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A priori error estimates of two monolithic schemes for Biot's consolidation model

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Abstract

This paper concentrates on a priori error estimates of two monolithic schemes for Biot's consolidation model based on the three-field formulation introduced by Oyarzúa et al. (SIAM J Numer Anal, 2016). The spatial discretizations are based on the Taylor–Hood finite elements combined with Lagrange elements for the three primary variables. We employ two different schemes to discretize the time domain. One uses the backward Euler method, and the other applies the combination of the backward Euler and Crank-Nicolson methods. A priori error estimates show that both schemes are unconditionally convergent with optimal error orders. Detailed numerical experiments are presented to validate the theoretical analysis.

KEYWORDS

a priori error estimates, Biot's model, finite element

1 | INTRODUCTION

Biot's consolidation model [1,2] is a valuable tool for understanding the interaction between fluid flow and mechanical deformation in a porous medium, which is a solid structure with pores. This model has widespread applications in various fields, including biomechanics [14], petroleum engineering [15], and others. However, obtaining an exact analytic solution is difficult, so various methods have been developed to approximate numerical solutions for the system. These methods include finite volume methods [20], virtual element methods [7], and mixed finite element methods [8,16], and so forth.

In [24], it is pointed out that standard finite element methods for solving the classical two-field Biot's model may suffer from Poisson locking and pressure oscillations. Therefore, various reformulations of Biot's model are proposed to overcome these numerical difficulties. For example, fluid flux arising from the inherent Darcy law is introduced as a new variable to obtain a three-field formulation of Biot's model in [13,24]. A new four-field formulation is proposed in [25]. Another three-field model is established in [10] by introducing two pseudo-pressures. In [18,22], an intermediate variable, called "total pressure", is introduced to derive a three-field reformulation of Biot's model. The advantages of such a three-field reformulation exist in that it avoids using $\boldsymbol{H}(\mathrm{div})$ space and the classical inf-sup stable Stokes finite elements combined with Lagrange elements can be applied for the spatial discretization. By taking the advantage of such a three-field formulation, some relevant algorithms and analyses are carried out. For instance, Qi et al. [23] derive optimal-order error estimates. A second order unconditionally convergent algorithm is proposed in [17].

In this work, we introduce two fully discrete monolithic schemes for solving the three-field formulation of Biot's model. Method 1 utilizes the backward Euler method [7,14,22,23] for the time discretization, which only achieves first-order convergence in time. To overcome this limitation, we propose Method 2 applying the combination of the backward Euler and Crank-Nicolson methods, drawing inspiration from [17]. This approach employs a unified scheme for all time steps and achieves second-order convergence in time. It is noteworthy that a special case with the Biot-Willis constant $\alpha = 1$ and specific storage coefficient $c_0 = 0$ is addressed in [23]. In comparison, we present rigorous analyses for both Methods 1 and 2 that are applicable to more general physical parameters $\alpha > 1$ and $c_0 \geq 0$. Instead of providing cumulative H^1 error estimates or rough H^1 error estimates for pressure [7,10], our research presents rigorous H^1 error estimates of pressure at the final time, drawing inspiration from [16,23]. Furthermore, the unconditional convergence of Method 2 suggests that such a second order method can also be applied to iterative schemes for Biot's model [3,4,11], significantly enhancing computational efficiency. We comment here that the theoretical framework in this work can be extended to provide a novel perspective for analyzing decoupled schemes [14,17] that offer high efficiency and superconvergence.

The rest of the paper is organized as follows. In Section 2, we present a three-field formulation of Biot's consolidation model and the corresponding weak formulation. In Section 3, we introduce finite element spaces, projection operators, two monolithic schemes, and some useful propositions. A priori estimates of Methods 1 and 2 are given in Section 4. Numerical experiments are carried out to validate the theoretical results in Section 5. Conclusions and outlook are given in Section 6.

2 | MATHEMATICAL FORMULATIONS

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded polygonal domain with boundary $\partial\Omega$. The classical Sobolev spaces are denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k(\Omega)}$. We denote $H^k_{0,\Gamma}(\Omega)$ for the subspace of $H^k(\Omega)$ with the vanishing trace on $\Gamma \subset \partial\Omega$, and use $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ to denote the standard $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products, respectively. In this paper, we use C to denote a generic positive constant independent of mesh sizes, and use $x \lesssim y$ to denote $x \leq Cy$. The governing equations describing the quasi-static Biot system are given as follows

$$-\mathrm{div}\sigma(\boldsymbol{u}) + \alpha\nabla p = \boldsymbol{f}, \tag{1}$$

$$\partial_t\left(c_0 p + \alpha \mathrm{div}\boldsymbol{u}\right) - \mathrm{div}K\left(\nabla p - \rho_f \boldsymbol{g}\right) = Q_s, \tag{2}$$

where

$$\sigma(\boldsymbol{u}) = 2\mu\varepsilon(\boldsymbol{u}) + \lambda(\mathrm{div}\boldsymbol{u})\boldsymbol{I}, \quad \varepsilon(\boldsymbol{u}) = \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right].$$

Here, the primary unknowns are the displacement vector of the solid $\boldsymbol{u}$ and the fluid pressure p, the coefficient $\alpha > 0$ is the Biot-Willis constant which is close to 1, $\boldsymbol{f}$ is the body force, $c_0 \geq 0$ is the specific storage coefficient, K represents the hydraulic conductivity, ρ_f is the fluid density, $\boldsymbol{g}$ is the gravitational acceleration, Q_s is a source or sink term, $\boldsymbol{I}$ is the identity matrix, and Lamé constants λ and μ are computed from the Young's modulus E and Poisson ratio ν:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Suitable boundary and initial conditions should be provided to complete the system. Assuming that $\partial\Omega = \Gamma_d \cup \Gamma_t = \Gamma_p \cup \Gamma_f$ with $|\Gamma_d| > 0$ and $|\Gamma_p| > 0$, for the simplicity of presentation, we consider the following boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_d, \tag{3}$$

$$\left(\sigma(\boldsymbol{u}) - \alpha p\boldsymbol{I}\right)\boldsymbol{n} = \boldsymbol{h} \quad \text{on } \Gamma_t, \tag{4}$$

$$p = 0 \quad \text{on } \Gamma_p, \tag{5}$$

$$K\left(\nabla p - \rho_f \boldsymbol{g}\right)\cdot\boldsymbol{n} = g_2 \quad \text{on } \Gamma_f, \tag{6}$$

where $\boldsymbol{n}$ is the unit outward normal to the boundary. We comment here that the discussion can be easily extended to nonhomogeneous boundary condition cases. For the initial conditions, we consider

$$\boldsymbol{u}(0) = \boldsymbol{u}^0, \quad p(0) = p^0. \tag{7}$$

Next, we introduce an intermediate variable called "total pressure":

$$\xi = \alpha p - \lambda \mathrm{div}\boldsymbol{u}.$$

Then, (1) and (2) can be rewritten as the following three-field formulation of Biot's consolidation model

$$-2\mu\mathrm{div}\left(\varepsilon(\boldsymbol{u})\right) + \nabla\xi = \boldsymbol{f}, \tag{8}$$

$$\mathrm{div}\boldsymbol{u} + \frac{1}{\lambda}\xi - \frac{\alpha}{\lambda}p = 0, \tag{9}$$

$$\left(c_0 + \frac{\alpha^2}{\lambda}\right)\partial_t p - \frac{\alpha}{\lambda}\partial_t\xi - \mathrm{div}K\left(\nabla p - \rho_f\boldsymbol{g}\right) = Q_s. \tag{10}$$

After the reformulation, we can still apply the boundary conditions (3)–(6) and initial conditions (7) with $\xi(0) = \alpha p^0 - \lambda\mathrm{div}\boldsymbol{u}^0$ here. For ease of presentation, we assume $\boldsymbol{g} = \boldsymbol{0}$ in the rest of the paper.

Let $\boldsymbol{V} = \boldsymbol{H}^1_{0,\Gamma_d}(\Omega)$, $W = L^2(\Omega)$ and $M = H^1_{0,\Gamma_p}(\Omega)$, and define the bilinear forms

$$a_1(\boldsymbol{u},\boldsymbol{v}) = 2\mu \int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}), \quad b(\boldsymbol{v},\phi) = \int_\Omega \phi \,\mathrm{div}\boldsymbol{v},$$
$$a_2(\xi,\phi) = \frac{1}{\lambda}\int_\Omega \xi\phi, \qquad c(p,\phi) = \frac{\alpha}{\lambda}\int_\Omega p\phi,$$
$$a_3(p,\psi) = \left(c_0 + \frac{\alpha^2}{\lambda}\right)\int_\Omega p\psi, \quad d(p,\psi) = K\int_\Omega \nabla p\cdot\nabla\psi.$$

Multiplying (8)–(10) by test functions, integrating by parts, and applying boundary conditions (3)–(6) lead to the following variational problem: for a given $t \geq 0$, find $(\boldsymbol{u},\xi,p) \in \boldsymbol{V}\times W\times M$ such that

$$a_1(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},\xi) = (\boldsymbol{f},\boldsymbol{v}) + \langle \boldsymbol{h},\boldsymbol{v}\rangle_{\Gamma_t}, \qquad \forall \boldsymbol{v}\in\boldsymbol{V}, \tag{11}$$

$$b(\boldsymbol{u},\phi) + a_2(\xi,\phi) - c(p,\phi) = 0, \qquad \forall \phi \in W, \tag{12}$$

$$a_3(\partial_t p,\psi) - c(\psi,\partial_t\xi) + d(p,\psi) = (Q_s,\psi) + \langle g_2,\psi\rangle_{\Gamma_f}, \qquad \forall \psi\in M. \tag{13}$$

The well-posedness of problem (11)–(13) is established in [22]. We note that the Korn's inequality [21] holds on $\boldsymbol{V}$, that is, there exists a constant $C_k = C_k(\Omega,\Gamma_d) > 0$ such that

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C_k\|\varepsilon(\boldsymbol{v})\|_{L^2(\Omega)}, \quad \forall \boldsymbol{v}\in\boldsymbol{V}. \tag{14}$$

Furthermore, the following inf-sup condition [5] holds: there exists a constant $\beta > 0$ depending only on Ω and Γ_d such that

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(\boldsymbol{v},\phi)}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \geq \beta\|\phi\|_{L^2(\Omega)}, \quad \forall \phi\in W.$$

3 | FINITE ELEMENT DISCRETIZATION AND NUMERICAL SCHEMES

Let $\mathcal{T}_h$ be a partition of the domain Ω into triangles in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$, and h be the maximum diameter over all elements in the mesh. We define finite element spaces on $\mathcal{T}_h$

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}^1_{0,\Gamma_d}(\Omega)\cap \boldsymbol{C}^0(\overline{\Omega});\ \boldsymbol{v}_h|_E \in \boldsymbol{P}_k(E),\ \forall E\in\mathcal{T}_h\},$$
$$W_h := \{\phi_h \in L^2(\Omega)\cap C^0(\overline{\Omega});\ \phi_h|_E \in P_{k-1}(E),\ \forall E\in\mathcal{T}_h\},$$
$$M_h := \{\psi_h \in H^1_{0,\Gamma_p}(\Omega)\cap C^0(\overline{\Omega});\ \psi_h|_E \in P_l(E),\ \forall E\in\mathcal{T}_h\},$$

where $k \geq 2$ and $l \geq 1$ are two integers. In this work, the Taylor-Hood element, which consists of the pair $(\boldsymbol{V}_h, W_h)$, and Lagrange finite element are adopted to the pair $(\boldsymbol{u},\xi)$ and p, respectively. Based on the three-field formulation (8)–(10) and discrete spaces $\boldsymbol{V}_h$, W_h, and M_h, we define two projection operators [19, 23]. First, we introduce the Stokes projection operator $\boldsymbol{R}_{\boldsymbol{u}}\times R_\xi : \boldsymbol{V}\times W \to \boldsymbol{V}_h\times W_h$, which is defined by

$$a_1(\boldsymbol{R}_{\boldsymbol{u}}\boldsymbol{u},\boldsymbol{v}_h) - b(\boldsymbol{v}_h,R_\xi\xi) = a_1(\boldsymbol{u},\boldsymbol{v}_h) - b(\boldsymbol{v}_h,\xi), \quad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{15}$$

$$b(\boldsymbol{R}_{\boldsymbol{u}}\boldsymbol{u},\phi_h) = b(\boldsymbol{u},\phi_h), \qquad \forall \phi_h\in W_h. \tag{16}$$

Second, we define the elliptic projection operator $R_p : M \to M_h$ as follows

$$d(R_p p,\psi_h) = d(p,\psi_h), \qquad \forall \psi_h \in M_h. \tag{17}$$

If $\boldsymbol{u} \in \boldsymbol{H}^{k+1}_{0,\Gamma_d}(\Omega)$, $\xi \in H^k(\Omega)$, and $p \in H^{l+1}_{0,\Gamma_p}(\Omega)$, then the following error estimates hold true for the Stokes projection operator and the elliptic projection operator [6].

$$\|\boldsymbol{u}-\boldsymbol{R}_{\boldsymbol{u}}\boldsymbol{u}\|_{H^1(\Omega)}+\|\xi-R_\xi\xi\|_{L^2(\Omega)}\le Ch^k\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+\|\xi\|_{H^k(\Omega)}\right), \tag{18}$$

$$\|p-R_pp\|_{H^1(\Omega)}\le Ch^l\|p\|_{H^{l+1}(\Omega)}. \tag{19}$$

Under the assumption that the domain Ω has the full elliptic regularity, there holds

$$\|p-R_pp\|_{L^2(\Omega)}\le Ch^{l+1}\|p\|_{H^{l+1}(\Omega)}. \tag{20}$$

As we use a stable Stokes element pair, the corresponding finite element spaces satisfy the following discrete inf-sup condition, that is, there exists a positive constant $\tilde{\beta}$ independent of h such that

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{b\left(\boldsymbol{v}_h,\phi_h\right)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}}\ge\tilde{\beta}\|\phi_h\|_{L^2(\Omega)},\quad\forall\phi_h\in W_h. \tag{21}$$

An equidistant partition $0=t_0<t_1<\cdots<t_{N+1}=T$ with a step size Δt is considered for the time discretization. For simplicity, we introduce the notations $\boldsymbol{u}^n=\boldsymbol{u}\left(t_n\right)$, $\xi^n=\xi\left(t_n\right)$ and $p^n=p\left(t_n\right)$. Suitable approximation of initial conditions $\boldsymbol{u}_h^0=\boldsymbol{R}_{\boldsymbol{u}}\boldsymbol{u}^0$, $\xi_h^0=R_\xi\xi^0$, and $p_h^0=R_pp^0$ is considered here. Following [7, 14, 22, 23], we present the first monolithic scheme using the backward Euler method for the time discretization as follows.

Method 1: Given $\left(\boldsymbol{u}_h^n,\xi_h^n,p_h^n\right)\in\boldsymbol{V}_h\times W_h\times M_h$, find $\left(\boldsymbol{u}_h^{n+1},\xi_h^{n+1},p_h^{n+1}\right)\in\boldsymbol{V}_h\times W_h\times M_h$ such that

$$a_1\left(\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h\right)-b\left(\boldsymbol{v}_h,\xi_h^{n+1}\right)=\left(\boldsymbol{f}^{n+1},\boldsymbol{v}_h\right)+\langle\boldsymbol{h}^{n+1},\boldsymbol{v}_h\rangle_{\Gamma_t},\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{22}$$

$$b\left(\boldsymbol{u}_h^{n+1},\phi_h\right)+a_2\left(\xi_h^{n+1},\phi_h\right)-c\left(p_h^{n+1},\phi_h\right)=0,\qquad\forall\phi_h\in W_h, \tag{23}$$

$$\begin{aligned}a_3&\left(\frac{p_h^{n+1}-p_h^n}{\Delta t},\psi_h\right)-c\left(\psi_h,\frac{\xi_h^{n+1}-\xi_h^n}{\Delta t}\right)\\&+d\left(p_h^{n+1},\psi_h\right)=\left(Q_s^{n+1},\psi_h\right)+\langle g_2^{n+1},\psi_h\rangle_{\Gamma_f},\qquad\forall\psi_h\in M_h.\end{aligned} \tag{24}$$

The second monolithic scheme considers the combination of the backward Euler and Crank-Nicolson methods for time discretization, which is inspired from [17]. In the entire time interval, we solve the problem as follows.

Method 2: Given $\left(\boldsymbol{u}_h^n,\xi_h^n,p_h^n\right)\in\boldsymbol{V}_h\times W_h\times M_h$, find $\left(\boldsymbol{u}_h^{n+1},\xi_h^{n+1},p_h^{n+1}\right)\in\boldsymbol{V}_h\times W_h\times M_h$ such that

$$a_1\left(\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h\right)-b\left(\boldsymbol{v}_h,\xi_h^{n+1}\right)=\left(\boldsymbol{f}^{n+1},\boldsymbol{v}_h\right)+\langle\boldsymbol{h}^{n+1},\boldsymbol{v}_h\rangle_{\Gamma_t},\quad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{25}$$

$$b\left(\boldsymbol{u}_h^{n+1},\phi_h\right)+a_2\left(\xi_h^{n+1},\phi_h\right)-c\left(p_h^{n+1},\phi_h\right)=0,\qquad\forall\phi_h\in W_h, \tag{26}$$

$$\begin{aligned}a_3&\left(\frac{p_h^{n+1}-p_h^n}{\Delta t},\psi_h\right)-c\left(\psi_h,\frac{\xi_h^{n+1}-\xi_h^n}{\Delta t}\right)+d\left(\frac{p_h^{n+1}+p_h^n}{2},\psi_h\right)\\&=\frac{1}{2}\left(Q_s^{n+1}+Q_s^n,\psi_h\right)+\frac{1}{2}\langle g_2^{n+1}+g_2^n,\psi_h\rangle_{\Gamma_f},\qquad\forall\psi_h\in M_h.\end{aligned} \tag{27}$$

We note that the computational cost for implementing Method 2 is nearly identical to that of Method 1, as the only change required is the substitution of (24) with (27). Furthermore, in practice, we have observed that the runtime overheads of both Methods 1 and 2 for solving each step are almost indistinguishable. Therefore, Method 2 is as efficient as Method 1, while providing more accurate numerical solutions without significantly increasing the computational burden.

Next, we state the following basic propositions.

Proposition 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has $k+1$ continuous derivatives on an open interval (a, b). For any $t_0, t \in (a, b)$, there holds*

$$f(t) = f(t_0) + f'(t_0)(t - t_0) + \cdots + \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k + \frac{1}{k!}\int_{t_0}^{t} f^{(k+1)}(s)(t - s)^k ds.$$

Then, the following estimate for the L^2-norm of the last term holds true.

$$\left\| \frac{1}{k!}\int_{t_0}^{t} f^{(k+1)}(s)(t - s)^k ds \right\|_{L^2(\Omega)}^2 \lesssim (b - a)^{2k+1} \left| \int_{t_0}^{t} \|f^{(k+1)}\|_{L^2(\Omega)}^2 ds \right|. \tag{28}$$

Proof. The first part of the conclusion comes from the Taylor expansion theorem. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left| \int_{t_0}^{t} f^{(k+1)}(s)(t - s)^k ds \right|^2 &\leq \int_{t_0}^{t} |t - s|^{2k} ds \int_{t_0}^{t} \left| f^{(k+1)}(s) \right|^2 ds \\ &\leq \frac{(t - t_0)^{2k+1}}{2k + 1} \int_{t_0}^{t} \left| f^{(k+1)}(s) \right|^2 ds, \end{aligned}$$

which implies (28) directly. ■

Proposition 3.2. *Let B be a symmetric bilinear form, there holds*

$$2B(u, u - v) = B(u, u) - B\big(v, v\big) + B\big(u - v, u - v\big), \tag{29}$$

$$B\big(u + v, u - v\big) = B\big(u, u\big) - B\big(v, v\big), \tag{30}$$

which immediately implies the following inequality

$$2B\big(u, u - v\big) \geq B\big(u, u\big) - B\big(v, v\big). \tag{31}$$

Instead of relying on Grönwall's inequality, we utilize an alternative lemma, which is valuable in estimating errors related to long-time stability. Further details can be found in [7, 16, 17, 19].

Lemma 3.3. *Let $\{X_n\}_{n=1}^N$, $\{D_n\}_{n=1}^N$ and $\{G_n\}_{n=1}^N$ be finite sequences of functions, and C_0, C_1 be non-negative constants such that*

$$X_n^2 \leq C_0 X_0^2 + C_1 X_0 + D_n + \sum_{j=1}^{n} G_j X_j \quad \text{for all } 1 \leq n \leq N.$$

Then, there holds

$$X_n^2 \lesssim X_0^2 + \max\left\{ C_1^2 + \sum_{j=1}^{n} G_j^2, D_n \right\} \quad \text{for all } 1 \leq n \leq N.$$

4 | MAIN RESULTS

In this section, we present our main results describing a priori error estimates of the proposed schemes. Here, we decompose error terms as

$$\begin{aligned} e_{\boldsymbol{u}}^n &= \boldsymbol{u}^n - \boldsymbol{u}_h^n = \big(\boldsymbol{u}^n - \boldsymbol{R}_{\boldsymbol{u}}\boldsymbol{u}^n\big) + \big(\boldsymbol{R}_{\boldsymbol{u}}\boldsymbol{u}^n - \boldsymbol{u}_h^n\big) =: e_{\boldsymbol{u}}^{I,n} + e_{\boldsymbol{u}}^{h,n}, \\ e_{\xi}^n &= \xi^n - \xi_h^n = \big(\xi^n - R_{\xi}\xi^n\big) + \big(R_{\xi}\xi^n - \xi_h^n\big) =: e_{\xi}^{I,n} + e_{\xi}^{h,n}, \\ e_p^n &= p^n - p_h^n = \big(p^n - R_p p^n\big) + \big(R_p p^n - p_h^n\big) =: e_p^{I,n} + e_p^{h,n}. \end{aligned}$$

We also define

$$D_{\boldsymbol{u}}^{n+1} := e_{\boldsymbol{u}}^{h,n+1} - e_{\boldsymbol{u}}^{h,n}, \quad D_{\xi}^{n+1} := e_{\xi}^{h,n+1} - e_{\xi}^{h,n}, \quad D_{p}^{n+1} := e_{p}^{h,n+1} - e_{p}^{h,n}.$$

Then, we present a priori error estimates for both schemes. The proof of each method consists of three parts. In the first part, we focus on the $\boldsymbol{H}^1$ norm of $e_{\boldsymbol{u}}^{h,N+1}$, L^2 norm of $e_{\xi}^{h,N+1}$, and L^2 norm of $e_{p}^{h,N+1}$. In the second part, we complete the H^1 norm estimate of $e_{p}^{h,N+1}$. In the third part, we draw our main conclusions describing the error bound.

4.1 | A priori error estimates for method 1

Theorem 4.1. *Let $(\boldsymbol{u},\xi,p)$ and $(\boldsymbol{u}_h^{n+1},\xi_h^{n+1},p_h^{n+1})$ be the solutions of Equations (11)–(13) and (22)–(24), respectively. Assume that $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}_{0,\Gamma_d}^{k+1}(\Omega))$, $\partial_t \boldsymbol{u} \in L^2(0,T;\boldsymbol{H}_{0,\Gamma_d}^{k+1}(\Omega))$, $\partial_{tt}\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}_{0,\Gamma_d}^{1}(\Omega))$, $\xi \in L^\infty(0,T;H^k(\Omega))$, $\partial_t\xi \in L^2(0,T;H^k(\Omega))$, $\partial_{tt}\xi \in L^2(0,T;L^2(\Omega))$, $p \in L^\infty(0,T;H_{0,\Gamma_p}^{l+1}(\Omega))$, $\partial_t p \in L^2(0,T;H_{0,\Gamma_p}^{l+1}(\Omega))$, $\partial_{tt}p \in L^2(0,T;L^2(\Omega))$. There holds*

$$\begin{aligned}
&\|\varepsilon(e_{\boldsymbol{u}}^{h,N+1})\|_{L^2(\Omega)}^2 + \|e_{\xi}^{h,N+1}\|_{L^2(\Omega)}^2 + \|e_{p}^{h,N+1}\|_{L^2(\Omega)}^2 + \Delta t\sum_{n=0}^{N}\|\nabla e_{p}^{h,n+1}\|_{L^2(\Omega)}^2 \\
&\lesssim (\Delta t)^2\int_0^T \left(\|\partial_{tt}\boldsymbol{u}\|_{H^1(\Omega)}^2 + \|\partial_{tt}\xi\|_{L^2(\Omega)}^2 + \|\partial_{tt}p\|_{L^2(\Omega)}^2\right) ds \\
&\quad + h^{2k}\int_0^T \left(\|\partial_t\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t\xi\|_{H^k(\Omega)}^2\right) ds + h^{2l+2}\int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds.
\end{aligned} \tag{32}$$

Proof. Setting $t = t^{n+1}$ in (11), (12), (13), and letting the test functions be the discrete test functions, then subtracting (22), (23), and (24) from these equations, we immediately derive the following error equations.

$$\begin{aligned}
&a_1(e_{\boldsymbol{u}}^{n+1},\boldsymbol{v}_h) - b(\boldsymbol{v}_h,e_{\xi}^{n+1}) = 0,\\
&b(e_{\boldsymbol{u}}^{n+1},\phi_h) + a_2(e_{\xi}^{n+1},\phi_h) - c(e_{p}^{n+1},\phi_h) = 0,\\
&a_3\left(\partial_t p^{n+1} - \frac{p_h^{n+1}-p_h^n}{\Delta t},\psi_h\right) - c\left(\psi_h,\partial_t\xi^{n+1} - \frac{\xi_h^{n+1}-\xi_h^n}{\Delta t}\right) + d(e_p^{n+1},\psi_h) = 0.
\end{aligned}$$

By using the assumptions of the projection operators (15), (16), and (17), the above equations can be rewritten as

$$a_1(e_{\boldsymbol{u}}^{h,n+1},\boldsymbol{v}_h) - b(\boldsymbol{v}_h,e_{\xi}^{h,n+1}) = 0, \tag{33}$$

$$b(e_{\boldsymbol{u}}^{h,n+1},\phi_h) + a_2(e_{\xi}^{n+1},\phi_h) - c(e_{p}^{n+1},\phi_h) = 0, \tag{34}$$

$$\begin{aligned}
&a_3(D_p^{n+1},\psi_h) - c(\psi_h,D_\xi^{n+1}) + \Delta t d(e_p^{h,n+1},\psi_h)\\
&= a_3(R_p p^{n+1} - R_p p^n - \Delta t\partial_t p^{n+1},\psi_h) - c(\psi_h,R_\xi\xi^{n+1} - R_\xi\xi^n - \Delta t\partial_t\xi^{n+1}).
\end{aligned} \tag{35}$$

Differentiating (12) with respect to t at the $(n+1)$-th time step, then multiplying the resulted equation by Δt, we derive that

$$b(\Delta t\partial_t\boldsymbol{u}^{n+1},\phi_h) + a_2(\Delta t\partial_t\xi^{n+1},\phi_h) - c(\Delta t\partial_t p^{n+1},\phi_h) = 0. \tag{36}$$

For (34), we write out the schemes for $t = t_{n+1}$ and $t = t_n$, then take a difference between the two resulted equations, we see that

$$b\left(D_{\boldsymbol{u}}^{n+1}, \phi_h\right) + a_2\left(e_\xi^{n+1} - e_\xi^n, \phi_h\right) - c\left(e_p^{n+1} - e_p^n, \phi_h\right) = 0. \tag{37}$$

Using the definitions of D_ξ^{n+1} and D_p^{n+1}, we can reformulate (37) as follows

$$\begin{aligned} &b\left(D_{\boldsymbol{u}}^{n+1}, \phi_h\right) + a_2\left(D_\xi^{n+1}, \phi_h\right) - c\left(D_p^{n+1}, \phi_h\right) \\ &\quad = -a_2\left(\xi^{n+1} - \xi^n, \phi_h\right) + c\left(p^{n+1} - p^n, \phi_h\right) \\ &\qquad + a_2\left(R_\xi \xi^{n+1} - R_\xi \xi^n, \phi_h\right) - c\left(R_p p^{n+1} - R_p p^n, \phi_h\right). \end{aligned} \tag{38}$$

After using the $(n+1)$-th and n-th time steps of (12) to obtain $b\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \phi_h\right) = -a_2\left(\xi^{n+1} - \xi^n, \phi_h\right) + c\left(p^{n+1} - p^n, \phi_h\right)$, we combine (36) and (38) to get

$$\begin{aligned} &b\left(D_{\boldsymbol{u}}^{n+1}, \phi_h\right) + a_2\left(D_\xi^{n+1}, \phi_h\right) - c\left(D_p^{n+1}, \phi_h\right) = b\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n - \Delta t \partial_t \boldsymbol{u}^{n+1}, \phi_h\right) \\ &\quad + a_2\left(R_\xi \xi^{n+1} - R_\xi \xi^n - \Delta t \partial_t \xi^{n+1}, \phi_h\right) - c\left(R_p p^{n+1} - R_p p^n - \Delta t \partial_t p^{n+1}, \phi_h\right). \end{aligned} \tag{39}$$

Choosing $\boldsymbol{v}_h = D_{\boldsymbol{u}}^{n+1}$ in (33), $\phi_h = e_\xi^{h,n+1}$ in (39), and $\psi_h = e_p^{h,n+1}$ in (35), we derive

$$a_1\left(e_{\boldsymbol{u}}^{h,n+1}, D_{\boldsymbol{u}}^{n+1}\right) - b\left(D_{\boldsymbol{u}}^{n+1}, e_\xi^{h,n+1}\right) = 0, \tag{40}$$

$$\begin{aligned} &b\left(D_{\boldsymbol{u}}^{n+1}, e_\xi^{h,n+1}\right) + a_2\left(D_\xi^{n+1}, e_\xi^{h,n+1}\right) - c\left(D_p^{n+1}, e_\xi^{h,n+1}\right) \\ &= b\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n - \Delta t \partial_t \boldsymbol{u}^{n+1}, e_\xi^{h,n+1}\right) + a_2\left(R_\xi \xi^{n+1} - R_\xi \xi^n - \Delta t \partial_t \xi^{n+1}, e_\xi^{h,n+1}\right) \\ &\quad - c\left(R_p p^{n+1} - R_p p^n - \Delta t \partial_t p^{n+1}, e_\xi^{h,n+1}\right), \end{aligned} \tag{41}$$

$$\begin{aligned} &a_3\left(D_p^{n+1}, e_p^{h,n+1}\right) - c\left(e_p^{h,n+1}, D_\xi^{n+1}\right) + \Delta t d\left(e_p^{h,n+1}, e_p^{h,n+1}\right) \\ &\quad = a_3\left(R_p p^{n+1} - R_p p^n - \Delta t \partial_t p^{n+1}, e_p^{h,n+1}\right) \\ &\qquad - c\left(e_p^{h,n+1}, R_\xi \xi^{n+1} - R_\xi \xi^n - \Delta t \partial_t \xi^{n+1}\right). \end{aligned} \tag{42}$$

Taking the summation of (40), (41), and (42) over the index n from 0 to N yields

$$\begin{aligned} \mathrm{LHS}_1 := \sum_{n=0}^{N} \Big[&a_1\left(e_{\boldsymbol{u}}^{h,n+1}, D_{\boldsymbol{u}}^{n+1}\right) + a_2\left(D_\xi^{n+1}, e_\xi^{h,n+1}\right) - c\left(D_p^{n+1}, e_\xi^{h,n+1}\right) \\ &+ a_3\left(D_p^{n+1}, e_p^{h,n+1}\right) - c\left(e_p^{h,n+1}, D_\xi^{n+1}\right) + \Delta t d\left(e_p^{h,n+1}, e_p^{h,n+1}\right)\Big] = \sum_{i=1}^{5} E_i, \end{aligned} \tag{43}$$

where

$$\begin{aligned} E_1 &= \sum_{n=0}^{N} b\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n - \Delta t \partial_t \boldsymbol{u}^{n+1}, e_\xi^{h,n+1}\right), \\ E_2 &= \sum_{n=0}^{N} a_2\left(R_\xi \xi^{n+1} - R_\xi \xi^n - \Delta t \partial_t \xi^{n+1}, e_\xi^{h,n+1}\right), \\ E_3 &= \sum_{n=0}^{N} c\left(\Delta t \partial_t p^{n+1} - R_p p^{n+1} + R_p p^n, e_\xi^{h,n+1}\right), \end{aligned}$$

$$E_4 = \sum_{n=0}^{N} a_3\left(R_p p^{n+1} - R_p p^n - \Delta t \partial_t p^{n+1}, e_p^{h,n+1}\right),$$
$$E_5 = \sum_{n=0}^{N} c\left(e_p^{h,n+1}, \Delta t \partial_t \xi^{n+1} - R_\xi \xi^{n+1} + R_\xi \xi^n\right).$$

Using the definitions of $a_2(\cdot,\cdot)$, $a_3(\cdot,\cdot)$, and $c(\cdot,\cdot)$, we can simplify LHS_1 by the identity

$$\begin{aligned} &a_2(D_\xi^{n+1}, e_\xi^{h,n+1}) - c\left(D_p^{n+1}, e_\xi^{h,n+1}\right) + a_3\left(D_p^{n+1}, e_p^{h,n+1}\right) - c\left(e_p^{h,n+1}, D_\xi^{n+1}\right) \\ &= \frac{1}{\lambda}\int_\Omega \left(\alpha D_p^{n+1} - D_\xi^{n+1}\right)\left(\alpha e_p^{h,n+1} - e_\xi^{h,n+1}\right) + c_0 \int_\Omega \left(e_p^{h,n+1} - e_p^{h,n}\right) e_p^{h,n+1}. \end{aligned} \tag{44}$$

Applying (31) and (44), we obtain the following lower bound estimate for LHS_1

$$\begin{aligned} &\frac{1}{2}\Big(2\mu\|\varepsilon(e_{\boldsymbol{u}}^{h,N+1})\|_{L^2(\Omega)}^2 - 2\mu\|\varepsilon(e_{\boldsymbol{u}}^{h,0})\|_{L^2(\Omega)}^2 + c_0\|e_p^{h,N+1}\|_{L^2(\Omega)}^2 \\ &- c_0\|e_p^{h,0}\|_{L^2(\Omega)}^2 + \frac{1}{\lambda}\|\alpha e_p^{h,N+1} - e_\xi^{h,N+1}\|_{L^2(\Omega)}^2 \\ &- \frac{1}{\lambda}\|\alpha e_p^{h,0} - e_\xi^{h,0}\|_{L^2(\Omega)}^2\Big) + K\Delta t \sum_{n=0}^{N}\|\nabla e_p^{h,n+1}\|_{L^2(\Omega)}^2 \le \mathrm{LHS}_1. \end{aligned} \tag{45}$$

Next, we bound the terms E_i for $i = 1, 2, \cdots, 5$. We use the Cauchy-Schwarz inequality, the Young's inequality, (18), (20), and (28) to estimate E_1, E_2, and E_3 with an $\epsilon_1 > 0$ as follows.

$$\begin{aligned} E_1 &\le \frac{\epsilon_1}{6}\Delta t \sum_{n=0}^{N}\|e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 + \frac{C}{2\epsilon_1}(\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{u}\|_{H^1(\Omega)}^2 ds, \\ E_2 &\le \frac{\epsilon_1}{6}\Delta t \sum_{n=0}^{N}\|e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 + \frac{C}{2\epsilon_1}\Bigg[(\Delta t)^2 \int_0^T \|\partial_{tt}\xi\|_{L^2(\Omega)}^2 ds \\ &\quad + h^{2k}\int_0^T \left(\|\partial_t \boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t \xi\|_{H^k(\Omega)}^2\right) ds\Bigg], \\ E_3 &\le \frac{\epsilon_1}{6}\Delta t \sum_{n=0}^{N}\|e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 + \frac{C}{2\epsilon_1}\left[(\Delta t)^2 \int_0^T \|\partial_{tt} p\|_{L^2(\Omega)}^2 ds + h^{2l+2}\int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right]. \end{aligned}$$

Using the Cauchy-Schwarz inequality, Young's inequality, the Poincaré inequality, (18), (20), and (28), we can bound E_4 and E_5 with an $\epsilon_2 > 0$ as follows.

$$\begin{aligned} E_4 &\le \frac{\epsilon_2}{4}\Delta t \sum_{n=0}^{N}\|\nabla e_p^{h,n+1}\|_{L^2(\Omega)}^2 + \frac{C}{2\epsilon_2}\left[(\Delta t)^2 \int_0^T \|\partial_{tt} p\|_{L^2(\Omega)}^2 ds + h^{2l+2}\int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right], \\ E_5 &\le \frac{\epsilon_2}{4}\Delta t \sum_{n=0}^{N}\|\nabla e_p^{h,n+1}\|_{L^2(\Omega)}^2 + \frac{C}{2\epsilon_2}\Bigg[(\Delta t)^2 \int_0^T \|\partial_{tt}\xi\|_{L^2(\Omega)}^2 ds \\ &\quad + h^{2k}\int_0^T \left(\|\partial_t \boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t \xi\|_{H^k(\Omega)}^2\right) ds\Bigg]. \end{aligned}$$

Combining (43), (45), and the bounds E_i for $i = 1, 2, \cdots, 5$, we derive that

$$
\begin{aligned}
&2\mu\|\varepsilon(e_{\boldsymbol{u}}^{h,N+1})\|_{L^2(\Omega)}^2 - 2\mu\|\varepsilon(e_{\boldsymbol{u}}^{h,0})\|_{L^2(\Omega)}^2 + c_0\|e_p^{h,N+1}\|_{L^2(\Omega)}^2 - c_0\|e_p^{h,0}\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{\lambda}\|\alpha e_p^{h,N+1} - e_\xi^{h,N+1}\|_{L^2(\Omega)}^2 - \frac{1}{\lambda}\|\alpha e_p^{h,0} - e_\xi^{h,0}\|_{L^2(\Omega)}^2 + 2K\Delta t\sum_{n=0}^{N}\|\nabla e_p^{h,n+1}\|_{L^2(\Omega)}^2 \\
&\leq \epsilon_1 \Delta t \sum_{n=0}^{N}\|e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 + \epsilon_2 \Delta t\sum_{n=0}^{N}\|\nabla e_p^{h,n+1}\|_{L^2(\Omega)}^2 \\
&\quad + \left(\frac{C}{\epsilon_1} + \frac{C}{\epsilon_2}\right)\left[(\Delta t)^2\int_0^T \left(\|\partial_{tt}\boldsymbol{u}\|_{H^1(\Omega)}^2 + \|\partial_{tt}\xi\|_{L^2(\Omega)}^2 + \|\partial_{tt}p\|_{L^2(\Omega)}^2\right) ds\right. \\
&\quad \left. + h^{2k}\int_0^T \left(\|\partial_t\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t\xi\|_{H^k(\Omega)}^2\right) ds + h^{2l+2}\int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right].
\end{aligned} \tag{46}
$$

Using the inf-sup condition (21), (33), and the Korn's inequality (14), we have

$$
\|e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 \lesssim \sup_{\boldsymbol{v}_h\in \boldsymbol{V}_h}\frac{b(\boldsymbol{v}_h, e_\xi^{h,n+1})}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} = \sup_{\boldsymbol{v}_h\in \boldsymbol{V}_h}\frac{a_1(e_{\boldsymbol{u}}^{h,n+1}, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} \lesssim \|\varepsilon(e_{\boldsymbol{u}}^{h,n+1})\|_{L^2(\Omega)}^2, \tag{47}
$$

which easily implies that

$$
\|e_p^{h,n+1}\|_{L^2(\Omega)}^2 \lesssim \|\alpha e_p^{h,n+1} - e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 + \|\varepsilon(e_{\boldsymbol{u}}^{h,n+1})\|_{L^2(\Omega)}^2. \tag{48}
$$

Then, we handle (46). Considering the fact $\varepsilon(e_{\boldsymbol{u}}^{h,0}) = 0$, $e_p^{h,0} = 0$, $e_\xi^{h,0} = 0$, ignoring the term $c_0\|e_p^{h,N+1}\|_{L^2(\Omega)}^2$, using (47) to choose a small enough positive ϵ_1 such that $\epsilon_1\|e_\xi^{h,k+1}\|_{L^2(\Omega)}^2 \leq 2\mu\|\varepsilon(e_{\boldsymbol{u}}^{h,k+1})\|_{L^2(\Omega)}^2$ and setting $\epsilon_2 = K$, we can apply Lemma 3.3 to obtain

$$
\begin{aligned}
&2\mu\|\varepsilon(e_{\boldsymbol{u}}^{h,N+1})\|_{L^2(\Omega)}^2 + \frac{1}{\lambda}\|\alpha e_p^{h,N+1} - e_\xi^{h,N+1}\|_{L^2(\Omega)}^2 + K\Delta t\sum_{n=0}^{N}\|\nabla e_p^{h,n+1}\|_{L^2(\Omega)}^2 \\
&\lesssim (\Delta t)^2\int_0^T \left(\|\partial_{tt}\boldsymbol{u}\|_{H^1(\Omega)}^2 + \|\partial_{tt}\xi\|_{L^2(\Omega)}^2 + \|\partial_{tt}p\|_{L^2(\Omega)}^2\right) ds \\
&\quad + h^{2k}\int_0^T \left(\|\partial_t\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t\xi\|_{H^k(\Omega)}^2\right) ds + h^{2l+2}\int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds.
\end{aligned} \tag{49}
$$

Finally, we come to the conclusion that the desired result (32) holds after applying (47), (48), and (49). This completes the proof. ∎

Theorem 4.2. *Let* $(\boldsymbol{u}, \xi, p)$ *and* $(\boldsymbol{u}_h^{n+1}, \xi_h^{n+1}, p_h^{n+1})$ *be the solutions of Equations (11)–(13) and (22)–(24), respectively. Under the assumptions of Theorem 4.1, there holds*

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{n=0}^{N}\left(\|\varepsilon(D_{\boldsymbol{u}}^{n+1})\|_{L^2(\Omega)}^2 + \|D_\xi^{n+1}\|_{L^2(\Omega)}^2 + \|D_p^{n+1}\|_{L^2(\Omega)}^2\right) + \|\nabla e_p^{h,N+1}\|_{L^2(\Omega)}^2 \\
&\lesssim (\Delta t)^2\int_0^T \left(\|\partial_{tt}\boldsymbol{u}\|_{H^1(\Omega)}^2 + \|\partial_{tt}\xi\|_{L^2(\Omega)}^2 + \|\partial_{tt}p\|_{L^2(\Omega)}^2\right) ds \\
&\quad + h^{2k}\int_0^T \left(\|\partial_t\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t\xi\|_{H^k(\Omega)}^2\right) ds + h^{2l+2}\int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds.
\end{aligned} \tag{50}
$$

Proof. Taking the difference of the $(n+1)$-th, n-th steps of (33) yields

$$a_1\left(D_{\boldsymbol{u}}^{n+1}, \boldsymbol{v}_h\right) - b\left(\boldsymbol{v}_h, D_\xi^{n+1}\right) = 0, \tag{51}$$

After using the definitions of $a_2(\cdot,\cdot)$, $a_3(\cdot,\cdot)$, and $c(\cdot,\cdot)$, one has

$$\begin{aligned} &a_2\left(D_\xi^{n+1}, D_\xi^{n+1}\right) + a_3\left(D_p^{n+1}, D_p^{n+1}\right) - 2c\left(D_p^{n+1}, D_\xi^{n+1}\right) \\ &= \frac{1}{\lambda}\|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2 + c_0\|D_p^{h,n+1}\|_{L^2(\Omega)}^2. \end{aligned} \tag{52}$$

Choosing $\boldsymbol{v}_h = D_{\boldsymbol{u}}^{n+1}$ in (51), $\phi_h = D_\xi^{n+1}$ in (39), $\psi_h = D_p^{n+1}$ in (35), applying the identity (52), and summing over the index n from 0 to N, we get

$$\begin{aligned} &\sum_{n=0}^{N}\left(2\mu\|\varepsilon\left(D_{\boldsymbol{u}}^{n+1}\right)\|_{L^2(\Omega)}^2 + c_0\|D_p^{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{\lambda}\|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2\right) \\ &\quad + \Delta t\sum_{n=0}^{N} d\left(e_p^{h,n+1}, D_p^{n+1}\right) = \sum_{i=1}^{5} T_i, \end{aligned} \tag{53}$$

where

$$\begin{aligned} T_1 &= \sum_{n=0}^{N} b\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n - \Delta t\partial_t \boldsymbol{u}^{n+1}, D_\xi^{n+1}\right), \\ T_2 &= \sum_{n=0}^{N} a_2\left(R_\xi \xi^{n+1} - R_\xi \xi^n - \Delta t\partial_t \xi^{n+1}, D_\xi^{n+1}\right), \\ T_3 &= \sum_{n=0}^{N} c\left(\Delta t\partial_t p^{n+1} - R_p p^{n+1} + R_p p^n, D_\xi^{n+1}\right), \\ T_4 &= \sum_{n=0}^{N} a_3\left(R_p p^{n+1} - R_p p^n - \Delta t\partial_t p^{n+1}, D_p^{n+1}\right), \\ T_5 &= \sum_{n=0}^{N} c\left(D_p^{n+1}, \Delta t\partial_t \xi^{n+1} - R_\xi \xi^{n+1} + R_\xi \xi^n\right). \end{aligned}$$

Based on our observation, we have found that

$$\begin{aligned} T_2 + T_5 &= \sum_{n=0}^{N}\frac{1}{\lambda}\left(\alpha D_p^{n+1} - D_\xi^{n+1}, \Delta t\partial_t \xi^{n+1} - R_\xi \xi^{n+1} + R_\xi \xi^n\right) =: \tilde{T}_2, \\ T_3 + T_4 &= \sum_{n=0}^{N}\frac{\alpha}{\lambda}\left(R_p p^{n+1} - R_p p^n - \Delta t\partial_t p^{n+1}, \alpha D_p^{n+1} - D_\xi^{n+1}\right) \\ &\quad + \sum_{n=0}^{N} c_0\left(R_p p^{n+1} - R_p p^n - \Delta t\partial_t p^{n+1}, D_p^{n+1}\right) =: \tilde{T}_3 + \tilde{T}_4. \end{aligned}$$

Next, we bound the terms T_1, $\tilde{T}_2$, $\tilde{T}_3$, and $\tilde{T}_4$. Applying the Cauchy-Schwarz inequality, the Young's inequality, (18), (20), and (28), we have the following estimates with

$\epsilon_1, \epsilon_2, \epsilon_3 > 0$.

$$T_1 \le \epsilon_1 \sum_{n=0}^{N} \|D_\xi^{n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_1}(\Delta t)^3 \int_0^T \|\partial_{tt}\boldsymbol{u}\|_{H^1(\Omega)}^2 ds,$$

$$\tilde{T}_2 \le \frac{\epsilon_2}{2} \sum_{n=0}^{N} \frac{1}{\lambda} \|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_2}\left[(\Delta t)^3 \int_0^T \|\partial_{tt}\xi\|_{L^2(\Omega)}^2 ds + h^{2k}\Delta t \int_0^T \left(\|\partial_t \boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t \xi\|_{H^k(\Omega)}^2\right) ds\right],$$

$$\tilde{T}_3 \le \frac{\epsilon_2}{2} \sum_{n=0}^{N} \frac{1}{\lambda} \|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_2}\left[(\Delta t)^3 \int_0^T \|\partial_{tt} p\|_{L^2(\Omega)}^2 ds + h^{2l+2}\Delta t \int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right],$$

$$\tilde{T}_4 \le \epsilon_3 \sum_{n=0}^{N} c_0 \|D_p^{n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_3}\left[(\Delta t)^3 \int_0^T \|\partial_{tt} p\|_{L^2(\Omega)}^2 ds + h^{2l+2}\Delta t \int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right].$$

Similarly, using the inf-sup condition (21), (51), and the Korn's inequality (14) yields

$$\|D_\xi^{n+1}\|_{L^2(\Omega)}^2 \lesssim \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b\left(\boldsymbol{v}_h, D_\xi^{n+1}\right)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} = \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{a_1\left(D_{\boldsymbol{u}}^{n+1}, \boldsymbol{v}_h\right)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} \lesssim \|\varepsilon\left(D_{\boldsymbol{u}}^{n+1}\right)\|_{L^2(\Omega)}^2, \tag{54}$$

which directly implies

$$\|D_p^{n+1}\|_{L^2(\Omega)}^2 \lesssim \|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2 + \|\varepsilon\left(D_{\boldsymbol{u}}^{n+1}\right)\|_{L^2(\Omega)}^2. \tag{55}$$

Using (31) and the fact $e_p^{h,0} = 0$, we obtain

$$\begin{aligned}\Delta t \sum_{n=0}^{N} d\left(e_p^{h,n+1}, D_p^{n+1}\right) &\ge \frac{\Delta t}{2} \sum_{n=0}^{N} \left[d\left(e_p^{h,n+1}, e_p^{h,n+1}\right) - d\left(e_p^{h,n}, e_p^{h,n}\right)\right] \\ &\ge \frac{\Delta t}{2} d\left(e_p^{h,N+1}, e_p^{h,N+1}\right).\end{aligned} \tag{56}$$

Based on (53), considering $\epsilon_2 = \frac{1}{2}$ and $\epsilon_3 = 1$, using (54) to choose a small enough positive ϵ_1 such that $\epsilon_1 \|D_\xi^{n+1}\|_{L^2(\Omega)}^2 \le \mu \|\varepsilon\left(D_{\boldsymbol{u}}^{n+1}\right)\|_{L^2(\Omega)}^2$, we can apply (56) and bounds of $T_1, \tilde{T}_2, \tilde{T}_3, \tilde{T}_4$ to obtain

$$\begin{aligned}&\frac{1}{\Delta t} \sum_{n=0}^{N} \left(\mu \|\varepsilon\left(D_{\boldsymbol{u}}^{n+1}\right)\|_{L^2(\Omega)}^2 + \frac{1}{2\lambda}\|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2\right) + \frac{K}{2}\|e_p^{h,N+1}\|_{L^2(\Omega)}^2 \\ &\le \left(\frac{C}{\epsilon_1} + \frac{C}{\epsilon_2} + \frac{C}{\epsilon_3}\right)\left[(\Delta t)^2 \int_0^T \left(\|\partial_{tt}\boldsymbol{u}\|_{H^1(\Omega)}^2 + \|\partial_{tt}\xi\|_{L^2(\Omega)}^2 + \|\partial_{tt} p\|_{L^2(\Omega)}^2\right) ds \right.\\ &\quad \left. + h^{2k} \int_0^T \left(\|\partial_t \boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t \xi\|_{H^k(\Omega)}^2\right) ds + h^{2l+2} \int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right].\end{aligned}$$

By using (54), (55), and the above estimate, the proof is completed. ■

Theorem 4.3. *Let $(\boldsymbol{u}, \xi, p)$ and $(\boldsymbol{u}_h^{n+1}, \xi_h^{n+1}, p_h^{n+1})$ be the solutions of Equations (11)–(13) and (22)–(24), respectively. Under the assumptions of Theorem 4.1, there holds*

$$\|\varepsilon(e_{\boldsymbol{u}}^{N+1})\|_{L^2(\Omega)} + \|e_\xi^{N+1}\|_{L^2(\Omega)} + \|e_p^{N+1}\|_{L^2(\Omega)} \lesssim \Delta t + h^k + h^{l+1}, \tag{57}$$

$$\|\nabla e_p^{N+1}\|_{L^2(\Omega)} \lesssim \Delta t + h^k + h^l. \tag{58}$$

Proof. We start with (18), (19), and (20). Applying the triangle inequality, Theorems 4.1 and 4.2, we see that the above error estimates readily follow. ∎

4.2 | A priori error estimates for method 2

Theorem 4.4. *Let $(\boldsymbol{u}, \xi, p)$ and $(\boldsymbol{u}_h^{n+1}, \xi_h^{n+1}, p_h^{n+1})$ be the solutions of Equations (11)–(13) and (25)–(27), respectively. Assume that $\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}_{0,\Gamma_d}^{k+1}(\Omega))$, $\partial_t \boldsymbol{u} \in L^2(0, T; \boldsymbol{H}_{0,\Gamma_d}^{k+1}(\Omega))$, $\partial_{tt} \boldsymbol{u} \in L^2(0, T; \boldsymbol{H}_{0,\Gamma_d}^{1}(\Omega))$, $\partial_{ttt} \boldsymbol{u} \in L^2(0, T; \boldsymbol{H}_{0,\Gamma_d}^{1}(\Omega))$, $\xi \in L^\infty(0, T; H^k(\Omega))$, $\partial_t \xi \in L^2(0, T; H^k(\Omega))$, $\partial_{tt} \xi \in L^2(0, T; L^2(\Omega))$, $\partial_{ttt} \xi \in L^2(0, T; L^2(\Omega))$, $p \in L^\infty(0, T; H_{0,\Gamma_p}^{l+1}(\Omega))$, $\partial_t p \in L^2(0, T; H_{0,\Gamma_p}^{l+1}(\Omega))$, $\partial_{tt} p \in L^2(0, T; L^2(\Omega))$, $\partial_{ttt} p \in L^2(0, T; L^2(\Omega))$. There holds*

$$\begin{aligned}
&\|\varepsilon(e_{\boldsymbol{u}}^{h,N+1})\|_{L^2(\Omega)}^2 + \|e_\xi^{h,N+1}\|_{L^2(\Omega)}^2 + \|e_p^{h,N+1}\|_{L^2(\Omega)}^2 + \Delta t \sum_{n=0}^{N} \|\nabla(e_p^{h,n+1} + e_p^{h,n})\|_{L^2(\Omega)}^2 \\
&\lesssim (\Delta t)^4 \int_0^T \left(\|\partial_{ttt} \boldsymbol{u}\|_{H^1(\Omega)}^2 + \|\partial_{ttt} \xi\|_{L^2(\Omega)}^2 + \|\partial_{ttt} p\|_{L^2(\Omega)}^2\right) ds \\
&\quad + h^{2k} \int_0^T \left(\|\partial_t \boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t \xi\|_{H^k(\Omega)}^2\right) ds + h^{2l+2} \int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds.
\end{aligned} \tag{59}$$

Proof. Firstly, we note that (33) and (34) still hold here. Summing up the $(n+1)$-th, n-th steps of (33), and following a similar argument of (39), we get

$$a_1(e_{\boldsymbol{u}}^{h,n+1} + e_{\boldsymbol{u}}^{h,n}, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, e_\xi^{h,n+1} + e_\xi^{h,n}) = 0, \tag{60}$$

$$\begin{aligned}
&b(D_{\boldsymbol{u}}^{n+1}, \phi_h) + a_2(D_\xi^{n+1}, \phi_h) - c(D_p^{n+1}, \phi_h) \\
&= b\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n - \frac{\Delta t \partial_t \boldsymbol{u}^{n+1} + \Delta t \partial_t \boldsymbol{u}^n}{2}, \phi_h\right) \\
&\quad + a_2\left(R_\xi \xi^{n+1} - R_\xi \xi^n - \frac{\Delta t \partial_t \xi^{n+1} + \Delta t \partial_t \xi^n}{2}, \phi_h\right) \\
&\quad - c\left(R_p p^{n+1} - R_p p^n - \frac{\Delta t \partial_t p^{n+1} + \Delta t \partial_t p^n}{2}, \phi_h\right).
\end{aligned} \tag{61}$$

After summing up the $(n+1)$-th, n-th time steps of (13), we multiply $\frac{1}{2}$ to get

$$\begin{aligned}
&a_3\left(\frac{\partial_t p^{n+1} + \partial_t p^n}{2}, \psi_h\right) - c\left(\psi_h, \frac{\partial_t \xi^{n+1} + \partial_t \xi^n}{2}\right) + d\left(\frac{p^{n+1} + p^n}{2}, \psi_h\right) \\
&= \frac{1}{2}(Q_s^{n+1} + Q_s^n, \psi_h) + \frac{1}{2}\langle g_2^{n+1} + g_2^n, \psi_h \rangle_{\Gamma_f}.
\end{aligned} \tag{62}$$

Subtracting (27) from (62) yields

$$a_3\left(\frac{\partial_t p^{n+1}+\partial_t p^n}{2}-\frac{p_h^{n+1}-p_h^n}{\Delta t},\psi_h\right)+d\left(\frac{e_p^{n+1}+e_p^n}{2},\psi_h\right)$$
$$=c\left(\psi_h,\frac{\partial_t\xi^{n+1}+\partial_t\xi^n}{2}-\frac{\xi_h^{n+1}-\xi_h^n}{\Delta t}\right). \tag{63}$$

Following the same argument as (35) in Theorem 4.1, we apply the projection operator (17) to (63) and derive

$$a_3\big(D_p^{n+1},\psi_h\big)-c\big(\psi_h,D_\xi^{n+1}\big)+\frac{\Delta t}{2}d\big(e_p^{h,n+1}+e_p^{h,n},\psi_h\big)$$
$$=a_3\left(R_pp^{n+1}-R_pp^n-\frac{\Delta t\partial_t p^{n+1}+\Delta t\partial_t p^n}{2},\psi_h\right)$$
$$-c\left(\psi_h,R_\xi\xi^{n+1}-R_\xi\xi^n-\frac{\Delta t\partial_t\xi^{n+1}+\Delta t\partial_t\xi^n}{2}\right). \tag{64}$$

Choosing $\boldsymbol{v}_h=D_{\boldsymbol{u}}^{n+1}$ in (60), $\phi_h=e_\xi^{h,n+1}+e_\xi^{h,n}$ in (61), $\psi_h=e_p^{h,n+1}+e_p^{h,n}$ in (64), and summing over the index n from 0 to N yield

$$\mathrm{LHS}_2:=\sum_{n=0}^{N}\Big[a_1\big(e_{\boldsymbol{u}}^{h,n+1}+e_{\boldsymbol{u}}^{h,n},D_{\boldsymbol{u}}^{n+1}\big)+a_2\big(D_\xi^{n+1},e_\xi^{h,n+1}+e_\xi^{h,n}\big)$$
$$-c\big(D_p^{n+1},e_\xi^{h,n+1}+e_\xi^{h,n}\big)+a_3\big(D_p^{n+1},e_p^{h,n+1}+e_p^{h,n}\big)-c\big(e_p^{h,n+1}+e_p^{h,n},D_\xi^{n+1}\big)$$
$$+\frac{\Delta t}{2}d\left(e_p^{h,n+1}+e_p^{h,n},e_p^{h,n+1}+e_p^{h,n}\right)\Big]=\sum_{i}^{5}J_i, \tag{65}$$

where

$$J_1=\sum_{n=0}^{N}b\left(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n-\frac{\Delta t\partial_t\boldsymbol{u}^{n+1}+\Delta t\partial_t\boldsymbol{u}^n}{2},e_\xi^{h,n+1}+e_\xi^{h,n}\right),$$
$$J_2=\sum_{n=0}^{N}a_2\left(R_\xi\xi^{n+1}-R_\xi\xi^n-\frac{\Delta t\partial_t\xi^{n+1}+\Delta t\partial_t\xi^n}{2},e_\xi^{h,n+1}+e_\xi^{h,n}\right),$$
$$J_3=\sum_{n=0}^{N}c\left(\frac{\Delta t\partial_t p^{n+1}+\Delta t\partial_t p^n}{2}-R_pp^{n+1}+R_pp^n,e_\xi^{h,n+1}+e_\xi^{h,n}\right),$$
$$J_4=\sum_{n=0}^{N}a_3\left(R_pp^{n+1}-R_pp^n-\frac{\Delta t\partial_t p^{n+1}+\Delta t\partial_t p^n}{2},e_p^{h,n+1}+e_p^{h,n}\right),$$
$$J_5=\sum_{n=0}^{N}c\left(e_p^{h,n+1}+e_p^{h,n},\frac{\Delta t\partial_t\xi^{n+1}+\Delta t\partial_t\xi^n}{2}-R_\xi\xi^{n+1}+R_\xi\xi^n\right).$$

Using the definitions of $a_2(\cdot,\cdot)$, $a_3(\cdot,\cdot)$, and $c(\cdot,\cdot)$, we can simplify LHS_2 by the identity

$$a_2\big(D_\xi^{n+1},e_\xi^{h,n+1}+e_\xi^{h,n}\big)-c\big(D_p^{n+1},e_\xi^{h,n+1}+e_\xi^{h,n}\big)$$
$$+a_3\big(D_p^{n+1},e_p^{h,n+1}+e_p^{h,n}\big)-c\big(e_p^{h,n+1}+e_p^{h,n},D_\xi^{n+1}\big)$$
$$=c_0\left(\|e_p^{h,n+1}\|_{L^2(\Omega)}^2-\|e_p^{h,n}\|_{L^2(\Omega)}^2\right)$$
$$+\frac{1}{\lambda}\left(\|\alpha e_p^{h,n+1}-e_\xi^{h,n+1}\|_{L^2(\Omega)}^2-\|\alpha e_p^{h,n}-e_\xi^{h,n}\|_{L^2(\Omega)}^2\right). \tag{66}$$

Applying (65) and (66), we obtain

$$\begin{aligned}\mathrm{LHS}_2 &= 2\mu\|\varepsilon(e_{\boldsymbol{u}}^{h,N+1})\|_{L^2(\Omega)}^2 - 2\mu\|\varepsilon(e_{\boldsymbol{u}}^{h,0})\|_{L^2(\Omega)}^2 + c_0\|e_p^{h,N+1}\|_{L^2(\Omega)}^2 \\ &\quad - c_0\|e_p^{h,0}\|_{L^2(\Omega)}^2 + \frac{1}{\lambda}\|\alpha e_p^{h,N+1} - e_\xi^{h,N+1}\|_{L^2(\Omega)}^2 \\ &\quad - \frac{1}{\lambda}\|\alpha e_p^{h,0} - e_\xi^{h,0}\|_{L^2(\Omega)}^2 + \frac{K\Delta t}{2}\sum_{n=0}^{N}\|\nabla(e_p^{h,n+1} + e_p^{h,n})\|_{L^2(\Omega)}^2. \end{aligned} \tag{67}$$

Assuming that f is three times differentiable with respect to t and f''' is continuous in $[0, T]$, the Taylor expansion Theorem implies

$$\begin{aligned} f(t_{n+\frac{1}{2}}) &= f(t_n) + \frac{\Delta t}{2}f'(t_n) + \frac{(\Delta t)^2}{8}f''(\eta_1), \\ f(t_{n+\frac{1}{2}}) &= f(t_{n+1}) - \frac{\Delta t}{2}f'(t_{t+1}) + \frac{(\Delta t)^2}{8}f''(\eta_2), \end{aligned}$$

where $\eta_1 \in (t_n, t_{n+\frac{1}{2}})$, and $\eta_2 \in (t_{n+\frac{1}{2}}, t_{n+1})$. It follows that

$$f(t_{n+1}) - f(t_n) - \frac{\Delta t f'(t_{n+1}) + \Delta t f'(t_n)}{2} = \frac{(\Delta t)^2}{8}\left[f''(\eta_2) - f''(\eta_1)\right]. \tag{68}$$

Next, we bound the terms J_i for $i = 1, 2, \cdots, 5$. Applying the Cauchy-Schwarz inequality, the Young's inequality, (18), (20), (68), and (28), we can bound J_1, J_2, and J_3 with an $\epsilon_1 > 0$ as follows

$$\begin{aligned} J_1 &\le \frac{\epsilon_1}{3}\Delta t\sum_{n=0}^{N}\|e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_1}(\Delta t)^4\int_0^T \|\partial_{ttt}\boldsymbol{u}\|_{H^1(\Omega)}^2 ds, \\ J_2 &\le \frac{\epsilon_1}{3}\Delta t\sum_{n=0}^{N}\|e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_1}\Bigg[(\Delta t)^4\int_0^T \|\partial_{ttt}\xi\|_{L^2(\Omega)}^2 ds \\ &\quad + h^{2k}\int_0^T \left(\|\partial_t\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t\xi\|_{H^k(\Omega)}^2\right) ds\Bigg], \\ J_3 &\le \frac{\epsilon_1}{3}\Delta t\sum_{n=0}^{N}\|e_\xi^{h,n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_1}\left[(\Delta t)^4\int_0^T \|\partial_{ttt}p\|_{L^2(\Omega)}^2 ds + h^{2l+2}\int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right]. \end{aligned}$$

Using the Cauchy-Schwarz inequality, the Young's inequality, the Poincaré inequality, (18), (20), (68), and (28), we can bound E_4 and E_5 with an $\epsilon_2 > 0$ as follows.

$$\begin{aligned} J_4 &\le \frac{\epsilon_2}{2}\Delta t\sum_{n=0}^{N}\|\nabla(e_p^{h,n+1} + e_p^{h,n})\|_{L^2(\Omega)}^2 \\ &\quad + \frac{C}{\epsilon_2}\left[(\Delta t)^4\int_0^T \|\partial_{ttt}p\|_{L^2(\Omega)}^2 ds + h^{2l+2}\int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right], \\ J_5 &\le \frac{\epsilon_2}{2}\Delta t\sum_{n=0}^{N}\|\nabla(e_p^{h,n+1} + e_p^{h,n})\|_{L^2(\Omega)}^2 \\ &\quad + \frac{C}{\epsilon_2}\left[(\Delta t)^4\int_0^T \|\partial_{ttt}\xi\|_{L^2(\Omega)}^2 ds + h^{2k}\int_0^T \left(\|\partial_t\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t\xi\|_{H^k(\Omega)}^2\right) ds\right]. \end{aligned}$$

We note that (47) and (48) still hold here, then we deal with (65) next. Considering the fact $\varepsilon\left(e_{\boldsymbol{u}}^{h,0}\right)=0$, $e_p^{h,0}=0$, $e_\xi^{h,0}=0$, ignoring the term $c_0\|e_p^{h,N+1}\|^2_{L^2(\Omega)}$, using (47) to choose a small enough positive ϵ_1 such that $\epsilon_1\|e_\xi^{h,k+1}\|^2_{L^2(\Omega)}\le 2\mu\|\varepsilon\left(e_{\boldsymbol{u}}^{h,k+1}\right)\|^2_{L^2(\Omega)}$ and setting $\epsilon_2=K/4$, we can apply Lemma 3.3 to derive

$$
\begin{aligned}
&2\mu\|\varepsilon\left(e_{\boldsymbol{u}}^{h,N+1}\right)\|^2_{L^2(\Omega)}+\frac{1}{\lambda}\|\alpha e_p^{h,N+1}-e_\xi^{h,N+1}\|^2_{L^2(\Omega)}+\frac{K\Delta t}{4}\sum_{n=0}^{N}\|\nabla\left(e_p^{h,n+1}+e_p^{h,n}\right)\|^2_{L^2(\Omega)}\\
&\lesssim(\Delta t)^4\int_0^T\left(\|\partial_{ttt}\boldsymbol{u}\|^2_{H^1(\Omega)}+\|\partial_{ttt}\xi\|^2_{L^2(\Omega)}+\|\partial_{ttt}p\|^2_{L^2(\Omega)}\right)ds\\
&\quad+h^{2k}\int_0^T\left(\|\partial_t\boldsymbol{u}\|^2_{H^{k+1}(\Omega)}+\|\partial_t\xi\|^2_{H^k(\Omega)}\right)ds+h^{2l+2}\int_0^T\|\partial_t p\|^2_{H^{l+1}(\Omega)}ds.
\end{aligned}\tag{69}
$$

Finally, applying (47) and (48) to (69) yields the desired result (59). ∎

Theorem 4.5. *Let $(\boldsymbol{u},\xi,p)$ and $\left(\boldsymbol{u}_h^{n+1},\xi_h^{n+1},p_h^{n+1}\right)$ be the solutions of Equations (11)–(13) and (25)–(27), respectively. Under the assumptions of Theorem 4.4, there holds*

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{n=0}^{N}\left(\|\varepsilon\left(D_{\boldsymbol{u}}^{n+1}\right)\|^2_{L^2(\Omega)}+\|D_\xi^{n+1}\|^2_{L^2(\Omega)}+\|D_p^{n+1}\|^2_{L^2(\Omega)}\right)+\|\nabla e_p^{h,N+1}\|^2_{L^2(\Omega)}\\
&\lesssim(\Delta t)^4\int_0^T\left(\|\partial_{ttt}\boldsymbol{u}\|^2_{H^1(\Omega)}+\|\partial_{ttt}\xi\|^2_{L^2(\Omega)}+\|\partial_{ttt}p\|^2_{L^2(\Omega)}\right)ds\\
&\quad+h^{2k}\int_0^T\left(\|\partial_t\boldsymbol{u}\|^2_{H^{k+1}(\Omega)}+\|\partial_t\xi\|^2_{H^k(\Omega)}\right)ds+h^{2l+2}\int_0^T\|\partial_t p\|^2_{H^{l+1}(\Omega)}ds.
\end{aligned}\tag{70}
$$

Proof. Firstly, we note that (33) holds here, which implies (51) can be used here. Choosing $\boldsymbol{v}_h=D_{\boldsymbol{u}}^{n+1}$ in (51), $\phi_h=D_\xi^{n+1}$ in (61), $\psi_h=D_p^{n+1}$ in (64), summing over the index n from 0 to N, and applying the identity (52), we can deduce that

$$
\begin{aligned}
&\sum_{n=0}^{N}\left(2\mu\|\varepsilon\left(D_{\boldsymbol{u}}^{n+1}\right)\|^2_{L^2(\Omega)}+c_0\|D_p^{n+1}\|^2_{L^2(\Omega)}+\frac{1}{\lambda}\|\alpha D_p^{n+1}-D_\xi^{n+1}\|^2_{L^2(\Omega)}\right)\\
&\quad+\Delta t\sum_{n=0}^{N}d\left(e_p^{h,n+1}+e_p^{h,n},D_p^{n+1}\right)=\sum_{i=1}^{5}L_i,
\end{aligned}\tag{71}
$$

where

$$
L_1=\sum_{n=0}^{N}b\left(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n-\frac{\Delta t\partial_t\boldsymbol{u}^{n+1}+\Delta t\partial_t\boldsymbol{u}^n}{2},D_\xi^{n+1}\right),
$$

$$
L_2=\sum_{n=0}^{N}a_2\left(R_\xi\xi^{n+1}-R_\xi\xi^n-\frac{\Delta t\partial_t\xi^{n+1}+\Delta t\partial_t\xi^n}{2},D_\xi^{n+1}\right),
$$

$$
L_3=\sum_{n=0}^{N}c\left(\frac{\Delta t\partial_t p^{n+1}+\Delta t\partial_t p^n}{2}-R_p p^{n+1}+R_p p^n,D_\xi^{n+1}\right),
$$

$$
L_4=\sum_{n=0}^{N}a_3\left(R_p p^{n+1}-R_p p^n-\frac{\Delta t\partial_t p^{n+1}+\Delta t\partial_t p^n}{2},D_p^{n+1}\right),
$$

$$
L_5=\sum_{n=0}^{N}c\left(D_p^{n+1},\frac{\Delta t\partial_t\xi^{n+1}+\Delta t\partial_t\xi^n}{2}-R_\xi\xi^{n+1}+R_\xi\xi^n\right).
$$

From our observation, it has been discovered that

$$L_2 + L_5 = \sum_{n=0}^{N} \frac{1}{\lambda}\left(\alpha D_p^{n+1} - D_\xi^{n+1}, \frac{\Delta t \partial_t \xi^{n+1} + \Delta t \partial_t \xi^n}{2} - R_\xi \xi^{n+1} + R_\xi \xi^n\right) =: \tilde{L}_2,$$

$$L_3 + L_4 = \sum_{n=0}^{N} \frac{\alpha}{\lambda}\left(R_p p^{n+1} - R_p p^n - \frac{\Delta t \partial_t p^{n+1} + \Delta t \partial_t p^n}{2}, \alpha D_p^{n+1} - D_\xi^{n+1}\right)$$
$$+ \sum_{n=0}^{N} c_0 \left(R_p p^{n+1} - R_p p^n - \frac{\Delta t \partial_t p^{n+1} + \Delta t \partial_t p^n}{2}, D_p^{n+1}\right) =: \tilde{L}_3 + \tilde{L}_4.$$

Next, we bound the terms L_1, $\tilde{L}_2$, $\tilde{L}_3$, and $\tilde{L}_4$. Applying the Cauchy-Schwarz inequality, the Young's inequality, (18), (20), (68), and (28), we obtain the following estimates with $\epsilon_1, \epsilon_2, \epsilon_3 > 0$.

$$L_1 \le \epsilon_1 \sum_{n=0}^{N} \|D_\xi^{n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_1}(\Delta t)^5 \int_0^T \|\partial_{ttt} \boldsymbol{u}\|_{H^1(\Omega)}^2 ds,$$

$$\tilde{L}_2 \le \frac{\epsilon_2}{2} \sum_{n=0}^{N} \frac{1}{\lambda} \|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2$$
$$+ \frac{C}{\epsilon_2}\left[(\Delta t)^5 \int_0^T \|\partial_{ttt}\xi\|_{L^2(\Omega)}^2 ds + h^{2k} \Delta t \int_0^T \left(\|\partial_t \boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t \xi\|_{H^k(\Omega)}^2\right) ds\right],$$

$$\tilde{L}_3 \le \frac{\epsilon_2}{2} \sum_{n=0}^{N} \frac{1}{\lambda} \|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2$$
$$+ \frac{C}{\epsilon_2}\left[(\Delta t)^5 \int_0^T \|\partial_{ttt} p\|_{L^2(\Omega)}^2 ds + h^{2l+2} \Delta t \int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right],$$

$$\tilde{L}_4 \le \epsilon_3 \sum_{n=0}^{N} c_0 \|D_p^{n+1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon_3}\left[(\Delta t)^5 \int_0^T \|\partial_{ttt} p\|_{L^2(\Omega)}^2 ds + h^{2l+2} \Delta t \int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right].$$

Here, we note that (54) and (55) still hold true. Then, we handle (71). Using the fact $e_p^{h,0} = 0$, choosing $\epsilon_2 = \frac{1}{2}$ and $\epsilon_3 = 1$, using (54) to choose a small enough positive ϵ_1 such that $\epsilon_1 \|D_\xi^{n+1}\|_{L^2(\Omega)}^2 \le \frac{\mu}{2} \|\varepsilon(D_{\boldsymbol{u}}^{n+1})\|_{L^2(\Omega)}^2$, we obtain

$$\frac{1}{\Delta t} \sum_{n=0}^{N} \left(\mu \|\varepsilon(D_{\boldsymbol{u}}^{n+1})\|_{L^2(\Omega)}^2 + \frac{1}{2\lambda} \|\alpha D_p^{n+1} - D_\xi^{n+1}\|_{L^2(\Omega)}^2\right) + K \|\nabla e_p^{h,N+1}\|_{L^2(\Omega)}^2$$
$$\le \left(\frac{C}{\epsilon_1} + \frac{C}{\epsilon_2} + \frac{C}{\epsilon_3}\right)\left[(\Delta t)^4 \int_0^T \left(\|\partial_{ttt} \boldsymbol{u}\|_{H^1(\Omega)}^2 + \|\partial_{ttt}\xi\|_{L^2(\Omega)}^2 + \|\partial_{ttt} p\|_{L^2(\Omega)}^2\right) ds\right.$$
$$\left. + h^{2k} \int_0^T \left(\|\partial_t \boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|\partial_t \xi\|_{H^k(\Omega)}^2\right) ds + h^{2l+2} \int_0^T \|\partial_t p\|_{H^{l+1}(\Omega)}^2 ds\right].$$

Applying (54) and (55) to the above equation, we claim that (70) holds true. The proof is completed. ■

Theorem 4.6. *Let $(\boldsymbol{u}, \xi, p)$ and $(\boldsymbol{u}_h^{n+1}, \xi_h^{n+1}, p_h^{n+1})$ be the solutions of Equations (11)–(13) and (25)–(27), respectively. Under the assumptions of Theorem 4.4, there holds*

$$\|\varepsilon\left(e_{\boldsymbol{u}}^{N+1}\right)\|_{L^2(\Omega)} + \|e_{\xi}^{N+1}\|_{L^2(\Omega)} + \|e_p^{N+1}\|_{L^2(\Omega)} \lesssim \left(\Delta t\right)^2 + h^k + h^{l+1}, \tag{72}$$

$$\|\nabla e_p^{N+1}\|_{L^2(\Omega)} \lesssim \left(\Delta t\right)^2 + h^k + h^l. \tag{73}$$

Proof. We start with (18), (19), and (20). Applying the triangle inequality, Theorems 4.4 and 4.5, we see that the above error estimates readily follow. ■

5 | BENCHMARK TESTS

In this section, we present numerical experiments in two dimensions to validate the theoretical predictions described in Section 4. All computations are implemented by using the open-source software FreeFEM++ [12].

Example 1. Let the domain $\Omega = [0,1]^2$ and the final time is $T = 1.0$. We choose the body force $\boldsymbol{f}$, the source/sink term Q_s, initial conditions and Dirichlet boundary data on $\partial\Omega = \Gamma_d = \Gamma_p$ such that the exact solution is as follows:

$$u_1 = \frac{1}{10}e^t\left(x + y^3\right), \quad u_2 = \frac{1}{10}t^2\left(x^3 + y^3\right), \quad p = 10e^{\frac{x+y}{10}}\left(1 + t^3\right).$$

Following [9], the physical parameters are:

$$\mu = 1.0, \quad \lambda = 1.0, \quad c_0 = 1.0, \quad \alpha = 1.0, \quad K = 1.0.$$

We apply a small mesh size $h = \frac{1}{64}$ and take polynomial orders $k = 3$, $l = 2$ for the spatial discretization so that the spatial error is not dominant. To check the orders of convergence in time, we only refine the time step size Δt. In Tables 1 and 2, we present the results of errors and convergence rates for Methods 1 and 2, respectively. We observe that the orders of $\boldsymbol{H}^1$ error of $\boldsymbol{u}$, L^2 error of ξ, L^2 and H^1 errors of p are all around 1 in Table 1, and are all around 2 in Table 2. The results in both tables illustrate that the time error order based on Method 1 is $\mathcal{O}\left(\Delta t\right)$ and the time error order based on Method 2 is $\mathcal{O}\left(\left(\Delta t\right)^2\right)$, which verify the theoretical predictions of error analyses in Theorems 4.3 and 4.6.

Example 2. Let the domain $\Omega = [0,1]^2$ with $\Gamma_1 = \{\left(1,y\right); 0 \le y \le 1\}$, $\Gamma_2 = \{\left(x,0\right); 0 \le x \le 1\}$, $\Gamma_3 = \{\left(0,y\right); 0 \le y \le 1\}$, $\Gamma_4 = \{\left(x,1\right); 0 \le x \le 1\}$ and the final time is $T = 1.0$. The Neumann boundary $\Gamma_t = \Gamma_f = \Gamma_1 \cup \Gamma_3$ and the Dirichlet boundary $\Gamma_d = \Gamma_p = \Gamma_2 \cup \Gamma_4$ are considered in this example. We take the body force $\boldsymbol{f}$, the source/sink term Q_s, and initial and boundary conditions such that the exact solution is as follows:

$$u_1 = e^{-t}\left(\sin\left(2\pi y\right)\left(\cos\left(2\pi x\right) - 1\right) + \frac{1}{\mu + \lambda}\sin\left(\pi x\right)\sin\left(\pi y\right)\right),$$

$$u_2 = e^{-t}\left(\sin\left(2\pi x\right)\left(1 - \cos\left(2\pi y\right)\right) + \frac{1}{\mu + \lambda}\sin\left(\pi x\right)\sin\left(\pi y\right)\right),$$

$$p = e^{-t}\sin\left(\pi x\right)\sin\left(\pi y\right).$$

TABLE 1 Errors and convergence rates of method 1 for Example 1.

Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	5.219e-02		2.754e-01		2.971e-01 & 1.386e+00	
1/8	2.735e-02	0.93	1.443e-01	0.93	1.557e-01 & 7.263e-01	0.93 & 0.93
1/16	1.399e-02	0.97	7.381e-02	0.97	7.963e-02 & 3.715e-01	0.97 & 0.97
1/32	7.076e-03	0.98	3.732e-02	0.98	4.026e-02 & 1.878e-01	0.98 & 0.98

TABLE 2 Errors and convergence rates of method 2 for Example 1.

Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	2.630e-03		1.266e-02		1.385e-02 & 6.333e-02	
1/8	6.426e-04	2.03	3.296e-03	1.94	3.570e-03 & 1.653e-02	1.96 & 1.94
1/16	1.587e-04	2.02	8.278e-04	1.99	8.944e-04 & 4.159e-03	2.00 & 1.99
1/32	3.959e-05	2.00	2.071e-04	2.00	2.237e-04 & 1.041e-03	2.00 & 2.00

TABLE 3 Errors and convergence rates of method 1 for Example 2 using $k = 2$ and $l = 1$ with $\nu = 0.3$ and $K = 1.0$.

h	Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	1/4	4.582e-01		3.657e-02		1.858e-02 & 2.919e-01	
1/8	1/16	1.252e-01	1.87	7.262e-03	2.33	5.258e-03 & 1.531e-01	1.82 & 0.93
1/16	1/64	3.237e-02	1.95	1.677e-03	2.11	1.361e-03 & 7.766e-02	1.95 & 0.98
1/32	1/256	8.191e-03	1.98	4.084e-04	2.04	3.437e-04 & 3.900e-02	1.99 & 0.99

TABLE 4 Errors and convergence rates of method 1 for Example 2 using $k = 3$ and $l = 2$ with $\nu = 0.3$ and $K = 1.0$.

h	Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	1/8	6.283e-02		4.146e-03		2.841e-03 & 3.325e-02	
1/8	1/64	8.465e-03	2.89	6.203e-04	2.74	3.502e-04 & 8.398e-03	3.02 & 1.99
1/16	1/512	1.054e-03	3.01	7.839e-05	2.98	4.397e-05 & 2.146e-03	2.99 & 1.97
1/32	1/4096	1.312e-04	3.01	9.789e-06	3.00	5.520e-06 & 5.433e-04	2.99 & 1.98

The fixed physical parameters are:

$$E = 1.0, \quad c_0 = 1.0, \quad \alpha = 1.0.$$

Other physical parameters will vary to test the robustness of our numerical schemes. Numerical results for this example are summarized in Tables 3–10. Among them, we examine Method 1 in Tables 3,4,7,8, and Method 2 in Tables 5,6,9,10. Since we have already verified the error orders in time for both schemes in Example 1, our focus here is the verification of the spatial error orders. To verify the spatial error orders as analyzed in Theorem 4.3, we take Δt of an order $\mathcal{O}(h^2)$ (Tables 3 and 7) or $\mathcal{O}(h^3)$ (Tables 4 and 8) for Method 1. Similarly, to verify the spatial error orders as analyzed in Theorem 4.6, we take Δt of order $\mathcal{O}(h)$ (Tables 5 and 9) or $\mathcal{O}(h^2)$ (Tables 6 and 10) for Method 2.

Firstly, we fix $\nu = 0.3$ and $K = 1.0$. The numerical results for errors and convergence orders using $k = 2$, $l = 1$ and $k = 3$, $l = 2$ are presented in Tables 3,4,5, and 6, respectively. We refine the mesh size (and vary the corresponding time step size) to present the numerical results. From Table 3, it is clearly shown that the convergence $\|e_{\boldsymbol{u}}^{N+1}\|_{H^1(\Omega)}$, $\|e_{\xi}^{N+1}\|_{L^2(\Omega)}$, $\|e_p^{N+1}\|_{L^2(\Omega)}$ are of order $\mathcal{O}(\Delta t+h^2)$, and $\|e_p^{N+1}\|_{H^1(\Omega)}$ is of order $\mathcal{O}(\Delta t+h)$ for Method 1. Similarly, from Table 4, we see that the convergence $\|e_{\boldsymbol{u}}^{N+1}\|_{H^1(\Omega)}$, $\|e_{\xi}^{N+1}\|_{L^2(\Omega)}$, $\|e_p^{N+1}\|_{L^2(\Omega)}$ are of order $\mathcal{O}(\Delta t + h^3)$, and $\|e_p^{N+1}\|_{H^1(\Omega)}$ is of order $\mathcal{O}(\Delta t + h^2)$. From Table 5, we see that the convergence $\|e_{\boldsymbol{u}}^{N+1}\|_{H^1(\Omega)}$, $\|e_{\xi}^{N+1}\|_{L^2(\Omega)}$, $\|e_p^{N+1}\|_{L^2(\Omega)}$ are of order $\mathcal{O}((\Delta t)^2 + h^2)$, and $\|e_p^{N+1}\|_{H^1(\Omega)}$ is of order $\mathcal{O}((\Delta t)^2 + h)$ for Method 2. Moreover, from Table 6, we see that the convergence $\|e_{\boldsymbol{u}}^{N+1}\|_{H^1(\Omega)}$, $\|e_{\xi}^{N+1}\|_{L^2(\Omega)}$, $\|e_p^{N+1}\|_{L^2(\Omega)}$ are of order $\mathcal{O}((\Delta t)^2 + h^3)$, and $\|e_p^{N+1}\|_{H^1(\Omega)}$ is of order $\mathcal{O}((\Delta t)^2 + h^2)$.

Secondly, we fix $\nu = 0.49999$ and $K = 10^{-6}$ to test the robustness of the proposed schemes with respect to the key physical parameters. The numerical results for errors and convergence orders using $k = 2$, $l = 1$ and $k = 3$, $l = 2$ are presented in Tables 7,8,9, and 10, respectively. By checking

TABLE 5 Errors and convergence rates of method 2 for Example 2 using $k = 2$ and $l = 1$ with $\nu = 0.3$ and $K = 1.0$.

h	Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	1/2	4.584e-01		3.744e-02		2.376e-02 & 3.725e-01	
1/8	1/4	1.252e-01	1.87	7.238e-03	2.37	5.259e-03 & 1.624e-01	2.18 & 1.20
1/16	1/8	3.237e-02	1.95	1.693e-03	2.10	1.376e-03 & 7.859e-02	1.93 & 1.05
1/32	1/16	8.191e-03	1.98	4.142e-04	2.03	3.520e-04 & 3.910e-02	1.97 & 1.01

TABLE 6 Errors and convergence rates of method 2 for Example 2 using $k = 3$ and $l = 2$ with $\nu = 0.3$ and $K = 1.0$.

h	Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	1/4	6.280e-02		3.891e-03		1.440e-03 & 4.175e-02	
1/8	1/16	8.460e-03	2.89	5.934e-04	2.71	1.580e-04 & 9.268e-03	3.19 & 2.17
1/16	1/64	1.054e-03	3.01	7.502e-05	2.98	1.848e-05 & 2.156e-03	3.10 & 2.10
1/32	1/256	1.312e-04	3.01	9.368e-06	3.00	2.336e-06 & 5.428e-04	2.98 & 1.99

TABLE 7 Errors and convergence rates of method 1 for Example 2 using $k = 2$ and $l = 1$ with $\nu = 0.49999$ and $K = 10^{-6}$.

h	Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	1/4	4.658e-01		7.691e-02		3.411e-02 & 3.831e-01	
1/8	1/16	1.252e-01	1.90	1.149e-02	2.74	9.063e-03 & 1.667e-01	1.91 & 1.20
1/16	1/64	3.229e-02	1.96	2.412e-03	2.25	2.336e-03 & 8.027e-02	1.96 & 1.05
1/32	1/256	8.163e-03	1.98	5.709e-04	2.08	5.921e-04 & 3.953e-02	1.98 & 1.02

TABLE 8 Errors and convergence rates of method 1 for Example 2 using $k = 3$ and $l = 2$ with $\nu = 0.49999$ and $K = 10^{-6}$.

h	Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	1/8	6.320e-02		8.826e-03		1.864e-02 & 9.813e-02	
1/8	1/64	8.546e-03	2.89	1.176e-03	2.91	2.426e-03 & 1.669e-02	2.94 & 2.56
1/16	1/512	1.063e-03	3.01	1.385e-04	3.09	3.067e-04 & 3.187e-03	2.98 & 2.39
1/32	1/4096	1.323e-04	3.01	1.632e-05	3.08	3.850e-05 & 6.651e-04	2.99 & 2.26

TABLE 9 Errors and convergence rates of method 2 for Example 2 using $k = 2$ and $l = 1$ with $\nu = 0.49999$ and $K = 10^{-6}$.

h	Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	1/2	4.658e-01		7.691e-02		7.259e-02 & 4.259e-01	
1/8	1/4	1.252e-01	1.90	1.149e-02	2.74	1.905e-02 & 1.738e-01	1.93 & 1.29
1/16	1/8	3.229e-02	1.96	2.412e-03	2.25	4.832e-03 & 8.122e-02	1.98 & 1.10
1/32	1/16	8.163e-03	1.98	5.709e-04	2.08	1.214e-03 & 3.965e-02	1.99 & 1.03

TABLE 10 Errors and convergence rates of method 2 for Example 2 using $k = 3$ and $l = 2$ with $\nu = 0.49999$ and $K = 10^{-6}$.

h	Δt	H^1 errors of u	Orders	L^2 errors of ξ	Orders	L^2 & H^1 errors of p	Orders
1/4	1/4	6.320e-02		8.826e-03		3.445e-03 & 5.186e-02	
1/8	1/16	8.546e-03	2.89	1.176e-03	2.91	3.177e-04 & 1.267e-02	3.44 & 2.03
1/16	1/64	1.063e-03	3.01	1.385e-04	3.09	3.092e-05 & 2.877e-03	3.36 & 2.14
1/32	1/256	1.323e-04	3.01	1.632e-05	3.08	3.160e-06 & 6.425e-04	3.29 & 2.16

the error results and convergence rates one table by one table, one can verify the theoretical analysis provided in Section 4. From these tables, it is shown clearly that all energy norm errors decrease with the optimal convergence orders. By comparing the results in Tables 3–6 with the corresponding results in Tables 7–10, we conclude that our schemes are robust with respect to the Poisson ratio ν and the hydraulic conductivity K.

6 | CONCLUSIONS AND OUTLOOK

In this paper, we present a priori estimates of the two monolithic schemes for the three-field formulation of Biot's consolidation model. The theoretical results show that both schemes are unconditionally convergent with optimal error orders. We comment here that Method 2 achieves a second-order convergence in time without significantly increasing the computational burden. Detailed numerical experiments are carried out to verify the predictions of error estimates. In future work, we plan to develop some decoupled algorithms [11,14] and the corresponding analysis based on the theory studied in this work.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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