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Composing arbitrarily many $SU(N)$ fundamentals

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Abstract

We compute the multiplicity of the irreducible representations in the decomposition of the tensor product of an arbitrary number n of fundamental representations of $SU(N)$, and we identify a duality in the representation content of this decomposition. Our method utilizes the mapping of the representations of $SU(N)$ to the states of free fermions on the circle, and can be viewed as a random walk on a multidimensional lattice. We also derive the large-n limit and the response of the system to an external non-abelian magnetic field. These results can be used to study the phase properties of non-abelian ferromagnets and to take various scaling limits.

Contents

1 Introduction

Symmetries in physics and in mathematics are often encoded in unitary groups. In quantum mechanics, in particular, many physical systems realize representations of unitary groups. The canonical example is spin, an irreducible representation of the group $SU(2)$ (a double cover of the group of spatial rotations $SO(3)$), with isospin (also $SU(2)$) and elementary particle "flavor" symmetry ($SU(3)$) constituting more exotic examples. Systems consisting of several components, each carrying an irreducible representation of a symmetry group, transform under the same symmetry, in the direct sum of the representations of the components. The decomposition of the states of the full system into irreducible representations of the symmetry group becomes, then, a problem of physical relevance.

The group theoretical techniques of decomposing a direct sum of two irreducible representations (irreps) are well established [1]. In particular, the irrep content of the decomposition of the tensor product of irreps of $SU(N)$ has been studied in the mathematics literature [2, 3]. The result for the tensor product of fundamental irreps, in particular, can be derived from Schur-Weyl duality [4].[1] The existing derivations, however, use a rather heavily mathematical language and the results are in a somewhat implicit form. A derivation in a physically motivated language and in a way accessible to physicists was desirable. In [5] we undertook this task for the simplest case of $SU(2)$. Specifically, we computed the multiplicity of spin j irreps arising from composing n spins s (related results were also derived in [6, 8] and used in [7], and were further elaborated in [9]). We also derived explicit expressions for the multiplicities arising from spins corresponding to indistinguishable particles by distilling the totally symmetric (for bosons) or antisymmetric (for fermions) parts of the decomposition. We could recast the composition problem as random walks on a one-dimensional lattice, or equivalently generalized Dyck or Lukasiewicz paths [10] with appropriate boundary conditions. Even for the $SU(2)$ case the results are interesting, exhibiting nontrivial scaling properties in the large-n and large-s limits, especially for bosonic or fermionic spins. In [5] we provided the asymptotic expressions in these limits, and explored phase transitions occuring in a system of interacting spins, possibly coupled to an external magnetic field.

An interesting question is whether the general $SU(N)$ case manifests similar phenomena and if it has qualitatively different properties. Investigating it requires having explicit expressions for the multiplicities in the decomposition of the direct sum of an arbitrary number of $SU(N)$ irreps. Since representations of $SU(N)$ are in general labeled by $N-1$ nonnegative integers (the Young tableau row lengths), the corresponding general formulas for the multiplicities are in general more complicated and implicit. In the present work we simplify somewhat the problem by restricting to the direct sum of n fundamental representations of $SU(N)$. As we shall see, this problem still has a rich structure.

[1] We thank Jules Lamers for pointing out this connection.

Our motivation for carrying out this analysis is manifold. Firstly, we wished to present a complete and explicit solution to the problem. Moreover, we wanted to present an intuitive and pedagogical approach, building on physics connections and intuition. In this vein, we used the fact that this problem can be related to a system of N free quantum mechanical fermions on a circle. Finally, systems of many $SU(N)$ degrees of freedom arise in condensed matter and ultracold systems [11, 12], spin chains [13, 14], particle physics [15, 16], matrix models [17], etc., and we wanted to pave the road for possible physical applications.

In this paper we present a full solution to the problem of decomposing the direct sum of an arbitrary number of $SU(N)$ irreps and apply it to the case of fundamental irreps, deriving explicit formulae for the representation content of the decomposition. The example of $SU(3)$ is analyzed in some detail. We point out a random walk connection, this time on a $N-1$-dimensional lattice, and identify a duality relation in the distribution of irreps. We also consider the limit of a large number n of fundamentals, as a prelude to more general and intricate limiting cases. Finally, we calculate the response of the system to coupling it with a nonabelian magnetic field, which is relevant in the study of the thermodynamics of interacting $SU(N)$ systems. The structure and phase transitions of the thermodynamic limit of such systems are treated in [20], and nontrivial scaling limits involving both N and n taken large will be investigated in forthcoming publications.

We present our results in the upcoming sections, hinting at additional physical applications and possible extensions of our analysis in the conclusions.

2 Fermion representation of $SU(N)$ group theory

We review here the description of irreducible representations (irreps) of $SU(N)$ as N-fermion energy eigenstates on the circle [18] and the corresponding composition rules in the fermion picture. This representation affords a conceptual standpoint that is well-suited to our considerations and will be useful throughout the rest of the paper. It also leads naturally to a duality relation between groups and representations.

2.1 Quantum mechanics on the $U(N)$ manifold

The correspondence of irreps of $SU(N)$ and N-fermion energy eigenstates can be most naturally established by considering the action of a particle moving freely on the group manifold $U(N)$ [19]. The Lagrangian is the kinetic energy of the particle

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}\left(U^{-1}\dot{U}\right)^2 , \tag{2.1}$$

where overdot signifies time derivative. This is essentially a unitary matrix model.

One approach to find the energy eigenstates of the corresponding Hamiltonian is to note that, upon defining momenta and quantizing, the Hermitian matrix operators $L = i\dot{U}U^{-1}$ and $R = -iU^{-1}\dot{U} = -U^{-1}LU$ satisfy the $U(N)$ algebra and mutually commute

$$[L_{ij}, L_{kl}] = \mathrm{i}L_{il}\delta_{kj} - \mathrm{i}L_{kj}\delta_{il} , \quad [R_{ij}, R_{kl}] = \mathrm{i}R_{il}\delta_{kj} - \mathrm{i}R_{kj}\delta_{il} , \quad [L_{ij}, R_{kl}] = 0 . \tag{2.2}$$

Left and right multiplications of U by unitary matrices are generated by L and R, respectively, which conforms with their commutation relations. L and R have equal and opposite $U(1)$ parts $\sum_k L_{kk} = -\sum_k R_{kk}$ and share a common quadratic Casimir. They are both constants of motion, and the Hamiltonian is their second Casimir. Energy eigenstates are matrix elements of U in arbitrary $U(N)$ irreps r: $r_{ab}(U)$, $a, b = 1, \ldots, d_r$, transforming in the $r \otimes \bar{r}$ representation under $L \otimes R$, with energy eigenvalue $C_2(r)$ and degeneracy d_r^2.

Now impose the additional first-class constraint $L + R = 0$, which commutes with the Hamiltonian. It amounts to choosing states that are invariant under the combined left and right unitary rotations $U \to VUV^{-1}$ generated by $L + R$, that is, under unitary conjugations. Energy eigenstates become the conjugation-invariant linear combinations $\sum_a r_{aa}(U) = \mathrm{Tr}_r(U) = \chi_r(U)$, the latter being the character of the irrep. We recover the same spectrum, with states corresponding to irreps r of $SU(N)$, but now with degeneracy one due to the trace.

The condition $L + R = 0$, or, classically, the vanishing of the matrix commutator $[U, \dot{U}] = 0$, implies (using the equation of motion for U) that $U(t)$ and $U(0)$ commute as matrices. Hence, $U(t)$ can be diagonalized by a time-independent unitary conjugation. Implementing this at the level of the Lagrangian by writing

$$U(t) = V\,\mathrm{diag}\{z_j\}\,V^{-1}\,, \quad z_j := e^{ix_j}\,, \quad x_j \sim x_j + 2\pi\,, \tag{2.3}$$

we see that (2.1) becomes the Lagrangian of N free particles on the unit circle with coordinates x_j. Since exchanging the eigenvalues is a special case of unitary conjugation, states $\phi(x_1, \ldots, x_N)$ are invariant under exchange of the x_j and are, in principle, bosonic. However, the change of variables from U to x_j introduces the standard measure, equal to the square of the Vandermonde determinant Δ^2. In terms of the exponential variables $\mathbf{z}$ (boldfaced $\mathbf{z}$ stands for the full set of variable $z_1, \ldots, z_N$, and similarly for other sets of N quantities), $\Delta(\mathbf{z})$ is given by

$$\Delta(\mathbf{z}) = (z_1 \cdots z_N)^{-\frac{N-1}{2}} \begin{vmatrix} z_1^{N-1} & z_1^{N-2} & \cdots & z_1 & 1 \\ z_2^{N-1} & z_2^{N-2} & \cdots & z_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_N^{N-1} & z_N^{N-2} & \cdots & z_N & 1 \end{vmatrix} = \prod_{i>j=1}^{N} 2\sin\frac{x_i - x_j}{2}\,. \tag{2.4}$$

Upon incorporating one factor $\Delta(\mathbf{z})$ into the wavefunction

$$\psi(\mathbf{z}) = \Delta(\mathbf{z})\,\phi(\mathbf{z})\,, \tag{2.5}$$

the Hamiltonian becomes the standard free N-particle Hamiltonian on the x_j (compare with the transformation $u(r) = r\phi(r)$ in the Schrödinger equation in spherical coordinates). Because of the prefactor $\Delta(\mathbf{z})$ the states $\psi(\mathbf{z})$ are antisymmetric upon interchanging any two of the x_i. This establishes the correspondence between the singlet sector of the model (2.1) and free fermions on the circle, and thus of irreps of $SU(N)$ and free fermion energy eigenstates. In particular, the fermion energy eigenstate corresponding to an irrep r and energy $E = C_2(r)$ is

$$\psi_r(\mathbf{z}) = \Delta(\mathbf{z})\,\chi_r(\mathbf{z})\,, \tag{2.6}$$

with $\chi_r(\mathbf{z}) = \mathrm{Tr}\, r(U)$ the character of irrep r in terms of the eigenvalues z_j of U.

2.2 Correspondence with Young tableaux

It will be useful to establish the correspondence of fermion states and Young tableaux of $SU(N)$. The single-particle spectrum on the circle consists of discrete momentum eigenstates with eigenvalue $k = 0, \pm 1, \pm 2, \dots$ and energy $E_k = k^2/2$. An N-fermion energy eigenstate corresponds to filling n of the single-particle states with fermions. Call $k_1 > k_2 > \cdots > k_N$ the momenta of these states in decreasing order (see figure 1):

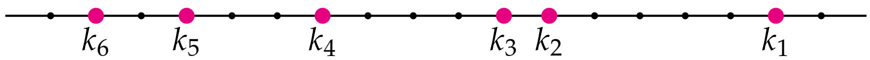

Figure 1: The state represents an $SU(6)$ irrep with $\ell_1 = 10$, $\ell_2 = \ell_3 = 6$, $\ell_4 = 3$, $\ell_5 = 1$. The corresponding Young tableau has five lines with $10, 6, 6, 3$ and 1 box, respectively.

The (unnormalized) wavefunction corresponding to this state is given by the Slater determinant

$$\psi_{\mathbf{k}}(\mathbf{z}) = \begin{vmatrix} z_1^{k_1} & z_1^{k_2} & \cdots & z_1^{k_{N-1}} & z_1^{k_N} \\ z_2^{k_1} & z_2^{k_2} & \cdots & z_2^{k_{N-1}} & z_2^{k_N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_N^{k_1} & z_N^{k_2} & \cdots & z_N^{k_{N-1}} & z_N^{k_N} \end{vmatrix} . \tag{2.7}$$

The total momentum of the fermions $k = k_1 + \cdots + k_n$ corresponds to the states picking up a phase e^{ick} upon the shift $x_j \to x_j + c$, that is, upon $U \to e^{ic}U$. It thus represents the $U(1)$ charge of the state. We may shift all momenta by a constant, changing k and the $U(1)$ charge without affecting the $SU(N)$ part of the states, which can then be labeled by the $N-1$ shift-invariant integers $k_1 - k_N > k_2 - k_N > \cdots > k_{N-1} - k_N > 0$. Alternatively, we can neutralize the $U(1)$ charge by introducing the prefactor $(z_1 \dots z_N)^{-\sum_i k_i/N}$ in (2.7) . Note that with this prefactor (2.7) maps to (2.4) for the singlet representation for which $k_i = N - i\,, i = 1, \dots, N$. The final step involves expressing $k_j - k_N$ in terms of new "bosonic" variables ℓ_j as

$$\ell_j = k_j - k_N + j - N\,, \qquad \ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{N-1} \geqslant 0\,. \tag{2.8}$$

The non-negative, ordered integers ℓ_j represent the lengh of rows $j = 1, 2, \dots, N-1$ of the Young tableau of the irrep corresponding to the fermionic state. The condition $\ell_N = 0$ is the statement that Young tableau columns of length N give a singlet, contributing at most to the $U(1)$ charge, and are eliminated. Fig. 1 depicts an example of a fermionic state equivalent to a Young tableau with five rows. As stated, the total momentum and energy of the fermion state map to the $U(1)$ charge and the quadratic Casimir of the representation.

The transition from k_j to ℓ_j is, in fact, bosonization, in the sense that two or more ℓ_i's can be equal. For reference, we also record the different parametrization

$$m_j = \ell_j - \ell_{j+1} = k_j - k_{j+1} - 1 \geqslant 0 \,, \qquad j = 1, 2, \ldots, N-1 \,. \tag{2.9}$$

The non-negative integers m_j are unrestricted and thus provide a "field theoretical" bosonization of the system. They can be thought of as corresponding to excitations in the first $N-1$ positive modes j of a second quantized boson field $\phi(x)$.

3 Composition of representations

We will now present the method of composing irreps and derive explicit expressions for the case of composition of fundamentals. We will also work out explicitly some examples of low rank groups.

3.1 General method

The composition of representations, and their decomposition into irreps, becomes particularly convenient in the fermion picture.

Consider the direct product $r_1 \times r_2$ of two (possibly reducible) representations r_1 and r_2. The basic relation

$$\mathrm{Tr}\,(r_1 \times r_2)(U) = \mathrm{Tr}\, r_1(U)\, \mathrm{Tr}\, r_2(U) = \chi_{r_1}(U)\, \chi_{r_2}(U) \,, \tag{3.1}$$

implies, through (2.5 and 2.6), that their corresponding fermion states are related as

$$\psi_{r_1 \times r_2}(\mathbf{z}) = \Delta(\mathbf{z}) \chi_{r_1}(\mathbf{z}) \chi_{r_2}(\mathbf{z}) = \frac{\psi_{r_1}(\mathbf{z})\, \psi_{r_2}(\mathbf{z})}{\Delta(\mathbf{z})} = \psi_{r_1}(\mathbf{z})\, \chi_{r_2}(\mathbf{z}) = \psi_{r_2}(\mathbf{z})\, \chi_{r_1}(\mathbf{z}) \,. \tag{3.2}$$

The advantage of the fermionic representation is that from the wavefunction $\psi(\mathbf{z})$ we can simply read off the irreducible components by expanding in monomials $z_1^{k_1} \cdots z_N^{k_N}$ and mapping them to Young tableaux. This will be demonstrated in detail in the upcoming sections for the composition of a large number of irreps. We record here the characters for the fundamental f, doubly symmetric (a single row with two boxes) s, doubly antisymmetric (two rows with one box each) a, and adjoint ad irreps

$$
\begin{aligned}
\chi_f(\mathbf{z}) &= \sum_{i=1}^{N} z_i \,, \\
\chi_s(\mathbf{z}) &= \sum_{i=1}^{N} z_i^2 + \sum_{j>i=1}^{N} z_i z_j = \frac{1}{2}\Big(\sum_{i=1}^{N} z_i\Big)^2 + \frac{1}{2}\sum_{i=1}^{N} z_i^2 \,, \\
\chi_a(\mathbf{z}) &= \sum_{j>i=1}^{N} z_i z_j = \frac{1}{2}\Big(\sum_{i=1}^{N} z_i\Big)^2 - \frac{1}{2}\sum_{i=1}^{N} z_i^2 \,, \\
\chi_{ad}(\mathbf{z}) &= \sum_{i,j=1}^{N} z_i^{-1} z_j - 1 = \Big(\sum_{i=1}^{N} z_i^{-1}\Big)\Big(\sum_{i=1}^{N} z_i\Big) - 1 \,.
\end{aligned}
\tag{3.3}
$$

To obtain the above note that the singlet corresponds to $k_i = N - i$, $i = 1, \ldots, N$. For the fundamental representation we increase k_1 by one, leaving the other k_i's intact. For the symmetric representation we change k_1 by two and for the antisymmetric representation we change k_1 as well as k_2 by unity. Then, from (2.6), (2.7) and (2.4) we obtain the first three lines above. The conjugate to the fundamental representation $\bar{\chi}_f$ is just χ_f with the z_i replaced by their inverses. By decomposing $\chi_f \bar{\chi}_f = \chi_{ad} + 1$ we obtain χ_{ad} as above. These will constitute basic building blocks when we examine the product of a large number of such irreps.

3.2 Composition of fundamentals

The composition of two irreps $r \otimes f$, one of which, specifically f, is the fundamental, is particularly intuitive in the fermionic picture. It corresponds to exciting one of the fermions in the state representing the original irrep by one unit of momentum. This gives, in principle, N resulting states, according to which of the N fermions is excited. However, excitations that would make two fermions occupy the same momentum level are impossible.

It is easy to see that this reproduces the usual rules for Young tableau composition, adding a single box. Adding one unit to k_i amounts to increasing $\ell_i = k_i - k_N + i - N$ by one, except when $k_i = k_{i-1} - 1$, that is, $\ell_i = \ell_{i-1}$, in which case this is not possible. Further, exciting the last fermion k_N by 1 (when this is possible) amounts to subtracting 1 from all ℓ_j, that is, removing a column of length N from the Young tableau.

The fermion wavefunction of the state corresponding to $r \otimes f$ is, using (3.2) and (3.3), simply

$$\psi_{r\otimes f}(\mathbf{z}) = \psi_r(\mathbf{z})\chi_f(\mathbf{z}) = \psi_r(\mathbf{z}) \sum_{i=1}^{N} z_i \,. \tag{3.4}$$

This can be used to obtain the fermionic state corresponding to the composition of several (n in number) fundamental irreps f. The original, singlet state is simply $\Delta(\mathbf{z})$ and an iteration of the above formula yields

$$\psi_{N,n}(\mathbf{z}) = \Delta(\mathbf{z})\Big(\sum_{i=1}^{N} z_i\Big)^n \,. \tag{3.5}$$

The irreducible components of the resulting representation can be "read off" from the terms of $\psi_{N,n}(\mathbf{z})$ expanded as a polnomial in z_j: the coefficient of the terms $z_1^{k_1}\cdots z_N^{k_N}$ such that $k_1 > k_2 > \cdots > k_N$, represents the multiplicity of the irrep corresponding to Young tableau lengths $\ell_j = k_j - k_N + j - N$, according to (2.8). The remaining terms, making $\psi_{N,n}$ properly fermionic, can be ignored.

3.3 Elementary example: Fundamental $SU(2)$ irreps

As a simple example, we can apply the above method to the case of $SU(2)$ and make contact with previous results. In this case (3.5) becomes simply

$$\psi_{2,n}(z_1, z_2) = (z_1 - z_2)(z_1 + z_2)^n \,. \tag{3.6}$$

Terms $z_1^{k_1} z_2^{k_2}$ with $k_1 > k_2$ can be isolated by writing $z_1 = zx$, $z_2 = zx^{-1}$, which gives $z_1^{k_1} z_2^{k_2} = z^{k_1+k_2} x^{k_1-k_2}$. So powers of z count total $U(1)$ charge, and focusing on terms x^k for $k > 0$ gives $k_1 - k_2 = k$. Clearly the single Young tableau row length is $\ell = k_1 - k_2 - 1 = k - 1 = 2j$, with j being the spin of the irrep. We obtain

$$\begin{aligned} \psi_{2,n} &= z^{n+1}\left(x - x^{-1}\right)\left(x + x^{-1}\right)^n \\ &= z^{n+1} \sum_{k=0}^{n} \binom{n}{k} \left(x^{2k-n+1} - x^{2k-n-1}\right) \\ &= z^{n+1} \sum_{j=-n/2-1}^{n/2} \frac{n!\,(2j+1)}{\left(\frac{n}{2}-j\right)!\,\left(\frac{n}{2}+j+1\right)!}\, x^{2j+1} \,, \end{aligned} \tag{3.7}$$

with the sum running in integer decrements from $n/2$. The coefficient of the term $x^k = x^{\ell+1} = x^{2j+1}$ for $j \geqslant 0$

$$d_{n,j} = \frac{n!\,(2j+1)}{\left(\frac{n}{2}-j\right)!\,\left(\frac{n}{2}+j+1\right)!}\,, \tag{3.8}$$

is the multiplicity of spin j in the decomposition, thus reproducing the result of [5].[2]

A direct interpretation of (3.7) is obtained by focusing on the factor $(x+x^{-1})^n$. Its expansion in x^k produces the number of states $D_{n,k}$ at spin $S_z = k/2$. Irreps with total spin $j \geqslant k/2$ contribute one state at $S_z = k/2$, so we can isolate the number of irreps at $j = k/2$ by subtracting the contribution from $j \geqslant k/2+1$, that is, $D_{n,k+2}$. Given that $\ell = 2j$, we have

$$d_{n,\ell} = D_{n,\ell} - D_{n,\ell+2}\,. \tag{3.9}$$

which is precisely what the prefactor $x - x^{-1}$ is achieving, upon identifying $k = \ell+1$.

The large-n, large-j limit is interesting, and it is instructive to compute it in two alternative ways. For $n \gg 1$ and j up to $\mathcal{O}(\sqrt{n})$, Stirling's formula

$$n! \simeq \sqrt{2\pi}\, e^{-n} n^{n+\frac{1}{2}}\,, \qquad n \gg 1\,. \tag{3.10}$$

applied to (3.8) yields

$$d_{n,j} \simeq \frac{2^n}{\sqrt{2\pi}}\left(\frac{4}{n}\right)^{3/2} j\, e^{-2j^2/n}\,, \tag{3.11}$$

which coincides, for $j \gg 1$, with the result of [8,5]. Note that the distribution (3.11), for fixed n, has a maximum at $j = \sqrt{n}/2$ with a value of order $\mathcal{O}\big(2^n/n\big)$.

An alternative method is based on random walks, leading to a "diffusion process," which does not rely on the exact formula for $d_{n,j}$ and is generalizable to higher groups. We focus on the factor $(x+x^{-1})^n$ in (3.7) leading to $D_{n,k}$. Each additional composition with a spin $\frac{1}{2}$ shifts the value of k by either $+1$ or -1 and similarly the corresponding values for ℓ. Explicitly,

$$D_{n+1,k} = D_{n,k+1} + D_{n,k-1}\,. \tag{3.12}$$

In our case $D_{n,k} = \begin{pmatrix} n \\ n/2+k/2 \end{pmatrix}$ which indeed satisfies (3.12).

We will view (3.12) as a discrete evolution process and, for large n and k, we will

[2]One may readily check that indeed $\sum\limits_{j=j_{\min}}^{n/2} (2j+1)d_{n,j} = 2^n$, where $j_{\min}$ equals 0 or $1/2$ depending on whether or not n is even or odd.

take the continuum limit. To do this, we set $D_{n,k}$ to a continuous function $D(n,k)$ end extract an explicit scaling factor 2^n to account for the doubling of states per extra added spin

$$D(n,k) = 2^n F(n,k) \, , \quad \sum_k F(n,k) \simeq \int_{-\infty}^{\infty} dk F(n,k) = 1 \, . \tag{3.13}$$

Then, a Taylor expansion of $F(n,k)$ in (3.12) to leading orders in n,k implies that this obeys the heat equation

$$2\,\partial_n F = \partial_k^2 F \, . \tag{3.14}$$

Note that the extra factor 2^n in (3.13) ensures that there is a smooth continuum limit of (3.12), and $F(n,k)$ becomes a continuous distribution. The solution of the above equation satisfying the initial condition $D_{0,k} = \delta_{0,k}$, that is, $F(0,k) = \delta(k)$, is given by

$$F(n,k) = \frac{1}{\sqrt{2\pi n}} e^{-k^2/(2n)} \, , \tag{3.15}$$

for $n \geqslant 0$ and $k \in (-\infty, \infty)$, and thus

$$D(n,k) = \frac{2^n}{\sqrt{2\pi n}} e^{-k^2/(2n)} \, . \tag{3.16}$$

Then (3.9) becomes in this limit

$$d(n,\ell) \simeq -2\partial_\ell D(n,\ell) = \frac{2^{n+1}}{\sqrt{2\pi n}} \frac{\ell}{n} e^{-\ell^2/(2n)} \, . \tag{3.17}$$

Finally, changing variable to $j = \ell/2$ and taking into account that in the continuum limit $d_{n,j}$ becomes a distribution, so that $d(n,j)dj = d(n,\ell)d\ell = 2d(n,\ell = 2j)dj$, giving an extra factor of 2, we reproduce (3.11). This implies the proper normalization

$$\int_0^\infty dj\,(2j+1)\,d(n,j) \simeq \int_0^\infty dj\,2j\,d(n,j) = 2^n \tag{3.18}$$

where now the integration is over physical (positive) values of j, with the multiplicity $2j+1 \simeq 2j$ (valid for large j) included.

3.4 Fundamental $SU(3)$ irreps

A more nontrivial case is that of the composition of n fundamental $SU(3)$ irreps. For $SU(3)$, the fermion state (3.5) becomes

$$\psi_{3,n} = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)(z_1 + z_2 + z_3)^n \,. \tag{3.19}$$

To isolate terms $z_1^{k_1} z_2^{k_2} z_3^{k_3}$ with $k_1 > k_2 > k_3 = 0$ we adopt the parametrization

$$z_1 = zx\,, \quad z_2 = zy\,, \quad z_3 = \frac{z}{xy}\,, \tag{3.20}$$

from which

$$z_1^{k_1} z_2^{k_2} z_3^{k_3} = x^{k_1 - k_3} y^{k_2 - k_3} z^{k_1 + k_2 + k_3} = x^{\ell_1 + 2} y^{\ell_2 + 1} z^{k_1 + k_2 + k_3} \tag{3.21}$$

and focus on terms in the wavefunction with positive, ordered powers of x and y. Clearly z counts the $U(1)$ charge, so we can put $z = 1$. We also divide the wavefunction by $x^2 y$ to make its terms proportional to $x^{\ell_1} y^{\ell_2}$. The reduced form of the wavefunction becomes

$$\Psi_n = x^{-2} y^{-1} (x - y) \left(x - \frac{1}{xy}\right) \left(y - \frac{1}{xy}\right) \left(x + y + \frac{1}{xy}\right)^n \,. \tag{3.22}$$

The generating function for the irrep content of the product of n fundamentals of $SU(3)$ can then be expressed as

$$\zeta_n(x,y) = \sum_{\ell_1 \geqslant \ell_2} d_{n;\ell_1,\ell_2} \, x^{\ell_1} y^{\ell_2} = \frac{(x-y)(x^2y-1)(xy^2-1)}{x^4 y^3} \left(x + y + \frac{1}{xy}\right)^n \Bigg|_+ \,, \tag{3.23}$$

with $d_{n;\ell_1,\ell_2}$ the multiplicity of the irrep $\{\ell_1, \ell_2\}$ in the product, and $|_+$ denoting that only positive, ordered powers $x^{\ell_1} y^{\ell_2}$ are kept. As a demonstration, we list the nonvanishing $d_{n;\ell_1,\ell_2}$ for the first few values of n

$$\begin{aligned}
\zeta_0 &= 1 & \quad & d_{0;00} = 1 \\
\zeta_1 &= x & & d_{1;10} = 1 \\
\zeta_2 &= x^2 + xy & & d_{2;20} = d_{2;11} = 1 \\
\zeta_3 &= 1 + x^3 + 2x^2 y & & d_{3;00} = d_{3;30} = 1\,,\ d_{3;21} = 2 \\
\zeta_4 &= 3x + x^4 + 3x^3 y + 2x^2 y^2 & & d_{4;10} = d_{4;31} = 3\,,\ d_{4;22} = 2\,,\ d_{4;40} = 1\,,
\end{aligned} \tag{3.24}$$

which reproduce the correct decomposition of $(\otimes f)^n$. The coefficients $d_{n;\ell_1,\ell_2}$ for general n can be calculated explicitly, but we defer their derivation to the general $SU(N)$ case, where we will present a complete treatment.

4 Composition of many fundamental $SU(N)$ irreps

The fermion method described in the previous sections can be used to derive explicit combinatorial expressions for the components of the decomposition of an arbitrary number n of fundamental $U(N)$ irreps. We will work directly with the fermion momenta k_i, leaving the transition to Young tableau row lengths ℓ_i in (2.8) for the end.

4.1 Combinatorial expressions

The fermion state for the product of n $SU(N)$ irreps is given by (3.5) and gives the generating function of irreps in the momentum parametrization $\mathbf{k}$ (recall that we use boldfaced letters for sets of N quantities such as $k_1, \dots, k_N$)

$$\psi_{N,n}(\mathbf{z}) = \Delta(\mathbf{z})\Big(\sum_{i=1}^{N} z_i\Big)^n := \sum_{\mathbf{k}} d_{n;\mathbf{k}} \prod_{i=1}^{N} z_i^{k_i} \,. \tag{4.1}$$

The $\psi_{N,n}$ is antisymmetric in the z_i, and as a consequence the $d_{n;\mathbf{k}}$ appearing above are fully antisymmetric in the k_i. When the k_i are in decreasing order, $d_{n;\mathbf{k}}$ gives the multiplicity of the irrep labeled by $k_1 > \cdots > k_N$.

To derive an explicit combinatorial expression for the multiplicity we first focus on the coefficients produced by the term $(z_1 + \cdots + z_N)^n$, denoted by $D_{n;\mathbf{k}}$. We have

$$\left(\sum_{i=1}^{N} z_i\right)^n := \sum_{k_1,\dots,k_N} D_{n;\mathbf{k}} \prod_{i=1}^{N} z_i^{k_i} = \sum_{k_1,\dots,k_N} \delta_{k_1+\cdots+k_N,n} \binom{n}{k_1, k_2, \dots, k_N} \prod_{i=1}^{N} z_i^{k_i} \,, \tag{4.2}$$

that is,

$$D_{n;\mathbf{k}} = \delta_{k_1+\cdots+k_N,n} \, \frac{n!}{\prod_{i=1}^{N} k_i!} \,, \qquad k_i \geqslant 0 \,, \tag{4.3}$$

where we used the standard expression for the multinomial coefficient. The Vandermonde prefactor in (4.1) contains $N!$ terms, each augmenting the powers of z_i in each term in the expansion. This corresponds to shifting the indices $k_1, \dots, k_N$ in the

$D_{n;k_1,\ldots,k_N}$ summation by the opposite of the corresponding powers; that is,

$$d_{n;\mathbf{k}} = \prod_{j>i=1}^{N} (S_i - S_j) D_{n;\mathbf{k}} \,, \tag{4.4}$$

with S_i being discrete shift operators acting as

$$S_i F(k_1, \ldots, k_i, \ldots, k_N) = F(k_1, \ldots, k_i - 1, \ldots, k_N) \,. \tag{4.5}$$

Substituting (4.3) yields

$$d_{n;\mathbf{k}} = \delta_{k_1+\cdots+k_N, n+N(N-1)/2}\, n! \prod_{j>i=1}^{N} (S_i - S_j) \frac{1}{\prod_{r=1}^{N} k_r!} \,. \tag{4.6}$$

Observing that

$$S_i \frac{P(k_1, \ldots, k_N)}{\prod_{r=1}^{N} k_r!} = \frac{k_i\, P(k_1, \ldots, k_i - 1, \ldots, k_N)}{\prod_{r=1}^{N} k_r!} \,, \tag{4.7}$$

we see that for any polynomial function $P(k_1, \ldots, k_N)$ the Vandermonde of shift operators $\prod_{j>i}(S_i - S_j)$ acting on the last term in (4.6) will produce a polynomial of degree $N(N-1)/2$ in the numerator with leading coefficient 1. Since it must also be fully antisymmetric in the k_i, the only such polynomial is the Vandermonde of k_i. We obtain

$$d_{n;\mathbf{k}} = n! \frac{\prod\limits_{j>i=1}^{N} (k_i - k_j)}{\prod\limits_{i=1}^{N} k_i!} \,, \quad \text{with} \quad \sum_{i=1}^{N} k_i = n + \frac{N(N-1)}{2} \,. \tag{4.8}$$

The above is written in the parametrization with a fixed total momentum. To restore invariance under total momentum shifts $k_i \to k_i + c$, we re-express the k_i as

$$k_i \to\; k_i + \frac{n-k}{N} + \frac{N-1}{2} \,, \quad k := \sum_{i=1}^{N} k_i \,, \tag{4.9}$$

which reduce to k_i when the constraint $k = n + N(N-1)/2$ is satisfied, but otherwise are invariant under the common shift $k_i \to k_i + c$, and thus $k \to k + cN$. We obtain

$$d_{n;\mathbf{k}} = \frac{n! \prod\limits_{j>i=1}^{N} (k_i - k_j)}{\prod\limits_{i=1}^{N} \Big(k_i + \frac{n-k}{N} + \frac{N-1}{2}\Big)!} \,. \tag{4.10}$$

The transcription in terms of Young tableau row lengths $\ell_i = k_i - k_N + i - N$ is immediate. Using $\sum_{i=1}^{N} \ell_i = k - Nk_N - N(N-1)/2$ we obtain

$$d_{n;\boldsymbol{\ell}} = n! \, \frac{\prod_{j>i=1}^{N-1} (\ell_i - \ell_j - i + j) \prod_{i=1}^{N-1} (\ell_i + N - i)}{\left(\frac{n-\ell}{N}\right)! \prod_{i=1}^{N-1} \left(\ell_i + \frac{n-\ell}{N} + N - i\right)!} \,, \qquad \ell := \sum_{i=1}^{N-1} \ell_i \,. \tag{4.11}$$

Formulae (4.10) and (4.11) hold whenever the N-ality condition $\ell = n \,(\text{mod } N)$, or $k{=}n + N(N{-}1)/2 \,(\text{mod } N)$, is satisfied.

The above multiplicities reproduce, as they should, the total number of states as

$$N^n = \sum_{k_1 > \cdots > k_N} \dim(\mathbf{k}) \, d_{n;\mathbf{k}} \,, \tag{4.12}$$

where $\dim(\mathbf{k})$ is the dimension of irrep $\{k_1, \ldots, k_n\}$, given by

$$\dim(\mathbf{k}) = \prod_{j>i=1}^{N} \frac{k_i - k_j}{j - i} = \frac{\Delta(\mathbf{k})}{\prod_{s=1}^{N-1} s!} \,. \tag{4.13}$$

To show this, we act on both sides of (4.1) with the Vandermonde of derivative operators $\Delta(\partial) = \prod_{j>i}(\partial_i - \partial_j)$, where $\partial_i = \partial/\partial z_i$, and then set all $z_i = 1$. On the left hand side, $\Delta(\partial)$ acts only on $\Delta(\mathbf{z})$ (the action of any single $\partial_i - \partial_j$ on the symmetric part gives zero). The term $\partial_1^{n_1} \cdots \partial_N^{n_N}$ in $\Delta(\partial)$, with $n_1, \ldots, n_N$ a permutation of $0, 1, \ldots, N{-}1$, gives a nonzero result only upon acting on the corresponding term $z_1^{n_1} \cdots z_N^{n_N}$ in $\Delta(\mathbf{z})$, equal to $0! \, 1! \cdots (N-1)!$. Accounting for all $N!$ such terms we have

$$\Delta(\partial)\Delta(\mathbf{z}) = N! \prod_{s=1}^{N-1} s! = \prod_{s=1}^{N} s! \tag{4.14}$$

and thus

$$\Delta(\partial)\psi_{N,n}(\mathbf{z})|_{z_i=1} = N^n \prod_{s=1}^{N} s! \,. \tag{4.15}$$

On the right hand side, we can antisymmetrize $\prod_i z_i^{k_i}$ into the full Slater determinant (2.7), taking advantage of the antisymmetry of $d_{n;k_1,\ldots,k_N}$, and restrict the summation to $k_1 > \cdots > k_N$. Acting with $\Delta(\partial)$ produces a symmetric polynomial in z_i with coefficients polynomials in k_i, and upon setting $z_i = 1$ it gives an antisymmetric polynomial in k_i of degree $N(N{-}1)/2$, which is necessarily proportional to the Vandermonde $\Delta(\mathbf{k})$.

By acting on the ground state $k_i = N - i$, in which case the Slater state is simply the Vandermonde $\Delta(\mathbf{z})$, and using (4.15), the overall coefficient is found to be $N!$, for a result of $N!\,\Delta(\mathbf{k})$. Equating left and right hand side results we obtain

$$N^n \prod_{s=1}^{N} s! = \sum_{k_1 > \cdots > k_N} d_{n;k_1,\dots,k_N} N!\,\Delta(\mathbf{k}) \;, \tag{4.16}$$

which, upon using (4.13), reproduces (4.12).

We conclude by pointing out that the above procedure can be interpreted as a random walk on the $(N-1)$-dimensional lattice spanned by $x_i = k_i - k_{i+1}$, $i = 1, \dots, N-1$ with starting point $x_1 = \cdots = x_{N-1} = 1$. Each additional term in (4.2) increasing n to $n+1$ can be interpreted as one of N possible steps on the lattice taking $x_{i-1} \to x_{i-1} - 1, x_i \to x_i + 1, i = 1, \dots, N$ ($x_0 = x_N := 0$), and steps violating the boundary condition $x_i \geqslant 1$ forbidden. Imposing the boundary condition can be achieved by leaving the walk unrestricted and summing over the $N!$ images of the starting point obtained by permuting the initial k_i, each with a sign equal to the parity of the permutation. Formula (4.4) is implementing this image method: the term $D_{n;k_1,\dots,k_N}$ corresponds to unrestricted walks starting from $x_1 = \cdots = x_N = 0$, and the $N!$ terms in the expansion of the product of shift operators generates the images. This is a useful picture, especially in the large-n limit where the random walk essentially becomes a Brownian motion obeying a n anisotropic diffusion equation, but we will not elaborate it further here.

4.2 Momentum density and a group duality

We conclude by giving a "second quantized" expression for the $d_{n,k_1,\dots,k_N}$ that is useful in the large-N, n limit. Thinking of the k_i as a distribution of fermions on the positive momentum lattice $s = 0, 1, \dots$, we define the discrete momentum density of fermions ρ_s equal to 1 on points s of the momentum lattice where there is a fermion and zero elsewhere; that is,

$$\rho_s = \sum_{i=1}^{N} \delta_{s,k_i} \;. \tag{4.17}$$

Clearly ρ_s satisfies

$$\sum_{s=0}^{M} \rho_s = N \;, \qquad \sum_{s=0}^{M} s\,\rho_s = k = n + \frac{N(N-1)}{2} \;. \tag{4.18}$$

In the above, M is a cutoff momentum that can be chosen arbitrarily as long as it is bigger than all the k_i. Then (4.8) can be written as

$$d_{n;k_1,\ldots,k_N} = n! \prod_{t>s=0}^{M} (t-s)^{(\rho_s-1)\rho_t} \ , \tag{4.19}$$

The integer M could in principle be taken to infinity. However, keeping it finite serves to demonstrate an interesting particle-hole duality of the formulae. Define

$$\tilde{\rho}_s = 1 - \rho_{M-s} \ , \quad s = 0, 1, \ldots, M \ . \tag{4.20}$$

Clearly $\tilde{\rho}_s$ is the density of holes on the lattice $[0, M]$ with the momentum reversed. Moreover, $\tilde{\rho}_s$ satisfies

$$\sum_{s=0}^{M} \tilde{\rho}_s = M - N + 1 \ , \quad \sum_{s=0}^{M} s\, \tilde{\rho}_s = n + \frac{(M-N+1)(M-N)}{2} \ . \tag{4.21}$$

Therefore, $\tilde{\rho}_s$ represents an irrep of $SU(M-N+1)$ with the same excitation n (total number of boxes) but with the rows in the Young tableau of the $SU(N)$ irrep turned into columns for $SU(M-N+1)$, which defines the dual irrep. One can check that

$$d_{n;k_1,\ldots,k_N} = n! \prod_{t>s=0}^{M} (t-s)^{(\tilde{\rho}_s-1)\tilde{\rho}_t} \ . \tag{4.22}$$

That is, in the decomposition of the tensor product of n fundamentals of $SU(N)$, the multiplicity of any given irrep is the same as the one for its dual irrep in the product of n fundamentals of $SU(M-N+1)$. Note that this relation holds for any M such that $M \geqslant k_1 > k_2 > \cdots > k_N$.

This duality between $SU(N)$ and $SU(M-N+1)$ can be turned into a self-duality if we choose $M = 2N-1$, which is possible if $k_1 \leqslant 2N-1$. This will be guaranteed, e.g., in the case $n < N$. Then (4.22) states that dual irreps in the decomposition come with equal multiplicities.

The above group duality manifests as a particle-hole duality in the fermion density picture, but it can also be understood in terms of Young tableau composition rules: when adding one additional box, the rules are symmetric with respect to rows or columns, the only difference arising when a column of length N is formed, which is then eliminated. Therefore, starting with the singlet and adding single boxes (fundamentals), the obtained set of irreps will be symmetric under row-column exchange as long as $n < N$. This also shows that a similar duality will emerge in the composition of any number of self-dual irreps, such as, e.g., $\ell_1 = 2, \ell_2 = 1$.

4.3 Large-n limit

The limit $n \gg 1$ is similar to the large-n limit in the $SU(2)$ cases. We can use either a Stirling approximation in the combinatorial formula for $d_{n;\ell_1,\ldots,\ell_{N-1}}$ or $d_{n;k_1,\ldots,k_N}$, or a random walk approach as in the $SU(2)$ case. Either method leads to the desired result, and we omit the details of the derivation. We obtain, in the k_i parametrization, a continuous distribution $d(n;\mathbf{k})$

$$d(n;\mathbf{k}) = \delta(k-n)\frac{N^{n+N^2/2}}{\sqrt{2\pi}^{N-1} n^{(N^2-1)/2}} \prod_{j>i=1}^{N} (k_i - k_j) \exp\Big[-\frac{N}{2n}\sum_{i=1}^{N} (k_i - k/N)^2\Big], \tag{4.23}$$

where we used $\delta_{k,n+N(N-1)/2} \simeq \delta(k-n)$.[3] The result can be expressed in terms of the dimension and quadratic Casimir of each irrep $r = (k_1, \ldots, k_N)$, given by

$$\dim(r) = \prod_{j>i=1}^{N} \frac{k_i - k_j}{j-i}\,, \qquad c_2(r) = \frac{1}{2}\left(\sum_{i=1}^{N} k_i^2 - \frac{k^2}{N}\right) - \frac{N(N^2-1)}{24}\,. \tag{4.25}$$

As a result, $d(n;k_1,\ldots,k_N) := d(N,n;r)$ rewrites as

$$d(N,n;r) = \prod_{s=1}^{N-1} s!\, \frac{N^{n+N^2/2}\, e^{\frac{N^2(N^2-1)}{24n}}}{\sqrt{2\pi}^{N-1} n^{(N^2-1)/2}}\, \dim(r)\, e^{-\frac{N}{n}c_2(r)}\,. \tag{4.26}$$

We see that in the large-n limit the distribution of row lengths becomes Gaussian, with a polynomial prefactor, and the typical row lengths of the obtained representations scale as $\sqrt{n}$, so that the exponent in the Gaussian factor be of order 1.

[3] We have also used $\prod_{i<j=1}^{N} (j-i) = \prod_{s=1}^{N-1} s!$, that $N \ll n$, and that k_i is at most of order $\sqrt{n}$. In addition, the limit

$$\lim_{m\to\infty} \prod_{i=1}^{N} \Big(1 + \frac{\alpha_i}{\sqrt{m}} + \frac{\beta_i}{m}\Big)^{m+\sqrt{m}\gamma_i} = \exp\Big(\sum_{i=1}^{N} (\alpha_i\gamma_i - \frac{1}{2}\alpha_i^2 + \beta_i)\Big)\,, \tag{4.24}$$

for some α_i, β_i and γ_i, provided $\sum_{i=1}^{N} \alpha_i = 0$, is useful.

5 Coupling to magnetic fields

As a physical application of the results of this paper, we examine the partition function of a system of many fundamental $SU(N)$ "spins" (or better, flavors) coupled to an external nonabelian magnetic field. This will require the evaluation of traces of exponentials of the total flavor operators in arbitrary irreps.

Consider n flavor multiplets, each transforming under the same irrep R of $SU(N)$, with generators $J_{s,a}$, $s = 1, \dots, n$, $a = 1, \dots, N^2-1$, in the presence of an external nonabelian magnetic field B_j, $j = 1, \dots N-1$ that couples to the Cartan generators of the flavors $H_{s,j}$ according to the Hamiltonian

$$H_{B,n} = -\sum_{s=1}^{n}\sum_{j=1}^{N-1} B_j H_{s,j} = -\sum_{j=1}^{N-1} B_j H_j \ , \tag{5.1}$$

where

$$J_a = \sum_{s=1}^{n} J_{s,a} \ , \quad H_j = \sum_{j=1}^{N-1} H_{s,j} \ , \tag{5.2}$$

are the total $SU(N)$ operators and the corresponding Cartan components. This is the $SU(N)$ analog of $SU(2)$ spins coupling to an external magnetic field through the unique Cartan generator of the total spin $H = J_3$. Note that this is the most general $SU(N)$ invariant coupling: any linear combination of the form $\sum_a B_a J_a$ can be brought to the form $\sum_h B'_j H_j$ by an appropriate $SU(N)$ conjugation (chance of basis in the flavor space), and if B_a are common to all n flavors this can be achieved with a global $SU(N)$ rotation.

If the full Hamiltonian is the one described by (5.1), the flavors are not interacting and the full partition function is the n^{th} power of the single-flavor partition function

$$\mathcal{Z}_n = \mathcal{Z}_1^n \ , \quad \mathcal{Z}_1 = \mathrm{Tr}_R\, e^{-\beta H_{B,1}} = \mathrm{Tr}_R\, e^{\beta B_i J_i} \ . \tag{5.3}$$

So we would need to calculate the trace in (5.3) only for the single irrep R. In general, however, the flavors could couple. We will consider interactions that preserve the global $SU(N)$ symmetry and, in fact, depend only on the total $SU(N)$ operators. For instance, there could be an additional ferromagnetic-type interaction between the flavors, similar to the ferromagnetic interaction of the $SU(2)$ spin case, for a full Hamiltonian of the form

$$H = -C \sum_{a=1}^{N^2-1} J_a^2 - \sum_{j=1}^{N-1} B_j H_j \ , \tag{5.4}$$

with C a constant. This extra ferromagnetic-type term is proportional to the quadratic Casimir C_2 of the total $SU(N)$. A more general interaction of this type would be any function $F(\{C_a\})$ of the full set of Casimirs C_a, $a = 2, \ldots, N$, of the total $SU(N)$

$$H = F(\{C_a\}) - \sum_{j=1}^{N-1} B_j H_j \ . \tag{5.5}$$

In the presence of such an interaction the partition function does not factorize. Decomposing the full flavor space into irreps r of the total $SU(N)$, the partition function becomes

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H} = \sum_r d_{n;R,r}\, e^{-\beta F(r)}\, \mathrm{Tr}_r \exp\Big(\beta \sum_j B_j H_j\Big) \tag{5.6}$$

with Tr the trace in the full n-flavor Hilbert space, $F(r)$ the value of $F(\{C_a\})$ for the irrep r, Tr_r the trace in the irrep r, and $d_{n;R,r}$ the multiplicity of r in the tensor product $(\otimes R)^n$. An evaluation of this partition function requires, apart from the multiplicity $d_{n;R,r}$, the calculation of the trace $\mathrm{Tr}_r \exp(\sum_j b_j H_j)$, where $b_j = \beta B_j$.

The calculation of the trace $\mathrm{Tr}_r \exp(\sum_j b_j H_j)$ for any $SU(N)$ group and any irrep r can be deduced from the matrix model analysis of section 2. As stated there, energy eigenstates ϕ_r are labeled by irreps r and have the form

$$\chi_r(U) = \mathrm{Tr}_r\, U \ . \tag{5.7}$$

The matrix U can be written as

$$U = \exp\big(i \sum_a \theta_a J_a\big) = V \exp\Big(i \sum_{j=1}^{N} x_j H_j\Big) V^{-1} \ , \tag{5.8}$$

where θ_a are group parameters, V is a diagonalizing matrix, and H_j are the Cartan generators, including the $U(1)$ part, in the diagonal basis with matrix elemens

$$(H_j)_{ik} = \delta_{ij}\, \delta_{ik} \ . \tag{5.9}$$

Therefore, the energy eigenstates provide the traces (characters)

$$\chi_r(U) = \mathrm{Tr}_r \exp\Big(i \sum_{j=1}^{N} x_j H_j\Big) \,. \tag{5.10}$$

On the other hand, as explained in section 2, the energy eigenstates are given as

$$\chi_r(\mathbf{z}) = \frac{\psi_r(\mathbf{z})}{\Delta(\mathbf{z})} \,, \qquad z_j = e^{ix_j} \,, \tag{5.11}$$

with $\Delta(\mathbf{z})$ the Vandermonde determinant (2.4) and $\psi_r(\mathbf{z})$ a fermionic wavefunction given by the Slater determinant (2.7). Upon setting $ix_j = b_j = \beta B_j$ we finally have

$$\mathrm{Tr}_r \exp\Big(\beta \sum_{j=1}^{N} B_j H_j\Big) = \frac{\psi_{\mathbf{k}}(\mathbf{z})}{\Delta(\mathbf{z})} \,, \quad z_j = e^{\beta B_j} \,. \tag{5.12}$$

The irrep r with Young tableau lengths ℓ_j maps to the fermionic momenta k_j as expressed in (2.8).

Note that, (5.12) contains one extra magnetic variable, since there is also a term coupling to the $U(1)$ charge of U (the identity matrix). This can easily be eliminated by shifting all k_j by a constant such that $k_1 + \cdots + k_N = 0$ which neutralizes the $U(1)$ charge. Alternatively, we can simply choose magnetic parameters B_j such that $B_1 + \cdots + B_j = 0$, ensuring that the coupling to the $U(1)$ part vanishes. Further, the magnetic variables B_j appearing in the trace (5.12) couple to H_j in the specific base (5.9). Moving to any other base, and in particular to one where the H_j are traceless, would simply amount to replacing B_j by appropriate linear combinations.

The above traces constitute the basic components needed to determine the statistical mechanics of any collection of flavors with Hamiltonian of the type considered in this section.

6 Concluding remarks

The $SU(N)$ decomposition problem is solvable in explicit form, at least for the direct product of many "flavors" (fundamental irreps). The results in this paper open the road to various applications and generalizations.

Perhaps the most physically relevant question is the thermodynamics and phase transitions of a large collection of $SU(N)$ flavors in the presence of an external nonabelian magnetic field. This is in principle accessible from the results of the present work. For interactions of ferromagnetic type, the system can be shown to exhibit ferromagnetic-like phase transition, but its phase structure and properties turn out to be quite complicated and nontrivial, exhibiting qualitatively different behavior compared to the standard $SU(2)$ ferromagnet. This system is treated in the follow-up paper [20].

Another interesting issue, relevant to various theoretical physics contexts, is the scaling properties of the system as the number of flavors n and the size of the group N both go to infinity. The simplifications and special properties of the limit of large N are known, and have been extensively used in particle physics [15–17]. In our case, the existence of the additional parameter n enriches the structure, and it turns out that different limits can be reached according to the relation between n and N as they both go to infinity. Again, in these limits phase transitions appear depending on the relation of the two parameters. This will also be investigated in a future publication.

Finally, the decomposition of the direct product of irreps other than the fundamental can be considered. The relevant formulae quickly grow complicated and unwieldy (see (3.3)), but the thermodynamic properties and the various large-n, N limits could still be analytically accessible, and are topics for future investigation.

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