 0$ follows immediately from (3-28). Since $\phi_a(x) \ge 0$ by criticality and since $a(x_0) < 0$ by assumption, we deduce from (3-28) that $Q_V(x_0) \le 0$ and that

$$\lambda^{-1} \le \frac{|Q_V(x_0)| + o(1)}{4\pi^2|a(x_0)| + o(1)}\varepsilon = \mathcal{O}(\varepsilon).$$

Reinserting this into (3-28), we find $\phi_a(x) = \mathcal{O}(\varepsilon)$. Inserting this into Proposition 3.4, we obtain the claimed bound on $\|\nabla r\|_2$, and inserting it into (3-24) and (3-13), we obtain the claimed expansion of α^4. □

The proof of (3-28) is based on the Pohozaev identity obtained by integrating the equation for u against $\partial_\lambda \psi_{x,\lambda}$. We write the resulting equality in the form

$$\begin{aligned}\int_\Omega (\nabla\psi_{x,\lambda}\cdot\nabla\partial_\lambda\psi_{x,\lambda} + (a+\varepsilon V)\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda} - 3\alpha^4\psi_{x,\lambda}^5\partial_\lambda\psi_{x,\lambda})\\ = -\int_\Omega (\nabla q\cdot\nabla\partial_\lambda\psi_{x,\lambda} + aq\partial_\lambda\psi_{x,\lambda} - 15\alpha^4 q\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda}) + 30\alpha^4\int_\Omega q^2\psi_{x,\lambda}^3\partial_\lambda\psi_{x,\lambda} + \mathcal{R}, \quad (3\text{-}30)\end{aligned}$$

with

$$\mathcal{R} = -\varepsilon\int_\Omega Vq\partial_\lambda\psi_{x,\lambda} + 3\alpha^4\sum_{k=3}^{5}\binom{5}{k}\int_\Omega \psi_{x,\lambda}^{5-k}q^k\partial_\lambda\psi_{x,\lambda}.$$

The involved terms can be expanded as follows.

Lemma 3.10. *As $\varepsilon\to 0$, the following hold*:

(a) $$\begin{aligned}\int_\Omega (\nabla\psi_{x,\lambda}\cdot\nabla\partial_\lambda\psi_{x,\lambda} + (a+\varepsilon V)\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda} - 3\alpha^4\psi_{x,\lambda}^5\partial_\lambda\psi_{x,\lambda})\\ = -2\pi\phi_a(x)\lambda^{-2} - \tfrac{1}{2}Q_V(x)\varepsilon\lambda^{-2} + (1-\alpha^4)4\pi\phi_a(x)\lambda^{-2} + (2\pi^2 a(x) + 15\pi^2\phi_a(x)^2)\lambda^{-3}\\ + o(\lambda^{-3}) + o(\varepsilon\lambda^{-2}).\end{aligned}$$

(b) $$\begin{aligned}\int_\Omega (\nabla q\cdot\nabla\partial_\lambda\psi_{x,\lambda} + aq\partial_\lambda\psi_{x,\lambda} - 15\alpha^4 q\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda})\\ = -(1-\alpha^4)2\pi(\phi_a(x)-\phi_0(x))\lambda^{-2} + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\varepsilon\lambda^{-2}) + o(\lambda^{-3}).\end{aligned}$$

(c) $30\alpha^4\int_\Omega q^2\psi_{x,\lambda}^3\partial_\lambda\psi_{x,\lambda} = \dfrac{15\pi^2}{16}\beta\gamma\lambda^{-3} + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\varepsilon\lambda^{-2}) + o(\lambda^{-3})$.

(d) $\mathcal{R} = \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\varepsilon\lambda^{-2}) + o(\lambda^{-3})$.

We emphasize that the proof of Lemma 3.10 is independent of the expansion of α^4 in (3-24). We only use the fact that $\alpha = 1 + o(1)$.

Proof. (a) Because of (3-26), the quantity of interest can be written as

$$\begin{aligned}\int_\Omega (\nabla\psi_{x,\lambda}\cdot\nabla\partial_\lambda\psi_{x,\lambda} + (a+\varepsilon V)\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda} - 3\alpha^4\psi_{x,\lambda}^5\partial_\lambda\psi_{x,\lambda})\\ = 3\int_\Omega (U_{x,\lambda}^5 - \alpha^4\psi_{x,\lambda}^5)\partial_\lambda\psi_{x,\lambda} - \int_\Omega a(f_{x,\lambda}+g_{x,\lambda})\partial_\lambda\psi_{x,\lambda} + \varepsilon\int_\Omega V\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda}. \quad (3\text{-}31)\end{aligned}$$

We discuss the three integrals on the right side separately. As a general rule, terms involving $f_{x,\lambda}$ will be negligible as a consequence of the bounds $\|f_{x,\lambda}\|_\infty = \mathcal{O}(\lambda^{-5/2})$ and $\|\partial_\lambda f_{x,\lambda}\|_\infty = \mathcal{O}(\lambda^{-7/2})$ in Lemma A.2. This will not always be carried out in detail.

We have

$$\int_\Omega (U_{x,\lambda}^5-\alpha^4\psi_{x,\lambda}^5)\partial_\lambda\psi_{x,\lambda}=(1-\alpha^4)\int_\Omega U_{x,\lambda}^5\partial_\lambda\psi_{x,\lambda}+\alpha^4\int_\Omega (U_{x,\lambda}^5-\psi_{x,\lambda}^5)\partial_\lambda\psi_{x,\lambda}. \tag{3-32}$$

The first integral is, since $\psi_{x,\lambda}=U_{x,\lambda}-\lambda^{-1/2}H_a(x,\cdot)-f_{x,\lambda}$,

$$\int_\Omega U_{x,\lambda}^5\partial_\lambda\psi_{x,\lambda}=\int_\Omega U_{x,\lambda}^5\partial_\lambda U_{x,\lambda}+\frac{1}{2}\lambda^{-3/2}\int_\Omega U_{x,\lambda}^5 H_a(x,\cdot)+\mathcal{O}(\lambda^{-4}). \tag{3-33}$$

Since $\int_{\mathbb{R}^3} U_{x,\lambda}^5\partial_\lambda U_{x,\lambda}=\frac{1}{6}\partial_\lambda\int_{\mathbb{R}^3}U_{x,\lambda}^6=0$, we have

$$\left|\int_\Omega U_{x,\lambda}^5\partial_\lambda U_{x,\lambda}\right|=\left|\int_{\mathbb{R}^3\setminus\Omega} U_{x,\lambda}^5\partial_\lambda U_{x,\lambda}\right|\lesssim\lambda^{-1}\int_{d\lambda}^\infty\left|\frac{r^2-r^4}{(1+r^2)^4}\right|dr=\mathcal{O}(\lambda^{-4}). \tag{3-34}$$

Next, by Lemma B.3,

$$\frac{1}{2}\lambda^{-3/2}\int_\Omega U_{x,\lambda}^5 H_a(x,\cdot)=\frac{2\pi}{3}\phi_a(x)\lambda^{-2}+\mathcal{O}(\lambda^{-3}).$$

This completes our discussion of the first term on the right side of (3-32). For the second term we have similarly,

$$\begin{aligned}
&\int_\Omega (U_{x,\lambda}^5-\psi_{x,\lambda}^5)\partial_\lambda\psi_{x,\lambda}\\
&=\int_\Omega (U_{x,\lambda}^5-(U_{x,\lambda}-\lambda^{-1/2}H_a(x,\cdot))^5)\partial_\lambda(U_{x,\lambda}-\lambda^{-1/2}H_a(x,\cdot))+o(\lambda^{-3})\\
&=5\lambda^{-1/2}\int_\Omega U_{x,\lambda}^4 H_a(x,\cdot)\partial_\lambda U_{x,\lambda}+\frac{5}{2}\lambda^{-2}\int_\Omega U_{x,\lambda}^4 H_a(x,\cdot)^2-10\lambda^{-1}\int_\Omega U_{x,\lambda}^3 H_a(x,\cdot)^2\partial_\lambda U_{x,\lambda}\\
&\quad+\sum_{k=3}^5\binom{5}{k}(-1)^k\lambda^{-k/2}\int_\Omega U_{x,\lambda}^{5-k}H_a(x,\cdot)^k\partial_\lambda U_{x,\lambda}\\
&\qquad-\frac{1}{2}\sum_{k=2}^5\binom{5}{k}(-1)^k\lambda^{-(k+3)/2}\int_\Omega U_{x,\lambda}^{5-k}H_a(x,\cdot)^{k+1}+o(\lambda^{-3}).
\end{aligned} \tag{3-35}$$

Again, by Lemma B.3,

$$\begin{aligned}
5\lambda^{-1/2}\int_\Omega U_{x,\lambda}^4 H_a(x,\cdot)\partial_\lambda U_{x,\lambda}+\frac{5}{2}\lambda^{-2}\int_\Omega U_{x,\lambda}^4 H_a(x,\cdot)^2-10\lambda^{-1}\int_\Omega U_{x,\lambda}^3 H_a(x,\cdot)^2\partial_\lambda U_{x,\lambda}\\
=-\frac{2\pi}{3}\phi_a(x)\lambda^{-2}+(2\pi a(x)+5\pi^2\phi_a(x)^2)\lambda^{-3}+o(\lambda^{-3}).
\end{aligned} \tag{3-36}$$

Finally, the two sums are bounded, in absolute value, by

$$\begin{aligned}
\int_\Omega (U_{x,\lambda}^2\lambda^{-3/2}|H_a(x,\cdot)|^3+\lambda^{-5/2}|H_a(x,\cdot)|^5)|\partial_\lambda U_{x,\lambda}|+\int_\Omega (U_{x,\lambda}^3\lambda^{-5/2}|H_a(x,\cdot)|^3+\lambda^{-4}|H_a(x,\cdot)|^6)\\
\lesssim\|\partial_\lambda U_{x,\lambda}\|_6(\|U_{x,\lambda}\|_{12/5}^2\lambda^{-3/2}+\lambda^{-5/2})+\|U_{x,\lambda}\|_3^3\lambda^{-5/2}+\lambda^{-4}=o(\lambda^{-3}).
\end{aligned}$$

This completes our discussion of the second term on the right side of (3-32) and therefore of the first term on the right side of (3-31).

For the second term on the right side of (3-31) we get, using $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2} H_a(x,\cdot) - f_{x,\lambda}$,

$$\int_\Omega a(f_{x,\lambda} + g_{x,\lambda})\partial_\lambda \psi_{x,\lambda} = \int_\Omega a g_{x,\lambda} \partial_\lambda U_{x,\lambda} + \frac{1}{2}\lambda^{-3/2} \int_\Omega a g_{x,\lambda} H_a(x,\cdot) + o(\lambda^{-3}).$$

The second integral is negligible since, by Lemma A.4,

$$\left| \frac{1}{2}\lambda^{-3/2} \int_\Omega a g_{x,\lambda} H_a(x,\cdot) \right| \lesssim \lambda^{-3/2} \int_\Omega g_{x,\lambda} \lesssim \lambda^{-4} \log \lambda.$$

Since a is differentiable, we can expand the first integral as

$$\int_\Omega a g_{x,\lambda} \partial_\lambda U_{x,\lambda} = a(x) \int_\Omega g_{x,\lambda} \partial_\lambda U_{x,\lambda} + \mathcal{O}\biggl(\int_\Omega |x-y| g_{x,\lambda} |\partial_\lambda U_{x,\lambda}| \biggr).$$

We have

$$\int_\Omega g_{x,\lambda} \partial_\lambda U_{x,\lambda} = \lambda^{-3} \int_{\lambda(\Omega - x)} g_{0,1} \partial_\lambda U_{0,1} = \lambda^{-3} \int_{\mathbb{R}^3} g_{0,1} \partial_\lambda U_{0,1} + o(\lambda^{-3})$$

and

$$\int_{\mathbb{R}^3} g_{0,1} \partial_\lambda U_{0,1} = 4\pi \int_0^\infty \left(\frac{1}{r} - \frac{1}{\sqrt{1+r^2}} \right) \frac{1-r^2}{2(1+r^2)^{3/2}} r^2 \, dr = 2\pi(3-\pi).$$

Using similar bounds one verifies that

$$\int_\Omega |x-y| g_{x,\lambda} |\partial_\lambda U_{x,\lambda}| \lesssim \lambda^{-4} \int_{\lambda(\Omega - x)} |z| g_{0,1} |\partial_\lambda U_{0,1}| \lesssim \lambda^{-4}.$$

This completes our discussion of the second term on the right side of (3-31).

For the third term on the right side of (3-31), we write

$$\psi_{x,\lambda} = \lambda^{-1/2} G_a(x,\cdot) - f_{x,\lambda} - g_{x,\lambda}$$

and get

$$\begin{aligned}
&\int_\Omega V \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} \\
&\quad = \int_\Omega V(\lambda^{-1/2} G_a(x,\cdot) - g_{x,\lambda}) \partial_\lambda (\lambda^{-1/2} G_a(x,\cdot) - g_{x,\lambda}) + o(\lambda^2) \\
&\quad = -\tfrac{1}{2}\lambda^{-2} Q_V(x) + \mathcal{O}\biggl(\lambda^{-3/2} \int_\Omega G_a(x,\cdot) g_{x,\lambda} + \lambda^{-1/2} \int_\Omega G_a(x,\cdot) |\partial_\lambda g_{x,\lambda}| + \int_\Omega g_{x,\lambda} |\partial_\lambda g_{x,\lambda}| \biggr) + o(\lambda^2) \\
&\quad = -\tfrac{1}{2}\lambda^{-2} Q_V(x) + \mathcal{O}\bigl(\lambda^{-3/2} \|G_a(x,\cdot)\|_2 \|g_{x,\lambda}\|_2 \\
&\qquad\qquad + \lambda^{-1} \|G_a(x,\cdot)\|_2 \|\partial_\lambda g_{x,\lambda}\|_2 + \|g_{x,\lambda}\|_2 \|\partial_\lambda g_{x,\lambda}\|_2 \bigr) + o(\lambda^{-2}) \\
&\quad = -\tfrac{1}{2}\lambda^{-2} Q_V(x) + o(\lambda^{-2}).
\end{aligned}$$

In the last equality we used the bounds from Lemma A.4 and the fact that $G_a(x,\cdot) \in L^2(\Omega)$. This completes our discussion of the third term on the right side of (3-31) and concludes the proof of (a).

(b) We note that (3-26) yields

$$-\Delta\partial_\lambda\psi_{x,\lambda}+a\partial_\lambda\psi_{x,\lambda}=15U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-a(\partial_\lambda f_{x,\lambda}+\partial_\lambda g_{x,\lambda}).$$

Because of this equation, the quantity of interest can be written as

$$\int_\Omega(\nabla q\cdot\nabla\partial_\lambda\psi_{x,\lambda}+aq\partial_\lambda\psi_{x,\lambda}-15\alpha^4 q\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda}) \\ =15\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-\alpha^4\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda})-\int_\Omega aq(\partial_\lambda f_{x,\lambda}+\partial_\lambda g_{x,\lambda}). \quad (3\text{-}37)$$

We discuss the two integrals on the right side separately.

We have

$$\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-\alpha^4\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda}) \\ =(1-\alpha^4)\int_\Omega qU_{x,\lambda}^4\partial_\lambda U_{x,\lambda}+\alpha^4\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda}). \quad (3\text{-}38)$$

The first integral is, by the orthogonality condition $0=\int_\Omega\nabla w\cdot\nabla\partial_\lambda PU_{x,\lambda}=15\int_\Omega wU_{x,\lambda}^4\partial_\lambda U_{x,\lambda}$,

$$\int_\Omega qU_{x,\lambda}^4\partial_\lambda U_{x,\lambda}=\lambda^{-1/2}\int_\Omega(H_a(x,\cdot)-H_0(x,\cdot))U_{x,\lambda}^4\partial_\lambda U_{x,\lambda} \\ =-\frac{2\pi}{15}(\phi_a(x)-\phi_0(x))\lambda^{-2}+\mathcal{O}(\lambda^{-3}). \quad (3\text{-}39)$$

For the second integral on the right side of (3-38), we have

$$\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda}) \\ =\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-(U_{x,\lambda}-\lambda^{-1/2}H_a(x,\cdot))^4\partial_\lambda(U_{x,\lambda}-\lambda^{-1/2}H_a(x,\cdot)))+o(\lambda^{-3}) \\ =\mathcal{O}(\phi_a(x)\lambda^{-3})+o(\varepsilon\lambda^{-2})+o(\lambda^{-3}). \quad (3\text{-}40)$$

Let us justify the claimed bound here for a typical term. We write $H_a(x,y)=\phi_a(x)+\mathcal{O}(|x-y|)$ and get

$$\int_\Omega qU_{x,\lambda}^4\lambda^{-3/2}H_a(x,\cdot)=\lambda^{-3/2}\phi_a(x)\int_\Omega qU_{x,\lambda}^4+\mathcal{O}\Big(\lambda^{-3/2}\int_\Omega qU_{x,\lambda}^4|x-y|\Big).$$

Using the bound (3-27) on q and Lemma A.1 we get

$$\left|\int_\Omega qU_{x,\lambda}^4\right|\le\|q\|_6\|U_{x,\lambda}\|_{24/5}^4\lesssim\lambda^{-3/2}+\varepsilon\lambda^{-1}.$$

The remainder term is better because of the additional factor of $|x-y|$. We gain a factor of λ^{-1} since

$$\||x-\cdot|^{1/4}U_{x,\lambda}\|_{24/5}^4\lesssim\lambda^{-3/2}.$$

Another typical term,

$$\int_\Omega qU_{x,\lambda}^3\lambda^{-1/2}H_a(x,\cdot)\partial_\lambda U_{x,\lambda},$$

can be treated in the same way, since the bounds for $\partial_\lambda U_{x,\lambda}$ are the same as for $\lambda^{-1} U_{x,\lambda}$; see Lemma A.1. The remaining terms are easier. This completes our discussion of the first term on the right side of (3-37).

The second term on the right side of (3-37) is negligible. Indeed,

$$\int_\Omega aq(\partial_\lambda f_{x,\lambda} + \partial_\lambda g_{x,\lambda}) = \mathcal{O}(\|q\|_6 \|\partial_\lambda g_{x,\lambda}\|_{6/5}) + o(\lambda^{-3}) = o(\lambda^{-3}), \tag{3-41}$$

where we used Lemma A.4 and the same bound on q as before. This completes our discussion of the second term on the right side of (3-37) and concludes the proof of (b).

(c) We use the form (3-8) of the zero modes s, as well as the bounds on $\|\nabla s\|_2$ and $\|\nabla r\|_2$ from (3-10) and (3-18), to find

$$\begin{aligned}\int_\Omega q^2 \psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda} &= \int_\Omega s^2 \psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda} + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\lambda^{-3}) + o(\varepsilon\lambda^{-2}) \\ &= \beta^2 \lambda^{-2} \int_\Omega U_{x,\lambda}^5 \partial_\lambda U_{x,\lambda} + 2\beta\gamma\lambda^{-1} \int_\Omega U_{x,\lambda}^4 (\partial_\lambda U_{x,\lambda})^2 + \gamma^2 \int_\Omega U_{x,\lambda}^3 (\partial_\lambda U_{x,\lambda})^3 \\ &\qquad + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\lambda^{-3}) + o(\varepsilon\lambda^{-2}). \end{aligned} \tag{3-42}$$

A direct calculation using (B-15) gives

$$\lambda^{-2} \int_\Omega U_{x,\lambda}^5 \partial_\lambda U_{x,\lambda} = o(\lambda^{-3}), \qquad \int_\Omega U_{x,\lambda}^3 (\partial_\lambda U_{x,\lambda})^3 = o(\lambda^{-3})$$

and

$$\begin{aligned}\int_\Omega U_{x,\lambda}^4 (\partial_\lambda U_{x,\lambda})^2 &= \frac{1}{4}\lambda^{-2} \int_\Omega U_{x,\lambda}^6 - \lambda^3 \int_\Omega \frac{|x-y|^2}{(1+\lambda^2|x-y|^2)^4} + \lambda^5 \int_\Omega \frac{|x-y|^4}{(1+\lambda^2|x-y|^2)^5} \\ &= \frac{\pi^2}{16}\lambda^{-2} - 4\pi\lambda^{-2} \int_0^\infty \frac{t^4\,dt}{(1+t^2)^4} + 4\pi\lambda^{-2} \int_0^\infty \frac{t^6\,dt}{(1+t^2)^5} + o(\lambda^{-2}) \\ &= \frac{\pi^2}{64}\lambda^{-2} + o(\lambda^{-2}). \end{aligned}$$

Inserting this into (3-42) gives the claimed expansion (c).

The proof of (d) uses similar bounds as in the rest of the proof and is omitted. □

Proof of Proposition 3.8. Combining (3-30) with Lemma 3.10 yields

$$0 = -4\pi\phi_a(x)\lambda^{-2} - Q_V(x)\varepsilon\lambda^{-2} + 4\pi^2 a(x)\lambda^{-3} + \lambda^{-3} R + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\lambda^{-3}) + o(\varepsilon\lambda^{-2}), \tag{3-43}$$

with

$$R = \lambda(1-\alpha^4)4\pi(\phi_a(x) + \phi_0(x)) + 30\pi^2\phi_a(x)^2 - \tfrac{15}{8}\beta\gamma\pi^2.$$

We now make use of the expansion (3-24) of $\alpha^4 - 1$ and obtain

$$R = 16\beta\pi\phi_0(x) - \tfrac{15}{8}\beta\gamma\pi^2 + \mathcal{O}(\phi_a(x) + \lambda^{-1} + \varepsilon).$$

Inserting the expansions (3-13) of β and γ, we find the cancellation

$$R = \mathcal{O}(\phi_a(x) + \lambda^{-1} + \varepsilon). \tag{3-44}$$

In particular, $R = \mathcal{O}(1)$ and, inserting this into (3-43), we obtain

$$\phi_a(x) = \mathcal{O}(\lambda^{-1} + \varepsilon).$$

In particular, for the error term in (3-43), we have $\phi_a(x)\lambda^{-3} = o(\lambda^{-3})$ and, moreover, by (3-44), we have $R = \mathcal{O}(\lambda^{-1} + \varepsilon)$. Inserting this bound into (3-43), we obtain the claimed expansion (3-28). □

3E. *Bounding* $\nabla\phi_a(x)$. In this subsection we prove the bound on $\nabla\phi_a(x)$ in Proposition 3.1.

Proposition 3.11. *For every* $\mu < 1$, *as* $\varepsilon \to 0$,

$$|\nabla\phi_a(x)| \lesssim \varepsilon^{\mu}. \tag{3-45}$$

The proof of this proposition is a refined version of the proof of Proposition 2.5. It is also based on expanding the Pohozaev identity (2-9). Abbreviating, for $v, z \in H^1(\Omega)$,

$$I[v, z] := \int_{\partial\Omega} \frac{\partial v}{\partial n}\frac{\partial z}{\partial n} n + \int_{\Omega} (\nabla a) v z \tag{3-46}$$

and writing $u = \alpha(\psi_{x,\lambda} + q)$, we can write identity (2-9) as

$$0 = I[\psi_{x,\lambda}] + 2I[\psi_{x,\lambda}, q] + I[q] + \varepsilon \int_{\Omega} (\nabla V)(\psi_{x,\lambda} + q)^2. \tag{3-47}$$

The following lemma extracts the leading contribution from the main term $I[\psi_{x,\lambda}]$.

Lemma 3.12. $I[\psi_{x,\lambda}] = 4\pi\nabla\phi_a(x)\lambda^{-1} + \mathcal{O}(\lambda^{-1-\mu})$ *for every* $\mu < 1$.

On the other hand, the next lemma allows us to control the error terms involving q.

Lemma 3.13.
$$\left\| \frac{\partial q}{\partial n} \right\|_{L^2(\partial\Omega)} \lesssim \varepsilon\lambda^{-1/2}.$$

Before proving these two lemmas, let us use them to give the proof of Proposition 3.11. In that proof, and later in this subsection, we will use the inequality

$$\|q\|_2 \lesssim \varepsilon\lambda^{-1/2}. \tag{3-48}$$

This follows from the bound (3-10) on s and the bounds in Corollary 3.9 on λ^{-1} and r. Note that (3-48) is better than the bound (3-27) in the L^6 norm.

Proof of Proposition 3.11. We shall make use of the bounds

$$\|\psi_{x,\lambda}\|_2 + \left\| \frac{\partial \psi_{x,\lambda}}{\partial n} \right\|_{L^2(\partial\Omega)} \lesssim \lambda^{-1/2}. \tag{3-49}$$

The first bound follows by writing $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2} H_a(x,\cdot) + f_{x,\lambda}$ and using the bounds in Lemmas A.1 and A.2 and in (B-1). We write $\psi_{x,\lambda} = PU_{x,\lambda} - \lambda^{-1/2}(H_a(x,\cdot) - H_0(x,\cdot))$ and use the bounds in Lemmas A.3 and B.1 for the second bound.

Combining the bounds (3-49) with the corresponding bounds for q from Lemma 3.13 and (3-48), we obtain

$$|I[\psi_{x,\lambda}, q]| \lesssim \varepsilon\lambda^{-1} \quad \text{and} \quad I[q] \lesssim \varepsilon^2\lambda^{-1}.$$

Moreover, by (3-48) and (3-49),

$$\varepsilon\left|\int_\Omega (\nabla V)(\psi_{x,\lambda}+q)^2\right| \lesssim \varepsilon\lambda^{-1}.$$

In view of these bounds, Lemma 3.12 and (3-47) imply $|\nabla\phi_a(x)| \lesssim \varepsilon + \lambda^{-\mu}$. Because of (3-29), this implies (3-45). □

It remains to prove Lemmas 3.12 and 3.13.

Proof of Lemma 3.12. We integrate (3-26) for $\psi_{x,\lambda}$ against $\nabla\psi_{x,\lambda}$ and obtain

$$-\tfrac{1}{2} I[\psi_{x,\lambda}] = 3\int_\Omega U_{x,\lambda}^5 \nabla\psi_{x,\lambda} - \int_\Omega a(f_{x,\lambda}+g_{x,\lambda})\nabla\psi_{x,\lambda}. \tag{3-50}$$

For the first integral on the right side we write $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2} H_a(x,\cdot) + f_{x,\lambda}$ and integrate by parts to obtain

$$3\int_\Omega U_{x,\lambda}^5 \nabla\psi_{x,\lambda} = 3\int_{\partial\Omega} U_{x,\lambda}^5 \bigl(\tfrac{1}{6}U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) + f_{x,\lambda}\bigr) n \\ + 15\int_\Omega U_{x,\lambda}^4 (\nabla U_{x,\lambda})(\lambda^{-1/2}H_a(x,\cdot) - f_{x,\lambda}).$$

By Lemma B.3 (see also Remark B.4) we have

$$\int_\Omega U_{x,\lambda}^4 (\nabla U_{x,\lambda}) H_a(x,\cdot) = -\int_\Omega U_{x,\lambda}^4 (\nabla_x U_{x,\lambda}) H_a(x,\cdot) = -\frac{2\pi}{15}\nabla\phi_a(x)\lambda^{-1/2} + \mathcal{O}(\lambda^{-1/2-\mu}).$$

Finally, since $U_{x,\lambda} \lesssim \lambda^{-1/2}$ on $\partial\Omega$ and by the bounds on $U_{x,\lambda}$, $f_{x,\lambda}$ and $H_a(x,\cdot)$ from Lemmas A.1 and A.2 and from (B-1), we have

$$3\int_{\partial\Omega} U_{x,\lambda}^5 \bigl(\tfrac{1}{6}U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) + f_{x,\lambda}\bigr) n + 15\int_\Omega U_{x,\lambda}^4 (\nabla U_{x,\lambda}) f_{x,\lambda} = \mathcal{O}(\lambda^{-2}).$$

This shows that the first term on the right side of (3-50) gives the claimed contribution.

On the other hand, for the second term on the right side of (3-50) we have

$$\begin{aligned}\int_\Omega a(f_{x,\lambda}+g_{x,\lambda})\nabla\psi_{x,\lambda} &= \int_\Omega a(f_{x,\lambda}+g_{x,\lambda})\nabla(U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot)) - \frac{1}{2}\int_\Omega (\nabla a) f_{x,\lambda}^2 \\ &\quad - \int_\Omega (a\nabla g_{x,\lambda} + g_{x,\lambda}\nabla a) f_{x,\lambda} + \frac{1}{2}\int_{\partial\Omega} a f_{x,\lambda}^2 + \int_{\partial\Omega} a f_{x,\lambda} g_{x,\lambda} \\ &= \int_\Omega a g_{x,\lambda}\nabla U_{x,\lambda} + \mathcal{O}(\lambda^{-3}).\end{aligned}$$

Here we used bounds from Lemmas A.2 and A.4 and from the proof of the latter. Finally, we write $a(y) = a(x) + \mathcal{O}(|x-y|)$ and use the oddness of $g_{x,\lambda}\nabla U_{x,\lambda}$ to obtain

$$\int_\Omega a g_{x,\lambda}\nabla U_{x,\lambda} = \mathcal{O}\left(\int_\Omega |x-y| g_{x,\lambda} |\nabla U_{x,\lambda}|\right) = \mathcal{O}(\lambda^{-2}).$$

This proves the claimed bound on the second term on the right side of (3-50). □

Proof of Lemma 3.13. The proof is analogous to that of Lemma 2.6. By combining (2-7) for w with $\Delta(H_a(x,\cdot)-H_0(x,\cdot)) = -aG_a(x,\cdot)$, we obtain $-\Delta q = F$ with

$$F := -3U_{x,\lambda}^5 + 3\alpha^4(\psi_{x,\lambda}+q)^5 - aq + a(f_{x,\lambda}+g_{x,\lambda}) - \varepsilon V(\psi_{x,\lambda}+q).$$

(We use the same notation as in the proof of Lemma 2.6 for analogous but different objects.)

We define the cut-off function ζ as before, but now in our bounds we do not make the dependence on d explicit, since we know already $d^{-1} = \mathcal{O}(1)$ by Proposition 2.5. Then $\zeta q \in H^2(\Omega)\cap H_0^1(\Omega)$ and

$$-\Delta(\zeta q) = \zeta F - 2\nabla\zeta\cdot\nabla q - (\Delta\zeta)q.$$

We claim that

$$\zeta|F| \lesssim \zeta|q|^5 + \varepsilon\zeta U_{x,\lambda} + |q| + \varepsilon\lambda^{-1/2}. \tag{3-51}$$

Indeed, on $\Omega\setminus B_{d/2}(x)$, we have $U_{x,\lambda}\lesssim\lambda^{-1/2}$ and $g_{x,\lambda}\lesssim\lambda^{-5/2}$. By Corollary 3.9, we have $\lambda^{-5/2} = \mathcal{O}(\varepsilon\lambda^{-1/2})$. Moreover, we write $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) + f_{x,\lambda}$ and use the bounds on $f_{x,\lambda}$ and $H_a(x,\cdot)$ from Lemma A.2 and (B-1).

Combining (3-51) with inequality (2-12), we obtain

$$\begin{aligned}\left\|\frac{\partial q}{\partial n}\right\|_{L^2(\partial\Omega)} &= \left\|\frac{\partial(\zeta q)}{\partial n}\right\|_{L^2(\partial\Omega)} \lesssim \|\Delta(\zeta q)\|_{3/2} = \|\zeta F - 2\nabla\zeta\cdot\nabla q - (\Delta\zeta)q\|_{3/2}\\ &\lesssim \|\zeta q^5\|_{3/2} + \varepsilon\|\zeta U_{x,\lambda}\|_{3/2} + \|q\|_{3/2} + \varepsilon\lambda^{-1/2} + \||\nabla\zeta||\nabla q|\|_{3/2} + \|(\Delta\zeta)q\|_{3/2}.\end{aligned}$$

It remains to bound the norms on the right side. All terms, except for the first one, are easily bounded. Indeed, by (3-48),

$$\|q\|_{3/2} + \|(\Delta\zeta)q\|_{3/2} \lesssim \|q\|_2 \lesssim \varepsilon\lambda^{-1/2}$$

and

$$\||\nabla\zeta||\nabla q|\|_{3/2} \lesssim \|\nabla q\|_{L^2(\Omega\setminus B_{d/2}(x))} \le \|\nabla s\|_{L^2(\Omega\setminus B_{d/2}(x))} + \|\nabla r\|_2 \lesssim \varepsilon\lambda^{-1/2},$$

where we used $\|\nabla s\|_{L^2(\Omega\setminus B_{d/2}(x))}\lesssim\lambda^{-3/2}$ by Equation (3-10) and $\|\nabla r\|_2\lesssim\varepsilon\lambda^{-1/2}$ by Corollary 3.9. (Notice that for the estimate on s it is crucial that the integral avoids $B_{d/2}(x)$.) Moreover, by Lemma A.1,

$$\|\zeta U_{x,\lambda}\|_{3/2} \lesssim \|U_{x,\lambda}\|_{L^{3/2}(\Omega\setminus B_{d/2}(x))} \lesssim \lambda^{-1/2}.$$

To bound the remaining term $\|\zeta q^5\|_{3/2}$ we argue as in Lemma 2.6 above and get

$$\begin{aligned}\|\zeta q^5\|_{3/2} &= \left(\int_\Omega |\zeta^{1/4}|q|^{1/4}q|^6\right)^{\frac{2}{3}} \lesssim \left(\int_\Omega |\nabla(\zeta^{1/4}|q|^{1/4}q)|^2\right)^2\\ &\lesssim \left(\int_\Omega |q|^{5/2}|\nabla(\zeta^{1/4})|^2\right)^2 + \left(\int_\Omega |F|\zeta^{1/2}|q|^{3/2}\right)^2 \lesssim \|q\|_6^5 + \left(\int_\Omega |F|\zeta^{1/2}|q|^{3/2}\right)^2.\end{aligned}$$

We use the pointwise estimate (3-51) on ζF, which is equally valid for $\zeta^{1/2}F$. The term coming from $|q|^5$ is bounded by

$$\left(\int_\Omega |q|^{5+3/2}\zeta^{1/2}\right)^2 = \left(\int_\Omega (\zeta|q|^5)^{1/2}q^4\right)^2 \le \|\zeta q^5\|_{3/2}\|q\|_6^8 = o(\|\zeta q^5\|_{3/2}),$$

which can be absorbed into the left side. The contributions from the remaining terms in the pointwise bound on $\zeta^{1/2}|F|$ can by easily controlled, and we obtain

$$\|\zeta q^5\|_{3/2} \lesssim \|q\|_6^5 + \lambda^{-5} + (\varepsilon\lambda^{-1/2})^5 \lesssim \varepsilon\lambda^{-1/2}.$$

Collecting all the estimates, we obtain the claimed bound. □

4. Proof of Theorems 1.5 and 1.6

4A. *The behavior of ϕ_a near x_0.* We are now in a position to complete the proof of Theorem 1.5. Our main remaining goal is to prove

$$\phi_a(x) = o(\varepsilon). \tag{4-1}$$

Once this is shown, we will be able to find a relation between λ and ε. The proof of (4-1) (and only this proof) relies on the nondegeneracy of critical points of ϕ_a.

We already know that $\phi_a(x_0) = 0$ and that $\phi_a(y) \geq 0$ for all $y \in \Omega$, hence x_0 is a critical point of ϕ_a. In this subsection we collect the necessary ingredients which exploit this fact.

Lemma 4.1. *The function ϕ_a is of class C^2 on Ω.*

Since we were unable to find a proof for this fact in the literature, we provide one in Section B2.

Thus, the following general lemma applies to ϕ_a.

Lemma 4.2. *Let u be C^2 near the origin and suppose that $u(0) = 0$, $\nabla u(0) = 0$ and that* Hess $u(0)$ *is invertible. Then, as $x \to 0$,*

$$u(x) = \tfrac{1}{2}\nabla u(x) \cdot (\operatorname{Hess} u(0))^{-1} \nabla u(x) + o(|x|^2). \tag{4-2}$$

Suppose additionally that Hess $u(0) \geq c$ *for some $c > 0$ in the sense of quadratic forms, i.e., the origin is a nondegenerate minimum of u. Then, as $x \to 0$,*

$$u(x) \lesssim |\nabla u(x)|^2. \tag{4-3}$$

Proof. We abbreviate $H(x) = \operatorname{Hess} u(x)$ and make a Taylor expansion around x to get

$$0 = u(0) = u(x) - \nabla u(x) \cdot x + \tfrac{1}{2} x \cdot H(x) x + o(|x|^2) \tag{4-4}$$

and

$$0 = \nabla u(0) = \nabla u(x) - H(x)x + o(|x|^2). \tag{4-5}$$

We infer from (4-5) and the invertibility of $H(0)$ that

$$x = H(x)^{-1} \nabla u(x) + o(|x|^2).$$

Inserting this into (4-4) gives

$$0 = u(x) - \tfrac{1}{2}\nabla u(x) \cdot H(x)^{-1} \nabla u(x) + o(|x|^2).$$

Since $H(x)^{-1} = H(0)^{-1} + o(|x|)$, this yields (4-2).

To prove (4-3), if zero is a nondegenerate minimum, then a Taylor expansion around zero shows

$$u(x) = \tfrac{1}{2}x \cdot H(0)x + o(|x|^2) \geq \tfrac{1}{4}c|x|^2 \tag{4-6}$$

for small enough $|x|$. Thus the $o(|x|^2)$ in (4-2) can be absorbed in the left side, and thus (4-3) holds. ☐

4B. ***Proof of Theorem 1.5.*** Equation (1-18) follows from Proposition 2.1, together with (3-2), (3-3) and (3-5). The facts that $x_0 \in \mathcal{N}_a$ and $Q_V(x_0) \leq 0$ follow from Corollary 3.9.

By Lemma 4.1 and the assumption that x_0 is a nondegenerate minimum of ϕ_a, we can apply Lemma 4.2 to the function $u(x) := \phi_a(x + x_0)$ to get

$$\phi_a(x) \lesssim |\nabla \phi_a(x)|^2.$$

Therefore, by the bound on $\nabla \phi_a(x)$ in Proposition 3.1 with some fixed $\mu \in \left(\frac{1}{2}, 1\right)$, we get

$$\phi_a(x) \lesssim |\nabla \phi_a(x)|^2 = o(\varepsilon). \tag{4-7}$$

This proves (1-20) and, by nondegeneracy of x_0, also (1-19). Moreover, inserting (4-7) into the expansion of $\phi_a(x)$ from Proposition 3.1, we find

$$0 = a(x)\pi\lambda^{-1} - \frac{\varepsilon}{4\pi} Q_V(x) + o(\lambda^{-1}) + o(\varepsilon),$$

that is,

$$\varepsilon\lambda = 4\pi^2 \frac{|a(x_0)| + o(1)}{|Q_V(x_0)| + o(1)}$$

with the understanding that this means $\varepsilon\lambda \to \infty$ if $Q_V(x_0) = 0$. This proves (1-21).

The remaining claims in Theorem 1.5 follow from Proposition 3.1.

4C. ***A bound on*** $\|w\|_\infty$**.** In this subsection, we prove a crude bound on the L^∞ norm of the first-order remainder w appearing in the decomposition $u = \alpha(PU_{x,\lambda} + w)$, and also on some of its L^p norms which cannot be controlled through Sobolev inequalities, i.e., $p > 6$. This bound was not needed in the proof of Theorem 1.5, but will be in that of Theorem 1.6.

Proposition 4.3. *As* $\varepsilon \to 0$,

$$\|w\|_p \lesssim \lambda^{-3/p} \quad \textit{for all } p \in (6, \infty). \tag{4-8}$$

Moreover, for every $\mu > 0$,

$$\|w\|_\infty = o(\lambda^\mu). \tag{4-9}$$

Our proof follows [Rey 1989, Proof of (25)], which concerns the case $N \geq 4$ and $a = 0$. Since some of the required modifications are rather complicated to state, we give details for the convenience of the reader.

Proof. We begin by proving the first bound in the proposition, which we write as

$$\|w\|_{3(r+1)}^{r+1} \lesssim \lambda^{-1} \quad \text{for all } r \in (1, \infty).$$

To prove this, we define F by (2-13), multiply (2-7) with $|w|^{r-1}w$ and integrate by parts to obtain

$$\frac{4r}{(r+1)^2}\int_\Omega \big|\nabla |w|^{\frac{r+1}{2}}\big|^2 = \int_\Omega F|w|^{r-1}w.$$

Thus, by Sobolev's inequality applied to $v=|w|^{(r+1)/2}$,

$$\|w\|_{3(r+1)}^{r+1} \lesssim \int_\Omega |F||w|^r. \tag{4-10}$$

In order to estimate the right side of (4-10), we make use of the bound

$$|F| \lesssim |\alpha^4-1|U_{x,\lambda}^5 + U_{x,\lambda}^4|w| + |w|^5 + U_{x,\lambda}^4\varphi_{x,\lambda} + U_{x,\lambda} + \varphi_{x,\lambda} + |w|. \tag{4-11}$$

This is a refinement of (3-51), which is obtained by writing $PU_{x,\lambda}=U_{x,\lambda}-\varphi_{x,\lambda}$ and using Lemma A.2 to bound $\varphi_{x,\lambda}^5 \lesssim \varphi_{x,\lambda}$.

We estimate the resulting terms separately. Using Hölder's inequality, Lemma A.1, Proposition 3.6 and the fact that for any $\eta, p, q>0$ with $p^{-1}+q^{-1}=1$ there is $C_\eta>0$ such that for any $a,b>0$ one has $ab\le \eta a^p + C_\eta b^q$, we obtain

$$\begin{aligned}
|\alpha^4-1|\int_\Omega U_{x,\lambda}^5|w|^r &\le \lambda^{-1}\|w\|_{3(r+1)}^r \|U\|_{5\cdot\frac{3r+3}{2r+3}}^5 \lesssim \lambda^{-1}\|w\|_{3(r+1)}^r \lambda^{\frac12\cdot\frac{r-1}{r+1}} \\
&= \|w\|_{3(r+1)}^r \lambda^{-\frac{r+3}{2(r+1)}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta \lambda^{-\frac{r+3}{2}};
\end{aligned}$$

$$\int_\Omega U_{x,\lambda}^4|w|^{r+1} \le \left(\int_\Omega U_{x,\lambda}^5|w|^r\right)^{\frac45}\left(\int_\Omega |w|^{r+5}\right)^{\frac15} \le \|w\|_{3(r+1)}^{r+\frac15}\lambda^{-\frac{4}{5(r+1)}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-1};$$

$$\int_\Omega |w|^{5+r} \le \|w\|_{3(r+1)}^{r+1}\|w\|_6^4 \lesssim \|w\|_{3(r+1)}^{r+1}\lambda^{-2};$$

$$\begin{aligned}
\int_\Omega U_{x,\lambda}^4|w|^r\varphi_{x,\lambda} &\le \lambda^{-\frac12}\|w\|_{3(r+1)}^r\|U_{x,\lambda}\|_{4\cdot\frac{3r+3}{2r+3}}^4 = \lambda^{-\frac12-\frac{1}{r+1}}\|w\|_{3(r+1)}^r = \lambda^{-\frac{r+3}{2(r+1)}}\|w\|_{3(r+1)}^r \\
&\le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+3}{2}};
\end{aligned}$$

$$\int_\Omega U_{x,\lambda}|w|^r \le \|w\|_{3(r+1)}^r\|U_{x,\lambda}\|_{\frac{3r+3}{2r+3}} \lesssim \|w\|_{3(r+1)}^r\lambda^{-\frac12} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+1}{2}};$$

$$\int_\Omega \varphi_{x,\lambda}|w|^r \lesssim \lambda^{-\frac12}\|w\|_{3(r+1)}^r \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+1}{2}};$$

$$\int_\Omega |w|^{r+1} \lesssim \left(\int_\Omega |w|^{5+r}\right)^{\frac{r+1}{r+5}} \lesssim \|w\|_{3(r+1)}^{\frac{(r+1)^2}{r+5}}\lambda^{-\frac{2(r+1)}{r+5}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+1}{2}}.$$

By choosing η small enough (but independent of λ), we can absorb the term $\eta\|w\|_{3(r+1)}^{r+1}$, as well as the term $\lambda^{-2}\|w\|_{3(r+1)}^{r+1}$, into the left-hand side of inequality (4-10) to get

$$\|w\|_{3(r+1)}^{r+1} \lesssim \lambda^{-\frac{r+3}{2}} + \lambda^{-1} + \lambda^{-\frac{r+1}{2}} \lesssim \lambda^{-1}.$$

This is the claimed bound.

We now turn to the bound of the L^∞ norm of w. We write (2-7) for w as

$$w(x) = \frac{1}{4\pi}\int_\Omega G_0(x,y)F(y). \tag{4-12}$$

By Hölder's inequality and the fact that $0 \le G_0(x,y) \le |x-y|^{-1}$, we have for every $\delta \in (0,2)$

$$\|w\|_\infty \le \sup_{x\in\Omega}\|G_0(x,\cdot)\|_{3-\delta}\|F\|_{\frac{3-\delta}{2-\delta}} \lesssim \|F\|_{\frac{3-\delta}{2-\delta}}. \tag{4-13}$$

Hence it suffices to estimate $\|F\|_q$ with some $q := (3-\delta)/(2-\delta) > \frac{3}{2}$.

We use again the bound (4-11). The L^q norms of the resulting terms are easy to estimate. Indeed, since $|\alpha^4-1| \lesssim \lambda^{-1}$ by Proposition 3.6, we have by Lemma A.1

$$|\alpha^4-1|\|U_{x,\lambda}^5\|_q \lesssim \lambda^{-1}\|U\|_{5q}^5 \lesssim \lambda^{\frac{3}{2}-\frac{3}{q}}.$$

Next, by Lemma A.1 and A.2,

$$\|U_{x,\lambda}^4\varphi_{x,\lambda}\|_q \lesssim \lambda^{-\frac{1}{2}}\|U_{x,\lambda}\|_{4q}^4 =\lesssim \lambda^{\frac{3}{2}-\frac{3}{q}}.$$

Using additionally the bound on $\|\nabla w\|$ from Proposition 2.1, we can estimate, for every $q<3$,

$$\|U_{x,\lambda}+\varphi_{x,\lambda}+|w|\|_q \le \|U_{x,\lambda}\|_q + \|\varphi_{x,\lambda}\|_\infty + \|\nabla w\|_6 \lesssim \lambda^{-\frac{1}{2}}.$$

Finally, using the bound (4-8),

$$\|U_{x,\lambda}^4 w\|_q \le \|U_{x,\lambda}\|_{5q}^4\|w\|_{5q} \lesssim \lambda^{2-\frac{12}{5q}}\|w\|_{5q} \lesssim \lambda^{2-\frac{3}{q}}$$

and

$$\|w^5\|_q = \|w\|_{5q}^5 \lesssim \lambda^{-\frac{3}{q}}.$$

Inserting these estimates into (4-13) yields

$$\|w\|_\infty \lesssim \lambda^{2-\frac{3}{q}} \quad \text{for every } q \in \left(\tfrac{3}{2}, 3\right).$$

As $\delta \searrow 0$ in (4-13), we have $q \searrow \frac{3}{2}$ and hence $2-\frac{3}{q} \searrow 0$. Thus (4-9) is proved. □

4D. ***Proof of Theorem 1.6.*** By Proposition 2.1, we have $u = \alpha(PU_{x,\lambda}+w)$ with $\alpha = 1+o(1)$. Moreover, by Proposition 4.3, $\|w\|_\infty = o(\lambda^{1/2})$. On the other hand, by Lemma A.2 we have

$$\|PU_{x,\lambda}\|_\infty = \|U_{x,\lambda}\|_\infty + \mathcal{O}(\|\varphi_{x,\lambda}\|_\infty) = \lambda^{\frac{1}{2}} + \mathcal{O}(\lambda^{-\frac{1}{2}}).$$

Putting these estimates together, we obtain

$$\varepsilon\|u_\varepsilon\|_\infty^2 = \varepsilon(\lambda^{\frac{1}{2}} + o(\lambda^{\frac{1}{2}}))^2 = \varepsilon\lambda(1+o(1)) = 4\pi^2\frac{|a(x_0)|}{|Q_V(x_0)|}(1+o(1))$$

by the relationship between ε and λ proved in Theorem 1.5. Moreover, $U_{x,\lambda}(x) = \lambda^{1/2} = \|U_{x,\lambda}\|_\infty$. This finishes the proof of part (a) in Theorem 1.6.

The proof of part (b) necessitates significantly fewer prerequisites. It only relies on the crude expansion of u given in Proposition 2.1 and the rough bounds on w from Proposition 4.3.

By applying $(-\Delta+a)^{-1}$, we write (1-3) as

$$u(z)=\frac{3}{4\pi}\int_\Omega G_a(z,y)u(y)^5-\frac{\varepsilon}{4\pi}\int_\Omega G_a(z,y)V(y)u(y). \tag{4-14}$$

We fix a sequence $\delta=\delta_\varepsilon=o(1)$ with $\lambda^{-1}=o(\delta_\varepsilon)$. This condition, together with the bounds from Proposition 2.1, easily implies $\frac{3}{4\pi}\int_{B_\delta(x)}u(y)^5=\lambda^{-1/2}+o(\lambda^{-1/2})$. Hence

$$\frac{3}{4\pi}\int_{B_\delta(x)}G_a(z,y)u(y)^5=\frac{3}{4\pi}\int_{B_\delta(x)}(G_a(z,x_0)+o(1))u(y)^5=\lambda^{-1/2}G_a(z,x_0)+o(\lambda^{-1/2}).$$

On the complement of $B_\delta(x)$, using Proposition 4.3 and Lemma A.1, we bound

$$\left|\int_{\Omega\setminus B_\delta(x)}G_a(z,y)u(y)^5\right|\lesssim\|G_a(z,\cdot)\|_2(\|U_{x,\lambda}\|^5_{L^{10}(\Omega\setminus B_\delta(x))}+\|w\|^5_{10})\lesssim\lambda^{-5/2}\delta^{-7/2}+\lambda^{-3/2}.$$

Choosing, e.g., $\delta=\lambda^{-2/7}$, the last bound is $o(\lambda^{-1/2})$.

The second term on the right side of (4-14) is easily bounded by

$$\varepsilon\left|\int_\Omega G_a(z,y)V(y)u(y)\right|\lesssim\varepsilon\|G_a(z,\cdot)\|_2(\|U\|_2+\|w\|_2)\lesssim\varepsilon\lambda^{-1/2}$$

using the bounds from Proposition 2.1 and from Lemma A.1. Collecting the above estimates, part (b) of Theorem 1.6 follows.

5. Subcritical case: a first expansion

In the remainder of the paper we will deal with the proof of Theorems 1.2 and 1.3. The structure of our argument is very similar to that leading to Theorems 1.5 and 1.6. Namely, in the present section we derive a preliminary asymptotic expansion of u_ε and the involved parameters, which is refined subsequently in Section 6 below. Because of the similarities to the above argument, we will not always give full details.

The following proposition summarizes the results of this section.

Proposition 5.1. *Let (u_ε) be a family of solutions to* (1-2) *satisfying* (1-5). *Then, up to the extraction of a subsequence, there are sequences $(x_\varepsilon)\subset\Omega$, $(\lambda_\varepsilon)\subset(0,\infty)$, $(\alpha_\varepsilon)\subset\mathbb{R}$ and $(w_\varepsilon)\subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ such that*

$$u_\varepsilon=\alpha_\varepsilon(PU_{x_\varepsilon,\lambda_\varepsilon}+w_\varepsilon) \tag{5-1}$$

and a point $x_0\in\Omega$ such that

$$|x_\varepsilon-x_0|=o(1),\quad \alpha_\varepsilon=1+o(1),\quad \lambda_\varepsilon\to\infty,\quad \|\nabla w_\varepsilon\|_2=\mathcal{O}(\lambda_\varepsilon^{-1/2}),\quad \varepsilon=\mathcal{O}(\lambda_\varepsilon^{-1}). \tag{5-2}$$

5A. ***A qualitative initial expansion.*** As a first step towards Proposition 5.1, we observe that the qualitative expansion from Proposition 2.2 still holds true, that is, there are sequences $(x_\varepsilon)\subset\Omega$, $(\lambda_\varepsilon)\subset(0,\infty)$, $(\alpha_\varepsilon)\subset\mathbb{R}$ and $(w_\varepsilon)\subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ such that (5-1) holds and a point $x_0\in\overline{\Omega}$ such that, along a subsequence,

$$|x_\varepsilon-x_0|=o(1),\quad \alpha_\varepsilon=1+o(1),\quad d_\varepsilon\lambda_\varepsilon\to\infty,\quad \|\nabla w_\varepsilon\|_2=o(1),$$

where, as before, $d_\varepsilon:=d(x_\varepsilon,\partial\Omega)$.

Indeed, as explained in the proof of Proposition 2.2, it suffices to prove $u_\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$ up to a subsequence. To achieve this, we first integrate (1-2) against u_ε to obtain

$$3\Big(\int_\Omega u_\varepsilon^{6-\varepsilon}\Big)^{\frac{4-\varepsilon}{6-\varepsilon}} = \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\big(\int_\Omega u_\varepsilon^{6-\varepsilon}\big)^{2/(6-\varepsilon)}} + \frac{\int_\Omega a u_\varepsilon^2}{\big(\int_\Omega u_\varepsilon^{6-\varepsilon}\big)^{2/(6-\varepsilon)}}.$$

By (1-5) and Hölder's inequality, the right side is bounded, hence $\|u_\varepsilon\|_{6-\varepsilon} \lesssim 1$. By (1-5) again, $\|\nabla u_\varepsilon\|_2 \lesssim 1$. On the other hand, the right side is bounded from below by a positive constant by coercivity of $-\Delta + a$, which is a consequence of criticality, and by Hölder's inequality. This gives $\|u_\varepsilon\|_{6-\varepsilon} \gtrsim 1$, and hence $\|\nabla u_\varepsilon\|_2 \gtrsim 1$ by the inequalities of Sobolev and Hölder. This completes the analogue of Step 1 in the proof of Proposition 2.2.

Let us now turn to Step 2 in that proof. We denote by u_0 a weak limit point of u_ε in $H_0^1(\Omega)$, which exists by Step 1. Still by Step 1, we may assume that the quantities $\|u_\varepsilon\|_{6-\varepsilon}$ and $\|\nabla u_\varepsilon\|_2$ have nonzero limits. The only difference to Proposition 2.2 is now that we modify the definition of $\mathcal{M}$ to

$$\mathcal{M} = \lim_{\varepsilon\to 0}\int_\Omega (u_\varepsilon - u_0)^{6-\varepsilon},$$

where the exponent is $6-\varepsilon$ instead of 6. Thanks to the uniform bound $\|u_\varepsilon\|_{6-\varepsilon} \lesssim 1$ by Step 1, it can be easily checked that the proof of the Brezis–Lieb lemma (see, e.g., [Lieb and Loss 1997]) still yields

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon^{6-\varepsilon} = \lim_{\varepsilon\to 0}\int_\Omega u_0^{6-\varepsilon} + \mathcal{M} = \int_\Omega u_0^6 + \mathcal{M}.$$

Then the modified assumption (1-5) can be used to conclude

$$S\Big(\int_\Omega u_0^6 + \mathcal{M}\Big)^{\frac{1}{3}} = \int_\Omega |\nabla u_0|^2 + \mathcal{T}.$$

The rest of the proof is identical to Proposition 2.2.

We again adopt the convention in the remainder of the proof that we only consider the above subsequence and we will drop the subscript ε.

In order to prove Proposition 5.1, we will prove in the following subsections that $x_0 \in \Omega$, $\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2})$ and $\varepsilon = \mathcal{O}(\lambda^{-1})$.

5B. *The bound on $\|\nabla w\|_2$.* The goal of this subsection is to prove:

Proposition 5.2. *As $\varepsilon \to 0$,*

$$\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2}) + \mathcal{O}((\lambda d)^{-1}) + \mathcal{O}(\varepsilon). \tag{5-3}$$

Note that, in contrast to Proposition 2.4, there appears an additional error $\mathcal{O}(\varepsilon)$. We will prove in an extra step (Proposition 5.5) that $\varepsilon = \mathcal{O}((\lambda d)^{-1})$, so this extra term will disappear later.

The proof of Proposition 5.2 is somewhat lengthy, and we precede it by an auxiliary result, which is a simple consequence of the fact that $\alpha \to 1$.

Lemma 5.3. *As $\varepsilon \to 0$,*

$$\varepsilon \log \lambda = o(1).$$

A useful consequence of this lemma is that

$$U_{x,\lambda}^{-\varepsilon} \lesssim 1 \quad \text{in } \Omega. \tag{5-4}$$

Indeed, this follows from the lemma together with the fact that $U_{x,\lambda} \gtrsim \lambda^{-1/2}$ in Ω.

Proof. We integrate (1-2) against u and use the decomposition (5-1). This gives

$$\int_\Omega |\nabla(PU_{x,\lambda}+w)|^2 + \int_\Omega a(PU_{x,\lambda}+w)^2 = 3\alpha^{4-\varepsilon}\int_\Omega (PU_{x,\lambda}+w)^{6-\varepsilon}. \tag{5-5}$$

By orthogonality

$$\int_\Omega |\nabla(PU_{x,\lambda}+w)|^2 = \int_\Omega |\nabla PU_{x,\lambda}|^2 + \int_\Omega |\nabla w|^2 = \frac{3\pi^2}{4} + o(1).$$

Moreover, using Lemmas A.1 and A.2 we find $\int_\Omega a(PU_{x,\lambda}+w)^2 = o(1)$. On the other hand,

$$\int_\Omega (PU_{x,\lambda}+w)^{6-\varepsilon} = \int_\Omega U_{x,\lambda}^{6-\varepsilon} + o(1).$$

Hence (5-5) combined with the fact that $\alpha \to 1$ implies

$$\int_\Omega U_{x,\lambda}^{6-\varepsilon} = \frac{\pi^2}{4} + o(1). \tag{5-6}$$

Since

$$\int_\Omega U_{x,\lambda}^{6-\varepsilon} = \lambda^{-\varepsilon/2}\lambda^3 \int_\Omega (1+\lambda^2|x-y|^2)^{-3+\varepsilon/2} = \lambda^{-\varepsilon/2}\frac{\pi^2}{4}(1+o(1)),$$

we have $\lambda^{-\varepsilon/2} \to 1$ and hence the claim. □

The next result quantifies the difference between $\int_\Omega U_{x,\lambda}^{5-\varepsilon} v$ and $\int_\Omega U_{x,\lambda}^5 v = 0$ for $v \in T_{x,\lambda}^\perp$.

Lemma 5.4. *For every $v \in T_{x,\lambda}^\perp$,*

$$\left|\int_\Omega U_{x,\lambda}^{5-\varepsilon} v\right| \lesssim \varepsilon \|v\|_6. \tag{5-7}$$

Proof. By orthogonality,

$$\int_\Omega U_{x,\lambda}^{5-\varepsilon} v = \lambda^{-\varepsilon/2}\int_\Omega U_{x,\lambda}^5 e^{\varepsilon \log\sqrt{1+\lambda^2|x-y|^2}} v = \lambda^{-\varepsilon/2}\int_\Omega U_{x,\lambda}^5 (e^{\varepsilon \log\sqrt{1+\lambda^2|x-y|^2}} - 1)v.$$

By Lemma 5.3,

$$\varepsilon \log\sqrt{1+\lambda^2|x-y|^2} = o(1) \tag{5-8}$$

uniformly in x and y. Hence

$$0 < e^{\varepsilon \log\sqrt{1+\lambda^2|x-y|^2}} - 1 \lesssim \varepsilon \log\sqrt{1+\lambda^2|x-y|^2} \le \varepsilon\lambda|x-y|, \tag{5-9}$$

where we have used the inequality $\log\sqrt{1+t^2} \le |t|$. Since

$$\| |x-y| U_{x,\lambda}^5 \|_{6/5} = \mathcal{O}(\lambda^{-1}),$$

the result follows from the Hölder inequality. □

We are now in position to give the following:

Proof of Proposition 5.2. From (1-2) for u we obtain the following equation for w:

$$-\Delta w + aw = -3U_{x,\lambda}^5 - aPU_{x,\lambda} + 3\alpha^{4-\varepsilon}(PU_{x,\lambda} + w)^{5-\varepsilon}. \tag{5-10}$$

Integrating this equation against w gives

$$\int_\Omega (|\nabla w|^2 + aw^2) = -\int_\Omega aPU_{x,\lambda} w + 3\alpha^{4-\varepsilon} \int_\Omega w(PU_{x,\lambda} + w)^{5-\varepsilon}. \tag{5-11}$$

As before, the first term on the right-hand side is controlled easily by Hölder's inequality,

$$\left| \int_\Omega aPU_{x,\lambda} w \right| \lesssim \|PU_{x,\lambda}\|_2 \|w\|_2 \lesssim \lambda^{-1/2} \|\nabla w\|_2.$$

In order to control the second term we use the fact that $PU_{x,\lambda} = U_{x,\lambda} - \varphi_{x,\lambda}$. Moreover, by a Taylor expansion and (5-4),

$$\begin{aligned}(PU_{x,\lambda} + w)^{5-\varepsilon} &= (U_{x,\lambda} - \varphi_{x,\lambda} + w)^{5-\varepsilon} \\ &= U_{x,\lambda}^{5-\varepsilon} + (5-\varepsilon)U_{x,\lambda}^{4-\varepsilon} w + \mathcal{O}(U_{x,\lambda}^4 \varphi_{x,\lambda} + U_{x,\lambda}^3 w^2 + |w|^{5-\varepsilon} + \varphi_{x,\lambda}^{5-\varepsilon}).\end{aligned} \tag{5-12}$$

Hence,

$$\begin{aligned}&\left| \int_\Omega (PU_{x,\lambda} + w)^{5-\varepsilon} w - (5-\varepsilon)\alpha^{4-\varepsilon} \int_\Omega U_{x,\lambda}^{4-\varepsilon} w^2 \right| \\ &\qquad \le \left| \int_\Omega U_{x,\lambda}^{5-\varepsilon} w \right| + \mathcal{O}\biggl(\int_\Omega U_{x,\lambda}^4 \varphi_{x,\lambda} |w| \biggr) + \mathcal{O}(\|\nabla w\|_2^3 + \|\nabla w\|_2 \|\varphi_{x,\lambda}\|_6^{5-\varepsilon}).\end{aligned}$$

We estimate the first term on the right side using Lemma 5.4. For the second term on the right side we argue as in the proof of Proposition 2.4 and obtain

$$\int_\Omega U_{x,\lambda}^4 \varphi_{x,\lambda} |w| = \mathcal{O}((\lambda d)^{-1} \|\nabla w\|_2).$$

For the last term on the right side we use $\|\varphi_{x,\lambda}\|_6^2 = \mathcal{O}((\lambda d)^{-1})$. Moreover, in view of (5-9),

$$\begin{aligned}\int_\Omega U_{x,\lambda}^{4-\varepsilon} w^2 &\le \lambda^{-\varepsilon/2} \int_\Omega U_{x,\lambda}^4 w^2 + C\varepsilon\lambda \int_\Omega U_{x,\lambda}^4 |x-y| w^2 \\ &\le (1+o(1)) \int_\Omega U_{x,\lambda}^4 w^2 + \mathcal{O}(\varepsilon \lambda^{-1/2} \|\nabla w\|_2^2).\end{aligned} \tag{5-13}$$

Altogether we obtain, from (5-11),

$$\int_\Omega (|\nabla w|^2 + aw^2 - 15\alpha^{4-\varepsilon} U_{x,\lambda}^4 w^2) \lesssim ((\lambda d)^{-1} + \lambda^{-1/2} + \varepsilon) \|\nabla w\|_2 + o(\|\nabla w\|_2^2).$$

An application of the coercivity inequality of Lemma 2.3 now implies (5-3). □

5C. *The bound on ε.* The goal of this subsection is to prove:

Proposition 5.5. *As* $\varepsilon \to 0$,

$$\varepsilon = \mathcal{O}((\lambda d)^{-1}). \tag{5-14}$$

We note that the analogue of this proposition is not needed in Section 2 when studying (1-3).

The proof of Proposition 5.5 is based on the Pohozaev-type identity

$$\int_\Omega \nabla PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} + \int_\Omega a(PU_{x,\lambda} + w)\partial_\lambda PU_{x,\lambda} = \alpha^{4-\varepsilon} 3 \int_\Omega (PU_{x,\lambda} + w)^{5-\varepsilon} \partial_\lambda PU_{x,\lambda}, \tag{5-15}$$

which arises from integrating (4-4) against $\partial_\lambda PU_{x,\lambda}$ and inserting the following bounds.

Lemma 5.6. *As* $\varepsilon \to 0$, *we have*

$$\int_\Omega \nabla PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} + \int_\Omega a(PU_{x,\lambda} + w)\partial_\lambda PU_{x,\lambda} = \mathcal{O}(\lambda^{-2} d^{-1} + \lambda^{-1} \|\nabla w\|_2^2) \tag{5-16}$$

and

$$3 \int_\Omega (PU_{x,\lambda} + w)^{5-\varepsilon} \partial_\lambda PU_{x,\lambda} = -\tfrac{1}{16}(1 + o(1))\varepsilon \lambda^{-1} + \mathcal{O}(\lambda^{-2} d^{-1} + \lambda^{-1} \|\nabla w\|_2^2). \tag{5-17}$$

Before proving Lemma 5.6, let us use it to deduce the main result of this subsection.

Proof of Proposition 5.5. Inserting (5-16) and (5-17) into (5-15) and applying the bound (5-3) on $\|\nabla w\|$ we obtain

$$(1 + o(1))\varepsilon \lesssim (\lambda d)^{-1} + \|\nabla w\|_2^2 \lesssim (\lambda d)^{-1} + \varepsilon^2.$$

Since $\varepsilon = o(1)$, (5-14) follows. □

In the proof of Lemma 5.6 we need the following auxiliary bound.

Lemma 5.7. *For every* $v \in T_{x,\lambda}^\perp$,

$$\left| \int_\Omega U_{x,\lambda}^{4-\varepsilon} \partial_\lambda U_{x,\lambda} v \right| \lesssim \varepsilon \lambda^{-1} \|\nabla v\|_2. \tag{5-18}$$

The proof of this lemma is analogous to that of Lemma 5.4 and is omitted.

Proof of Lemma 5.6. We begin with proving (5-16). First, by [Rey 1990, (B.5)],

$$\int_\Omega \nabla PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} = \mathcal{O}(\lambda^{-2} d^{-1}).$$

Writing $PU_{x,\lambda} = U_{x,\lambda} - \varphi_{x,\lambda}$, the second term in (5-16) is bounded by

$$\begin{aligned} \left| \int_\Omega a(PU_{x,\lambda} + w)\partial_\lambda PU_{x,\lambda} \right| &\lesssim (\|U_{x,\lambda}\|_2 + \|w\|_2)(\|\partial_\lambda U_{x,\lambda}\|_2 + \|\partial_\lambda \varphi_{x,\lambda}\|_2) \\ &\lesssim \lambda^{-2} d^{-1/2} + \lambda^{-3/2} d^{-1/2} \|\nabla w\|_2 \lesssim \lambda^{-2} d^{-1} + \lambda^{-1} \|\nabla w\|_2^2, \end{aligned}$$

by Lemma A.1 and (A-3), followed by Young's inequality.

Next, we prove (5-17). Using (5-12) and (5-4) we bound pointwise

$$\begin{aligned}(PU_{x,\lambda}+w)^{5-\varepsilon}\partial_\lambda PU_{x,\lambda} &= U_{x,\lambda}^{5-\varepsilon}\partial_\lambda U_{x,\lambda}+(5-\varepsilon)U_{x,\lambda}^{4-\varepsilon}\partial_\lambda U_{x,\lambda}w \\ &\quad +\mathcal{O}((U_{x,\lambda}^4\varphi_{x,\lambda}+U_{x,\lambda}^3w^2+|w|^{5-\varepsilon}+\varphi_{x,\lambda}^{5-\varepsilon})|\partial_\lambda U_{x,\lambda}|) \\ &\quad +\mathcal{O}((U_{x,\lambda}^5+|w|^{5-\varepsilon}+\varphi_{x,\lambda}^{5-\varepsilon})|\partial_\lambda\varphi_{x,\lambda}|). \end{aligned} \tag{5-19}$$

The integral over Ω of the two remainder terms is bounded by a constant times

$$\begin{aligned}&\|\varphi_{x,\lambda}\|_\infty\|U_{x,\lambda}\|_5^4\|\partial_\lambda U_{x,\lambda}\|_5+(\|U_{x,\lambda}\|_6^3\|w\|_6^2+\|w\|_6^{5-\varepsilon}+\|\varphi_{x,\lambda}\|_6^{5-\varepsilon})\|\partial_\lambda U_{x,\lambda}\|_6 \\ &\qquad +\|U_{x,\lambda}\|_5^5\|\partial_\lambda\varphi_{x,\lambda}\|_\infty+(\|w\|_6^{5-\varepsilon}+\|\varphi_{x,\lambda}\|_6^{5-\varepsilon})\|\partial_\lambda\varphi_{x,\lambda}\|_6\lesssim\lambda^{-2}d^{-1}+\lambda^{-1}\|w\|_6^2,\end{aligned}$$

where in the last inequality we used the bounds from Lemmas A.1 and A.2.

By Lemma 5.7, the integral over Ω of the second term on the right side of (5-19) is bounded by a constant times $\varepsilon\lambda^{-1}\|\nabla w\|_2=o(\varepsilon\lambda^{-1})$.

Finally, by an explicit calculation,

$$\begin{aligned}\int_\Omega U_{x,\lambda}^{5-\varepsilon}\partial_\lambda U_{x,\lambda} &= \int_\Omega U_{x,\lambda}^{5-\varepsilon}\Big(\frac{U_{x,\lambda}}{2\lambda}-\frac{\lambda^{3/2}|x-y|^2}{(1+\lambda^2|x-y|^2)^{3/2}}\Big) \\ &= \pi\lambda^{-1-\varepsilon/2}\bigg[\frac{\Gamma\big(\frac32\big)\Gamma\big(\frac{3-\varepsilon}{2}\big)}{\Gamma\big(3-\frac{\varepsilon}{2}\big)}-\frac{2\Gamma\big(\frac52\big)\Gamma\big(\frac{3-\varepsilon}{2}\big)}{\Gamma\big(4-\frac{\varepsilon}{2}\big)}\bigg]+\mathcal{O}(\lambda^{-4}d^{-3}) \\ &= -\frac{\pi^{3/2}}{4}\varepsilon\lambda^{-1-\varepsilon/2}\frac{\Gamma\big(\frac{3-\varepsilon}{2}\big)}{\Gamma\big(4-\frac{\varepsilon}{2}\big)}+\mathcal{O}(\lambda^{-4}d^{-3}) \\ &= -\frac{\pi^2}{48}\varepsilon\lambda^{-1}(1+o(1))+\mathcal{O}(\lambda^{-4}d^{-3}),\end{aligned} \tag{5-20}$$

where, in the last step, we used Lemma 5.3. This completes the proof of (5-17). ☐

5D. ***Excluding boundary concentration.*** The goal of this subsection is to prove:

Proposition 5.8. $d^{-1}=\mathcal{O}(1)$.

Proof. The proof is very similar to that of Proposition 2.5 and we will be brief. Integrating the first equation in (1-2) against ∇u implies the Pohozaev-type identity

$$-\int_\Omega(\nabla a)u^2=\int_{\partial\Omega}n\Big(\frac{\partial u}{\partial n}\Big)^2. \tag{5-21}$$

The volume integral on the left side can be estimated as before, since by Propositions 5.2 and 5.5 we have the same bound

$$\|\nabla w\|_2^2\lesssim\lambda^{-1}+(\lambda d)^{-2}$$

as before. To bound the surface integral, we use the fact that

$$\int_{\partial\Omega}\Big(\frac{\partial w}{\partial n}\Big)^2=\mathcal{O}(\lambda^{-1}d^{-1})+o(^{-1}d^{-2}).$$

This is the analogue of Lemma 2.6. We only note that by (5-10) we have

$$F := -\Delta w = 3\alpha^{4-\varepsilon}(PU_{x,\lambda}+w)^{5-\varepsilon} - 3U_{x,\lambda}^5 - a(PU_{x,\lambda}+w) \tag{5-22}$$

and that this function satisfies (2-15). Therefore, using the above bound on $\|\nabla w\|_2$ we can proceed exactly in the same way as in the proof of Lemma 2.6.

Thus, as before, we obtain

$$C\lambda^{-1}\nabla\phi_0(x) = \mathcal{O}(\lambda^{-1}d^{-3/2}) + o(\lambda^{-1}d^{-2})$$

and then from $|\nabla\phi_0(x)| \gtrsim d^{-2}$ we conclude that $d^{-1} = \mathcal{O}(1)$, as claimed. □

5E. ***Proof of Proposition 5.1.*** The existence of the expansion is discussed in Section 5A. Proposition 5.8 implies that $d^{-1} = \mathcal{O}(1)$, which implies that $x_0 \in \Omega$. Moreover, inserting the bound $d^{-1} = \mathcal{O}(1)$ into Propositions 5.2 and 5.5, we obtain $\varepsilon = \mathcal{O}(\lambda^{-1})$ and $\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2})$, as claimed in Proposition 5.1. This completes the proof of the proposition. □

6. Subcritical case: refining the expansion

As in the additive case, we refine the analysis of the remainder term w_ε in Proposition 5.1, which we write as $w_\varepsilon = \lambda_\varepsilon^{-1/2}(H_0(x_\varepsilon,\cdot) - H_a(x_\varepsilon,\cdot)) + s_\varepsilon + r_\varepsilon$ with s_ε and r_ε as in (3-4).

The following proposition summarizes the main results of this section.

Proposition 6.1. *Let* (u_ε) *be a family of solutions to* (1-2) *satisfying* (1-5). *Then, up to the extraction of a subsequence, there are sequences* $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$, $(s_\varepsilon) \subset T_{x_\varepsilon,\lambda_\varepsilon}$ *and* $(r_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ *such that*

$$u_\varepsilon = \alpha_\varepsilon(\psi_{x_\varepsilon,\lambda_\varepsilon} + s_\varepsilon + r_\varepsilon) \tag{6-1}$$

and a point $x_0 \in \Omega$ *such that, in addition to Proposition 5.1,*

$$\|\nabla r_\varepsilon\|_2 = \mathcal{O}(\varepsilon + \lambda_\varepsilon^{-3/2} + \phi_a(x_\varepsilon)\lambda_\varepsilon^{-1}), \tag{6-2}$$

$$\phi_a(x_\varepsilon) = \pi a(x_\varepsilon)\lambda_\varepsilon^{-1} + \frac{\pi}{32}\varepsilon\lambda_\varepsilon(1+o(1)) + o(\lambda_\varepsilon^{-1}), \tag{6-3}$$

$$\nabla\phi_a(x) = \mathcal{O}(\varepsilon\lambda_\varepsilon^{1/2} + \lambda_\varepsilon^{-\mu} + \phi_a(x_\varepsilon)\lambda_\varepsilon^{-1/2}) \quad \textit{for any } \mu < 1, \tag{6-4}$$

$$\alpha_\varepsilon^{4-\varepsilon} = 1 + \frac{\varepsilon}{2}\log\lambda_\varepsilon - 4\beta\lambda_\varepsilon^{-1} + \mathcal{O}(\varepsilon + \phi_a(x_\varepsilon)\lambda_\varepsilon^{-1}) + o(\lambda_\varepsilon^{-1}). \tag{6-5}$$

We will prove Proposition 6.1 through a series of propositions in the following subsections.

6A. ***The bound on*** $\|\nabla r\|_2$**.** The following proposition contains the bound on $\|\nabla r\|_2$ from Proposition 6.1.

Proposition 6.2. *As* $\varepsilon \to 0$,

$$\|\nabla r\|_2 = \mathcal{O}(\varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}). \tag{6-6}$$

Proof. Notice that

$$-\Delta r = -3U_{x,\lambda}^5 + 3\alpha^{4-\varepsilon}(\psi_{x,\lambda}+s+r)^{5-\varepsilon} + a(g_{x,\lambda}+f_{x,\lambda}) - a(s+r) + \Delta s,$$

with $g_{x,\lambda}$ as in (A-4). Hence

$$\int_\Omega (|\nabla r|^2 + ar^2) = 3\alpha^{4-\varepsilon} \int_\Omega (\psi_{x,\lambda} + s + r)^{5-\varepsilon} r - \int_\Omega a\Big(U_{x,\lambda} - \frac{\lambda^{-1/2}}{|x-y|} + s - f_{x,\lambda}\Big) r. \tag{6-7}$$

By Lemma 3.5(b)

$$\left|\int_\Omega a(g_{x,\lambda} + f_{x,\lambda} - s) r\right| \lesssim \lambda^{-3/2} \|r\|_6.$$

Now,

$$\int_\Omega (\psi_{x,\lambda} + s + r)^{5-\varepsilon} r = \int_\Omega U_{x,\lambda}^{5-\varepsilon} r + (5-\varepsilon) \int_\Omega U_{x,\lambda}^{4-\varepsilon} r^2 + (5-\varepsilon) \int_\Omega U_{x,\lambda}^{4-\varepsilon} rs$$
$$- (5-\varepsilon) \int_\Omega U_{x,\lambda}^{4-\varepsilon} (\lambda^{-1/2} H_a(x,\cdot) + f_{x,\lambda}) r + T_{3,\varepsilon}, \tag{6-8}$$

where similarly as in the proof Lemma 3.5 we find that

$$|T_{3,\varepsilon}| \lesssim \lambda^{-2} \|r\|_6 + \|r\|_6^3.$$

Moreover, similarly as in (5-13) we obtain

$$3\alpha^{4-\varepsilon}(5-\varepsilon) \int_\Omega U_{x,\lambda}^{4-\varepsilon} r^2 \le 15 \int_\Omega U_{x,\lambda}^4 r^2 + o(\|r\|_6^2).$$

Next, we write

$$\int_\Omega U_{x,\lambda}^{4-\varepsilon} rs = \lambda^{-\varepsilon/2} \Big(\int_\Omega U_{x,\lambda}^4 rs + \int_\Omega U_{x,\lambda}^4 (e^{\varepsilon \log \sqrt{1+\lambda^2|x-y|^2}} - 1) rs \Big).$$

The prefactor $\lambda^{-\varepsilon/2}$ on the right side tends to 1 by Lemma 5.3. The first integral in the parentheses is bounded in (3-22). For the second integral we proceed again as in (5-13) and obtain

$$\left|\int_\Omega U_{x,\lambda}^4 (e^{\varepsilon \log \sqrt{1+\lambda^2|x-y|^2}} - 1) rs\right| \lesssim \lambda\varepsilon \|U^4|x-y|\|_{3/2} \|r\|_6 \|s\|_6 \lesssim \varepsilon\lambda^{-1} \|r\|_6,$$

where we used (3-10) in the last inequality. Thus, recalling the bound on ε in (5-2),

$$\left|\int_\Omega U_{x,\lambda}^{4-\varepsilon} rs\right| \lesssim \lambda^{-3/2} \|r\|_6.$$

The fourth term on the right side of (6-8) is bounded, in absolute value, by a constant times

$$\int_\Omega U_{x,\lambda}^4 (\lambda^{-1/2} |H_a(x,\cdot)| + |f_{x,\lambda}|) |r| \lesssim (\lambda^{-1} \phi_a(x) + \lambda^{-2}) \|r\|_6,$$

where we used (3-23).

Using Lemma 5.4 to control the first term on the right-hand side of (6-8) and putting all the estimates into (6-7) we finally get

$$\int_\Omega (|\nabla r|^2 + ar^2 - 15 U_{x,\lambda}^4 r^2) \lesssim (\varepsilon + \lambda^{-1}\phi_a(x) + \lambda^{-3/2}) \|r\|_6 + o(\|r\|_6^2).$$

This, in combination with the coercivity inequality of Lemma 2.3, implies the claim. □

6B. ***Expanding*** $\boldsymbol{\alpha^{4-\varepsilon}}$**.** In this subsection, we prove the expansion of $\alpha^{4-\varepsilon}$ in Proposition 6.1.

Proposition 6.3. *As* $\varepsilon \to 0$,

$$\alpha^{4-\varepsilon} = 1 + \frac{\varepsilon}{2}\log\lambda - 4\beta\lambda^{-1} + \mathcal{O}(\varepsilon + \phi_a(x)\lambda^{-1}) + o(\lambda^{-1}). \tag{6-9}$$

Proof. As in the proof of Lemma 5.3 we will integrate (1-2) against u. However, this time we write $u = \alpha(\psi_{x,\lambda} + q)$ and obtain

$$\int_\Omega |\nabla(\psi_{x,\lambda} + q)|^2 + \int_\Omega a(\psi_{x,\lambda} + q)^2 = 3\alpha^{4-\varepsilon}\int_\Omega (\psi_{x,\lambda} + q)^{6-\varepsilon},$$

which we write as

$$\int_\Omega (|\nabla\psi_{x,\lambda}|^2 + a\psi_{x,\lambda}^2 - 3\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{6-\varepsilon}) \\ + 2\int_\Omega \left(\nabla q\cdot\nabla\psi_{x,\lambda} + aq\psi_{x,\lambda} - \frac{3(6-\varepsilon)}{2}\alpha^{4-\varepsilon} q|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda}\right) = \mathcal{R}_0, \tag{6-10}$$

with

$$\mathcal{R}_0 := -\int_\Omega (|\nabla q|^2 + aq^2) + 3\alpha^{4-\varepsilon}\int_\Omega ((\psi_{x,\lambda} + q)^{6-\varepsilon} - |\psi_{x,\lambda}|^{6-\varepsilon} - (6-\varepsilon)|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} q).$$

We discuss separately the three terms that are involved in (6-10).

First, we claim that

$$\int_\Omega (|\nabla\psi_{x,\lambda}|^2 + a\psi_{x,\lambda}^2 - 3\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{6-\varepsilon}) = (1-\alpha^{4-\varepsilon})\frac{3\pi^2}{4} + \frac{3\pi^2}{8}\alpha^{4-\varepsilon}\varepsilon\log\lambda + \mathcal{O}(\varepsilon + \phi_a(x)\lambda^{-1} + \lambda^{-2}).$$

Indeed, this follows in the same way as in the proof of Lemma 3.7(a) together with the fact that

$$\int_\Omega (|\psi_{x,\lambda}|^{6-\varepsilon} - \psi_{x,\lambda}^6) = -\frac{\pi^2}{8}\varepsilon\log\lambda + \mathcal{O}(\varepsilon + \phi_a(x)\lambda^{-1} + \lambda^{-5/2}).$$

To prove the latter expansion, we write $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) - f_{x,\lambda}$ and expand, recalling (5-4),

$$|\psi_{x,\lambda}|^{6-\varepsilon} - \psi_{x,\lambda}^6 = U_{x,\lambda}^{6-\varepsilon} - U_{x,\lambda}^6 + \mathcal{O}(U_{x,\lambda}^5(\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|) + \lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5).$$

Using the bounds from Lemma A.2, (B-1) and proceeding as in the proof of Lemma B.3, we obtain

$$\int_\Omega (U_{x,\lambda}^5(\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|) + \lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5) = \mathcal{O}(\phi_a(x)\lambda^{-1} + \lambda^{-5/2}).$$

On the other hand, by an explicit computation,

$$\begin{aligned}\int_\Omega (U_{x,\lambda}^{6-\varepsilon} - U_{x,\lambda}^6) = \int_{\mathbb{R}^3} (U_{x,\lambda}^{6-\varepsilon} - U_{x,\lambda}^6) + \mathcal{O}(\lambda^{-3}) &= \pi^{3/2}\left(\lambda^{-\varepsilon/2}\frac{\Gamma\left(\frac{3-\varepsilon}{2}\right)}{\Gamma\left(3-\frac{\varepsilon}{2}\right)} - \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}\right) + \mathcal{O}(\lambda^{-3}) \\ &= -\frac{\pi^2}{8}\varepsilon\log\lambda + \mathcal{O}(\varepsilon + \lambda^{-3}),\end{aligned}$$

proving the claimed expansion of the first term on the left side of (6-10).

We turn now to the second term on the left side of (6-10) and claim that

$$\int_\Omega \Big(\nabla q \cdot \nabla \psi_{x,\lambda} + a q \psi_{x,\lambda} - \frac{3(6-\varepsilon)}{2} \alpha^{4-\varepsilon} q |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} \Big) = (1 - 3\alpha^{4-\varepsilon}) \frac{3\pi^2}{4} \beta \lambda^{-1} + \mathcal{O}(\lambda^{-2}).$$

To show this, we proceed as in the proof of Lemma 3.7(b) and use the equation for $\psi_{x,\lambda}$ to write

$$\begin{aligned}\int_\Omega \Big(\nabla q \cdot \nabla \psi_{x,\lambda} + a q \psi_{x,\lambda} - \frac{3(6-\varepsilon)}{2} \alpha^{4-\varepsilon} q |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} \Big) \\ = 3\Big(1 - \frac{6-\varepsilon}{2}\alpha^{4-\varepsilon}\Big) \int_\Omega q U_{x,\lambda}^5 - \frac{3(6-\varepsilon)}{2} \int_\Omega q (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) \\ - \int_\Omega q \Big(\frac{3(6-\varepsilon)}{2} (|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} - U_{x,\lambda}^{5-\varepsilon}) + a(f_{x,\lambda} + g_{x,\lambda}) \Big).\end{aligned}$$

The first term on the right side was already computed in the proof of Lemma 3.7(b), and the last term on the right side can be bounded in the same way as there, except that now, instead of (3-27), we use the bound

$$\|\nabla q\|_2 \lesssim \lambda^{-1}, \tag{6-11}$$

which follows from the bounds on s and r in Propositions 3.2 and (6-6). For the second term on the right side we proceed as in the proof of Lemma 5.4 and obtain

$$\Big| \int_\Omega q (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) \Big| \lesssim \varepsilon \lambda^{1-\varepsilon/2} \int_\Omega |q| U_{x,\lambda}^5 |x-y| \le \varepsilon \lambda^{1-\varepsilon/2} \|U^5 |x-y|\|_{6/5} \|q\|_6 \lesssim \varepsilon \|q\|_6 \lesssim \varepsilon \lambda^{-1}.$$

By Proposition 5.5, this is $\mathcal{O}(\lambda^{-2})$.

Finally, we bound $\mathcal{R}_0$, the term on the right side of (6-10). Because of (6-11), the first integral in the definition of $\mathcal{R}_0$ is $\mathcal{O}(\lambda^{-2})$. The second integral is bounded, in absolute value, by a constant times

$$\int_\Omega (|\psi_{x,\lambda}|^{4-\varepsilon} q^2 + |q|^{6-\varepsilon}) \lesssim \|\psi_{x,\lambda}\|_6^{4-\varepsilon} \|q\|_6^2 + \|q\|_6^{6-\varepsilon} \lesssim \lambda^{-2}.$$

Inserting all the bounds in (6-10), we obtain the claimed bound. ☐

6C. *Expanding $\phi_a(x)$.* In this subsection we prove the following important expansion.

Proposition 6.4. *As $\varepsilon \to 0$,*

$$\phi_a(x) = \pi a(x) \lambda^{-1} + \frac{\pi}{32} \varepsilon \lambda (1 + o(1)) + o(\lambda^{-1}). \tag{6-12}$$

The proof of this proposition, which is the analogue of Proposition 3.8, is a refined version of the proof of Proposition 5.5. We integrate (1-2) for u against $\partial_\lambda \psi_{x,\lambda}$, and we write the resulting equality in the form

$$\begin{aligned}\int_\Omega (\nabla \psi_{x,\lambda} \cdot \nabla \partial_\lambda \psi_{x,\lambda} + a \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} - 3\alpha^{4-\varepsilon} |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda}) \\ = -\int_\Omega (\nabla q \cdot \nabla \partial_\lambda \psi_{x,\lambda} + a q \partial_\lambda \psi_{x,\lambda} - 3(5-\varepsilon) \alpha^{4-\varepsilon} q |\psi_{x,\lambda}|^{4-\varepsilon} \partial_\lambda \psi_{x,\lambda}) \\ + \frac{3(5-\varepsilon)(4-\varepsilon)}{2} \alpha^{4-\varepsilon} \int_\Omega q^2 |\psi_{x,\lambda}|^{2-\varepsilon} \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} + \mathcal{R},\end{aligned} \tag{6-13}$$

with

$$\mathcal{R} = 3\alpha^{4-\varepsilon} \int_\Omega \bigg((\psi_{x,\lambda} + q)^{5-\varepsilon} - |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} - (5-\varepsilon) |\psi_{x,\lambda}|^{4-\varepsilon} q - \frac{(5-\varepsilon)(4-\varepsilon)}{2} |\psi_{x,\lambda}|^{2-\varepsilon} \psi_{x,\lambda} q^2 \bigg) \partial_\lambda \psi_{x,\lambda}.$$

Lemma 6.5. *As* $\varepsilon \to 0$, *the following hold*:

(a) $$\int_\Omega (\nabla \psi_{x,\lambda} \cdot \nabla \partial_\lambda \psi_{x,\lambda} + a \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} - 3\alpha^{4-\varepsilon} |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda}) = -2\pi \phi_a(x) \lambda^{-2} (1+o(1)) + \frac{\pi^2}{16} \varepsilon \lambda^{-1} (1+o(1)) + 2\pi^2 a(x) \lambda^{-3} + o(\lambda^{-3}).$$

(b) $$\int_\Omega (\nabla q \cdot \nabla \partial_\lambda \psi_{x,\lambda} + a q \partial_\lambda \psi_{x,\lambda} - 3(5-\varepsilon) \alpha^{4-\varepsilon} q |\psi_{x,\lambda}|^{4-\varepsilon} \partial_\lambda \psi_{x,\lambda}) = -(1-\alpha^{4-\varepsilon}) 2\pi (\phi_a(x) - \phi_0(x)) \lambda^{-2} + \mathcal{O}(\varepsilon \lambda^{-2} \log \lambda + \phi_a(x) \lambda^{-3}) + o(\lambda^{-3}).$$

(c) $$\int_\Omega q^2 |\psi_{x,\lambda}|^{2-\varepsilon} \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} = \frac{\pi^2}{32} \beta \gamma \lambda^{-3} + \mathcal{O}(\varepsilon \lambda^{-2} + \phi_a(x) \lambda^{-3}) + o(\lambda^{-3}).$$

(d) $\mathcal{R} = o(\lambda^{-3})$.

The proof of Lemma 6.5 is independent of the expansion of $\alpha^{4-\varepsilon}$ in Proposition 6.3. We only use the fact that $\alpha = 1 + o(1)$.

Proof. (a) As in the proof of Lemma 3.10(a), see (3-31), we have

$$\int_\Omega (\nabla \psi_{x,\lambda} \cdot \nabla \partial_\lambda \psi_{x,\lambda} + a \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} - 3\alpha^{4-\varepsilon} |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda}) = 3 \int_\Omega (U_{x,\lambda}^5 - \alpha^{4-\varepsilon} |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda}) \partial_\lambda \psi_{x,\lambda} - \int_\Omega a (f_{x,\lambda} + g_{x,\lambda}) \partial_\lambda \psi_{x,\lambda}.$$

The second integral on the right side was shown in the proof of Lemma 3.10(a) to satisfy

$$\int_\Omega a (f_{x,\lambda} + g_{x,\lambda}) \partial_\lambda \psi_{x,\lambda} = 2\pi (3-\pi) a(x) \lambda^{-3} + o(\lambda^{-3}).$$

We write the first integral on the right side as

$$\int_\Omega (U_{x,\lambda}^5 - \alpha^{4-\varepsilon} |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda}) \partial_\lambda \psi_{x,\lambda} = (1-\alpha^{4-\varepsilon}) \int_\Omega U_{x,\lambda}^5 \partial_\lambda \psi_{x,\lambda} - \alpha^{4-\varepsilon} \int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) \partial_\lambda \psi_{x,\lambda} - \alpha^{4-\varepsilon} \int_\Omega (|\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} - U_{x,\lambda}^{5-\varepsilon}) \partial_\lambda \psi_{x,\lambda}. \tag{6-14}$$

As shown in the proof of Lemma 3.10(a),

$$\int_\Omega U_{x,\lambda}^5 \partial_\lambda \psi_{x,\lambda} = \frac{2\pi}{3} \phi_a(x) \lambda^{-2} + \mathcal{O}(\lambda^{-3}).$$

Next, by Lemma A.2,

$$\int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) \partial_\lambda \psi_{x,\lambda} = \int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) \partial_\lambda U_{x,\lambda} + \tfrac{1}{2} \lambda^{-3/2} \int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) H_a(x,\cdot) + o(\lambda^{-3}).$$

For the first term, we use (5-20) and the bounds from the proof of Lemma 3.10(a) to get

$$\int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5)\partial_\lambda U_{x,\lambda} = -\frac{\pi^2}{48}\varepsilon\lambda^{-1}(1+o(1)) + \mathcal{O}(\lambda^{-4}).$$

For the second term, we use the bound $\|U_{x,\lambda}^{-\varepsilon} - 1\|_\infty = \mathcal{O}(\varepsilon \log \lambda)$ and compute

$$\lambda^{-3/2}\left|\int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5)H_a(x,\cdot)\right| \lesssim \varepsilon\lambda^{-3/2}\log\lambda \int_\Omega U_{x,\lambda}^5 H_a(x,\cdot) \lesssim \varepsilon\lambda^{-2}\log\lambda = o(\varepsilon\lambda^{-1}).$$

Concerning the last term on the right-hand side of (6-14), we will prove

$$\int_\Omega (|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} - U_{x,\lambda}^{5-\varepsilon})\partial_\lambda \psi_{x,\lambda} = \frac{2\pi}{3}\phi_a(x)\lambda^{-2}(1+o(1)) - 2\pi a(x)\lambda^{-3} + \mathcal{O}(\phi_a(x)^2\lambda^{-3}) + o(\lambda^{-3}). \quad (6\text{-}15)$$

This will complete our discussion of the right-hand side of (6-14) and hence the proof of (a).

The proof of (6-15) is similar to the corresponding argument in the proof of Lemma 3.10(a), but we include some details. We bound pointwise

$$|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} - U_{x,\lambda}^{5-\varepsilon} = -(5-\varepsilon)\lambda^{-1/2}U_{x,\lambda}^{4-\varepsilon}H_a(x,\cdot) + \tfrac{1}{2}(5-\varepsilon)(4-\varepsilon)\lambda^{-1}U_{x,\lambda}^{3-\varepsilon}H_a(x,\cdot)^2 + \mathcal{O}(\lambda^{-3/2}U_{x,\lambda}^2|H_a(x,\cdot)|^3 + \lambda^{-5/2}|H_a(x,\cdot)|^5 + U_{x,\lambda}^4|f_{x,\lambda}| + |f_{x,\lambda}|^5).$$

Using the bounds from Lemmas A.1 and A.2, we easily find that the remainder term, when integrated against $|\partial_\lambda \psi_{x,\lambda}|$, is $o(\lambda^{-3})$. Using expansion (B-5) we obtain, by an explicit calculation similar to (B-11) and (B-13),

$$\begin{aligned}\int_\Omega U_{x,\lambda}^{4-\varepsilon}H_a(x,\cdot)\partial_\lambda\psi_{x,\lambda} &= \int_\Omega U_{x,\lambda}^{4-\varepsilon}\partial_\lambda U_{x,\lambda}H_a(x,\cdot) + \mathcal{O}(\lambda^{-5/2}\phi_a(x)^2) + o(\lambda^{-5/2}) \\ &= -\Big(\frac{2\pi}{15} + \mathcal{O}(\varepsilon)\Big)\phi_a(x)\lambda^{-(3+\varepsilon)/2} + \frac{2\pi}{5}a(x)\lambda^{-5/2} + \mathcal{O}(\lambda^{-5/2}\phi_a(x)^2) + o(\lambda^{-5/2}) \\ &= -\frac{2\pi}{15}\phi_a(x)\lambda^{-3/2}(1+o(1)) + \frac{2\pi}{5}a(x)\lambda^{-5/2} + \mathcal{O}(\lambda^{-5/2}\phi_a(x)^2) + o(\lambda^{-5/2}),\end{aligned}$$

where we used Lemma 5.3. In the same way, we get

$$\int_\Omega U_{x,\lambda}^{3-\varepsilon}H_a(x,\cdot)^2\partial_\lambda\psi_{x,\lambda} = \mathcal{O}(\lambda^{-2}\phi_a^2(x)) + o(\lambda^{-2}).$$

This proves (6-15).

(b) As in the proof of Lemma 3.10(b) we have

$$\int_\Omega (\nabla q\cdot\nabla\partial_\lambda\psi_{x,\lambda} + aq\partial_\lambda\psi_{x,\lambda} - 3(5-\varepsilon)\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}q\partial_\lambda\psi_{x,\lambda}) = 3\int_\Omega q(5U_{x,\lambda}^4\partial_\lambda U_{x,\lambda} - (5-\varepsilon)\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}\partial_\lambda\psi_{x,\lambda}) - \int_\Omega aq(\partial_\lambda f_{x,\lambda} + \partial_\lambda g_{x,\lambda}).$$

According to (3-41), the second term on the right side is $o(\lambda^{-3})$. (Note that we now use the bound (6-11) instead of (3-27).) We write the first integral as

$$\int_\Omega q(5U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} - (5-\varepsilon)\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}\partial_\lambda \psi_{x,\lambda})$$
$$= (5(1-\alpha^{4-\varepsilon}) + \varepsilon\alpha^{4-\varepsilon}) \int_\Omega qU_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} + (5-\varepsilon)\alpha^{4-\varepsilon} \int_\Omega q(U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} - \psi_{x,\lambda}^4 \partial_\lambda \psi_{x,\lambda})$$
$$+ (5-\varepsilon)\alpha^{4-\varepsilon} \int_\Omega q(\psi_{x,\lambda}^4 - |\psi_{x,\lambda}|^{4-\varepsilon})\partial_\lambda \psi_{x,\lambda}.$$

According to (3-39),

$$(5(1-\alpha^{4-\varepsilon})+\varepsilon\alpha^{4-\varepsilon}) \int_\Omega qU_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} = (5(1-\alpha^{4-\varepsilon})+\varepsilon\alpha^{4-\varepsilon})\Big(-\frac{2\pi}{15}(\phi_a(x)-\phi_0(x))\lambda^{-2}+\mathcal{O}(\lambda^{-3})\Big)$$
$$= -\frac{2\pi}{3}(1-\alpha^{4-\varepsilon})(\phi_a(x)-\phi_0(x))\lambda^{-2}+\mathcal{O}(\varepsilon\lambda^{-2})+o(\lambda^{-3}),$$

and according to (3-40), using (6-11) instead of (3-27),

$$\int_\Omega q(U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} - \psi_{x,\lambda}^4 \partial_\lambda \psi_{x,\lambda}) = \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\lambda^{-3}).$$

Finally, for any fixed $\delta \in (0, d(x))$ and for any $p > 1$ we have, by Lemma A.2,

$$\|\psi_{x,\lambda}^p \partial_\lambda \psi_{x,\lambda}\|_{L^\infty(B_\delta(x)^c \cap \Omega)} = \mathcal{O}(\lambda^{-(3+p)/2}). \tag{6-16}$$

On the other hand, taking δ sufficiently small (but independent of ε) we obtain $U_{x,\lambda} \lesssim \psi_{x,\lambda} \lesssim U_{x,\lambda}$ on $B_\delta(x)$. The latter implies $\psi_{x,\lambda}^{-\varepsilon} = U_{x,\lambda}^{-\varepsilon}(1+\mathcal{O}(\varepsilon))$ on $B_\delta(x)$, and therefore

$$\|1-\psi_{x,\lambda}^{-\varepsilon}\|_{L^\infty(B_\delta(x))} = \mathcal{O}(\varepsilon \log \lambda).$$

Consequently, using (6-11) and (6-16),

$$\left|\int_\Omega q(\psi_{x,\lambda}^4 - |\psi_{x,\lambda}|^{4-\varepsilon})\partial_\lambda \psi_{x,\lambda}\right| \lesssim \|q\|_6(\varepsilon \log\lambda \|\psi_{x,\lambda}^4 \partial_\lambda \psi_{x,\lambda}\|_{6/5} + \lambda^{-7/2}) \lesssim \varepsilon\lambda^{-2}\log\lambda + \lambda^{-9/2}.$$

Collecting all the bounds, we arrive at the claimed expansion in (b).

(c) The relevant term with exponent $2-\varepsilon$ replaced by 2 was computed in Lemma 3.10(c). The same computation, but with Proposition 6.2 instead of Proposition 3.4, gives

$$\int_\Omega q^2 \psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda} = \frac{\pi^2}{32}\beta\gamma\lambda^{-3} + \mathcal{O}(\varepsilon\lambda^{-2} + \phi_a(x)\lambda^{-3}) + o(\lambda^{-3}).$$

(The $\mathcal{O}(\varepsilon\lambda^{-2})$ term comes from bounding $\int_\Omega rs\psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda}$.)

We bound the difference similarly as at the end of the previous part (b), namely,

$$\left|\int_\Omega q^2(|\psi_{x,\lambda}|^{2-\varepsilon}\psi_{x,\lambda} - \psi_{x,\lambda}^3)\partial_\lambda \psi_{x,\lambda}\right| \lesssim \|q\|_6^2(\varepsilon\log\lambda\|\psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda}\|_{3/2} + \lambda^{-3})$$
$$\lesssim \varepsilon\lambda^{-3}\log\lambda + \lambda^{-5} = o(\lambda^{-3}).$$

The proof of (d) uses similar bounds as in the rest of the proof and is omitted. □

Proof of Proposition 6.4. Inserting the bounds from Lemma 6.5 into (6-13), we obtain

$$\phi_a(x)(1+o(1)) - \frac{\pi}{32}\varepsilon\lambda(1+o(1)) - \pi a(x)\lambda^{-1} - (1-\alpha^{4-\varepsilon})\phi_0(x) + \frac{15\pi}{32}\beta\gamma\lambda^{-1} = o(\lambda^{-1}).$$

Inserting the expansion of $\alpha^{4-\varepsilon}$ from Proposition 6.3, this becomes

$$\phi_a(x)(1+o(1)) - \frac{\pi}{32}\varepsilon\lambda(1+o(1)) - \pi a(x)\lambda^{-1} - 4\beta\phi_0(x)\lambda^{-1} + \frac{15\pi}{32}\beta\gamma\lambda^{-1} = o(\lambda^{-1}).$$

Using the expansions (3-13) of β and γ, this can be simplified to

$$\phi_a(x)(1+o(1)) - \frac{\pi}{32}\varepsilon\lambda(1+o(1)) - \pi a(x)\lambda^{-1} = o(\lambda^{-1}),$$

which is the assertion. □

6D. *Bounding $\nabla\phi_a$.* In this subsection we prove the bound on $\nabla\phi_a(x)$ in Proposition 6.1.

Proposition 6.6. *For every $\mu < 1$, as $\varepsilon \to 0$,*

$$|\nabla\phi_a(x)| \lesssim \varepsilon\lambda^{1/2} + \lambda^{-\mu} + \phi_a(x)\lambda^{-1/2}. \tag{6-17}$$

Note that together with (5-2) it follows from Proposition 6.6 that x_0 is a critical point of ϕ_a.

The proof of Proposition 6.6 is a refined version of the proof of Proposition 5.8 and is again based on the Pohozaev identity (5-21). The latter reads, in the notation of (3-46),

$$0 = I[\psi_{x,\lambda}] + 2I[\psi_{x,\lambda}, q] + I[q]. \tag{6-18}$$

To control the boundary integrals involving q in this identity, we need the following lemma, which is the analogue of Lemma 3.13.

Lemma 6.7.
$$\left\|\frac{\partial q}{\partial n}\right\|_{L^2(\partial\Omega)} \lesssim \varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}.$$

Before proving this lemma, let us use it to complete the proof of Proposition 6.6. In that proof, and later in this subsection, we will use the inequality

$$\|q\|_2 \lesssim \varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}. \tag{6-19}$$

This follows from the bound (3-10) on s and the bound in Proposition 6.2 on r.

Proof of Proposition 6.6. It follows from Lemma 6.7 and the bounds (6-19) and (3-49) that

$$|I[\psi_{x,\lambda}, q]| \lesssim \varepsilon\lambda^{-1/2} + \lambda^{-2} + \phi_a(x)\lambda^{-3/2}, \quad |I[q]| \lesssim \varepsilon^2 + \lambda^{-3} + \phi_a(x)^2\lambda^{-2}.$$

The claim thus follows from Lemma 3.12 and (6-18). □

Proof of Lemma 6.7. Note that $-\Delta q = F$, with

$$F := -3U_{x,\lambda}^5 + 3\alpha^{4-\varepsilon}(\psi_{x,\lambda} + q)^{5-\varepsilon} - aq + a(f_{x,\lambda} + g_{x,\lambda}).$$

With the cut-off function ζ defined as in the proof of Lemma 2.6, we have

$$-\Delta(\zeta q) = \zeta F - 2\nabla\zeta\cdot\nabla q - (\Delta\zeta)q.$$

Arguing as in (3-51) we deduce that

$$\zeta|F| \lesssim \zeta|q|^{5-\varepsilon} + |q| + \lambda^{-5/2}. \tag{6-20}$$

Now we follow the line of arguments in the proof of Lemma 3.13. The only difference is that instead of (3-48) we have the bound

$$\|q\|_2 \lesssim \varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}, \tag{6-21}$$

which follows from (3-10) and Proposition 6.2. Using this estimate we find

$$\|\Delta(\zeta q)\|_{3/2} \lesssim \varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}.$$

In combination with (2-12), this proves the claim. □

7. Proof of Theorems 1.2 and 1.3

7A. ***Proof of Theorem 1.2.*** Equation (1-10) follows from Proposition 5.1, together with (3-2), (3-3) and (3-5). Proposition 5.1 gives also $|x_\varepsilon - x_0| = o(1)$. Moreover, the bound on λ in (5-2) together with (6-4) gives $\nabla\phi_a(x_0) = 0$, and (6-2) gives $\|\nabla r\|_2 = \mathcal{O}(\varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1})$. By the bound on λ in (5-2), this proves the claimed bound on $\|\nabla r\|_2$ if $\phi_a(x_0) \neq 0$. In the case $\phi_a(x_0) = 0$, we will see below that $\phi_a(x) = o(\lambda^{-1})$ and $\varepsilon = \mathcal{O}(\lambda^{-2})$, so we again obtain the claimed bound.

Next, (6-3) shows that

$$\lim_{\varepsilon\to 0} \varepsilon\lambda = \frac{32}{\pi}\phi_a(x_0), \tag{7-1}$$

which is (1-12).

Equation (1-13) follows from (6-5). In the case $\phi_a(x_0) \neq 0$ this is immediate, and in the case $\phi_a(x_0) = 0$ we use, in addition, the expansion of β from Proposition 3.3 and the fact that $\varepsilon = o(\lambda^{-1})$ by (7-1).

Finally, let us assume $\phi_a(x_0) = 0$ and prove (1-15). We apply Lemma 4.2 to the function $u(x) := \phi_a(x + x_0)$ and get $\phi_a(x) \lesssim |\nabla\phi_a(x)|^2$. From (6-4), together with the fact that $\varepsilon = o(\lambda^{-1})$ by (7-1), we then get

$$\phi_a(x) = o(\lambda^{-1}). \tag{7-2}$$

Inserting this into (6-3), we obtain

$$\pi a(x)\lambda^{-1} + \frac{\pi}{32}\varepsilon\lambda(1 + o(1)) = o(\lambda^{-1}),$$

which is (1-15). This completes the proof of Theorem 1.2. □

7B. ***A bound on $\|w\|_\infty$.*** To complete the proof of Theorem 1.3 it remains to establish a suitable bound on $\|w\|_\infty$, as well as on $\|w\|_p$ for $p > 6$. This is provided by the following modification of Proposition 4.3.

Proposition 7.1. *As $\varepsilon \to 0$,*

$$\|w\|_p \lesssim \lambda^{-3/p} \quad \textit{for every } p \in (6, \infty). \tag{7-3}$$

Moreover, for every $\mu > 0$,

$$\|w\|_\infty = o(\lambda^\mu). \tag{7-4}$$

Proof. To prove the bound (7-3), let $r > 1$ and F be given by (5-22). As in the proof of Proposition 4.3, we obtain the same bound (4-10), where, similarly to (4-11), F satisfies

$$|F| \lesssim U_{x,\lambda}^{5-\varepsilon}|\alpha^{4-\varepsilon}-1| + |U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5| + U_{x,\lambda}^4(|w| + \varphi_{x,\lambda}) + |w|^5 + \varphi_{x,\lambda} + U_{x,\lambda} + |w|. \tag{7-5}$$

Using the bounds $\varepsilon \lesssim \lambda^{-1}$ from Proposition 5.1 and $|\alpha^{4-\varepsilon}-1| \lesssim \varepsilon \log\lambda$ by Proposition 6.3, we can estimate, for every $r > 1$,

$$\begin{aligned}\int_\Omega (U_{x,\lambda}^{5-\varepsilon}|\alpha^{4-\varepsilon}-1| + |U_{x,\lambda}^5 - U_{x,\lambda}^{5-\varepsilon}|)|w|^r \\ \lesssim \|w\|_{3(r+1)}^r (\|U_{x,\lambda}^{5-\varepsilon}\|_{\frac{3r+3}{2r+3}}|\alpha^{4-\varepsilon}-1| + \|U_{x,\lambda}^5 - U_{x,\lambda}^{5-\varepsilon}\|_{\frac{3r+3}{2r+3}}) \lesssim \|w\|_{3(r+1)}^r \varepsilon \log\lambda \|U_{x,\lambda}\|_{5\cdot\frac{3r+3}{2r+3}}^5 \\ \lesssim \|w\|_{3(r+1)}^r \varepsilon \log\lambda\, \lambda^{\frac{1}{2}\cdot\frac{r-1}{r+1}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta(\log\lambda)^{r+1}\lambda^{-\frac{r+3}{2}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-1}.\end{aligned}$$

Hence the right side of (4-10) fulfills the same estimate as in the proof of Proposition 4.3, and we conclude (7-3) as we did there.

We now turn to the bound (7-4). From (5-10) we deduce that

$$w(x) = \frac{1}{4\pi}\int_\Omega G_0(x,y)F(y). \tag{7-6}$$

As in Proposition 4.3, we need to estimate $\|F\|_q$ for some $q > \frac{3}{2}$ using (7-5). We bound

$$\|U_{x,\lambda}^{5-\varepsilon}|\alpha^{4-\varepsilon}-1|\|_q \lesssim (\varepsilon\log\lambda + \lambda^{-1})\|U_{x,\lambda}\|_{5q}^5 \lesssim \lambda^{3/2-3/q}\log\lambda$$

for every $q > \frac{3}{2}$. Similarly,

$$\|U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5\|_q \lesssim \varepsilon\log\lambda\|U_{x,\lambda}\|_{5q}^5 \lesssim \lambda^{3/2-3/q}\log\lambda$$

for every $q > \frac{3}{2}$. The other terms resulting from (7-5) are identical to those already estimated in Proposition 4.3. As there, we thus obtain $\|F\|_q \lesssim \lambda^{2-3/q}\log\lambda$. Letting $q \searrow \frac{3}{2}$ yields (7-4). □

7C. ***Proof of Theorem 1.3.*** At this point, the proof of Theorem 1.3 is almost identical to the proof of Theorem 1.6. We provide some details nevertheless.

By the bound $\|w\|_\infty = o(\lambda^{1/2})$ from Proposition 7.1 and Proposition 2.1, we have $\|u_\varepsilon\|_\infty = \lambda^{1/2} + o(\lambda^{1/2})$. Thus part (a) of Theorem 1.3 follows from (1-12) and (1-15), respectively.

To prove part (b), we rewrite (1-3) as

$$u(z) = \frac{3}{4\pi}\int_\Omega G_a(z,y)u(y)^{5-\varepsilon}.$$

Fix again $\delta = \delta_\varepsilon = o(1)$ with $\lambda^{-1} = o(\delta_\varepsilon)$, so that $\frac{3}{4\pi}\int_{B_{\delta_\varepsilon(x)}} u(y)^5 = 1 + o(1)$. Then

$$\frac{3}{4\pi}\int_{B_\delta(x)} G_a(z,y)u(y)^5 = \frac{3}{4\pi}\int_{B_\delta(x)} (G_a(z,x_0)+o(1))u(y)^5 = \lambda^{-1/2-\varepsilon/2}G_a(z,x_0) + o(\lambda^{-1/2-\varepsilon/2}).$$

On the other hand, by Lemmas 7.1 and A.1,

$$\left|\int_{\Omega\setminus B_\delta(x)} G_a(z,y)u(y)^{5-\varepsilon}\right| \lesssim \|G_a(z,\cdot)\|_2(\|U_{x,\lambda}\|^{5-\varepsilon}_{L^{10}(\Omega\setminus B_\delta(x))}+\|w\|_{10}^{5-\varepsilon}) \lesssim \lambda^{-5/2}\delta^{-7/2}+\lambda^{-3/2}.$$

Choosing $\delta=\lambda^{-c}$ with $c>0$ small enough and observing that $\lambda^{-\varepsilon/2}=1+o(1)$ by Lemma 5.3, the proof of part (b) of Theorem 1.3 is complete. □

Appendix A: Some useful bounds

In this section, we collect some bounds which will be of frequent use in our estimates.

Lemma A.1. *Let $x\in\Omega$ and let $1\le q<\infty$. As $\lambda\to\infty$, we have*

$$\|U_{x,\lambda}\|_{L^q(\Omega)} \lesssim \begin{cases} \lambda^{-1/2}, & 1\le q<3,\\ \lambda^{-1/2}(\log\lambda)^{1/3}, & q=3,\\ \lambda^{1/2-3/q}, & q>3. \end{cases} \tag{A-1}$$

Moreover, we have

$$\partial_{x_i}U_{x,\lambda}(y)=\lambda^{5/2}\frac{y_i-x_i}{(1+\lambda^2|x-y|^2)^{3/2}},$$

with

$$\|\partial_{x_i}U_{x,\lambda}\|_{L^q(\Omega)} \lesssim \begin{cases} \lambda^{-1/2}, & 1\le q<\frac32,\\ \lambda^{-1/2}(\log\lambda)^{2/3}, & q=\frac32,\\ \lambda^{3/2-3/q}, & q>\frac32, \end{cases}$$

and

$$\partial_\lambda U_{x,\lambda}(y)=\tfrac12\lambda^{-1/2}\frac{1-\lambda^2|x-y|^2}{(1+\lambda^2|x-y|^2)^{3/2}},$$

with

$$\|\partial_\lambda U\|_q\le\lambda^{-1}\|U\|_q \quad \textit{for any } 1\le q\le\infty.$$

Moreover, for any $\rho=\rho_\lambda$ with $\rho\lambda\to\infty$,

$$\|U\|_{L^q(\Omega\setminus B_\rho(x))} \lesssim \begin{cases} \lambda^{-1/2}, & 1\le q<3,\\ \lambda^{-1/2}(\log\lambda)^{1/3}, & q=3,\\ \lambda^{-1/2}\rho^{(3-q)/q}, & q>3, \end{cases}$$

and

$$\|\partial_\lambda U\|_{L^q(\Omega\setminus B_\rho(x))} \lesssim \begin{cases} \lambda^{-3/2}, & 1\le q<3,\\ \lambda^{-3/2}(\log\lambda)^{1/3}, & q=3,\\ \lambda^{-3/2}\rho^{(3-q)/q}, & q>3, \end{cases}$$

and

$$\|\partial_{x_i}U\|_{L^q(\Omega\setminus B_\rho(x))} \lesssim \begin{cases} \lambda^{-1/2}, & 1\le q<\frac32,\\ \lambda^{-1/2}(\log\lambda)^{2/3}, & q=\frac32,\\ \lambda^{-1/2}\rho^{(3-2q)/q}, & q>\frac32. \end{cases}$$

Proof. Taking $R > 0$ such that $\Omega \subset B_R(x)$, we have

$$\int_\Omega U_{x,\lambda}^q \lesssim \lambda^{-3+q/2} \int_0^{\lambda R} \frac{r^2}{(1+r^2)^{q/2}} \lesssim \lambda^{-3+q/2} \int_1^{\lambda R} r^{2-q} \lesssim \begin{cases} \lambda^{-q/2}, & 1 \le q < 3, \\ \lambda^{-q/2}(\log \lambda)^{1/3}, & q = 3, \\ \lambda^{q/2-3}, & q > 3. \end{cases}$$

This proves (A-1). The remaining bounds follow by analogous explicit computations, which we omit. ☐

Lemma A.2. *We have*

$$U_{x,\lambda} = PU_{x,\lambda} + \lambda^{-1/2} H_0(x,\cdot) + f_{x,\lambda},$$

with

$$\|f_{x,\lambda}\|_\infty \lesssim \lambda^{-5/2} d^{-3}, \quad \|\partial_\lambda f_{x,\lambda}\|_\infty \lesssim \lambda^{-7/2} d^{-3}, \quad \|\partial_{x_i} f_{x,\lambda}\|_\infty \lesssim \lambda^{-5/2} d^{-4}. \tag{A-2}$$

The function $\varphi_{x,\lambda} := \lambda^{-1/2} H_0(x,\cdot) + f_{x,\lambda}$ *satisfies* $0 \le \varphi_{x,\lambda} \le U_{x,\lambda}$ *as well as*

$$\|\varphi_{x,\lambda}\|_6 \lesssim \lambda^{-1/2} d^{-1/2}, \quad \|\varphi_{x,\lambda}\|_\infty \lesssim \lambda^{-1/2} d^{-1}. \tag{A-3}$$

Moreover,

$$\|\partial_\lambda \varphi_{x,\lambda}\|_6 \lesssim \lambda^{-3/2} d^{-1/2}, \quad \|\partial_\lambda \varphi_{x,\lambda}\|_\infty \lesssim \lambda^{-3/2} d^{-1}$$

and

$$\|\partial_{x_i} \varphi_{x,\lambda}\|_6 \lesssim \lambda^{-1/2} d^{-1/2}, \quad \|\partial_{x_i} \varphi_{x,\lambda}\|_\infty \lesssim \lambda^{-1/2} d^{-2}.$$

Proof. Everything, except for the L^∞ bounds on $\varphi_{x,\lambda}$, $\partial_{x_i}\varphi_{x,\lambda}$ and $\partial_\lambda \varphi_{x,\lambda}$, is taken from [Rey 1990, Proposition 1]. Since these functions are harmonic, the remaining bounds follow from the maximum principle. ☐

Lemma A.3. *We have*

(a) $\displaystyle \int_{\partial\Omega} n \left(\frac{\partial PU_{x,\lambda}}{\partial n} \right)^2 = C \lambda^{-1} \nabla \phi_0(x) + o(\lambda^{-1} d^{-2})$ *for some constant* $C > 0$,

(b) $\displaystyle \int_{\partial\Omega} y \cdot n \left(\frac{\partial PU_{x,\lambda}}{\partial n} \right)^2 = \mathcal{O}(\lambda^{-1} d^{-2})$,

(c) $\displaystyle \int_{\partial\Omega} \left(\frac{\partial PU_{x,\lambda}}{\partial n} \right)^2 = \mathcal{O}(\lambda^{-1} d^{-2})$.

For the proof of Lemma A.3 we refer to [Rey 1990] Equations (2.7), (2.10), and (B.25), respectively.

We define the function

$$g_{x,\lambda}(y) := \frac{\lambda^{-1/2}}{|x-y|} - U_{x,\lambda}(y). \tag{A-4}$$

Lemma A.4. *As* $\lambda \to \infty$,

$$\|g_{x,\lambda}\|_p \lesssim \lambda^{1/2-3/p} \quad \text{and} \quad \|\partial_\lambda g_{x,\lambda}\|_p \lesssim \lambda^{-1/2-3/p}$$

hold if $1 \le p < 3$. *Moreover,* $\nabla g_{x,\lambda} \in L^p(\mathbb{R}^3)$ *for all* $1 \le p < \frac{3}{2}$.

Proof. We have $g_{x,\lambda}(y) = \lambda^{1/2} g_{0,1}(\lambda(x-y))$ with $g_{0,1}(z) = |z|^{-1} - (1+|z|^2)^{-1/2}$. As $|z| \to \infty$,

$$g_{0,1}(z) = |z|^{-1}(1-(1+|z|^{-2})^{-1/2}) \lesssim |z|^{-3}.$$

Hence $g_{0,1} \in L^p(\mathbb{R}^3)$ for all $1 \le p < 3$, which yields $\|g_{x,\lambda}\|_p \le \lambda^{1/2-3/p} \|g_{0,1}\|_{L^p(\mathbb{R}^3)}$.

Next, by direct calculation,

$$\nabla g_{0,1}(z) = -\frac{z}{|z|^3} + \frac{z}{(1+|z|^2)^{3/2}} \lesssim |z|^{-4} \quad \text{as } |z| \to \infty.$$

Hence $\nabla g_{0,1} \in L^p(\mathbb{R}^3)$ for all $1 \le p < \frac{3}{2}$ and since $\nabla g_{x,\lambda}(x,y) = \lambda^{3/2}(\nabla g_{0,1})(\lambda(x-y))$, we conclude that $\nabla g_{x,\lambda} \in L^p(\mathbb{R}^3)$ for all $1 \le p < \frac{3}{2}$.

Finally, we observe

$$\partial_\lambda g_{x,\lambda}(y) = \lambda^{-1} g_{x,\lambda} + \lambda^{1/2}(x-y)\cdot(\nabla g_{0,1})(\lambda(x-y)).$$

By the above, we have $z \cdot \nabla g_{0,1} \in L^p(\mathbb{R}^3)$ for all $1 \le p < 3$ and thus

$$\|\partial_\lambda g_{x,\lambda}\|_p \le \lambda^{-1}\|g_{x,\lambda}\|_p + \lambda^{-1/2-3/p}\|z\cdot\nabla g_{0,1}\|_{L^p(\mathbb{R}^3)}$$

for all $1 \le p < 3$. □

Appendix B: Properties of the functions $H_a(x, y)$

In this appendix, we prove some properties of $H_a(x, y)$ needed in the proofs of the main results. Since these properties hold independently of the criticality of a, we state them for a generic function b which satisfies the same regularity conditions as a, namely,

$$b \in C(\overline{\Omega}) \cap C^{2,\sigma}_{\mathrm{loc}}(\Omega) \quad \text{for some } 0 < \sigma < 1.$$

(In fact, in Section B1 we only use $b \in C(\overline{\Omega}) \cap C^{1,\sigma}_{\mathrm{loc}}(\Omega)$ for some $0 < \sigma < 1$.) In addition, we assume that $-\Delta + b$ is coercive in Ω with Dirichlet boundary conditions. Note that the choice $b = 0$ is allowed.

B1. ***Estimates on $H_b(x,\cdot)$.*** We start by recalling the bound

$$\|H_b(x,\cdot)\|_\infty \lesssim d(x)^{-1} \quad \text{for all } x \in \Omega, \tag{B-1}$$

see [Frank et al. 2021, Equation (2.6)]. We next prove a similar bound for the derivatives of $H_b(x,\cdot)$.

Lemma B.1. *Let $x, y \in \Omega$ with $x \ne y$. Then $\nabla_x H_b(x, y)$ and $\nabla_y H_b(x, y)$ exist and satisfy*

$$\sup_{y \in \Omega\setminus\{x\}} |\nabla_x H_b(x,y)| \le C, \tag{B-2}$$

$$\sup_{y \in \Omega\setminus\{x\}} |\nabla_y H_b(x,y)| \le C, \tag{B-3}$$

with C uniform for x in compact subsets of Ω.

Proof. Step 1: We first prove the bounds for the special case $b = 0$, which we shall need as an ingredient for the general proof. Since $H_0(x,\cdot)$ is harmonic, we have $\Delta_y \nabla_y H_0(x, y) = 0$. Moreover, we have the bound $\nabla_y G_0(x, y) \lesssim |x-y|^{-2}$ uniformly for $x, y \in \Omega$ [Widman 1967, Theorem 2.3]. This implies that for x in a compact subset of Ω and for $y \in \partial\Omega$,

$$|\nabla_y H_0(x, y)| = |\nabla_y(|x-y|^{-1}) - \nabla_y G_0(x, y)| \leq C.$$

We now conclude by the maximum principle.

The proof for the bound on $\nabla_x H_0(x, y)$ is analogous, but simpler, because $\nabla_x G_0(x, y) = 0$ for $y \in \partial\Omega$.

Step 2: For general b, we first prove the bounds for both x and y lying in a compact subset of Ω. By [Frank et al. 2021, Proof of Lemma 2.5] we have

$$H_b(x, y) = \phi_b(x) + \Psi_x(y) - \tfrac{1}{2} b(x)|y-x|,$$

with $\|\Psi_x\|_{C^{1,\mu}(K)} \leq C$ for every $0 < \mu < 1$ and every compact subset K of Ω, and with C uniform for x in compact subsets. This shows that $|\nabla_y H_b(x, y)| \leq C$ uniformly for x, y in compact subsets of Ω. By symmetry of H_b, this also implies $|\nabla_x H_b(x, y)| \leq C$ uniformly for x, y in compact subsets of Ω.

Step 3: We complete the proof of the lemma by treating the case when x remains in a compact subset but y is close to the boundary. In particular, for what follows we may assume

$$|x-y|^{-1} \lesssim 1. \tag{B-4}$$

By the resolvent formula, we write

$$H_b(x, y) = H_0(x, y) + \frac{1}{4\pi} \int_\Omega G_0(x, z) b(z) G_b(z, y)\, dz.$$

By Step 1, the derivatives of $H_0(x, y)$ are uniformly bounded.

We thus only need to consider the integral term. Its ∂_{x_i}-derivative equals

$$\int_\Omega \partial_{x_i}\left(\frac{1}{|x-z|}\right) b(z) G_b(z, y)\, dz - \int_\Omega \partial_{x_i} H_0(x, z) b(z) G_b(z, y)\, dz \\ \lesssim \int_\Omega \frac{1}{|x-z|^2} \frac{1}{|z-y|}\, dz + 1 \lesssim \frac{1}{|x-y|^2} + 1 \lesssim 1,$$

where we again used the fact that (B-2) holds for $b = 0$, together with (B-4). This completes the proof of (B-2).

The proof of (B-3) can be completed analogously. It suffices to write the resolvent formula as

$$H_b(x, y) = H_0(x, y) + \frac{1}{4\pi} \int_\Omega G_b(x, z) b(z) G_0(z, y)\, dz$$

in order to ensure that the ∂_{y_i}-derivative falls on G_0 and we can use (B-3) for $b = 0$. □

We now prove an expansion of $H_b(x, y)$ on the diagonal which improves upon [Frank et al. 2021, Lemma 2.5].

Lemma B.2. *Let $0 < \mu < 1$. If $y \to x$, then uniformly for x in compact subsets of Ω,*

$$H_b(x, y) = \phi_b(x) + \tfrac{1}{2} \nabla\phi_b(x) \cdot (y-x) - \tfrac{1}{2} b(x)|y-x| + \mathcal{O}(|y-x|^{1+\mu}). \tag{B-5}$$

Proof. In [Frank et al. 2021, Lemma 2.5], it is proved that

$$\Psi_x(y) := H_b(x,y) - \phi_b(x) + \tfrac{1}{2} b(x)|y-x| \tag{B-6}$$

is in $C^{1,\mu}_{\mathrm{loc}}(\Omega)$ (as a function of y) for any $\mu < 1$. Thus, by expanding $\Psi_x(y)$ near $y = x$,

$$H_b(x,y) = \phi_b(x) + \nabla\Psi_x(x)\cdot(y-x) - \tfrac{1}{2} b(x)|y-x| + \mathcal{O}(|y-x|^{1+\mu}). \tag{B-7}$$

This gives (B-5) provided we can show that, for each fixed $x \in \Omega$,

$$\nabla\Psi_x(x) = \tfrac{1}{2}\nabla\phi_b(x). \tag{B-8}$$

Indeed, by using (B-7) twice with the roles of x and y exchanged, subtracting and recalling $H_b(x,y) = H_b(y,x)$, we get

$$\begin{aligned}\phi_b(y) - \phi_b(x) &= (\nabla\Psi_y(y) + \nabla\Psi_x(x))(y-x) + \tfrac{1}{2}(b(y)-b(x))|x-y| + \mathcal{O}(|x-y|^{1+\mu}) \\ &= (\nabla\Psi_y(y) + \nabla\Psi_x(x))(y-x) + \mathcal{O}(|x-y|^{1+\mu}),\end{aligned} \tag{B-9}$$

because $b \in C^{0,\mu}_{\mathrm{loc}}(\Omega)$. We now argue that $\Psi_y \to \Psi_x$ in $C^1_{\mathrm{loc}}(\Omega)$, which implies $\nabla\Psi_y(y) \to \nabla\Psi_x(x)$. Together with this, (B-8) follows from (B-9).

To justify the convergence of Ψ_y we argue similarly as in [Frank et al. 2021, Lemma 2.5]. We note that $-\Delta_z\Psi_y = F_y(z)$, with

$$F_y(z) := \frac{b(z)-b(y)}{|z-y|} - b(z)H_b(y,z).$$

We claim that $F_y \to F_x$ in $L^p_{\mathrm{loc}}(\Omega)$ for any $p < \infty$. Indeed, the first term in the definition of F_y converges pointwise to F_x in $\Omega \setminus \{x\}$ and is locally bounded, independently of y, since $b \in C^{0,1}_{\mathrm{loc}}(\Omega)$. Thus, by dominated convergence it converges in $L^p_{\mathrm{loc}}(\Omega)$ for any $p < \infty$. Convergence in $L^\infty_{\mathrm{loc}}(\Omega)$ of the second term in the definition of F_y follows from the bound on the gradient of H_b in Lemma B.1. This proves the claim.

By elliptic regularity, the convergence $F_y \to F_x$ in $L^p_{\mathrm{loc}}(\Omega)$ implies the convergence $\Psi_y \to \Psi_x$ in $C^{1,1-3/p}_{\mathrm{loc}}(\Omega)$. This completes the proof. ☐

Lemma B.3. *For any $x \in \Omega$ we have, as $\lambda \to \infty$,*

$$\int_\Omega U_{x,\lambda}^5 H_b(x,\cdot) = \frac{4\pi}{3}\phi_b(x)\lambda^{-1/2} - \frac{4\pi}{3} b(x)\lambda^{-3/2} + o(\lambda^{-3/2}), \tag{B-10}$$

$$\int_\Omega U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} H_b(x,\cdot) = -\frac{2\pi}{15}\phi_b(x)\lambda^{-3/2} + \frac{2\pi}{5} b(x)\lambda^{-5/2} + o(\lambda^{-5/2}), \tag{B-11}$$

$$\int_\Omega U_{x,\lambda}^4 \partial_{x_i} U_{x,\lambda} H_b(x,\cdot) = \frac{2\pi}{15}\nabla\phi_b(x)\lambda^{-1/2} + o(\lambda^{-1/2}), \tag{B-12}$$

$$\int_\Omega U_{x,\lambda}^4 H_b(x,\cdot)^2 = \pi^2\phi_b(x)^2\lambda^{-1} + o(\lambda^{-1}), \tag{B-13}$$

$$\int_\Omega U_{x,\lambda}^3 \partial_\lambda U_{x,\lambda} H_b(x,\cdot)^2 = -\frac{\pi^2}{4}\phi_b(x)^2\lambda^{-2} + o(\lambda^{-2}). \tag{B-14}$$

The implied constants can be chosen uniformly for x in compact subsets of Ω.

Proof. Equalities (B-10) and (B-13) are proved in [Frank et al. 2021, Lemmas 2.5 and 2.6]. To prove (B-11), we write

$$\partial_\lambda U_{x,\lambda} = \frac{U_{x,\lambda}}{2\lambda} - \lambda^{3/2}\frac{|x-y|^2}{(1+\lambda^2|x-y|^2)^{3/2}}, \tag{B-15}$$

and therefore, using (B-10),

$$\int_\Omega H_b(x,y)U_{x,\lambda}^4\partial_\lambda U_{x,\lambda} = \frac{2\pi}{3}\phi_b(x)\lambda^{-3/2} - \frac{2\pi}{3}b(x)\lambda^{-5/2} - \lambda^{7/2}\int_\Omega H_b\frac{|x-y|^2}{(1+\lambda^2|x-y|^2)^{7/2}} + o(\lambda^{-5/2}).$$

With the help of (B-5) and the bound (B-1) we get

$$\begin{aligned}\int_\Omega H_b\frac{|x-y|^2}{(1+\lambda^2|x-y|^2)^{7/2}} &= 4\pi\phi_b(x)\lambda^{-5}\int_0^\infty \frac{t^4\,dt}{(1+t^2)^{7/2}} - 2\pi b(x)\lambda^{-6}\int_0^\infty \frac{t^5\,dt}{(1+t^2)^{7/2}} + o(\lambda^{-6}) \\ &= \frac{4\pi}{5}\phi_b(x)\lambda^{-5} - \frac{16\pi}{15}b(x)\lambda^{-6} + o(\lambda^{-6}).\end{aligned}$$

Combining the last two equations gives (B-11).

For the proof of (B-14) we again use (B-15), but now we use (B-13) instead of (B-10). The constant comes from

$$\int_0^\infty \frac{t^4\,dt}{(1+t^2)^3} = \frac{3\pi}{16}.$$

We omit the details.

For the proof of (B-12) we use the explicit formula for $\partial_{x_i}U_{x,\lambda}$ in Lemma A.1. We split the integral into $B_d(x)$ and $\Omega\setminus B_d(x)$. In the first one, we used the bound (B-1) and the expansion (B-5). By oddness, the contribution coming from $\phi_a(x)$ cancels, as does the contribution from $\sum_{k\neq i}\partial_k\phi_b(x)(y_k-x_k)$. For the remaining term we use

$$\int_{B_d(x)} U_{x,\lambda}^4(y)\partial_{x_i}U_{x,\lambda}(y)(y_i-x_i) = \frac{4\pi}{3}\lambda^{-1/2}\int_0^{\lambda d}\frac{t^4\,dt}{(1+t^2)^{7/2}} = \frac{4\pi}{15}\lambda^{-1/2} + \mathcal{O}(\lambda^{-5/2}).$$

A similar computation shows that the contribution from the error $|x-y|^{1+\mu}$ on $B_d(x)$ is $\mathcal{O}(\lambda^{-1/2-\mu})$. Finally, the bounds from Lemma A.1 show that the contribution from $\Omega\setminus B_d(x)$ is $\mathcal{O}(\lambda^{-5/2})$. This completes the proof. □

Remark B.4. The proof just given shows that (B-12) holds with the error bound $\mathcal{O}(\lambda^{-1/2-\mu})$ for any $0<\mu<1$ instead of $o(\lambda^{-1/2})$.

B2. ***C^2 differentiability of ϕ_a.*** In this subsection, we prove Lemma 4.1. The argument is independent of the criticality of a, and we give the proof for a general function $b\in C^{0,1}(\overline{\Omega})\cap C^{2,\sigma}_{\mathrm{loc}}(\Omega)$ for some $0<\sigma<1$. The following argument is similar to [Frank et al. 2021, Lemma 2.5], where a first-order differentiability result is proved, and to [del Pino et al. 2004, Lemma A.1], where it is shown that $\phi_b\in C^\infty(\Omega)$ for constant b.

Let

$$\Psi(x,y) := H_b(x,y) + \tfrac{1}{4}(b(x)+b(y))|x-y|, \quad (x,y)\in\Omega\times\Omega. \tag{B-16}$$

Then $\phi_b(x)=\Psi(x,x)$, so it suffices to show that $\Psi\in C^2(\Omega\times\Omega)$.

Using $-\Delta_y|x-y| = -2|x-y|^{-1}$ and $-\Delta_y H_b(x,y) = b(y)G_b(x,y)$, we have

$$-\Delta_y \Psi(x,y) = -b(y)H_b(x,y) - \frac{1}{2}\frac{b(x)-b(y)-\nabla b(y)\cdot(x-y)}{|x-y|} - \frac{1}{4}\Delta b(y)|x-y|.$$

Since $b \in C^{2,\sigma}_{\text{loc}}(\Omega)$ and since H_b is Lipschitz by Lemma B.1, the right side is in $C^{0,\sigma}_{\text{loc}}(\Omega)$ as a function of y. By elliptic regularity, $\Psi(x,y)$ is in $C^{2,\sigma}_{\text{loc}}(\Omega)$ as a function of y. Since $\Psi(x,y)$ is symmetric in x and y, we infer that $\Psi(x,y)$ is in $C^{2,\sigma}_{\text{loc}}(\Omega)$ as a function of x.

It remains to justify the existence of mixed derivatives $\partial_{y_j}\partial_{x_i}\Psi(x,y)$. For this, we carry out a similar elliptic regularity argument for the function $\partial_{x_i}\Psi(x,y)$. We have

$$-\Delta_y \partial_{x_i}\Psi(x,y) = -b(y)\partial_{x_i}H_b(x,y) - \frac{1}{4}\Delta b(y)\frac{x_i-y_i}{|x-y|} - \frac{1}{2}\frac{\partial_i b(x)-\partial_i b(y)}{|x-y|} + \frac{1}{2}\frac{x_i-y_i}{|x-y|^3}(b(x)-b(y)-\nabla b(y)\cdot(x-y)).$$

Since $b \in C^{1,1}_{\text{loc}}(\Omega)$ and since $\partial_{x_i}H_b$ is bounded by Lemma B.1, the right side is in $L^{\infty}_{\text{loc}}(\Omega)$ as a function of y. By elliptic regularity, $\partial_{x_i}\Psi(x,y) \in C^{1,\mu}(\Omega)$ for every $\mu < 1$ as a function of y. In particular, the mixed derivative $\partial_{y_j}\partial_{x_i}\Psi(x,y)$ is in $C^{0,\mu}_{\text{loc}}(\Omega)$ as a function of y. By symmetry, the same argument shows that the mixed derivative $\partial_{x_j}\partial_{y_i}\Psi(x,y)$ is in $C^{0,\mu}_{\text{loc}}(\Omega)$ as a function of x.

The proof of Lemma 4.1 is therefore complete. □

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Added in proof

The topic of this paper has been further pursued in [König and Laurain 2022; König and Laurain 2023], where the case of several blow-up points is analyzed.

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