IMA Journal of Numerical Analysis (2023) **00**, 1–39
https://doi.org/DOI HERE

Monolithic and local time-stepping decoupled algorithms for transport problems in fractured porous media

YANZHAO CAO, THI-THAO-PHUONG HOANG*, AND PHUOC-TOAN HUYNH
Department of Mathematics and Statistics, Auburn University, 221 Roosevelt Concourse, Auburn, 36849, Alabama, USA
*Corresponding author: tzh0059@auburn.edu

[Received on Date Month Year; revised on Date Month Year; accepted on Date Month Year]

The objective of this paper is to develop efficient numerical algorithms for the linear advection-diffusion equation in fractured porous media. A reduced fracture model is considered where the fractures are treated as interfaces between subdomains and the interactions between the fractures and the surrounding porous medium are taken into account. The model is discretized by a backward Euler upwind-mixed hybrid finite element method in which the flux variable represents both the advective and diffusive fluxes. The existence, uniqueness, as well as optimal error estimates in both space and time for the fully discrete coupled problem are established. Moreover, to facilitate different time steps in the fracture-interface and the subdomains, global-in-time, non-overlapping domain decomposition is utilized to derive two implicit iterative solvers for the discrete problem. The first method is based on the time-dependent Steklov-Poincaré operator, while the second one employs the optimized Schwarz waveform relaxation (OSWR) approach with Ventcel-Robin transmission conditions. A discrete space-time interface system is formulated for each method and is solved iteratively with possibly variable time step sizes. The convergence of the OSWR-based method with conforming time grids is also proved. Finally, numerical results in two dimensions are presented to verify the optimal order of convergence of the monolithic solver and to illustrate the performance of the two decoupled schemes with local time-stepping on problems of high Péclet numbers.

Keywords: Advection-diffusion; reduced fracture model; mixed-hybrid finite elements; local time-stepping; time-dependent Steklov-Poincaré operator; optimized Schwarz waveform relaxation.

1. Introduction

The presence of fractures in porous media often has a strong influence on the fluid flow in the rock matrix, which greatly complicates the modeling of flow and transport problems. The permeability in the fractures may be significantly higher or lower than the one in the surrounding regions. Therefore, the time scales in the fractures can be much faster or much slower than those in the subdomains. Additionally, the widths of the fractures are usually much smaller than the size of the domain of calculation and any reasonable parameter of spatial discretization. Thus, to accurately represent the fractures, one must refine the grids locally around the fractures, which is computationally costly. To address this difficulty, we consider a reduced fracture model in which the fractures are treated as lower dimensional interfaces embedded in the rock matrix. The model consists of d-dimensional problems in the subdomains coupled with $(d-1)$-dimensional problems on the fractures via suitable interface conditions. For a thorough review of such reduced fracture models, we refer to [2, 3, 6, 29, 37, 51, 54, 57] and the references therein.

In this paper, we are concerned with numerical algorithms for the reduced fracture models of the linear advection-diffusion equation written in mixed form. The fracture is assumed to have larger permeability than the surrounding porous medium; as a consequence, the physical processes in the fracture occur faster than those in the rock matrix. Thus, it is computationally inefficient to use a single time step size throughout the entire domain of calculation. Additionally, when the advection is strongly dominant, standard numerical methods tend to produce numerical instabilities, typically due to the improper resolution of sharp layers in the approximate solution [19, 59]. To overcome these challenges, we propose to use non-overlapping, global-in-time domain decomposition (DD) methods and upwind-mixed hybrid finite elements to facilitate different time step sizes on the fracture and the subdomains and obtain a stable numerical solution.

Global-in-time DD methods provide a powerful tool to perform parallel simulations of time-dependent physical phenomena with different time steps across the domain. This approach has been extensively studied for the flow and transport problems without fractures (see, e.g., [43, 45, 46, 47, 52] and the references therein), in which an artificial interface and additional equations on that interface are introduced to write the transmission conditions. When a fracture is present and the reduced fracture model is considered, the fracture serves as a physical interface which decomposes the rock matrix into non-overlapping subdomains. Moreover, the tangential PDEs on the fracture obtained from the averaging process and the interactions between the fracture and the rock matrix provide a natural representation of the physical transmission conditions. Recently, different global-in-time DD methods have been constructed to find a numerical solution of reduced fracture models for the compressible fluid flow [44, 49] and linear transport problems with operator splitting [50]. Among those methods, the global-in-time fracture-based Schur (GTF-Schur) and the global-in-time optimized Schwarz (GTO-Schwarz) methods are shown to be most efficient; in particular, numerical results suggested that both methods give fast convergence without using any preconditioners and GTF-Schur preserves the accuracy in time on the fracture when smaller time steps are used in the fracture than in the subdomains [49, 50]. However, with operator splitting, the advection is treated explicitly, which increases the computational time for problems with high Péclet numbers as the time step is constrained by the CFL condition. We also remark that due to the complexity in the structure of reduced fracture models, the convergence analysis of global-in-time DD methods for such models in general remains an open problem.

The upwind-mixed hybrid finite element scheme for the transport problem (with no fractures) was first introduced in [59] and analyzed in [19]. Unlike the standard upwind-mixed schemes [23, 24] where the flux variable only represents the diffusive flux, the upwind-mixed hybrid scheme employs a mixed hybrid finite element method for spatial discretization in which the flux variable approximates the total flux consisting of both advective and diffusive fluxes. To define the upwind weights for the scheme, the Lagrange multipliers arising in the hybrid formulation are utilized to give an approximation for the advective flux. A similar idea was also employed in [62] for the discretization with Raviart–Thomas elements of lowest order and in [18] with Brezzi–Douglas–Marini elements of lowest order. Optimal first-order convergence in both spatial and temporal errors for the upwind-mixed scheme was proved in [19]. It was shown in [19, 59] that the upwind-mixed hybrid scheme is fully mass conservative and provides the same accuracy as the upwind-mixed method [23], while being more robust and less costly for problems with high Péclet numbers.

The goal of this work is to design and analyze efficient numerical methods for the reduced fracture model of strongly advection-dominated transport problems. While there is a rich literature on the numerical methods for the reduced fracture model of the flow problems and their convergence analysis with or without providing the order of convergence [1, 5, 6, 13, 22, 28, 37, 51, 54, 56, 57], there has been little work that explores these aspects for the transport problems [4, 38]. In this work, we first introduce a fully discrete upwind-mixed hybrid finite element scheme and demonstrate the existence, uniqueness, and optimal first-order convergence for the scheme with conforming spatial discretization.

Due to the presence of the fracture, the main difficulty of proving the optimal error estimate lies in the terms representing the traces on the fracture of the normal fluxes from the subdomains. In particular, if one uses the inverse inequality directly to handle these terms, only sub-optimal order convergence in space is obtained. However, these normal fluxes can be eliminated if we utilize the properties of the L^2-projections associated with the Raviart-Thomas mixed finite element spaces on conforming spatial meshes (see the proof of Theorem 3). Moreover, since solving the reduced model monolithically only allows a single time grid to be imposed on the fracture and the subdomains, we propose two global-in-time DD methods, namely GTF-Schur and GTO-Schwarz, for the fully discrete problem to enhance computational efficiency with nonconforming temporal discretization. These methods were previously used in [50] for the same reduced fracture model. However, unlike [50] where the advection and the diffusion are separated and treated differently due to operator splitting, the fully discrete interface problem formulated for each method in this work requires no separate unknowns or equations for the advection and the diffusion. In addition, the methods proposed here are fully implicit with no CFL constraints for the time step size.

The main contributions of this work include four aspects. Firstly, we derive a fully discrete upwind-mixed scheme for the reduced fracture model and establish optimal first-order convergence in both space and time discretizations. Secondly, to allow local time-stepping in the subdomains and in the fracture, we formulate two global-in-time DD methods, GTF-Schur and GTO-Schwarz, in the context of upwind-mixed hybrid finite element method. In particular, we impose smaller time steps in the fracture and larger ones in the rock matrix via an L^2-projection in time [33, 34]. Thirdly, we prove the convergence of the GTO-Schwarz method with conforming discretization in time. To the best of our knowledge, this is the first time optimal order convergence as well as the convergence of GTO-Schwarz for the reduced fracture model of linear transport problems with mixed hybrid finite elements have been established. Lastly, we carry out numerical experiments with various Péclet numbers to verify and compare the performance of the two proposed DD methods with conforming and nonconforming time grids in the fracture and in the rock matrix. It should be noted that while the numerical methods and analysis are presented for problems in two spatial dimensions, the results can be straightforwardly extended to the three dimensional case, except for Theorem 2 as discussed in Remark 1.

The rest of this paper is organized as follows: in the next section, we present the reduced fracture model of the linear advection-diffusion equation, its fully discrete formulation, and the corresponding upwind-mixed hybrid scheme. Existence, uniqueness, and convergence analysis for the upwind-mixed hybrid scheme is carried out in Section 3. In Section 4, the fully discrete interface equations for the GTF-Schur and GTO-Schwarz methods are derived; convergence of the OSWR algorithm is also proved where conforming time steps are imposed in the subdomains and in the fracture. In Section 5, we describe how to formulate the two global-in-time DD methods when nonconforming time grids are used via L^2-projection operators. We present numerical results in Section 6 to illustrate and compare the performance of the proposed methods. Finally, some concluding remarks are given in Section 7.

2. Upwind-mixed hybrid finite element method for the reduced fracture model

2.1. *Reduced fracture model of the linear transport problem*

Let Ω be any bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and $T > 0$ be some fixed time. We assume that Ω is separated into two non-overlapping subdomains $\Omega_i,\ i = 1,2$, by a fracture Ω_f of thickness δ as depicted in Figure 1. For simplicity, we assume further that Ω_f can be expressed as

$$\Omega_f = \left\{ \boldsymbol{x} \in \Omega : \boldsymbol{x} = \boldsymbol{x}_\gamma + s\boldsymbol{n},\ \text{where } \boldsymbol{x}_\gamma \in \gamma \text{ and } s \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \right\},$$

where γ is the intersection between a line and Ω.

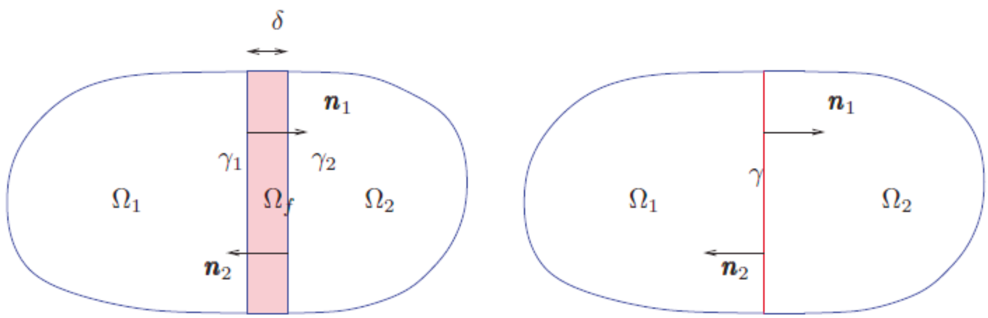

FIG. 1. The domain Ω with the fracture Ω_f (left) and the fracture-interface γ (right).

We consider the linear advection-diffusion problem written in mixed formulation as follows:

$$\begin{array}{rcll} \phi\partial_t c + \operatorname{div}\boldsymbol{\varphi} & = & q & \text{in } \Omega\times(0,T), \\ \boldsymbol{\varphi} & = & \boldsymbol{u}c - \boldsymbol{D}\nabla c & \text{in } \Omega\times(0,T), \\ c & = & 0 & \text{on } \partial\Omega\times(0,T), \\ c(\cdot,0) & = & c_0 & \text{in } \Omega, \end{array} \tag{2.1}$$

where c is the concentration of a contaminant dissolved in a fluid, q is the source term, ϕ is the porosity, $\boldsymbol{u}$ is the Darcy velocity (assume to be given and time-independent), and $\boldsymbol{D}$ is a symmetric time-independent diffusion tensor.

For $i=1,2$, we denote by γ_i the part of the boundary of Ω_i shared with the boundary of the fracture Ω_f: $\gamma_i = (\partial\Omega_i \cap \partial\Omega_f)\cap\Omega$. Moreover, let $\boldsymbol{n}_i$ be the unit, outward-pointing, normal vector field on $\partial\Omega_i$, where $\boldsymbol{n}=\boldsymbol{n}_1=-\boldsymbol{n}_2$. For $i=1,2,f$, and for any scalar, vector, or tensor valued function g defined on Ω, we denote by g_i the restriction of g to Ω_i. The original problem (2.1) can be rewritten as the following transmission problems:

$$\begin{array}{rcllr} \phi_i\partial_t c_i + \operatorname{div}\boldsymbol{\varphi}_i & = & q_i & \text{in } \Omega_i\times(0,T), & i=1,2,f, \\ \boldsymbol{\varphi}_i & = & \boldsymbol{u}_i c_i - \boldsymbol{D}_i\nabla c_i & \text{in } \Omega_i\times(0,T), & i=1,2,f, \\ c_i & = & 0 & \text{on } (\partial\Omega_i\cap\partial\Omega)\times(0,T), & i=1,2,f, \\ c_i & = & c_f & \text{on } \gamma_i\times(0,T), & i=1,2, \\ \boldsymbol{\varphi}_i\cdot\boldsymbol{n}_i & = & \boldsymbol{\varphi}_f\cdot\boldsymbol{n}_i & \text{on } \gamma_i\times(0,T), & i=1,2, \\ c_i(\cdot,0) & = & c_{0,i} & \text{in } \Omega_i, & i=1,2,f. \end{array} \tag{2.2}$$

In this work, we consider a reduce fracture model for (2.2) under the assumptions that the fracture Ω_f has a small width compared to the size of Ω and higher permeability than that in the subdomains. The model was first derived in [2, 3] by averaging across the transversal cross sections of the two-dimensional fracture Ω_f. Denote by ∇_τ and div_τ the tangential gradient and tangential divergence, respectively, and let $\phi_\gamma := \delta\phi_f$ and $\boldsymbol{D}_\gamma := \delta\boldsymbol{D}_{f,\tau}$, where $\boldsymbol{D}_{f,\tau}$ is the tangential component of $\boldsymbol{D}_f$. The reduced model for (2.2) consists of equations in the subdomains,

$$\begin{array}{rcll} \phi_i\partial_t c_i + \operatorname{div}\boldsymbol{\varphi}_i & = & q_i & \text{in } \Omega_i\times(0,T), \\ \boldsymbol{\varphi}_i & = & \boldsymbol{u}_i c_i - \boldsymbol{D}_i\nabla c_i & \text{in } \Omega_i\times(0,T), \\ c_i & = & 0 & \text{on } (\partial\Omega_i\cap\partial\Omega)\times(0,T), \\ c_i & = & c_\gamma & \text{on } \gamma\times(0,T), \\ c_i(\cdot,0) & = & c_{0,i} & \text{in } \Omega_i, \end{array} \tag{2.3}$$

for $i=1,2$, coupled with the following equation in the one-dimensional fracture,

$$\begin{array}{rcll}\phi_\gamma \partial_t c_\gamma + \mathrm{div}_\tau \boldsymbol{\varphi}_\gamma &=& q_\gamma + \sum\limits_{i=1}^{2} \boldsymbol{\varphi}_i \cdot \boldsymbol{n}_{i|\gamma} & \text{in } \gamma \times (0,T), \\ \boldsymbol{\varphi}_\gamma &=& \boldsymbol{u}_\gamma c_\gamma - \boldsymbol{D}_\gamma \nabla_\tau c_\gamma & \text{in } \gamma \times (0,T), \\ c_\gamma &=& 0 & \text{on } \partial\gamma \times (0,T), \\ c_\gamma(\cdot,0) &=& c_{0,\gamma} & \text{in } \gamma. \end{array} \tag{2.4}$$

Throughout the paper, we assume that:

(A1) The coefficient matrices $\boldsymbol{D}_i^{-1}$, $i=1,2$ and $\boldsymbol{D}_\gamma^{-1}$ are symmetric and uniformly positive definite. Furthermore, there exist two pairs of positive numbers (D^-, D^+) and $\left(D_\gamma^-, D_\gamma^+\right)$ such that

$$\begin{array}{l} D^-|\eta|^2 \le \eta^T \boldsymbol{D}_i^{-1}(x)\,\eta \le D^+|\eta|^2, \text{ for a.e. } x\in\Omega_i,\ \forall \eta\in\mathbb{R}^2,\ i=1,2, \\ D_\gamma^-|\varsigma|^2 \le \varsigma^T \boldsymbol{D}_\gamma^{-1}(s)\,\varsigma \le D_\gamma^+|\varsigma|^2, \text{ for a.e. } s\in\gamma,\ \forall\varsigma\in\mathbb{R}. \end{array}$$

(A2) There exist two positive numbers ϕ^- and ϕ^+ such that

$$\phi^- \le \phi_i(x) \le \phi^+, \text{ for a.e. } x\in\Omega_i,\ i=1,2, \text{ and } \phi^- \le \phi_\gamma(s) \le \phi^+, \text{ for a.e. } s\in\gamma.$$

(A3) Let $J=(0,T)$ and $H_*^1(\Omega_i) = \left\{g\in H^1(\Omega_i):\ g=0 \text{ on } \partial\Omega_i\cap\partial\Omega\right\}$, the following regularity conditions hold: $\boldsymbol{u}_i \in C\left(\bar{J};\left(W^{1,\infty}(\Omega_i)\right)^2\right)$, $q_i\in C\left(J, L^2(\Omega_i)\right)$ and $c_{0,i}\in H_*^1(\Omega_i)$, for $i=1,\ 2$, and $\boldsymbol{u}_\gamma \in C\left(\bar{J};\left(W^{1,\infty}(\gamma)\right)^2\right)$, $q_\gamma \in C\left(J, L^2(\gamma)\right)$ and $c_{0,\gamma}\in H_0^1(\gamma)$.

We utilize the following notation to derive the weak formulations of (2.3)-(2.4). For any measurable subset $\mathscr{O}$ of $\mathbb{R}^2$, let $(\cdot,\cdot)_{\mathscr{O}}$ and $\|\cdot\|_{0,\mathscr{O}}$ denote the inner product and norm on $L^2(\mathscr{O})$ or $\left(L^2(\mathscr{O})\right)^2$, respectively, and let $\|\cdot\|_{k,\mathscr{O}}$ stand for the norm on $H^k(\mathscr{O}) := W^{k,2}(\mathscr{O})$ ($H^k(\mathscr{O})$ concides with $L^2(\mathscr{O})$ when $k=0$). Let $H(\mathrm{div},\mathscr{O})$ denote the space of functions in $\left(L^2(\mathscr{O})\right)^2$ having the divergence in $L^2(\mathscr{O})$. We next define the following Hilbert spaces:

$$M = \left\{\mu = (\mu_1,\mu_2,\mu_\gamma) \in L^2(\Omega_1)\times L^2(\Omega_2)\times L^2(\gamma)\right\},$$

$$\boldsymbol{\Sigma} = \left\{\boldsymbol{v} = (\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_\gamma)\in (L^2(\Omega_1))^2\times(L^2(\Omega_2))^2\times L^2(\gamma) : \mathrm{div}\,\boldsymbol{v}_i\in L^2(\Omega_i),\ i=1,2, \text{ and } \mathrm{div}_\tau \boldsymbol{v}_\gamma - \sum_{n=1}^{2}(\boldsymbol{v}_i\cdot\boldsymbol{n}_i)_{|\gamma}\in L^2(\gamma)\right\}.$$

Finally, we introduce the following bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $r_\phi(\cdot,\cdot)$, $d_{\boldsymbol{u}}(\cdot,\cdot)$ and $e(\cdot,\cdot)$ on $\boldsymbol{\Sigma}\times\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}\times M$, $M\times M$, $M\times\boldsymbol{\Sigma}$ and $M\times\boldsymbol{\Sigma}$, respectively,

$$\begin{array}{l} a(\boldsymbol{w},\boldsymbol{v}) = \sum\limits_{i=1}^{2}\left(\boldsymbol{D}_i^{-1}\boldsymbol{w}_i,\boldsymbol{v}_i\right)_{\Omega_i} + \left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_\gamma,\boldsymbol{v}_\gamma\right)_\gamma, \quad b(\boldsymbol{w},\mu) = \sum\limits_{i=1}^{2}(\mathrm{div}\,\boldsymbol{w}_i,\mu_i)_{\Omega_i} + \left(\mathrm{div}_\tau\,\boldsymbol{w}_\gamma,\mu_\gamma\right)_\gamma, \\ r_\phi(\eta,\mu) = \sum\limits_{i=1}^{2}(\phi_i\eta_i,\mu_i)_{\Omega_i} + \left(\phi_\gamma\eta_\gamma,\mu_\gamma\right)_\gamma, \quad d_{\boldsymbol{u}}(\mu,\boldsymbol{w}) = \sum\limits_{i=1}^{2}\left(\boldsymbol{D}_i^{-1}\boldsymbol{u}_i\mu_i,\boldsymbol{w}_i\right)_{\Omega_i} + \left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{u}_\gamma\mu_\gamma,\boldsymbol{w}_\gamma\right)_\gamma, \\ e(\boldsymbol{w},\mu) = \sum\limits_{i=1}^{2}\left\langle \boldsymbol{w}_i\cdot\boldsymbol{n}_{i|\gamma},\mu_\gamma\right\rangle_\gamma, \end{array} \tag{2.5}$$

and the linear form L_q on M: $L_q(\mu) = \sum\limits_{i=1}^{2}(q_i,\mu_i)_{\Omega_i} + (q_\gamma,\mu_\gamma)_\gamma$. With these spaces and forms, the weak form of (2.3)-(2.4) can be written as follows:

Find $c=(c_1,c_2,c_\gamma)\in H^1(0,T;M)$ and $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,\boldsymbol{\varphi}_\gamma)\in L^2(0,T;\boldsymbol{\Sigma})$ such that

$$\begin{array}{rcll} a(\boldsymbol{\varphi},\boldsymbol{v}) - b(\boldsymbol{v},c) + e(\boldsymbol{v},c) - d_{\boldsymbol{u}}(c,\boldsymbol{v}) &=& 0 & \forall \boldsymbol{v}\in\boldsymbol{\Sigma}, \\ r_\phi(\partial_t c,\mu) + b(\boldsymbol{\varphi},\mu) - e(\boldsymbol{\varphi},\mu) &=& L_q(\mu) & \forall\mu\in M, \end{array} \tag{2.6}$$

together with the initial conditions:

$$c_i(\cdot,0) = c_{0,i}, \text{ in } \Omega_i,\ i=1,2, \quad \text{and} \quad c_\gamma(\cdot,0) = c_{0,\gamma}, \text{ in } \gamma. \tag{2.7}$$

For error analysis purpose, we shall assume that the solution $(\boldsymbol{\varphi}, c)$ of (2.6)-(2.7) satisfies the following regularity condition:

(A4) $(\boldsymbol{\varphi}, c) \in C\left(\bar{J}, \boldsymbol{\mathscr{H}}^1\right) \times \left(H^1\left(J;\mathscr{H}^1\right) \cap H^2\left(J;M\right) \cap C\left(J,\mathscr{H}^1\right)\right)$, where $\mathscr{H}^k := H^k(\Omega_1) \times H^k(\Omega_2) \times H^k(\gamma)$ and $\boldsymbol{\mathscr{H}}^k := (H^k(\Omega_1))^2 \times (H^k(\Omega_2))^2 \times H^k(\gamma)$, $k = 0, 1$.

2.2. *Upwind-mixed hybrid finite element method for the monolithic problem*

We now derive the fully discrete upwind-mixed hybrid finite element algorithm to find a numerical solution to (2.6)-(2.7). We begin with discretizing the equations in (2.6) in space based on the lowest-order Raviart-Thomas mixed finite element method. For simplicity, assume Ω is a rectangular domain. Let $\mathscr{K}_{h,i},\ i = 1, 2$ be a finite element partition of each Ω_i into rectangles such that they match on γ and their union $\mathscr{K}_h = \cup_{i=1}^{2} \mathscr{K}_{h,i}$ forms a finite element partition of Ω. Note that the analysis presented below also holds for triangular meshes that satisfy assumptions (M1) - (M6) in [19] and match on the interface.

For $i = 1, 2$, let $\mathscr{E}_{h,i}^I$ be the set of all interior edges and $\mathscr{E}_{h,i}^D$ be the set of edges of the external boundary $\partial\Omega_i \cap \partial\Omega$. Moreover, we denote by $\mathscr{E}_h^\gamma$ the set of edges of elements in $\mathscr{K}_{h,1}$ or $\mathscr{K}_{h,2}$ that lie on γ. We then denote by $\mathscr{E}_{h,i}$ the set of all edges of elements in $\mathscr{K}_{h,i}$:

$$\mathscr{E}_{h,i} = \mathscr{E}_{h,i}^I \cup \mathscr{E}_{h,i}^D \cup \mathscr{E}_h^\gamma,\ i = 1, 2.$$

We also denote by $\mathscr{P}_h^\gamma$ the set of endpoints P of all interface edges $E \in \mathscr{E}_h^\gamma$. For any $K \in \mathscr{K}_{h,i},\ i = 1, 2$, let $\boldsymbol{n}_K$ denote the unit, normal, outward-pointing vector field on the boundary ∂K; for each edge E on ∂K, let $\boldsymbol{n}_E$ denote the unit normal vector of E, outward to K and let $\boldsymbol{n}_{\partial E}$ be the unit tangential vector field of E at the two endpoints of E, outward to E. Let $h_K = \operatorname{diam}(K), h_E = |E|$, $h_i = \max\limits_{K \in \mathscr{K}_{h,i}} h_K, i = 1, 2$, $h_\gamma = \max\limits_{E \in \mathscr{E}_h^\gamma} h_E$, and $h = \max\{h_1, h_2, h_\gamma\}$. With the given notation, the lowest-order Raviart-Thomas mixed finite element spaces on Ω_i are defined as follows:

$$\begin{aligned} M_{h,i} &:= \left\{\mu_i \in L^2(\Omega_i) : \mu_{i|K} = \text{constant}, \forall K \in \mathscr{K}_{h,i}\right\}, \\ \boldsymbol{\Sigma}_{h,i} &:= \left\{\boldsymbol{v}_{h,i} \in H(\operatorname{div}, \Omega_i) : \boldsymbol{v}_{h,i|K} \in \boldsymbol{\Sigma}_K, \forall K \in \mathscr{K}_{h,i}\right\}, i = 1, 2, \end{aligned}$$

where $\boldsymbol{\Sigma}_K := \left\{\boldsymbol{w} : K \to \mathbb{R}^2,\ \boldsymbol{w}(x,y) = (a_K + b_K x, a_K' + b_K' y), (a_K, b_K, a_K', b_K') \in \mathbb{R}^4\right\}$ is the local Raviart –Thomas space of lowest order on $K \in \mathscr{K}_{h,i}$. Similarly, for the discretization on γ, we have the following mixed finite element spaces:

$$\begin{aligned} M_{h,\gamma} &:= \left\{\mu_\gamma \in L^2(\gamma) : \mu_{\gamma|E} = \text{ constant on } E,\ \forall E \in \mathscr{E}_h^\gamma\right\}, \\ \boldsymbol{\Sigma}_{h,\gamma} &:= \left\{\boldsymbol{v}_\gamma \in H(\operatorname{div}_\tau, \gamma) : \boldsymbol{v}_{\gamma|E} \in \boldsymbol{\Sigma}_{\gamma,E}, \forall E \in \mathscr{E}_h^\gamma\right\}, \end{aligned}$$

where $\boldsymbol{\Sigma}_{\gamma,E} := \left\{\boldsymbol{v}_\gamma : E \to \mathbb{R},\ \boldsymbol{v}_\gamma(s) = a_E + b_E s,\ (a_E, b_E) \in \mathbb{R}^2\right\}$, for $E \in \mathscr{E}_h^\gamma$.

Instead of using these classical mixed finite element spaces, in this work we apply a hybridization technique to obtain an equivalent hybrid formulation, namely the mixed hybrid finite element method [17, 61]. For this approach, the continuity constraint of the normal fluxes across inter-element boundaries is relaxed and is imposed by virtue of an additional equation involving Lagrange multipliers. The finite element spaces related to the mixed hybrid finite element scheme are defined as

$$\begin{aligned} &\widetilde{\boldsymbol{\Sigma}}_{h,i} := \left\{\boldsymbol{v}_i \in (L^2(\Omega_i))^2 : \boldsymbol{v}_{i|K} \in \boldsymbol{\Sigma}_K, \forall K \in \mathscr{K}_{h,i}\right\}, i = 1, 2, \\ &\widetilde{\boldsymbol{\Sigma}}_{h,\gamma} := \left\{\boldsymbol{v}_\gamma \in L^2(\gamma) : \boldsymbol{v}_{\gamma|E} \in \boldsymbol{\Sigma}_{\gamma,E}, \forall E \in \mathscr{E}_h^\gamma\right\}, \\ &\Theta_{h,i} := \left\{\eta \in L^2(\mathscr{E}_{h,i}) : \eta_{|E} = \text{ constant on } E,\ \forall E \in \mathscr{E}_{h,i}^I \cup \mathscr{E}_h^\gamma \text{ and } \eta_{|E} = 0,\ \forall E \in \mathscr{E}_{h,i}^D\right\}, i = 1,\ 2, \\ &\Theta_{h,\gamma} := \left\{\varsigma : \mathscr{P}_h^\gamma \to \mathbb{R},\ \varsigma(P) = 0 \text{ if } P \in \partial\gamma\right\}, \end{aligned}$$

where the last two spaces are for the Lagrange multipliers of the two-dimensional problems on the subdomains and the one-dimensional equations on the fracture, respectively. Next, we introduce several products of these finite element spaces:

$$\begin{aligned} M_h &= M_{h,1} \times M_{h,2} \times M_{h,\gamma}, \quad & \mathbf{\Sigma}_h &= \mathbf{\Sigma}_{h,1} \times \mathbf{\Sigma}_{h,2} \times \mathbf{\Sigma}_{h,\gamma}, \\ \widetilde{\mathbf{\Sigma}}_h &= \widetilde{\mathbf{\Sigma}}_{h,1} \times \widetilde{\mathbf{\Sigma}}_{h,2} \times \widetilde{\mathbf{\Sigma}}_{h,\gamma}, \quad & \Theta_h &= \Theta_{h,1} \times \Theta_{h,2} \times \Theta_{h,\gamma}. \end{aligned} \tag{2.8}$$

For $i = 1,2$, and any $c_{h,i}(t) \in M_{h,i}$, we have the unique representation:

$$c_{h,i}(t,x,y) = \sum_{K \in \mathscr{K}_{h,i}} c_{i,K}(t)\chi_K(x,y),$$

where χ_K is the characteristic function of element $K \in \mathscr{K}_{h,i}$ and $c_{i,K}$ represents the average of $c_{h,i}$ on K. Similarly, for the Lagrange multipliers $\theta_{h,i} \in \Theta_{h,i}$ of $c_{h,i}$, it can be represented uniquely as

$$\theta_{h,i}(t,x,y) = \sum_{E \in \mathscr{E}_{h,i}} \theta_{i,E}(t)\chi_E(x,y),$$

where χ_E is the characteristic function of edge $E \in \mathscr{E}_{h,i}$ and $\theta_{i,E}$ is the average value of $\theta_{h,i}$ on E. The velocity $\boldsymbol{\varphi}_{h,i}(t) \in \widetilde{\mathbf{\Sigma}}_{h,i}$ is defined locally as

$$\boldsymbol{\varphi}_{h,i}(t,x,y)_{|K} = \sum_{E \subset \partial K} \varphi_{i,KE}(t)\boldsymbol{w}_{i,KE}(x,y), \ \forall K \in \mathscr{K}_{h,i},$$

where $\varphi_{i,KE}$ is the normal flux leaving $K \in \mathscr{K}_{h,i}$ through the edge E and $\{\boldsymbol{w}_{i,KE}\}_{E \subset \partial K}$ are the basis functions of the local Raviart-Thomas space $\mathbf{\Sigma}_K$ satisfying

$$\int_{E'} \boldsymbol{w}_{i,KE} \cdot \boldsymbol{n}_K = \delta_{E,E'}, \ \forall E' \subset \partial K.$$

Similarly, for any $c_{h,\gamma}(t) \in M_{h,\gamma}$ and $\boldsymbol{\varphi}_{h,\gamma}(t) \in \widetilde{\mathbf{\Sigma}}_{h,\gamma}$, we have the following unique expressions

$$c_{h,\gamma}(t,y) = \sum_{E \in \mathscr{E}_h^\gamma} c_{\gamma,E}(t)\chi_E(y), \ \text{ and } \boldsymbol{\varphi}_{h,\gamma}(t,y)_{|E} = \sum_{P \in \partial E} \varphi_{\gamma,EP}(t)\boldsymbol{w}_{\gamma,EP}(y), \ \forall E \in \mathscr{E}_h^\gamma,$$

where $\{\boldsymbol{w}_{\gamma,EP}\}_{P \in \partial E}$ are the local basis functions of $\mathbf{\Sigma}_{\gamma,E}$. We also denote by $\boldsymbol{u}_h = (\boldsymbol{u}_{h,1}, \boldsymbol{u}_{h,2}, \boldsymbol{u}_{h,\gamma})$ the projections of $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_\gamma)$ on $\mathbf{\Sigma}_{h,1} \times \mathbf{\Sigma}_{h,2} \times \mathbf{\Sigma}_{h,\gamma}$:

$$\boldsymbol{u}_{h,i}(x,y) := \sum_{K \in \mathscr{K}_{h,i}} \sum_{E \subset \partial K} u_{i,KE}\boldsymbol{w}_{i,KE}(x,y), \quad \boldsymbol{u}_{h,\gamma}(y) := \sum_{E \in \mathscr{E}_h^\gamma} \sum_{P \in \partial E} u_{\gamma,EP}\boldsymbol{w}_{\gamma,EP}(y), \tag{2.9}$$

where $u_{i,KE} = \frac{1}{|E|}\int_E \boldsymbol{u}_i \cdot \boldsymbol{n}_K$ and $u_{\gamma,EP} = (\boldsymbol{u}_\gamma \cdot \boldsymbol{n}_{\partial E})_{|P}$.

The classical mixed finite element scheme for the problem (2.6)-(2.7) is given by:

Find $(c_h(t), \boldsymbol{\varphi}_h(t)) \in M_h \times \mathbf{\Sigma}_h$ for a.e. $t \in (0,T)$ such that

$$\begin{array}{rcll} a(\boldsymbol{\varphi}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, c_h) + e(\boldsymbol{v}_h, c_h) - d_{\boldsymbol{u}_h}(c_h, \boldsymbol{v}_h) &=& 0 & \forall \boldsymbol{v}_h \in \mathbf{\Sigma}_h, \\ r_\phi(\partial_t c_h, \mu_h) + b(\boldsymbol{\varphi}_h, \mu_h) - e(\boldsymbol{\varphi}_h, \mu_h) &=& L_q(\mu_h) & \forall \mu_h \in M_h, \end{array} \tag{2.10}$$

with the initial condition

$$(c_h(0), \mu_h) = (c_0, \mu_h), \ \forall \mu_h \in M_h, \tag{2.11}$$

where $c_h = (c_{h,1}, c_{h,2}, c_{h,\gamma})$, $c_0 = (c_{0,1}, c_{0,2}, c_{0,\gamma})$ and $\boldsymbol{\varphi}_h = (\boldsymbol{\varphi}_{h,1}, \boldsymbol{\varphi}_{h,2}, \boldsymbol{\varphi}_{h,\gamma})$.

It is well-known that if we employ the basis functions of M_h and $\mathbf{\Sigma}_h$ in the system (2.10), the resulting linear algebra system is in general indefinite [19, 59, 62]. Moreover, it is not guaranteed that the scheme works for strongly advection-dominated problems [59]. To overcome these difficulties, we apply a hybridization process to replace $\mathbf{\Sigma}_h$ by $\widetilde{\mathbf{\Sigma}}_h$, thus, the continuity conditions of the normal components of the flux variables across element interfaces are no longer required. These conditions are later imposed by introducing the Lagrange multipliers from the space Θ_h to ensure the equivalence of both algorithms. Towards this end, we define, in addition to the forms in (2.5), the following mapping on $\Theta_h \times \widetilde{\mathbf{\Sigma}}_h$:

$$l(\eta_h, \boldsymbol{v}_h) = \sum_{i=1}^{2} \sum_{K \in \mathscr{K}_{h,i}} \sum_{\substack{E \subset \partial K \\ E \in \mathscr{E}_{h,i} \setminus \mathscr{E}_h^{\gamma}}} \left\langle \eta_{h,i}, \boldsymbol{v}_{h,i} \cdot \boldsymbol{n}_K \right\rangle_E + \sum_{E \in \mathscr{E}_h^{\gamma}} \left\langle \eta_{h,\gamma}, \, \boldsymbol{v}_{h,\gamma} \cdot \boldsymbol{n}_{\partial E} \right\rangle_{\partial E}, \tag{2.12}$$

where $\eta_h = (\eta_{h,1}, \eta_{h,2}, \eta_{h,\gamma}) \in \Theta_h$ and $\boldsymbol{v}_h = (\boldsymbol{v}_{h,1}, \boldsymbol{v}_{h,2}, \boldsymbol{v}_{h,\gamma}) \in \widetilde{\mathbf{\Sigma}}_h$. To take into account the interface as part of the subdomain boundary, we define the space

$$\Theta_{h,i}^0 := \left\{ \eta \in \Theta_{h,i} : \eta_{|E} = 0, \, \forall E \in \mathscr{E}_h^{\gamma} \right\}, \; i = 1, 2,$$

and denote by Θ_h^0 the product space $\Theta_{h,1}^0 \times \Theta_{h,2}^0 \times \Theta_{h,\gamma}$. Altogether, the semi-discrete mixed hybrid formulation associated with (2.10) is written as follows:

Find $(c_h(t), \boldsymbol{\varphi}_h(t), \theta_h(t)) \in M_h \times \widetilde{\mathbf{\Sigma}}_h \times \Theta_h$ for a.e. $t \in (0, T)$ such that

$$\begin{aligned} & a(\boldsymbol{\varphi}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, c_h) + e(\boldsymbol{v}_h, c_h) - d_{\boldsymbol{u}_h}(c_h, \boldsymbol{v}_h) + l(\theta_h, \boldsymbol{v}_h) = 0, \\ & r_{\phi}(\partial_t c_h, \mu_h) + b(\boldsymbol{\varphi}_h, \mu_h) - e(\boldsymbol{\varphi}_h, \mu_h) = L(q, \mu_h), \qquad \forall (\mu_h, \boldsymbol{v}_h, \eta_h) \in M_h \times \widetilde{\mathbf{\Sigma}}_h \times \Theta_h^0, \\ & l(\eta_h, \boldsymbol{\varphi}_h) = 0, \end{aligned} \tag{2.13}$$

with the initial conditions (2.11), where $\theta_h = (\theta_{h,1}, \theta_{h,2}, \theta_{h,\gamma})$.

For advection-diffusion equations, the Lagrange multipliers can also be used to discretize the advective terms via upwind operators, which leads to an upwind-mixed hybrid scheme [19, 59] that can handle strongly advection-dominated problems. Specifically, we define an approximation of the advective flux $d_{\boldsymbol{u}_h}(\cdot,\cdot)$ as follows: for $\mu_h = (\mu_{h,1}, \mu_{h,2}, \mu_{h,\gamma}) \in M_h$, $\eta_h = (\eta_{h,1}, \eta_{h,2}, \eta_{h,\gamma}) \in \Theta_h$ and $\boldsymbol{v}_h = (\boldsymbol{v}_{h,1}, \boldsymbol{v}_{h,2}, \boldsymbol{v}_{h,\gamma}) \in \widetilde{\mathbf{\Sigma}}_h$,

$$\begin{aligned} \widetilde{d}_{\boldsymbol{u}_h}((\mu_h, \eta_h), \boldsymbol{v}_h) = & \sum_{i=1}^{2} \sum_{K \in \mathscr{K}_{h,i}} \Bigg(\sum_{\substack{E \subset \partial K \\ E \in \mathscr{E}_{h,i} \setminus \mathscr{E}_h^{\gamma}}} u_{i,KE} \mathscr{U}_{i,KE}(\mu_{i,K}, \eta_{i,E}) (\boldsymbol{D}_i^{-1} \boldsymbol{w}_{i,KE}, \boldsymbol{v}_{h,i})_K \\ & + \sum_{\substack{E \subset \partial K \\ E \in \mathscr{E}_h^{\gamma}}} u_{i,KE} \mathscr{U}_{i,KE}^{\gamma}(\mu_{i,K}, \mu_{\gamma,E}) (\boldsymbol{D}_i^{-1} \boldsymbol{w}_{i,KE}, \boldsymbol{v}_{h,i})_K \Bigg) \\ & + \sum_{E \in \mathscr{E}_h^{\gamma}} \sum_{P \in \partial E} u_{\gamma,EP} \mathscr{U}_{\gamma,EP}(\mu_{\gamma,E}, \eta_{\gamma,P}) (\boldsymbol{D}_{\gamma}^{-1} \boldsymbol{w}_{\gamma,EP}, \boldsymbol{v}_{h,\gamma})_{\gamma}, \end{aligned} \tag{2.14}$$

where, for any $K \in \mathscr{K}_{h,i}$ and $E \subset \partial K$:

i) if $E \in \mathscr{E}_{h,i} \setminus \mathscr{E}_h^{\gamma}$, the upwind value $\mathscr{U}_{i,KE}$ is computed by

$$\mathscr{U}_{i,KE}(\mu_{i,K}, \eta_{i,E}) = \begin{cases} \mu_{i,K}, & \text{if } u_{i,KE} \geq 0, \\ 2\eta_{i,E} - \mu_{i,K}, & \text{if } E \in \mathscr{E}_{h,i}^{I} \text{ and } u_{i,KE} < 0, \\ 0, & \text{if } E \in \mathscr{E}_{h,i}^{D} \text{ and } u_{i,KE} < 0, \end{cases} \quad i = 1, 2, \tag{2.15}$$

ii) if $E \in \mathscr{E}_h^{\gamma}$, the upwind value $\mathscr{U}_{i,KE}^{\gamma}$ is computed by

$$\mathscr{U}_{i,KE}^{\gamma}(\mu_{i,K}, \mu_{\gamma,E}) = \begin{cases} \mu_{i,K}, & \text{if } u_{i,KE} \geq 0, \\ \mu_{\gamma,E}, & \text{if } u_{i,KE} < 0, \end{cases} \quad i = 1, 2, \tag{2.16}$$

while for any $E \in \mathscr{E}_h^\gamma$ and $P \in \partial E$, the upwind value $\mathscr{U}_{\gamma,EP}$ is given by

$$\mathscr{U}_{\gamma,EP}(\mu_{\gamma,E},\eta_{\gamma,P}) = \begin{cases} \mu_{\gamma,E}, & \text{if } u_{\gamma,EP} \geq 0, \\ 2\eta_{\gamma,P} - \mu_{\gamma,E}, & \text{if } P \notin \partial\gamma \text{ and } u_{\gamma,EP} < 0, \\ 0, & \text{if } P \in \partial\gamma \text{ and } u_{\gamma,EP} < 0. \end{cases} \tag{2.17}$$

By replacing $d_{\boldsymbol{u}_h}(\cdot,\cdot)$ in (2.13) with the new operator defined by (2.14), we obtain the following semi-discrete upwind-mixed hybrid scheme:

Find $(c_h(t),\boldsymbol{\varphi}_h(t),\theta_h(t)) \in M_h \times \widetilde{\boldsymbol{\Sigma}}_h \times \Theta_h$ for a.e. $t \in (0,T)$ such that

$$\begin{aligned} & a(\boldsymbol{\varphi}_h,\boldsymbol{v}_h) - b(\boldsymbol{v}_h,c_h) + e(\boldsymbol{v}_h,c_h) - \widetilde{d}_{\boldsymbol{u}_h}((c_h,\theta_h),\boldsymbol{v}_h) + l(\theta_h,\boldsymbol{v}_h) = 0, \\ & r_\phi(\partial_t c_h,\mu_h) + b(\boldsymbol{\varphi}_h,\mu_h) - e(\boldsymbol{\varphi}_h,\mu_h) = L_q(\mu_h), \qquad \forall(\mu_h,\boldsymbol{v}_h,\eta_h) \in M_h \times \widetilde{\boldsymbol{\Sigma}}_h \times \Theta_h^0. \\ & l(\eta_h,\boldsymbol{\varphi}_h) = 0, \end{aligned} \tag{2.18}$$

Finally, we discretize (2.18) in time by the backward Euler scheme and obtain the fully discrete upwind-mixed hybrid finite element method. We define the time step size $\Delta t = T/N$ and the discrete times $t_n = n\Delta t$, $n = 1,\dots,N$, where N is a positive integer. The time derivatives are approximated by the backward difference quotient

$$\bar{\partial} c^n = \frac{c^n - c^{n-1}}{\Delta t}, n = 1,\dots,N,$$

where the superscript n indicates the evaluation of a function at the discrete time $t = t^n$. The fully-discrete version of (2.18) reads as follows:

For $n = 1,\dots,N$, find $(c_h^n,\boldsymbol{\varphi}_h^n,\theta_h^n) \in M_h \times \widetilde{\boldsymbol{\Sigma}}_h \times \Theta_h$ satisfying

$$\begin{aligned} & a(\boldsymbol{\varphi}_h^n,\boldsymbol{v}_h) - b(\boldsymbol{v}_h,c_h^n) + e(\boldsymbol{v}_h,c_h^n) - \widetilde{d}_{\boldsymbol{u}_h}((c_h^n,\theta_h^n),\boldsymbol{v}_h) + l(\theta_h^n,\boldsymbol{v}_h) = 0, \\ & r_\phi(\bar{\partial} c_h^n,\mu_h) + b(\boldsymbol{\varphi}_h^n,\mu_h) - e(\boldsymbol{\varphi}_h^n,\mu_h) = L_{q^n}(\mu_h), \qquad \forall(\mu_h,\boldsymbol{v}_h,\eta_h) \in M_h \times \widetilde{\boldsymbol{\Sigma}}_h \times \Theta_h^0, \\ & l(\eta_h,\boldsymbol{\varphi}_h^n) = 0, \end{aligned} \tag{2.19}$$

where the initial conditions $(c_{h,1}^0,c_{h,2}^0,c_{h,\gamma}^0)$ are given by

$$c_{h,i|K_i}^0 := \frac{1}{|K_i|}\int_{K_i} c_{0,i}, \quad \forall K_i \in \mathscr{K}_{h,i}, i = 1,2, \text{ and } c_{h,\gamma|E}^0 := \frac{1}{|E|}\int_E c_{0,\gamma}, \quad \forall E \in \mathscr{G}_h. \tag{2.20}$$

That means $c_{h,i}^0$ is the L^2-projection of $c_{0,i}$ onto $M_{h,i}$, for $i = 1,2$, and $c_{h,\gamma}^0$ is the L^2-projection of $c_{0,\gamma}$ onto $M_{h,\gamma}$.

In the next section, we establish the existence, uniqueness and derive a priori error estimates for the solution of the upwind-mixed hybrid scheme (2.19).

3. Analysis of the upwind-mixed hybrid finite element method

For analysis purpose, we make use of the Raviart-Thomas projection operators $\Pi_{h,i} \times P_{h,i} : (H^1(\Omega_i))^2 \times L^2(\Omega_i) \to \boldsymbol{\Sigma}_{h,i} \times M_{h,i}, i = 1,2$, and $\Pi_{h,\gamma} \times P_{h,\gamma} : H^1(\gamma) \times L^2(\gamma) \to \boldsymbol{\Sigma}_{h,\gamma} \times M_{h,\gamma}$. The following properties hold for these operators [28, 60]:

(P1) $P_{h,1}, P_{h,2}$ and $P_{h,\gamma}$ are the L^2-orthogonal projections onto $M_{h,1}$, $M_{h,2}$ and $M_{h,\gamma}$, respectively.

(P2) For any $(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_\gamma) \in (H^1(\Omega_1))^2 \times (H^1(\Omega_2))^2 \times H^1(\gamma)$ and $(\mu_1,\mu_2,\mu_\gamma) \in L^2(\Omega_1) \times L^2(\Omega_2) \times L^2(\gamma)$,

$$\begin{aligned} & \big(\operatorname{div}(\boldsymbol{v}_i - \Pi_{h,i}\boldsymbol{v}_i), w_{h,i}\big)_{\Omega_i} = 0, \ \forall\, w_{h,i} \in M_{h,i}, \quad \big(\operatorname{div}_\tau(\boldsymbol{v}_\gamma - \Pi_{h,\gamma}\boldsymbol{v}_\gamma), w_{h,\gamma}\big)_\gamma = 0, \ \forall\, w_{h,\gamma} \in M_{h,\gamma}, \\ & \big(\operatorname{div}\, \boldsymbol{v}_{h,i}, P_{h,i}\mu_i - \mu_i\big)_{\Omega_i} = 0, \ \forall\, \boldsymbol{v}_{h,i} \in \boldsymbol{\Sigma}_{h,i}, \quad \big(\operatorname{div}_\tau \boldsymbol{v}_{h,\gamma}, P_{h,\gamma}\mu_\gamma - \mu_\gamma\big)_\gamma = 0, \ \forall\, \boldsymbol{v}_{h,\gamma} \in \boldsymbol{\Sigma}_{h,\gamma}. \end{aligned} \tag{3.1}$$

(P3) The following approximation properties hold:

$$\begin{aligned}&\left\|\boldsymbol{v}_i-\Pi_{h,i}\boldsymbol{v}_i\right\|_{0,\Omega_i}\leq Ch\left\|\boldsymbol{v}_i\right\|_{1,\Omega_i},\ \forall\boldsymbol{v}_i\in(H^1(\Omega_i))^2,\ \left\|\boldsymbol{v}_\gamma-\Pi_{h,\gamma}\boldsymbol{v}_\gamma\right\|_{0,\gamma}\leq Ch\left\|\boldsymbol{v}_\gamma\right\|_{1,\gamma},\ \forall\boldsymbol{v}_i\in H^1(\gamma),\\&\left\|\mu_i-P_{h,i}\mu_i\right\|_{0,\Omega_i}\leq Ch\left\|\mu_i\right\|_{1,\Omega_i},\ \forall\mu_i\in H^1(\Omega_i),\ \left\|\mu_\gamma-P_{h,\gamma}\mu_\gamma\right\|_{0,\gamma}\leq Ch\left\|\mu_\gamma\right\|_{1,\gamma},\ \forall\mu_\gamma\in H^1(\gamma).\end{aligned}\tag{3.2}$$

(P4) For sufficiently smooth $\boldsymbol{v}_i\in(H^1(\Omega_i))^2$, we also have [28]

$$\left\|(\boldsymbol{v}_i-\Pi_{h,i}\boldsymbol{v}_i)\cdot\boldsymbol{n}_i\right\|_{0,\gamma}\leq Ch\left\|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\right\|_{1,\gamma}, i=1,2.\tag{3.3}$$

Finally, we define the following norms for any function g in $\mathscr{H}^k$ or $\boldsymbol{\mathscr{H}}^k$: $\|g\|_k^2:=\|g_1\|_{k,\Omega_1}^2+\|g_2\|_{k,\Omega_2}^2+\left\|g_\gamma\right\|_{k,\gamma}^2$, $k=0,1$.

3.1. *Well-posedness analysis*

Theorem 1 *For every $n\in\{1,\ldots,N\}$ and sufficiently small Δt and h, problem (2.19) has a unique solution.*

Proof Since problem (2.19) is linear, it suffices to show its uniqueness. For this purpose, we consider the corresponding homogeneous system:

$$\begin{aligned}&a(\boldsymbol{\varphi}_h^n,\boldsymbol{v}_h)-b(\boldsymbol{v}_h,c_h^n)-\widetilde{d}_{\boldsymbol{u}_h}((c_h^n,\theta_h^n),\boldsymbol{v}_h)+e(\boldsymbol{v}_h,c_h^n)+l(\theta_h^n,\boldsymbol{v}_h)=0,\\&r_\phi(\bar{\partial}c_h^n,\mu_h)+b(\boldsymbol{\varphi}_h^n,\mu_h)-e(\boldsymbol{\varphi}_h^n,\mu_h)=0, \qquad \forall(\mu_h,\boldsymbol{v}_h,\eta_h)\in M_h\times\widetilde{\boldsymbol{\Sigma}}_h\times\Theta_h^0,\\&l(\eta_h,\boldsymbol{\varphi}_h^n)=0,\end{aligned}\tag{3.4}$$

for $n=1,2,\ldots,N$, given that the initial condition $\left(c_h^{n-1},\boldsymbol{\varphi}_h^{n-1},\theta_h^{n-1}\right)$ is zero. We show that the only solution $\left(c_h^n,\ \boldsymbol{\varphi}_h^n,\theta_h^n\right)$ to (3.4) is zero. Let $\mu_h=c_h^n$, $\boldsymbol{v}_h=\Delta t\boldsymbol{\varphi}_h^n$ and $\eta_h=(\eta_{h,1},\eta_{h,2},\eta_{h,\gamma})$ in (3.4) where, for $i=1,2$,

$$\left(\eta_{h,i}\right)_{|E}=\begin{cases}-\Delta t\theta_{i,E}^n, & \text{on } E\in\mathscr{E}_{h,i}^I\\0, & \text{otherwise,}\end{cases},\quad\left(\eta_{h,\gamma}\right)_{|P}=\begin{cases}-\Delta t\theta_{\gamma,P}^n, & \text{at interior point } P,\\0, & \text{otherwise ,}\end{cases}$$

and adding the resulting equations, we obtain

$$r_\phi(c_h^n,c_h^n)+\Delta ta(\boldsymbol{\varphi}_h^n,\boldsymbol{\varphi}_h^n)=\Delta t\widetilde{d}_{\boldsymbol{u}_h}((c_h^n,\theta_h^n),\boldsymbol{\varphi}_h^n).\tag{3.5}$$

We next provide an estimate for the error $\left\|\theta_h^n\right\|_{0,E}$ on each edge $E\in\mathscr{E}_{h,i}$, $i=1,2$ by utilizing the technique in [8]. Fix $i\in\{1,2\}$, for $K\in\mathscr{K}_{h,i}$, $E\subset\partial K$, let τ_E denote the unique element of $\widetilde{\boldsymbol{\Sigma}}_{h,i}$ with $\mathrm{supp}(\tau_E)\subseteq K$ and

$$\tau_E\cdot\boldsymbol{n}_{E'}=\begin{cases}\theta_{i,E}^n, & \text{on } E=E',\\0, & \text{otherwise.}\end{cases}$$

Then, it follows from a scaling argument [8] that

$$h_K\left\|\tau_E\right\|_{1,K}+\left\|\tau_E\right\|_{0,K}\leq Ch_K^{1/2}\left\|\theta_{h,i}^n\right\|_{0,E}.\tag{3.6}$$

By using $\boldsymbol{v}_h=(v_{h,1},v_{h,2},0)$ where $v_{h,i}=\tau_E$, $v_{h,j}=0,\ j=3-i$ as a test function in the first equation of (3.4), utilizing (3.6) and the weighted Cauchy-Schwarz inequality, we obtain

$$\left\|\theta_{h,i}\right\|_{0,E}\leq C\left(h_K^{1/2}\left\|\boldsymbol{\varphi}_{h,i}^n\right\|_{0,K}+h_K^{-1/2}\left\|c_{h,i}^n\right\|_{0,K}+h_K\sum_{E'\in\mathscr{E}(K)}\left\|\theta_{h,i}\right\|_{0,E'}\right).\tag{3.7}$$

Summing this estimate over all edges of K and pushing back the last term for h sufficiently small yields

$$\|\theta_{h,i}\|_{0,E} \le C\left(h_K^{1/2}\left\|\boldsymbol{\varphi}_{h,i}^n\right\|_{0,K} + h_K^{-1/2}\left\|c_{h,i}^n\right\|_{0,K}\right). \tag{3.8}$$

Similarly, for the Lagrange multipliers on the fracture, we have

$$\left|(\theta_{h,\gamma})_{|P}\right| \le C\left(h_E^{1/2}\left\|\boldsymbol{\varphi}_{h,\gamma}^n\right\|_{0,E} + h_E^{-1/2}\left\|c_{h,\gamma}^n\right\|_{0,E}\right). \tag{3.9}$$

By using (3.8) and Young's inequality, we have, for $i = 1,2$,

$$\begin{aligned}
&\Delta t \sum_{K\in\mathscr{K}_{h,i}}\left(\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}} u_{i,KE}\mathscr{U}_{i,KE}(c_{i,K}^n,\theta_{i,K}^n)(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{\varphi}_{h,i}^n)_K + \sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_h^\gamma}} u_{i,KE}\mathscr{U}_{i,KE}^\gamma(c_{i,K}^n,c_{\gamma,E}^n)(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{\varphi}_{h,i}^n)_K\right)\\
&\le C_i\Delta t\sum_{K\in\mathscr{K}_{h,i}}\left\|c_{h,i}^n\right\|_{0,K}\left\|\boldsymbol{\varphi}_{h,i}^n\right\|_{0,K} + C_i\Delta t\sum_{K\in\mathscr{K}_{h,i}}\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_h^\gamma}}\left\|c_{h,\gamma}^n\right\|_{0,E}\left\|\boldsymbol{\varphi}_{h,i}^n\right\|_{0,K} + C_i\Delta t\sum_{K\in\mathscr{K}_{h,i}} h_K\left\|\boldsymbol{\varphi}_{h,i}^n\right\|_{0,K}^2\\
&\le C_i\varepsilon\left\|c_{h,i}^n\right\|_{0,\Omega_i}^2 + C_i\varepsilon\left\|c_{h,\gamma}^n\right\|_{0,\gamma}^2 + C_i\frac{\Delta t^2}{4\varepsilon}\left\|\boldsymbol{\varphi}_{h,i}^n\right\|_{0,\Omega_i}^2 + C_i\Delta t h_i\left\|\boldsymbol{\varphi}_{h,i}^n\right\|_{0,\Omega_i}^2.
\end{aligned} \tag{3.10}$$

Similarly, from (3.9) and Young's inequality, we have

$$\Delta t\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E} u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^n,\theta_{\gamma,P}^n)(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_{\gamma,EP},\boldsymbol{\varphi}_{h,\gamma}^n)_\gamma \le C_\gamma\varepsilon\left\|c_{h,\gamma}^n\right\|_{0,\gamma}^2 + C_\gamma\frac{\Delta t^2}{4\varepsilon}\left\|\boldsymbol{\varphi}_{h,\gamma}^n\right\|_{0,\gamma}^2 + C_\gamma\Delta t h_\gamma\left\|\boldsymbol{\varphi}_{h,\gamma}^n\right\|_{0,\gamma}^2. \tag{3.11}$$

Altogether, we combine (3.5), (3.10), and (3.11) to find

$$r_\phi(c_h^n,c_h^n) + \Delta t a(\boldsymbol{\varphi}_h^n,\boldsymbol{\varphi}_h^n) \le C\varepsilon\left\|c_h^n\right\|_0^2 + \left(\frac{C(\Delta t)^2}{4\varepsilon} + C\Delta t h\right)\left\|\boldsymbol{\varphi}_h^n\right\|_0^2, \tag{3.12}$$

where $C = \max\{C_1, C_2, C_\gamma\}$. To give a lower bound for the left-hand side of (3.12), we use the assumptions (A1) - (A2) to find

$$r_\phi(c_h^n,c_h^n) \ge \phi^-\left\|c_h^n\right\|_0^2, \quad a(\boldsymbol{\varphi}_h^n,\boldsymbol{\varphi}_h^n) \ge D_{\min}^-\left\|\boldsymbol{\varphi}_h^n\right\|_0^2, \tag{3.13}$$

where $D_{\min}^- = \min\{D^-,\ D_\gamma^-\}$. By substituting (3.13) into (3.12),we have

$$\phi^-\|c_h^n\|_0^2 + D_{\min}^-\Delta t\|\boldsymbol{\varphi}_h^n\|_0^2 \le C\varepsilon\|c_h^n\|_0^2 + \left(\frac{C(\Delta t)^2}{4\varepsilon} + C\Delta t h\right)\|\boldsymbol{\varphi}_h^n\|_0^2. \tag{3.14}$$

By taking Δt, h and ε in (3.14) sufficiently small such that $\phi^- - C\varepsilon > 0,\ D_{\min}^- - Ch - C\dfrac{\Delta t}{4\varepsilon} > 0$, we have c_h^n, $\boldsymbol{\varphi}_h^n$ vanish. Then θ_h^n vanishes according to (3.8)-(3.9). $\square$

3.2. *A priori error estimates*

We first state some preliminary lemmas: Lemma 1 is a direct consequence of the Bochner's inequality [27], Lemma 2 is a discrete version of integration by parts, Lemma 3 demonstrates the discrete Gronwall's inequality, and Lemma 4 is a generalization of [19, Lemma 4.2] to the reduced fracture model of transport problems. We remark that the key result needed for our proof of error estimates is Lemma 4 to control the advective terms. The proof of Lemma 4 relies on [8, Lemma 2.1] and can be found in Appendix A.

Lemma 1 *Let X be any Banach space with norm $\|\cdot\|_X$ and let $f:[0,T]\to X$ be a measurable mapping such that the mapping $t\mapsto \|f(t)\|_X$ is also measurable. Then, we have*

$$\left\|\int_0^T f(s)ds\right\|_X \leq \int_0^T \|f(s)\|_X\, ds. \tag{3.15}$$

Lemma 2 *[19, Lemma 4.3] Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be real sequences. Then, for any $m\in\mathbb{N}_0$,*

$$\sum_{n=1}^{m}(a_n-a_{n-1})b_n = a_m b_m - a_0 b_0 - \sum_{n=1}^{m} a_{n-1}(b_n-b_{n-1}).$$

Lemma 3 *[58] Let $\tau>0, B\geq 0$, and let $a_m, b_m, c_m, d_m, m\geq 0$, be non-negative sequences such that $a_0\leq B$ and*

$$a_m+\tau\sum_{l=1}^{m} b_l \leq \tau\sum_{l=1}^{m-1} d_l a_l + \tau\sum_{l=1}^{m} c_l + B,\ m\geq 1.$$

Then

$$a_m+\tau\sum_{l=1}^{m} b_l \leq exp\left(\tau\sum_{l=1}^{m-1} d_l\right)\left(\tau\sum_{l=1}^{m} c_l + B\right),\ m\geq 1.$$

Lemma 4 *Assume that the solution $(c,\boldsymbol{\varphi})$ of (2.6)-(2.7) satisfies (A4). Let $(c_h^n,\boldsymbol{\varphi}_h^n,\theta_h^n)\in M_h\times\widetilde{\boldsymbol{\Sigma}}_h\times\Theta_h$ be the solution of (2.19). Then, for h sufficiently small, there exists a constant $C>0$, independent of n and h, such that*

$$\begin{aligned}\sum_{i=1}^{2}\sum_{K\in\mathscr{K}_{h,i}}\left(\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}|E|^2(\theta_{i,E}^n-c_{i,K}^n)^2+\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_h^\gamma}}|E|^2(c_{\gamma,E}^n-c_{i,K}^n)^2\right)+\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E}(\theta_{\gamma,P}^n-c_{\gamma,E}^n)^2\\ \leq C\left(\left\|c^n-c_h^n\right\|_0^2+\left\|\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n\right\|_0^2+h^2\left\|\boldsymbol{u}c^n-\boldsymbol{u}_h c_h^n\right\|_0^2+h^2\left\|c^n\right\|_1^2\right).\end{aligned} \tag{3.16}$$

We now state the first-order convergence in both space and time of the upwind-mixed hybrid algorithm (2.19). Unlike [19], here the reduced fracture model consists of an extra term representing the total normal flux across the fracture which may cause the loss in spatial accuracy if it is not treated carefully. In the following analysis, we eliminate that total normal flux term in the formulation by employing the orthogonality property of the L^2-projection $P_{h,\gamma}$ since with conforming spatial discretization, the traces on the fracture of the discrete normal fluxes belong to $M_{h,\gamma}$, the same space as the scalar variable in the fracture.

Theorem 2 *Assume that Δt and h are sufficiently small and the solution of problem (2.6)-(2.7) satisfies (A4). Let $(c_h^n,\boldsymbol{\varphi}_h^n,\theta_h^n)$ be the solution of problem (2.19), then there exists a constant $C>0$ independent of Δt and h, such that*

$$
\begin{aligned}
&\max_{n=1,\ldots,N} \left\|c(t^n)-c_h^n\right\|_0^2+\Delta t \sum_{n=1}^{N} \left\|\boldsymbol{\varphi}(t^n)-\boldsymbol{\varphi}_h^n\right\|_0^2 \\
&\quad \le C\left(\|\partial_{tt}c\|^2_{L^2(0,T;\mathscr{H}^0)}\Delta t^2+\|c\|^2_{L^\infty(0,T;\mathscr{H}^0)}h^2+\|c\|^2_{L^\infty(0,T;\mathscr{H}^1)}h^2+\|\partial_t c\|^2_{L^2(0,T;\mathscr{H}^1)}h^2\right. \\
&\qquad \left.+\|\boldsymbol{\varphi}\|^2_{L^\infty(0,T;\mathscr{H}^1)}h^2+\sum_{i=1}^{2}\|\boldsymbol{\varphi}_i\cdot\boldsymbol{n}_i\|^2_{L^\infty(0,T;H^1(\gamma))}h^2\right).
\end{aligned}
\tag{3.17}
$$

Proof We first take $v_h \in \boldsymbol{\Sigma}_h$ in (2.19) and use the continuity of concentration across the interface to obtain

$$
\begin{aligned}
&a(\boldsymbol{\varphi}_h^n,\boldsymbol{v}_h)-b(\boldsymbol{v}_h,c_h^n)+e(\boldsymbol{v}_h,c_h^n)-\widetilde{d}_{\boldsymbol{u}_h}((c_h^n,\theta_h^n),\boldsymbol{v}_h)=0,\\
&r_\phi(\bar\partial c_h^n,\mu_h)+b(\boldsymbol{\varphi}_h^n,\mu_h)-e(\boldsymbol{\varphi}_h^n,\mu_h)=L_{q^n}(\mu_h),
\end{aligned}
\qquad \forall(\mu_h,\boldsymbol{v}_h)\in M_h\times\boldsymbol{\Sigma}_h.
\tag{3.18}
$$

In (2.6), we take $t=t^n$ and $\boldsymbol{v}=\boldsymbol{v}_h\in\boldsymbol{\Sigma}_h$ and substract (3.18) from the resulting equations to have the following error equations

$$
\begin{aligned}
&a(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\boldsymbol{v}_h)-b(\boldsymbol{v}_h,c^n-c_h^n)+e(\boldsymbol{v}_h,c^n-c_h^n)-d_{\boldsymbol{u}}(c^n,\boldsymbol{v}_h)+\widetilde{d}_{\boldsymbol{u}_h}((c_h^n,\theta_h^n),\boldsymbol{v}_h)=0,\\
&r_\phi(\partial_t c^n-\bar\partial c_h^n,\mu_h)+b(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\mu_h)-e(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\mu_h)=0,
\end{aligned}
\quad \forall(\mu_h,\boldsymbol{v}_h)\in M_h\times\boldsymbol{\Sigma}_h.
\tag{3.19}
$$

Let $\boldsymbol{v}_h=\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n, \mu_h=P_hc^n-c_h^n$ in (3.19), where $\Pi_h\boldsymbol{\varphi}=(\Pi_{h,1}\boldsymbol{\varphi}_1,\Pi_{h,2}\boldsymbol{\varphi}_2,\Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma)$ and $P_hc=(P_{h,1}c_1,P_{h,2}c_2,P_{h,\gamma}c_\gamma)$, we have

$$
\begin{aligned}
&a(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)-b(\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,c^n-c_h^n)+e(\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,c^n-c_h^n)\\
&\qquad\qquad -d_{\boldsymbol{u}}(c^n,\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)+\widetilde{d}_{\boldsymbol{u}_h}((c_h^n,\theta_h^n),\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)=0,\\
&r_\phi(\partial_t c^n-\bar\partial c_h^n,P_hc^n-c_h^n)+b(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,P_hc^n-c_h^n)-e(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,P_hc^n-c_h^n)=0.
\end{aligned}
\tag{3.20}
$$

By adding both equations in (3.20) and using property (3.1) of the Raviart-Thomas projection operators, we find that

$$
\begin{aligned}
&a(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)+r_\phi(\partial_t c^n-\bar\partial c_h^n,P_hc^n-c_h^n)\\
&=d_{\boldsymbol{u}}(c^n,\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)-\widetilde{d}_{\boldsymbol{u}_h}((c_h^n,\theta_h^n),\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)-e(\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,P_hc^n-c^n)+e(\boldsymbol{\varphi}^n-\Pi_h\boldsymbol{\varphi}^n,P_hc^n-c_h^n).
\end{aligned}
\tag{3.21}
$$

Equivalently, (3.21) can be rewritten as

$$
\begin{aligned}
&a(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)+r_\phi(\bar\partial(c^n-c_h^n),c^n-c_h^n)\\
&\quad=-a(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}^n)-r_\phi(\partial_t c^n-\bar\partial c^n,P_hc^n-c^n)-r_\phi(\partial_t c^n-\bar\partial c^n,c^n-c_h^n)\\
&\qquad -r_\phi(\bar\partial(c^n-c_h^n),P_hc^n-c^n)+d_{\boldsymbol{u}}(c^n,\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)-d_{\boldsymbol{u}_h}(c_h^n,\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)+d_{\boldsymbol{u}_h}(c_h^n,\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)\\
&\qquad -\widetilde{d}_{\boldsymbol{u}_h}((c_h^n,\theta_h^n),\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)+e(\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,P_hc^n-c^n)+e(\boldsymbol{\varphi}^n-\Pi_h\boldsymbol{\varphi}^n,P_hc^n-c_h^n).
\end{aligned}
\tag{3.22}
$$

Fix any $1\le m\le N$, by summing both sides of (3.22) from $n=1,\ldots,m$, and multiplying by $2\Delta t$, we have

$$
2\Delta t\sum_{n=1}^{m}a(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)+2\Delta t\sum_{n=1}^{m}c_\phi(\bar\partial(c^n-c_h^n),c^n-c_h^n)=T_1+T_2+\ldots T_7,
\tag{3.23}
$$

where

$$T_1 = -2\Delta t \sum_{n=1}^{m} a(\boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n, \Pi_h \boldsymbol{\varphi}^n - \boldsymbol{\varphi}^n), \quad T_2 = -2\Delta t \sum_{n=1}^{m} r_\phi(\partial_t c^n - \bar{\partial} c^n, P_h c^n - c^n),$$

$$T_3 = -2\Delta t \sum_{n=1}^{m} r_\phi(\partial_t c^n - \bar{\partial} c^n, c^n - c_h^n), \quad T_4 = -2\Delta t \sum_{n=1}^{m} r_\phi(\bar{\partial}(c^n - c_h^n), P_h c^n - c^n),$$

$$T_5 = 2\Delta t \sum_{n=1}^{m} d_{\boldsymbol{u}}(c^n, \Pi_h \boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n) - 2\Delta t \sum_{n=1}^{m} d_{\boldsymbol{u}_h}(c_h^n, \Pi_h \boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n),$$

$$T_6 = 2\Delta t \sum_{n=1}^{m} d_{\boldsymbol{u}_h}(c_h^n, \Pi_h \boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n) - 2\Delta t \sum_{n=1}^{m} \widetilde{d}_{\boldsymbol{u}_h}((c_h^n, \theta_h^n), \Pi_h \boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n),$$

$$T_7 = 2\Delta t \sum_{n=1}^{m} e(\Pi_h \boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n, P_h c^n - c^n) + 2\Delta t \sum_{n=1}^{m} e(\boldsymbol{\varphi}^n - \Pi_h \boldsymbol{\varphi}^n, P_h c^n - c_h^n).$$

Our next step is to give an upper bound for each term $T_i, 1 \le i \le 7$. By using the Cauchy-Schwarz inequality and Young's inequality, we first obtain

$$\begin{aligned}
|T_1| &\le 2\Delta t \sum_{i=1}^{2}\sum_{n=1}^{m} \left| \left(\boldsymbol{D}_i^{-1}(\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_{h,i}^n), \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right)_{\Omega_i} \right| + 2\Delta t \sum_{n=1}^{m} \left| \left(\boldsymbol{D}_\gamma^{-1}(\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_{h,\gamma}^n), \Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_\gamma^n \right)_\gamma \right| \\
&\le 2\Delta t \sum_{i=1}^{2}\sum_{n=1}^{m} \left\| \boldsymbol{D}_i^{-1}\left(\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_{h,i}^n \right) \right\|_{0,\Omega_i} \left\| \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right\|_{0,\Omega_i} + 2\Delta t \sum_{n=1}^{m} \left\| \boldsymbol{D}_\gamma^{-1}\left(\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_{h,\gamma}^n \right) \right\|_{0,\gamma} \left\| \Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_\gamma^n \right\|_{0,\gamma} \\
&\le D^+ \Delta t \varepsilon \sum_{n=1}^{m} \left\| \boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_{h,i}^n \right\|_{0,\Omega_i}^2 + \frac{D^+\Delta t}{\varepsilon} \sum_{n=1}^{m} \left\| \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right\|_{0,\Omega_i}^2 + D_\gamma^+ \Delta t \varepsilon \sum_{n=1}^{m} \left\| \boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_{h,\gamma}^n \right\|_{0,\gamma}^2 + \frac{D_\gamma^+\Delta t}{\varepsilon} \sum_{n=1}^{m} \left\| \Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_\gamma^n \right\|_{0,\gamma}^2 \\
&\le C_1 \Delta t \varepsilon \sum_{n=1}^{m} \left\| \boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n \right\|_0 + \frac{C_1 \Delta t}{\varepsilon} \sum_{n=1}^{m} \left\| \Pi_h \boldsymbol{\varphi}^n - \boldsymbol{\varphi}^n \right\|_0^2.
\end{aligned} \tag{3.24}$$

where $C_1 = \max\{D^+, D_\gamma^+\}$. Similarly, it follows from the Cauchy-Schwarz inequality, Bochner's inequality (3.15) and the regularity of c that

$$\begin{aligned}
|T_2| &\le \Delta t \sum_{n=1}^{m} \left\| \partial_t c^n - \frac{c^n - c^{n-1}}{\Delta t} \right\|_0^2 + \Delta t \sum_{n=1}^{m} \|P_h c^n - c^n\|_0^2 \le \|\partial_{tt} c\|_{L^2(0,T;\mathscr{H}^0)}^2 (\Delta t)^2 + \Delta t \sum_{n=1}^{m} \|P_h c^n - c^n\|_0^2, \\
|T_3| &\le \Delta t \sum_{n=1}^{m} \left\| \partial_t c^n - \frac{c^n - c^{n-1}}{\Delta t} \right\|_0^2 + \Delta t \sum_{n=1}^{m} \|c^n - c^n\|_0^2 \le \|\partial_{tt} c\|_{L^2(0,T;\mathscr{H}^0)}^2 (\Delta t)^2 + \Delta t \sum_{n=1}^{m} \left\| c^n - c_h^n \right\|_0^2.
\end{aligned} \tag{3.25}$$

To obtain an upper bound for the term T_4, we first use Lemma 2 and get

$$\begin{aligned}
T_4 &= 2 \sum_{n=1}^{m} c_\phi \left((c^n - c_h^n) - (c^{n-1} - c_h^{n-1}), P_h c^n - c^n \right) \\
&= 2c_\phi \left(c^m - c_h^m, P_h c^m - c^m \right) - 2c_\phi \left(c^0 - c_h^0, P_h c^0 - c^0 \right) - 2 \sum_{n=1}^{m} c_\phi \left(c^{n-1} - c_h^{n-1}, (P_h - I)(c^n - c^{n-1}) \right).
\end{aligned} \tag{3.26}$$

We then apply the Cauchy-Schwarz inequality, Bochner's inequality (3.15) and the regularity of c on the right-hand side of (3.26):

$$\begin{aligned}
|T_4| &\le \varepsilon \left\| c^m - c_h^m \right\|_0^2 + \frac{1}{\varepsilon} \|P_h c^m - c^m\|_0^2 + C_4 \left\| P_h c^0 - c^0 \right\|_0^2 + \Delta t \sum_{n=1}^{m-1} \left\| c^n - c_h^n \right\|_0^2 + \frac{1}{\Delta t} \sum_{n=1}^{m} \left\| (P_h - I)(c^n - c^{n-1}) \right\|_0^2 \\
&\le \varepsilon \left\| c^m - c_h^m \right\|_0^2 + \frac{h^2}{\varepsilon} \|c^m\|_1^2 + \Delta t \sum_{n=1}^{m-1} \left\| c^n - c_h^n \right\|_0^2 + C_4 \left\| c^0 \right\|_1^2 h^2 + \|\partial_t c\|_{L^2(0,T;\mathscr{H}^1)}^2 h^2.
\end{aligned} \tag{3.27}$$

For T_5, we decompose it into three subterms $T_{5,1}, T_{5,2}$ and $T_{5,\gamma}$, where

$$T_{5,i} = 2\Delta t \sum_{n=1}^{m} \left(\boldsymbol{D}_i^{-1}(\boldsymbol{u}_{h,i}c_{h,i}^n - \boldsymbol{u}_i c_i^n), \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right)_{\Omega_i} + 2\Delta t \sum_{n=1}^{m} \left(\boldsymbol{D}_i^{-1}(\boldsymbol{u}_{h,i}c_{h,i}^n - \boldsymbol{u}_i c_i^n), \boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_{h,i}^n \right)_{\Omega_i}, \ i = 1,2,$$
$$T_{5,\gamma} = 2\Delta t \sum_{n=1}^{m} \left(\boldsymbol{D}_\gamma^{-1}(\boldsymbol{u}_{h,\gamma}c_{h,\gamma}^n - \boldsymbol{u}_\gamma c_\gamma^n),\ \Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_\gamma^n \right)_{\gamma} + 2\Delta t \sum_{n=1}^{m} \left(\boldsymbol{D}_\gamma^{-1}(\boldsymbol{u}_{h,\gamma}c_{h,\gamma}^n - \boldsymbol{u}_\gamma c_\gamma^n),\ \boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_{h,\gamma}^n \right)_{\gamma}.$$

By using the Cauchy-Schwarz inequality, Young's inequality and the $L^\infty-$approximation properties of $\Pi_{h,i}, i = 1,2$ [25], we have

$$\begin{aligned}
2\Delta t \sum_{n=1}^{m} &\left| \left(\boldsymbol{D}_i^{-1}(\boldsymbol{u}_{h,i}c_{h,i}^n - \boldsymbol{u}_i c_i^n), \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right)_{\Omega_i} \right| \le 2\Delta t \sum_{n=1}^{m} \left\| \boldsymbol{D}_i^{-1}(\boldsymbol{u}_{h,i}c_{h,i}^n - \boldsymbol{u}_i c_i^n) \right\|_{0,\Omega_i} \left\| \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right\|_{0,\Omega_i} \\
&\le 2\Delta t \sum_{n=1}^{m} \left(\left\| \boldsymbol{D}_i^{-1}\boldsymbol{u}_{h,i}(c_{h,i}^n - c_i^n) \right\|_{0,\Omega_i} \left\| \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right\|_{0,\Omega_i} + \left\| \boldsymbol{D}_i^{-1}(\boldsymbol{u}_{h,i} - \boldsymbol{u}_i)c_i^n \right\|_{0,\Omega_i} \left\| \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right\|_{0,\Omega_i} \right) \\
&\le C_5\Delta t \sum_{n=1}^{m} \left\| c_{h,i}^n - c_i^n \right\|_{0,\Omega_i}^2 + C_5\Delta t h^2 \sum_{n=1}^{m} \|c_i^n\|_{0,\Omega_i}^2 + C_5\Delta t \sum_{n=1}^{m} \left\| \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n \right\|_{0,\Omega_i}^2 .
\end{aligned} \tag{3.28}$$

Similarly,

$$\begin{aligned}
2\Delta t \sum_{n=1}^{m} &\left| \left(\boldsymbol{D}_\gamma^{-1}(\boldsymbol{u}_{h,\gamma}c_{h,\gamma}^n - \boldsymbol{u}_\gamma c_\gamma^n),\ \Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_\gamma^n \right)_{\gamma} \right| \\
&\le C_5\Delta t \sum_{n=1}^{m} \left\| c_{h,\gamma}^n - c_\gamma^n \right\|_{0,\gamma}^2 + C_5\Delta t h^2 \sum_{n=1}^{m} \|c_\gamma^n\|_{0,\gamma}^2 + C_5\Delta t \sum_{n=1}^{m} \left\| \Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_\gamma^n \right\|_{0,\gamma}^2 .
\end{aligned} \tag{3.29}$$

By repeating the steps in (3.28) and (3.29) for the second terms of $T_{5,i},\ i = 1,2,\gamma$, we obtain the following upper bound for the term T_5:

$$\begin{aligned}
|T_5| \le C_5\Delta t \sum_{n=1}^{m} \left\| c^n - c_h^n \right\|_0^2 + C_5\Delta t h^2 \sum_{n=1}^{m} \|c^n\|_0^2 + C_5\Delta t \sum_{n=1}^{m} \|\Pi_h\boldsymbol{\varphi}^n - \boldsymbol{\varphi}^n\|_0^2 \\
+ \frac{C_5\Delta t}{\varepsilon} \sum_{n=1}^{m} \left\| c^n - c_h^n \right\|_0^2 + \frac{C_5\Delta t h^2}{\varepsilon} \sum_{n=1}^{m} \|c^n\|_0^2 + C_5\varepsilon\Delta t \sum_{n=1}^{m} \left\| \boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n \right\|_0^2 .
\end{aligned} \tag{3.30}$$

To handle T_6, we first perform the decomposition $T_6 = T_{6,1} + T_{6,2} + T_{6,\gamma}$, where

$$\begin{aligned}
T_{6,i} = 2\Delta t \sum_{n=1}^{m} \sum_{K\in\mathscr{K}_{h,i}} &\left(\sum_{\substack{E\subset\partial K \\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}} u_{i,KE} \left(\mathscr{U}_{i,KE}(c_{i,K}^n, \theta_{i,E}^n) - c_{i,K}^n \right) \left(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\ \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_{h,i}^n \right)_K \right. \\
&\left. + \sum_{\substack{E\subset\partial K \\ E\in\mathscr{E}_h^\gamma}} u_{i,KE} \left(\mathscr{U}_{i,KE}^\gamma(c_{i,K}^n, c_{\gamma,E}^n) - c_{i,K}^n \right) \left(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\ \Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_{h,i}^n \right)_K \right), \ i = 1,2, \\
T_{6,\gamma} = 2\Delta t \sum_{n=1}^{m} \sum_{E\in\mathscr{E}_h^\gamma} \sum_{P\in\partial E} &u_{\gamma,EP} \left(\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^n, \theta_{\gamma,P}^n) - c_{\gamma,E}^n \right) \left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_{\gamma,EP},\ \Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma^n - \boldsymbol{\varphi}_{h,\gamma}^n \right)_\gamma .
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, Young's inequality, and Lemma 4 yields

$$
\begin{aligned}
|T_6| \leq{}& \frac{C_6\Delta t}{\varepsilon}\sum_{n=1}^{m}\sum_{i=1}^{2}\sum_{K\in\mathscr{K}_{h,i}}\left(\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}|E|^2(\theta_{i,E}^n-c_{i,K}^n)^2+\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_h^\gamma}}|E|^2(c_{\gamma,E}^n-c_{i,K}^n)^2\right)\\
&+C_6\Delta t\varepsilon\sum_{n=1}^{m}\sum_{i=1}^{2}\left\|\Pi_{h,i}\boldsymbol{\varphi}_i^n-\boldsymbol{\varphi}_{h,i}^n\right\|_{0,\Omega_i}^2+\frac{C_6\Delta t}{\varepsilon}\sum_{n=1}^{m}\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E}(\theta_{\gamma,P}^n-c_{\gamma,E}^n)^2+C_6\Delta t\varepsilon\sum_{n=1}^{m}\left\|\Pi_{h,\gamma}\boldsymbol{\varphi}_\gamma^n-\boldsymbol{\varphi}_{h,\gamma}^n\right\|_{0,\gamma}^2\\
\leq{}& \frac{C_6h^2\Delta t}{\varepsilon}\sum_{n=1}^{m}\|c^n\|_1^2+\frac{C_6h^4\Delta t}{\varepsilon}\sum_{n=1}^{m}\|c^n\|_0^2+C_6\varepsilon\Delta t\sum_{n=1}^{m}\|\Pi_h\boldsymbol{\varphi}^n-\boldsymbol{\varphi}^n\|_0^2\\
&+\frac{C_6\Delta t}{\varepsilon}\left(1+h^2\right)\sum_{n=1}^{m}\left\|c^n-c_h^n\right\|_0^2+C_6\left(\frac{h^2}{\varepsilon}+\varepsilon\right)\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n\right\|_0^2.
\end{aligned}
\tag{3.31}
$$

By collecting the estimates (3.24)-(3.31) and plugging them in the right-hand side of (3.23), then using the L^2-approximation properties (3.2), we deduce that

$$
\begin{aligned}
&2\Delta t\sum_{n=1}^{m}a(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)+2\Delta t\sum_{n=1}^{m}c_\phi(\bar\partial(c^n-c_h^n),c^n-c_h^n)\\
&\quad\leq\left(C_1\varepsilon+C_5\varepsilon+C_6\left(\frac{h^2}{\varepsilon}+\varepsilon\right)\right)\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n\right\|_0^2+\varepsilon\left\|c^m-c_h^m\right\|_0^2\\
&\qquad+\left(2+C_5+\frac{C_5}{\varepsilon}+\frac{C_6}{\varepsilon}\left(1+h^2\right)\right)\Delta t\sum_{n=1}^{m}\left\|c^n-c_h^n\right\|+\frac{h^2}{\varepsilon}\|c^m\|_1^2+C_4h^2\left\|c^0\right\|_1^2\\
&\qquad+\left(C_5+\frac{C_5}{\varepsilon}+\frac{C_6h^2}{\varepsilon}\right)h^2\Delta t\sum_{n=1}^{m}\|c^n\|_0^2+\left(\frac{C_1}{\varepsilon}+C_5+C_6\varepsilon\right)h^2\Delta t\sum_{n=1}^{m}\|\boldsymbol{\varphi}^n\|_1^2\\
&\qquad+\left(\frac{C_6}{\varepsilon}+1\right)h^2\Delta t\sum_{n=1}^{m}\|c^n\|_1^2+\|\partial_t c\|_{L^2(0,T;\mathscr{H}^1)}^2h^2+\|\partial_{tt}c\|_{L^2(0,T;\mathscr{H}^0)}^2(\Delta t)^2+|T_7|.
\end{aligned}
\tag{3.32}
$$

For the left-hand side of (3.32), by using assumptions $(\mathrm{A1})-(\mathrm{A2})$ and Lemma 2, we have

$$
\begin{aligned}
&2\Delta t\sum_{n=1}^{m}a(\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n,\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n)+2\Delta t\sum_{n=1}^{m}c_\phi(\bar\partial(c^n-c_h^n),c^n-c_h^n)\\
&\qquad\geq 2D_{\min}^-\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n\right\|_0^2+\phi^-\left\|c^m-c_h^m\right\|_0^2-\phi^+\left\|c^0-c_h^0\right\|_0^2.
\end{aligned}
\tag{3.33}
$$

From (3.32)-(3.33) and (3.2), it is implied that

$$
\begin{aligned}
&2D_{\min}^-\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n\right\|_0^2+\phi^-\left\|c^m-c_h^m\right\|_0^2\\
&\quad\leq\left(C_1\varepsilon+C_5\varepsilon+C_6\left(\frac{h^2}{\varepsilon}+\varepsilon\right)\right)\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n-\boldsymbol{\varphi}_h^n\right\|_0^2+\varepsilon\left\|c^m-c_h^m\right\|_0^2\\
&\qquad+\left(2+C_5+\frac{C_5}{\varepsilon}+\frac{C_6}{\varepsilon}\left(1+h^2\right)\right)\Delta t\sum_{n=1}^{m}\left\|c^n-c_h^n\right\|_0^2+\frac{h^2}{\varepsilon}\|c^m\|_1^2+(C_4+\phi^+)h^2\left\|c^0\right\|_1^2\\
&\qquad+\left(C_5+\frac{C_5}{\varepsilon}+\frac{C_6h^2}{\varepsilon}\right)h^2\Delta t\sum_{n=1}^{m}\|c^n\|_0^2+\left(\frac{C_1}{\varepsilon}+C_5+C_6\varepsilon\right)h^2\Delta t\sum_{n=1}^{m}\|\boldsymbol{\varphi}^n\|_1^2\\
&\qquad+\left(\frac{C_6}{\varepsilon_6}+1\right)h^2\Delta t\sum_{n=1}^{m}\|c^n\|_1^2+\|\partial_t c\|_{L^2(0,T;\mathscr{H}^1)}^2h^2+\|\partial_{tt}c\|_{L^2(0,T;\mathscr{H}^0)}^2(\Delta t)^2+|T_7|.
\end{aligned}
\tag{3.34}
$$

For the last term T_7, we recall that

$$T_7 = 2\Delta t \sum_{n=1}^{m}\sum_{i=1}^{2}\left(\left(\Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_{h,i}^n\right)\cdot \boldsymbol{n}_{i|\gamma}, P_{h,\gamma}c_\gamma^n - c_\gamma^n\right)_\gamma - 2\Delta t \sum_{n=1}^{m}\sum_{i=1}^{2}\left(\left(\Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n\right)\cdot \boldsymbol{n}_{i|\gamma}, P_{h,\gamma}c_\gamma^n - c_{h,\gamma}^n\right)_\gamma . \tag{3.35}$$

It follows from [17, Proposition 3.2] that for any local lowest-order Raviart-Thomas mixed finite element space $\boldsymbol{\Sigma}_K$ in which K is a triangle, the traces on ∂K of the normal component of any function in $\boldsymbol{\Sigma}_K$ are constant. This result also holds when K is a rectangle [17]. Therefore, for any $K \in \mathscr{K}_{h,i}, i = 1,2$, we have $\boldsymbol{\varphi}_{h,i}^n \cdot \boldsymbol{n}_{i|E}$ is constant for any $E \subset \partial K$. Moreover, since the spatial mesh grids on the two subdomains are conforming and match on the fracture γ, we have $\boldsymbol{\varphi}_{h,i}^n \cdot \boldsymbol{n}_{i|E}$ is also constant for any $E \in \mathscr{E}_h^\gamma$. In other words, we have $\boldsymbol{\varphi}_{h,i}^n \cdot \boldsymbol{n}_{i|\gamma} \in M_{h,\gamma}$ for $i = 1,2$. Similarly, we have $\Pi_{h,i}\boldsymbol{\varphi}_i^n \cdot \boldsymbol{n}_{i|\gamma} \in M_{h,\gamma}$ for $i = 1,2$. Therefore, the first term in (3.35) vanishes due to the orthogonality property of the L^2-projection operator $P_{h,\gamma}$. By using the approximation property (3.3) and Young's inequality, we have

$$\begin{aligned}
|T_7| &\le \sum_{i=1}^{2}\Delta t \sum_{n=1}^{m}\left(\frac{1}{\varepsilon}\left\|\left(\Pi_{h,i}\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_i^n\right)\cdot \boldsymbol{n}_{i|\gamma}\right\|_{0,\gamma}^2 + \varepsilon\left\|P_{h,\gamma}c_\gamma^n - c_{h,\gamma}^n\right\|_{0,\gamma}^2\right)\\
&\le \frac{C_7h^2}{\varepsilon}\sum_{i=1}^{2}\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}_i^n\cdot\boldsymbol{n}_{i|\gamma}\right\|_{1,\gamma}^2 + \Delta t\varepsilon\sum_{n=1}^{m}\left\|P_{h,\gamma}c_\gamma^n - c_\gamma^n\right\|_{0,\gamma}^2 + \Delta t\varepsilon\sum_{n=1}^{m}\left\|c_\gamma^n - c_{h,\gamma}^n\right\|_{0,\gamma}^2\\
&\le \frac{C_7h^2}{\varepsilon}\sum_{i=1}^{2}\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}_i^n\cdot\boldsymbol{n}_{i|\gamma}\right\|_{1,\gamma}^2 + C_7\varepsilon h^2\sum_{n=1}^{m}\left\|c_h^n\right\|_{1,\gamma}^2 + \Delta t\varepsilon\sum_{n=1}^{m}\left\|c_\gamma^n - c_{h,\gamma}^n\right\|_{0,\gamma}^2.
\end{aligned} \tag{3.36}$$

From (3.34) and (3.36), we deduce that

$$\begin{aligned}
&2D_{\min}^-\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n\right\|_0^2 + \phi^-\left\|c^m - c_h^m\right\|_0^2\\
&\quad\le \left(C_1\varepsilon + C_5\varepsilon + C_6\left(\frac{h^2}{\varepsilon}+\varepsilon\right)\right)\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n\right\|_0^2 + \varepsilon\left\|c^m - c_h^m\right\|_0^2 + \frac{h^2}{\varepsilon}\left\|c^m\right\|_1^2 + (C_4+\phi^+)h^2\left\|c^0\right\|_1^2\\
&\qquad + \left(2 + C_5 + \frac{C_5}{\varepsilon} + \frac{C_6}{\varepsilon}\left(1+h^2\right)\right)\Delta t\sum_{n=1}^{m}\left\|c^n - c_h^n\right\|_0^2 + \left(C_5 + \frac{C_5}{\varepsilon} + \frac{C_6h^2}{\varepsilon}\right)h^2\Delta t\sum_{n=1}^{m}\left\|c^n\right\|_0^2\\
&\qquad + \left(\frac{C_1}{\varepsilon} + C_5 + C_6\varepsilon\right)h^2\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n\right\|_1^2 + \left(\frac{C_6}{\varepsilon} + 1 + C_7\varepsilon\right)h^2\Delta t\sum_{n=1}^{m}\left\|c^n\right\|_1^2\\
&\qquad + \left\|\partial_t c\right\|_{L^2(0,T;\mathscr{H}^1)}^2 h^2 + \left\|\partial_{tt} c\right\|_{L^2(0,T;\mathscr{H}^0)}^2(\Delta t)^2 + \frac{C_7h^2\Delta t}{\varepsilon}\sum_{i=1}^{2}\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}_i^n\cdot\boldsymbol{n}_{i|\gamma}\right\|_{1,\gamma}^2.
\end{aligned} \tag{3.37}$$

In (3.37), we fix ε small enough such that $\phi^- - \varepsilon > 0,\ 2-(C_1+C_5+C_6)\varepsilon > 0$, then choose Δt and h sufficiently small to find

$$\begin{aligned}
&\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n\right\|_0^2 + \left\|c^m - c_h^m\right\|_0^2\\
&\quad\le C\Delta t\sum_{n=1}^{m-1}\left\|c^n - c_h^n\right\|_0^2 + Ch^2\Delta t\sum_{n=1}^{m}\left\|c^n\right\|_0^2 + Ch^2\Delta t\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}^n\right\|_1^2 + Ch^2\Delta t\sum_{n=1}^{m}\left\|c^n\right\|_1^2 + Ch^2\left\|c^m\right\|_1^2\\
&\qquad + Ch^2\left\|c^0\right\|_1^2 + Ch^2\Delta t\sum_{i=1}^{2}\sum_{n=1}^{m}\left\|\boldsymbol{\varphi}_i^n\cdot\boldsymbol{n}_{i|\gamma}\right\|_{1,\gamma}^2 + \left\|\partial_t c\right\|_{L^2(0,T;\mathscr{H}^1)}^2 h^2 + \left\|\partial_{tt} c\right\|_{L^2(0,T;\mathscr{H}^0)}^2(\Delta t)^2\\
&\quad\le C\left(\Delta t\sum_{n=1}^{m-1}\left\|c^n - c_h^n\right\|_0^2 + \left\|c\right\|_{L^\infty(0,T;\mathscr{H}^0)}^2 h^2 + \left\|\boldsymbol{\varphi}\right\|_{L^\infty(0,T;\boldsymbol{\mathscr{H}}^1)}^2 h^2 + \left\|c\right\|_{L^\infty(0,T;\mathscr{H}^1)}^2 h^2\right.\\
&\qquad \left. + \sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_i\cdot\boldsymbol{n}_{i|\gamma}\right\|_{L^\infty(0,T;H^1(\gamma))h^2}^2 + \left\|\partial_t c\right\|_{L^2(0,T;\mathscr{H}^1)}^2 h^2 + \left\|\partial_{tt} c\right\|_{L^2(0,T;\mathscr{H}^0)}^2(\Delta t)^2\right).
\end{aligned} \tag{3.38}$$

Let $B = \|c\|^2_{L^\infty(0,T;\mathscr{H}^0)} h^2 + \|\boldsymbol{\varphi}\|^2_{L^\infty(0,T;\boldsymbol{\mathscr{H}}^1)} h^2 + \|c\|^2_{L^\infty(0,T;\mathscr{H}^1)} h^2 + \sum_{i=1}^{2} \left\|\boldsymbol{\varphi}_i \cdot \boldsymbol{n}_{i|\gamma}\right\|^2_{L^\infty(0,T;H^1(\gamma))} h^2 + \|\partial_t c\|^2_{L^2(0,T;\mathscr{H}^1)} h^2 + \|\partial_{tt} c\|^2_{L^2(0,T;\mathscr{H}^0)} (\Delta t)^2$. By applying Lemma 3 in (3.38) with $a_m = \left\|c^m - c_h^m\right\|_0^2$, $b_m = \left\|\boldsymbol{\varphi}^m - \boldsymbol{\varphi}_h^m\right\|_0^2$, $c_m = 0$, $d_m = 1$, we obtain

$$\left\|c^m - c_h^m\right\|_0^2 + \Delta t \sum_{n=1}^{m} \left\|\boldsymbol{\varphi}^n - \boldsymbol{\varphi}_h^n\right\|_0^2 \le \exp\left(\Delta t(m-1)\right) B \le \exp\left(\Delta t N\right) B \le \exp(T) B \le CB. \tag{3.39}$$

Since (3.39) holds for any $1 \le m \le N$, we obtain (3.17). □

Remark 1 *The proof of Theorem 2 relies on Lemma 4 to bound the advection terms $T_{6,i}$, $i = 1, 2, \gamma$, which involve the upwind operators and Lagrange multipliers arising from the hybridization. For simplicity of presentation, we have considered only the lowest order Raviart-Thomas RT_0 space on rectangular grids. The results are also valid for any two-dimensional RT_k spaces of arbitrary order k by invoking [8, Lemma 2.1] if k is even (cf. (A.4) – (A.7) in Appendix A) or using the arguments (3.25) – (3.32) in [17, pp. 189-190] if k is odd. Extension to other mixed finite element spaces such as BDM_k and $BDFM_k$ [17] can be obtained by using the results in [14, 15, 16] which are analogous to [8, Lemma 2.1] and valid for both two- and three-dimensional cases.*

The upwind-mixed scheme (2.19) can be solved directly to find an approximate solution to (2.6)-(2.7). However, as $\boldsymbol{D}_f \gg \boldsymbol{D}_i, i = 1, 2$, it would be more efficient to have a smaller time step on the fracture than on the subdomains. In the next section, we use global-in-time non-overlapping DD [42, 43, 44, 45, 46, 49, 50] to decouple (2.19) and enforce local time-stepping.

4. Fully-discrete, global-in-time nonoverlapping domain decomposition methods

We first decompose (2.19) into local problems on the subdomains:

For $i = 1, 2$, and for $n = 1, \ldots, N$, find $(c_{h,i}^n, \boldsymbol{\varphi}_{h,i}^n, \theta_{h,i}^n) \in M_{h,i} \times \widetilde{\boldsymbol{\Sigma}}_{h,i} \times \Theta_{h,i}$ such that

$$\begin{aligned}
&\left(\boldsymbol{D}_i^{-1} \boldsymbol{\varphi}_{h,i}^n, \boldsymbol{v}_{h,i}\right)_{\Omega_i} - \left(\operatorname{div} \boldsymbol{v}_{h,i}, c_{h,i}^n\right)_{\Omega_i} - \sum_{K \in \mathscr{K}_{h,i}} \Bigg(\sum_{\substack{E \subset \partial K \\ E \in \mathscr{E}_{h,i} \setminus \mathscr{E}_h^\gamma}} u_{i,KE} \mathscr{U}_{i,KE}(c_{i,K}^n, \theta_{i,E}^n) (\boldsymbol{D}_i^{-1} \boldsymbol{w}_{i,KE}, \boldsymbol{v}_{h,i})_K \\
&\quad + \sum_{\substack{E \subset \partial K \\ E \in \mathscr{E}_h^\gamma}} u_{i,KE} \mathscr{U}_{i,KE}^\gamma (c_{i,K}^n, (c_{h,i}^n)_{|E}) (\boldsymbol{D}_i^{-1} \boldsymbol{w}_{i,KE}, \boldsymbol{v}_{h,i})_K \Bigg) + \left\langle \boldsymbol{v}_{h,i} \cdot \boldsymbol{n}_{i|\gamma}, (c_{h,i}^n)_{|\gamma} \right\rangle_\gamma \\
&\qquad + \sum_{K \in \mathscr{K}_{h,i}} \sum_{\substack{E \subset \partial K \\ E \in \mathscr{E}_{h,i} \setminus \mathscr{E}_h^\gamma}} \left\langle \theta_{h,i}^n, \boldsymbol{v}_{h,i} \cdot \boldsymbol{n}_K \right\rangle_E = 0, \quad \forall \boldsymbol{v}_{h,i} \in \widetilde{\boldsymbol{\Sigma}}_{h,i}, \\
&\left(\phi_i \bar{\partial} c_{h,i}^n, \mu_{h,i}\right)_{\Omega_i} + \left(\operatorname{div} \boldsymbol{\varphi}_{h,i}^n, \mu_{h,i}\right)_{\Omega_i} = \left(q_i^n, \mu_{h,i}\right)_{\Omega_i}, \quad \forall \mu_{h,i} \in M_{h,i}, \\
&\sum_{K \in \mathscr{K}_{h,i}} \sum_{\substack{E \subset \partial K \\ E \in \mathscr{E}_{h,i} \setminus \mathscr{E}_h^\gamma}} \left\langle \eta_{h,i}, \boldsymbol{\varphi}_{h,i}^n \cdot \boldsymbol{n}_K \right\rangle_E = 0, \ \forall \eta_{h,i} \in \Theta_{h,i}^0,
\end{aligned} \tag{4.1}$$

where the initial data $c_{h,i}^0$ is given by (2.20). Moreover, to recover the solution of (2.19), the solutions of (4.1) are required to satisfy the following transmission conditions across the space-time interface $\gamma \times (0,T)$: for $n = 1, \ldots, N$,

$$\int_E c_{h,i}^n = \int_E c_{h,\gamma}^n, \ \forall E \in \mathscr{E}_h^{\gamma}, \tag{4.2}$$

$$\begin{aligned}
&\left(\boldsymbol{D}_{\gamma}^{-1}\boldsymbol{\varphi}_{h,\gamma}^n, \boldsymbol{v}_{h,\gamma}\right)_{\gamma} - \left(\operatorname{div}_{\tau}\boldsymbol{v}_{h,\gamma}, c_{h,\gamma}^n\right)_{\gamma} - \sum_{E\in\mathscr{E}_h^{\gamma}}\sum_{P\in\partial E} u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^n, \theta_{\gamma,P}^n)(\boldsymbol{D}_{\gamma}^{-1}\boldsymbol{w}_{\gamma,EP}, \boldsymbol{v}_{h,\gamma})_{\gamma} \\
&\qquad + \sum_{E\in\mathscr{E}_h^{\gamma}}\left\langle \theta_{h,\gamma}^n, \ \boldsymbol{v}_{h,\gamma}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E} = 0, \quad \forall \boldsymbol{v}_{h,\gamma} \in \widetilde{\boldsymbol{\Sigma}}_{h,\gamma}, \\
&\left(\phi_{\gamma}\bar{\partial}c_{h,\gamma}^n, \mu_{h,\gamma}\right)_{\gamma} + \left(\operatorname{div}_{\tau}\boldsymbol{\varphi}_{h,\gamma}^n, \mu_{h,\gamma}\right)_{\gamma} = \left(q_{\gamma}^n, \mu_{h,\gamma}\right)_{\gamma} + \sum_{i=1}^{2}\left\langle \boldsymbol{\varphi}_{h,i}^n\cdot\boldsymbol{n}_{i|\gamma}, \mu_{h,\gamma}\right\rangle_{\gamma}, \quad \forall \mu_{h,\gamma}\in M_{h,\gamma}, \\
&\qquad \sum_{E\in\mathscr{E}_h^{\gamma}}\left\langle \eta_{h,\gamma}, \ \boldsymbol{\varphi}_{h,\gamma}^n\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E} = 0, \quad \forall \eta_{h,\gamma}\in\Theta_{h,\gamma}.
\end{aligned} \tag{4.3}$$

Based on these transmission conditions, we develop two global-in-time DD methods: GTF-Schur and GTO-Schwarz. The former is derived using directly the equations (4.2)-(4.3), while the latter is constructed based on more general transmission conditions, namely Ventcel-to-Robin transmission conditions which will be derived in Subsection 4.2. For each method, a fully discrete interface system is formulated on the space-time fracture $\gamma\times(0,T)$ and is solved iteratively. Throughout this section for any mixed finite element space $\mathscr{O}_h$ defined in the previous section, we write $\varsigma_h := \left(\varsigma_h^n\right)_{n=1}^N \in (\mathscr{O}_h)^N$.

4.1. *Global-in-time fracture-based (GTF) Schur method*

The idea of GTF-Schur is to construct an interface operator which is close to the identity operator by making use of the presence of the fracture. In particular, the contribution of the traces on γ of the discrete normal fluxes from both subdomains is considered as the interface unknown, and is denoted by $\psi_{h,\gamma} = \left(\psi_{h,\gamma}^n\right)_{n=1}^N \in \left(M_{h,\gamma}\right)^N$, where

$$\int_E \psi_{h,\gamma}^n := \int_E \sum_{i=1}^{2}\boldsymbol{\varphi}_{h,i}^n\cdot\boldsymbol{n}_{i|\gamma}, \quad \text{for } n=1,\dots N, \quad E\in\mathscr{E}^{\gamma}. \tag{4.4}$$

We also use (4.4) to write the discrete space-time interface system for GTF-Schur. For the pure diffusion problems, this approach has been shown to work effectively without the need of any preconditioner [49, 50]. To formulate the interface problem for GTF-Schur, we first introduce the solution operator $\mathscr{R}_{\gamma}$:

$$\begin{array}{rccc}
\mathscr{R}_{\gamma}: & \left(M_{h,\gamma}\right)^N \times L^2(0,T;L^2(\gamma))\times H_0^1(\gamma) & \longrightarrow & \left(M_{h,\gamma}\right)^N \\
& (\psi_{h,\gamma}, \ q_{\gamma}, \ c_{0,\gamma}) & \mapsto & c_{h,\gamma},
\end{array}$$

where $\left(c_{h,\gamma}, \boldsymbol{\varphi}_{h,\gamma}, \theta_{h,\gamma}\right) \in \left(M_{h,\gamma}\right)^N \times \left(\widetilde{\boldsymbol{\Sigma}}_{h,\gamma}\right)^N \times \left(\Theta_{h,\gamma}\right)^N$ is the solution to the following time-dependent problem on the fracture:

For $n=1,\dots,N$, find $(c_{h,\gamma}^n, \boldsymbol{\varphi}_{h,\gamma}^n, \theta_{h,\gamma}^n) \in M_{h,\gamma}\times\widetilde{\boldsymbol{\Sigma}}_{h,\gamma}\times\Theta_{h,\gamma}$ such that

$$\begin{aligned}
&\left(\boldsymbol{D}_{\gamma}^{-1}\boldsymbol{\varphi}_{h,\gamma}^n, \boldsymbol{v}_{h,\gamma}\right)_{\gamma} - \left(\operatorname{div}_{\tau}\boldsymbol{v}_{h,\gamma}, c_{h,\gamma}^n\right)_{\gamma} - \sum_{E\in\mathscr{E}_h^{\gamma}}\sum_{P\in\partial E} u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^n, \theta_{\gamma,P}^n)(\boldsymbol{D}_{\gamma}^{-1}\boldsymbol{w}_{\gamma,EP}, \boldsymbol{v}_{h,\gamma})_{\gamma} \\
&\qquad + \sum_{E\in\mathscr{E}_h^{\gamma}}\left\langle \theta_{h,\gamma}^n, \ \boldsymbol{v}_{h,\gamma}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E} = 0, \ \forall \boldsymbol{v}_{h,\gamma} \in \widetilde{\boldsymbol{\Sigma}}_{h,\gamma},
\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\left(\phi_\gamma\bar{\partial}c^n_{h,\gamma},\mu_{h,\gamma}\right)_\gamma+\left(\mathrm{div}_\tau\boldsymbol{\varphi}^n_{h,\gamma},\mu_{h,\gamma}\right)_\gamma=\left(q^n_\gamma,\mu_{h,\gamma}\right)_\gamma+\left(\psi^n_{h,\gamma},\mu_{h,\gamma}\right)_\gamma,\ \forall\mu_{h,\gamma}\in M_{h,\gamma},\\&\sum_{E\in\mathscr{E}^\gamma_h}\left\langle\eta_{h,\gamma},\ \boldsymbol{\varphi}^n_{h,\gamma}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E}=0,\ \forall\eta_{h,\gamma}\in\Theta_{h,\gamma}.\end{aligned}\tag{4.6}$$

where the initial data $c^0_{h,\gamma}$ is given by (2.20). To compute the right-hand side of (4.4), we define the space-time Dirichlet-to-Neumann operators $\mathscr{S}^{\mathrm{DtN}}_i$, $i=1,2$:

$$\begin{array}{llll}\mathscr{S}^{\mathrm{DtN}}_i: & \left(M_{h,\gamma}\right)^N\times L^2(0,T;L^2(\Omega_i))\times H^1_*(\Omega_i) & \longrightarrow & \left(M_{h,\gamma}\right)^N\\ & (\lambda_{h,\gamma},\ q_i,\ c_{0,i}) & \mapsto & \left(\boldsymbol{\varphi}_{h,i}\cdot\boldsymbol{n}_i\right)_{|\gamma},\end{array}\tag{4.7}$$

where $\left(c_{h,i},\boldsymbol{\varphi}_{h,i},\theta_{h,i}\right)\in\left(M_{h,i}\right)^N\times\left(\widetilde{\boldsymbol{\Sigma}}_{h,i}\right)^N\times\left(\Theta_{h,i}\right)^N$ is the solution of the local problem (4.1) with Dirichlet boundary conditions

$$(c^n_{h,i})_{|\gamma}=\lambda^n_{h,\gamma},\quad\text{for } n=1,2,\ldots,N.\tag{4.8}$$

Altogether, the fully-discrete interface problem for GTF-Schur is obtained by enforcing (4.4):

Find $\psi_{h,\gamma}\in\left(M_{h,\gamma}\right)^N$ such that

$$\int_{t^{n-1}}^{t^n}\int_E\psi_{h,\gamma}\,d\gamma dt=\int_{t^{n-1}}^{t^n}\int_E\sum_{i=1}^2\mathscr{S}^{\mathrm{DtN}}_i(\mathscr{R}_\gamma(\psi_{h,\gamma},q_\gamma,c_{0,\gamma}),q_i,c_{0,i})\,d\gamma dt,\ \ \forall n=1,\ldots,N,\ \forall E\in\mathscr{E}^\gamma_h,$$

or, equivalently, find $\psi_{h,\gamma}\in\left(M_{h,\gamma}\right)^N$ such that

$$\mathscr{S}_{\mathrm{F}}\psi_{h,\gamma}=\chi_{\mathrm{F}},\tag{4.9}$$

where

$$\mathscr{S}_{\mathrm{F}}\psi_{h,\gamma}=\left(\int_{t^{n-1}}^{t^n}\int_E\psi_{h,\gamma}\,d\gamma dt-\int_{t^{n-1}}^{t^n}\int_E\sum_{i=1}^2\mathscr{S}^{\mathrm{DtN}}_i(\mathscr{R}_\gamma(\psi_{h,\gamma},0,0),0,0)\,d\gamma dt\right)_{n=1,\ldots,N,\,E\in\mathscr{E}^\gamma_h},$$

and

$$\chi_{\mathrm{F}}=\left(\int_{t^{n-1}}^{t^n}\int_E\sum_{i=1}^2\mathscr{S}^{\mathrm{DtN}}_i(\mathscr{R}_\gamma(0,q_\gamma,c_{0,\gamma}),q_i,c_{0,i})\,d\gamma dt\right)_{n=1,\ldots,N,\,E\in\mathscr{E}^\gamma_h}.$$

The interface problem (4.9) is then solved iteratively by using GMRES without applying any preconditioner.

4.2. *Global-in-time Optimized Schwarz (GTO-Schwarz) method*

To derive the interface problem for GTO-Schwarz, we first transform the transmission conditions (4.2)-(4.3) into more general ones, namely Ventcel-to-Robin transmission conditions. For each $i=1,2$, let $c_{i,\gamma}=\left(c_{i,\gamma,E}\right)_{E\in\mathscr{E}^\gamma_h}\in M_{h,\gamma}$ be the trace of $c_{h,i}$ on γ and $\theta_{i,\gamma}=\left(\theta_{i,\gamma,P}\right)_{P\in\mathscr{P}^\gamma_h}\in\Theta_{h,\gamma}$ the Lagrange multipliers of $c_{i,\gamma}$ at the endpoints of each edge $E\in\mathscr{E}^\gamma_h$. We denote by $\boldsymbol{\varphi}_{\gamma,i}$ the tangential velocity associated with $c_{i,\gamma}$ through the second equation of (2.19). Due to the continuity of the concentration across the discrete counterpart of $\gamma\times(0,T)$, we have:

$$\boldsymbol{\varphi}^n_{\gamma,1}=\boldsymbol{\varphi}^n_{\gamma,2}=\boldsymbol{\varphi}^n_{h,\gamma},\quad\text{for } n=1,\ldots,N.$$

Under sufficient regularity, the transmission conditions (4.2)-(4.3) can be replaced by the following Ventcel-to-Robin transmission conditions: for $i=1,2$, $j=3-i$, and for $n=1,\ldots,N$,

$$
\begin{aligned}
&\left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{\varphi}_{\gamma,i}^n,\boldsymbol{v}_{h,\gamma}\right)_\gamma-\left(\operatorname{div}_\tau\boldsymbol{v}_{h,\gamma},c_{i,\gamma}^n\right)_\gamma-\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E}u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^n,\theta_{i,\gamma,P}^n)(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_{\gamma,EP},\boldsymbol{v}_{h,\gamma})_\gamma\\
&\qquad+\sum_{E\in\mathscr{E}_h^\gamma}\left\langle\theta_{i,\gamma}^n,\ \boldsymbol{v}_{h,\gamma}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E}=0,\ \forall\boldsymbol{v}_{h,\gamma}\in\widetilde{\boldsymbol{\Sigma}}_{h,\gamma},\\
&\left(-\boldsymbol{\varphi}_{h,i}^n\cdot\boldsymbol{n}_{i|\gamma}+\alpha c_{i,\gamma}^n,\mu_{h,\gamma}\right)_\gamma+(\phi_\gamma\bar{\partial}c_{i,\gamma}^n,\mu_{h,\gamma})_\gamma+(\operatorname{div}_\tau\boldsymbol{\varphi}_{\gamma,i}^n,\mu_{h,\gamma})_\gamma\\
&\qquad\qquad=\left(q_\gamma,\mu_{h,\gamma}\right)_\gamma+\left(\boldsymbol{\varphi}_{h,j}^n\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^n,\mu_{h,\gamma}\right)_\gamma,\ \forall\mu_{h,\gamma}\in M_{h,\gamma},\\
&\sum_{E\in\mathscr{E}_h^\gamma}\left\langle\eta_{h,\gamma},\ \boldsymbol{\varphi}_{\gamma,i}^n\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E}=0,\ \forall\eta_{h,\gamma}\in\Theta_{h,\gamma},
\end{aligned}
\tag{4.10}
$$

where $\alpha>0$. We denote by $\zeta_{h,i}=\left(\zeta_{h,i}^n\right)_{n=1}^N\in\left(M_{h,\gamma}\right)^N,\ \ i=1,2$, the space-time discrete Robin data transmitted from one sub-domain to the neighboring sub-domain at each time step:

$$
\int_E\zeta_{h,i}^n=\int_E\left(\boldsymbol{\varphi}_{h,j}^n\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^n\right),\ \forall n=1,\dots,N,\ j=3-i. \tag{4.11}
$$

We then define the Ventcel-Robin operators S_i^{VtR}, for $i=1,2$:

$$
\begin{aligned}
\mathscr{S}_i^{\mathrm{VtR}}:\left(M_{h,\gamma}\right)^N\times L^2(0,T;L^2(\Omega_i))\times H_*^1(\Omega_i)\times H_0^1(\gamma)&\longrightarrow\left(M_{h,\gamma}\right)^N\\
(\varsigma_h,\ q_i,\ c_{0,i},\ c_{0,\gamma})&\mapsto\boldsymbol{\varphi}_{h,i}\cdot\boldsymbol{n}_{i|\gamma}+\alpha c_{i,\gamma},
\end{aligned}
$$

where $(c_{h,i},\boldsymbol{\varphi}_{h,i},\theta_{h,i},c_{i,\gamma},\varphi_{\gamma,i},\theta_{i,\gamma})\in\left(M_{h,i}\right)^N\times\left(\widetilde{\boldsymbol{\Sigma}}_{h,i}\right)^N\times\left(\Theta_{h,i}\right)^N\times\left(M_{h,\gamma}\right)^N\times\left(\widetilde{\boldsymbol{\Sigma}}_{h,\gamma}\right)^N\times\left(\Theta_{h,\gamma}\right)^N$ is the solution to the time-dependent subdomain problem with Ventcel boundary conditions (4.10)-(4.11) on Ω_i:

For $n=1,\dots,N$, find $(c_{h,i}^n,\boldsymbol{\varphi}_{h,i}^n,\theta_{h,i}^n,c_{i,\gamma}^n,\varphi_{\gamma,i}^n,\theta_{i,\gamma}^n)$ such that

$$
\begin{aligned}
&\left(\boldsymbol{D}_i^{-1}\boldsymbol{\varphi}_{h,i}^n,\boldsymbol{v}_{h,i}\right)_{\Omega_i}+\left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{\varphi}_{\gamma,i}^n,\boldsymbol{v}_{h,\gamma}\right)_\gamma-\left(\operatorname{div}\boldsymbol{v}_{h,i},c_{h,i}^n\right)_{\Omega_i}-\left(\operatorname{div}_\tau\boldsymbol{v}_{h,\gamma},c_{i,\gamma}^n\right)_\gamma\\
&-\sum_{K\in\mathscr{K}_{h,i}}\left(\sum_{\substack{E\subset\partial K\\E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}u_{i,KE}\mathscr{U}_{i,KE}(c_{i,K}^n,\theta_{i,E}^n)(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{v}_{h,i})_K+\sum_{\substack{E\subset\partial K\\E\in\mathscr{E}_h^\gamma}}u_{i,KE}\mathscr{U}_{i,KE}^\gamma(c_{i,K}^n,c_{i,\gamma,E}^n)(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{v}_{h,i})_K\right)\\
&\quad-\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E}u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^n,\theta_{i,\gamma,P}^n)(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_{\gamma,EP},\boldsymbol{v}_{h,\gamma})_\gamma+\left\langle\boldsymbol{v}_{h,i}\cdot\boldsymbol{n}_{i|\gamma},c_{i,\gamma}^n\right\rangle_\gamma\\
&\quad+\sum_{K\in\mathscr{K}_{h,i}}\sum_{\substack{E\subset\partial K\\E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}\left\langle\theta_{h,i}^n,\boldsymbol{v}_{h,i}\cdot\boldsymbol{n}_K\right\rangle_E+\sum_{E\in\mathscr{E}_h^\gamma}\left\langle\theta_{i,\gamma}^n,\ \boldsymbol{v}_{h,\gamma}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E}=0,\ \forall(\boldsymbol{v}_{h,i},\boldsymbol{v}_{h,\gamma})\in\widetilde{\boldsymbol{\Sigma}}_{h,i}\times\widetilde{\boldsymbol{\Sigma}}_{h,\gamma},\\
&\left(-\boldsymbol{\varphi}_{h,i}^n\cdot\boldsymbol{n}_{i|\gamma}+\alpha c_{i,\gamma}^n,\mu_{h,\gamma}\right)_\gamma+(\phi_i\bar{\partial}c_{h,i}^n,\mu_{h,i})_{\Omega_i}+(\phi_\gamma\bar{\partial}c_{i,\gamma}^n,\mu_{h,\gamma})_\gamma+(\operatorname{div}\boldsymbol{\varphi}_{h,i}^n,\mu_{h,i})_{\Omega_i}+(\operatorname{div}_\tau\boldsymbol{\varphi}_{\gamma,i}^n,\mu_{h,\gamma})_\gamma\\
&\qquad=(q_i^n,\mu_i)_{\Omega_i}+(q_\gamma^n,\mu_{h,\gamma})_\gamma+\left(\varsigma_h^n,\mu_{h,\gamma}\right)_\gamma,\ \forall(\mu_{h,i},\mu_{h,\gamma})\in M_{h,i}\times M_{h,\gamma},\\
&\sum_{K\in\mathscr{K}_{h,i}}\sum_{\substack{E\subset\partial K\\E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}\left\langle\eta_{h,i},\boldsymbol{\varphi}_{h,i}^n\cdot\boldsymbol{n}_K\right\rangle_E+\sum_{E\in\mathscr{E}_h^\gamma}\left\langle\eta_{h,\gamma},\ \boldsymbol{\varphi}_{\gamma,i}^n\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E}=0,\ \forall(\eta_{h,i},\eta_{h,\gamma})\in\Theta_{h,i}^0\times\Theta_{h,\gamma},
\end{aligned}
\tag{4.12}
$$

where the initial data $c_{h,i}^0$ and $c_{i,\gamma}^0$ are given by (2.20). The fully-discrete interface problem for the GTO-Schwarz method may be written as:

Find $(\zeta_{h,1},\zeta_{h,2})\in\left(M_{h,\gamma}\right)^{2N}$ such that

$$\left.\begin{aligned}\int_{t^{n-1}}^{t^n}\int_E \zeta_{h,1}\,d\gamma dt &= \int_{t^{n-1}}^{t^n}\int_E \mathscr{S}_2^{\mathrm{VtR}}(\zeta_{h,2},q_2,c_{0,2},c_{0,\gamma})\,d\gamma dt,\\ \int_{t^{n-1}}^{t^n}\int_E \zeta_{h,2} d\gamma dt &= \int_{t^{n-1}}^{t^n}\int_E \mathscr{S}_1^{\mathrm{VtR}}(\zeta_{h,1},q_1,c_{0,1},c_{0,\gamma})\,d\gamma dt,\end{aligned}\right\}\ \forall n=1,\dots,N,\ \forall E\in\mathscr{E}_h^\gamma,$$

or, in a more compact form,

$$\mathscr{S}_{\mathrm{O}}\begin{pmatrix}\zeta_{h,1}\\ \zeta_{h,2}\end{pmatrix}=\chi_{\mathrm{O}}, \tag{4.13}$$

where

$$\mathscr{S}_{\mathrm{O}}\begin{pmatrix}\zeta_{h,1}\\ \zeta_{h,2}\end{pmatrix}=\begin{pmatrix}\int_{t^{n-1}}^{t^n}\int_E \zeta_{h,1}\,d\gamma dt-\int_{t^{n-1}}^{t^n}\int_E \mathscr{S}_2^{\mathrm{VtR}}(\zeta_{h,2},0,0,0)\,d\gamma dt\\ \int_{t^{n-1}}^{t^n}\int_E \zeta_{h,2} d\gamma dt-\int_{t^{n-1}}^{t^n}\int_E \mathscr{S}_1^{\mathrm{VtR}}(\zeta_{h,1},0,0,0)\,d\gamma dt\end{pmatrix}_{n=1,\dots,N,\,E\in\mathscr{E}_h^\gamma},$$

and

$$\chi_{\mathrm{O}}=\begin{pmatrix}\int_{t^{n-1}}^{t^n}\int_E \mathscr{S}_2^{\mathrm{VtR}}(0,q_2,c_{0,2},c_{0,\gamma})\,d\gamma dt\\ \int_{t^{n-1}}^{t^n}\int_E \mathscr{S}_1^{\mathrm{VtR}}(0,q_1,c_{0,1},c_{0,\gamma})\,d\gamma dt\end{pmatrix}_{n=1,\dots,N,\,E\in\mathscr{E}_h^\gamma}.$$

The interface problem (4.13) can be solved iteratively using either Jacobi iterations or GMRES. Performing Jacobi iterations leads to the following Optimized Schwarz waveform relaxation (OSWR) algorithm: at the kth iteration, solve in parallel the following time-dependent subdomain problems on $\Omega_i\times(0,T),\ i=1,2$: for $n=1,\dots,N$,

$$\begin{aligned}&\left(\boldsymbol{D}_i^{-1}\boldsymbol{\varphi}_{h,i}^{k,n},\boldsymbol{v}_{h,i}\right)_{\Omega_i}+\left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{\varphi}_{\gamma,i}^{k,n},\boldsymbol{v}_{h,\gamma}\right)_\gamma-\left(\operatorname{div}\boldsymbol{v}_{h,i},c_{h,i}^{k,n}\right)_{\Omega_i}-\left(\operatorname{div}_\tau\boldsymbol{v}_{h,\gamma},c_{i,\gamma}^{k,n}\right)_\gamma\\
&-\sum_{K\in\mathscr{K}_{h,i}}\left(\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}u_{i,KE}\mathscr{U}_{i,KE}(c_{i,K}^{k,n},\theta_{i,E}^{k,n})(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{v}_{h,i})_K+\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_h^\gamma}}u_{i,KE}\mathscr{U}_{i,KE}^\gamma(c_{i,K}^{k,n},c_{i,\gamma,E}^{k,n})(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{v}_{h,i})_K\right)\\
&\quad-\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E}u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^{k,n},\theta_{i,\gamma,P}^{k,n})(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_{\gamma,EP},\boldsymbol{v}_{h,\gamma})_\gamma+\left\langle\boldsymbol{v}_{h,i}\cdot\boldsymbol{n}_{i|\gamma},c_{i,\gamma}^{k,n}\right\rangle_\gamma\\
&\qquad+\sum_{K\in\mathscr{K}_{h,i}}\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}\left\langle\theta_{h,i}^{k,n},\boldsymbol{v}_{h,i}\cdot\boldsymbol{n}_K\right\rangle_E+\sum_{E\in\mathscr{E}_h^\gamma}\left\langle\theta_{i,\gamma}^{k,n},\,\boldsymbol{v}_{h,\gamma}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E}=0,\ \forall(\boldsymbol{v}_{h,i},\boldsymbol{v}_{h,\gamma})\in\widetilde{\boldsymbol{\Sigma}}_{h,i}\times\widetilde{\boldsymbol{\Sigma}}_{h,\gamma},\\
&\left(-\boldsymbol{\varphi}_{h,i}^{k,n}\cdot\boldsymbol{n}_{i|\gamma}+\alpha c_{i,\gamma}^{k,n},\mu_{h,\gamma}\right)_\gamma+(\phi_i\bar\partial c_{h,i}^{k,n},\mu_{h,i})_{\Omega_i}+(\phi_\gamma\bar\partial c_{i,\gamma}^{k,n},\mu_{h,\gamma})_\gamma+(\operatorname{div}\boldsymbol{\varphi}_{h,i}^{k,n},\mu_{h,i})_{\Omega_i}+(\operatorname{div}_\tau\boldsymbol{\varphi}_{\gamma,i}^{k,n},\mu_{h,\gamma})_\gamma\\
&\qquad=(q_i^n,\mu_i)_{\Omega_i}+(q_\gamma^n,\mu_{h,\gamma})_\gamma+\left(\boldsymbol{\varphi}_{h,j}^{k-1,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{k-1,n},\mu_{h,\gamma}\right)_\gamma,\ \forall(\mu_{h,i},\mu_{h,\gamma})\in M_{h,i}\times M_{h,\gamma},\\
&\sum_{K\in\mathscr{K}_{h,i}}\sum_{\substack{E\subset\partial K\\ E\in_{h,i}\setminus\mathscr{E}_h^\gamma}}\left\langle\eta_{h,i},\boldsymbol{\varphi}_{h,i}^{k,n}\cdot\boldsymbol{n}_K\right\rangle_E+\sum_{E\in\mathscr{E}_h^\gamma}\left\langle\eta_{h,\gamma},\,\boldsymbol{\varphi}_{\gamma,i}^{k,n}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E}=0,\ \forall(\eta_{h,i},\eta_{h,\gamma})\in\Theta_{h,i}^0\times\Theta_{h,\gamma},\end{aligned}\tag{4.14}$$

with given initial guesses $g_{i,j}(t) := \boldsymbol{\varphi}^0_{h,i}\cdot\boldsymbol{n}_i + \alpha\theta^0_{i,\gamma} \in M_{h,\gamma}$, for $i = 1,2,\ j = (3-i)$, to start the first iterate. We next show that for the OSWR iterative algorithm (4.14) converges. The following lemma is needed in our proof.

Lemma 5 *[7, Lemma 4.1] For $i = 1,2$, there exists a constant $\bar{C}$ independent of h_i such that:*

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{0,\partial\Omega_i} \le \bar{C}h_i^{-1/2}\|\boldsymbol{v}\|_{0,\Omega_i}, \quad \text{for any } \boldsymbol{v}\in\boldsymbol{\Sigma}_{h,i}.$$

Theorem 3 *For any sufficiently small but fixed Δt and h such that $\Delta t/\bar{h} < (\phi^- D^-_{min})/(16\bar{C})$ where $\bar{h} = \min\{h_1,h_2\}$ and $\bar{C}$ is provided in Lemma 5, Algorithm (4.14), initialized by $(g_{i,j}), i = 1,2,\ j = (3-i)$, defines a unique sequence of iterates*

$$(c^k_{h,i},\boldsymbol{\varphi}^k_{h,i},\theta^k_{h,i},c^k_{i,\gamma},\boldsymbol{\varphi}^k_{\gamma,i},\theta^k_{i,\gamma}) \in (M_{h,i})^N\times(\widetilde{\boldsymbol{\Sigma}}_{h,i})^N\times(\Theta_{h,i})^N\times(M_{h,\gamma})^N\times(\widetilde{\boldsymbol{\Sigma}}_{h,\gamma})^N\times(\Theta_{h,\gamma})^N,$$

that converges, as $k\to\infty$, to the solution of the problem

$$\begin{aligned}
&\left(\boldsymbol{D}_i^{-1}\boldsymbol{\varphi}^n_{h,i},\boldsymbol{v}_{h,i}\right)_{\Omega_i} + \left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{\varphi}^n_{\gamma,i},\boldsymbol{v}_{h,\gamma}\right)_\gamma - \left(\operatorname{div}\boldsymbol{v}_{h,i},c^n_{h,i}\right)_{\Omega_i} - \left(\operatorname{div}_\tau\boldsymbol{v}_{h,\gamma},c^n_{i,\gamma}\right)_\gamma \\
&\quad - \sum_{K\in\mathscr{K}_{h,i}}\left(\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}} u_{i,KE}\mathscr{U}_{i,KE}(c^n_{i,K},\theta^n_{i,E})(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{v}_{h,i})_K + \sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_h^\gamma}} u_{i,KE}\mathscr{U}^\gamma_{i,KE}(c^n_{i,K},c^n_{i,\gamma,E})(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{v}_{h,i})_K\right) \\
&\quad - \sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E} u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c^n_{\gamma,E},\theta^n_{i,\gamma,P})(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_{\gamma,EP},\boldsymbol{v}_{h,\gamma})_\gamma + \left\langle\boldsymbol{v}_{h,i}\cdot\boldsymbol{n}_{i|\gamma},c^n_{i,\gamma}\right\rangle_\gamma \\
&\quad + \sum_{K\in\mathscr{K}_{h,i}}\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}\left\langle\theta^n_{h,i},\boldsymbol{v}_{h,i}\cdot\boldsymbol{n}_K\right\rangle_E + \sum_{E\in\mathscr{E}_h^\gamma}\left\langle\theta^n_{i,\gamma},\ \boldsymbol{v}_{h,\gamma}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E} = 0,\ \forall(\boldsymbol{v}_{h,i},\boldsymbol{v}_{h,\gamma})\in\widetilde{\boldsymbol{\Sigma}}_{h,i}\times\widetilde{\boldsymbol{\Sigma}}_{h,\gamma}, \\
&\left(-\boldsymbol{\varphi}^n_{h,i}\cdot\boldsymbol{n}_{i|\gamma}+\alpha c^n_{i,\gamma},\mu_{h,\gamma}\right)_\gamma + (\phi_i\bar{\partial}c^n_{h,i},\mu_{h,i})_{\Omega_i} + (\phi_\gamma\bar{\partial}c^n_{i,\gamma},\mu_{h,\gamma})_\gamma + (\operatorname{div}\boldsymbol{\varphi}^n_{h,i},\mu_{h,i})_{\Omega_i} + (\operatorname{div}_\tau\boldsymbol{\varphi}^n_{\gamma,i},\mu_{h,\gamma})_\gamma \\
&\qquad = (q^n_i,\mu_i)_{\Omega_i} + (q^n_\gamma,\mu_{h,\gamma})_\gamma + \left(\boldsymbol{\varphi}^n_{h,j}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c^n_{j,\gamma},\mu_{h,\gamma}\right)_\gamma,\ \forall(\mu_{h,i},\mu_{h,\gamma})\in M_{h,i}\times M_{h,\gamma}, \\
&\sum_{K\in\mathscr{K}_{h,i}}\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}\left\langle\eta_{h,i},\boldsymbol{\varphi}^n_{h,i}\cdot\boldsymbol{n}_K\right\rangle_E + \sum_{E\in\mathscr{E}_h^\gamma}\left\langle\eta_{h,\gamma},\ \boldsymbol{\varphi}^n_{\gamma,i}\cdot\boldsymbol{n}_{\partial E}\right\rangle_{\partial E} = 0,\ \forall(\eta_{h,i},\eta_{h,\gamma})\in\Theta^0_{h,i}\times\Theta_{h,\gamma}.
\end{aligned} \tag{4.15}$$

Proof As the equations are linear, we take $q_i = 0$, $q_\gamma = 0$ and $c_{0,i} = 0$, $c_{0,\gamma} = 0$, and prove the sequence of iterates converges to zero. Fix i, for any $n = 1,\ldots,N$, let $(\eta_{h,i},\eta_{h,\gamma})$ in $\Theta_{h,i}\times\Theta_{h,\gamma}$ be such that

$$\left(\eta_{h,i}\right)_{|E} = \begin{cases}\theta^{k,n}_{i,E}, & \text{on } E\in\mathscr{E}^I_{h,i},\\ 0 & \text{otherwise},\end{cases}, \quad \left(\eta_{h,\gamma}\right)_{|P} = \begin{cases}\theta^{k,n}_{i,\gamma,P}, & \text{on } P\notin\partial\gamma,\\ 0 & \text{otherwise}\end{cases}. \tag{4.16}$$

We then substitute $(\boldsymbol{v}_{h,i},\boldsymbol{v}_{h,\gamma}) = \left(\boldsymbol{\varphi}^{k,n}_{h,i},\boldsymbol{\varphi}^{k,n}_{\gamma,i}\right)$, $(\mu_{h,i},\mu_{h,\gamma}) = \left(c^{k,n}_{h,i},c^{k,n}_{i,\gamma}\right)$ and $(\eta_{h,i},\eta_{h,\gamma})$ defined by (4.16) into the first two equations of (4.14) and add the resulting equations to obtain

$$\begin{aligned}
&\left(\boldsymbol{D}_i^{-1}\boldsymbol{\varphi}_{h,i}^{k,n},\boldsymbol{\varphi}_{h,i}^{k,n}\right)_{\Omega_i}+\left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{\varphi}_{\gamma,i}^{k,n},\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right)_\gamma+\alpha\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2+\left(\phi_i\bar{\partial}c_{h,i}^{k,n},c_{h,i}^{k,n}\right)_{\Omega_i}+\left(\phi_\gamma\bar{\partial}c_{i,\gamma}^{k,n},c_{i,\gamma}^{k,n}\right)_\gamma\\
&=\sum_{K\in\mathscr{K}_{h,i}}\left(\sum_{\substack{E\subset\partial K\\E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}u_{i,KE}\mathscr{U}_{i,KE}(c_{i,K}^{k,n},\theta_{i,E}^{k,n})(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{\varphi}_{h,i}^{k,n})_K+\sum_{\substack{E\subset\partial K\\E\in\mathscr{E}_h^\gamma}}u_{i,KE}\mathscr{U}_{i,KE}^\gamma(c_{i,K}^{k,n},c_{i,\gamma,E}^{k,n})(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{\varphi}_{h,i}^{k,n})_K\right)\\
&+\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E}u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^{k,n},\theta_{i,\gamma,P}^{k,n})(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_{\gamma,EP},\boldsymbol{\varphi}_{\gamma,i}^{k,n})_\gamma+\left(\boldsymbol{\varphi}_{h,j}^{k-1,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{k-1,n},c_{i,\gamma}^{k,n}\right)_\gamma.
\end{aligned}\tag{4.17}$$

By summing (4.17) from $n=1,\ldots,N$ and then multiplying both sides of the resulting equation by $2\Delta t$, we have

$$\begin{aligned}
&2\Delta t\sum_{n=1}^N\left(\boldsymbol{D}_i^{-1}\boldsymbol{\varphi}_{h,i}^{k,n},\boldsymbol{\varphi}_{h,i}^{k,n}\right)_{\Omega_i}+2\Delta t\sum_{n=1}^N\left(\boldsymbol{D}_\gamma^{-1}\boldsymbol{\varphi}_{\gamma,i}^{k,n},\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right)_\gamma+2\alpha\Delta t\sum_{n=1}^N\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2+2\Delta t\sum_{n=1}^N\left(\phi_i\bar{\partial}c_{h,i}^{k,n},c_{h,i}^{k,n}\right)_{\Omega_i}\\
&+2\Delta t\sum_{n=1}^N\left(\phi_\gamma\bar{\partial}c_{i,\gamma}^{k,n},c_{i,\gamma}^{k,n}\right)_\gamma=2\Delta t\sum_{n=1}^N\sum_{K\in\mathscr{K}_{h,i}}\left(\sum_{\substack{E\subset\partial K\\E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}}u_{i,KE}\mathscr{U}_{i,KE}(c_{i,K}^{k,n},\theta_{i,E}^{k,n})(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{\varphi}_{h,i}^{k,n})_K\right.\\
&\qquad\left.+\sum_{\substack{E\subset\partial K\\E\in\mathscr{E}_h^\gamma}}u_{i,KE}\mathscr{U}_{i,KE}^\gamma(c_{i,K}^{k,n},c_{i,\gamma,E}^{k,n})(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE},\boldsymbol{\varphi}_{h,i}^{k,n})_K\right)\\
&\qquad+2\Delta t\sum_{n=1}^N\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E}u_{\gamma,EP}\mathscr{U}_{\gamma,EP}(c_{\gamma,E}^{k,n},\theta_{i,\gamma,P}^{k,n})(\boldsymbol{D}_\gamma^{-1}\boldsymbol{w}_{\gamma,EP},\boldsymbol{\varphi}_{\gamma,i}^{k,n})_\gamma\\
&\qquad+2\Delta t\sum_{n=1}^N\left(\boldsymbol{\varphi}_{h,j}^{k-1,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{k-1,n},c_{i,\gamma}^{k,n}\right)_\gamma
\end{aligned}\tag{4.18}$$

By proceeding in the same manner as in (3.10), (3.11) and (3.13), we obtain from (4.18) that

$$\begin{aligned}
&\phi^-\left\|c_{h,i}^{k,N}\right\|_{0,\Omega_i}^2+2\Delta tD_{\min}^-\sum_{n=1}^N\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+\phi^-\left\|c_{i,\gamma}^{k,N}\right\|_{0,\gamma}^2+2\Delta tD_{\min}^-\sum_{n=1}^N\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2+2\alpha\Delta t\sum_{n=1}^N\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\\
&\le\frac{C\Delta t}{\varepsilon}\sum_{n=1}^N\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+\frac{C\Delta t\varepsilon}{4}\sum_{n=1}^N\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+C\Delta th\sum_{n=1}^N\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+\frac{C\Delta t}{\varepsilon}\sum_{n=1}^N\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\\
&\quad+\frac{C\Delta t\varepsilon}{4}\sum_{n=1}^N\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2+C\Delta th\sum_{n=1}^N\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2+2\Delta t\sum_{n=1}^N\left(\boldsymbol{\varphi}_{h,j}^{k-1,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{k-1,n},c_{i,\gamma}^{k,n}\right)_\gamma.
\end{aligned}\tag{4.19}$$

We choose ε small enough such that $2D_{\min}^- - \dfrac{C\varepsilon}{4} > D_{\min}^-$, and then Δt, and h small enough such that

$$\phi^- - \frac{C\Delta t}{\varepsilon} > \frac{\phi^-}{2},\quad D_{\min}^- - Ch > \frac{D_{\min}^-}{2}.\tag{4.20}$$

From (4.19) and (4.20), we find

$$\begin{aligned}
&\frac{\phi^-}{2}\left\|c_{h,i}^{k,N}\right\|_{0,\Omega_i}^2+\frac{D_{\min}^-}{2}\Delta t\sum_{n=1}^N\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+\frac{\phi^-}{2}\left\|c_{i,\gamma}^{k,N}\right\|_{0,\gamma}^2+\frac{D_{\min}^-}{2}\Delta t\sum_{n=1}^N\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2+2\alpha\Delta t\sum_{n=1}^N\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\\
&\le\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left\|\theta_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2+2\Delta t\sum_{n=1}^N\left(\boldsymbol{\varphi}_{h,j}^{k-1,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{k-1,n},c_{i,\gamma}^{k,n}\right)_\gamma.
\end{aligned}\tag{4.21}$$

By summing over the iterates k from 1 to K in (4.21), we obtain,

$$\begin{aligned}
&\frac{\phi^-}{2}\sum_{k=1}^{K}\left\|c_{h,i}^{k,N}\right\|_{0,\Omega_i}^2+\frac{D_{\min}^-}{2}\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\frac{\phi^-}{2}\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,N}\right\|_{0,\gamma}^2+\frac{D_{\min}^-}{2}\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2\right)\\
&+2\alpha\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)\leq\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)+\\
&\qquad 2\Delta t\sum_{n=1}^{N}\sum_{k=1}^{K-1}\left(\boldsymbol{\varphi}_{h,j}^{k,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{k,n},c_{i,\gamma}^{k+1,n}\right)_\gamma+2\Delta t\sum_{n=1}^{N}\left(\boldsymbol{\varphi}_{h,j}^{0,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{0,n},c_{i,\gamma}^{1,n}\right)_\gamma.
\end{aligned}\tag{4.22}$$

By applying the weighted Cauchy-Schwarz inequality and Lemma 5, we obtain, for $i=1,2,\ j=3-i$:

$$\begin{aligned}
&2\Delta t\sum_{n=1}^{N}\left(\boldsymbol{\varphi}_{h,j}^{k-1,n}\cdot\boldsymbol{n}_{j|\gamma},c_{i,\gamma}^{k,n}\right)_\gamma\leq 2\Delta t\sum_{n=1}^{N}\left\|\boldsymbol{\varphi}_{h,j}^{k-1,n}\cdot\boldsymbol{n}_{j|\gamma}\right\|_{0,\gamma}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}\\
&\qquad\leq\Delta t\sum_{n=1}^{N}\left(\rho h_i\left\|\boldsymbol{\varphi}_{h,j}^{k-1,n}\cdot\boldsymbol{n}_{j|\gamma}\right\|_{0,\gamma}^2+\frac{1}{\rho h_i}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)\leq\Delta t\sum_{n=1}^{N}\left(\bar{C}\rho\left\|\boldsymbol{\varphi}_{h,j}^{k-1,n}\right\|_{0,\Omega_j}^2+\frac{1}{\rho\bar{h}}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right).
\end{aligned}$$

Using this and the weighted Cauchy-Schwarz inequality on the last two terms on the right-hand side of (4.22) yields

$$\begin{aligned}
&\frac{\phi^-}{2}\sum_{k=1}^{K}\left\|c_{h,i}^{k,N}\right\|_{0,\Omega_i}^2+\frac{D_{\min}^-\Delta t}{2}\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\frac{\phi^-}{2}\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,N}\right\|_{0,\gamma}^2+\frac{D_{\min}^-\Delta t}{2}\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2\right)\\
&+2\alpha\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)\leq\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)\\
&\quad+\alpha\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K-1}\left\|c_{j,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)+\alpha\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K-1}\left\|c_{i,\gamma}^{k+1,n}\right\|_{0,\gamma}^2\right)+\bar{C}\rho\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K-1}\left\|\boldsymbol{\varphi}_{h,j}^{k,n}\right\|_{0,\Omega_j}^2\right)\\
&\quad+\frac{\Delta t}{\rho\bar{h}}\sum_{n=1}^{N}\left(\sum_{k=1}^{K-1}\left\|c_{i,\gamma}^{k+1,n}\right\|_{0,\gamma}^2\right)+\frac{\Delta t}{\alpha}\sum_{n=1}^{N}\left\|\boldsymbol{\varphi}_{h,j}^{0,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{0,n}\right\|_{0,\gamma}^2+\alpha\Delta t\sum_{n=1}^{N}\left\|c_{i,\gamma}^{1,n}\right\|_{0,\gamma}^2.
\end{aligned}$$

We then fix $\rho=\dfrac{D_{\min}^-}{4\bar{C}}$ and use the assumption $\dfrac{\Delta t}{\bar{h}}<\dfrac{\phi^-D_{\min}^-}{16\bar{C}}$ to deduce that

$$\begin{aligned}
&\frac{\phi^-}{4}\sum_{k=1}^{K}\left\|c_{h,i}^{k,N}\right\|_{0,\Omega_i}^2+\frac{D_{\min}^-\Delta t}{2}\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\frac{\phi^-}{4}\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,N}\right\|_{0,\gamma}^2+\frac{D_{\min}^-\Delta t}{2}\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2\right)\\
&+\alpha\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)\leq\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\left(\frac{C\Delta t}{\varepsilon}+\frac{4\Delta t\bar{C}}{D_{\min}^-\bar{h}}\right)\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)\\
&\qquad+\alpha\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K-1}\left\|c_{j,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)+\frac{D_{\min}^-\Delta t}{4}\sum_{n=1}^{N}\left(\sum_{k=1}^{K-1}\left\|\boldsymbol{\varphi}_{h,j}^{k,n}\right\|_{0,\Omega_j}^2\right)\\
&\qquad+\frac{\Delta t}{\alpha}\sum_{n=1}^{N}\left\|\boldsymbol{\varphi}_{h,j}^{0,n}\cdot\boldsymbol{n}_{j|\gamma}+\alpha c_{j,\gamma}^{0,n}\right\|_{0,\gamma}^2.
\end{aligned}\tag{4.23}$$

By summing over the index i for i from 1 to 2, we obtain from (4.23),

$$\begin{aligned}
&\frac{\phi^-}{4}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{h,i}^{k,N}\right\|_{0,\Omega_i}^2+\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{i,\gamma}^{k,N}\right\|_{0,\gamma}^2\right)+\frac{D_{\min}^-\Delta t}{2}\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\frac{D_{\min}^-\Delta t}{2}\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2\right)\\
&+\alpha\Delta t\sum_{n=1}^{N}\left(\sum_{i=1}^{2}\left\|c_{i,\gamma}^{K,n}\right\|_{0,\gamma}^2\right)\leq\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\left(\frac{C\Delta t}{\varepsilon}+\frac{4\Delta t\bar{C}}{D_{\min}^-\bar{h}}\right)\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)\\
&\qquad+\frac{D_{\min}^-}{4}\Delta t\sum_{n=1}^{N}\left(\sum_{k=1}^{K-1}\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\frac{\Delta t}{\alpha}\sum_{n=1}^{N}\left(\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{h,i}^{0,n}\cdot\boldsymbol{n}_{i|\gamma}+\alpha c_{i,\gamma}^{0,n}\right\|_{0,\gamma}^2\right).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\frac{\phi^-}{4}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{h,i}^{k,N}\right\|_{0,\Omega_i}^2+\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{i,\gamma}^{k,N}\right\|_{0,\gamma}^2\right)+\frac{D_{\min}^-\Delta t}{4}\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2\right)\\
&+\alpha\Delta t\sum_{n=1}^{N}\left(\sum_{i=1}^{2}\left\|c_{i,\gamma}^{K,n}\right\|_{0,\gamma}^2\right)\leq\frac{C\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}^2\right)+\left(\frac{C}{\varepsilon}+\frac{4\bar{C}}{D_{\min}^-\bar{h}}\right)\Delta t\sum_{n=1}^{N-1}\left(\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2\right)\\
&+\frac{\Delta t}{\alpha}\sum_{n=1}^{N}\left(\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{h,i}^{0,n}\cdot\boldsymbol{n}_{i|\gamma}+\alpha c_{i,\gamma}^{0,n}\right\|_{0,\gamma}^2\right),
\end{aligned}\tag{4.24}$$

Denote the positive numbers

$$R_{\phi,\boldsymbol{D},\alpha}=\min\left\{\frac{\phi^-}{4},\frac{D_{\min}^-}{4},\alpha\right\},\quad L_h=\frac{\max\left\{\frac{C}{\varepsilon},C+\frac{4\bar{C}}{D_{\min}^-\bar{h}}\right\}}{R_{\phi,\boldsymbol{D},\alpha}},$$

and the sequences

$$\begin{aligned}
a_n&=\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}^2, \qquad b_n=\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}^2+\sum_{k=1}^{K}\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}^2,\\
c_n&=\frac{1}{\alpha R_{\phi,\boldsymbol{D},\alpha}}\sum_{i=1}^{2}\left\|\boldsymbol{\varphi}_{h,i}^{0,n}\cdot\boldsymbol{n}_{i|\gamma}+\alpha c_{i,\gamma}^{0,n}\right\|_{0,\gamma}^2.
\end{aligned}$$

We obtain from (4.24) the following inequality

$$a_N+\Delta t\sum_{n=1}^{N}b_n\leq\Delta t\sum_{n=1}^{N-1}L_h a_n+\Delta t\sum_{n=1}^{N}c_n.\tag{4.25}$$

By using Lemma 3 in (4.25) with $\tau=\Delta t$, $B=0$, and $d_l=L_h$, we obtain, (using $n\Delta t\leq N\Delta t=T$ for any n)

$$a_N+\Delta t\sum_{n=1}^{N}b_n\leq\exp\left(\Delta t(N-1)L_h\right)\left(\Delta t\sum_{n=1}^{N}c_n\right)\leq\exp\left(L_hT\right)\left(\frac{1}{\alpha}\int_0^{t_N}\sum_{i=1}^{2}\left\|g_{i,j}\right\|_{0,\Omega_i}^2\right).\tag{4.26}$$

From (4.26), we deduce that a_N and b_n are bounded since the right-hand side of (4.26) does not depend on k. Hence, $\left\|c_{h,i}^{k,N}\right\|_{0,\Omega_i}$, $\left\|\boldsymbol{\varphi}_{h,i}^{k,n}\right\|_{0,\Omega_i}$, $\left\|c_{i,\gamma}^{k,N}\right\|_{0,\gamma}$, and $\left\|\boldsymbol{\varphi}_{\gamma,i}^{k,n}\right\|_{0,\gamma}$ converge to 0 as $k\to\infty$, for $i=1,2$ and for $n=1,\ldots,N$. Note that (4.26) can be established for any $1\leq n\leq N$, hence, we have $\left\|c_{h,i}^{k,n}\right\|_{0,\Omega_i}$, $\left\|c_{i,\gamma}^{k,n}\right\|_{0,\gamma}$ converge to 0 as $k\to\infty$ for any $1\leq n\leq N$.

To show the well-posedness of (4.14) for $i=1,2$, it suffices to show uniqueness which can be obtained by repeating similar steps as above. □

Remark 2 *In our convergence analysis, we assumed some relation between Δt and h to handle the traces on the fracture of the normal fluxes $\boldsymbol{\varphi}_{h,i}^n\cdot\boldsymbol{n}_{i|\gamma}, i=1,2$ from both subdomains. However, such an assumption is not needed when one has Robin-Robin or Ventcel-Ventcel transmission conditions since for these cases, the boundary terms from both sides of the fracture can be manipulated in such a way that they cancel each other (e.g., [40, 47]). Thus, it is possible to show the convergence of the OSWR algorithm with nonconforming temporal discretization in the absence of the fracture. For the reduced fracture model, this remains an open question.*

The space-time interface system derived for each method is global-in-time, thus one can impose different time steps on the fracture and on the subdomains. In the next section, we show how to formulate the interface problem for each method with nonconforming discretization in time.

5. Nonconforming discretization in time

Let $\mathscr{T}_1, \mathscr{T}_2$, and $\mathscr{T}_\gamma$ be different partitions of the time interval $(0,T]$ into subintervals $J_m^i = \left(t_{m-1}^i, t_m^i\right]$ for $m = 1,\ldots,N_i$, and $i = 1,2,\gamma$, respectively (see Figure 2). For simplicity, we consider uniform partitions and denote by $\Delta t_i,\ i = 1,2,\gamma$, the corresponding time steps such that $\Delta t_\gamma \ll \Delta t_i,\ i = 1,2$ (note that the fracture is assumed to have much larger permeability than the surrounding rock matrix).

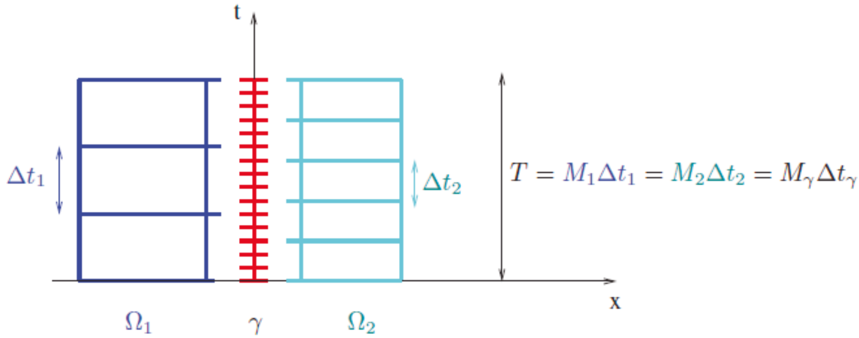

FIG. 2. Nonconforming time grids in the rock matrix and in the fracture.

We denote by $P_0(\mathscr{T}_i, \mathscr{W})$ the space of functions which are piecewise constant in time on grid $\mathscr{T}_i$ with values in $\mathscr{W}$: $P_0(\mathscr{T}_i, \mathscr{W}) = \{\psi : (0,T) \to \mathscr{W}, \psi \text{ is constant on } J,\ \forall J \in \mathscr{T}_i\}$. In order to exchange data on the space-time interface between different time grids $\mathscr{T}_i$ and $\mathscr{T}_j$ (for i,j in $\{1,2,\gamma\}$), we use the L^2 projection Π_{ji} from $P_0(\mathscr{T}_i, \mathscr{W})$ to $P_0(\mathscr{T}_j, \mathscr{W})$: for $\psi \in P_0(\mathscr{T}_i, \mathscr{W})$, $\Pi_{ji}\psi_{|J_m^j}$ is the average value of ψ on J_m^j, for $m = 1,\ldots,N_j$.

To write the interface equations for GTF-Schur and GTO-Schwarz with nonconconforming time grids, we enforce the transmission conditions weakly over the fracture time subintervals. More details can also be found in [44, 49, 50].

5.1. *GTF-Schur method*

We choose $\psi_{h,\gamma} = \left(\psi_{h,\gamma}^n\right)_{n=1}^{N_\gamma} \in P_0(\mathscr{T}_\gamma, M_{h,\gamma})$ to be piecewise constant in time on the time grid imposed on the fracture. The interface system (4.9) is then rewritten as:

$$\int_{t_\gamma^{n-1}}^{t_\gamma^n}\int_E \psi_{h,\gamma}\, d\gamma dt = \int_{t_\gamma^{n-1}}^{t_\gamma^n}\int_E \sum_{i=1}^{2} \Pi_{\gamma i}\mathscr{S}_i^{\mathrm{DtN}}(\Pi_{i\gamma}\mathscr{R}_\gamma(\psi_{h,\gamma}, q_\gamma, c_{0,\gamma}), q_i, c_{0,i})\, d\gamma dt, \forall n = 1,\ldots,N_\gamma, \forall E \in \mathscr{E}_h^\gamma. \tag{5.1}$$

5.2. *GTO-Schwarz method*

The two interface unknowns represent the Ventcel term on each subdomain, thus, we let $\zeta_{h,i} = \left(\zeta_{h,i}^n\right)_{n=1}^{N_\gamma} \in P_0\left(\mathscr{T}_\gamma, M_{h,\gamma}\right), i = 1,2$. The interface problem (4.13) of GTO-Schwarz is rewritten as:

$$\left.\begin{aligned}\int_{t_\gamma^{n-1}}^{t_\gamma^n}\int_E \zeta_{h,1}\, d\gamma dt &= \int_{t_\gamma^{n-1}}^{t_\gamma^n}\int_E \Pi_{\gamma 2} S_2^{\mathrm{VtR}}(\Pi_{2\gamma}\zeta_{h,2}, q_2, c_{0,2}, c_{0,\gamma})\, d\gamma dt,\\ \int_{t_\gamma^{n-1}}^{t_\gamma^n}\int_E \zeta_{h,2} d\gamma dt &= \int_{t_\gamma^{n-1}}^{t_\gamma^n}\int_E \Pi_{\gamma 1} S_1^{\mathrm{VtR}}(\Pi_{1\gamma}\zeta_{h,1}, q_1, c_{0,1}, c_{0,\gamma})\, d\gamma dt,\end{aligned}\right\} \quad \forall n = 1,\ldots,N_\gamma,\ \forall E \in \mathscr{E}_h^\gamma. \tag{5.2}$$

6. Numerical results

We consider an adapted version of the test case used in [3] which is a leaking contaminant repository, located in a rock with low permeability (Figure 3). The repository is crossed by a fracture and transported mostly upward. The rock is covered by an aquifer and the contaminant is assumed to be moved away instantly at the top boundary of the domain calculation so the boundary condition there is a vanishing concentration. The actual physical parameters are given in Table 1, where the diffusion $\boldsymbol{D}_i = d_i\mathbf{I}$ ($i = 1,2,\gamma$) is isotropic and constant in each subdomain and on the fracture, where $\mathbf{I}$ is the 2D identity matrix. The velocity $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_\gamma)$ presented in (2.3)-(2.4) is obtained by solving the steady-state flow problem on the subdomains

$$\begin{array}{rcll} \operatorname{div}\boldsymbol{u}_i &=& 0 & \text{in } \Omega_i\times(0,\ T),\\ \boldsymbol{u}_i &=& -k_i\nabla p_i & \text{in } \Omega_i\times(0,\ T),\\ p_i &=& g_i & \text{on } (\partial\Omega_i\cap\partial\Omega)\times(0,\ T), \quad i=1,2,\\ p_i &=& p_\gamma & \text{on } \gamma\times(0,\ T),\\ p_i(\cdot,0) &=& p_{0,i} & \text{in } \Omega_i, \end{array} \tag{6.1}$$

and in the fracture

$$\begin{array}{rcll} \operatorname{div}_\tau\boldsymbol{u}_\gamma &=& \sum\limits_{i=1}^{2}\boldsymbol{u}_i\cdot\boldsymbol{n}_{i|\gamma} & \text{in } \gamma\times(0,T),\\ \boldsymbol{u}_\gamma &=& -k_\gamma\delta\nabla_\tau p_\gamma & \text{in } \gamma\times(0,T),\\ p_\gamma &=& g_\gamma & \text{on } \partial\gamma\times(0,T),\\ p_\gamma(\cdot,0) &=& p_{0,\gamma} & \text{in } \gamma, \end{array} \tag{6.2}$$

where, for $i = 1,2,\gamma$, q_i is the source term, p_i the pressure, $\boldsymbol{u}_i$ the Darcy velocity, and k_i the time-independent hydraulic conductivity in the subdomains and in the fracture, respectively. The global Péclect (Pe) numbers on each subdomain and on the fracture are defined as

$$\mathrm{Pe}_i = \max_{K\in\mathscr{K}_{h,i}} \frac{H_i \max\limits_{(x,y)\in K}|\boldsymbol{u}_{i,K}(x,y)|}{d_i}, i=1,2,\ \ \mathrm{Pe}_\gamma = \max_{E\in\mathscr{E}_h^\gamma}\frac{H_\gamma \max\limits_{y\in E}|\boldsymbol{u}_{\gamma,E}(y)|}{d_\gamma}, \tag{6.3}$$

where $H_i, i=1,2,\gamma$ are the size of the subdomains Ω_i, respectively, and $\boldsymbol{u}_{i,K}, i = 1,2$ and $\boldsymbol{u}_{\gamma,E}$ are the restrictions of $\boldsymbol{u}_i$ and $\boldsymbol{u}_\gamma$ on the element K and the edge E, respectively. We also include in Table 1 the values of the Péclet numbers corresponding to the given parameters.

Boundary conditions are as follows: for the velocity, we assume that there is no horizontal flow on the lateral sides of the domain while a pressure drop constant in time is given between the top and bottom boundaries. At the top, the pressure is constant in space while at the bottom it is increasing slightly from the fracture toward the lateral sides. For the concentration, it is given, constant, at the top and bottom boundaries, vanishing at the top. On the lateral sides we assume that there is no exchange with the outside. We show in Figure 4 the snapshots of the concentration c and the flux field $\boldsymbol{\varphi}$ at the final time $T = 4$.

We fix $T = 1$ and verify numerically the optimal first-order error estimates (cf. Theorem 2) of the monolithic scheme (2.19). Table 2 reports the errors in the $L^2(0,T;\mathscr{O})$-norm (where $\mathscr{O}$ is either Ω_1,Ω_2, or γ) of the concentration and velocity with decreasing uniform spatial and time step sizes. These errors are computed by comparing with a reference solution on a fine mesh $h_{\text{ref}} = 1/256$ and fine time step $\Delta t_{\text{ref}} = T/512$. First-order convergence is observed in the subdomains as well as on the fracture for both the concentration and velocity.

Next we consider global-in-time DD methods to enforce nonconforming temporal discretizations. We examine the accuracy in time of both GTF-Schur and GTO-Schwarz where smaller time step

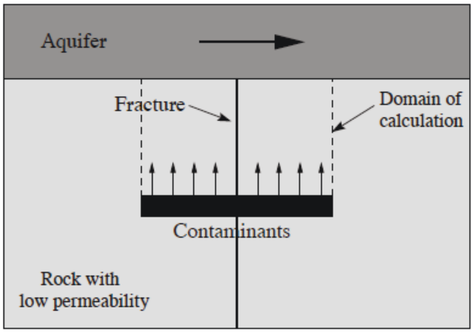

FIG. 3. A contaminant storage crossed by a fracture.

TABLE 1 *Physical parameters for the experiment shown in Figure 3.*

Parameters	*Subdomains*	*Fracture*
Hydraulic conductivity k_i	3.15×10^{-8}	10^{-7}
Molecular diffusion d_i	10^{-5}	3.15×10^{-4}
Porosity ϕ_i	0.05	0.1
Subdomains dimensions	10×10	-
Fracture width	-	1
Péclet numbers Pe_i	6.77e-02	3.00e-04

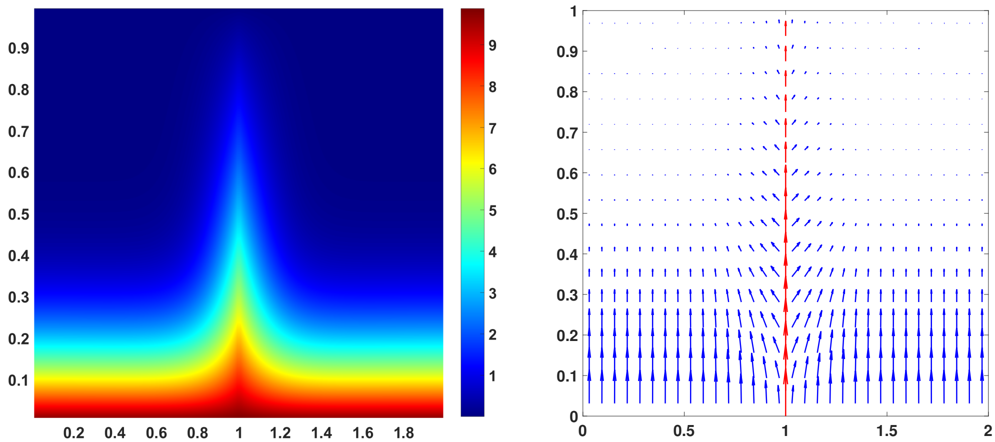

FIG. 4. Snapshots of the concentration c (right) and the flux field $\boldsymbol{\varphi}$ (left) at $T = 3$.

sizes are used on the fracture and larger ones in the subdomains. The space-time L^2 errors of the concentration and velocity are computed using the reference solution obtained from (2.19) on a fine time grid $dt_{\text{ref}} = T/512$ with $T = 1$. We report the errors for both methods in Tables 3 and 4; the corresponding convergence rates are shown in the square brackets. We first notice that the two DD methods preserve the first-order convergence in time in nonconforming time grids. By checking the columns corresponding to γ in Tables 3 and 4, we find that the errors on the fracture by GTF-Schur

TABLE 2 *Converge in both space and time for the monolithic upwind-mixed hybrid scheme with* conforming *time steps. The corresponding convergence rates are shown in square brackets.*

		Errors for concentration			Errors for velocity		
h	Δt	Ω_1	Ω_2	γ	Ω_1	Ω_2	γ
1/8	T/4	8.15e-01	8.15e-01	4.74e-01	2.39e-01	2.39e-01	1.59e-03
1/16	T/8	4.07e-01 [1.00]	4.07e-01 [1.00]	2.38e-01 [0.99]	1.12e-01 [1.09]	1.12e-01 [1.09]	7.90e-04 [1.01]
1/32	T/16	2.04e-01 [0.99]	2.04e-01 [0.99]	1.18e-01 [1.01]	5.16e-02 [1.12]	5.16e-02 [1.12]	3.90e-04 [1.02]
1/64	T/32	1.01e-01 [1.01]	1.01e-01 [1.01]	5.72e-02 [1.04]	2.20e-02 [1.23]	2.20e-02 [1.23]	1.91e-04 [1.03]

are approximately half the values of those by GTO-Schwarz (note that $\Delta t_f = \Delta t_i/2$). This behavior has also been observed in previous works [49, 50], only GTF-Schur preserves the accuracy in time with nonconforming temporal discretization.

TABLE 3 *Converge in time of the* concentration *with nonconforming time grids. The corresponding convergence rates are shown in square brackets.*

		GTO-Schwarz			GTF-Schur		
Δt_i	Δt_γ	Ω_1	Ω_2	γ	Ω_1	Ω_2	γ
T/4	T/8	1.14e-01	1.14e-01	1.47e-01	1.14e-01	1.14e-01	6.42e-02
T/8	T/16	5.80e-02 [0.97]	5.80e-02 [0.97]	7.31e-02 [1.01]	5.78e-02 [0.98]	5.78e-02 [0.98]	3.16e-02 [1.02]
T/16	T/32	2.90e-02 [1.00]	2.90e-02 [1.00]	3.60e-02 [1.02]	2.88e-02 [1.01]	2.88e-02 [1.01]	1.52e-02 [1.06]
T/32	T/64	1.41e-02 [1.04]	1.41e-02 [1.04]	1.74e-02 [1.05]	1.40e-02 [1.04]	1.40e-02 [1.04]	7.02e-03 [1.11]

TABLE 4 *Convergence in time of the* velocity *with nonconforming time grids. The corresponding convergence rates are shown in square brackets.*

		GTO-Schwarz			GTF-Schur		
Δt_i	Δt_γ	Ω_1	Ω_2	γ	Ω_1	Ω_2	γ
T/4	T/8	6.14e-04	6.14e-04	7.27e-04	6.14e-04	6.14e-04	3.58e-04
T/8	T/16	3.06e-04 [1.00]	3.06e-04 [1.00]	3.49e-04 [1.06]	3.04e-04 [1.01]	3.04e-04 [1.01]	1.71e-04 [1.07]
T/16	T/32	1.51e-04 [1.02]	1.51e-04 [1.02]	1.69e-04 [1.05]	1.50e-04 [1.02]	1.50e-04 [1.02]	8.15e-05 [1.07]
T/32	T/64	7.29e-05 [1.05]	7.29e-05 [1.05]	8.08e-05 [1.06]	7.24e-05 [1.05]	7.24e-05 [1.05]	3.77e-05 [1.11]

We now increase Péclet numbers and investigate the convergence of both DD methods with either conforming or nonconforming time grids. We vary the values of the hydraulic conductivity k_i, $i = 1, 2, \gamma$, while keeping other physical parameters as in Table 1. Four sets of Péclet numbers corresponding to different choices of k_i are shown in Table 5. Again, the final time is $T = 1$. We first consider the uniform time step $\Delta t = T/N$ in the fracture and in the subdomains, where $N = 32$. The convergence speed of GTF-Schur and GTO-Schwarz are illustrated via the relative residuals versus the number of subdomain solves as shown in Figure 5. We observe that both GTF-Schur and GTO-Schwarz exhibit nearly the same fast convergence speed. In addition, similarly to the results in [49, 50], they converge quickly without the need for preconditioners, which highlights the efficiency of both methods. Moreover, GTF-Schur and GTO-Schwarz are insensitive to the effect of the advection, which can be observed from the consistency of their convergence curves as the Péclet number increases. Such robustness with respect to the Péclet number is obtained due to the construction of the interface problem for GTF-Schur and the optimized parameters for GTO-Schwarz; the use of upwind operators does not affect this behavior of the proposed DD methods. We remark that when operator splitting is used [50], i.e., the advection is treated explicitly and the diffusion implicitly, the convergence of GTF-Schur and GTO-Schwarz is also independent of the Péclet number. However, unlike [50], here no CFL conditions are imposed on the time step size.

TABLE 5 *Parameters for different cases.*

Parameters					
k_i	Ω_1	$6.5e-06$	$6.5e-05$	$1.4e-04$	$2.4e-03$
	Ω_2	$6.5e-06$	$6.5e-05$	$1.4e-04$	$2.4e-03$
	Ω_f	$4.4e-02$	$4.4e-01$	$9e-01$	$9e-00$
Pe_i	Ω_1	≈ 0.45	≈ 4.45	≈ 9.6	≈ 165
	Ω_2	≈ 0.45	≈ 4.45	≈ 9.6	≈ 165
	Ω_f	≈ 4.4	≈ 44	≈ 91	≈ 907

Finally, we consider *nonconforming* time grids on the subdomains and on the fracture with different Péclet numbers in Table 5. Since we have the same diffusion coefficients in the subdomains, which are smaller than that in the fracture, we impose the same large time step in the subdomains and a smaller one in the fracture: $\Delta t_1 = \Delta t_2 = 2\Delta t_\gamma$. For this experiment, we fix $\Delta t_1 = \Delta t_2 = \Delta t = T/N$, $\Delta t_f = T/N_f$ where $N = 32$ and $N_f = 64$. Figure 6 shows the residual curves versus the number of subdomain solves with increasing Peclét numbers. From these curves, we deduce that the GTF-Schur and the GTO-Schwarz methods preserve their fast convergence speed and remain unaffected by the magnitudes of the advection when nonconforming temporal discretization is employed.

7. Conclusion

In this work, we have investigated both monolithic and decoupled numerical methods for the reduced fracture model of the linear advection-diffusion equation in a fractured porous medium. The Euler-implicit upwind-mixed hybrid finite element algorithm was first introduced to discretize the coupled system in space and time, in which a mixed finite element method with a hybridization technique is considered and Lagrange multipliers are used for the discretization of the advection terms. We proved the existence and uniqueness of the discrete solution, as well as optimal first-order convergence in both temporal and spatial errors of the monolithic solver. To accommodate different time steps on the fracture and on the subdomains, we then proposed two non-overlapping global-in-time DD methods, namely

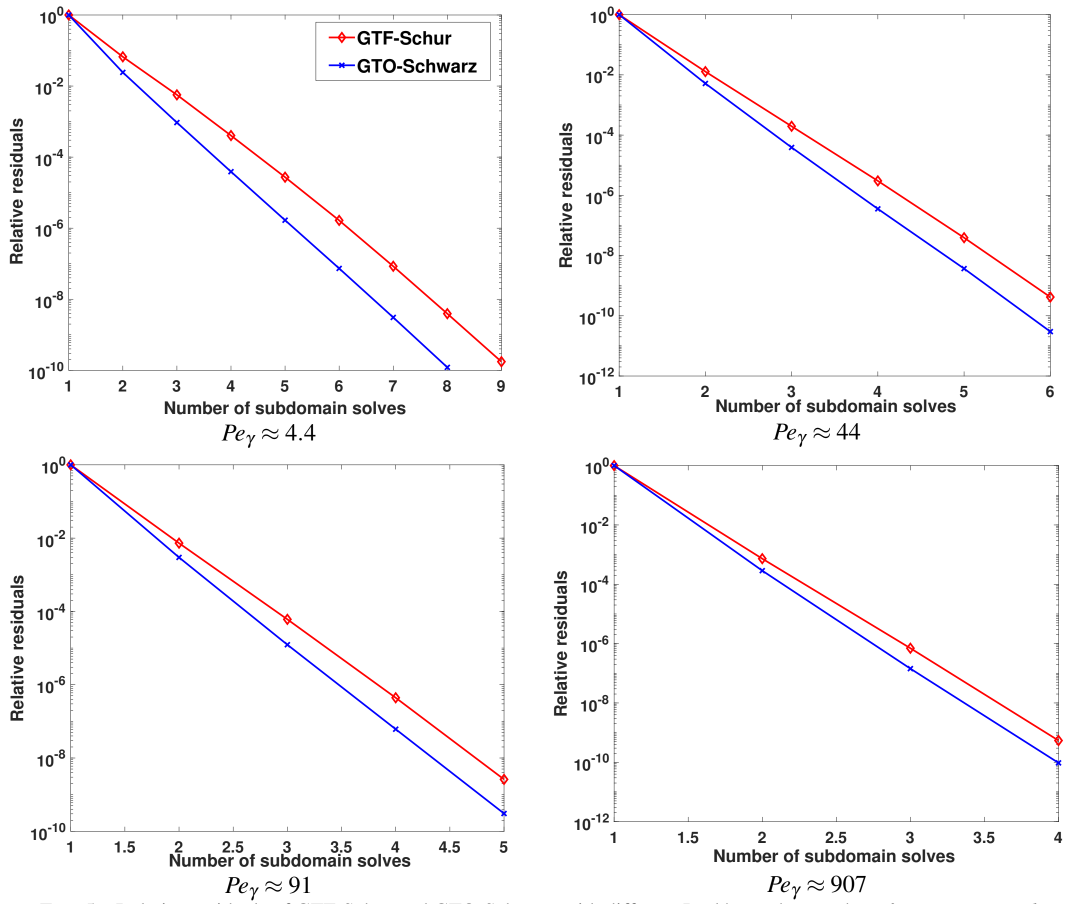

FIG. 5. Relative residuals of GTF-Schur and GTO-Schwarz with different Peclét numbers and *conforming time grid.*

GTF-Schur and GTO-Schwarz in the context of mixed hybrid finite elements. The convergence of GTO-Schwarz with conforming temporal discretization was also proved. Several numerical experiments were conducted to verify the accuracy of the monolithic solver and to compare the performance of the two DD methods with different Péclet numbers and with both conforming and nonconforming temporal discretizations. The results demonstrate that both GTF-Schur and GTO-Schwarz are capable of handling strongly advection-dominated problems as they maintain the same fast convergence speed regardless of the values of the Péclet numbers. Importantly, they achieved such fast convergence without applying any preconditioners. Moreover, the methods are fully implicit and have no CFL constraints on the time step size. Finally, GTF-Schur provided better accuracy in time on the fracture than GTO-Schwarz with nonconforming temporal discretization as the errors on the fracture obtained from GTF-Schur in such case were smaller than those of GTO-Schwarz. Thus, we conclude that among the two DD methods, GTF-Schur is the most efficient method in terms of accuracy and convergence speed. Future work includes extending the error estimates (3.17) to the case with both nonconforming temporal and spatial discretizations, proving the convergence of GTO-Schwarz with nonconforming discretization in time, and developing local time-stepping algorithms for multiphysics problems in fractured porous media.

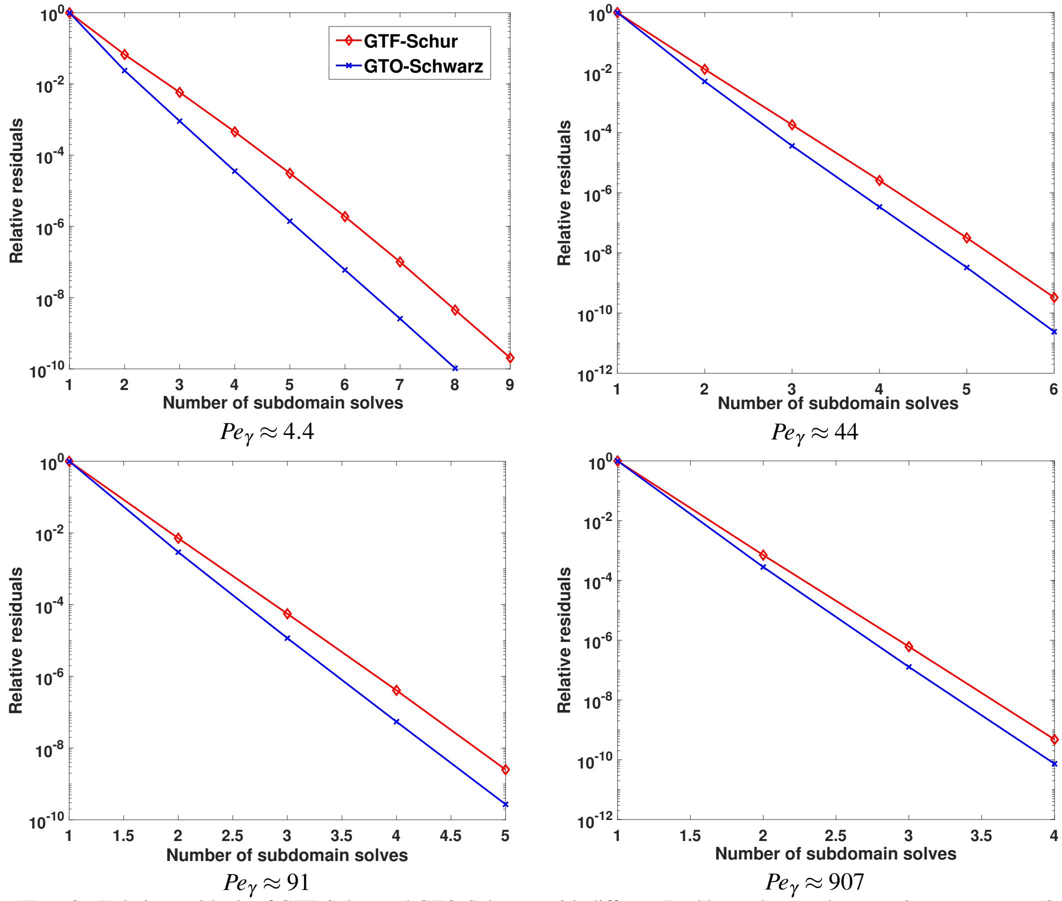

FIG. 6. Relative residuals of GTF-Schur and GTO-Schwarz with different Peclét numbers and *nonconforming time grid*.

Acknowledgements. This work is partially supported by the US National Science Foundation under grant numbers DMS-1912626.

A. Proof of Lemma 4

We now present the proof of Lemma 4. The proof follows a similar idea as in [19, Lemma 4.2]. For$i = 1,2$, let $\boldsymbol{v}_h = \left(\boldsymbol{v}_{h,1}, \boldsymbol{v}_{h,2}, 0\right) \in \widetilde{\boldsymbol{\Sigma}}_h$ be such that $\boldsymbol{v}_{h,j} = 0$ if $j \neq i$. By taking $\boldsymbol{v}_h$ as a test function in the first equation of (2.19), we obtain

$$
\begin{aligned}
&\left(\boldsymbol{D}_i^{-1}\boldsymbol{\varphi}_{h,i}^n, \boldsymbol{v}_{h,i}\right)_{\Omega_i} - \left(\operatorname{div} \boldsymbol{v}_{h,i}, c_{h,i}^n\right)_{\Omega_i} + \left\langle \boldsymbol{v}_{h,i}\cdot \boldsymbol{n}_{i|\gamma}, c_{h,\gamma}^n \right\rangle_\gamma + \sum_{K\in\mathscr{K}_{h,i}} \sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}} \left\langle \theta_{h,i}^n, \boldsymbol{v}_{h,i}\cdot\boldsymbol{n}_K \right\rangle_E \\
&- \sum_{K\in\mathscr{K}_{h,i}} \left(\sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}} u_{i,KE}\,\mathscr{U}_{i,KE}(c_{i,K}^n, \theta_{i,E}^n)(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE}, \boldsymbol{v}_{h,i})_K - \sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_h^\gamma}} u_{i,KE}\,\mathscr{U}_{i,KE}^\gamma(c_{i,K}^n, c_{\gamma,E}^n)(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE}, \boldsymbol{v}_{h,i})_K \right) \\
&= 0, \ \forall \boldsymbol{v}_{h,i} \in \widetilde{\boldsymbol{\Sigma}}_{h,i}.
\end{aligned} \tag{A.1}
$$

For any $K \in \mathscr{K}_{h,i}$, we divide K into two triangles T_1^K and T_2^K by drawing a diagonal E^K:

$$K = T_1^K \cup T_2^K \cup E^K, \;\; T_1^K \cap T_2^K = \emptyset, \;\; \partial T_1^K \cap \partial T_2^K = E^K.$$

Let $\mathscr{T}_{h,i} = \left\{T_1^K, T_2^K\right\}_{K\in\mathscr{K}_{h,i}}$ be a partition of Ω_i into triangles, and let $\tilde{\mathscr{E}}_{h,i} = \mathscr{E}_{h,i} \cup \left\{E^K\right\}_{K\in\mathscr{K}_{h,i}}$. Moreover, let $\mathscr{Q}^0_{h,i}$ be the L^2-projection from $L^2\left(\tilde{\mathscr{E}}_{h,i}\right)$ onto $\mathscr{P}^0\left(\tilde{\mathscr{E}}_{h,i}\right)$ where $\mathscr{P}^0\left(\tilde{\mathscr{E}}_{h,i}\right)$ is the space of piecewise constant functions on $\tilde{\mathscr{E}}_{h,i}$. We then denote a new element $\xi^n_{h,i} \in \mathscr{P}^0\left(\tilde{\mathscr{E}}_{h,i}\right)$ as follows

$$\xi^n_{h,i|E} = \xi^n_{i,E} := \begin{cases} \theta^n_{i,E}, \;\; \text{if } E \in \mathscr{E}_{h,i}, \\ c^n_{\gamma,E}, \;\; \text{if } E \in \mathscr{E}^\gamma_h, \\ \mathscr{Q}^0_{h,i} c^n_{i|E}, \;\; \text{if } E \in \left\{E^K\right\}_{K\in\mathscr{K}_{h,i}}. \end{cases} \tag{A.2}$$

We begin with providing the following estimate:

$$\begin{aligned} &\sum_{K\in\mathscr{K}_{h,i}} \sum_{\substack{E\subset\partial K \\ E\in\mathscr{E}_{h,i}}} |E|^2 (\xi^n_{i,E} - c^n_{i,K})^2 \\ &\quad \le C\left(\left\|c^n_i - c^n_{h,i}\right\|^2_{0,\Omega_i} + \left\|\boldsymbol{\varphi}^n_i - \boldsymbol{\varphi}^n_{h,i}\right\|^2_{0,\Omega_i} + h^2\left\|\boldsymbol{u}_i c^n{}_i - \boldsymbol{u}_{h,i} c^n_{h,i}\right\|^2_{0,\Omega_i} + h^2\left\|c^n_i\right\|^2_{1,\Omega_i}\right). \end{aligned} \tag{A.3}$$

By [8, Lemma 2.1], there exists unique elements $\tilde{c}^n_{h,i}, \hat{c}^n_{h,i}$ in the space $\mathscr{P}^{\mathrm{CR}}_1\left(\mathscr{T}_{h,i}\right)$ of linear Crouzeix–Raviart elements [21] by means of

$$\mathscr{Q}^0_{h,i}\tilde{c}^n_{h,i} = \xi^n_{h,i}, \quad \mathscr{Q}^0_{h,i}\hat{c}^n_i = \mathscr{Q}^0_{h,i} c^n_{h,i}. \tag{A.4}$$

Then, by standard arguments, it follows that

$$\left\|\hat{c}^n_{h,i} - c^n_i\right\|_{0,\Omega_i} \le Ch\left\|c^n_i\right\|_{1,\Omega_i}. \tag{A.5}$$

From (A.4), we have $\mathscr{Q}^0_{h,i}(\tilde{c}^n_{h,i} - \hat{c}^n_{h,i}) = \xi^n_{h,i} - \mathscr{Q}^0_{h,i} c^n_i$. Thus, it follows from [8, Lemma 2.1] and the definition of $\xi^n_{h,i}$ on $E \in \left\{E^K\right\}_{K\in\mathscr{K}_{h,i}}$ that for any $K \in \mathscr{K}_{h,i}$,

$$\begin{aligned} \left\|\tilde{c}^n_{h,i} - \hat{c}^n_{h,i}\right\|_{0,K} &\le \left\|\tilde{c}^n_{h,i} - \hat{c}^n_{h,i}\right\|_{0,H^K_1} + \left\|\tilde{c}^n_{h,i} - \hat{c}^n_{h,i}\right\|_{0,H^K_2} \\ &\le Ch_K^{1/2} \sum_{\substack{E\subset\partial H^K_1 \\ E\in\tilde{\mathscr{E}}_{h,i}}} \left\|\xi^n_{h,i} - \mathscr{Q}^0_{h,i} c^n_i\right\|_{0,E} + Ch_K^{1/2} \sum_{\substack{E\subset\partial H^K_2 \\ E\in\tilde{\mathscr{E}}_{h,i}}} \left\|\xi^n_{h,i} - \mathscr{Q}^0_{h,i} c^n_i\right\|_{0,E} \\ &\le Ch_K^{1/2} \sum_{\substack{E\subset\partial K \\ E\in\mathscr{E}_{h,i}}} \left\|\xi^n_{h,i} - \mathscr{Q}^0_{h,i} c^n_i\right\|_{0,E}. \end{aligned} \tag{A.6}$$

By combining (A.5)- (A.6) with the triangle inequality and Cauchy-Schwarz inequality, we find

$$\begin{aligned} \left\|\tilde{c}^n_{h,i} - c^n_{h,i}\right\|^2_{0,\Omega_i} &\le 3\left\|\tilde{c}^n_{h,i} - \hat{c}^n_h\right\|^2_{0,\Omega_i} + 3\left\|\hat{c}^n_{h,i} - c^n_i\right\|^2_{0,\Omega_i} + 3\left\|c^n_i - c^n_{h,i}\right\|^2_{0,\Omega_i} \\ &\le C\left(h\sum_{K\in\mathscr{K}_{h,i}} \sum_{\substack{E\subset\partial K \\ E\in\mathscr{E}_{h,i}}} \left\|\xi^n_{h,i} - \mathscr{Q}^0_{h,i} c^n_i\right\|^2_{0,E} + \left\|c^n_i - c^n_{h,i}\right\|^2_{0,\Omega_i} + h^2\left\|c^n_i\right\|^2_{0,\Omega_i}\right). \end{aligned} \tag{A.7}$$

We now use that for any piecewise linear polynomial $p_h \in \mathscr{P}_1(\mathscr{T}_{h,i})$, the following estimate holds [17, p. 112]:

$$\|p_h\|_{L^1(\partial H_j^K)} \leq C\|p_h\|_{0,H_j^K},\ j=1,2. \tag{A.8}$$

By applying (A.8) and the definition of $\tilde{c}_{h,i}^n$, we obtain

$$|E|(\xi_{i,E}^n - c_{i,K}^n) = \left\langle \tilde{c}_{h,i}^n - c_{i,K}^n, 1\right\rangle_E \leq C\left\|\tilde{c}_{h,i}^n - c_{i,K}^n\right\|_{L^1(\partial H_1^K)} \leq C\left\|\tilde{c}_{h,i}^n - c_{i,K}^n\right\|_{0,H_1^K} \leq \left\|\tilde{c}_{h,i}^n - c_{h,i}^n\right\|_{0,K}. \tag{A.9}$$

It follows from (A.7) and (A.9) that

$$\begin{aligned}\sum_{K\in\mathscr{K}_{h,i}} \sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}}} |E|^2(\xi_{i,E}^n - c_{i,K}^n)^2 &\leq 3\left\|\tilde{c}_{h,i}^n - c_{h,i}^n\right\|_{0,\Omega_i}^2 \\ &\leq C\left(h\sum_{K\in\mathscr{K}_{h,i}} \sum_{\substack{E\subset\partial K\\ E\in\mathscr{E}_{h,i}}} \left\|\xi_{h,i}^n - \mathscr{Q}_{h,i}^0 c_i^n\right\|_{0,E}^2 + \left\|c_i^n - c_{h,i}^n\right\|_{0,\Omega_i}^2 + h^2\|c_i^n\|_{0,\Omega_i}^2\right).\end{aligned} \tag{A.10}$$

Finally, we show that

$$\begin{aligned}\left\|\xi_{h,i}^n - \mathscr{Q}_{h,i}^0 c_i^n\right\|_{0,E} \leq C\Bigg(& h_K^{1/2}\left\|\boldsymbol{\varphi}_i^n - \boldsymbol{\varphi}_{h,i}^n\right\|_{0,K} + h_K^{1/2}\sum_{E'\subset\partial K}|E'||\xi_{i,E'}^n - c_{i,K}^n| \\ &+ h_K^{1/2}\left\|\boldsymbol{u}_i c_i^n - \boldsymbol{u}_{h,i}c_{h,i}^n\right\|_{0,K} + h_K^{-1/2}\left\|c_i^n - c_{h,i}^n\right\|_{0,K}\Bigg), \forall\, K\in\mathscr{K}_{h,i}, \forall\, E\in\mathscr{E}_{h,i}, E\subset\partial K.\end{aligned} \tag{A.11}$$

For any $K\in\mathscr{K}_{h,i}$ and $E\subset\partial K, E\in\mathscr{E}_{h,i}$, it follows from [8] that there exist a unique element $\boldsymbol{\tau}_{i,E}\in\widetilde{\boldsymbol{\Sigma}}_{h,i}$ such that $\operatorname{supp}(\boldsymbol{\tau})_{i,E}\subseteq K$ and

$$\boldsymbol{\tau}_{i,E}\cdot\boldsymbol{n}_E = \begin{cases} \xi_{h,i}^n - \mathscr{Q}_{h,i}^0 c_i^n, & \text{on } E,\\ 0, \text{ on } \partial K\backslash E.\end{cases}$$

It follows from a scaling argument [8] that

$$h_K\|\boldsymbol{\tau}_{i,E}\|_{1,K} + \|\boldsymbol{\tau}_{i,E}\|_{0,K} \leq Ch_K^{1/2}\left\|\xi_{h,i}^n - \mathscr{Q}_{h,i}^0 c_i^n\right\|_{0,E}. \tag{A.12}$$

By using $\boldsymbol{v}_{h,i} = \boldsymbol{\tau}_{i,E}$ in the first equation of (A.1), we obtain

$$\begin{aligned}&\left(\boldsymbol{D}_i^{-1}\boldsymbol{\varphi}_{h,i}^n, \boldsymbol{\tau}_{i,E}\right)_{\Omega_i} - \left(\operatorname{div}\boldsymbol{\tau}_{i,E}, c_{h,i}^n\right)_{\Omega_i} - \sum_{\substack{E'\subset\partial K\\ E'\in\mathscr{E}_{h,i}\backslash\mathscr{E}_h^\gamma}} u_{i,KE'}\mathscr{U}_{i,KE'}(c_{i,K}^n, \theta_{i,E'}^n)\left(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE'}, \boldsymbol{\tau}_{i,E}\right)_K \\ &\sum_{\substack{E'\subset\partial K\\ E'\in\mathscr{E}_h^\gamma}} u_{i,KE'}\mathscr{U}_{i,KE'}^\gamma(c_{i,K}^n, c_{\gamma,E'}^n)(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE'}, \boldsymbol{\tau}_{i,E})_K = -\left\langle \xi_{h,i}^n, \xi_{h,i}^n - \mathscr{Q}_{h,i}^0 c_i^n\right\rangle_E.\end{aligned} \tag{A.13}$$

From the relation $q_i^n = \boldsymbol{u}_i c_i^n - \boldsymbol{D}_i\nabla c_i^n$, by apply Green's formula, we obtain

$$\left(\boldsymbol{D}_i^{-1}\boldsymbol{\varphi}_i^n, \boldsymbol{\tau}_{i,E}\right)_{\Omega_i} - (\operatorname{div}\boldsymbol{\tau}_{i,E}, c_i^n)_{\Omega_i} - \left(\boldsymbol{D}_i^{-1}\boldsymbol{u}_i c_i^n, \boldsymbol{\tau}_{i,E}\right)_{\Omega_i} = -\left\langle c_i^n, \xi_{h,i}^n - \mathscr{Q}_{h,i}^0 c_i^n\right\rangle_E. \tag{A.14}$$

By subtracting (A.13) from (A.14) and using the definition of $\mathscr{Q}^0_{h,i}$, we find

$$\begin{aligned}
\left\|\xi^n_{h,i}-\mathscr{Q}^0_{h,i}c^n_i\right\|^2_{0,E} &= \left\langle \xi^n_{h,i}-c^n_i, \xi^n_{h,i}-\mathscr{Q}^0_{h,i}c^n_i\right\rangle_E \\
&= \left(\boldsymbol{D}_i^{-1}(\boldsymbol{\varphi}^n_i-\boldsymbol{\varphi}^n_{h,i}), \boldsymbol{\tau}_{i,E}\right)_{\Omega_i} - \left(\operatorname{div}\boldsymbol{\tau}_{i,E}, c^n_i-c^n_{h,i}\right)_{\Omega_i} - \left(\boldsymbol{D}_i^{-1}\boldsymbol{u}_i(c^n_i-c^n_{h,i}), \boldsymbol{\tau}_{i,E}\right)_{\Omega_i} \\
&- \sum_{\substack{E'\subset\partial K\\ E'\in\mathscr{E}_{h,i}\setminus\mathscr{E}_h^\gamma}} u_{i,KE'}\left(\mathscr{U}_{i,KE'}(c^n_{i,K},\theta^n_{i,E'})-c^n_{i,K}\right)\left(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE'},\boldsymbol{\tau}_{i,E}\right)_K \\
&- \sum_{\substack{E'\subset\partial K\\ E'\in\mathscr{E}_h^\gamma}} u_{i,KE'}\left(\mathscr{U}^\gamma_{i,KE'}(c^n_{i,K},c^n_{\gamma,E'})-c^n_{i,K}\right)\left(\boldsymbol{D}_i^{-1}\boldsymbol{w}_{i,KE'},\boldsymbol{\tau}_{i,E}\right)_K.
\end{aligned} \tag{A.15}$$

By applying the Cauchy-Schwarz inequality, we obtain from (A.15) that

$$\begin{aligned}
\left\|\xi^n_{h,i}-\mathscr{Q}^0_{h,i}c^n\right\|^2_{0,E} \le C\Bigg(&\left\|\boldsymbol{\varphi}^n_i-\boldsymbol{\varphi}^n_{h,i}\right\|_{0,K}\|\boldsymbol{\tau}_{i,E}\|_{0,K}+\left\|c^n_i-c^n_{h,i}\right\|_{0,K}\|\boldsymbol{\tau}_{i,E}\|_{1,K} \\
&+\sum_{E'\subset\partial K}|E'||\xi^n_{i,E'}-c^n_{i,K}|\,\|\boldsymbol{\tau}_{i,E}\|_{0,K}+\left\|\boldsymbol{D}_i^{-1}\boldsymbol{u}_i(c^n_i-c^n_{h,i})\right\|_{0,K}\|\boldsymbol{\tau}_{i,E}\|_{0,K}\Bigg), \\
&\qquad \forall\, K\in\mathscr{K}_{h,i}, \forall\, E\in\mathscr{E}_{h,i}, E\subset\partial K.
\end{aligned} \tag{A.16}$$

We then obtain (A.11) by combining (A.16) with (A.12) and applying the Cauchy-Schwarz inequality. By combining (A.10) with (A.11), we obtain

$$\begin{aligned}
\sum_{K\in\mathscr{K}_{h,i}}\sum_{E\subset\partial K}|E|^2(\xi^n_{i,E}-c^n_{i,K})^2 \le C\Bigg(& h^2\left\|\boldsymbol{\varphi}^n_i-\boldsymbol{\varphi}^n_{h,i}\right\|^2_{0,\Omega_i}+h^2\sum_{K\in\mathscr{K}_{h,i}}\sum_{E'\subset\partial K}|E'|^2\left(\xi^n_{i,E'}-c^n_{i,K}\right)^2 \\
&+h^2\left\|\boldsymbol{u}_ic^n_i-\boldsymbol{u}_{h,i}c^n_{h,i}\right\|^2_{0,\Omega_i}+\left\|c^n_i-c^n_{h,i}\right\|^2_{0,\Omega_i}+h^2\|c^n_i\|^2_{0,\Omega_i}\Bigg).
\end{aligned} \tag{A.17}$$

By choosing h small enough and pushing back the second term on the right-hand side of (A.17), we obtain (A.3). For the advection term on the fracture, since we can derive analogous result to Lemma 2 for the 1-dimensional case, we can follow the steps in [19] and arrive at

$$\begin{aligned}
\sum_{E\in\mathscr{E}_h^\gamma}\sum_{P\in\partial E}(\theta^n_{\gamma,P}-c^n_{\gamma,E})^2 \le C\Bigg(& h^2\left\|\boldsymbol{\varphi}^n_\gamma-\boldsymbol{\varphi}^n_{h,\gamma}\right\|^2_{0,\Omega_i}+h^2\left\|\boldsymbol{u}_\gamma c^n_\gamma-\boldsymbol{u}_{h,\gamma}c^n_{h,\gamma}\right\|^2_{0,\gamma} \\
&+\left\|c^n_\gamma-c^n_{h,\gamma}\right\|^2_{0,\gamma}+h^2\|c^n_\gamma\|^2_{0,\gamma}\Bigg).
\end{aligned} \tag{A.18}$$

Then (3.16) follows from the combination of (A.3) and (A.18). □

REFERENCES

1. Ahmed, E., Fumagalli, A., Budiša, A., Keilegavlen, E., Nordbotten, J. M. & Radu, F. A. (2021) Robust linear domain decomposition schemes for reduced non-linear fracture flow models, *SIAM J. Numer. Anal.,* **51** (1), 583 – 612.
2. Alboin, C., Jaffré, J., Roberts, J. E. & Serres, C. (1999) Domain decomposition for flow in fractured porous media, in Domain Decomposition Methods in Science and Engineering, C. H. Lai, P. E. Bjorstad, M. Cross, and O. B. Widlund, eds., *Domain Decomposition Press*, Bergen, Norway, 365 – 373.

3. Alboin, C., Jaffre, J., Roberts, J. E. & Serres, C. (2002) Modeling fractures as interfaces for flow and transport in porous media, in *Fluid flow and transport in porous media: mathematical and numerical treatment* (South Hadley, MA, 2001), vol. 295 of Contemp. Math., Amer. Math. Soc., Providence, RI, 13 – 24.
4. Ambartsumyan, I., Khattatov, E., Nguyen, T. & Yotov, I. (2019) Flow and transport in fractured poroelastic media, *GEM - Int. J. Geomath.*, **10**, 11.
5. Amir, L., Kern, M., Mghazli, Z. & Roberts, J. E. (2021) Intersecting fractures in porous media: mathematical and numerical analysis, *Appl. Anal.*, https: //doi.org/10.1080/00036811.2021.1981878.
6. Angot, P., Boyer, F. & Hubert, F. (2009) Asymptotic and numerical modelling of flows in fractured porous media, *M2AN Math. Model. Numer. Anal.* **43**(2), 239 – 275.
7. Arbogast, T., Cowsar, L. C., Wheeler, M.F. & Yotov, I. (2000) Mixed finite element methods on non-matching multiblock grids, *SIAM J. Numer. Anal.* **37**, 1295–1315.
8. Arnold, D. N. & Brezzi, F. (1985) Mixed and nonconforming finite element methods: Implementation, postprocessing and error estimates, *RAIRO Modél. Math. Anal. Numér.*, **19**, 7 – 32.
9. Bause, M. & Knabner, P. (2004) Computation of variably saturated subsurface flow by adaptive mixed hybrid finite element methods, *Adv. Water Resour.*, **27**, 565–81.
10. Bennequin, D., Gander, M. J., Gouarin, L. & Halpern, L. (2009) A homographic best approximation problem with application to optimized Schwarz waveform relaxation, *Math. Comp.* **78**(265), 185 – 223.
11. Blayo, E., Debreu, L. & Lemarié, F. (2009) Toward an optimized global-in-time Schwarz algorithm for diffusion equation with discontinuous and spatially variable coefficients. Part 1: the constant coefficients case, *Electron. Trans. Numer. Anal.* **40**, 170 – 186.
12. Boffi, D., Brezzi, F. & Fortin, M. (2013) *Mixed Finite elements methods and applications*, Springer, Heidelberg.
13. Boon, W. M., Nordbotten, J. M. & Yotov, I. (2018) Robust discretization of flow in fractured porous media, *SIAM J. Numer. Anal.* **56** (4), 2203–2233.
14. Brezzi, F., Douglas Jr., J. & Martini, L. D. (1985), Two families of mixed finite elements for second order elliptic problems, *Numer. Math.*, **47**, pp. 217 – 235.
15. Brezzi, F., Douglas, J., Fortin, M. & Marini, L. D. (1987) Efficient rectangular mixed finite elements in two and three space variables, *Math. Model. Numer. Anal.*, **21**, pp. 581 – 604.
16. Brezzi, F., Douglas Jr., J.,Durán, R. & Fortin, M. (1987) Mixed finite elements for second order elliptic problems in three variables, *Numer. Math.*, **51**, pp. 237 – 250.
17. Brezzi, F. & Fortin, M. (1991) *Mixed and hybrid finite element methods,* volume 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York.
18. Brunner, F., Radu, F.A., Bause, M. & Knabner, P. (2012) Optimal order convergence of a modified BDM_1 mixed finite element scheme for reactive transport in porous media, *Adv. Water Resources*, 35, 163–171.
19. Brunner, F., Radu, F. A. & Knarbner, P. (2014) Analysis of an upwind-mixed hybrid finite element method for transport problems, *SIAM J. Numer. Anal.*, vol **52** (1), 83 – 102.
20. Cowsar, L. C., Mandel, J. & Wheeler, M. F. (1995) Balancing domain decomposition for mixed finite elements, *Math. Comp.* **64**, 989 – 1015.
21. Crouzeix, M., Raviart, P. A. (1973), Conforming and nonconforming finite element methods for the stationary Stokes equations I, *R.A.I.R.O.*, vol. **7**, 33–76.
22. D'Angelo, C. & Scotti, A. (2012) A mixed finite element method for Darcy flow in fractured porous media with non-matching grids, *Math. Model. Anal.*, **46** (02), **245** 465–489.
23. Dawson, C. (1998) Analysis of an upwind-mixed finite element method for nonlinear contaminant transport equations, *SIAM J. Numer. Anal.*, **35**, 1709–1724.
24. Dawson, C. & Aizinger, V. (1999) Upwind-mixed methods for transport equations, *Comput. Geosci.*, **3**, 93–110.
25. Durán, R. G. (1998) Error analysis in L^p, $1 \leq p \leq \infty$, for mixed finite element methods for linear and quasi-linear elliptic problems, RAIRO Modél. Math. Anal. Numér., **22**, 371–387.
26. Durán, R. G. (2006) Mixed finite element methods, in *Mixed Finite Elements, Compatibility Conditions, and Applications*, 1 – 44.

27. Evans, L. C. (1998) *Partial differential equations*, Providence, RI: American Mathematical Society.
28. Frih, N., Martin, V., Roberts, J. E. & Saâda, A. (2010) Modeling fractures as interfaces with non-matching grids, *Comput. Geosci.*, **16** (4) , 1043 – 1060.
29. A. Fumagalli, A. Scotti, Numerical modeling of multiphase subsurface flow in the presence of fractures, *Commun. Appl. Ind. Math.* **3**, (2011) 1 – 23.
30. Gander, M. J., Halpern, L. & Nataf, F. (1999) Optimal convergence for overlapping and non-overlapping Schwarz waveform relaxation, in *Proceedings of the* 11*th International Conference on Domain Decomposition Methods*, C-H. Lai, P. Bjørstad, M. Cross, and O. Widlund, eds.,27 – 36.
31. Gander, M. J. & Halpern, L. (2007*a*) Optimized Schwarz waveform relaxation for advection reaction diffusion problems, *SIAM J. Numer. Anal.* **45**(2), 666 – 697.
32. Gander, M. J., Halpern, L. & Kern, M. (2007*b*) A Schwarz waveform relaxation method for advection-diffusion-reaction problems with continuous coefficients and non-matching grids, in *Domain decomposition methods in science and engineering XVI*, vol. 55 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 283 – 290.
33. Gander, M. J., Japhet, C., Maday, Y. & Nataf, F. (2007*c*) A new cement to glue nonconforming grids with Robin interface conditions: The finite element case, in *Domain Decomposition Methods in Science and Engineering, Lect. Notes Comput. Sci. Eng.* **40**, Springer, Berlin, 259 – 266.
34. Gander, M. J., Japhet, C. (2013) Algorithm 932: PANG: software for nonmatching grid projections in 2D and 3D with linear complexity, *ACM Trans. Math. Software* **40** (1), 1 – 25.
35. Gander, M. J., Kwok, F. & Mandal, B. C. (2016) Dirichlet-Neumann and Neumann-Neumann waveform relaxation algorithms for parabolic problems, *Electron, Trans. Numer. Anal.* **45**, 424 – 456.
36. Gander, M. J., Kwok, F. & Mandal, B. C. (2020) Dirichlet-Neumann waveform relaxation methods for parabolic and hyperbolic problems in multiple subdomains, *BIT Numerical Mathematics*, 1 – 35.
37. Gander, M. J, Hennicker, J. & Masson, R. (2021) Modeling and analysis of the coupling in discrete fracture matrix models, *SIAM J. Numer. Anal.* **59**(1), 195 – 218.
38. Gross, S., Olshanskii, M. A. & Reusken, A. (2015) A trace finite element method for a class of coupled bulk-interface transport problems, *ESAIM: Math. Model Numer. Anal.*, 49.5, 1303–1330
39. Halpern, L., Japhet, C. & Omnes, P. (2010) *Nonconforming in time domain decomposition methods for porous method applications*, in Proceedings of the 5th European Conference on Computational Fluid Dynamics ECCOMAS CFD, J. C. F. Pereira and A. Sequeira, eds., Lisbon, Portugal, 2010.
40. Halpern, L., Japhet, C. & Szeftel, J. (2012) Optimized Schwarz waveform relaxation and discontinuous Galerkin time stepping for heterogeneous problems, *SIAM J. Numer. Anal.* **50**(5), 2588 – 2611.
41. Hecht, F., Mghazli, Z., Naji, I. & Roberts, J. E. (2019) A residual *a posteriori* error estimators for a model for flow in porous media with fractures, *J. Sci. Comput.* **79**, 935 – 968.
42. Hoang, T.T.P. (2013*a*) Space-time domain decomposition methods for mixed formulations of flow and transport problems in porous media (Ph.D. thesis), University Paris 6, 2013.
43. Hoang, T. T. P., Jaffré, J., Japhet, C., Kern, M. & Roberts, J. E (2013*b*) Space-time domain decomposition methods for diffusion problems in mixed formulations, *SIAM J. Numer. Anal.* **51**(6),3532 – 3559.
44. Hoang, T. T. P., Japhet, C., Kern, M. & Roberts, J. E. (2016) Space-time domain decomposition for reduced fracture models in mixed formulation, *SIAM J. Numer. Anal.* **54**(1), 288 – 316.
45. Hoang, T. T. P., Japhet, C., Kern, M. & Roberts, J. E (2017) Space-time domain decomposition for advection-diffusion problems in mixed formulations, *Math. Comput. Simulat.* **137**, 366 – 389.
46. Hoang, T. T. P., Lee, H. (2021) A Global-in-time Domain Decomposition Method for the Coupled Nonlinear Stokes and Darcy Flows, *J Sci Comput* **87** (1), 1 – 22 .
47. Hoang, T. T. P. (2022) Fully implicit local time-stepping methods for advection-diffusion problems in mixed formulations, *Comput. Math. with Appl.* **118**, 248 – 264.
48. Hundsdorfer, W. & Verwer, J. (2010) *Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations*, Springer, Berlin, Heidelberg New York etc.
49. Huynh, P. T., Hoang,T. T. P. & Cao, Y. (2023) Fast and Accuracy-Preserving Domain Decomposition Methods for Reduced Fracture Models with Nonconforming Time Grids, *J. Sci. Comput.*, **96** (23).

50. Huynh, P. T., Hoang,T. T. P. & Cao, Y. (2023) Operator splitting and local time-stepping methods for transport problems in fractured porous media, *Commun. Comput. Phys.*, **34** (5), pp. 1215 – 1246.
51. Jaffré, J., Martin, V. & Roberts, J. E. (2005) Modeling Fractures and Barriers as Interfaces for Flow in Porous Media, *SIAM. J. Sci. Comput.* **26**, 1667 – 1691.
52. Jayadharan, M., Kern, M. Vohralík, M. & Yotov, I. (2023) A space-time multiscale mortar mixed finite element method for parabolic equations, **61** (2).
53. Kwok, F. (2014) Neumann-Neumann waveform relaxation for the time-dependent heat equation. In: *J. Erhel, M.J. Gander, L. Halpern, G. Pichot, T. Sassi, O.B. Widlund (eds.) Domain Decomposition in Science and Engineering XXI*, vol. 98, Springer-Verlag, 189 – 198.
54. List, F., Kumar, K., Pop, I. S. & Radu, F. A. (2020) Rigorous upscaling of unsaturated flow in fractured porous media, *SIAM J. Numer. Anal.* **52**(1), 239 – 276.
55. Mandel, J. & Brezina, M. (1996) Balancing doamin decomposition for problems with large jumps in coefficients, *Math. Comp.* **65**, 1387 – 1401.
56. Morales, F. & Showalter, R. E. (2010) The narrow fracture approximation by channeled flow, *J. Math. Anal. Appl.*, **365** (1), 320 – 331.
57. Morales, F. & Showalter, R. E. (2012) Interface approximation of Darcy flow in a narrow channel, *Math. Methods Appl. Sci.* **35**, 182 – 195.
58. Quarteroni, A. & Valli, A. (2008) *Numerical Approximation of Partial Differential Equations*, Springer, Berlin, Heidelberg.
59. Radu, F. A., Suciu, N., Hoffmann, J., Vogel, A., Kolditz, O., Park, C. -H. & Attinger, S. (2011) Accuracy of numerical simulations of contaminant transport in heterogeneous aquifers: A comparative study, *Adv. Water Resour.* **34**, 47 – 61.
60. Raviart, P. A. & Thomas, J. M. (1977) A mixed finite element method for 2nd order elliptic problems, *in Mathematical Aspects of the Finite Element Method, Lecture Notes in Math.* 606, Springer, New York, 292–315.
61. Robert, J. E & Thomas, J. M. (1991) Mixed and Hybrid Methods, in *Handbook of Numerical Analysis*, Vol. II, Handb. Numer. Anal., II, North–Holland, Amsterdam, 523–639.
62. Vohralik, M. (2007) A posteriori error estimates for lowest-order mixed finite element discretizations of convection-diffusion-reaction equations, *SIAM J. Numer. Anal.*, **45**, 1570–1599.