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Scattering of identical particles by a one-dimensional Dirac delta function barrier potential: The role of statistics

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Scattering of non-interacting, identical bosons or fermions by a one-dimensional Dirac delta function barrier potential underlines the importance of the role of statistics (that is, whether the particles obey Fermi–Dirac or Bose–Einstein statistics) in the scattering. We consider an initial wave function for the system that corresponds to one particle incident from the left and one from the right of the potential barrier. For bosons, both particles are scattered either to the left or to the right if the intensity reflection coefficient is 1/2, provided the left and right propagating wave packets fully overlap in the scattering region. For fermions, the particles "pass through" one another, provided the left and right propagating wave packets fully overlap in the scattering region, with zero probability that both particles are scattered to the left or right, consistent with the Pauli exclusion principle.
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I. INTRODUCTION

In an introductory quantum mechanics course, students often learn about the concepts of spin and statistics. Specifically, they are told that the wave function for a system of identical bosons must be symmetric under the interchange of any two particles, and the wave function for identical fermions must be antisymmetric under the interchange of any two particles. Feynman *et al.*[1] provided an elementary discussion of how the scattering cross section of two indistinguishable particles by one another depends on whether they are bosons or fermions. For non-interacting particles, examples are sometimes given in which the energy levels are calculated for identical bosons or fermions placed in a one-dimensional, infinite potential well.[2] On the other hand, introductory courses do not usually include examples involving the scattering of a pair of bosons or fermions by a one-dimensional potential. This is unfortunate, since the results clearly demonstrate the importance of the scattering amplitudes on the nature (bosonic or fermionic) of the particles. Moreover, the resulting physics is qualitatively different from that encountered in particle–particle scattering.

In this paper, we consider scattering of two non-interacting identical bosons or fermions by a delta function barrier potential. Instead of using a time-independent approach from which the reflection and transmission amplitudes could be obtained, we prefer to use a time-dependent approach in which an initial wave packet is incident upon and scattered by the potential. This approach allows us to explore the dynamics of the scattering. In effect, the delta function potential acts as a "beam-splitter." The particles are incident from opposite sides of the barrier. If the particle wave packets do not overlap as they are scattered by the barrier, the scattering is the same as it would be had the particles been distinguishable—there is no difference for bosons and fermions. In other words, although, in principle, it is always necessary to symmetrize or anti-symmetrize the wave functions for identical particles, as long as the wave packets of the particles never overlap, they can be treated as distinguishable. On the other hand, the scattering differs dramatically for fermions and bosons when the wave packets of the two particles *do* overlap in the potential region. For a 50–50 beam splitter and perfect overlap, both bosons are scattered to either the right or the left, while the fermions are scattered one to each side of the potential. These results constitute another of the amazing predictions of quantum mechanics. Even if the particles are separated *initially* by a distance much larger than the extent of their wave packets, it is still necessary to symmetrize or anti-symmetrize the composite wave function if the wave packets are destined to meet in the scattering region.

The results of this calculation for bosons are fully analogous to those involving Hong–Ou–Mandel (HOM) interference[3] of photons in an interferometric scheme involving a beam splitter. In HOM interference, two single-photon pulses are sent into the two input channels of a 50–50 beam splitter and overlap at the beam splitter. It is observed that the photons emerge in one or the other output arms of the beam splitter—the probability to observe one photon in each of the output arms is identically equal to zero, an effect attributed to the bosonic nature of photons. Analogous calculations for particles, both bosons and fermions, were carried out by Loudon.[4] In his calculation, Loudon uses an input-output theory of the beam-splitter involving creation and annihilation operators for the various modes, an approach which is used commonly in the quantum optics community. In contrast, our calculation is formulated in terms of quantum-mechanical, one-dimensional scattering by a potential barrier. As such, it should be accessible to upper-level undergraduate or beginning graduate students.

II. QUALITATIVE PICTURE OF THE ROLE OF STATISTICS ON THE SCATTERING

Before launching into a formal calculation of scattering by a delta function potential barrier, we consider first the scattering of two *distinguishable* particles (1 and 2) by a localized potential barrier centered at the origin whose intensity reflection and transmission coefficients are equal (50–50 beam splitter). This will allow us to understand the role of statistics (that is, whether the particles obey Fermi–Dirac or Bose–Einstein statistics) in the scattering process. In this section, we take the initial wave function for our two particles as

$$\psi(x_1, x_2; t=0) = e^{ik_0 x_1} f(x_1 + x_0) \times e^{-ik_0 x_2} f(x_2 - x_0), \tag{1}$$

with $f(x)$ being a real, symmetric envelope function centered at $x = 0$, $x_0 > 0$, and propagation constant $k_0 > 0$. One component of the initial wave function is localized to the left of the potential barrier, centered at $x_1 = -x_0$ and moves towards the barrier with average speed $v_0 = \hbar k_0/m$, while the other component is localized to the right of the potential, centered at $x_2 = x_0$ and moves towards the barrier with the same average speed. The function $f(x)$ has been taken to be an even function of x to ensure that the two components of the wave packet would fully overlap at time $t = x_0/v_0$ if the barrier were not present. We assume that each wave packet component has a width that is much smaller than the initial distance to the potential barrier and neglect wave packet spreading (see Fig. 1 for the geometry). None of these restrictions are critical to the calculations, but they simplify the discussion in this section.

We will show formally in Sec. III that, apart from an overall phase factor and neglecting any time delays, the wave function $\psi^+(x_1, x_2; t)$ following the scattering (that is, for times $t \gtrsim 2x_0/v_0$) is

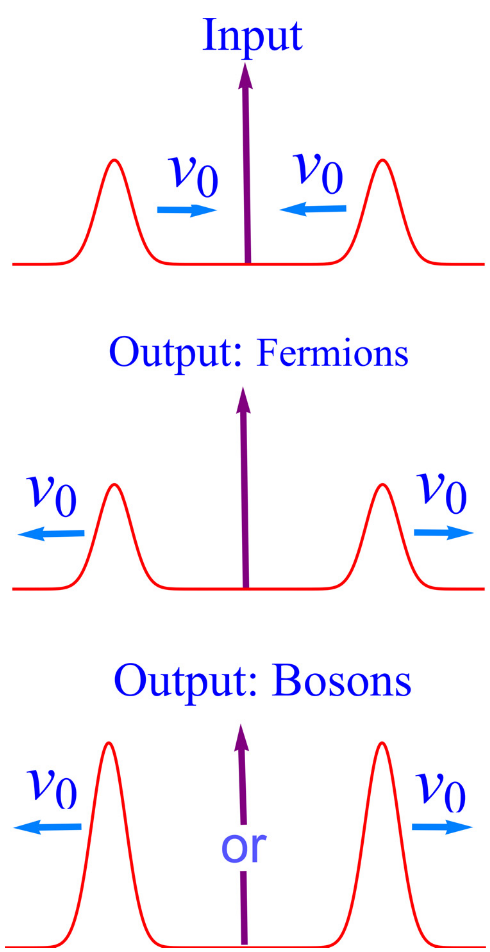

Fig. 1. Schematic representation of the scattering of two identical fermions or bosons by a localized, symmetric potential barrier having $|\mathcal{T}|^2 = |\mathcal{R}|^2 = 1/2$.

$$\begin{aligned}\psi^+(x_1, x_2; t) = {} & \left[e^{-ik_0 x_1} \mathcal{R}(k_0) f(-x_1 + x_0 - v_0 t)\right. \\ & \left. + e^{ik_0 x_1} \mathcal{T}(k_0) f(x_1 + x_0 - v_0 t)\right] \\ & \times \left[e^{-ik_0 x_2} \mathcal{T}(k_0) f(x_2 - x_0 + v_0 t)\right. \\ & \left. + e^{ik_0 x_2} \mathcal{R}(k_0) f(-x_2 - x_0 + v_0 t)\right],\end{aligned} \tag{2}$$

where $\mathcal{R}(k)$ and $\mathcal{T}(k)$ are, respectively, the amplitude reflection and transmission coefficients for a particle of mass m having energy $E_k = \hbar^2 k^2/2m$. Since $f(x) = f(-x)$, this can be rewritten as

$$\begin{aligned}\psi^+(x_1, x_2; t) = {} & \left[e^{-ik_0 x_1} \mathcal{R}(k_0) f(x_1 - x_0 + v_0 t)\right. \\ & \left. + e^{ik_0 x_1} \mathcal{T}(k_0) f(x_1 + x_0 - v_0 t)\right] \\ & \times \left[e^{-ik_0 x_2} \mathcal{T}(k_0) f(x_2 - x_0 + v_0 t)\right. \\ & \left. + e^{ik_0 x_2} \mathcal{R}(k_0) f(x_2 + x_0 - v_0 t)\right].\end{aligned} \tag{3}$$

For a 50–50 beam splitter, it follows from probability conservation that

$$|\mathcal{T}(k)| = |\mathcal{R}(k)| = 1/\sqrt{2}, \tag{4a}$$

$$\mathcal{R}(k) = \pm i \mathcal{T}(k), \tag{4b}$$

and, therefore,

$$\begin{aligned}\psi^+(x_1, x_2; t) = {} & \frac{1}{2}\left[ie^{-ik_0 x_1} f(x_1 - x_0 + v_0 t)\right. \\ & \left. + e^{ik_0 x_1} f(x_1 + x_0 - v_0 t)\right] \\ & \times \left[e^{-ik_0 x_2} f(x_2 - x_0 + v_0 t)\right. \\ & \left. + ie^{ik_0 x_2} f(x_2 + x_0 - v_0 t)\right],\end{aligned} \tag{5}$$

where, for the sake of definiteness, we have chosen the plus sign in Eq. (4b).

If we expand the above equation and take into account that $t \gtrsim 2x_0/v_0$, we obtain four terms, one that corresponds to both particles to the right of the potential barrier—*RR*, one that corresponds to both particles to the left of the barrier—*LL*, one that corresponds to particle 1 to the right and particle 2 to the left—*RL* and one that corresponds to particle 2 to the right and particle 1 to the left—*LR*. Explicitly,

$$\begin{aligned}\psi^+(x_1, x_2; t) = {} & \psi_{RR}(x_1, x_2; t) + \psi_{LL}(x_1, x_2; t) \\ & + \psi_{RL}(x_1, x_2; t) + \psi_{LR}(x_1, x_2; t),\end{aligned}$$

with

$$\begin{aligned}\psi_{RR}(x_1, x_2; t) = {} & (i/2) e^{ik_0(x_1 + x_2)} f(x_1 + x_0 - v_0 t) \\ & \times f(x_2 + x_0 - v_0 t),\end{aligned} \tag{6a}$$

$$\begin{aligned}\psi_{LL}(x_1, x_2; t) = {} & (i/2) e^{-ik_0(x_1 + x_2)} f(x_1 - x_0 + v_0 t) \\ & \times f(x_2 - x_0 + v_0 t),\end{aligned} \tag{6b}$$

$$\begin{aligned}\psi_{RL}(x_1, x_2; t) = {} & (1/2) e^{ik_0(x_1 - x_2)} f(x_1 + x_0 - v_0 t) \\ & \times f(x_2 - x_0 + v_0 t),\end{aligned} \tag{6c}$$

$$\begin{aligned}\psi_{LR}(x_1, x_2; t) = {} & -(1/2) e^{-ik_0(x_1 - x_2)} f(x_1 - x_0 + v_0 t) \\ & \times f(x_2 + x_0 - v_0 t).\end{aligned} \tag{6d}$$

To go over into scattering by identical bosons or fermions, we must symmetrize or anti-symmetrize the wave function.

For scattering corresponding to one particle incident from the left and one from the right, this implies that the total scattered wave functions are

$$\psi^{+}_{\text{Bosons}}(x_1,x_2;t) = \left(1/\sqrt{2}\right)\left[\psi^{+}(x_1,x_2;t)+\psi^{+}(x_2,x_1;t)\right] \tag{7a}$$

$$= \sqrt{2}[\psi_{LL}(x_1,x_2;t)+\psi_{RR}(x_1,x_2;t)], \tag{7b}$$

$$\psi^{+}_{\text{Fermions}}(x_1,x_2;t) = \left(1/\sqrt{2}\right)\left[\psi^{+}(x_1,x_2;t)-\psi^{+}(x_2,x_1;t)\right] \tag{7c}$$

$$= \sqrt{2}[\psi_{LR}(x_1,x_2;t)+\psi_{RL}(x_1,x_2;t)]. \tag{7d}$$

To arrive at this result, we used the fact that, for our specific choice of initial wave packet and for $f(x)=f(-x)$, the combination $\psi_{LL}(x_1,x_2;t)+\psi_{RR}(x_1,x_2;t)$ is symmetric under particle exchange $(x_1 \leftrightarrow x_2)$, whereas $\psi_{RL}(x_1,x_2;t)+\psi_{LR}(x_2,x_1;t)$ is anti-symmetric. This means that for bosons only the LL + RR term survives when we symmetrize the wave function, and the particles "bunch" on each side of the potential after scattering. For fermions, on the other hand, the particles exclude themselves as a result of the anti-symmetrization and are scattered one to each side of the potential. The results are represented schematically in Fig. 1.

We now carry out a specific calculation for a delta-function potential barrier. We do not limit the calculation to times $t \gtrsim 2x_0/v_0$, enabling us to map out the time evolution of the scattering process.

III. SINGLE-PARTICLE SCATTERING

Although the calculation can be carried out for arbitrary 1D localized potential barriers, we restrict our discussion to a 1D delta function barrier at the origin,

$$V(x) = G\delta(x), \tag{8}$$

where $G>0$ is a constant that has units of energy $\times$ length. The use of a delta function barrier simplifies the calculation while still illustrating the relevant physics.

A. Eigenfunctions

For a particle having mass m and energy $E(k)=\hbar^2k^2/2m$ with $k>0$, the two degenerate eigenfunctions can be taken as[5]

$$\psi_L(k,x) = \sqrt{\frac{1}{2\pi}}\begin{cases} e^{ikx}+\mathcal{R}(k)e^{-ikx}, & \mathrm{x}<0, \\ \mathcal{T}(k)e^{ikx}, & \mathrm{x}>0, \end{cases} \tag{9}$$

$$\psi_R(k,x) = \sqrt{\frac{1}{2\pi}}\begin{cases} \mathcal{T}(k)e^{-ikx}, & \mathrm{x}<0, \\ e^{-ikx}+\mathcal{R}(k)e^{ikx}, & \mathrm{x}>0, \end{cases} \tag{10}$$

where

$$\mathcal{R}(k) = -i\frac{\alpha}{k+i\alpha}, \tag{11a}$$

$$\mathcal{T}(k) = \frac{k}{k+i\alpha}, \tag{11b}$$

and

$$\alpha = \frac{mG}{\hbar^2}. \tag{12}$$

For $k=\alpha$, $|\mathcal{T}(k)|=|\mathcal{R}(k)|=1/\sqrt{2}$, and the barrier acts as a 50–50 beam splitter; moreover, in this limit, $\mathcal{R}(k)=-i\mathcal{T}(k)$, consistent with Eq. (4b).

For potentials that are invariant under reflection such as the delta function potential centered at the origin considered in this work, the reflection and transmission coefficients satisfy[6]

$$|\mathcal{T}(k)|^2+|\mathcal{R}(k)|^2=1, \tag{13a}$$

$$\mathcal{T}(k)\mathcal{R}^*(k)+\mathcal{R}(k)\mathcal{T}^*(k)=0. \tag{13b}$$

If we write

$$\mathcal{R}(k)=|\mathcal{R}(k)|e^{i\phi_{\mathcal{R}}(k)};\quad \mathcal{T}(\mathrm{k})=|\mathcal{T}(\mathrm{k})|\mathrm{e}^{\mathrm{i}\phi_{\mathcal{T}}(\mathrm{k})}, \tag{14}$$

then

$$|\mathcal{R}(k)|=\frac{\alpha}{\sqrt{k^2+\alpha^2}};\quad |\mathcal{T}(\mathrm{k})|=\frac{\mathrm{k}}{\sqrt{\mathrm{k}^2+\alpha^2}}, \tag{15a}$$

$$\phi_{\mathcal{R}}(k)=\tan^{-1}\left(\frac{k}{\alpha}\right);\quad \phi_{\mathcal{T}}(\mathrm{k})=-\tan^{-1}\left(\frac{\alpha}{\mathrm{k}}\right), \tag{15b}$$

such that

$$\phi_{\mathcal{R}}(k)-\phi_{\mathcal{T}}(k)=\pi/2. \tag{16}$$

At a given energy, the delta function barrier acts as a beam-splitter, with the characteristic phase difference of $\pi/2$ between the reflection and amplitude coefficients.

In some sense, $\psi_L(k,x)$ corresponds to a wave incident from the left and $\psi_R(k,x)$ corresponds to a wave incident from the right. If a convergence factor of $e^{\epsilon x/2}$ is introduced into the wave functions for $x<0$ and $e^{-\epsilon x/2}$ for $x>0$, with $\epsilon>0$, one can show that the wave functions are both orthogonal and (delta-function) normalized; that is,

$$\int_{-\infty}^{\infty}\psi_L(k,x)\left[\psi_L(k',x)\right]^*dx=\int_{-\infty}^{\infty}\psi_R(k,x)\left[\psi_R(k',x)\right]^*dx = \delta(k-k'); \tag{17a}$$

$$\int_{-\infty}^{\infty}\psi_L(k,x)\left[\psi_R(k',x)\right]^*dx=0. \tag{17b}$$

B. Scattering

One-dimensional scattering of wave packets is a topic covered in some introductory quantum mechanics texts.[5,7] Here, we review the basic features of the problem. We take as our initial condition

$$\psi(x,0)=\frac{e^{ik_0x}}{\pi^{1/4}\sqrt{\sigma}}e^{-(x+x_L)^2/2\sigma^2}, \tag{18}$$

where $k_0>0$ is a propagation constant. The initial probability distribution $|\psi(x,0)|^2$ is a Gaussian centered at $x=-x_L<0$ and has full-width at half maximum $1.67\sigma \ll x_L$.

This wave packet is localized to the left of the potential and moves towards the barrier with average speed

$$v_0 = \hbar k_0/m. \tag{19}$$

At any time $t > 0$, the wave function $\psi(x,t)$ can be expanded in terms of the eigenfunctions as

$$\psi(x,t) = \int_0^\infty dk\,[\Phi_L(k)\psi_L(k,x) + \Phi_R(k)\psi_R(k,x)]e^{-i\hbar k^2 t/2m}, \tag{20}$$

where $\Phi_L(k)$ and $\Phi_R(k)$ are expansion coefficients. The integral is restricted to positive values of k since the eigenfunctions $\psi_{L,R}(k,x)$ given in Eqs. (9) and (10) are so restricted. Since the incident wave packet is localized to the left of the origin and moving to the right, $\Phi_R(k) \approx 0$. Moreover, the e^{-ikx} component of $\psi_L(k,x)$ does not contribute significantly to the expansion at time $t = 0$ since it corresponds to a particle moving to the left. As a consequence, it is a good approximation to expand the initial wave packet in terms of *free-particle plane wave eigenfunctions,*

$$\psi(x,0) \approx \int_0^\infty dk\,\Phi_L(k)\psi_L(k,x) \approx \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty dk\,\Phi_L(k)e^{ikx}. \tag{21}$$

The integral has been extended to $-\infty$ based on the assumption that $\Phi_L(k) \approx 0$ for $k < 0$. From Eqs. (18) and (21), it then follows that

$$\Phi_L(k) \approx \frac{\sqrt{\sigma}e^{i(k-k_0)\sigma}}{\pi^{1/4}}e^{-(k-k_0)^2\sigma^2/2} \tag{22}$$

has a Gaussian envelope that is peaked at $k = k_0$, corresponding to energy

$$E_0 = \hbar^2 k_0^2/2m. \tag{23}$$

At any time, the wave function $\psi(x,t)$ can be calculated using

$$\begin{aligned}\psi(x,t) &\approx \int_0^\infty dk\,\Phi_L(k)\psi_L(k,x)e^{-i\hbar k^2 t/2m}\\ &= \sqrt{\frac{1}{2\pi}}\int_0^\infty dk\,\Phi_L(k)e^{-i\hbar k^2 t/2m}\\ &\quad\times\begin{cases} e^{ikx} + |\mathcal{R}(k)|e^{i\phi_R(k)}e^{-ikx}, & x<0,\\ |\mathcal{T}(k)|e^{i\phi_T(k)}e^{ikx}, & x>0.\end{cases}\end{aligned} \tag{24}$$

Although it is not difficult to evaluate the integral in Eq. (24) numerically, the wave packet evolution simplifies considerably if we are able to neglect wave packet spreading and take the initial packet to be quasi-monoenergetic (conditions that are compatible, as we will see below). We are interested in times of order $t_s = 2x_L/v_0$ for which the scattering is complete and there is a component of the wave packet to the left of the potential moving to the left and a component of the wave packet to the right of the potential moving to the right. On this time scale, spreading of the wave packet can be neglected if

$$\frac{\hbar\Delta k^2}{2m}t \approx \frac{\hbar}{2m\sigma^2}\frac{2x_L}{v_0} = \frac{\hbar x_L}{m v_0\sigma^2} \ll 1; \tag{25}$$

that is, if $\sigma^2 \gg \hat{\lambda}_{dB}x_L$, where $\hat{\lambda}_{dB} = \hbar/mv_0$. Note this condition can be written as

$$\frac{x_L}{k_0\sigma^2} = \frac{1}{k_0\sigma}\frac{x_L}{\sigma} \ll 1, \tag{26}$$

implying that

$$k_0\sigma \gg \frac{x_L}{\sigma} \gg 1; \tag{27}$$

that is, our calculation is valid only in this limit.[8] Since $\Delta k = \sigma/\sqrt{2}$, inequality (27) implies that $\Delta k/(\sqrt{2}k_0) \ll 1$; the wave packet is quasi-monoenergetic.

In the limit that inequalities (25) and (27) are valid, $|\mathcal{R}(k)| \approx |\mathcal{R}(k_0)|$, $|\mathcal{T}(k)| \approx |\mathcal{T}(k_0)|$, while the phases $\phi_R(k)$ and $\phi_T(k)$ can be expanded in a Taylor series about $k = k_0$. Using these expansions and setting the $e^{-i\hbar k^2 t/2m}$ factor in Eq. (24) equal to unity (neglect of spreading), we obtain

$$\begin{aligned}\psi(x,t) \approx\ & \frac{e^{-i\hbar k_0^2 t/2m}}{(\pi\sigma^2)^{1/4}}\\ &\times\begin{cases} e^{ik_0x}e^{-(x+x_L-v_0t)^2/2\sigma^2} + \mathcal{R}(k_0)e^{-ik_0x} & \\ \times e^{-[-x+x_L-v_0(t-\tau_R)]^2/2\sigma^2}, & x<0,\\ \mathcal{T}(k_0)e^{ik_0x}e^{-[x+x_L-v_0(t-\tau_T)]^2/2\sigma^2}, & x\ge 0,\end{cases}\end{aligned} \tag{28}$$

where the time delays are given by

$$\tau_{\mathcal{R}} = \frac{1}{v_0}\frac{d\phi_{\mathcal{R}}(k_0)}{dk_0} = \tau_{\mathcal{T}} = \frac{1}{v_0}\frac{d\phi_{\mathcal{T}}(k_0)}{dk_0} = \frac{1}{v_0}\frac{\alpha}{k_0^2+\alpha^2}. \tag{29}$$

Equation (28) is intuitively satisfying. The first term corresponds to the incident packet and will vanish approximately once the scattering is complete, that is for times $t \gtrsim x_L/v_0 + 3\sigma$. The second term is the reflected packet and the third the transmitted packet, both of which are nonvanishing only for $t \gtrsim x_L/v_0 - 3\sigma$, that is, only after the incident packet has reached the barrier. Following the scattering, there are reflected and transmitted components of the wave function, each of which has the same envelope as the incident packet (because spreading has been neglected).

On time scales $t \ge t_s/2 = 2x_L/v_0$, the time delay

$$\begin{aligned}\frac{\tau_{\mathcal{R}}}{t_s/2} \le \frac{1}{x_L}\frac{\alpha}{k_0^2+\alpha^2} &= \frac{1}{x_L\sqrt{k_0^2+\alpha^2}}\frac{\alpha}{\sqrt{k_0^2+\alpha^2}}\\ &< \frac{1}{k_0 x_L} \ll \frac{1}{k_0\sigma} \ll 1.\end{aligned} \tag{30}$$

As a consequence of this inequality and the fact that the time delay is the same for reflection and transmission, we will set the time delay equal to zero in all that follows.

A MATHEMATICA notebook and MATHEMATICA cdf file (which can be viewed even if you don't have access to MATHEMATICA) showing wave packet propagation is provided in the supplementary material.[9] At the barrier, there are interference fringes between the incident and reflected components of the wave function. In the simulation and in all graphs to follow, all quantities are converted to dimensionless variables in units of σ, e.g., $\tilde{x} = x/\sigma$, $\tilde{k}_0 = k_0\sigma$, $\tilde{\alpha} = \alpha\sigma$, $\tilde{t} \to v_0 t/\sigma$, and $\tilde{\psi} = \psi\sqrt{\sigma}$. In terms of these variables and with the time delays neglected,

$$\tilde{\psi}(\tilde{x},\tilde{t}) \approx \frac{e^{-i\tilde{k}_0\tilde{t}}}{(\pi)^{1/4}} \begin{cases} e^{i\tilde{k}_0\tilde{x}} e^{-(\tilde{x}+\tilde{x}_L-\tilde{t})^2/2} + \mathcal{R}(\tilde{k}_0\sigma) e^{-i\tilde{k}_0\tilde{x}} & \\ \times e^{-(-\tilde{x}+\tilde{x}_L-\tilde{t})^2/2}, & \tilde{x} < 0, \\ \mathcal{T}(\tilde{k}_0\sigma) e^{i\tilde{k}_0\tilde{x}} e^{-(\tilde{x}+\tilde{x}_L-\tilde{t})^2/2}, & \tilde{x} \geq 0. \end{cases} \tag{31}$$

We can also calculate the probabilities $P_L(\tilde{t})$ and $P_R(\tilde{t})$ that the particle is found to the left or to the right of the barrier, given by

$$P_L(\tilde{t}) = \int_{-\infty}^{0} |\psi(\tilde{x},\tilde{t})|^2 dx, \tag{32a}$$

$$P_R(\tilde{t}) = \int_{0}^{\infty} |\psi(\tilde{x},\tilde{t})|^2 dx. \tag{32b}$$

Analytic expressions for these quantities, obtained in terms of error functions, are included in the MATHEMATICA notebooks provided in the supplementary material. For times $\tilde{t} \geq \tilde{t}_s = 2\tilde{x}_L$,

$$P_L \sim |\mathcal{R}(k_0)|^2; \quad P_\mathrm{R} \sim |\mathcal{T}(\mathrm{k}_0)|^2. \tag{33}$$

A plot of P_L and P_R as a function of $\tilde{t}$ is shown in Fig. 2 for $\tilde{k}_0 = \tilde{\alpha} = 40$ and $\tilde{x}_L = 4$. As can be seen, the particle is first localized to the left of the barrier, but eventually is scattered by the barrier, leading to reflected and transmitted components. Since we have taken $\tilde{k}_0 \gg 1$, any interference effects between the incident and scattered waves are washed out when integrating over position. As a result, the probabilities undergo smooth transitions from their initial to final values. The MATHEMATICA files enable you to vary $\tilde{\alpha}$.

IV. SCATTERING OF TWO IDENTICAL BOSONS OR FERMIONS

We now extend the calculation to the scattering of two non-interacting identical bosons or fermions, with one particle incident from the left and one from the right. By "identical," we mean the two particles have the same spin and $z-$component of spin. We take as our initial wave packet

$$\psi_\pm(x_1,x_2;t=0) = \frac{1}{\sqrt{2}}[\psi_L(x_1,0)\psi_R(x_2,0) \pm \psi_L(x_2,0)\psi_R(x_1,0)], \tag{34}$$

with

$$\psi_L(x,0) = \frac{e^{ik_0x}}{\pi^{1/4}\sqrt{\sigma}} e^{-(x+x_L)^2/2\sigma^2}, \tag{35a}$$

$$\psi_R(x,0) = \frac{e^{-ik_0x}}{\pi^{1/4}\sqrt{\sigma}} e^{-(x-x_R)^2/2\sigma^2}. \tag{35b}$$

The $+$ subscript is for bosons, and the $-$ subscript is for fermions. We have allowed for the distance from the origin of the peaks of the initial left and right probability distributions to differ, that is $|\psi_L(x,0)|^2$, is peaked at $x = -x_L$ while $|\psi_R(x,0)|^2$ is peaked at $x = x_R$. One particle is localized to the left of the potential and one to the right, but the composite wave function is symmetric or antisymmetric on exchange of the particles. To simplify matters, we have assumed that the central velocities of the two components are equal in magnitude but opposite in direction and that the envelope functions of the two components are the same. These restrictions can be relaxed without changing any of the general results that are to be derived, but the resulting calculations are somewhat more complicated.

The calculation then proceeds as in the single particle case, and we find

$$\psi_\pm(x_1,x_2;t) = \frac{1}{\sqrt{2}}\left[\psi_L(x_1,t)\psi_R(x_2,t) \pm \psi_L(x_2,t)\psi_R(x_1,t)\right], \tag{36}$$

where

$$\psi_L(x,t) \approx \frac{e^{-i\hbar k_0^2 t/2m}}{(\pi\sigma^2)^{1/4}} \begin{cases} e^{ik_0x} e^{-(x+x_L-v_0t)^2/2\sigma^2} + \mathcal{R}(k_0) e^{-ik_0x} & \\ \times e^{-(-x+x_L-v_0t)^2/2\sigma^2}, & x < 0, \\ \mathcal{T}(k_0) e^{ik_0x} e^{-(x+x_L-v_0t)^2/2\sigma^2}, & x \geq 0 \end{cases} \tag{37}$$

and

$$\psi_R(x,t) \approx \frac{e^{-i\hbar k_0^2 t/2m}}{(\pi\sigma^2)^{1/4}} \begin{cases} e^{-ik_0x} e^{-(x-x_R+v_0t)^2/2\sigma^2}, + \mathcal{R}(k_0) e^{ik_0x} & \\ \times e^{-(-x-x_R+v_0t)^2/2\sigma^2}, & x > 0, \\ \mathcal{T}(k_0) e^{-ik_0x} e^{-(x-x_R+v_0t)^2/2\sigma^2}, & x \leq 0. \end{cases} \tag{38}$$

We have neglected the time delays in these expressions.

To analyze the scattering, we want to calculate the probability $P_{RR}^{\pm}(t)$ that both particles are on the right,

$$P_{RR}^{\pm}(t) = \int_0^\infty dx_1 \int_0^\infty dx_2 |\psi_\pm(x_1,x_2;t)|^2, \tag{39}$$

the probability $P_{LL}^{\pm}(t)$ that both particles are on the left,

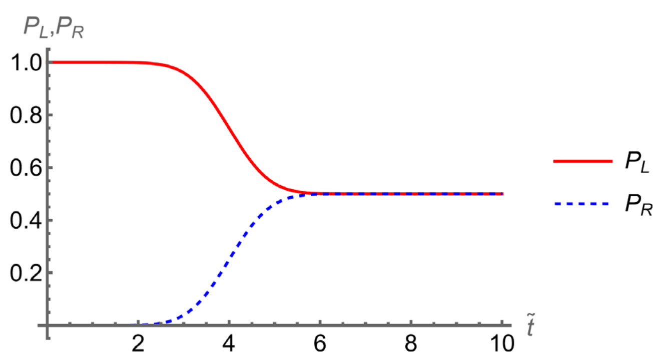

Fig. 2. Single-particle scattering. For a Gaussian particle wave packet incident from the left side of the delta-function barrier, the probability to find the particle to the left of the barrier (P_L) and to the right of the barrrier (P_R) are plotted as a function of dimensionless time $\tilde{t}$ for $\tilde{k}_0 = \tilde{\alpha} = 40$ and $\tilde{x}_L = 4$. The full width at half maximum of the incident packet is equal to 1.67 in dimensionless units.

$$P_{LL}^{\pm}(t) = \int_{-\infty}^{0} dx_1 \int_{-\infty}^{0} dx_2 |\psi_{\pm}(x_1, x_2; t)|^2, \tag{40}$$

and the probability $P_{RL}^{\pm}(t)$ that one particle is on the right and one on the left,

$$\begin{aligned} P_{RL}^{\pm}(t) &= \left[\int_{-\infty}^{0} dx_1 \int_{0}^{\infty} dx_2 + \int_{0}^{\infty} dx_1 \int_{-\infty}^{0} dx_2 \right] \\ &\quad \times |\psi_{\pm}(x_1, x_2; t)|^2 \\ &= 2 \int_{-\infty}^{0} dx_1 \int_{0}^{\infty} dx_2 |\psi_{\pm}(x_1, x_2; t)|^2, \end{aligned} \tag{41}$$

the last line following since $|\psi_{\pm}(x_1, x_2; t)|^2 = |\psi_{\pm}(x_2, x_1; t)|^2$.

When Eqs. (36)–(38) are substituted into Eqs. (39)–(41), the resulting integrals for $P_{RR}^{\pm}(t)$, $P_{LL}^{\pm}(t)$, and $P_{RL}^{\pm}(t)$ can be evaluated analytically. Explicit expressions for these (rather lengthy) quantities are given in the MATHEMATICA notebooks provided in the supplementary material. In the limit that $k_0\sigma \gg 1$ and for $t > 2(x_R + x_L)/v_0$, the leading terms in Eqs. (37) and (38) no longer contribute and the expressions take on relatively simple forms. For bosons, we find

$$P_{LL}^{+} = P_{RR}^{+} \sim |\mathcal{T}\mathcal{R}|^2 \left[1 + e^{-(x_L - x_R)^2/2\sigma^2}\right], \tag{42a}$$

$$P_{RL}^{+} \sim |\mathcal{T}|^4 + |\mathcal{R}|^4 - 2e^{-(x_L - x_R)^2/2\sigma^2} |\mathcal{R}\mathcal{T}|^2, \tag{42b}$$

and, for fermions,

$$P_{LL}^{-} = P_{RR}^{-} \sim |\mathcal{T}\mathcal{R}|^2 \left[1 - e^{-(x_L - x_R)^2/2\sigma^2}\right], \tag{43a}$$

$$P_{RL}^{-} \sim |\mathcal{T}|^4 + |\mathcal{R}|^4 + 2e^{-(x_L - x_R)^2/2\sigma^2} |\mathcal{R}\mathcal{T}|^2, \tag{43b}$$

where $\mathcal{T} = \mathcal{T}(k_0)$ and $\mathcal{R} = |\mathcal{R}(k_0)|$. The exponential factor corresponds to the overlap of the envelopes of the reflected and transmitted wave packets on a given side of the barrier.

The results are very interesting. In the limit that $x_L = x_R$ and $|\mathcal{R}| = |\mathcal{T}|$, $P_{RL}^{+} = 0$, $P_{LL}^{+} = P_{RR}^{+} = 1/2$, $P_{RL}^{-} = 1$, and $P_{LL}^{-} = P_{RR}^{-} = 0$. In other words, for a symmetric beam splitter, the bosons are scattered either to the left or the right as in HOM interference, while fermions are scattered one particle to the left and one to the right. These results depend on the symmetry of the total wave function and the fact that there is a $\pi/2$ difference between the phases of the amplitude reflection and transmission coefficients. Note that $P_{RL}^{+}(t) \neq 0$ and $P_{LL}^{-}(t) = P_{RR}^{-}(t) \neq 0$, in general, for times of order $x_L/v_0 = x_R/v_0$, when the wave packets are localized near the origin.

We plot $P_{RR}^{\pm}$, $P_{LL}^{\pm}$, and $P_{RL}^{\pm}$ as a function of $\tilde{t}$ for several values of $\tilde{x}_R$ for $\tilde{k}_0 = \tilde{\alpha} = 40$ and $\tilde{x}_L = 4$. In Figs. 3 and 4, we show the results for bosons and fermions when $x_L = x_R$. As can be seen $P_{RL}^{+}(\tilde{t}) \neq 0$ and $P_{LL}^{-}(\tilde{t}) = P_{RR}^{-}(\tilde{t}) \neq 0$ for times of order $\tilde{t} = 4$. However, following the scattering, $P_{RL}^{+} = 0$ and $P_{LL}^{-} = P_{RR}^{-} = 0$. The provided MATHEMATICA files allow you to plot these probabilities for different $\tilde{\alpha}$, $\tilde{x}_R$ as a function of $\tilde{t}$.

We now consider the changes that occur when $x_R \neq x_L$. Clearly, if $x_R > 2x_L$, the wave packets from the left and right reach the barrier at different times so the scattering is that of individual particles—statistics play no role. In this limit, the exponentials in Eqs. (42) and (43) go to zero and

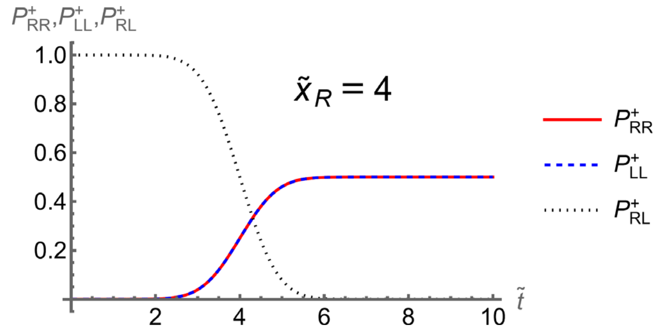

Fig. 3. Scattering of two bosons. For an initial, symmetrized wave packet that is a superposition of Gaussian components incident from both the left and right sides of the delta-function barrier and centered at equal distances from the barrier, the probability to find both bosons to the right of the barrier (P_{RR}^{+}), to find both bosons to the left of the barrier (P_{LL}^{+}), and to find one boson to the left and one to the right (P_{RR}^{+}) as a function of dimensionless time $\tilde{t}$ with $\tilde{k}_0 = \tilde{\alpha} = 40, \tilde{x}_L = 4$, and $\tilde{x}_R = 4$.

$$\begin{aligned} &P_{LL}^{+} = P_{RR}^{+} = P_{LL}^{-} = P_{RR}^{-} \sim |\mathcal{T}\mathcal{R}|^2, \\ &P_{RL}^{+} = P_{RL}^{-} = |\mathcal{T}|^4 + |\mathcal{R}|^4. \end{aligned} \tag{44}$$

These are precisely the results that would have been obtained if we had assumed the particles were distinguishable and had taken the wave function as

$$\psi(x_1, x_2; t) = \psi_L(x_1, t)\psi_R(x_2, t). \tag{45}$$

In Fig. 5, we illustrate this situation by taking $\tilde{x}_R = 12$. The results are the same for bosons and fermions. For times $5 \lesssim \tilde{t} \lesssim 10$, $P_{RR}^{\pm}(t)$ is non-zero owing to the transmitted wave from the left. As such, the total probability $P_{RR}^{\pm}(t)$ is equal to the probability that this component has traversed the barrier which is equal to $|\mathcal{T}|^2 = 1/2$, while $P_{LL}^{\pm}(t) = 0$, since the wave from the right has not yet reached the barrier. For longer times, $P_{RR}^{\pm}(t)$ is equal to the probability that the wave from the left was transmitted times the probability that the wave from the right was reflected, $P_{RR}^{\pm} \sim |\mathcal{T}\mathcal{R}|^2 = 1/4$. A similar argument gives $P_{LL}^{\pm} \sim |\mathcal{T}\mathcal{R}|^2 = 1/4$. On the other hand, $P_{RL}^{\pm}$ has two contributions, one from the joint probability that both particles were transmitted, equal to $|\mathcal{T}|^4$ and one from the joint probability that both particles were reflected, equal to $|\mathcal{R}|^4$ (recall that the transmitted and reflected

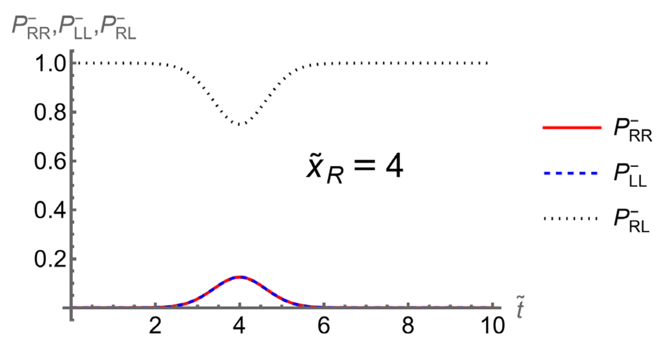

Fig. 4. Scattering of two fermions. For an initial, anti-symmetrized wave packet that is a superposition of Gaussian components incident from both the left and right sides of the delta-function barrier and centered at equal distances from the barrier, the probability to find both fermions to the right of the barrier (P_{RR}^{-}), to find both fermions to the left of the barrier (P_{LL}^{-}), and to find one fermion to the left and one to the right (P_{RR}^{-}) as a function of dimensionless time $\tilde{t}$ with $\tilde{k}_0 = \tilde{\alpha} = 40, \tilde{x}_L = 4$, and $\tilde{x}_R = 4$.

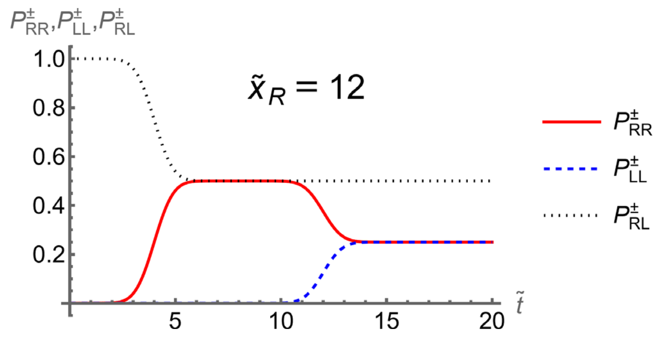

Fig. 5. Scattering of two bosons or two fermions when $\tilde{x}_L \neq \tilde{x}_R$. For an initial, symmetrized or anti-symmetrized wave packet that is a superposition of Gaussian components incident from both the left and right sides of the delta-function barrier and centered at different distances from the barrier, the probability to find both particles to the right of the barrier ($P^{\pm}_{RR}$), to find both particles to the left of the barrier ($P^{\pm}_{LL}$), and to find one particle to the left and one to the right ($P^{\pm}_{RR}$) as a function of dimensionless time $\tilde{t}$ with $\tilde{k}_0 = \tilde{\alpha} = 40, \tilde{x}_L = 4$, and $\tilde{x}_R = 12$. In this case, the incident wave packets do not overlap in the scattering region so the scattering is the same for bosons and fermions.

packets do not overlap so the total probability is just the sum of the two individual probabilities), giving a total probability $P^{\pm}_{RL} = |\mathcal{T}|^4 + |\mathcal{R}|^4 = 1/2$.

The more interesting case is when $0 \leq |\tilde{x}_R - \tilde{x}_L| \lesssim 1$. In Figs. 6 and 7, we plot the probabilities as a function of $\tilde{t}$ for $\tilde{x}_R = 5$ and $\tilde{x}_L = 4$. If either $x_R \neq x_L$ or $k_0 \neq \alpha$, the symmetry of the scattering process is broken and, following the scattering, $P^{+}_{RL} \neq 0$ and $P^{-}_{LL} = P^{-}_{RR} \neq 0$.

V. CONCLUSION

We have illustrated the importance of particle statistics on the scattering of two identical bosons or fermions by a one-dimensional potential barrier. If the particles are destined to meet and overlap in the potential region, the overall wave function must be symmetrized or anti-symmetrized to properly describe the scattering. Aside from the pedagogical value of the calculation, it should be noted that the predictions of the theory for bosons are consistent with experiments that have been carried out using single-photon pulses[3] or a source of atom pairs.[10] In both cases, it is

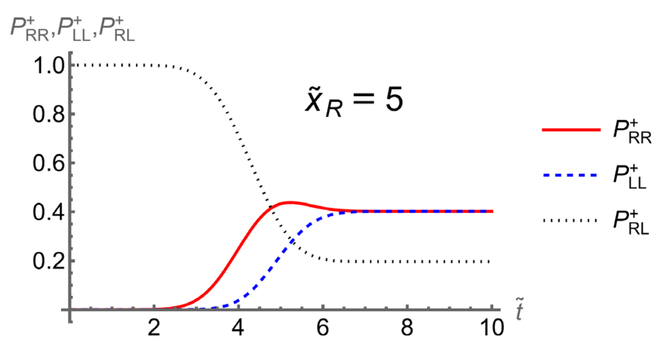

Fig. 6. Scattering of two bosons when $\tilde{x}_L \neq \tilde{x}_R$. For an initial, symmetrized wave packet that is a superposition of Gaussian components incident from both the left and right sides of the delta-function barrier and centered at different distances from the barrier, the probability to find both bosons to the right of the barrier (P^{+}_{RR}), to find both particles to the left of the barrier (P^{+}_{LL}), and to find one fermion to the left and one to the right (P^{+}_{RR}) as a function of dimensionless time $\tilde{t}$ with $\tilde{k}_0 = \tilde{\alpha} = 40, \tilde{x}_L = 4$, and $\tilde{x}_R = 5$. In this case, the incident wave packets partially overlap in the scattering region.

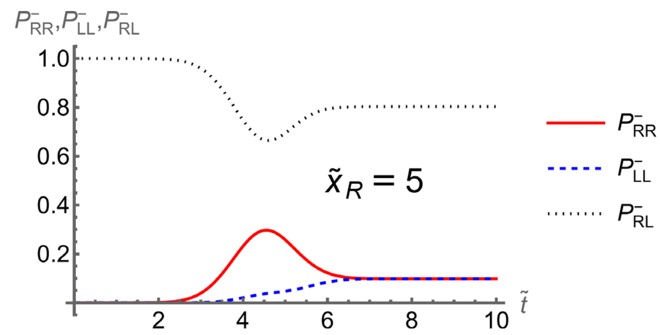

Fig. 7. Scattering of two fermions when $\tilde{x}_L \neq \tilde{x}_R$. For an initial, anti-symmetrized wave packet that is a superposition of Gaussian components incident from both the left and right sides of the delta-function barrier and centered at different distances from the barrier, the probability to find both fermions to the right of the barrier (P^{+}_{RR}), to find both fermions to the left of the barrier (P^{+}_{LL}), and to find one fermion to the left and one to the right (P^{+}_{RR}) as a function of dimensionless time $\tilde{t}$ with $\tilde{k}_0 = \tilde{\alpha} = 40, \tilde{x}_L = 4$, and $\tilde{x}_R = 5$. In this case, the incident wave packets partially overlap in the scattering region.

necessary that the radiation pulses or atomic wave packets that are incident on the beam splitter are indistinguishable. This was achieved using down conversion to produce pairs of single-photon pulses[3] or a He Bose condensate to produce atom pairs.[10]

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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[7]E. Merzbacher, *Quantum Mechanics*, 3rd ed. (John Wiley and Sons, New York, 1998), Sec. 13.2.

[8]If the inequality $k_0\sigma \gg 1$ is not satisfied, our approach fails and can lead to unphysical results, such as violation of conservation of probability.

[9]If you do not have MATHEMATICA, you can access the MATHEMATICA cdc file provided in the supplementary materials and use the Wolfram cdf Player to view it. The Wolfram cdf Player can be downloaded for free at <https://www.wolfram.com/player/>.

[10]R. Lopes, A. Imanaliev, A. Aspect, M. Cheneau, D. Boiron, and C. I. Westbrook, "Atomic Hong–Ou–Mandel experiment," Nature **520**, 66–68 (2015).