

Hardy inequalities for large fermionic systems

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Abstract. Given $0 < s < \frac{d}{2}$ with $s \leq 1$, we are interested in the large N -behavior of the optimal constant κ_N in the Hardy inequality $\sum_{n=1}^N (-\Delta_n)^s \geq \kappa_N \sum_{n < m} |X_n - X_m|^{-2s}$, when restricted to antisymmetric functions. We show that $N^{1-\frac{2s}{d}} \kappa_N$ has a positive, finite limit given by a certain variational problem, thereby generalizing a result of Lieb and Yau related to the Chandrasekhar theory of gravitational collapse.

Dedicated to Brian Davies, in admiration, on the occasion of his 80th birthday

1. Introduction and main result

A prototypical form of Hardy's inequality states that

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx,$$

when $d \geq 3$ and $u \in \dot{H}^1(\mathbb{R}^d)$, the homogeneous Sobolev space. This and other forms of Hardy's inequality are fundamental tools in many questions in PDE, harmonic analysis, spectral theory and mathematical physics. We refer to the survey paper by Davies [6] and the books of Maz'ya [34] and Opic and Kufner [35] for extensive results, as well as background and further references.

In [18], the second and third authors and their coauthors studied what they called *many-particle Hardy inequalities*. These are inequalities for functions defined on \mathbb{R}^{dN} with coordinates denoted by $X = (X_1, \dots, X_N)$ with $X_1, \dots, X_N \in \mathbb{R}^d$. Here, $N \geq 2$ can be interpreted as the number of (quantum) particles in \mathbb{R}^d and the X_n , $n = 1, \dots, N$, as their positions. The Hardy weight takes the form $\sum_{1 \leq n < m \leq N} |X_n - X_m|^{-2}$. It is shown in [18] that

$$\sum_{n=1}^N \int_{\mathbb{R}^{dN}} |\nabla_n u(X)|^2 dX \geq \beta_N^{(d)} \sum_{1 \leq n < m \leq N} \int_{\mathbb{R}^{dN}} \frac{|u(X)|^2}{|X_n - X_m|^2} dX \quad (1)$$

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for all $u \in \dot{H}^1(\mathbb{R}^{dN})$ with a certain explicit lower bound on the optimal constant $\beta_d^{(N)}$ that is positive, provided $d \geq 3$ and $N \geq 2$. What is of interest is the behavior of the optimal constant $\beta_d^{(N)}$ as $N \rightarrow \infty$ for fixed d . This corresponds to the many-particle limit, a classical topic in mathematical physics. The explicit lower bound for $\beta_N^{(d)}$ obtained in [18] shows that $\liminf_{N \rightarrow \infty} N^{-1} \beta_N^{(d)} > 0$. It is also noted in [18] that $\limsup_{N \rightarrow \infty} N^{-1} \beta_N^{(d)} < \infty$. The methods of the present paper allow us to prove that $\lim_{N \rightarrow \infty} N^{-1} \beta_N^{(d)}$ exists and to give an explicit expression for it in terms of a variational problem for functions on \mathbb{R}^d ; see Section 2.7 for more details.

Our main interest, however, lies in a variant of inequality (1), namely, its restriction to *antisymmetric functions*. A function u on \mathbb{R}^{dN} is called antisymmetric if for any permutation σ of $\{1, \dots, N\}$ and a.e. $X \in \mathbb{R}^{dN}$ one has

$$u(X_1, \dots, X_N) = (\operatorname{sgn} \sigma) u(X_{\sigma(1)}, \dots, X_{\sigma(N)}).$$

Here, $\operatorname{sgn} \sigma \in \{+1, -1\}$ denotes the sign of σ . The antisymmetry requirement appears naturally in physics in the description of fermions. (Note that we restrict ourselves here to scalar functions, corresponding to the spinless or spin-polarized situation, although in our results for dimension $d \geq 3$ spin could be incorporated; however, for $d = 1, 2$, when $|x|^{-2}$ is not locally integrable, the spin-polarization is crucial.)

We denote by $\kappa_N^{(d)}$ the optimal constant in inequality (1) when restricted to antisymmetric functions, that is,

$$\kappa_N^{(d)} := \inf_{\substack{0 \neq u \in \dot{H}^1(\mathbb{R}^{dN}) \\ \text{antisymmetric}}} \frac{\sum_{n=1}^N \int_{\mathbb{R}^{dN}} |\nabla_n u(X)|^2 dX}{\sum_{1 \leq n < m \leq N} \int_{\mathbb{R}^{dN}} \frac{|u(X)|^2}{|X_n - X_m|^2} dX}.$$

As emphasized in [18], there are significant differences between the inequality on all functions in $\dot{H}^1(\mathbb{R}^{dN})$ and its restriction to antisymmetric ones. One important difference is that $\kappa_N^{(d)} > 0$ for all $d \geq 1$ and $N \geq 2$, while $\beta_N^{(d)} = 0$ for $d = 1, 2$ and $N \geq 2$; see [18, Remarks 2.2(i) and Theorem 2.8]. Remarkably, for $d = 1$ the explicit value of the sharp constant $\kappa_N^{(1)}$ is known, namely, $\kappa_N^{(1)} = \frac{1}{2}$ for all $N \geq 2$ [18, Theorem 2.5]. For $d \geq 2$, as far as we know, only the lower bound $\kappa_N^{(d)} \geq d^2/N$ is known. This displays the same N^{-1} behavior as $\beta_N^{(d)}$, but, as we will see in the present paper, this is not optimal, at least when $d \geq 3$.

In fact, our main result states that $\lim_{N \rightarrow \infty} N^{1-\frac{2}{d}} \kappa_N^{(d)}$ exists as a positive and finite number when $d \geq 3$, and gives an explicit expression for it in terms of a variational problem for functions on \mathbb{R}^d .

This is the special case of a more general result, which concerns the inequality

$$\sum_{n=1}^N \int_{\mathbb{R}^{dN}} |(-\Delta_n)^{\frac{s}{2}} u(X)|^2 dX \geq \beta_N^{(d,s)} \sum_{1 \leq n < m \leq N} \int_{\mathbb{R}^{dN}} \frac{|u(X)|^2}{|X_n - X_m|^{2s}} dX \quad (2)$$

for all $u \in \dot{H}^s(\mathbb{R}^{dN})$. Here, s is a real number satisfying $0 < s < \frac{d}{2}$ and the operator $(-\Delta_n)^{\frac{s}{2}}$ acts on the n th variable of $X = (X_1, \dots, X_N)$ by multiplication by $|\xi_n|^s$ in Fourier space. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^{dN})$ is the completion of $C_c^\infty(\mathbb{R}^{dN})$ with respect to the quadratic form on the left-hand side of (2). It is relatively straightforward to see that

$$\inf_{N \geq 2} N \beta_N^{(d,s)} > 0. \quad (3)$$

Indeed, for each pair (n, m) with $n \neq m$, we have by the ordinary fractional Hardy inequality (see Lemma 9 below),

$$\int_{\mathbb{R}^d} |(-\Delta_n)^{\frac{s}{2}} u(X)|^2 dX_n \gtrsim \int_{\mathbb{R}^d} \frac{|u(X)|^2}{|X_n - X_m|^{2s}} dX_n. \quad (4)$$

Integrating this inequality with respect to the remaining variables and summing over n and m give (3). As an aside, we mention that the optimal constant in (4) is known; see [16, 19] and also [13, 38].

Our interest is again in the sharp constant in (2) *when restricted to antisymmetric functions*, that is, in

$$\kappa_N^{(d,s)} := \inf_{\substack{0 \neq u \in \dot{H}^s(\mathbb{R}^{dN}) \\ \text{antisymmetric}}} \frac{\sum_{n=1}^N \int_{\mathbb{R}^{dN}} |(\Delta_n)^{\frac{s}{2}} u(X)|^2 dX}{\sum_{1 \leq n < m \leq N} \int_{\mathbb{R}^{dN}} \frac{|u(X)|^2}{|X_n - X_m|^{2s}} dX}.$$

To state the limiting variational problem, we introduce for nonnegative, measurable functions ρ on \mathbb{R}^d

$$D_\lambda[\rho] := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho(x)\rho(x')}{|x - x'|^\lambda} dx dx'.$$

Moreover, we denote

$$\tau_{d,s} := \inf_{0 \leq \rho \in L^{1+\frac{2s}{d}} \cap L^1(\mathbb{R}^d)} \frac{\int_{\mathbb{R}^d} \rho(x)^{1+\frac{2s}{d}} dx \left(\int_{\mathbb{R}^d} \rho(x) dx \right)^{1-\frac{2s}{d}}}{D_{2s}[\rho]}.$$

The fact that $\tau_{d,s} > 0$ follows from the Hardy–Littlewood–Sobolev inequality [24, Theorem 4.3], together with Hölder’s inequality. Finally, let

$$c_{d,s}^{\text{TF}} := \frac{(4\pi)^s}{1 + \frac{2s}{d}} \Gamma(1 + \frac{d}{2})^{\frac{2s}{d}}.$$

The superscript TF stands for “Thomas–Fermi” and it will become clear in the proof that this constant is related to the Thomas–Fermi approximation for the kinetic energy.

The following is our main result.

Theorem 1. *Let $d \geq 1$ and $0 < s < \frac{d}{2}$ with $s \leq 1$. Then,*

$$\lim_{N \rightarrow \infty} N^{1-\frac{2s}{d}} \kappa_N^{(d,s)} = \tau_{d,s} c_{d,s}^{\text{TF}}. \quad (5)$$

Remarks 2. (a) This result in the special case $s = \frac{1}{2}$, $d = 3$ is due to Lieb and Yau [31], following earlier work by Lieb and Thirring [30]. While our overall strategy is similar to theirs, there are some significant differences, which we explain below.

(b) Our proof of the asymptotics (5) comes with remainder bounds. We show that $N^{1-\frac{2s}{d}} \kappa_N^{(d,s)}$ is equal to its limit up to a relative error of $\mathcal{O}(N^{-\frac{s(d-2s)}{d^2}})$; see equations (6) and (13).

(c) We believe that Theorem 1 remains valid without the extra assumption $s \leq 1$. This would probably require significant additional effort at various places and, since our main interest is the case $s = 1$, we decided to impose this simplifying assumption.

(d) Theorem 1 extends to the case where spin is taken into account, except that the limiting expression in (5) is multiplied by a power of the number of spin states. We refer to [26] for an explanation of this terminology and to [31] for proofs where spin is taken into account.

(e) Finding the asymptotic behavior of $\kappa_N^{(d,s)}$ in the case $d = 2s$ is an *open problem*. In Appendix B, we discuss a conjecture of what might be the right order and prove the corresponding upper bound.

Let us give some background on Theorem 1 and explain some aspects of its proof. The basic idea is that it is a combined semiclassical and mean-field limit. Such a limit is behind what is called Thomas–Fermi approximation for Coulomb systems and has first been made rigorous by Lieb and Simon in [26]. Parts of this proof were simplified through the use of coherent states [23, 37] and the Lieb–Thirring inequality [23, 29], and these tools will also play an important role for us. For a recent study of this combined semiclassical and mean-field limit for quite general systems we refer to [8].

One difficulty that we face here, compared to the analysis of nonrelativistic Coulomb systems [27] or the systems in [8], is that the kinetic energy and the potential energy scale in the same way, so that there is no natural length scale. This problem was overcome by Lieb and Yau [31], following earlier work of Lieb and Thirring [30], in their rigorous derivation of Chandrasekhar theory of gravitational collapse of stars. An important ingredient in the proofs of [30, 31] and also in the more recent [8], is the *Lévy-Leblond method*. This method will also play a crucial role in our proof. It consists in dividing the N particles into two groups, treating one part as “electrons” and the other part as “nuclei”. The electrons repel each other, and similarly the nuclei, while electrons and nuclei attract each other. The construction involves a further, free parameter that corresponds to the quotient between the charges of the electrons and the nuclei. At the end one averages over all such partitions.

There is one important structural property, however, that Lieb and Yau can take advantage of and we cannot. They deal with the case $s = \frac{1}{2}$, $d = 3$, where the interaction potential $|x|^{-1}$ is, up to a constant, the fundamental solution of the Laplacian and the corresponding (sub/super)harmonicity properties enter into the proof of [31, Lemma 1]. The same phenomenon occurs, for instance, for $s = 1$, $d = 4$, or in general for $s = \frac{d-2}{2}$, $d \geq 3$, but in the general case the interaction $|x|^{-2s}$ is not harmonic outside of the origin. Therefore, some effort goes into proving a bound for systems interacting through Riesz potentials $|x|^{-2s}$ for general exponents $0 < 2s < d$; see Proposition 3. For this we rely on the Fefferman–de la Llave decomposition of this interaction potential. This decomposition is also the main tool in the proof of Proposition 11 and, as a curiosity, we mention that also the Cwikel–Lieb–Rozenblum inequality, and therefore the Lieb–Thirring inequality, which is another important ingredient in our proof, can be established using the Fefferman–de la Llave decomposition [10]. For more on this decomposition, see also [15].

Neither the Lévy–Leblond method nor the Fefferman–de la Llave decomposition seem to work for $d \leq 2s$ and this case remains *open* (except for $s = d = 1$). In Appendix B, we give a suggestion of what might be the relevant mechanism in the borderline case $d = 2s$.

Finally, we mention that the results in this paper, with the exception of those in Section 2, are contained in a preprint with the same title, dated October 30, 2006, that was circulated among colleagues. The present paper corrects some minor mistakes therein and adds a proof of the sharp asymptotic lower bound.

It is our pleasure to dedicate this paper to Brian Davies in admiration of his many profound contributions to spectral theory and mathematical physics and, in particular, to the topic of Hardy inequalities. Happy birthday, Brian!

2. Lower bound

Our goal in this section is to prove the lower bound in Theorem 1. That is, we will show that

$$\liminf_{N \rightarrow \infty} N^{1-\frac{2s}{d}} \kappa_N^{(d,s)} \geq \tau_{d,s} c_{d,s}^{\text{TF}}.$$

More precisely, we will prove the following quantitative version of it:

$$N^{1-\frac{2}{d}} \kappa_N \geq \tau_{d,s} c_{d,s}^{\text{TF}} \left(1 - \text{const } N^{-\frac{s(d-2s)}{d^2}} \right). \quad (6)$$

As explained in the introduction, we mostly follow the method in [31], but an important new ingredient, which replaces their [31, equation (2.21)], is the electrostatic inequality in Proposition 3.

2.1. An electrostatic inequality

For a (Borel) probability measure μ on \mathbb{R}^{dM} , we denote by ρ_μ the nonnegative measure on \mathbb{R}^d obtained by summing the M marginals of μ . That is, for any bounded continuous function f on \mathbb{R}^d , we have

$$\sum_{m=1}^M \int_{\mathbb{R}^{dN}} f(Y_m) d\mu(Y) = \int_{\mathbb{R}^d} f(y) d\rho_\mu(y).$$

The definition of D_λ is extended to nonnegative measures on \mathbb{R}^d in a natural way, namely, by

$$D_\lambda[v] := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dv(y) dv(y')}{|y - y'|^\lambda}.$$

Next, for $R = (R_1, \dots, R_K) \in \mathbb{R}^{dK}$ and $y \in \mathbb{R}^d$, we set

$$\delta_R(y) := \min_{k=1, \dots, K} |y - R_k|.$$

Proposition 3. *Let $d \geq 1$ and $0 < \lambda < d$. Then, for any $M, K \in \mathbb{N}$, $R = (R_1, \dots, R_K) \in \mathbb{R}^{dK}$, $Z > 0$ and any probability measure μ on \mathbb{R}^{dM}*

$$\int_{\mathbb{R}^{dM}} \sum_{m,k} \frac{Z}{|Y_m - R_k|^\lambda} d\mu(Y) - \sum_{k < l} \frac{Z^2}{|R_k - R_l|^\lambda} - D_\lambda[\rho_\mu] \lesssim Z \int_{\mathbb{R}^d} \frac{d\rho_\mu(y)}{\delta_R(y)^\lambda},$$

where the implied constant depends only on d and λ .

The following proof uses some ideas from that of [7, Corollary 1].

Proof. According to the Fefferman–de la Llave formula [7], we have for all $y, y' \in \mathbb{R}^d$

$$\frac{1}{|y - y'|^\lambda} = \text{const} \int_0^\infty \frac{dr}{r^{d+\lambda+1}} \int_{\mathbb{R}^d} da \mathbb{1}_{B_r(a)}(y) \mathbb{1}_{B_r(a)}(y') \quad (7)$$

with a constant depending only on d and λ . This implies that

$$\begin{aligned} & \int_{\mathbb{R}^{dM}} \sum_{m,k} \frac{Z}{|Y_m - R_k|^\lambda} d\mu(Y) - \sum_{k < l} \frac{Z^2}{|R_k - R_l|^\lambda} - D_\lambda[\rho_\mu] \\ &= \text{const} \int_0^\infty \frac{dr}{r^{d+\lambda+1}} \int_{\mathbb{R}^d} da \left(\int_{\mathbb{R}^{dM}} Z \sum_{m,k} \mathbb{1}_{B_r(a)}(Y_m) \mathbb{1}_{B_r(a)}(R_k) d\mu(Y) \right. \\ & \quad \left. - Z^2 \sum_{k < l} \mathbb{1}_{B_r(a)}(R_k) \mathbb{1}_{B_r(a)}(R_l) \right. \\ & \quad \left. - \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \rho_\mu(y) \mathbb{1}_{B_r(a)}(y) \mathbb{1}_{B_r(a)}(y') \rho_\mu(y') \right) \\ &= \text{const} \int_0^\infty \frac{dr}{r^{d+\lambda+1}} \int_{\mathbb{R}^d} da \left(Z n_{B_r(a)} K_{B_r(a)} - \frac{1}{2} Z^2 K_{B_r(a)} (K_{B_r(a)} - 1) - \frac{1}{2} n_{B_r(a)}^2 \right), \end{aligned}$$

where we have introduced, for any ball B ,

$$n_B := \rho_\mu(B) \quad \text{and} \quad K_B := \sum_k \mathbb{1}_B(R_k).$$

Note that K_B is a nonnegative integer. We claim that for any $n \geq 0$ and any $K' \in \mathbb{N}_0$,

$$ZnK' - \frac{1}{2}Z^2K'(K' - 1) - \frac{1}{2}n^2 \leq Zn \mathbb{1}(K' \geq 1).$$

Indeed, this is true when $K' = 0$, and when $K' \geq 1$, we write the left-hand side as

$$-\frac{1}{2}(n - Z\sqrt{K'(K' - 1)})^2 + nZ(K' - \sqrt{K'(K' - 1)})$$

and bound $K' - \sqrt{K'(K' - 1)} \leq 1$.

Thus, we have shown that

$$\begin{aligned} & \int_{\mathbb{R}^{dM}} \sum_{m,k} \frac{Z}{|Y_m - R_k|^\lambda} d\mu(Y) - \sum_{k < l} \frac{Z^2}{|R_k - R_l|^\lambda} - D_\lambda[\rho_\mu] \\ & \leq \text{const } Z \int_0^\infty \frac{dr}{r^{d+\lambda+1}} \int_{\mathbb{R}^d} da n_{B_r(a)} \mathbb{1}(K_{B_r(a)} \geq 1) \\ & = \text{const } Z \int_{\mathbb{R}^d} d\rho_\mu(y) \int_0^\infty \frac{dr}{r^{d+\lambda+1}} \int_{\mathbb{R}^d} da \mathbb{1}_{B_r(a)}(y) \mathbb{1}(K_{B_r(a)} \geq 1). \end{aligned}$$

Next, we bound

$$\mathbb{1}_{B_r(a)}(y) \mathbb{1}(K_{B_r(a)} \geq 1) \leq \mathbb{1}_{B_r(a)}(y) \mathbb{1}(\delta_R(y) < 2r).$$

Indeed, when the left-hand side does not vanish, we have $|y - a| < r$ and there is a $k \in \{1, \dots, K\}$ such that $|R_k - a| < r$. Consequently,

$$\delta_R(y) \leq |y - R_k| < 2r.$$

By performing first the a and then the r integration, we obtain for each $y \in \mathbb{R}^d$,

$$\begin{aligned} & \int_0^\infty \frac{dr}{r^{d+\lambda+1}} \int_{\mathbb{R}^d} da \mathbb{1}_{B_r(a)}(y) \mathbb{1}(K_{B_r(a)} \geq 1) \\ & \leq \int_0^\infty \frac{dr}{r^{d+\lambda+1}} \int_{\mathbb{R}^d} da \mathbb{1}_{B_r(a)}(y) \mathbb{1}(\delta_R(y) < 2r) \\ & = \text{const} \int_0^\infty \frac{dr}{r^{\lambda+1}} \mathbb{1}(\delta_R(y) < 2r) \\ & = \text{const} \frac{1}{\delta_R(y)^\lambda}. \end{aligned}$$

This implies the claimed inequality. ■

2.2. Lieb–Thirring inequality

Associated to a normalized function $\psi \in L^2(\mathbb{R}^{dM})$ is a probability measure $d\mu(Y) = |\psi(Y)|^2 dY$, and, therefore, we can consider the measure $d\rho_\mu$ on \mathbb{R}^d as in the previous subsection. In the present case, this measure turns out to be absolutely continuous and we denote its density by ρ_ψ . Explicitly,

$$\rho_\psi(y) := \sum_{m=1}^M \int_{\mathbb{R}^{3(M-1)}} |\psi(\dots, Y_{m-1}, y, Y_{m+1}, \dots)|^2 dY_1 \cdots dY_{m-1} dY_{m+1} \cdots dY_M.$$

This density appears in the following famous Lieb–Thirring inequality.

Lemma 4. *Let $d \geq 1$ and $s > 0$. Then, for any $M \in \mathbb{N}$ and any antisymmetric and L^2 -normalized $\psi \in H^s(\mathbb{R}^{dM})$,*

$$\sum_{m=1}^M \int_{\mathbb{R}^{dM}} |(-\Delta_m)^{\frac{s}{2}} \psi|^2 dY \gtrsim \int_{\mathbb{R}^d} \rho_\psi^{1+\frac{2s}{d}} dy,$$

where the implied constant depends only on d and s .

For $s = 1$, this inequality is due to Lieb and Thirring [29]. Their original proof generalizes readily to the full regime $s > 0$; see also [26, Chapter 4], as well as [12, Theorem 4.60 and Section 7.4] and [9]. For the currently best-known values of the constants, see [11].

2.3. Coherent states

The following lemma is a rigorous version of the Thomas–Fermi approximation for the kinetic energy. It is proved with the help of coherent states.

Lemma 5. *Let $d \geq 1$ and $0 < s \leq 1$. Let $g \in H^s(\mathbb{R}^d)$ be L^2 -normalized and, when $s > \frac{1}{2}$, assume that $|\hat{g}|$ is even under $\xi \mapsto -\xi$. Then, for any antisymmetric, L^2 -normalized $\psi \in H^s(\mathbb{R}^{dM})$,*

$$\sum_{m=1}^M \int_{\mathbb{R}^{dM}} |(-\Delta_m)^{\frac{s}{2}} \psi|^2 dY \geq c_{d,s}^{\text{TF}} \int_{\mathbb{R}^d} (\rho_\psi * |g|^2)^{1+\frac{2s}{d}} dy - M \|(-\Delta)^{\frac{s}{2}} g\|^2.$$

Here,

$$\hat{g}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} g(x) dx$$

denotes the Fourier transform of g .

Bounds of the same type as in the lemma appear in [23, equations (5.14)–(5.22)] in the special case $s = 1$ and $d = 3$; a general version is formulated in [31, Lemma B.4]. Because of a subtlety in the application of that lemma, we sketch the proof.

Proof. We will show that for any L^2 -normalized $v \in H^s(\mathbb{R}^d)$, one has

$$\|(-\Delta)^{\frac{s}{2}} v\|^2 \geq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\eta|^{2s} |\tilde{v}(\eta, y)|^2 \frac{d\eta dy}{(2\pi)^d} - \|(-\Delta)^{\frac{s}{2}} g\|^2, \quad (8)$$

where

$$\tilde{v}(\eta, y) := \int_{\mathbb{R}^d} e^{-i\eta \cdot x} \overline{g(y-x)} v(x) dx.$$

(Compared to [24, Section 12.7] and other presentations, we find it convenient to use $y-x$ instead of $x-y$ in the definition of \tilde{v} .) Once (8) is shown, the inequality in the lemma follows as in [31, Lemma B.3].

To prove (8), we observe that

$$\int_{\mathbb{R}^d} |\tilde{v}(\eta, y)|^2 dy = (2\pi)^d \int_{\mathbb{R}^d} |\hat{v}(\xi)|^2 |\hat{g}(\eta - \xi)|^2 d\xi.$$

We multiply this identity by $|\eta|^{2s}$ and integrate with respect to η .

In case $s \leq \frac{1}{2}$, we use the subadditivity of $t \mapsto t^{2s}$ to bound on the right-hand side

$$|\eta|^{2s} \leq |\xi|^{2s} + |\eta - \xi|^{2s}$$

and obtain the claimed inequality (8). (This is essentially the argument in [31, Lemma B.3].)

In case $\frac{1}{2} < s \leq 1$, we use the evenness of $|\hat{g}|$ to replace $|\eta|^{2s} = |\xi + (\eta - \xi)|^{2s}$ with $\frac{1}{2}(|\xi + (\eta - \xi)|^{2s} + |\xi - (\eta - \xi)|^{2s})$. We then apply the elementary inequality

$$\frac{1}{2}(|\xi + \zeta|^{2s} + |\xi - \zeta|^{2s}) \leq (|\xi|^2 + |\zeta|^2)^s \leq |\xi|^{2s} + |\zeta|^{2s} \quad (9)$$

and argue similarly as for $s \leq \frac{1}{2}$ to obtain the claimed inequality (8). The second inequality in (9) follows from the subadditivity of $t \mapsto t^s$, and to prove the first inequality, we write

$$\frac{1}{2}(|\xi + \zeta|^{2s} + |\xi - \zeta|^{2s}) = \frac{1}{2}((1 + t_*)^s + (1 - t_*)^s)(|\xi|^2 + |\zeta|^2)^s$$

with

$$t_* = 2\xi \cdot \zeta / (|\xi|^2 + |\zeta|^2) \in [-1, 1]$$

and note that

$$[-1, 1] \ni t \mapsto (1 + t)^s + (1 - t)^s$$

attains its maximum at $t = 0$. ■

We expect a similar bound as in Lemma 5 to hold for $s > 1$ as well, but the structure of the remainder term will probably be more complicated.

2.4. Summary so far

Let us combine the bounds from this section.

Corollary 6. *Let $d \geq 1$, $0 < s < \frac{d}{2}$ with $s \leq 1$, $M \in \mathbb{N}$, $K \geq 2$, $R = (R_1, \dots, R_K) \in \mathbb{R}^{dK}$ and $Z > 0$. Then, for any antisymmetric and L^2 -normalized $\psi \in H^s(\mathbb{R}^{dM})$,*

$$\begin{aligned} & \left\langle \psi, \sum_{m,k} \frac{Z}{|Y_m - R_k|^{2s}} \psi \right\rangle - \sum_{k < l} \frac{Z^2}{|R_k - R_l|^{2s}} \\ & \leq \left(1 + \text{const } M^{-\frac{s(d-2s)}{d^2}} \right) (\tau_{d,s} c_{d,s}^{\text{TF}})^{-1} M^{1-\frac{2s}{d}} \sum_{m=1}^M \int_{\mathbb{R}^{dM}} |(-\Delta_m)^{\frac{s}{2}} \psi|^2 dY \\ & \quad + \text{const } Z \int_{\mathbb{R}^d} \frac{\rho_\psi(y)}{\delta_R(y)^{2s}} dy. \end{aligned}$$

Proof. Let $\psi \in H^s(\mathbb{R}^{dM})$ be antisymmetric and L^2 -normalized. We recall that according to Proposition 3, we have

$$\left\langle \psi, \sum_{m,k} \frac{Z}{|Y_m - R_k|^{2s}} \psi \right\rangle - \sum_{k < l} \frac{Z^2}{|R_k - R_l|^{2s}} \leq D_{2s}[\rho_\psi] + \text{const } Z \int_{\mathbb{R}^d} \frac{\rho_\psi(y)}{\delta_R(y)^{2s}} dy. \quad (10)$$

The second term on the right-hand side appears in the claimed error bound. To bound the first term, let $g \in H^s(\mathbb{R}^d)$ be L^2 -normalized. Using the definition of $\tau_{d,s}$, Young's convolution inequality and Lemma 5, we find

$$\begin{aligned} D_{2s}[\rho_\psi] & \leq \tau_{d,s}^{-1} \int_{\mathbb{R}^d} (\rho_\psi * |g|^2)^{1+\frac{2s}{d}} dy \left(\int_{\mathbb{R}^d} \rho_\psi * |g|^2 dy \right)^{1-\frac{2s}{d}} \\ & \quad + (D_{2s}[\rho_\psi] - D_{2s}[\rho_\psi * |g|^2]) \\ & \leq \tau_{d,s}^{-1} M^{1-\frac{2s}{d}} \int_{\mathbb{R}^d} (\rho_\psi * |g|^2)^{1+\frac{2s}{d}} dy \\ & \quad + \frac{1}{2} \|\rho_\psi\|_{1+\frac{2s}{d}}^2 \| |x|^{-2s} - |g|^2 * |x|^{-2s} * |g|^2 \|_{\frac{d+2s}{4s}} \\ & \leq (\tau_{d,s} c_{d,s}^{\text{TF}})^{-1} M^{1-\frac{2s}{d}} \sum_{m=1}^M \int_{\mathbb{R}^{dM}} |(-\Delta_m)^{\frac{s}{2}} \psi|^2 dY + \mathcal{R} \end{aligned}$$

with

$$\begin{aligned} \mathcal{R} & := (\tau_{d,s} c_{d,s}^{\text{TF}})^{-1} M^{2-\frac{2s}{d}} \|(-\Delta)^{\frac{s}{2}} g\|^2 \\ & \quad + \frac{1}{2} \|\rho_\psi\|_{1+\frac{2s}{d}}^2 \| |x|^{-2s} - |g|^2 * |x|^{-2s} * |g|^2 \|_{\frac{d+2s}{4s}}. \end{aligned}$$

We now assume that $g(x) = \ell^{-\frac{d}{2}} G(\ell^{-1}x)$ for an L^2 -normalized function $G \in H^s(\mathbb{R}^d)$ and a parameter $\ell > 0$ to be chosen. We have

$$\begin{aligned} \mathcal{R} &= \ell^{-2s} (\tau_{d,s} c_{d,s}^{\text{TF}})^{-1} M^{2-\frac{2s}{d}} \|(-\Delta)^{\frac{s}{2}} G\|^2 \\ &\quad + \ell^{\frac{2s(d-2s)}{d+2s}} \frac{1}{2} \|\rho_\psi\|_{1+\frac{2s}{d}}^2 \| |x|^{-2s} - |G|^2 * |x|^{-2s} * |G|^2 \|_{\frac{d+2s}{4s}}. \end{aligned}$$

We note that the function $|x|^{-2s} - |G|^2 * |x|^{-2s} * |G|^2$ behaves like $|x|^{-2s}$ as $|x| \rightarrow 0$. Moreover, assuming that $|G|$ is even and that $|x|^2 |G|^2$ is integrable, it is easy to see that $|x|^{-2s} - |G|^2 * |x|^{-2s} * |G|^2 = O(|x|^{-2s-2})$ as $|x| \rightarrow \infty$. A tedious, but elementary analysis shows that $(2s+2)\frac{d+2s}{4s} > d$. (Indeed, this is equivalent to $2s^2 - s(d-2) + d > 0$. This is always satisfied when $d = 1, 2$. For $d \geq 3$, the left-hand side is decreasing with respect to s for $s \leq \frac{d-2}{4}$, so its validity for our parameter values follows from its validity at $s = \min\{1, \frac{d-2}{4}\}$.) The conclusion of this discussion is that $|x|^{-2s} - |G|^2 * |x|^{-2s} * |G|^2 \in L^{\frac{d+2}{4}}(\mathbb{R}^d)$. We consider G as fixed and choose

$$\ell = M^{\frac{(d-s)(d+2s)}{2sd^2}} \|\rho_\psi\|_{1+\frac{2s}{d}}^{-\frac{d+2s}{2sd}}$$

in order to balance the two error terms and obtain

$$\mathcal{R} \lesssim M^{-\frac{s(d-2s)}{d^2}} M^{1-\frac{2s}{d}} \|\rho_\psi\|_{1+\frac{2s}{d}}^{1+\frac{2s}{d}}.$$

Using the Lieb–Thirring inequality (Proposition 4), we can further bound the right-hand side and arrive at

$$\mathcal{R} \lesssim M^{-\frac{s(d-2s)}{d^2}} M^{1-\frac{2s}{d}} \sum_{m=1}^M \int_{\mathbb{R}^{dM}} |(-\Delta_m)^{\frac{s}{2}} \psi|^2 dY.$$

This implies the claimed bound ■

2.5. Domination of the nearest neighbor attraction

For $X = (X_1, \dots, X_N)$ and $n \in \{1, \dots, N\}$, let

$$\delta_n(X) := \min_{m \neq n} |X_n - X_m|.$$

Proposition 7. *Let $d \geq 1$ and $0 < s < \frac{d}{2}$ with $s \leq 1$. Then, for any antisymmetric $u \in \dot{H}^s(\mathbb{R}^{dN})$, we have*

$$\sum_{n=1}^N \int_{\mathbb{R}^{dN}} |(-\Delta_n)^{\frac{s}{2}} u(X)|^2 dX \gtrsim \sum_{n=1}^N \int_{\mathbb{R}^{dN}} \frac{|u(X)|^2}{\delta_n(X)^{2s}} dX$$

with an implicit constant that only depends on d and s .

This bound appears as [32, Theorem 5] in the cases $d = 3$ and $s \in \{\frac{1}{2}, 1\}$, but the proof readily generalizes to the stated parameter regime and is omitted. We also mention an alternative proof in [7, Corollary 2], which is based on a Fefferman–de la Llave-type formula for the $\dot{H}^s(\mathbb{R}^d)$ -seminorm and which generalizes to the regime $s < 1$.

Probably, Proposition 7 remains valid for $1 < s < \frac{d}{2}$, but this would require an argument and for the sake of brevity we do not consider this case. The IMS localization formula in [33, Lemma 14] might be useful.

2.6. Proof of the lower bound in Theorem 1

We turn to the proof of (6), for which we use the Levy-Leblond method [21], similarly as in [30, 31]. Given $N \geq 3$, we choose an integer $M \in \{1, \dots, N-2\}$ and a real number $Z > 0$. We set $K := N - M$ and consider partitions $\pi = (\pi_1, \pi_2)$ of $\{1, \dots, N\}$ into two disjoint sets π_1 and π_2 with M and K elements, respectively. We have

$$\begin{aligned} \sum_{n < m} \frac{1}{|X_n - X_m|^{2s}} &= \frac{M(N-1)}{2ZMK - Z^2K(K-1)} \frac{N}{M} \binom{N}{M}^{-1} \\ &\quad \times \sum_{\pi} \left(\sum_{m \in \pi_1} \sum_{k \in \pi_2} \frac{Z}{|X_m - X_k|^{2s}} - \sum_{k < l \in \pi_2} \frac{Z^2}{|X_k - X_l|^{2s}} \right). \end{aligned} \quad (11)$$

Let $u \in \dot{H}^s(\mathbb{R}^{dN})$ be antisymmetric. Our goal is to bound the Hardy quotient for u . By density we may assume that $u \in H^s(\mathbb{R}^{dN})$ and by homogeneity we may assume that u is L^2 -normalized. We integrate the left-hand side of (11) against $|u(X)|^2$. Correspondingly, on the right-hand side, we obtain a sum over partitions and we bound the integral for each fixed such partition P . We first carry out the integral over the variables in π_1 . Denoting these variables by (Y_1, \dots, Y_M) and the variables in π_2 by (R_1, \dots, R_K) we infer from Corollary 6 that

$$\begin{aligned} &\int_{\mathbb{R}^{dM}} \left(\sum_{m \in \pi_1} \sum_{k \in \pi_2} \frac{Z}{|X_m - X_k|^{2s}} - \sum_{k < l \in \pi_2} \frac{Z^2}{|X_k - X_l|^{2s}} \right) |u(X_1, \dots, X_N)|^2 d\pi_1(X) \\ &\leq \left(1 + \text{const } M^{-\frac{s(d-2s)}{d^2}} \right) (\tau_{d,s} c_{d,s}^{\text{TF}})^{-1} M^{1-\frac{2s}{d}} \\ &\quad \times \sum_{m \in \pi_1} \int_{\mathbb{R}^{dM}} |(\Delta_m)^{\frac{s}{2}} u(X_1, \dots, X_N)|^2 d\pi_1(X) \\ &\quad + \text{const } Z \sum_{m \in \pi_1} \int_{\mathbb{R}^{dM}} \frac{|u(X_1, \dots, X_N)|^2}{\delta_{\pi_2(X)}(X_m)^{2s}} d\pi_1(X). \end{aligned}$$

Here, $d\pi_1(X)$ denotes integration with respect to the variables X_m with $m \in \pi_1$ and, for $m \in \pi_1$, we have

$$\delta_{\pi_2(X)}(X_m) = \min_{k \in \pi_2} |X_m - X_k| \geq \min_{k \neq m} |X_m - X_k| = \delta_m(X).$$

Inserting this into the above bound and carrying out the integration over the variables in π_2 , we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{dN}} \left(\sum_{m \in \pi_1} \sum_{k \in \pi_2} \frac{Z}{|X_m - X_k|^{2s}} - \sum_{k < l \in \pi_2} \frac{Z^2}{|X_k - X_l|^{2s}} \right) |u(X)|^2 dX \\ & \leq \left(1 + \text{const } M^{-\frac{s(d-2s)}{d^2}} \right) (\tau_{d,s} c_{d,s}^{\text{TF}})^{-1} M^{1-\frac{2s}{d}} \sum_{m \in \pi_1} \int_{\mathbb{R}^{dN}} |(-\Delta_m)^{\frac{s}{2}} u(X)|^2 dX \\ & \quad + \text{const } Z \sum_{m \in \pi_1} \int_{\mathbb{R}^{dN}} \frac{|u(X_1, \dots, X_N)|^2}{\delta_m(X)^{2s}} dX. \end{aligned}$$

According to (11), summing this bound over π gives

$$\begin{aligned} & \int_{\mathbb{R}^{dN}} \frac{|u(X)|^2}{|X_n - X_m|^{2s}} dX \\ & \leq \frac{M(N-1)}{2ZMK - Z^2K(K-1)} \left(1 + \text{const } M^{-\frac{s(d-2s)}{d^2}} \right) \\ & \quad \times (\tau_{d,s} c_{d,s}^{\text{TF}})^{-1} M^{1-\frac{2s}{d}} \sum_{n=1}^N \int_{\mathbb{R}^{dN}} |(-\Delta_n)^{\frac{s}{2}} u|^2 dX \\ & \quad + \text{const } \frac{M(N-1)}{2ZMK - Z^2K(K-1)} Z \sum_{n=1}^N \int_{\mathbb{R}^{dN}} \frac{|u(X)|^2}{\delta_n(X)^{2s}} dX. \end{aligned}$$

Using Proposition 7, the right-hand side can be bounded by

$$C (\tau_{d,s} c_{d,s}^{\text{TF}})^{-1} N^{1-\frac{2s}{d}} \sum_{n=1}^N \int_{\mathbb{R}^{dN}} |(-\Delta_n)^{\frac{s}{2}} u|^2 dX$$

with

$$C := \frac{M(N-1)}{2ZMK - Z^2K(K-1)} \left(1 + \text{const } M^{-\frac{s(d-2s)}{d^2}} + \text{const } Z M^{-1+\frac{2s}{d}} \right) \left(\frac{M}{N} \right)^{1-\frac{2s}{d}}.$$

Our goal is to choose the parameters M and Z (depending on N) in such a way that $C \rightarrow 1$ as $N \rightarrow \infty$. We choose $Z = M/K$ and obtain

$$C = \frac{1 + M^{-1}(K-1)}{1 + K^{-1}} \left(1 + \text{const } M^{-\frac{s(d-2s)}{d^2}} + \text{const } K^{-1} M^{\frac{2s}{d}} \right) \left(\frac{M}{N} \right)^{1-\frac{2s}{d}}.$$

With the choice

$$K := \left[N^{\frac{s}{d} + \frac{1}{2}} \right],$$

we find

$$C \leq 1 + \text{const } N^{-\frac{s(d-2s)}{d^2}}.$$

This completes the proof of (6). \blacksquare

Remark 8. Under the additional assumption $d > 4s$, one can prove (6) (with a worse remainder bound) without using Proposition 7. Indeed, inserting the bound from Lemma 9 below into the bound in Corollary 6, we can drop the last term there at the expense of replacing the factor in front of the first term by

$$1 + \text{const } M^{-\frac{s(d-2s)}{d^2}} + \text{const } Z K^{\frac{2s}{d}} M^{-1 + \frac{2s}{d}}.$$

Choosing again $Z = M/K$, we can choose $K \sim N^{\frac{d+2s}{2(d-s)}}$ and arrive at (6) with the remainder $1 - \text{const } N^{-\frac{d-4s}{2(d-s)}}$.

Lemma 9. Let $0 < s < \frac{d}{2}$. Then, for all $v \in \dot{H}^s(\mathbb{R}^d)$, $K \in \mathbb{N}$ and $R \in \mathbb{R}^{3K}$,

$$\|(-\Delta)^{\frac{s}{2}} v\|_2^2 \gtrsim K^{-\frac{2s}{d}} \int_{\mathbb{R}^d} \frac{|v(x)|^2}{\delta_R(x)^{2s}} dx$$

with an implicit constant depending only on d and s .

The following proof has some similarities with [31, Lemma B.1].

Proof. We use the improved Sobolev embedding in Lorentz spaces [36] (see also [20, Theorem 17.49]),

$$\|(-\Delta)^{\frac{s}{2}} v\|_2^2 \gtrsim \|v\|_{L^{\frac{2d}{d-2s}, 2}(\mathbb{R}^d)}^2,$$

together with Hölder's inequality in Lorentz spaces [20, Exercise 15.22],

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|v(x)|^2}{\delta_R(x)^{2s}} dx &\lesssim \|\delta_R^{-2s}\|_{L^{\frac{d}{2s}, \infty}(\mathbb{R}^d)} \| |v|^2 \|_{L^{\frac{d}{d-2s}, 1}(\mathbb{R}^d)} \\ &= \|\delta_R^{-2s}\|_{L^{\frac{d}{2s}, \infty}(\mathbb{R}^d)} \|v\|_{L^{\frac{2d}{d-2s}, 2}(\mathbb{R}^d)}^2. \end{aligned}$$

It remains to bound the weak $L^{\frac{d}{2s}}$ norm of δ_R^{-2s} . We have

$$|\{\delta_R < \lambda\}| \leq \sum_{k=1}^K |\{|\cdot - R_k| < \lambda\}| = \text{const } K \lambda^d$$

so that

$$\|\delta_R^{-2s}\|_{L^{\frac{d}{2s}, \infty}(\mathbb{R}^d)} = \sup_{\mu > 0} \mu |\{\delta_R^{-2s} > \mu\}|^{\frac{2s}{d}} \lesssim K^{\frac{2s}{d}}.$$

This gives the claimed bound. \blacksquare

2.7. The case without antisymmetry

In this subsection, we explain how the proof of (13) can be modified to give a lower bound on the optimal constant $\beta_N^{(d,s)}$ in (2). We denote

$$\omega_{d,s} := \inf_{0 \neq \sqrt{\rho} \in H^s(\mathbb{R}^d)} \frac{\|(-\Delta)^{\frac{s}{2}} \sqrt{\rho}\|_2^2 \|\rho\|_1}{D_{2s}[\rho]}.$$

It is not difficult to show that $\omega_{d,s} > 0$ when $0 < s < \frac{d}{2}$ and that there is an optimizer ρ_* ; see, e.g., [31, Theorem 4] in the case $s = \frac{1}{2}$, $d = 3$. Then, one can show that

$$\beta_N^{(d,s)} \leq \frac{1}{N-1} \omega_{d,s} \quad \text{for all } N \geq 2 \quad (12)$$

by taking $u(X) = \prod_{n=1}^N \sqrt{\rho_*(X_n)}$. For $s = 1$, $d \geq 3$ this argument appears in [18, Theorem 2.3]. We now state the lower bound corresponding to (12).

Proposition 10. *Let $d \geq 1$ and $0 < s < \frac{d}{2}$ with $s \leq 1$. Then,*

$$N\beta_N^{(d,s)} \geq \omega_{d,s} \left(1 - \text{const } N^{-1+\frac{2s}{d}}\right).$$

Proof. We proceed from inequality (10), which did not use the antisymmetry of ψ . Using the definition of $\omega_{d,s}$ and Lemma 9, we can bound the right-hand side of (10) by

$$\begin{aligned} D_{2s}[\rho_\psi] + \text{const } Z \int_{\mathbb{R}^d} \frac{\rho_\psi(y)}{\delta_R(y)^{2s}} dy \\ \leq \omega_{d,s}^{-1} \|(-\Delta)^{\frac{s}{2}} \sqrt{\rho_\psi}\|_2^2 \|\rho_\psi\|_1 + \text{const } Z K^{\frac{2s}{d}} \|(-\Delta)^{\frac{s}{2}} \sqrt{\rho_\psi}\|_2^2 \\ \leq \left(1 + \text{const } Z M^{-1} K^{\frac{2s}{d}}\right) \omega_{d,s}^{-1} M \sum_{m=1}^M \int_{\mathbb{R}^{dM}} |(-\Delta_m)^{\frac{s}{2}} \psi|^2 dY. \end{aligned}$$

The second inequality here is the Hoffmann-Ostenhof inequality [17] for $s = 1$ and its generalization to $s < 1$ by Conlon [5]. Following the Lévy-Leblond method, we deduce from this bound that

$$\int_{\mathbb{R}^{dN}} \frac{|u(X)|^2}{|X_n - X_m|^{2s}} dX \leq C \omega_{d,s}^{-1} N \sum_{n=1}^N \int_{\mathbb{R}^{dN}} |(-\Delta_n)^{\frac{s}{2}} \psi|^2 dX$$

with

$$C := \frac{M(N-1)}{2ZMK - Z^2K(K-1)} \left(1 + \text{const } Z M^{-1} K^{\frac{2s}{d}}\right) \frac{M}{N}.$$

We choose again $Z = M/K$ and obtain

$$C := \frac{1 + M^{-1}(K-1)}{1 + K^{-1}} \left(1 + \text{const } K^{-1+\frac{2s}{d}}\right) \frac{M}{N}.$$

Choosing $M = 1$, we arrive at the claimed bound. ■

3. Upper bound

Our goal in this section is to prove the upper bound in Theorem 1. That is, we will show that

$$\limsup_{N \rightarrow \infty} N^{1-\frac{2s}{d}} \kappa_N^{(d,s)} \leq \tau_{d,s} c_{d,s}^{\text{TF}}.$$

More precisely, we will prove the following quantitative version of it:

$$N^{1-\frac{2s}{d}} \kappa_N^{(d,s)} \leq \tau_{d,s} c_{d,s}^{\text{TF}} \left(1 + \text{const } N^{-\frac{s(d-2s)}{d^2}} \right). \quad (13)$$

For the proof, we follow rather closely the method in [30, Section 3], combined with an exchange inequality, which appears in Proposition 11.

3.1. A bound on the indirect part of the Riesz energy

Here, we return to the setting of Section 2.1 and consider probability measures μ on \mathbb{R}^{dN} and their marginals ρ_μ .

Proposition 11. *Let $d \geq 1$ and $0 < \lambda < d$. Then, for any $N \in \mathbb{N}$ and for any non-negative probability measure μ on \mathbb{R}^{dN} with $\rho_\mu \in L^{1+\frac{1}{\lambda}}(\mathbb{R}^d)$,*

$$\sum_{1 \leq n < m \leq N} \int_{\mathbb{R}^{dN}} \frac{d\mu(X)}{|X_n - X_m|^\lambda} - D_\lambda[\rho_\mu] \gtrsim - \int_{\mathbb{R}^d} \rho_\mu(x)^{1+\frac{1}{\lambda}} dx$$

with an implicit constant depending only on d and λ .

This bound for $\lambda = 1$ and $d = 3$ is due to [22] with an improved constant in [25]. For the general, case one can adapt the proof strategy from [28], which is based on the Fefferman–de la Llave formula and does not use (sub/super)harmonicity properties of the interaction potential. The details appear in [33, Lemma 16]. (The proof given there for absolutely continuous measures extends to the general case.)

3.2. Relaxation to density matrices

We use the result from the previous subsection to make the next step towards (13), namely, by proving an upper bound in terms of density matrices.

We recall that a nonnegative trace class operator γ on $L^2(\mathbb{R}^d)$ has a well-defined density $\rho_\gamma \in L^1(\mathbb{R}^d)$. Indeed, if we decompose $\gamma = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ with orthonormal ψ_i , then $\rho_\gamma = \sum_i \lambda_i |\psi_i|^2$. (In the case of a non-simple eigenvalue λ_i one can convince oneself easily that this is independent of the choice of the eigenfunction ψ_i .)

We claim that for any operator γ on $L^2(\mathbb{R}^d)$ satisfying

$$0 \leq \gamma \leq 1 \quad \text{and} \quad N := \text{Tr } \gamma \in \mathbb{N}, \quad (14)$$

we have

$$\mathrm{Tr}(-\Delta)^s \gamma \geq \kappa_N^{(d,s)} \left(D_{2s}[\rho_\gamma] - \mathrm{const} \int_{\mathbb{R}^d} \rho_\gamma(x)^{1+\frac{2s}{d}} dx \right). \quad (15)$$

Here, as usual, we write $\mathrm{Tr}(-\Delta)^s \gamma$ instead of $\mathrm{Tr}(-\Delta)^{\frac{s}{2}} \gamma (-\Delta)^{\frac{s}{2}}$. Of course, the bound is only meaningful if the latter quantity is finite.

Given Proposition 11, the proof of this assertion is relatively standard (see, e.g., [30, Section 3]), but we include some details for the sake of completeness. First, there is a nonnegative operator Γ on the antisymmetric subspace of $L^2(\mathbb{R}^{dN})$ satisfying

$$\mathrm{Tr}_{N-1} \Gamma = \gamma,$$

where Tr_{N-1} denotes the partial trace with respect to $N-1$ variables. This is due to [4]; see also [26, Theorem 3.2]. It follows that

$$\mathrm{Tr} \Gamma = N^{-1} \mathrm{Tr} \gamma = 1 \quad \text{and} \quad \sum_{n=1}^N \mathrm{Tr}(-\Delta_n)^s \Gamma = \mathrm{Tr}(-\Delta)^s \gamma.$$

Expanding

$$\Gamma = \sum_i p_i |u_i\rangle \langle u_i|$$

with orthonormal antisymmetric functions $u_i \in L^2(\mathbb{R}^{dN})$ and nonnegative numbers p_i , we obtain

$$\begin{aligned} \sum_i p_i &= 1, \\ \sum_i p_i \sum_{n=1}^N \int_{\mathbb{R}^{dN}} |(-\Delta_n)^{\frac{s}{2}} u_i|^2 dX &= \mathrm{Tr}(-\Delta)^s \gamma. \end{aligned}$$

Therefore, the definition of $\kappa_N^{(d,s)}$, applied to u_i , yields the inequality

$$\mathrm{Tr}(-\Delta)^s \gamma \geq \kappa_N^{(d,s)} \sum_i p_i \sum_{n < m} \int_{\mathbb{R}^{dN}} \frac{|u_i(X)|^2}{|X_n - X_m|^{2s}} dX.$$

We now apply Proposition 11 to the measure

$$d\mu(X) = \sum_i p_i |u_i(X)|^2 dX,$$

which, by the above properties, is indeed a probability measure. Moreover, using the partial trace relation between Γ and γ , we find $\rho_\mu = \rho_\gamma$. Thus, the claimed inequality (15) follows from Proposition 11.

3.3. Construction of γ using coherent states

Let $0 \leq \rho \in L^1 \cap L^{1+\frac{2s}{d}}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \rho \, dx = N$ and let $g \in H^s(\mathbb{R}^d)$ be L^2 -normalized. We consider the operator

$$\gamma(x, x') = \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(y-x) e^{i\eta \cdot (x-x')} \mathbb{1}(|\eta|^{2s} < c\rho(y)^{\frac{2s}{d}}) \overline{g(y-x')} \frac{dy \, d\eta}{(2\pi)^d}$$

with $c = (2\pi)^{2s} \omega_d^{-\frac{2s}{d}}$, where ω_d is the volume of the unit ball in \mathbb{R}^d .

It is easy to see that this operator satisfies (14). Indeed, the bound $\gamma \geq 0$ follows immediately by estimating $\mathbb{1}(|\xi|^{2s} < c\rho(x)^{\frac{2s}{d}}) \geq 0$ and the bound $\gamma \leq 1$ follows by estimating $\mathbb{1}(|\xi|^{2s} < c\rho(x)^{\frac{2s}{d}}) \leq 1$ and using Plancherel and the normalization of g . To prove $\text{Tr} \gamma = N$, we integrate the kernel on the diagonal, using the choice of c and, again, the normalization of g . In this connection, we also note that the density of γ is

$$\rho_\gamma(x) = \int_{\mathbb{R}^d} \rho(x) |g(y-x)|^2 \, dy = \rho * |g|^2(x).$$

Assuming that $|\hat{g}|$ is even, we claim that

$$\text{Tr}(-\Delta)^s \gamma \leq c_{d,s}^{\text{TF}} \int_{\mathbb{R}^d} (\rho * |g|^2)^{1+\frac{2s}{d}} \, dx + N \|(-\Delta)^{\frac{s}{2}} g\|_2^2.$$

This is shown in the special case $d = 3, s = \frac{1}{2}$ in [30, Section 3] (see also [24, Theorem 12.10]). The proof generalizes to the general case, the underlying estimates being the same as in the proof of Lemma 5.

If we insert these facts into (15), we obtain

$$\begin{aligned} & c_{d,s}^{\text{TF}} \int_{\mathbb{R}^d} (\rho * |g|^2)^{1+\frac{2s}{d}} \, dx + N \|(-\Delta)^{\frac{s}{2}} g\|_2^2 \\ & \geq \kappa_N^{(d,s)} \left(D_{2s}[\rho * |g|^2] - \text{const} \int_{\mathbb{R}^d} (\rho * |g|^2)^{1+\frac{2s}{d}} \, dx \right). \end{aligned}$$

By the normalization of g and Minkowski's inequality, we have

$$\int_{\mathbb{R}^d} (\rho * |g|^2)^{1+\frac{2s}{d}} \, dx \leq \int_{\mathbb{R}^d} \rho^{1+\frac{2s}{d}} \, dx.$$

Moreover, as in the proof of (6), Young's convolution inequality shows that

$$D_{2s}[\rho * |g|^2] \geq D_{2s}[\rho] - \frac{1}{2} \|\rho\|_{1+\frac{2s}{d}}^2 \| |x|^{-2s} - |g|^2 * |x|^{-2s} * |g|^2 \|_{\frac{d+2s}{4s}}.$$

To summarize, we have

$$c_{d,s}^{\text{TF}} \int_{\mathbb{R}^d} \rho^{1+\frac{2s}{d}} \, dx + N \|(-\Delta)^{\frac{s}{2}} g\|_2^2 \geq \kappa_N^{(d,s)} (D_{2s}[\rho] - \mathcal{R})$$

with

$$\mathcal{R} := \frac{1}{2} \|\rho\|_{1+\frac{2s}{d}}^2 \| |x|^{-2s} - |g|^2 * |x|^{-2s} * |g|^2 \|_{\frac{d+2s}{4s}} + \text{const} \int_{\mathbb{R}^d} \rho^{1+\frac{2s}{d}} dx.$$

Similarly, in the proof of the lower bound, we now assume that $g(x) = \ell^{-\frac{d}{2}} G(\ell^{-1}x)$ for an L^2 -normalized function $G \in H^s(\mathbb{R}^d)$ and a parameter $\ell > 0$ to be chosen. We consider G as fixed and obtain, as before,

$$\mathcal{R} \lesssim \ell^{\frac{2s(d-2s)}{d+2s}} \|\rho\|_{1+\frac{2s}{d}}^2 + \|\rho\|_{1+\frac{2s}{d}}^{1+\frac{2s}{d}}.$$

Thus,

$$\begin{aligned} c_{d,s}^{\text{TF}} \int_{\mathbb{R}^d} \rho^{1+\frac{2s}{d}} dx + \text{const} \ell^{-2s} N \\ \geq \kappa_N^{(d,s)} \left(D_{2s}[\rho] - \text{const} \left(\ell^{\frac{2s(d-2s)}{d+2s}} \|\rho\|_{1+\frac{2s}{d}}^2 + \|\rho\|_{1+\frac{2s}{d}}^{1+\frac{2s}{d}} \right) \right). \end{aligned} \quad (16)$$

3.4. The semiclassical problem

The following result states that the variational problem defining $\tau_{d,s}$ has an optimizer. This result is not strictly necessary for our proof of the upper bound in Theorem 1, but it is readily available and makes the proof more transparent.

Lemma 12. *Let $d \geq 1$ and $0 < s < \frac{d}{2}$. Then, there is a $0 \leq \rho_* \in L^{1+\frac{2s}{d}} \cap L^1(\mathbb{R}^d)$, $\rho_* \neq 0$, such that*

$$\frac{\int_{\mathbb{R}^d} \rho_*(x)^{1+\frac{2s}{d}} dx \left(\int_{\mathbb{R}^d} \rho_*(x) dx \right)^{1-\frac{2s}{d}}}{D_{2s}[\rho_*]} = \tau_{d,s}.$$

In the special case $s = \frac{1}{2}$, $d = 3$, this appears in [25, Appendix A]. The proof in the general case is exactly the same.

For the sake of completeness, we mention that the uniqueness (up to translations, dilations, and multiplication by a constant) of ρ_* has been studied in [25] (see also [31]), as well as in the recent papers [2, 3].

3.5. Proof of the upper bound in Theorem 1

Let ρ_* be the optimizer from Lemma 12. After a dilation and a multiplication by a constant, we may assume that

$$\int_{\mathbb{R}^d} \rho_* dx = 1 = \int_{\mathbb{R}^d} \rho_*^{1+\frac{2s}{d}} dx, \quad D_{2s}[\rho_*] = \tau_{d,s}^{-1}.$$

We then apply the construction outlined in this section with the choice $\rho = N\rho_*$. Inequality (16) turns into

$$c_{d,s}^{\text{TF}} \left(1 + \text{const } \ell^{-2s} N^{-\frac{2s}{d}} \right) \geq \kappa_N^{(d,s)} \tau_{d,s}^{-1} N^{1-\frac{2s}{d}} \left(1 - \text{const} \left(\ell^{\frac{2s(d-2s)}{d+2s}} + N^{-1+\frac{2s}{d}} \right) \right).$$

Choosing

$$\ell = N^{-\frac{d+2s}{2d^2}},$$

we obtain, for all sufficiently large N , the claimed bound (13).

Appendix A. An order of magnitude bound

Our goal in this appendix is to prove the lower bound

$$\inf_{N \geq 2} N^{1-\frac{2s}{d}} \kappa_N^{(d,s)} > 0 \quad (17)$$

for $0 < s < \frac{d}{2}$ with $s \leq 1$. This is weaker than the asymptotics in Theorem 1, but it does capture the right order of magnitude as $N \rightarrow \infty$, and we feel that the argument is robust and may be useful in other contexts as well.

The main step in the proof of (17) is the following bound, which is similar to the sought-after Hardy inequality, but with an additional positive term on the left-hand side.

Proposition 13. *Let $d \geq 1$, $0 < s < \frac{d}{2}$ and $\tau > 0$. Then, there is a constant $C(\tau) > 0$ such that for any $N \geq 2$ and any antisymmetric function $u \in \dot{H}^s(\mathbb{R}^{dN})$,*

$$\begin{aligned} \sum_{n=1}^N \int_{\mathbb{R}^{dN}} \left(|(-\Delta_n)^{\frac{s}{2}} u|^2 + \frac{\tau |u|^2}{\delta_n^2} \right) dX \\ \geq C(\tau) N^{-1+\frac{2s}{d}} \sum_{1 \leq n < m \leq N} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{|X_n - X_m|^{2s}} dX. \end{aligned}$$

Proof of (17) given Proposition 13. Denoting by c the implicit constant in Proposition 7, we infer from that proposition and from Proposition 13 that for any $0 < \theta < 1$

$$\begin{aligned} \sum_{n=1}^N (-\Delta_n)^{\frac{s}{2}} &\geq (1-\theta) \sum_{n=1}^N \left((-\Delta_n)^s + \frac{\theta c}{1-\theta} \delta_n^{-2s} \right) \\ &\geq (1-\theta) C \left(\frac{\theta c}{1-\theta} \right) N^{-1+\frac{2s}{d}} \sum_{1 \leq n < m \leq N} |X_n - X_m|^{-2s}. \end{aligned}$$

Hence, $N^{1-\frac{2s}{d}} \kappa_N^{(d,s)} \geq \sup_{0 < \theta < 1} (1-\theta) C \left(\frac{\theta c}{1-\theta} \right) > 0$, as claimed. \blacksquare

We emphasize that, while Proposition 13 does not require $s \leq 1$, our proof of (17) does, since we apply Proposition 7.

It remains to prove Proposition 13, and to do so, we proceed again with the help of the Lévy-Leblond method [21]. We split the N variables $X = (X_1, \dots, X_N)$ into a group of “electronic” variables $Y = (Y_1, \dots, Y_M)$ and a group of “nuclear” variables $R = (R_1, \dots, R_K)$ with $M + K = N$. For a fixed $R \in \mathbb{R}^{dK}$ with $R_k \neq R_l$ for $k \neq l$ we define the function on \mathbb{R}^d ,

$$V_R(y) := \sum_{k=1}^M \frac{1}{|y - R_k|^{2s}} - \frac{1}{\delta_R(y)^{2s}},$$

and the constant

$$U_R := \sum_{1 \leq k < l \leq K} \frac{1}{|R_k - R_l|^{2s}} + \sum_{k=1}^K \frac{1}{\delta_k(R)^{2s}}.$$

Here, as before,

$$\delta_R(y) = \min \{|y - R_k| : 1 \leq k \leq K\}$$

and

$$\delta_k(R) = \min \{|R_k - R_l| : 1 \leq k \leq K, l \neq k\}.$$

We will estimate the sum of the negative eigenvalues of the (one-particle) operator $(-\Delta)^s - \lambda V_R$ in $L^2(\mathbb{R}^d)$ in terms of U_R .

Lemma 14. *Let $d \geq 1$ and $0 < s < \frac{d}{2}$. Then, for all $K \geq 2$, $R \in \mathbb{R}^{dK}$ and $\lambda > 0$,*

$$\mathrm{Tr}((-\Delta)^s - \lambda V_R)_- \lesssim \lambda^{1+\frac{d}{2s}} K^{\frac{d}{2s}-1} U_R \quad (18)$$

with an implicit constant that only depends on d and s .

Proof. By the Lieb–Thirring inequality (see, e.g., [12, Theorem 4.60]), we have

$$\mathrm{Tr}((-\Delta)^s - \lambda V_R)_- \lesssim \lambda^{1+\frac{d}{2s}} \int_{\mathbb{R}^d} V_R(y)^{1+\frac{d}{2s}} dy.$$

To estimate the latter integral, we write

$$V_R(y) = \sum_{k=1}^K \chi_k(y) |y - R_k|^{-2s},$$

where $1 - \chi_k$ is the characteristic function of the Voronoi cell $\Gamma_k := \{y : |y - R_k| = \min_l |y - R_l|\}$. Note that Hölder’s inequality implies that

$$V_R(y)^{\frac{1}{2}(1+\frac{d}{2s})} \leq K^{\frac{d-2s}{4s}} \sum_{k=1}^K \chi_k(y) |y - R_k|^{-\frac{d+2s}{2}}.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^d} V_R(y)^{1+\frac{d}{2s}} dy &\leq K^{\frac{d-2s}{2s}} \sum_{k,l} \int_{\mathbb{R}^d} \chi_k(y) \chi_l(y) |y - R_k|^{-\frac{d+2s}{2}} |y - R_l|^{-\frac{d+2s}{2}} dy \\ &\leq K^{\frac{d-2s}{2s}} \left(2 \sum_{k < l} \int_{\mathbb{R}^d} |y - R_k|^{-\frac{d+2s}{2}} |y - R_l|^{-\frac{d+2s}{2}} dy \right. \\ &\quad \left. + \sum_k \int_{\mathbb{R}^d} \chi_k(y) |y - R_k|^{-d-2s} dy \right). \end{aligned}$$

The first integral is easily found to equal a constant times $|R_k - R_l|^{-2s}$. To estimate the second integral, we note that

$$\{y : |y - R_k| \leq \delta_k(R)/2\} \subset \Gamma_k.$$

Extending the domain of integration, we find that

$$\int_{\mathbb{R}^d} \chi_k(y) |y - R_k|^{-d-2s} dy \leq \int_{\{|y - R_k| > \delta_k(R)/2\}} |y - R_k|^{-d-2s} dy = \text{const } \delta_k(R)^{-2s}.$$

This proves the assertion. ■

Now, everything is in place for the proof of the following proposition.

Proof of Proposition 13. In view of (3), it suffices to prove the bound for sufficiently large N . In fact, we will prove the bound for $N \geq N(\tau)$ for some $N(\tau)$ to be determined later.

For given $N \geq 3$, $\tau > 0$, and $\kappa > 0$, we choose an integer $M \in \{1, \dots, N-2\}$ and parameters $\lambda, \alpha > 0$. Setting $K := N - M$, we write

$$\sum_{n=1}^N ((-\Delta_n)^s + \tau \delta_n^{-2s}) - \kappa \sum_{1 \leq n < m \leq N} |X_n - X_m|^{-2s} = \frac{N}{M} \binom{N}{M}^{-1} \sum_{\pi} h_{\pi}. \quad (19)$$

Here, the sum runs over all partitions $\pi = (\pi_1, \pi_2)$ of $\{1, \dots, N\}$ into two disjoint sets π_1, π_2 of sizes M and K , respectively, and for any such partition the operator h_{π} is defined by

$$\begin{aligned} h_{\pi} &:= \sum_{m \in \pi_1} \left((-\Delta_m)^s - \lambda \sum_{k \in \pi_2} |X_m - X_k|^{-2s} + \lambda \delta_m^{-2s} \right) \\ &\quad + \alpha \sum_{k < l \in \pi_2} |X_k - X_l|^{-2s} + \alpha \sum_{k \in \pi_2} \delta_k^{-2s}. \end{aligned}$$

In order that (19) be an identity, we require that

$$\lambda M + \alpha K = \tau M \quad (20)$$

and

$$2\lambda MK - \alpha K(K-1) = \kappa M(N-1). \quad (21)$$

It suffices to prove that for $\kappa \leq C(\tau)N^{-1+\frac{2s}{d}}$ one has $h_\pi \geq 0$ for all partitions π as above. We denote the variables in π_1 by $Y = (Y_1, \dots, Y_M)$ and those in π_2 by $R = (R_1, \dots, R_K)$. Then, one has the estimates

$$\delta_m(x) \geq \delta_R(Y_j), \quad m \in \pi_1,$$

and

$$\delta_k(x) \geq \delta_k(R), \quad k \in \pi_2.$$

These two estimates lead to the lower bound

$$h_\pi \geq \sum_{m=1}^M ((-\Delta_{Y_m})^s - \lambda V_R(Y_j)) + \alpha U_R. \quad (22)$$

The right-hand side is an operator in $L^2(\mathbb{R}^{dN})$, but there is no kinetic energy associated with the R variables. Hence, if we define for fixed $R \in \mathbb{R}^{dK}$ an operator h^R in the antisymmetric subspace of $L^2(\mathbb{R}^{dM})$ by the expression on the right-hand side of (22), then one has the estimate

$$h_\pi \geq \inf_{R \in \mathbb{R}^{dK}} \inf \text{spec } h^R.$$

Further, since h^R acts on antisymmetric functions, one has

$$\inf \text{spec } h^R \geq -\text{Tr}(-\Delta - \lambda V_R)_- + \alpha U_R,$$

and hence, by Lemma 14,

$$\inf_{R \in \mathbb{R}^{dK}} \inf \text{spec } h^R \geq 0$$

provided that

$$\alpha - C\lambda^{\frac{d}{2s}+1} K^{\frac{d}{2s}-1} \geq 0. \quad (23)$$

It remains to choose the parameters M, α, λ such that (20), (21), and (23) are satisfied. With the choice $\alpha/\lambda = M/K$, equation (21) becomes $\lambda = \frac{\tau}{2}$, (20) becomes

$$\kappa = \frac{\tau(K+1)}{2(N-1)}, \quad (24)$$

and (23) becomes

$$M \geq C 2^{-\frac{d}{2s}} \tau^{\frac{d}{2s}} K^{\frac{d}{2s}}. \quad (25)$$

We choose $K = \lceil \varepsilon \tau^{-1} N^{\frac{2s}{d}} \rceil$, where $\varepsilon > 0$ will be determined below (depending only on d and s). As we mentioned at the beginning of the proof, we may assume

that $N^{1-\frac{2s}{d}} \geq 2\varepsilon\tau^{-1} =: N(\tau)^{1-\frac{2s}{d}}$, which guarantees that $K \leq \frac{N}{2}$ and consequently $M \geq \frac{N}{2}$. This implies that (25) is satisfied, provided $\varepsilon > 0$ is chosen small enough depending on d and s . Then, κ given by (24) is easily seen to satisfy $\kappa \leq C(\tau)N^{-1+\frac{2s}{d}}$ for all $N \geq N(\tau)$. This completes the proof of Proposition 13. ■

Appendix B. The borderline case $s = 1, d = 2$

In this appendix for the sake of definiteness, we focus on the case $s = 1$. Our main result assumes $d \geq 3$ and its proof breaks down in several places in dimensions $d = 1, 2$. Meanwhile, for $d = 1$ we know from [18] that $\kappa_N^{(1)} = \frac{1}{2}$ for all N . In particular, this constant is independent of N . In the remaining case $d = 2$, we only know that $\kappa_N^{(2)} \geq 4N^{-1}$ for all N , but this does probably not capture the correct large N -behavior. The following result gives an upper bound.

Proposition 15. *Let $d = 2$ and $s = 1$. Then,*

$$\limsup_{N \rightarrow \infty} (\ln N) \kappa_N^{(2)} \leq 4. \quad (26)$$

It is a tantalizing question whether the right-hand side of (26) is, in fact, the limit of $(\ln N) \kappa_N^{(2)}$. We would like to express our gratitude to Robert Seiringer for first suggesting (26) and for several discussions related to it.

Proof. Our construction depends on two main parameters, L and μ . Given a sequence of N 's tending to infinity, these parameters will be chosen such that $N = \#\mathcal{N}$ for a certain set \mathcal{N} satisfying

$$\left\{ p \in \frac{2\pi}{L} \mathbb{Z}^2 : |p|^2 < \mu \right\} \subset \mathcal{N} \subset \left\{ p \in \frac{2\pi}{L} \mathbb{Z}^2 : |p|^2 \leq \mu \right\}.$$

This implies that

$$\frac{1}{4\pi} \mu L^2 \sim N \rightarrow \infty.$$

The antisymmetric function u on \mathbb{R}^{2N} that we will use as a trial function to bound $\kappa_N^{(2)}$ from above will be a Slater determinant of functions that are essentially plane waves restricted to $Q_L := (-L/2, L/2)^2$ with momenta in \mathcal{N} . There are several ways to construct such functions and here we use a method that we learned from [14].

Let $0 \leq \zeta \in C_c^1(Q_\ell)$ with $\int_{Q_\ell} \zeta \, dx = 1$, where $\ell > 0$ is a parameter satisfying $\ell \ll L$. (We keep track of it only for dimensional consistency.) For $p \in \frac{2\pi}{L} \mathbb{Z}^2$, let

$$\varphi_p(x) := L^{-1} \sqrt{\mathbb{1}_{Q_L} * \zeta} e^{ip \cdot x} \quad \text{for all } x \in \mathbb{R}^2.$$

A computation [14] (see also [12, Lemma 7.21]), based on the Fourier transform of the characteristic function of an interval, shows that the φ_p are orthonormal in $L^2(\mathbb{R}^2)$. We define u as their Slater determinant,

$$u(X) = (N!)^{-\frac{1}{2}} \det(\varphi_{p_n}(X_{n'}))_{p_n, p_{n'} \in \mathcal{N}},$$

where p_1, \dots, p_N is an enumeration of \mathcal{N} . We have [14] (see also [12, Lemma 7.21])

$$\sum_{n=1}^N \int_{\mathbb{R}^{2N}} |\nabla_n u|^2 dX = \sum_{p \in \mathcal{N}} \int_{\mathbb{R}^2} |\nabla \varphi_p|^2 dx = \frac{1}{8\pi} \mu^2 L^2 (1 + o(1)) \quad (27)$$

in the asymptotic regime that we are considering.

Our task is to bound from below

$$\sum_{1 \leq n < m \leq N} \int_{\mathbb{R}^{2N}} \frac{|u(X)|^2}{|X_n - X_m|^2} dX = \frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_u(x) \rho_u(x') - |\gamma_u(x, x')|^2}{|x - x'|^2} dx dx, \quad (28)$$

where

$$\begin{aligned} \rho_u(x) &= \sum_{p \in \mathcal{N}} |\varphi_p(x)|^2 = L^{-2} N \mathbb{1}_{Q_L} * \zeta, \\ \gamma_u(x, x') &= \sum_{p \in \mathcal{N}} \varphi_p(x) \overline{\varphi_p(x')} = L^{-2} \sqrt{\mathbb{1}_{Q_L} * \zeta(x)} \sqrt{\mathbb{1}_{Q_L} * \zeta(x')} \sum_{p \in \mathcal{N}} e^{ip \cdot (x - x')}. \end{aligned}$$

Note that the integrand on the right-hand side of (28) is nonnegative. Consequently, we obtain a lower bound by restricting it to

$$\Omega := \{(x, x') \in Q_{L-\ell} \times Q_{L-\ell} : \sqrt{\mu}|x - x'| > C\}$$

for a certain constant C , independent of L and μ , and to be chosen below. Note that for $x \in Q_{L-\ell}$, we have

$$\mathbb{1}_{Q_L} * \zeta(x) = 1.$$

Therefore, the ρ -part of the integral on the right-hand side of (28), restricted to Ω , is bounded from below by

$$\begin{aligned} L^{-4} N^2 \iint_{Q_{L-\ell} \times Q_{L-\ell}} \frac{\mathbb{1}(\sqrt{\mu}|x - x'| > C)}{|x - x'|^2} dx dx' \\ = L^{-4} N^2 (L - \ell)^2 \mathcal{I}(C\mu^{-\frac{1}{2}}(L - \ell)^{-1}), \end{aligned}$$

where

$$\mathcal{I}(\varepsilon) := \iint_{Q_1 \times Q_1} \frac{\mathbb{1}(|y - y'| \geq \varepsilon)}{|y - y'|^2} dy dy'.$$

An elementary computation shows that

$$\mathcal{I}(\varepsilon) = 2\pi \left(\ln \frac{1}{\varepsilon} \right) (1 + o(1)).$$

Using $\ell \ll L$ and $N \sim (4\pi)^{-1} \mu L^2$, we deduce that

$$\begin{aligned} \iint_{\Omega} \frac{\rho_u(x) \rho_u(x')}{|x - x'|^2} dx dx' &= 2\pi L^{-2} N^2 (\ln(\sqrt{\mu} L)) (1 + o(1)) \\ &= \frac{1}{16\pi} \mu^2 L^2 (\ln N) (1 + o(1)). \end{aligned}$$

Comparing this with (27) we arrive at the constant 4 on the right-hand side of (26).

Thus, it remains to prove that the γ -part of the right-hand side of (28), restricted to Ω , is negligible for some (and, in fact, any) choice of C . Before giving a complete proof, let us explain the heuristics. The nonrigorous step is that we approximate

$$L^{-2} \sum_{p \in \mathcal{N}} e^{ip \cdot (x - x')} \approx \int_{|p|^2 < \mu} e^{ip \cdot (x - x')} \frac{dp}{(2\pi)^2}.$$

(Since $\mu L^2 \gg 1$, this is justified for fixed $x - x'$, but we will use it uniformly for $\sqrt{\mu}|x - x'| \geq C > 0$ with $|x - x'| \leq L$.) It is known that

$$\int_{|p|^2 < \mu} e^{ip \cdot (x - x')} \frac{dp}{(2\pi)^2} = (2\pi)^{-1} \sqrt{\mu} |x - x'|^{-1} J_1(\sqrt{\mu}|x - x'|),$$

where J_1 is a Bessel function [1, Chapter 9]. Using the decay bound on Bessel functions, $|J_1(t)| \lesssim t^{-1/2}$, [1, equation (9.2.1)], we obtain

$$\left| \int_{|p|^2 < \mu} e^{ip \cdot (x - x')} \frac{dp}{(2\pi)^2} \right| \lesssim \mu^{\frac{1}{4}} |x - x'|^{-\frac{3}{2}}.$$

From this, we arrive at the expectation that for $x, x' \in Q_{L-\ell}$, at least on average, one has

$$|\gamma_u(x, x')| \lesssim \mu^{\frac{1}{4}} |x - x'|^{-\frac{3}{2}}. \quad (29)$$

Accepting this bound, we obtain by straightforward estimates

$$\iint_{\Omega} \frac{|\gamma_u(x, x')|^2}{|x - x'|^2} dx dx' \lesssim \sqrt{\mu} \iint_{\Omega} \frac{dx dx'}{|x - x'|^5} \lesssim \mu^2 L^2.$$

Recalling that our lower bound on the ρ -term in (28) is of size $\mu^2 L^2 \ln(\mu L^2)$, we see that the γ -term is indeed negligible.

We now present a rigorous proof that the γ -term is negligible. We will not be able to prove the bound in (29), but we will be able to prove that

$$|\gamma_u(x, x')| \lesssim \mu^{\frac{1}{2}} |x - x'|^{-1} \quad \text{if } (x, x') \in Q_{L-\ell} \times Q_{L-\ell}. \quad (30)$$

Accepting this bound and combining it with the trivial bound

$$|\gamma_u(x, x')|^2 \leq \rho_u(x)\rho_u(x'),$$

we obtain

$$\begin{aligned} & \iint_{\Omega} \frac{|\gamma_u(x, x')|^2}{|x - x'|^2} dx dx' \\ & \lesssim L^2 \left(\mu \int_{C\mu^{-\frac{1}{2}} < |r| \leq \frac{1}{2}L} \frac{dr}{|r|^4} + \mu^2 \int_{\frac{1}{2}L < |r| < L} \frac{dr}{|r|^2} \right) \lesssim \mu^2 L^2, \end{aligned}$$

which is the same as if the heuristic bound (29) was true.

It remains to prove (30). We bound

$$\left| \sum_{p \in \mathcal{N}} e^{ip \cdot r} \right| \leq \min \left\{ \sum_{p_1^2 \leq \mu} \left| \sum_{p_2: (p_1, p_2) \in \mathcal{N}} e^{ip_2 r_2} \right|, \sum_{p_2^2 \leq \mu} \left| \sum_{p_1: (p_1, p_2) \in \mathcal{N}} e^{ip_1 r_1} \right| \right\}$$

and use the elementary inequality, valid for any interval $I \subset \mathbb{R}$ and $t \in \mathbb{R} \setminus L\mathbb{Z}$,

$$\left| \sum_{\tau \in I \cap \frac{2\pi}{L}\mathbb{Z}} e^{i\tau t} \right| \lesssim \frac{L}{\text{dist}(t, L\mathbb{Z})}.$$

The latter follows by summing a geometric series. Since the sum over $p_j^2 \leq \mu$ contains $\lesssim \mu^{\frac{1}{2}} L$ elements, we deduce that

$$\left| \sum_{p \in \mathcal{N}} e^{ip \cdot r} \right| \lesssim \mu^{\frac{1}{2}} L \min \left\{ \frac{L}{\text{dist}(r_2, L\mathbb{Z})}, \frac{L}{\text{dist}(r_1, L\mathbb{Z})} \right\}.$$

If $|r| \leq \frac{L}{2}$, then

$$\text{dist}(r_j, L\mathbb{Z}) = |r_j|$$

for $j = 1, 2$. Moreover,

$$\max_j r_j^2 \geq \frac{1}{2} |r|^2.$$

Therefore, the right-hand side is $\lesssim \mu^{\frac{1}{2}} L^2 |r|^{-1}$. This, applied to $r = x - x'$, yields (30) and concludes the proof of the proposition. ■

Remarks 16. (a) In physics, the vanishing of $\rho_u(x)\rho_u(x') - |\gamma_u(x, x')|^2$ near $x = x'$ is called the *exchange hole*. The intuition is that it is this exchange hole that leads to the Hardy inequality for (spin-polarized) fermions in two dimensions. This hole, which is of size $\mu^{-\frac{1}{2}}$ in the above example, mitigates the logarithmic divergence of the integral $|x - x'|^{-2}$ and leads to the logarithmic behavior of the constant $\kappa_N^{(2)}$.

(b) It is essential for the validity of $\kappa_N^{(2)} > 0$ that the fermionic particles are spinless (or spin-polarized). If u has two or more spin states, there is no reason that

$$\sum_{n < m} \sum_{\sigma} \iint_{\mathbb{R}^{2N}} |X_n - X_m|^{-2} |u(X, \sigma)|^2 dX$$

is finite. (Here, \sum_{σ} denotes the sum over spin-states.)

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References

- [1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, No. 55, U.S. Government Printing Office, Washington, DC, 1964
Zbl 0171.38503 MR 167642
- [2] V. Calvez, J. A. Carrillo, and F. Hoffmann, [The geometry of diffusing and self-attracting particles in a one-dimensional fair-competition regime](#). In *Nonlocal and nonlinear diffusions and interactions: new methods and directions*, pp. 1–71, Lecture Notes in Math. 2186, Springer, Cham, 2017 Zbl 1384.35134 MR 3588121
- [3] V. Calvez, J. A. Carrillo, and F. Hoffmann, [Uniqueness of stationary states for singular Keller–Segel type models](#). *Nonlinear Anal.* **205** (2021), article no. 112222
Zbl 1458.35004 MR 4190637
- [4] A. J. Coleman, [Structure of fermion density matrices](#). *Rev. Modern Phys.* **35** (1963), 668–689 MR 0155637
- [5] J. G. Conlon, [The ground state energy of a classical gas](#). *Comm. Math. Phys.* **94** (1984), no. 4, 439–458 Zbl 0946.82501 MR 0763746
- [6] E. B. Davies, [A review of Hardy inequalities](#). In *The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998)*, pp. 55–67, Oper. Theory Adv. Appl. 110, Birkhäuser, Basel, 1999
Zbl 0936.35121 MR 1747888
- [7] C. Fefferman and R. de la Llave, [Relativistic stability of matter. I](#). *Rev. Mat. Iberoamericana* **2** (1986), no. 1–2, 119–213 Zbl 0602.58015 MR 0864658
- [8] S. Fournais, M. Lewin, and J. P. Solovej, [The semi-classical limit of large fermionic systems](#). *Calc. Var. Partial Differential Equations* **57** (2018), no. 4, article no. 105
Zbl 1395.81313 MR 3814648
- [9] R. L. Frank, [The Lieb–Thirring inequalities: Recent results and open problems](#). In *Nine mathematical challenges—an elucidation*, pp. 45–86, Proc. Sympos. Pure Math. 104, American Mathematical Society, Providence, RI, 2021 Zbl 1518.35002 MR 4337417

- [10] R. L. Frank, [Lieb–Thirring inequalities and other functional inequalities for orthonormal systems](#). In *ICM—International Congress of Mathematicians. Vol. 5. Sections 9–11*, pp. 3756–3774, EMS Press, Berlin, 2023 Zbl [07823084](#) MR [4680381](#)
- [11] R. L. Frank, D. Hundertmark, M. Jex, and P. T. Nam, [The Lieb–Thirring inequality revisited](#). *J. Eur. Math. Soc. (JEMS)* **23** (2021), no. 8, 2583–2600 Zbl [1467.35014](#) MR [4269422](#)
- [12] R. L. Frank, A. Laptev, and T. Weidl, [Schrödinger operators: eigenvalues and Lieb–Thirring inequalities](#). Cambridge Stud. Adv. Math. 200, Cambridge University Press, Cambridge, 2023 Zbl [7595814](#) MR [4496335](#)
- [13] R. L. Frank and R. Seiringer, [Non-linear ground state representations and sharp Hardy inequalities](#). *J. Funct. Anal.* **255** (2008), no. 12, 3407–3430 Zbl [1189.26031](#) MR [2469027](#)
- [14] D. Gontier, M. Lewin, and F. Q. Nazar, [The nonlinear Schrödinger equation for orthonormal functions: existence of ground states](#). *Arch. Ration. Mech. Anal.* **240** (2021), no. 3, 1203–1254 Zbl [1470.35328](#) MR [4264945](#)
- [15] C. Hainzl and R. Seiringer, [General decomposition of radial functions on \$\mathbb{R}^n\$ and applications to \$N\$ -body quantum systems](#). *Lett. Math. Phys.* **61** (2002), no. 1, 75–84 Zbl [1016.81059](#) MR [1930084](#)
- [16] I. W. Herbst, [Spectral theory of the operator \$\(p^2 + m^2\)^{1/2} - Ze^2/r\$](#) . *Comm. Math. Phys.* **53** (1977), no. 3, 285–294 Zbl [0375.35047](#) MR [0436854](#)
- [17] M. Hoffmann-Ostenhof and T. Hoffmann-Ostenhof, [“Schrödinger inequalities” and asymptotic behavior of the electron density of atoms and molecules](#). *Phys. Rev. A* (3) **16** (1977), no. 5, 1782–1785 MR [0471726](#)
- [18] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, and J. Tidblom, [Many-particle Hardy inequalities](#). *J. Lond. Math. Soc. (2)* **77** (2008), no. 1, 99–114 Zbl [1135.35012](#) MR [2389919](#)
- [19] V. F. Kovalenko, M. A. Perel'muter, and Y. A. Semenov, [Schrödinger operators with \$L^{1/2}_w\(\mathbf{R}^l\)\$ -potentials](#). *J. Math. Phys.* **22** (1981), no. 5, 1033–1044 Zbl [0463.47027](#) MR [0622855](#)
- [20] G. Leoni, [A first course in Sobolev spaces](#). 2nd edn., Grad. Stud. Math. 181, American Mathematical Society, Providence, RI, 2017 Zbl [1382.46001](#) MR [3726909](#)
- [21] J.-M. Lévy-Leblond, [Nonsaturation of gravitational forces](#). *J. Math. Phys.* **10** (1969), 806–812 Zbl [0181.56602](#)
- [22] E. H. Lieb, [A lower bound for Coulomb energies](#). *Phys. Lett. A* **70** (1979), no. 5-6, 444–446 MR [0588128](#)
- [23] E. H. Lieb, [Thomas–Fermi and related theories of atoms and molecules](#). *Rev. Modern Phys.* **53** (1981), no. 4, 603–641 Zbl [1049.81679](#) MR [0629207](#)
- [24] E. H. Lieb and M. Loss, [Analysis](#). 2nd edn., Grad. Stud. Math. 14, American Mathematical Society, Providence, RI, 2001 Zbl [0966.26002](#) MR [1817225](#)
- [25] E. H. Lieb and S. Oxford, [Improved lower bound on the indirect Coulomb energy](#). *Int. J. Quantum Chem.* **19** (1981), 427–439
- [26] E. H. Lieb and R. Seiringer, [The stability of matter in quantum mechanics](#). Cambridge University Press, Cambridge, 2010 Zbl [1179.81004](#) MR [2583992](#)

- [27] E. H. Lieb and B. Simon, [The Thomas–Fermi theory of atoms, molecules and solids](#). *Advances in Math.* **23** (1977), no. 1, 22–116 Zbl [0938.81568](#) MR [0428944](#)
- [28] E. H. Lieb, J. P. Solovej, and J. Yngvason, [Ground states of large quantum dots in magnetic fields](#). *Phys. Rev. B* **51** (1995), 10646–10665
- [29] E. H. Lieb and W. E. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. *Stud. Math. Phys.* (1976), 269–303 Zbl [0342.35044](#)
- [30] E. H. Lieb and W. E. Thirring, [Gravitational collapse in quantum mechanics with relativistic kinetic energy](#). *Ann. Physics* **155** (1984), no. 2, 494–512 MR [0753345](#)
- [31] E. H. Lieb and H.-T. Yau, [The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics](#). *Comm. Math. Phys.* **112** (1987), no. 1, 147–174 Zbl [0641.35065](#) MR [0904142](#)
- [32] E. H. Lieb and H.-T. Yau, [The stability and instability of relativistic matter](#). *Comm. Math. Phys.* **118** (1988), no. 2, 177–213 Zbl [0686.35099](#) MR [0956165](#)
- [33] D. Lundholm, P. T. Nam, and F. Portmann, [Fractional Hardy–Lieb–Thirring and related inequalities for interacting systems](#). *Arch. Ration. Mech. Anal.* **219** (2016), no. 3, 1343–1382 Zbl [1332.81292](#) MR [3448930](#)
- [34] V. Maz'ya, [Sobolev spaces with applications to elliptic partial differential equations](#). Grundlehren Math. Wiss. 342, Springer, Heidelberg, 2011 Zbl [1217.46002](#) MR [2777530](#)
- [35] B. Opic and A. Kufner, *Hardy-type inequalities*. Pitman Res. Notes Math. Ser. 219, Longman Scientific & Technical, Harlow, 1990 Zbl [0698.26007](#) MR [1069756](#)
- [36] J. Peetre, [Espaces d'interpolation et théorème de Soboleff](#). *Ann. Inst. Fourier (Grenoble)* **16** (1966), no. fasc. 1, 279–317 Zbl [0151.17903](#) MR [0221282](#)
- [37] W. Thirring, [A lower bound with the best possible constant for Coulomb Hamiltonians](#). *Comm. Math. Phys.* **79** (1981), no. 1, 1–7 MR [0609223](#)
- [38] D. Yafaev, [Sharp constants in the Hardy–Rellich inequalities](#). *J. Funct. Anal.* **168** (1999), no. 1, 121–144 Zbl [0981.26016](#) MR [1717839](#)

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