

Relatively Anosov representations via flows II:
Examples

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Abstract

This is the second in a series of two papers that devel-
ops a theory of relatively Anosov representations using
the original “contracting flow on a bundle” definition
of Anosov representations introduced by Labourie and
Guichard–Wienhard. In this paper, we focus on building
families of examples.

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1 | INTRODUCTION

Anosov representations were introduced by Labourie [31], and further developed by Guichard–Wienhard [22], as a generalization of convex cocompact representations into the isometry group of real hyperbolic space. Informally speaking, an Anosov representation is a representation of a word-hyperbolic group into a semisimple Lie group that has an equivariant boundary map into a flag manifold with good dynamical properties.

This is the second in a series of two papers whose purpose is to develop a theory of relatively Anosov representations, extending the theory of Anosov representations to relatively hyperbolic groups, using the original “contracting flow on a bundle” definition of Labourie and Guichard–Wienhard. The general theory was developed in the first paper. In this paper, we will focus on examples.

Throughout the paper, we will let $\mathbb{K}$ denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

1.1 | Some results from the first paper

We briefly recall some of the results from the first paper. Relatively Anosov representations are perhaps most naturally defined using the following boundary map definition (which is equivalent to being “asymptotically embedded” in the sense of Kapovich–Leeb [26] and “relatively dominated” in the sense of [43], see [45, Sec. 4] for details).

Definition 1.1. Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic with Bowditch boundary $\partial(\Gamma, \mathcal{P})$. A representation $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}$ if there exists a continuous map

$$\xi = (\xi^k, \xi^{d-k}) : \partial(\Gamma, \mathcal{P}) \rightarrow \mathrm{Gr}_k(\mathbb{K}^d) \times \mathrm{Gr}_{d-k}(\mathbb{K}^d),$$

which is

- (1) *ρ -equivariant*: if $\gamma \in \Gamma$, then $\rho(\gamma) \circ \xi = \xi \circ \gamma$,
- (2) *transverse*: if $x, y \in \partial(\Gamma, \mathcal{P})$ are distinct, then $\xi^k(x) \oplus \xi^{d-k}(y) = \mathbb{K}^d$,
- (3) *strongly dynamics-preserving*: if $(\gamma_n)_{n \geq 1}$ is a sequence of elements in Γ where $\gamma_n \rightarrow x \in \partial(\Gamma, \mathcal{P})$ and $\gamma_n^{-1} \rightarrow y \in \partial(\Gamma, \mathcal{P})$, then

$$\lim_{n \rightarrow \infty} \rho(\gamma_n)V = \xi^k(x)$$

for all $V \in \mathrm{Gr}_k(\mathbb{K}^d)$ transverse to $\xi^{d-k}(y)$.

One of the main results in the first paper shows that the definition above can be recast in terms of a contracting flow on a certain vector bundle associated to the representation.

Given a relatively hyperbolic group $(\Gamma, \mathcal{P})$, we can realize Γ as a subgroup of $\text{Isom}(X)$ where X is a proper geodesic Gromov-hyperbolic metric space such that every point in X is within a uniformly bounded distance of a geodesic, Γ acts geometrically finitely on the Gromov boundary $\partial_\infty X$ of X , and the stabilizers of the parabolic fixed points are exactly the conjugates of $\mathcal{P}$. Following the terminology in [6], we call such an X a *weak cusp space* for $(\Gamma, \mathcal{P})$.

Given such an X , let $\mathcal{G}(X)$ denote the space of parametrized geodesic lines in X and for $\sigma \in \mathcal{G}(X)$, let $\sigma^\pm := \lim_{t \rightarrow \pm\infty} \sigma(t) \in \partial_\infty X$. The space $\mathcal{G}(X)$ has a natural flow ϕ^t given by $\phi^t(\sigma) = \sigma(\cdot + t)$ which descends to a flow, which we also denote by ϕ^t , on the quotient $\widehat{\mathcal{G}}(X) := \Gamma \backslash \mathcal{G}(X)$.

Given a representation $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{K})$, let

$$E(X) := \mathcal{G}(X) \times \mathbb{K}^d \quad \text{and} \quad \widehat{E}_\rho(X) := \Gamma \backslash E(X),$$

where Γ acts on $E(X)$ by $\gamma \cdot (\sigma, Y) = (\gamma \circ \sigma, \rho(\gamma)Y)$. The flow ϕ^t extends to a flow on $E(X)$, which we call φ^t , which acts trivially on the second factor. This, in turn, descends to a flow on $\widehat{E}_\rho(X)$ which we also call φ^t .

Given a continuous, ρ -equivariant, transverse map

$$\xi = (\xi^k, \xi^{d-k}) : \partial(\Gamma, \mathcal{P}) \rightarrow \text{Gr}_k(\mathbb{K}^d) \times \text{Gr}_{d-k}(\mathbb{K}^d),$$

we can define vector bundles $\Theta^k, \Xi^{d-k} \rightarrow \mathcal{G}(X)$ by setting $\Theta^k(\sigma) := \xi^k(\sigma^+)$ and $\Xi^{d-k}(\sigma) := \xi^{d-k}(\sigma^-)$. Since ξ is transverse, we have $E(X) = \Theta^k \oplus \Xi^{d-k}$. Since ξ is ρ -equivariant, this descends to a vector bundle decomposition $\widehat{E}_\rho(X) = \widehat{\Theta}^k \oplus \widehat{\Xi}^{d-k}$. Also, by construction, these subbundles are φ^t -invariant. We can then consider the bundle $\text{Hom}(\widehat{\Xi}^{d-k}, \widehat{\Theta}^k) \rightarrow \widehat{\mathcal{G}}(X)$ and, since the subbundles are φ^t -invariant, we can define a flow on $\text{Hom}(\widehat{\Xi}^{d-k}, \widehat{\Theta}^k)$ by $\psi^t(f) := \varphi^t \circ f \circ \varphi^{-t}$. Finally, we note that any metric on $\widehat{E}_\rho(X) \rightarrow \widehat{\mathcal{G}}(X)$ induces, via the operator norm, a continuous family of norms on the fibers of $\text{Hom}(\widehat{\Xi}^{d-k}, \widehat{\Theta}^k) \rightarrow \widehat{\mathcal{G}}(X)$.

Definition 1.2. With the notation above, we say that ρ is P_k -Anosov relative to X if there exists a metric $\|\cdot\|$ on the vector bundle $\widehat{E}_\rho(X) \rightarrow \widehat{\mathcal{G}}(X)$ such that the flow ψ^t on $\text{Hom}(\widehat{\Xi}^{d-k}, \widehat{\Theta}^k)$ is exponentially contracting (with respect to the associated operator norms).

In [45], we proved that these two definitions are equivalent, and indeed, one can always make a particular choice of weak cusp space. These are what are often called *Groves–Manning cusp spaces* and they are formed by attaching so-called combinatorial horoballs to a Cayley graph of the group (see Definition 2.3). These spaces are perhaps the most canonical choice of weak cusp space, see [6, 21].

Theorem 1.3 [45, Th. 1.3]. *Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{K})$ is a representation. Then the following are equivalent:*

- (1) ρ is P_k -Anosov relative to $\mathcal{P}$,
- (2) there is a weak cusp space X for $(\Gamma, \mathcal{P})$ such that ρ is P_k -Anosov relative to X ,
- (3) if X is any Groves–Manning cusp space of $(\Gamma, \mathcal{P})$, then ρ is P_k -Anosov relative to X .

Remark 1.4. Theorem 1.3 leaves open the question if the above conditions are equivalent to ρ being P_k -Anosov relative to any weak cusp space. Using a different flow space (which is equivalent to ours when X is CAT(−1)), Wang showed that this is the case [40].

As a consequence of Theorem 1.3, standard dynamical arguments can be used to prove a relative stability result. Given a representation $\rho_0 : (\Gamma, \mathcal{P}) \rightarrow \mathrm{SL}(d, \mathbb{K})$, we let $\mathrm{Hom}_{\rho_0}(\Gamma, \mathrm{SL}(d, \mathbb{K}))$ denote the set of representations $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ such that for each $P \in \mathcal{P}$, the representations $\rho|_P$ and $\rho_0|_P$ are conjugate.

Theorem 1.5 [45, Th. 1.6]. *Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and X is a weak cusp space for $(\Gamma, \mathcal{P})$. If $\rho_0 : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -Anosov relative to X , then there exists an open neighborhood $\mathcal{O}$ of ρ_0 in $\mathrm{Hom}_{\rho_0}(\Gamma, \mathrm{SL}(d, \mathbb{K}))$ such that every representation in $\mathcal{O}$ is P_k -Anosov relative to X .*

Remark 1.6. In recent work, Weisman [41] introduces a new class of representations of relatively hyperbolic groups, called extended geometrically finite representations which includes the class of relatively Anosov representations. For this class of representations, Weisman proves a general stability result which implies, in the context of Theorem 1.5, that being P_k -Anosov relative to $\mathcal{P}$ is an open condition in $\mathrm{Hom}_{\rho_0}(\Gamma, \mathrm{SL}(d, \mathbb{K}))$.

In the relatively hyperbolic case, the space $\widehat{\mathcal{G}}(X)$ will be noncompact, and thus, it is possible for a metric on the vector bundle $\widehat{E}_\rho(X) \rightarrow \widehat{\mathcal{G}}(X)$ to be quite badly behaved. In [45], we introduced a subclass of relatively Anosov representations where the metric is assumed to have additional regularity properties and proved that this special class has nicer properties. This class is defined as follows.

Definition 1.7. Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic, X is a weak cusp space for $(\Gamma, \mathcal{P})$, and $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is a representation.

- A metric $\|\cdot\|$ on $\widehat{E}_\rho(X) \rightarrow \widehat{\mathcal{G}}(X)$ is *locally uniform* if its lift to $\mathcal{G}(X) \times \mathbb{K}^d \rightarrow \mathcal{G}(X)$ has the following property: For any $r > 0$, there exists $L_r > 1$ such that

$$\frac{1}{L_r} \|\cdot\|_{\sigma_1} \leq \|\cdot\|_{\sigma_2} \leq L_r \|\cdot\|_{\sigma_1}$$

- for all $\sigma_1, \sigma_2 \in \mathcal{G}(X)$ with $d_X(\sigma_1(0), \sigma_2(0)) \leq r$.
- ρ is *uniformly P_k -Anosov relative to X* if it is P_k -Anosov relative to $\mathcal{P}$ and there exists a locally uniform metric $\|\cdot\|$ on $\widehat{E}_\rho(X) \rightarrow \widehat{\mathcal{G}}(X)$ such that the flow ψ^t on $\mathrm{Hom}(\widehat{\mathbb{E}}^{d-k}, \widehat{\mathbb{O}}^k)$ is exponentially contracting (with respect to the associated operator norms).

In [45], we proved that uniformly relatively Anosov representations are very nicely behaved. In particular, one can construct an equivariant quasi-isometric map of the entire weak cusp space into the symmetric space associated to $\mathrm{SL}(d, \mathbb{K})$, and the boundary map is Hölder relative to any visual metric on the Bowditch boundary and Riemannian distance on the Grassmannian [45, Th. 1.13]. We also proved that the uniformly Anosov representations form an open set in the constrained space of representations considered in Theorem 1.5.

1.2 | Results of this paper

The main aim of this paper is to produce classes of examples of relatively Anosov representations. Just as Anosov representations can be thought of as a generalization of convex cocompact representations, so relatively Anosov representations can be thought of as a generalization of geometrically finite representations into rank-one semisimple Lie groups.

In fact, essentially by definition, these two notions coincide for rank-one semisimple Lie groups. More precisely, in [45, Sec. 13], we extended Definition 1.1 to relatively Anosov representations into general semisimple Lie groups and with that definition, we have the following observation.

Observation 1.8. Suppose that X is a negatively curved symmetric space and P^+, P^- is a pair of opposite parabolic subgroups in $\text{Isom}_0(X)$, the connected component of the identity in the isometry group of X .

If $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho : \Gamma \rightarrow \text{Isom}_0(X)$ is a representation, then the following are equivalent.

- (1) ρ is $P^\pm$ -Anosov relative to $\mathcal{P}$ (in the sense of [45, Def. 13.1]).
- (2) $\ker \rho$ is finite, $\rho(\Gamma)$ is geometrically finite, and $\rho(\mathcal{P})$ is a set of representatives of the conjugacy classes of maximal parabolic subgroups in $\rho(\Gamma)$.

Proof. This follows directly from the “F2” definition in [8] of geometrically finite subgroups in $\text{Isom}(X)$ and [45, Def. 13.1]. $\square$

Remark 1.9. $\text{Isom}_0(X)$ only contains one conjugacy class of opposite parabolic subgroups and so by definition (see [45, Def. 13.1]) a representation is $P^\pm$ -Anosov relative to $\mathcal{P}$ if and only if it is $Q^\pm$ -Anosov relative to $\mathcal{P}$ for any choice of opposite parabolic subgroups in $Q^\pm \leq \text{Isom}_0(X)$.

Motivated by this observation, we construct additional examples of relatively Anosov representations. The first set of examples come from considering representations of geometrically finite subgroups of rank-one semisimple Lie groups.

The second set of examples are motivated by the Klein–Beltrami model of hyperbolic geometry. In particular, this model realizes real hyperbolic n -space as a convex domain of $\mathbf{P}(\mathbb{R}^{n+1})$ in such a way that the hyperbolic metric coincides with the Hilbert metric on the convex domain. We observe that one can consider “geometrically finite” subgroups acting on more general convex domains to construct additional examples of relatively Anosov representations.

We also consider additional classes of examples, described in Section 1.2.3.

1.2.1 | Geometric finiteness in rank one

For the rest of this subsection, suppose that X is a negatively curved symmetric space and let $G := \text{Isom}_0(X)$ denote the connected component of the identity in the isometry group of X . Let $\partial_\infty X$ denote the geodesic boundary of X . Then, given a discrete group $\Gamma \leq G$, let $\Lambda_X(\Gamma) \subset \partial_\infty X$ denote the limit set of Γ and let $C_X(\Gamma)$ denote the convex hull of the limit set in X .

When $\Gamma \leq G$ is geometrically finite, we will let $\mathcal{P}(\Gamma)$ denote a set of representatives of the conjugacy classes of maximal parabolic subgroups in Γ . Then, $(\Gamma, \mathcal{P}(\Gamma))$ is relatively hyperbolic and $C_X(\Gamma)$ is a weak cusp space for $(\Gamma, \mathcal{P}(\Gamma))$.

We will observe that restricting a proximal linear representation of G to a geometrically finite subgroup produces a uniformly relatively Anosov representation.

Proposition 1.10 ((See Proposition 4.2). *Suppose that $\tau : G \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -proximal (i.e., the image of τ contains a P_k -proximal element). If $\Gamma \leq G$ is geometrically finite, then $\tau|_\Gamma$ is uniformly P_k -Anosov relative to $C_X(\Gamma)$.*)

Remark 1.11. A version of Proposition 1.10 also holds for representations into general semisimple Lie groups, in fact using [45, Proposition 13.4] the general case follows immediately from the $\mathrm{SL}(d, \mathbb{K})$ case.

In the context of Proposition 1.10, we can obtain additional examples by starting with the representation $\rho_0 := \tau|_\Gamma$ and deforming it in $\mathrm{Hom}_{\rho_0}(\Gamma, \mathrm{SL}(d, \mathbb{K}))$. By Theorem 1.5, any sufficiently small deformation will be a uniformly relatively Anosov representation.

Using Proposition 1.10, we will also construct the following example.

Example 1.12 (see Section 7). Let $X := \mathbb{H}_{\mathbb{C}}^2$ denote complex hyperbolic 2-space. There exists a geometrically finite subgroup $\Gamma \leq \mathrm{Isom}_0(X)$ and a representation $\rho : \Gamma \rightarrow \mathrm{SL}(3, \mathbb{C})$ that is uniformly P_1 -Anosov relative to $C_X(\Gamma)$, but not uniformly P_1 -Anosov relative to any Groves–Manning cusp space associated to $(\Gamma, \mathcal{P}(\Gamma))$.

We remark that the example makes crucial use of the fact that for horoballs in complex hyperbolic space, distances decay at different exponential rates as we approach the cusp. In fact, in real hyperbolic geometry, one can show that the convex hull of the limit set of a geometrically finite group is quasi-isometric to the associated Groves–Manning cusp space.

This example shows that there is value in studying bundles associated to general weak cusp spaces and not just the Groves–Manning cusp spaces. In future work, we will further explore how to select the “best” weak cusp spaces to study a given relatively Anosov representation.

We can relax the condition in Proposition 1.10 to only assuming that the representation extends on each peripheral subgroup. More precisely, if $\Gamma \leq G$ is geometrically finite and $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}(\Gamma)$, then we say that ρ has *almost homogeneous cusps* if there exists a finite cover $\pi : \tilde{G} \rightarrow G$ such that for each $P \in \mathcal{P}(\Gamma)$, there is a representation $\tau_P : \tilde{G} \rightarrow \mathrm{SL}(d, \mathbb{K})$ where

$$\{\tau_P(g)(\rho \circ \pi)(g)^{-1} : g \in \pi^{-1}(P)\}$$

is relatively compact in $\mathrm{SL}(d, \mathbb{K})$. This technical definition informally states that the representation restricted to each peripheral subgroup extends to a representation of G .

Theorem 1.13 (See Theorem 6.1). *Suppose that $\Gamma \leq G$ is geometrically finite and $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}(\Gamma)$. If ρ has almost homogeneous cusps, then ρ is uniformly P_k -Anosov relative to $C_X(\Gamma)$.*

Proposition 3.6 in [16] implies that every relatively Anosov representation of a geometrically finite Fuchsian group has almost homogeneous cusps and hence is uniform. This also follows directly from the construction of canonical norms in [16, Sec. 3.1].

Corollary 1.14. *If $X = \mathbb{H}_{\mathbb{R}}^2$ is real hyperbolic 2-space, $\Gamma \leq \text{Isom}_0(X)$ is geometrically finite, and $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}(\Gamma)$, then ρ is uniformly P_k -Anosov relative to $C_X(\Gamma)$.*

Allowing representations of finite covers in the definition of almost homogeneous cusps is motivated by the following examples.

Example 1.15. Identify $\text{Isom}_0(\mathbb{H}_{\mathbb{R}}^2)$ with $\text{PSL}(2, \mathbb{R})$ and let $\pi : \text{SL}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R})$ denote the double cover. Let $P \leq \text{PSL}(2, \mathbb{R})$ be the cyclic subgroup generated by the projection of

$$u := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{R})$$

to $\text{PSL}(2, \mathbb{R})$. Also, let $\tau_d : \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(d, \mathbb{R})$ denote the standard irreducible representation.

- The representation $\rho_1 : P \rightarrow \text{SL}(5, \mathbb{R})$ defined by

$$\rho_1([u]) = (\tau_2 \oplus \tau_3)(u)$$

does not extend to a representation of $\text{PSL}(2, \mathbb{R})$ since $(\tau_2 \oplus \tau_3)(-\text{id}_2) \neq \text{id}_5$. However,

$$\{(\tau_2 \oplus \tau_3)(g) \cdot (\rho_1 \circ \pi)(g)^{-1} : g \in \pi^{-1}(P)\} = \{(-\text{id}_2) \oplus \text{id}_3\}$$

is compact.

- The representation $\rho_2 : P \rightarrow \text{SL}(4, \mathbb{R})$ defined by

$$\rho_2([u]) = (-\tau_2(u)) \oplus \tau_2(u)$$

also does not extend to a representation of $\text{PSL}(2, \mathbb{R})$. However,

$$\{(\tau_2 \oplus \tau_2)(g) \cdot (\rho_2 \circ \pi)(g)^{-1} : g \in \pi^{-1}(P)\} = \{(-\text{id}_2) \oplus \text{id}_2\}$$

is compact.

1.2.2 | Geometric finiteness in convex projective geometry

We will also apply our general results to the setting of convex real projective geometry.

Given a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$, the *automorphism group of Ω* , denoted as $\text{Aut}(\Omega)$, is the subgroup of $\text{PGL}(d, \mathbb{R})$ that preserves Ω . Such a domain also has a natural $\text{Aut}(\Omega)$ -invariant metric, the Hilbert metric d_Ω (see Section 8.1 for the definition). The *limit set* of a subgroup $\Gamma \leq \text{Aut}(\Omega)$ is defined to be

$$\Lambda_\Omega(\Gamma) := \partial\Omega \cap \bigcup_{p \in \Omega} \overline{\Gamma \cdot p}.$$

Following [17], we say that Γ is a *projectively visible subgroup of $\text{Aut}(\Omega)$* if

- (1) for all $p, q \in \Lambda_\Omega(\Gamma)$ distinct, the open line segment in $\overline{\Omega}$ joining p to q is contained in Ω , and
- (2) every point in $\Lambda_\Omega(\Gamma)$ is a C^1 -smooth point of $\partial\Omega$.

Example 1.16. The Klein–Beltrami model identifies real hyperbolic n -space with the properly convex domain

$$\mathbb{B} := \left\{ [1 : x_1 : \dots : x_n] \in \mathbf{P}(\mathbb{R}^{n+1}) : \sum x_j^2 < 1 \right\}$$

endowed with its Hilbert metric $d_{\mathbb{B}}$. The domain $\mathbb{B}$ is strictly convex and has C^∞ -smooth boundary, so any discrete subgroup in $\text{Aut}(\mathbb{B})$ is a projectively visible subgroup.

A projectively visible subgroup acts as a convergence group on its limit set and if, in addition, the action on the limit set is geometrically finite, then the inclusion representation is relatively P_1 -Anosov. These assertions follow from [17, Prop. 3.5], see Proposition 8.6 below.

Conversely, we characterize exactly when the image of a relatively P_1 -Anosov representation is a projectively visible subgroup that acts geometrically finitely on its limit set. This characterization is in terms of a lifting property of the Anosov boundary map, see Definition 9.1 below.

Proposition 1.17 (See Proposition 9.2). *Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho : \Gamma \rightarrow \text{PGL}(d, \mathbb{R})$ is P_1 -Anosov relative to $\mathcal{P}$. Then the following are equivalent:*

- (1) ρ has the lifting property (in the sense of Definition 9.1),
- (2) there exists a properly convex domain $\Omega_0 \subset \mathbf{P}(\mathbb{R}^d)$ where $\rho(\Gamma) \leq \text{Aut}(\Omega_0)$,
- (3) there exists a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ where $\rho(\Gamma) \leq \text{Aut}(\Omega)$ is a projectively visible subgroup that acts geometrically finitely on its limit set.

We will also prove that the lifting property is an open and closed condition in the following sense.

Proposition 1.18 (See Proposition 10.1). *Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho_0 : \Gamma \rightarrow \text{PGL}(d, \mathbb{R})$ is a representation. Let $\mathcal{A}_1(\rho_0)$ denote the set of representations in $\text{Hom}_{\rho_0}(\Gamma, \text{PGL}(d, \mathbb{R}))$ that are P_1 -Anosov relative to $\mathcal{P}$. Then the subset $\mathcal{A}_1^+(\rho_0) \subset \mathcal{A}_1(\rho_0)$ of representations with the lifting property is open and closed in $\mathcal{A}_1(\rho_0)$.*

Remark 1.19. In the case when $\mathcal{P} = \emptyset$ (i.e., Γ is word hyperbolic), the above proposition follows from [37, Prop. 1.2]. In fact, in [37], they consider lifting properties for Anosov representations into general semisimple Lie groups. It seems likely that some version of their result should hold in the relative case as well.

As a corollary to [45, Cor. 13.6] and Proposition 1.18, we obtain the following stability result.

Corollary 1.20. *Suppose that $\Gamma \leq \text{Aut}(\Omega)$ is a projectively visible subgroup acting geometrically finitely on its limit set and $\iota : \Gamma \hookrightarrow \text{PGL}(d, \mathbb{R})$ is the inclusion representation. Then there is an open neighborhood $\mathcal{O} \subset \text{Hom}_\iota(\Gamma, \text{PGL}(d, \mathbb{R}))$ of ι such that: if $\rho \in \mathcal{O}$, then there exists a properly convex domain $\Omega_\rho \subset \mathbf{P}(\mathbb{R}^d)$ where $\rho(\Gamma) \leq \text{Aut}(\Omega_\rho)$ is a projectively visible subgroup acting geometrically finitely on its limit set.*

Remark 1.21. For other stability results in the context of convex real projective geometry, see [4, 12, 13, 29, 33].

Using the methods in [18] and [44], we will construct the following examples, which brings the examples in Sections 1.2.1 into the convex real projective setting.

Proposition 1.22 (See Propositions 11.1 and 11.3). *Suppose that X is a negatively curved symmetric space that is not isometric to real hyperbolic 2-space and $G := \text{Isom}_0(X)$. If $\tau : G \rightarrow \text{PGL}(d, \mathbb{R})$ is P_1 -proximal, then there exists a $\tau(G)$ -invariant properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ such that: if $\Gamma \leq G$ is geometrically finite, then*

- (1) $\tau(\Gamma)$ is a projectively visible subgroup of $\text{Aut}(\Omega)$ and acts geometrically finitely on its limit set.
- (2) If $C_\Gamma := C_\Omega(\tau(\Gamma))$, then (C_Γ, d_Ω) is Gromov-hyperbolic.

Remark 1.23. We also characterize the P_1 -proximal representations of $\text{Isom}_0(\mathbb{H}_{\mathbb{R}}^2)$ that satisfy the conclusion of Proposition 1.22, see Proposition 11.2 below.

In the context of Proposition 1.22, we can obtain additional examples in the convex real projective setting by starting with the representation $\rho_0 := \tau|_\Gamma$ and deforming it in $\text{Hom}_{\rho_0}(\Gamma, \text{PGL}(d, \mathbb{R}))$. By Corollary 1.20, any sufficiently small deformation will be a projectively visible subgroup of some properly convex domain that acts geometrically finitely on its limit set.

1.2.3 | Examples beyond geometric finiteness

We also describe three more families of examples that do not clearly fit within either of the two geometric finiteness frameworks above.

In Section 12, we use a ping-pong argument to show that certain free products of linear discrete groups give rise to relatively Anosov representations. This effort is motivated by the following question: which linear discrete groups appear as the image of a peripheral subgroup under a relatively P_k -Anosov representation? Delaying definitions until later, it follows fairly easily from the definition that any such linear group is

- (1) weakly unipotent,
- (2) P_k -divergent, and
- (3) has $(k, d - k)$ -limit set consisting of a single point

(see Proposition 2.6 and Observation 12.1). Using a ping-pong argument, we will show that these properties are essentially the only constraints. More precisely, we have the following.

Proposition 1.24 (See Proposition 12.2). *Suppose that $U \leq \text{SL}(d, \mathbb{K})$ is a discrete group that is weakly unipotent, P_k -divergent, and whose $(k, d - k)$ -limit set is a single point. Then there is a relatively hyperbolic group $(\Gamma, \mathcal{P})$, a P_k -Anosov representation $\rho : \Gamma \rightarrow \text{PSL}(d, \mathbb{K})$, and $P \in \mathcal{P}$ such that $\rho(P) \leq U$ has finite index.*

This allows us to construct new examples of relatively Anosov representations where the peripherals are non-abelian nilpotent groups, for instance, using the linear representation of the integer Heisenberg group constructed in [14].

In Section 13, we show that certain representations of $\text{PSL}(2, \mathbb{Z})$ into $\text{PGL}(3, \mathbb{R})$ constructed by Rich Schwartz [36] are P_1 -Anosov relative to certain cyclic subgroups. Schwartz' beautiful construction comes from iterating Pappus's theorem [36], and he also showed that these

representations have many of the properties that relatively Anosov representations (not yet defined at the time) have. We should also note that Barbot–Lee–Valério proved that these representations are limits of families of Anosov representations of word hyperbolic groups [7].

Finally, in Section 14, we show that if a representation $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}$, then so is any semisimplification $\rho^{ss} : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ of ρ . On the other hand, we exhibit a counterexample to the statement that if some semisimplification ρ^{ss} of ρ is P_k -Anosov relative to $\mathcal{P}$, then ρ is P_k -Anosov relative to $\mathcal{P}$. In particular, the notion of relative Anosovness is not well defined on the level of the character variety of Γ in $\mathrm{SL}(d, \mathbb{K})$, which can be viewed as the quotient of $\mathrm{Hom}(\Gamma, \mathrm{SL}(d, \mathbb{K}))$ by the relation “having the same semisimplification.” One can ask if there is some finer equivalence relation on the space of representations, such that the notion of relative Anosovness is well defined with respect to this equivalence relation.

2 | PRELIMINARIES

2.1 | Ambiguous notation

Here, we fix any possibly ambiguous notation.

- We let $\|\cdot\|_2$ denote the standard Euclidean norm on $\mathbb{K}^d$.
- A *metric* $\|\cdot\|$ on a vector bundle $V \rightarrow B$ is a continuous varying family of norms on the fibers each of which is induced by an inner product.
- Given a metric space X , we will use $B_X(p, r)$ to denote the open ball of radius r centered at $p \in X$ and $\mathcal{N}_X(A, r)$ to denote the r -neighborhood of a subset $A \subset X$.
- Given functions $f, g : S \rightarrow [0, \infty)$, we write $f \lesssim g$ or equivalently $g \gtrsim f$ if there exists a constant $C > 0$ such that $f(s) \leq Cg(s)$ for all $s \in S$. If $f \lesssim g$ and $g \lesssim f$, then we write $f \asymp g$.
- Except where otherwise specified, all logarithms are taken to base e .
- Note that constants often carry over between statements in the same section, but not across sections.

2.2 | Weak cusp spaces

Here, we recall facts about weak cusp spaces that are used in the paper. For a more in-depth discussion of relative hyperbolicity using the same notation/perspective, we refer the reader to Section 3 in [45].

Definition 2.1. Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and Γ acts properly discontinuously and by isometries on a proper geodesic Gromov-hyperbolic metric space X . If

- (1) Γ acts on $\partial_\infty X$ as a geometrically finite convergence group and the maximal parabolic subgroups are exactly $\{\gamma P \gamma^{-1} : P \in \mathcal{P}, \gamma \in \Gamma\}$,
- (2) every point in X is within a uniformly bounded distance of a geodesic line,

then X is a *weak cusp space* of $(\Gamma, \mathcal{P})$.

The main result in [42] implies that any relatively hyperbolic group has a weak cusp space.

By work of Bowditch [9] (also see the exposition in [6, Sec. 3]), one can alternatively define weak cusp spaces in terms of the action of Γ on X .

A relatively hyperbolic group can have non-quasi-isometric weak cusp spaces, see [23]. Perhaps, the most canonical is the construction due to Groves–Manning, obtained by attaching combinatorial horoballs to a standard Cayley graph. The precise construction is described as follows.

Definition 2.2. Suppose that Y is a graph with the simplicial distance d_Y . The *combinatorial horoball* $\mathcal{H}(Y)$ is the graph, also equipped with the simplicial distance, that has vertex set $Y^{(0)} \times \mathbb{N}$ and two types of edges:

- *vertical edges* joining vertices (v, n) and $(v, n + 1)$,
- *horizontal edges* joining vertices (v, n) and (w, n) when $d_Y(v, w) \leq 2^{n-1}$.

Definition 2.3. Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic group. A finite symmetric generating set $S \subset \Gamma$ is *adapted* if $S \cap P$ is a generating set of P for every $P \in \mathcal{P}$. Given such an S , we let $C(\Gamma, S)$ and $C(P, S \cap P)$ denote the associated Cayley graphs. Then the associated *Groves–Manning cusp space*, denoted as $C_{GM}(\Gamma, \mathcal{P}, S)$, is obtained from the Cayley graph $C(\Gamma, S)$ by attaching, for each $P \in \mathcal{P}$ and $\gamma \in \Gamma$, a copy of the combinatorial horoball $\mathcal{H}(\gamma C(P, S \cap P))$ by identifying $\gamma C(P, S \cap P)$ with the $n = 1$ level of $\mathcal{H}(\gamma C(P, S \cap P))$.

Theorem 2.4 [21, Th. 3.25]. *If $(\Gamma, \mathcal{P})$ is relatively hyperbolic and S is an adapted finite generating set, then $C_{GM}(\Gamma, \mathcal{P}, S)$ is a weak cusp space for $(\Gamma, \mathcal{P})$.*

2.3 | The geometry of the Grassmanians

Throughout the paper, we will let $d_{\mathbf{P}(\mathbb{K}^d)}$ denote the *angle distance* on $\mathbf{P}(\mathbb{K}^d)$, that is, if $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{K}^d$, then

$$d_{\mathbf{P}(\mathbb{K}^d)}([v], [w]) = \cos^{-1} \left(\frac{|\langle v, w \rangle|}{\sqrt{\langle v, v \rangle} \sqrt{\langle w, w \rangle}} \right)$$

for all nonzero $v, w \in \mathbb{K}^d$.

Using the Plücker embedding, we can view $\text{Gr}_k(\mathbb{K}^d)$ as a subset of $\mathbf{P}(\wedge^k \mathbb{K}^d)$. Let $d_{\mathbf{P}(\wedge^k \mathbb{K}^d)}$ denote the angle distance associated to the inner product on $\wedge^k \mathbb{K}^d$ that makes

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}$$

an orthonormal basis. We then let $d_{\text{Gr}_k(\mathbb{K}^d)}$ denote the distance on $\text{Gr}_k(\mathbb{K}^d)$ obtained by restricting $d_{\mathbf{P}(\wedge^k \mathbb{K}^d)}$.

2.4 | The singular value decomposition

Given $g \in \text{SL}(d, \mathbb{K})$, we let

$$\mu_1(g) \geq \cdots \geq \mu_d(g)$$

denote the singular values of g . By the singular value decomposition, we can write $g = m a \ell$ where $m, \ell \in \mathrm{SU}_{\mathbb{K}}(d)$ and a is a diagonal matrix with $\mu_1(g) \geq \dots \geq \mu_d(g)$ down the diagonal. In general, this decomposition is not unique, but when $\mu_k(g) > \mu_{k+1}(g)$ the subspace

$$U_k(g) := m \langle e_1, \dots, e_k \rangle$$

is well defined. Geometrically, $U_k(g)$ is the subspace spanned by the k largest axes of the ellipse $g \cdot \{x \in \mathbb{K}^d : \|x\|_2 = 1\}$.

We will frequently use the following observation.

Observation 2.5. Suppose that $(g_n)_{n \geq 1}$ is a sequence in $\mathrm{SL}(d, \mathbb{K})$, $V_0 \in \mathrm{Gr}_k(\mathbb{K}^d)$, and $W_0 \in \mathrm{Gr}_{d-k}(\mathbb{K}^d)$. Then the following are equivalent:

(1) $g_n(V) \rightarrow V_0$ uniformly on compact subsets of

$$\{V \in \mathrm{Gr}_k(\mathbb{K}^d) : V \text{ transverse to } W_0\}.$$

(2) $\frac{\mu_k}{\mu_{k+1}}(g_n) \rightarrow \infty$, $U_k(g_n) \rightarrow V_0$, and $U_{d-k}(g_n^{-1}) \rightarrow W_0$.

(3) There exist open sets $\mathcal{O} \subset \mathrm{Gr}_k(\mathbb{K}^d)$ and $\mathcal{O}' \subset \mathrm{Gr}_{d-k}(\mathbb{K}^d)$ such that $g_n(V) \rightarrow V_0$ for all $V \in \mathcal{O}$ and $g_n^{-1}(W) \rightarrow W_0$ for all $W \in \mathcal{O}'$.

Proof. See, for instance, Appendix A in [45]. □

2.5 | Eigenvalues and proximal/weakly unipotent elements

Given $g \in \mathrm{SL}(d, \mathbb{K})$, we let

$$\lambda_1(g) \geq \dots \geq \lambda_d(g)$$

denote the absolute values of the eigenvalues of g .

An element $g \in \mathrm{SL}(d, \mathbb{K})$ is P_k -proximal if $\lambda_k(g) > \lambda_{k+1}(g)$. In this case, g has a unique attracting fixed point $V_g^+ \in \mathrm{Gr}_k(\mathbb{K}^d)$, namely, the space corresponding to $\lambda_1(g), \dots, \lambda_k(g)$, and a unique repelling point $W_g^- \in \mathrm{Gr}_{d-k}(\mathbb{K}^d)$, namely, the space corresponding to $\lambda_{k+1}(g), \dots, \lambda_d(g)$. By writing g in its normal form, it is easy to see that

$$g^n(V) \rightarrow V_g^+$$

for all $V \in \mathrm{Gr}_k(\mathbb{K}^d)$ transverse to W_g^- . Further, $V_g^+ \oplus W_g^- = \mathbb{K}^d$.

An element $g \in \mathrm{SL}(d, \mathbb{K})$ is *weakly unipotent* if $\lambda_j(g) = 1$ for all j and a subgroup $U \leq \mathrm{SL}(d, \mathbb{K})$ is *weakly unipotent* if every element in U is weakly unipotent.

In [45], we observed the following.

Proposition 2.6 [45, Prop. 4.2]. Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}$.

- (1) If $P \in \mathcal{P}$, then $\rho(P)$ is weakly unipotent.
- (2) If $\gamma \in \Gamma$ is non-peripheral and has infinite order, then $\rho(\gamma)$ is P_k -proximal.

Remark 2.7. Recall that an element $\gamma \in \Gamma$ of a relatively hyperbolic group $(\Gamma, \mathcal{P})$ is non-peripheral if it is not contained in $\bigcup_{\gamma \in \Gamma} \bigcup_{P \in \mathcal{P}} \gamma P \gamma^{-1}$.

2.6 | The symmetric space associated to the special linear group

We will consider the symmetric spaces $M := \mathrm{SL}(d, \mathbb{K}) / \mathrm{SU}(d, \mathbb{K})$ normalized so that the distance is given by

$$d_M(g \mathrm{SU}(d, \mathbb{K}), h \mathrm{SU}(d, \mathbb{K})) = \sqrt{\sum_{j=1}^d (\log \mu_j(g^{-1}h))^2}, \quad (1)$$

see [5, Chap. II.10] for more details.

2.7 | Dominated splitting and contraction on Hom bundles

In this section, we observe that the exponential contraction of the flow on the Hom bundle described in Section 1.1 can be recast in terms of a dominated splitting condition. This is well known in the word-hyperbolic case [2, 10] and the same arguments work in the relative case as well.

Suppose, for the rest of this section, that $(\Gamma, \mathcal{P})$ is a relatively hyperbolic group, $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is a representation, X is a weak cusp space for $(\Gamma, \mathcal{P})$, and $\|\cdot\|$ is a metric on the vector bundle $\widehat{E}_\rho(X) \rightarrow \widehat{\mathcal{G}}(X)$.

If $V, W \subset \widehat{E}_\rho(X)$ are subbundles, we can consider the bundle $\mathrm{Hom}(V, W) \rightarrow \widehat{\mathcal{G}}(X)$ with the associated family of operator norms defined by

$$\|f\|_\sigma := \max \{ \|f(Y)\|_\sigma : Y \in V|_\sigma, \|Y\|_\sigma = 1 \}$$

when $f \in \mathrm{Hom}(V, W)|_\sigma$. In particular, given a continuous ρ -equivariant transverse map

$$\xi : \partial(\Gamma, \mathcal{P}) \rightarrow \mathrm{Gr}_k(\mathbb{K}^d) \times \mathrm{Gr}_{d-k}(\mathbb{K}^d)$$

let $\widehat{\Theta}^k, \widehat{\Xi}^{d-k} \subset \widehat{E}_\rho(X)$ denote the subbundles defined in Section 1.1 and endow

$$\mathrm{Hom}(\widehat{\Xi}^{d-k}, \widehat{\Theta}^k) \rightarrow \widehat{\mathcal{G}}(X)$$

with the operator norm. We then have the following connection between the dynamics on these bundles.

Proposition 2.8. *With the notation above and $c, C > 0$ fixed, the following are equivalent.*

(1) For all $t \geq 0$, $\sigma \in \widehat{\mathcal{G}}(X)$, $Y \in \widehat{\Theta}^k|_\sigma$, and $Z \in \widehat{\Xi}^{d-k}|_\sigma$ nonzero,

$$\frac{\|\varphi^t(Y)\|_{\phi^t(\sigma)}}{\|\varphi^t(Z)\|_{\phi^t(\sigma)}} \leq C e^{-ct} \frac{\|Y\|_\sigma}{\|Z\|_\sigma}.$$

(2) For all $t \geq 0$, $\sigma \in \widehat{\mathcal{G}}(X)$, and $f \in \text{Hom}(\widehat{\Xi}^{d-k}, \widehat{\Theta}^k)|_\sigma$,

$$\|\psi^t(f)\|_{\phi^t(\sigma)} \leq C e^{-ct} \|f\|_\sigma.$$

Proof. One can argue exactly as in Proposition 2.3 in [2]. □

PART 1. REPRESENTATIONS OF GEOMETRICALLY FINITE GROUPS IN NEGATIVELY CURVED SYMMETRIC SPACES

3 | REMINDERS ON NEGATIVELY CURVED SYMMETRIC SPACES

Suppose that G is a connected simple non-compact Lie group with rank one and finite center. Fix a maximal compact subgroup $K \leq G$, then the quotient manifold $X = G/K$ is simply connected and has a G -invariant negatively-curved symmetric Riemannian metric. The possible spaces X are described in [35, Chap. 19].

Since X is simply connected and has pinched negative curvature, it is Gromov-hyperbolic, and we will let $\partial_\infty X$ denote the Gromov boundary of X . We will also let T^1X denote the unit tangent bundle of X and let $\pi : T^1X \rightarrow X$ denote the natural projection. We will use ϕ^t to denote the geodesic flow on T^1X . Also, for $v \in T^1X$, we let $v^+, v^- \in \partial_\infty X$ denote the forward/backward endpoint of the geodesic line tangent to v , equivalently

$$v^\pm = \lim_{t \rightarrow \pm\infty} \pi(\phi^t(v)).$$

By construction, G acts isometrically on X . The induced homomorphism $\Phi : G \rightarrow \text{Isom}(X)$ maps onto $\text{Isom}_0(X)$, the connected component of the identity, and has kernel $Z(G)$, the center of G . Given a sequence $(g_n)_{n \geq 1}$ and $x \in \partial_\infty X$, we write

$$g_n \rightarrow x \tag{2}$$

if $g_n(p) \rightarrow x$ for some (any) $p \in X$.

An element of G is either

- *elliptic*, that is, it fixes a point in X ,
- *parabolic*, that is, it is not elliptic and fixes exactly one point in $\partial_\infty X$, or
- *loxodromic*, that is, it is not elliptic and fixes exactly two points in $\partial_\infty X$.

Parabolic and loxodromic elements have the following behavior.

(1) If $g \in G$ is parabolic and x_g^+ is the unique fixed point of g , then

$$\lim_{n \rightarrow \pm\infty} g^n(y) = x_g^+$$

for all $y \in (X \cup \partial_\infty X) \setminus \{x_g^+\}$.

(2) If $g \in G$ is loxodromic, then it is possible to label the fixed points of g as x_g^+, x_g^- so that

$$\lim_{n \rightarrow \pm\infty} g^n(y) = x_g^\pm$$

for all $y \in (X \cup \partial_\infty X) \setminus \{x_g^\pm\}$.

In both cases, the limits are locally uniform.

Given a discrete subgroup $\Gamma \leq G$, we can consider the limit set $\Lambda_X(\Gamma) \subset \partial_\infty X$ of all accumulation points of any Γ -orbit in X . We then define $C_X(\Gamma)$ to be the convex hull of $\Lambda_X(\Gamma)$ in X , that is, the smallest closed geodesically convex subset of X whose closure in $X \cup \partial_\infty X$ contains $\Lambda_X(\Gamma)$. Finally, we define $\mathcal{U}(\Gamma)$ to be the subspace of the unit tangent bundle T^1X consisting of vectors tangent to geodesics with both endpoints in the limit set $\Lambda_X(\Gamma)$ and let $\widehat{\mathcal{U}}(\Gamma) = \Gamma \backslash \mathcal{U}(\Gamma)$.

Example 3.1. If Γ is a lattice in G , then $\Lambda_X(\Gamma) = \partial_\infty X$, $C_X(\Gamma) = X$, and $\mathcal{U}(\Gamma) = T^1X$.

A discrete group $\Gamma \leq G$ acts as a convergence group on $\partial_\infty X$ and such a group is *geometrically finite* if it acts its limit set $\Lambda_X(\Gamma)$ as a geometrically finite convergence group (for definitions, see, e.g., [45, Sec. 3.3]). There are also equivalent characterizations in terms of the action of Γ on X , see [8].

In this case, if $\mathcal{P}$ is a set of representatives of the conjugacy classes of maximal parabolic subgroups in Γ , then $(\Gamma, \mathcal{P})$ is a relatively hyperbolic group. Moreover, $C_X(\Gamma)$ is a weak cusp space of $(\Gamma, \mathcal{P})$ (see the “F4” definition and Section 3.5 in [8]). The flow space $\mathcal{U}(\Gamma)$ then naturally identifies with the space of geodesics $\mathcal{G}(C_X(\Gamma))$ in $C_X(\Gamma)$. When considering a relatively Anosov representation ρ of Γ , it is more convenient to view the bundles in Definition 1.2 as having base $\widehat{\mathcal{U}}(\Gamma)$.

4 | REPRESENTATIONS OF RANK ONE GROUPS

Let G, K , and $X = G/K$ be as in Section 3. In this section, we will prove the following expanded version of Proposition 1.10 from the introduction. First we present a definition.

Definition 4.1. Given a representation $\tau : G \rightarrow \mathrm{SL}(d, \mathbb{K})$, we say that a continuous τ -equivariant map $\zeta : \partial_\infty X \rightarrow \mathrm{Gr}_k(\mathbb{K}^d) \times \mathrm{Gr}_{d-k}(\mathbb{K}^d)$ is

- (1) *transverse*: if $x, y \in \partial_\infty X$ are distinct, then $\zeta^k(x) \oplus \zeta^{d-k}(y) = \mathbb{K}^d$,
- (2) *strongly dynamics-preserving*: if $(g_n)_{n \geq 1}$ is a sequence of elements in G where $\gamma_n \rightarrow x \in \partial_\infty X$ and $\gamma_n^{-1} \rightarrow y \in \partial_\infty X$ (here we use the notation from Equation (2)), then

$$\lim_{n \rightarrow \infty} \tau(\gamma_n)V = \zeta^k(x)$$

for all $V \in \mathrm{Gr}_k(\mathbb{K}^d)$ transverse to $\zeta^{d-k}(y)$.

Proposition 4.2. If $\tau : G \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -proximal (i.e., $\tau(G)$ contains a P_k -proximal element) and $\|\cdot\|_{v \in T^1X}$ is a τ -equivariant family of norms on $\mathbb{K}^d$, then the following statements hold:

(1) *There exists a continuous τ -equivariant, transverse, strongly dynamics-preserving map*

$$\zeta_\tau = (\zeta_\tau^k, \zeta_\tau^{d-k}) : \partial_\infty X \rightarrow \mathrm{Gr}_k(\mathbb{K}^d) \times \mathrm{Gr}_{d-k}(\mathbb{K}^d).$$

(2) *There exist $C, c > 0$ such that: if $t \geq 0$, $v \in T^1 X$, $Y \in \zeta_\tau^k(v^+)$, and $Z \in \zeta_\tau^{d-k}(v^-)$ is nonzero, then*

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} \leq C e^{-ct} \frac{\|Y\|_v}{\|Z\|_v}.$$

(3) *For any $r > 0$, there exists $L_r > 1$ such that: if $v, w \in T^1 X$ satisfy $d_X(\pi(v), \pi(w)) \leq r$, then*

$$\frac{1}{L_r} \|\cdot\|_v \leq \|\cdot\|_w \leq L_r \|\cdot\|_v.$$

In particular, if $\Gamma \leq G$ is geometrically finite, then $\rho = \tau|_\Gamma$ is uniformly P_k -Anosov relative to $C_X(\Gamma)$.

Remark 4.3. To be precise, a family of norms $\|\cdot\|_{v \in T^1 X}$ is τ -equivariant if

$$\|\cdot\|_v = \|\tau(g)(\cdot)\|_{g(v)}$$

for all $v \in T^1 X$ and $g \in G$.

The rest of the section is devoted to the proof of the proposition. So, fix a representation $\tau : G \rightarrow \mathrm{SL}(d, \mathbb{K})$ as in the statement.

Let $p_0 := [K] \in X$ and notice that $K = \mathrm{Stab}_G(p_0)$. Fix a unit vector $v_0 \in T_{p_0}^1 X$ and a Cartan subgroup $A = \{a_t\}$ of G such that $t \mapsto a_t(p_0)$ parametrizes the geodesic through p_0 with initial velocity v_0 . Let M denote the centralizer of A in K .

We can conjugate τ so that $\tau(A)$ is a subgroup of the diagonal matrices and $\tau(K) \leq \mathrm{SU}(d, \mathbb{K})$, see, for instance, [34].

The next two lemmas are used to define the maps in part (1) of the proposition.

Lemma 4.4. *If $t > 0$, then $\tau(a_t)$ is P_k -proximal.*

Proof. By hypothesis, there exists $g \in G$ such that $\tau(g)$ is P_k -proximal. By the Cartan decomposition, there exist $m_n, \ell_n \in K$ and $t_n \rightarrow \infty$ such that $g^n = m_n a_{t_n} \ell_n$. Then

$$\begin{aligned} 0 < \log \frac{\lambda_k}{\lambda_{k+1}}(\tau(g)) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\mu_k}{\mu_{k+1}}(\tau(g^n)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\mu_k}{\mu_{k+1}}(\tau(a_{t_n})) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\lambda_k}{\lambda_{k+1}}(\tau(a_{t_n})) \end{aligned}$$

(the first equality follows from Gelfand's formula for the spectral radius applied to the linear operators $\wedge^k g$ and $\wedge^{k+1} g$; in the last equality, we use the fact that $\tau(a_t)$ is diagonal). So, when n is large, $\lambda_k(\tau(a_{t_n})) > \lambda_{k+1}(\tau(a_{t_n}))$. Since $\tau(a_t)$ is diagonal, this implies that $\lambda_k(\tau(a_t)) > \lambda_{k+1}(\tau(a_t))$ for all $t > 0$. $\square$

Let $V^+ \in \text{Gr}_k(\mathbb{K}^d)$ and $V^- \in \text{Gr}_{d-k}(\mathbb{K}^d)$ denote the attracting and repelling fixed points of $\tau(a_t)$ when $t > 0$. Then $\mathbb{K}^d = V^+ \oplus V^-$ and

$$\lim_{t \rightarrow \infty} \tau(a_t)V = V^+ \quad (3)$$

for all $V \in \text{Gr}_k(\mathbb{K}^d)$ transverse to V^- . Let $P^\pm$ denote the stabilizer of $v_0^\pm \in \partial_\infty X$ in G .

Lemma 4.5. $\tau(P^\pm)V^\pm = V^\pm$.

Proof. Fix $g \in P^+$. Then

$$g' := \lim_{t \rightarrow \infty} a_{-t}ga_t$$

exists and is contained in MA , see, for instance, [19, Prop. 2.17.3]. Since M commutes with A , $\tau(M)$ fixes V^+ . Hence, $\tau(g')V^+ = V^+$. So,

$$\lim_{t \rightarrow \infty} \tau(a_{-t}ga_t)V^+ = \tau(g')V^+ = V^+,$$

which implies, by Equation (3), that

$$\tau(g)V^+ = \lim_{t \rightarrow \infty} \tau(a_t)\tau(a_{-t}ga_t)V^+ = V^+.$$

Thus, $\tau(P^+)V^+ = V^+$.

Similar reasoning shows that $\tau(P^-)V^- = V^-$. □

Since G acts transitively on $\partial_\infty X$ and $\text{Stab}_G(v_0^\pm) = P^\pm$, the last lemma implies that the expressions

$$\zeta^k(gv_0^+) = \tau(g)V^+ \quad \text{and} \quad \zeta^{d-k}(gv_0^-) = \tau(g)V^- \quad \text{for all } g \in G$$

define a smooth τ -equivariant map $\zeta = (\zeta^k, \zeta^{d-k}) : \partial_\infty X \rightarrow \text{Gr}_k(\mathbb{K}^d) \times \text{Gr}_{d-k}(\mathbb{K}^d)$.

Lemma 4.6. ζ is transverse.

Proof. Fix $x, y \in \partial_\infty X$ distinct. Since G acts transitively on pairs of distinct points in $\partial_\infty X$, there exists $g \in G$ such that $(x, y) = g \cdot (v_0^+, v_0^-)$. Then,

$$\zeta^k(x) + \zeta^{d-k}(y) = \tau(g)(\zeta^k(v_0^+) + \zeta^{d-k}(v_0^-)) = \tau(g)(V^+ + V^-) = \mathbb{K}^d. \quad \square$$

Lemma 4.7. ζ is strongly dynamics-preserving.

Proof. Suppose that $(g_n)_{n \geq 1}$ is a sequence in G such that $g_n \rightarrow x \in \partial_\infty X$ and $g_n^{-1} \rightarrow y \in \partial_\infty X$. By the Cartan decomposition, there exist $m_n, \ell_n \in K$ and $t_n \rightarrow \infty$ such that $g_n = m_n a_{t_n} \ell_n$. Passing to a subsequence, we can suppose that $m_n \rightarrow m$ and $\ell_n \rightarrow \ell$. Then $m_n(v_0^+) \rightarrow m(v_0^+) = x$ and $\ell_n^{-1}(v_0^-) \rightarrow \ell^{-1}(v_0^-) = y$. Then, by Equation (3), if $V \in \text{Gr}_k(\mathbb{K}^d)$ is transverse to $\xi^{d-k}(y) =$

$\tau(\ell)^{-1}V^-$, then $\tau(\ell_n)V$ is transverse to $\tau(\ell_n)\tau(\ell)^{-1}V^-$ and hence, for large n , to V^- , and so,

$$\lim_{n \rightarrow \infty} \tau(g_n)V = \tau(m) \lim_{n \rightarrow \infty} \tau(a_{t_n})\tau(\ell_n)V = \tau(m)V^+ = \xi^k(x). \quad \square$$

Next, we prove parts (2) and (3). Since any two families of τ -equivariant norms are bi-Lipschitz, it is enough to consider the norms

$$\|\cdot\|_{g(v_0)} := \|\tau(g)^{-1}(\cdot)\|_2,$$

where $\|\cdot\|_2$ is the standard Euclidean norm. Since $\tau(K) \leq U(d, \mathbb{K})$ and $K = \text{Stab}_G(p_0)$, this is indeed a well-defined family.

Since each $\tau(a_t)$ is P_k -proximal and diagonal, there exists $\lambda > 0$ such that

$$\frac{\lambda_k}{\lambda_{k+1}}(\tau(a_t)) = e^{\lambda t}$$

when $t \geq 0$.

Lemma 4.8. *If $t \geq 0$, $v \in T^1X$, $Y \in \zeta^k(v^+)$, and $Z \in \zeta^{d-k}(v^-)$ is nonzero, then*

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} \leq e^{-\lambda t} \frac{\|Y\|_v}{\|Z\|_v}.$$

Proof. Fix $g \in G$ such that $g(v_0) = v$. Then, $\phi^t(v) = g\phi^t(v_0) = ga_t(v_0)$ for all t and $g(V^+, V^-) = (\zeta^k(v^+), \zeta^{d-k}(v^-))$. Since $\tau(A)$ is a subgroup of the diagonal matrices and V^+, V^- are the attracting, repelling spaces of $\tau(a_t)$ when $t > 0$, then

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} = \frac{\|\tau(a_{-t}g^{-1})Y\|_2}{\|\tau(a_{-t}g^{-1})Z\|_2} \leq \frac{\frac{1}{\lambda_k(\tau(a_t))} \|\tau(g^{-1})Y\|_2}{\frac{1}{\lambda_{k+1}(\tau(a_t))} \|\tau(g^{-1})Z\|_2} = e^{-\lambda t} \frac{\|Y\|_v}{\|Z\|_v}. \quad \square$$

Since $\tau(A) = \{\tau(a_t)\}$ is a one-parameter group of diagonal matrices, there exists $\mu > 0$ such that

$$\frac{\mu_1}{\mu_d}(\tau(a_t)) = e^{\mu t}$$

when $t \geq 0$.

Lemma 4.9. *If $v_1, v_2 \in T^1X$, then*

$$e^{-\mu d_X(\pi(v_1), \pi(v_2))} \|\cdot\|_{v_1} \leq \|\cdot\|_{v_2} \leq e^{\mu d_X(\pi(v_1), \pi(v_2))} \|\cdot\|_{v_1}.$$

Proof. Since the family of norms is τ -equivariant, it is enough to consider the case where $v_1 = v_0$ and $v_2 = g(v_0)$. By the Cartan decomposition, there exist $m, \ell \in K$ and $t \geq 0$ such that $g = ma_t\ell$.

Notice that

$$d_X(\pi(v_1), \pi(v_2)) = d_X(p_0, ma_t \ell(p_0)) = d_X(p_0, a_t(p_0)) = t$$

since $t \mapsto a_t(p_0)$ is a unit speed geodesic. Further, $\|\cdot\|_{v_1} = \|\cdot\|_2$ and

$$\|\cdot\|_{v_2} = \|\tau(g^{-1})(\cdot)\|_2 = \|\tau(ma_t)^{-1}(\cdot)\|_2.$$

Hence,

$$\frac{\mu_d}{\mu_1}(\tau(ma_t)^{-1})\|\cdot\|_{v_1} \leq \|\cdot\|_{v_2} \leq \frac{\mu_1}{\mu_d}(\tau(ma_t)^{-1})\|\cdot\|_{v_1}.$$

Since $\frac{\mu_1}{\mu_d}(\tau(ma_t)^{-1}) = \frac{\mu_1}{\mu_d}(\tau(a_{-t})) = e^{\mu t}$, the lemma follows. $\square$

Lemma 4.10. *If $\Gamma \leq G$ is geometrically finite, then $\rho = \tau|_\Gamma$ is uniformly P_k -Anosov relative to $C_X(\Gamma)$.*

Proof. Recall that $C_X(\Gamma)$ is a weak cusp space of Γ and $\mathcal{U}(\Gamma)$ naturally identifies with the space of geodesic lines in $C_X(\Gamma)$. Then Lemma 4.8, Lemma 4.9, and Proposition 2.8 imply that $\rho = \tau|_\Gamma$ is uniformly P_k -Anosov relative to $C_X(\Gamma)$. $\square$

5 | ALMOST HOMOGENEOUS CUSPS

Let G , K , and $X = G/K$ be as in Section 3. In this section, we consider the following setup.

- (1) $\Gamma_0 \leq G$ is a finitely generated discrete group that fixes a horoball $H \subset X$ and $\overline{H} \cap \partial_\infty X = \{\eta^+\}$.
- (2) $\tau : G \rightarrow \mathrm{SL}(d, \mathbb{K})$ is a P_k -proximal representation and

$$\zeta_\tau : \partial_\infty X \rightarrow \mathrm{Gr}_k(\mathbb{K}^d) \times \mathrm{Gr}_{d-k}(\mathbb{K}^d)$$

is the boundary map constructed in Proposition 4.2.

- (3) $\rho : \Gamma_0 \rightarrow \mathrm{SL}(d, \mathbb{K})$ is a representation where $\{\tau(g)\rho(g)^{-1} : g \in \Gamma_0\}$ is relatively compact in $\mathrm{SL}(d, \mathbb{K})$.
- (4) $\mathcal{L} \subset \partial_\infty X$ is a closed, Γ_0 -invariant set where the quotient $\Gamma_0 \backslash (\mathcal{L} \setminus \{\eta^+\})$ is compact.
- (5) $\xi : \mathcal{L} \rightarrow \mathrm{Gr}_k(\mathbb{K}^d) \times \mathrm{Gr}_{d-k}(\mathbb{K}^d)$ is continuous, ρ -equivariant, transverse, and $\xi(\eta^+) = \zeta_\tau(\eta^+)$.
- (6) $\mathcal{U} := \{v \in T^1 X : v^+, v^- \in \mathcal{L}\}$.

In the next section, we will apply the results of this section to the case where Γ_0 is a peripheral subgroup in a geometrically finite group $\Gamma \leq G$, $\mathcal{L}$ is the limit set of Γ , and $\mathcal{U}$ is the flow space $\mathcal{U}(\Gamma)$.

The first result establishes a type of infinitesimal homogeneity of a limit curve at the fixed point of a peripheral subgroup.

Proposition 5.1. *If*

- $\gamma \in G$ is a hyperbolic element with $\gamma^+ = \eta^+$, and
- $(x_n)_{n \geq 1} \subset \partial_\infty X$ is a sequence where $\{\gamma^n(x_n)\} \subset \mathcal{L}$ and $x_n \rightarrow x \in \partial_\infty X$,

then

$$\lim_{n \rightarrow \infty} \tau(\gamma)^{-n} \circ \xi \circ \gamma^n(x_n) = \zeta_\tau(x).$$

Remark 5.2. In the case when $\mathcal{L} = \partial_\infty X$, this says that $\tau(\gamma)^{-n} \circ \xi \circ \gamma^n$ converges uniformly to ζ_τ .

The second result constructs good norms over the horoball H . It will be helpful to use the following notation: given a subset $S \subset X$, let

$$\mathcal{U}|_S := \mathcal{U} \cap \bigcup_{p \in S} T_p^1 X.$$

Proposition 5.3. *There exists a ρ -equivariant family of norms $\|\cdot\|_{v \in T^1 X}$ on $\mathbb{K}^d$ with the following properties.*

- (1) *Each $\|\cdot\|_v$ is induced by an inner product.*
- (2) *For any $r > 1$, there exists $L_r > 1$ such that: if $v, w \in \mathcal{U}|_H$ satisfy $d_X(\pi(v), \pi(w)) \leq r$, then*

$$\frac{1}{L_r} \|\cdot\|_v \leq \|\cdot\|_w \leq L_r \|\cdot\|_v.$$

- (3) *There exist $C_1, c_1 > 0$ and a horoball $H' \subset H$ such that: if $t \geq 0$ and $v, \phi^t(v) \in \mathcal{U}|_{H'}$, then*

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} \leq C_1 e^{-c_1 t} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \xi^k(v^+)$ and nonzero $Z \in \xi^{d-k}(v^-)$.

5.1 | Proof of Proposition 5.1

The following argument is similar to the proof of [16, Prop. 5.3].

Fix a Riemannian distance $d_{\mathcal{F}}$ on $\text{Gr}_k(\mathbb{K}^d) \times \text{Gr}_{d-k}(\mathbb{K}^d)$. Suppose that the proposition is false. Then there exist $x \in \partial_\infty X$, a sequence $(x_j)_{j \geq 1}$ in $\partial_\infty X$, and a sequence $(n_j)_{j \geq 0}$ in $\mathbb{N}$ such that $x_j \rightarrow x$, $n_j \rightarrow \infty$, $\{\gamma^{n_j}(x_j)\} \subset \mathcal{L}$, and $\tau(\gamma)^{-n_j} \circ \xi \circ \gamma^{n_j}(x_j)$ does not converge to $\zeta_\tau(x)$. After passing to a subsequence, there exists $\epsilon > 0$ such that

$$\inf_{j \geq 1} d_{\mathcal{F}}(\tau(\gamma)^{-n_j} \circ \xi \circ \gamma^{n_j}(x_j), \zeta_\tau(x)) \geq \epsilon.$$

Notice that

$$\tau(\gamma)^{-n_j} \circ \xi \circ \gamma^{n_j}(\eta^+) = \tau(\gamma)^{-n_j} \circ \xi(\eta^+) = \tau(\gamma)^{-n_j} \circ \zeta_\tau(\eta^+) = \zeta_\tau(\eta^+)$$

and so after possibly passing to a subsequence $x_j \neq \eta^+$ for all j . Then there exists a sequence $(h_j)_{j \geq 1}$ in Γ_0 such that $y_j := h_j \gamma^{n_j}(x_j)$ is relatively compact in $\mathcal{L} \setminus \{\eta^+\}$. Passing to a subsequence,

we can suppose that $y_j \rightarrow y \in \mathcal{L} \setminus \{\eta^+\}$ and

$$g := \lim_{j \rightarrow \infty} \tau(h_j) \rho(h_j)^{-1} \in \mathrm{SL}(d, \mathbb{K}).$$

Notice that

$$g \xi(\eta^+) = \lim_{j \rightarrow \infty} \tau(h_j) \rho(h_j)^{-1} \xi(\eta^+) = \lim_{j \rightarrow \infty} \tau(h_j) \xi(\eta^+) = \lim_{j \rightarrow \infty} \tau(h_j) \zeta_\tau(\eta^+) = \zeta_\tau(\eta^+).$$

Then since $\xi(y)$ is transverse to $\xi(\eta^+)$, we see that $g \xi(y)$ is transverse to $\zeta_\tau(\eta^+)$.

Also, by construction, $h_j \gamma^{n_j}(p) \rightarrow \eta^+$ for all $p \in X$. By passing to a subsequence, we can suppose that

$$z := \lim_{j \rightarrow \infty} \gamma^{-n_j} h_j^{-1}(p) \in \partial_\infty X$$

for all $p \in (X \cup \partial_\infty X) \setminus \{\eta^+\}$ and the convergence is locally uniform. Since $\gamma^{-n_j} h_j^{-1}(y_j) = x_j \rightarrow x$, and $\{y_j\}$ is relatively compact in $(X \cup \partial_\infty X) \setminus \{\eta^+\}$, we must have $z = x$. So, by the strongly dynamics-preserving property of τ ,

$$\lim_{j \rightarrow \infty} \tau(\gamma^{-n_j} h_j^{-1}) F = \zeta_\tau(x)$$

for all $F = (F^k, F^{d-k}) \in \mathrm{Gr}_k(\mathbb{K}^d) \times \mathrm{Gr}_{d-k}(\mathbb{K}^d)$ transverse to $\zeta_\tau(\eta^+)$.

Finally,

$$\lim_{j \rightarrow \infty} \tau(\gamma)^{-n_j} \circ \xi \circ \gamma^{n_j}(x_j) = \lim_{j \rightarrow \infty} \tau(\gamma^{-n_j} h_j^{-1}) \tau(h_j) \rho(h_j)^{-1} \xi(y_j) = \zeta_\tau(x)$$

since $\tau(h_j) \rho(h_j)^{-1} \xi(y_j) \rightarrow g \xi(y)$ and $g \xi(y)$ is transverse to $\zeta_\tau(\eta^+)$. Thus, we have a contradiction.

5.2 | Proof of Proposition 5.3

Let $\pi_{\partial H} : X \rightarrow \partial H$ be the map where $\pi_{\partial H}(p)$ is the unique point in ∂H contained in the geodesic line passing through p and limiting to η^+ .

Lemma 5.4. *There exists a smooth function $\chi : X \rightarrow [0, 1]$ such that*

- (1) $\chi \circ \pi_{\partial H} = \chi$,
- (2) $\{\chi \circ g\}_{g \in \Gamma_0}$ is a partition of unity (i.e., $\sum_{g \in \Gamma_0} \chi \circ g$ is a locally finite sum that equals one everywhere).

Proof. By Selberg's lemma, there exists a finite-index torsion-free subgroup $\Gamma'_0 \leq \Gamma_0$. Let $n = [\Gamma_0 : \Gamma'_0]$.

Consider the manifold quotient $p : \partial H \rightarrow \Gamma'_0 \backslash \partial H$. Fix an open cover $\{U_i\}_{i \in I}$ of $\Gamma'_0 \backslash \partial H$ such that for all $i \in I$, there is a local inverse $U_i \rightarrow \tilde{U}_i \subset \partial H$ to p . Fix a partition of unity $\{\chi_i\}_{i \in I}$ of $\Gamma'_0 \backslash \partial H$

subordinate to $\{U_i\}_{i \in I}$. Then for each $i \in I$, let $\tilde{\chi}_i : \partial H \rightarrow [0, 1]$ be the lift of χ_i to $\tilde{U}_i$. Finally, let

$$\chi := \frac{1}{n} \sum_{i \in I} \tilde{\chi}_i \circ \pi_{\partial H}.$$

By construction,

$$\sum_{g \in \Gamma'_0} \sum_{i \in I} \tilde{\chi}_i \circ \pi_{\partial H} \circ g$$

is a locally finite sum that equals one everywhere. So, if $\{\Gamma'_0 g_1, \dots, \Gamma'_0 g_n\} = \Gamma'_0 \setminus \Gamma_0$, then

$$\sum_{g \in \Gamma_0} \chi \circ g = \frac{1}{n} \sum_{k=1}^n \left(\sum_{g \in \Gamma'_0} \sum_{i \in I} \tilde{\chi}_i \circ \pi_{\partial H} \circ g \right) \circ g_k$$

is a locally finite sum that equals one everywhere. □

Fix $v_0 \in \mathcal{U}$ with $p_0 := \pi(v_0) \in \partial H$ and $v_0^+ = \eta^+$. By conjugating K , we may assume that

$$K = \text{Stab}_G(p_0).$$

Since K is compact, there exists a $\tau(K)$ -invariant norm $\|\cdot\|^{(0)}$ on $\mathbb{K}^d$ that is induced by an inner product. Then

$$\|\cdot\|_{gv_0}^{(0)} := \|\tau(g)^{-1}(\cdot)\|^{(0)}$$

defines a smooth τ -equivariant family of norms indexed by $T^1 X$ where each norm is induced by an inner product.

Then given $v \in T^1 X$ define

$$\|\cdot\|_v = \sqrt{\sum_{g \in \Gamma_0} (\chi \circ g)(\pi(v)) \left(\|\rho(g)(\cdot)\|_{gv}^{(0)} \right)^2}.$$

Since $\{\chi \circ g\}_{g \in \Gamma_0}$ is a partition of unity, $\|\cdot\|_{v \in T^1 X}$ is a smooth family of norms where each $\|\cdot\|_v$ is induced by an inner product. One can check that it is ρ -equivariant. We will show that this family of norms satisfies the remaining conditions in the proposition.

We start by showing some useful compactness/cocompactness properties. Let $\{a_t\} \leq G$ be a Cartan subgroup such that $a_t(v_0) = \phi^t(v_0)$ for all $t \in \mathbb{R}$.

Lemma 5.5. *The set*

$$\{\tau(a_{-t})\tau(g)\rho(g)^{-1}\tau(a_t) : g \in \Gamma_0, t \geq 0\}$$

is relatively compact in $\text{SL}(d, \mathbb{K})$.

Proof. By [34] and conjugating τ and ρ , we may assume that

$$\tau(a_t) = \begin{pmatrix} e^{\lambda_1 t} \text{id}_{d_1} & & \\ & \ddots & \\ & & e^{\lambda_{m+1} t} \text{id}_{d_{m+1}} \end{pmatrix}$$

where $\lambda_1 > \dots > \lambda_{m+1}$. Since $\tau(a_{-t})$ is conjugate to $\tau(a_t)$, notice that $d_k = d_{m-k}$ and $\lambda_k = -\lambda_{m-k}$. For $1 \leq n \leq m$, let $k_n = \sum_{j=1}^n d_j$. Then, τ is P_{k_n} -proximal for all $1 \leq n \leq m$. Consider the partial flag manifold

$$\mathcal{F} = \left\{ (F^{k_n})_{n=1}^m : F^{k_1} \subset \dots \subset F^{k_m} \text{ and } \dim F^{k_n} = k_n \text{ for } n = 1, \dots, m \right\}$$

and let

$$F^+ = \left(\langle e_1, \dots, e_{k_n} \rangle \right)_{n=1}^m \in \mathcal{F}.$$

Since the boundary map constructed in Proposition 4.2 is equivariant and strongly dynamics-preserving, $\tau(\Gamma_0)$ fixes F^+ , and if $(g_n)_{n \geq 1}$ is an escaping sequence in Γ_0 , then

$$\lim_{n \rightarrow \infty} \tau(g_n)F = F^+$$

for all $F \in \mathcal{F}$ transverse to F^+ and the convergence is locally uniform.

We claim that $\rho(\Gamma_0)$ fixes F^+ . Fix $g \in \Gamma_0$ and fix an escaping sequence $(g_n)_{n \geq 1}$ in Γ_0 . Passing to a subsequence, we can suppose that

$$\rho(g_n)^{-1} \tau(g_n) \rightarrow h_1 \quad \text{and} \quad \tau(gg_n)^{-1} \rho(gg_n) \rightarrow h_2.$$

Fix $F \in \mathcal{F}$ transverse to $(h_2 h_1)^{-1} F^+$ and F^+ . Then

$$\begin{aligned} \rho(g)F^+ &= \lim_{n \rightarrow \infty} \rho(g) \tau(g_n)F = \lim_{n \rightarrow \infty} \tau(gg_n) \tau(gg_n)^{-1} \rho(gg_n) \rho(g_n)^{-1} \tau(g_n)F \\ &= F^+. \end{aligned}$$

Since $g \in \Gamma_0$ was arbitrary, $\rho(\Gamma_0)$ fixes F^+ .

Finally, since $\rho(\Gamma_0)$, $\tau(\Gamma_0)$ both fix F^+ and

$$\{ \tau(g) \rho(g)^{-1} : g \in \Gamma_0 \}$$

is relatively compact in $\text{SL}(d, \mathbb{K})$, for every $1 \leq i \leq j \leq m+1$, there exist compact subsets $K_{i,j}$ of d_i -by- d_j matrices such that

$$\{ \tau(g) \rho(g)^{-1} : g \in \Gamma_0 \} \subset \left\{ \begin{pmatrix} A_{1,1} & \dots & A_{1,m+1} \\ & \ddots & \vdots \\ & & A_{m+1,m+1} \end{pmatrix} : A_{i,j} \in K_{i,j} \right\}.$$

Then

$$\{\tau(a_{-t})\tau(g)\rho(g)^{-1}\tau(a_t) : g \in \Gamma_0, t \geq 0\}$$

is relatively compact in $\mathrm{SL}(d, \mathbb{K})$. □

Since X is Gromov-hyperbolic, there exists $\delta > 0$ such that every geodesic triangle, including every ideal geodesic triangle, is δ -slim.

Lemma 5.6. Γ_0 acts cocompactly on $\mathcal{U}|_{\partial H}$.

Proof. Fix a compact subset $K_0 \subset \mathcal{L} \setminus \{\eta^+\}$ such that $\Gamma_0 \cdot K_0 = \mathcal{L} \setminus \{\eta^+\}$. Then let

$$K_1 := \{v \in \mathcal{U}|_{\partial H} : v^- \in K_0 \text{ and } v^+ = \eta^+\}$$

and

$$K_2 := \{v \in \mathcal{U}|_{\partial H} : d_X(\pi(v), \pi(w)) \leq 2\delta \text{ for some } w \in K_1\}.$$

Notice that both K_1 and K_2 are compact subsets.

We claim that $\Gamma_0 \cdot K_2 = \mathcal{U}|_{\partial H}$. Fix $v \in \mathcal{U}|_{\partial H}$. By our choice of δ , the ideal geodesic triangle with vertices η^+, v^+, v^- is δ -slim. So, there exists $s \in \{-, +\}$ such that $\pi(v)$ is within δ of the geodesic line joining v^s and η^+ . Then let $w \in \mathcal{U}$ be the vector with $\pi(w) \in \partial H$, $w^- = v^s$, and $w^+ = \eta^+$. Fix $T \in \mathbb{R}$ such that

$$d_X(\pi(\phi^T(w)), \pi(v)) \leq \delta.$$

Since $w^+ = \eta^+$ and $\pi(w), \pi(v) \in \partial H$, then

$$|T| \leq d_X(\pi(\phi^T(w)), \pi(v)) \leq \delta.$$

So $d_X(\pi(v), \pi(w)) \leq 2\delta$. By our choice of K_1 , we have $w \in \Gamma_0 \cdot K_1$ which implies that $v \in \Gamma_0 \cdot K_2$. □

Lemma 5.7. There exists a compact subset $\mathcal{K} \subset G$ such that

$$\mathcal{U}|_H \subset \Gamma_0 \cdot \{a_t\}_{t \geq 0} \cdot \mathcal{K} \cdot v_0.$$

Proof. By the previous lemma, there exists $R > 0$ such that

$$\pi(\mathcal{U}|_{\partial H}) \subset \Gamma_0 \cdot B_X(p_0, R).$$

Then let

$$\mathcal{K} := \{g \in G : d_X(g(p_0), p_0) \leq R + \delta\}.$$

Fix $v \in \mathcal{U}|_H$. By our choice of δ , the ideal geodesic triangle with vertices η^+, v^+, v^- is δ -slim. So, there exists $s \in \{-, +\}$ such that $\pi(v)$ is within δ of the geodesic line joining v^s and η^+ . Then let $w \in \mathcal{U}$ be the vector with $\pi(w) \in \partial H$, $w^- = v^s$, and $w^+ = \eta^+$. Fix $T \geq 0$ such that

$$d_X(\pi(\phi^T(w)), \pi(v)) \leq \delta$$

and fix $\beta \in \Gamma_0$ such that $d_X(\beta(p_0), \pi(w)) \leq R$.

Then,

$$\begin{aligned} d_X(p_0, a_{-T}\beta^{-1}\pi(v)) &= d_X(\beta\pi(\phi^T(v_0)), \pi(v)) \\ &\leq d_X(\beta\pi(\phi^T(v_0)), \pi(\phi^T(w))) + d_X(\pi(\phi^T(w)), \pi(v)) \leq d_X(\beta(p_0), \pi(w)) + \delta \\ &\leq R + \delta. \end{aligned}$$

So, we can pick $\alpha \in \mathcal{K}$ such that $\alpha(v_0) = a_{-T}\beta^{-1}v$ or equivalently

$$v = \beta a_T \alpha(v_0) \in \Gamma_0 \cdot \{a_t\}_{t \geq 0} \cdot \mathcal{K} \cdot v_0. \quad \square$$

Lemma 5.8. *There exists $C > 1$ such that: If $v \in \mathcal{U}|_H$, then*

$$\frac{1}{C} \|\cdot\|_v^{(0)} \leq \|\cdot\|_v \leq C \|\cdot\|_v^{(0)}.$$

Proof. By Lemma 5.7, there exist $\beta \in \Gamma_0$, $T \geq 0$, and $\alpha \in \mathcal{K}$ such that $v = \beta a_T \alpha(v_0)$. Then,

$$\begin{aligned} \|\cdot\|_v &= \sqrt{\sum_{g \in \Gamma_0} (\chi \circ g)(\pi(v)) \left(\|\rho(g)(\cdot)\|_{gv}^{(0)} \right)^2} \\ &= \sqrt{\sum_{g \in \Gamma_0} (\chi \circ g)(\pi(v)) \left(\|\tau(\alpha)^{-1} \tau(a_{-T}) \tau(g\beta)^{-1} \rho(g)(\cdot)\|_{v_0}^{(0)} \right)^2}. \end{aligned}$$

Using the compactness of $\mathcal{K}$ and Lemma 5.5

$$\begin{aligned} \|\tau(\alpha)^{-1} \tau(a_{-T}) \tau(g\beta)^{-1} \rho(g)(\cdot)\|_{v_0}^{(0)} &\asymp \|\tau(a_{-T}) \tau(g\beta)^{-1} \rho(g)(\cdot)\|_{v_0}^{(0)} \\ &= \|\tau(a_{-T}) \tau(g\beta)^{-1} \rho(g\beta) \tau(a_T) \tau(a_{-T}) \rho(\beta)^{-1} \tau(\beta) \tau(a_T) \tau(\beta a_T)^{-1}(\cdot)\|_{v_0}^{(0)} \\ &\asymp \|\tau(\beta a_T)^{-1}(\cdot)\|_{v_0}^{(0)} \asymp \|\tau(\beta a_T)^{-1}(\cdot)\|_{\alpha(v_0)}^{(0)} = \|\cdot\|_v^{(0)}. \end{aligned}$$

Thus,

$$\|\cdot\|_v \asymp \sqrt{\sum_{g \in \Gamma_0} (\chi \circ g)(\pi(v)) \left(\|\cdot\|_v^{(0)} \right)^2} = \|\cdot\|_v^{(0)}. \quad \square$$

We can now establish part (2) of the proposition.

Lemma 5.9. *For any $r > 1$, there exists $L_r > 1$ such that: if $v, w \in \mathcal{U}|_H$ and $d_X(\pi(v), \pi(w)) \leq r$, then*

$$\frac{1}{L_r} \|\cdot\|_v \leq \|\cdot\|_w \leq L_r \|\cdot\|_v.$$

Proof. This follows immediately from Proposition 4.2 and Lemma 5.8, since $\|\cdot\|_v^{(0)}$ is a τ -equivariant family of norms. $\square$

We next establish part (3) of the proposition. By Proposition 4.2, there exist $C_\lambda, \lambda > 0$ such that:

$$\frac{\|Y\|_{\phi^t(v)}^{(0)}}{\|Z\|_{\phi^t(v)}^{(0)}} \leq C_\lambda e^{-\lambda t} \frac{\|Y\|_v^{(0)}}{\|Z\|_v^{(0)}} \quad (4)$$

for all $t \geq 0$, $v \in T^1X$, $Y \in \xi_\tau^k(v^+)$, and nonzero $Z \in \xi_\tau^{d-k}(v^-)$.

Lemma 5.10. *There exist $C_1 > 0$ and a horoball $H' \subset H$ such that: if $t \geq 0$ and $v, \phi^t(v) \in \mathcal{U}|_{H'}$, then*

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} \leq C_1 e^{-\frac{\lambda}{2}t} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \xi^k(v^+)$ and nonzero $Z \in \xi^{d-k}(v^-)$.

Proof. The following argument is similar to the proof of [16, Prop. 6.4]. Fix $T > 0$ such that

$$C^4 C_\lambda < e^{\frac{\lambda}{2}T}, \quad (5)$$

where C is the constant from Lemma 5.8.

We first claim that there exists a horoball $H' \subset H$ such that: if $t \in [T, 2T]$ and $v, \phi^t(v) \in \mathcal{U}|_{H'}$, then

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} \leq e^{-\frac{\lambda}{2}t} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \xi^k(v^+)$ and nonzero $Z \in \xi^{d-k}(v^-)$.

Suppose not. Then there exist sequences $(v_n)_{n \geq 1}$ in $\mathcal{U}$, $(t_n)_{n \geq 1}$ in $[T, 2T]$, and $(Y_n)_{n \geq 1}, (Z_n)_{n \geq 1}$ in $\mathbb{K}^d$ such that $d_X(\pi(v_n), \partial H) \rightarrow \infty$, $Y_n \in \xi^k(v_n^+)$, $Z_n \in \xi^{d-k}(v_n^-) \setminus \{0\}$, and

$$\frac{\|Y_n\|_{\phi^{t_n}(v_n)}}{\|Z_n\|_{\phi^{t_n}(v_n)}} > e^{-\frac{\lambda}{2}t_n} \frac{\|Y_n\|_{v_n}}{\|Z_n\|_{v_n}}.$$

By scaling, we may assume that

$$\|Y_n\|_{v_n} = \|Z_n\|_{v_n} = 1. \quad (6)$$

Using Lemma 5.7 and possibly replacing each v_n with an Γ_0 -translate, we can find a sequence $m_n \rightarrow \infty$ and a relatively compact sequence $(\alpha_n)_{n \geq 1}$ in G such that

$$v_n = a_1^{m_n} \alpha_n(v_0).$$

Let $Y'_n := \tau(a_1^{m_n} \alpha_n)^{-1} Y_n$ and $Z'_n := \tau(a_1^{m_n} \alpha_n)^{-1} Z_n$. Then, by Lemma 5.8 and Equation (6),

$$\|Y'_n\|_{v_0}^{(0)} = \|Y_n\|_{v_n}^{(0)} \in [C^{-1}, C].$$

Likewise $\|Z'_n\|_{v_0}^{(0)} \in [C^{-1}, C]$.

Passing to a subsequence, we can suppose that $t_n \rightarrow t \in [T, 2T]$, $\alpha_n \rightarrow \alpha \in G$, $Y'_n \rightarrow Y'$, and $Z'_n \rightarrow Z'$. Proposition 5.1 implies that

$$\begin{aligned} Y' &= \tau(\alpha)^{-1} \lim_{n \rightarrow \infty} \tau(a_1)^{-m_n} Y_n \in \tau(\alpha)^{-1} \lim_{n \rightarrow \infty} \tau(a_1)^{-m_n} \xi^k(v_n^+) \\ &= \tau(\alpha)^{-1} \lim_{n \rightarrow \infty} \tau(a_1)^{-m_n} \circ \xi^k \circ a_1^{m_n}(\alpha_n(v_0^+)) = \tau(\alpha)^{-1} \circ \zeta_\tau^k \circ \alpha(v_0^+) \\ &= \zeta_\tau^k(v_0^+). \end{aligned}$$

Likewise, $Z' \in \zeta_\tau^{d-k}(v_0^-)$.

Then Lemma 5.8 and Equation (4) imply that

$$\begin{aligned} e^{-\frac{\lambda}{2}t} &\leq \liminf_{n \rightarrow \infty} \frac{\|Y_n\|_{\phi^{t_n}(v_n)}^{(0)}}{\|Z_n\|_{\phi^{t_n}(v_n)}^{(0)}} \leq C^2 \liminf_{n \rightarrow \infty} \frac{\|Y_n\|_{\phi^{t_n}(v_n)}^{(0)}}{\|Z_n\|_{\phi^{t_n}(v_n)}^{(0)}} = C^2 \liminf_{n \rightarrow \infty} \frac{\|Y'_n\|_{\phi^{t_n}(v_0)}^{(0)}}{\|Z'_n\|_{\phi^{t_n}(v_0)}^{(0)}} \\ &= C^2 \frac{\|Y'\|_{\phi^t(v_0)}^{(0)}}{\|Z'\|_{\phi^t(v_0)}^{(0)}} \leq C^2 C_\lambda e^{-\lambda t} \frac{\|Y'\|_{v_0}^{(0)}}{\|Z'\|_{v_0}^{(0)}} \leq C^4 C_\lambda e^{-\lambda t}. \end{aligned}$$

Then $e^{\frac{\lambda}{2}T} \leq e^{\frac{\lambda}{2}t} \leq C^4 C_\lambda$ and we have a contradiction with Equation (5). So, the claim is true.

Now suppose that $t \geq 0$, $v, \phi^t(v) \in \mathcal{U}|_{H'}$, $Y \in \xi^k(v^+)$, and $Z \in \xi^{d-k}(v^-) \setminus \{0\}$. If $t \leq T$, then

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} \leq L_T e^{\frac{\lambda}{2}T} e^{-\frac{\lambda}{2}t} \frac{\|Y\|_v}{\|Z\|_v}$$

by Lemma 5.9. If $t \geq T$, then we can break $[0, t]$ into subintervals each with length between T and $2T$, then apply the claim on each subinterval to obtain

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} \leq e^{-\frac{\lambda}{2}t} \frac{\|Y\|_v}{\|Z\|_v}.$$

So, $C_1 := L_T e^{\frac{\lambda}{2}T}$ suffices. □

6 | REPRESENTATIONS WITH ALMOST HOMOGENEOUS CUSPS

Let G , K , and $X = G/K$ be as in Section 3. In this section, we prove Theorem 1.13, restated in the following form.

Theorem 6.1. *Suppose that*

- $\Gamma \leq G$ is geometrically finite and $\mathcal{P}$ is a set of representatives of the conjugacy classes of maximal parabolic subgroups of Γ ,
- $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}$, and
- for each $P \in \mathcal{P}$, there exists a representation $\tau_P : G \rightarrow \mathrm{SL}(d, \mathbb{K})$ such that

$$\{\tau_P(g)\rho(g)^{-1} : g \in P\}$$

is relatively compact in $\mathrm{SL}(d, \mathbb{K})$.

Then, ρ is uniformly P_k -Anosov relative to $C_X(\Gamma)$.

The rest of the section is devoted to the proof of theorem, so fix Γ , $\mathcal{P}$, ρ , and representations $\{\tau_P : P \in \mathcal{P}\}$ as in the statement. Let $E_\rho := \mathcal{U}(\Gamma) \times \mathbb{K}^d$ and $\hat{E}_\rho := \Gamma \backslash (\mathcal{U}(\Gamma) \times \mathbb{K}^d)$.

For each $P \in \mathcal{P}$, fix an open horoball H_P centered at the fixed point of P such that: if $\gamma \in \Gamma$, then $\gamma \bar{H}_P \cap \bar{H}_P \neq \emptyset$ if and only if $\gamma \in P$. This is possible by the “F1” definition of geometrically finite subgroups in [8]. Let

$$\hat{\mathcal{U}}_P := \Gamma \backslash \{v \in \mathcal{U}(\Gamma) : \pi(v) \in H_P\},$$

$\hat{\mathcal{U}}_{thin} := \bigcup_{P \in \mathcal{P}} \hat{\mathcal{U}}_P$, and $\hat{\mathcal{U}}_{thick} := \hat{\mathcal{U}}(\Gamma) \setminus \hat{\mathcal{U}}_{thin}$. Then $\hat{\mathcal{U}}_{thick}$ is compact by the “F1” definition of geometrically finite subgroups in [8].

Lemma 6.2. *After possibly replacing each H_P with a smaller horoball, there exist $C_0, c_0 > 0$ and a metric $\|\cdot\|_{v \in \hat{\mathcal{U}}(\Gamma)}$ on the vector bundle $\hat{E}_\rho \rightarrow \hat{\mathcal{U}}(\Gamma)$ such that:*

- (1) $\|\cdot\|_{v \in \hat{\mathcal{U}}(\Gamma)}$ is locally uniform,
- (2) if $t \geq 0$, $v \in \hat{\mathcal{U}}(\Gamma)$, and $\phi^s(v) \in \hat{\mathcal{U}}_{thin}$ for all $s \in [0, t]$, then

$$\frac{\|\phi^t(Y)\|_{\phi^t(v)}}{\|\phi^t(Z)\|_{\phi^t(v)}} \leq C_0 e^{-c_0 t} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \hat{\Theta}^k(v)$ and nonzero $Z \in \hat{\Xi}^{d-k}(v)$.

Proof. Fix a partition of unity $\{\chi_0\} \cup \{\chi_P : P \in \mathcal{P}\}$ of $\hat{\mathcal{U}}(\Gamma)$ such that $\mathrm{supp}(\chi_0)$ is compact and $\mathrm{supp}(\chi_P) \subset \hat{\mathcal{U}}_P$ for all $P \in \mathcal{P}$.

Let $\|\cdot\|_{v \in \hat{\mathcal{U}}(\Gamma)}^{(0)}$ be any metric on $\hat{E}_\rho \rightarrow \hat{\mathcal{U}}(\Gamma)$. For each $P \in \mathcal{P}$, let $\|\cdot\|_{v \in T^1 X}^P$ be a family of $\rho|_P$ -equivariant norms satisfying Proposition 5.3. Then $\|\cdot\|_{v \in T^1 X}^P$ descends to a metric on the fibers of

$\widehat{E}_\rho$ above $\widehat{U}_P$ which we denote by $\|\cdot\|_{v \in \widehat{U}_P}^P$. Then,

$$\|\cdot\|_v = \sqrt{\chi_0(v) \left(\|\cdot\|_v^{(0)} \right)^2 + \sum_{P \in \mathcal{P}} \chi_P(v) \left(\|\cdot\|_v^P \right)^2}$$

defines a metric with the desired properties. $\square$

Lemma 6.3. *There exists $T_0 > 0$ such that: if $t \geq T_0$ and $v, \phi^t(v) \in \widehat{U}_{thick}$, then*

$$\frac{\|\phi^t(Y)\|_{\phi^t(v)}}{\|\phi^t(Z)\|_{\phi^t(v)}} \leq \frac{1}{2C_0^2} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \widehat{\Theta}^k(v)$ and nonzero $Z \in \widehat{\Xi}^{d-k}(v)$.

Proof. Lift $\|\cdot\|_{v \in \widehat{U}(\Gamma)}$ to a ρ -equivariant family of norms $\|\cdot\|_{v \in \mathcal{U}(\Gamma)}$. Let

$$\mathcal{U}_{thick} := \mathcal{U}(\Gamma) \cap \pi^{-1}(\widehat{U}_{thick}).$$

Then fix a compact set $K \subset \mathcal{U}_{thick}$ such that $\Gamma \cdot K = \mathcal{U}_{thick}$. Finally, fix some $p_0 \in \pi(K)$ and let $R := \text{diam}_X(\pi(K))$.

Arguing as in the proof of [45, Lem. 9.4], there exists $C > 1$ such that: if $v \in K$, $t \geq 0$, and $\phi^t(v) \in g(K)$ for some $g \in \Gamma$, then

$$\frac{\|Y\|_{\phi^t(v)}}{\|Z\|_{\phi^t(v)}} \leq C \frac{\mu_{k+1}(\rho(g))}{\mu_k} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \xi^k(v^+)$ and nonzero $Z \in \xi^{d-k}(v^-)$. Notice that in this case

$$d_X(p_0, g(p_0)) \geq t - 2R.$$

Also, by the strongly dynamics-preserving property and Observation 2.5, there exists $T'_0 > 0$ such that: if $g \in \Gamma$ and $d_X(p_0, g(p_0)) \geq T'_0$, then

$$\frac{\mu_{k+1}(\rho(g))}{\mu_k} \leq \frac{1}{2CC_0^2}.$$

So $T_0 := T'_0 + 2R$ suffices. $\square$

Lemma 6.4. *There exists $T > 1$ such that: if $t \geq T$ and $v \in \widehat{U}(\Gamma)$, then*

$$\frac{\|\phi^t(Y)\|_{\phi^t(v)}}{\|\phi^t(Z)\|_{\phi^t(v)}} \leq \frac{1}{2} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \widehat{\Theta}^k(v)$ and nonzero $Z \in \widehat{\Xi}^{d-k}(v)$.

Proof. The following argument is similar to an argument in [16, pp. 33-35]. From Lemma 6.2(1), there exists $C_2 > 1$ such that: if $v \in \hat{\mathcal{U}}(\Gamma)$ and $t \in [0, T_0]$, then

$$\frac{\|\varphi^t(Y)\|_{\phi^t(v)}}{\|\varphi^t(Z)\|_{\phi^t(v)}} \leq C_2 \frac{\|Y\|_v}{\|Z\|_v} \quad (7)$$

for all $Y \in \hat{\Theta}^k(v)$ and nonzero $Z \in \hat{\Xi}^{d-k}(v)$.

Fix $T > 1$ so that

$$C_0 e^{-c_0 T} \leq \frac{1}{2} \quad \text{and} \quad C_0^2 C_2 e^{-c_0(T-T_0)} \leq \frac{1}{2}.$$

Suppose $t \geq T$ and $v \in \hat{\mathcal{U}}(\Gamma)$. If $\phi^s(v) \in \hat{\mathcal{U}}_{thin}$ for all $s \in [0, t]$, then Lemma 6.2(2) implies that

$$\frac{\|\varphi^t(Y)\|_{\phi^t(v)}}{\|\varphi^t(Z)\|_{\phi^t(v)}} \leq C_0 e^{-c_0 t} \frac{\|Y\|_v}{\|Z\|_v} \leq \frac{1}{2} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \hat{\Theta}^k(v)$ and nonzero $Z \in \hat{\Xi}^{d-k}(v)$. Otherwise, the set $\mathcal{R} := \{s \in [0, t] : \phi^s(v) \in \hat{\mathcal{U}}_{thick}\}$ is nonempty. Let $s_1 := \min \mathcal{R}$ and $s_2 := \max \mathcal{R}$. If $s_2 - s_1 \geq T_0$, then applying Lemma 6.2(2) to the intervals $[0, s_1]$, $[s_2, t]$ and Lemma 6.3 to the interval $[s_1, s_2]$ yields

$$\frac{\|\varphi^t(Y)\|_{\phi^t(v)}}{\|\varphi^t(Z)\|_{\phi^t(v)}} \leq C_0 e^{-c_0(t-s_2)} \frac{1}{2C_0^2} C_0 e^{-c_0 s_1} \frac{\|Y\|_v}{\|Z\|_v} \leq \frac{1}{2} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \hat{\Theta}^k(v)$ and nonzero $Z \in \hat{\Xi}^{d-k}(v)$. Otherwise, if $s_2 - s_1 \leq T_0$, then applying Lemma 6.2(2) to the intervals $[0, s_1]$, $[s_2, t]$ and Equation (7) to the interval $[s_1, s_2]$ yields

$$\frac{\|\varphi^t(Y)\|_{\phi^t(v)}}{\|\varphi^t(Z)\|_{\phi^t(v)}} \leq C_0 e^{-c_0(t-s_2)} C_2 C_0 e^{-c_0 s_1} \frac{\|Y\|_v}{\|Z\|_v} \leq C_0^2 C_2 e^{-c_0(T-T_0)} \frac{\|Y\|_v}{\|Z\|_v} \leq \frac{1}{2} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \hat{\Theta}^k(v)$ and nonzero $Z \in \hat{\Xi}^{d-k}(v)$. □

Proof of Theorem 6.1. By Lemma 6.2(1), we have locally uniform norms, and it remains only to verify the dominated splitting condition in Proposition 2.8. Already, from Lemma 6.2(1), there exists $C_3 > 1$ such that: if $v \in \hat{\mathcal{U}}(\Gamma)$ and $t \in [0, T]$, then

$$\frac{\|\varphi^t(Y)\|_{\phi^t(v)}}{\|\varphi^t(Z)\|_{\phi^t(v)}} \leq C_3 \frac{\|Y\|_v}{\|Z\|_v}$$

for all $Y \in \hat{\Theta}^k(v)$ and nonzero $Z \in \hat{\Xi}^{d-k}(v)$. Lemma 6.4 then implies that:

$$\frac{\|\varphi^t(Y)\|_{\phi^t(v)}}{\|\varphi^t(Z)\|_{\phi^t(v)}} \leq 2C_3 e^{-\frac{\ln(2)}{T}t} \frac{\|Y\|_v}{\|Z\|_v}$$

for all $v \in \hat{\mathcal{U}}(\Gamma)$, $t \geq 0$, $Y \in \hat{\Theta}^k(v)$, and nonzero $Z \in \hat{\Xi}^{d-k}(v)$. □

7 | NOT UNIFORM RELATIVE TO THE GROVES–MANNING CUSP SPACE

In this section, we construct the representation described in Example 1.12 above. In particular, we construct a relatively P_1 -Anosov representation that is uniform relative to some weak cusp space, but is not uniformly P_1 -Anosov relative to any Groves–Manning cusp space.

We consider the Siegel model of complex hyperbolic 2-space

$$\mathbb{H}_{\mathbb{C}}^2 = \{[z_1 : z_2 : 1] : \operatorname{Im}(z_1) > |z_2|^2\} \subset \mathbf{P}(\mathbb{C}^3).$$

Then $\operatorname{Isom}_0(\mathbb{H}_{\mathbb{C}}^2)$ coincides with the subgroup of $\operatorname{PSL}(3, \mathbb{C})$ that preserves $\mathbb{H}_{\mathbb{C}}^2$. Let $G \rightarrow \operatorname{Isom}_0(\mathbb{H}_{\mathbb{C}}^2)$ denote the preimage in $\operatorname{SL}(3, \mathbb{C})$.

For $m, n \in \mathbb{Z}$, define

$$u(m, n) := \begin{pmatrix} 1 & m & \frac{1}{2}m^2 + in \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \in \operatorname{SL}(3, \mathbb{C}).$$

Then let $P := \{u(m, n) : m, n \in \mathbb{Z}\} \leq G$. Notice that

$$(m, n) \in \mathbb{Z}^2 \mapsto u(m, n) \in P$$

is a group isomorphism. Using ping-pong, we can find a hyperbolic element $h \in G$ such that $\Gamma := \langle h \rangle * P$ is a geometrically finite subgroup of G isomorphic to $\mathbb{Z} * \mathbb{Z}^2$.

Let $\Lambda(\Gamma) \subset \partial_{\infty} \mathbb{H}_{\mathbb{C}}^2$ denote the limit set of Γ and let $\mathcal{C}(\Gamma)$ denote the convex hull of $\Lambda(\Gamma)$ in $\mathbb{H}_{\mathbb{C}}^2$. Then by Proposition 4.2, the inclusion representation $\rho : \Gamma \hookrightarrow \operatorname{SL}(3, \mathbb{C})$ is uniformly P_1 -Anosov relative to $\mathcal{C}(\Gamma)$.

Let $\mathcal{P} := \{P\}$ and $S := \{h, h^{-1}, u(1, 0), u(-1, 0), u(0, 1), u(0, -1)\}$. Then consider the associated Groves–Manning cusp space $X := C_{GM}(\Gamma, \mathcal{P}, S)$.

The main result of this section is the following.

Proposition 7.1. *There does not exist a ρ -equivariant quasi-isometric embedding of X into $M := \operatorname{SL}(3, \mathbb{C})/\operatorname{SU}(3, \mathbb{C})$.*

When combined with results in [45], this yields the following corollary.

Corollary 7.2. *ρ is not uniformly P_1 -Anosov relative to any Groves–Manning cusp space.*

Proof of Corollary. Suppose for a contradiction that ρ is uniformly P_1 -Anosov relative to some Groves–Manning cusp space Y . By [45, Th. 1.12], there exists a ρ -equivariant quasi-isometric embedding of $F : Y \rightarrow M$. However, the identity map on vertices extends to a Γ -equivariant quasi-isometry $G : X \rightarrow Y$, see [6, Th. 1.1], and so, we obtain a ρ -equivariant quasi-isometric embedding $F \circ G : X \rightarrow M$. Hence, we have a contradiction. $\square$

The rest of the section is devoted to the proof of the proposition. Suppose for a contradiction that there exists a ρ -equivariant quasi-isometric embedding $F : X \rightarrow M$. Let d_M denote the standard

symmetric distance on M defined in Equation (1) and let $K := \mathrm{SU}(3, \mathbb{C})$. Then

$$d_M(gK, K) \asymp \log \frac{\mu_1}{\mu_3}(g)$$

for all $g \in \mathrm{SL}(3, \mathbb{C})$.

Using the Iwasawa decomposition, for every $n \in \mathbb{N}$, we can write

$$F((\mathrm{id}_P, n)) = w_n a_n K$$

where a_n is a diagonal matrix with positive diagonal entries and w_n is upper triangular matrix with ones on the diagonal. Then, for all $g \in P$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} d_M(F((g, n)), F((\mathrm{id}_P, n))) &= d_M(\rho(g)w_n a_n K, w_n a_n K) \\ &= d_M(a_n^{-1} w_n^{-1} \rho(g) w_n a_n K, K) \asymp \log \frac{\mu_1}{\mu_3}(a_n^{-1} w_n^{-1} \rho(g) w_n a_n). \end{aligned}$$

Further, since $F: X \rightarrow M$ is a quasi-isometric embedding, there exist $\alpha > 1, \beta > 0$ such that: if $g \in P$ and $n \in \mathbb{N}$, then

$$\begin{aligned} \frac{1}{\alpha} d_X((g, n), (\mathrm{id}_P, n)) - \beta &\leq \log \frac{\mu_1}{\mu_3}(a_n^{-1} w_n^{-1} \rho(g) w_n a_n) \\ &\leq \alpha d_X((g, n), (\mathrm{id}_P, n)) + \beta. \end{aligned} \quad (8)$$

Suppose

$$a_n = \begin{pmatrix} \lambda_{n,1} & 0 & 0 \\ 0 & \lambda_{n,2} & 0 \\ 0 & 0 & \lambda_{n,3} \end{pmatrix} \quad \text{and} \quad w_n = \begin{pmatrix} 1 & s_n & r_n \\ 0 & 1 & t_n \\ 0 & 0 & 1 \end{pmatrix}.$$

We will obtain a contradiction by estimating $\lambda_{n,1}^{-1} \lambda_{n,3}$ in two ways.

We start with the following distance estimate in the Groves–Manning cusp space.

Lemma 7.3. *There exists $n_0 > 0$ such that: if $k \geq n \geq n_0$, then*

$$2k - 2n - 2 \leq d_X((u(0, 2^k), n), (\mathrm{id}_P, n)).$$

Proof. For $L \geq 1$, let $\mathcal{H}(L) \subset X$ denote the induced subgraph of X with vertex set

$$\{(g, n) : g \in P, n \geq L\}.$$

By [21, Lem. 3.26], there exists $\delta \geq 1$ such that $\mathcal{H}(\delta)$ is geodesically convex in X .

Fix $k \geq n \geq \delta$. By [21, Lem. 3.10], there exists a geodesic in $\mathcal{H}(\delta)$ joining $(u(0, 2^k), n)$ to (id_P, n) which consists of m vertical edges, followed by no more than three horizontal edges, followed by m vertical edges. Then

$$2^k = |u(0, 2^k)|_{S \cap P} \leq 3 \cdot 2^{n+m-1} \leq 2^{n+m+1}$$

and since $\mathcal{H}(\delta)$ is geodesically convex

$$\begin{aligned} d_X((u(0, 2^k), n), (\text{id}_P, n)) &= d_{\mathcal{H}(\delta)}((u(0, 2^k), n), (\text{id}_P, n)) \geq 2m \\ &\geq 2k - 2n - 2. \end{aligned}$$

So $n_0 := \delta$ suffices. □

In the arguments that follow, given a matrix $g \in \text{GL}(d, \mathbb{C})$, let

$$\|g\|_\infty := \max_{1 \leq i, j \leq d} |g_{i,j}|.$$

Then,

$$\|g\|_\infty \leq \mu_1(g) \leq d\|g\|_\infty \quad (9)$$

for all $g \in \text{GL}(d, \mathbb{C})$.

Lemma 7.4. $\lambda_{n,1}^{-1} \lambda_{n,3} \gtrsim 2^{-n}$.

Proof. For every $n \geq n_0$, let

$$k_n := \left\lceil \frac{1}{2} \alpha(\beta + 6) + n + 1 \right\rceil$$

and let $g_n := u(0, 2^{k_n})$. Then

$$a_n^{-1} w_n^{-1} \rho(g_n)^{\pm 1} w_n a_n = \begin{pmatrix} 1 & 0 & \pm i 2^{k_n} \lambda_{n,1}^{-1} \lambda_{n,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, by Equation (9),

$$\begin{aligned} \log \frac{\mu_1}{\mu_3}(a_n^{-1} w_n^{-1} \rho(g_n) w_n a_n) &= \log(\mu_1(a_n^{-1} w_n^{-1} \rho(g_n) w_n a_n) \mu_1(a_n^{-1} w_n^{-1} \rho(g_n)^{-1} w_n a_n)) \\ &\leq \max \left\{ 0, 6 \log \left(2^{k_n} \lambda_{n,1}^{-1} \lambda_{n,3} \right) \right\}. \end{aligned}$$

So, by Lemma 7.3 and Equation (8),

$$6 \leq \frac{1}{\alpha} (2k_n - 2n - 2) - \beta \leq \frac{1}{\alpha} d_X((g_n, n), (\text{id}_P, n)) - \beta \leq \max \left\{ 0, 6 \log \left(2^{k_n} \lambda_{n,1}^{-1} \lambda_{n,3} \right) \right\}.$$

Then

$$1 \leq \log \left(2^{k_n} \lambda_{n,1}^{-1} \lambda_{n,3} \right) \leq \log \left(2^{\frac{1}{2} \alpha(\beta+6) + n + 1} \lambda_{n,1}^{-1} \lambda_{n,3} \right),$$

or equivalently,

$$\frac{e}{2^{\frac{1}{2}\alpha(\beta+6)+1}} 2^{-n} \leq \lambda_{n,1}^{-1} \lambda_{n,3}.$$

□

Lemma 7.5. $\lambda_{n,1}^{-1} \lambda_{n,3} \lesssim 4^{-n}$.

Proof. Let $g_n := u(2^{n-1}, 0)$. Then

$$d_X((g_n, n), (\text{id}_P, n)) = 1.$$

Further,

$$a_n^{-1} \omega_n^{-1} \rho(g_n) \omega_n a_n = \begin{pmatrix} 1 & \lambda_{n,1}^{-1} \lambda_{n,2} 2^{n-1} & * \\ 0 & 1 & \lambda_{n,2}^{-1} \lambda_{n,3} 2^{n-1} \\ 0 & 0 & 1 \end{pmatrix}.$$

So, by Equations (9) and (8),

$$\begin{aligned} \max \left\{ \log \left(\lambda_{n,1}^{-1} \lambda_{n,2} 2^{n-1} \right), \log \left(\lambda_{n,2}^{-1} \lambda_{n,3} 2^{n-1} \right) \right\} &\leq \log \mu_1(a_n^{-1} \omega_n^{-1} \rho(g_n) \omega_n a_n) \\ &\leq \alpha d_X((g_n, n), (\text{id}_P, n)) + \beta = \alpha + \beta, \end{aligned}$$

which implies that

$$\lambda_{n,1}^{-1} \lambda_{n,3} = \lambda_{n,1}^{-1} \lambda_{n,2} \lambda_{n,2}^{-1} \lambda_{n,3} \lesssim 4^{-n}.$$

□

Then by Lemmas 7.4 and 7.5, we obtain the estimate $2^{-n} \lesssim 4^{-n}$, which is impossible. Hence, there does not exist a ρ -equivariant quasi-isometric embedding of X into M .

PART 2. GEOMETRICALLY FINITE GROUPS IN CONVEX REAL PROJECTIVE GEOMETRY

8 | CONVEX REAL PROJECTIVE GEOMETRY

In this expository section, we recall the definitions and results in convex real projective geometry that we will need in Sections 9–11. We also briefly discuss relatively Anosov representations into the projective linear group.

8.1 | Convexity and the Hilbert metric

A subset of $\mathbf{P}(\mathbb{R}^d)$ is called *convex* if it is a convex subset of some affine chart of $\mathbf{P}(\mathbb{R}^d)$ and called *properly convex* if it is a bounded convex subset of some affine chart $\mathbf{P}(\mathbb{R}^d)$. A *properly convex domain* is an open properly convex subset of $\mathbf{P}(\mathbb{R}^d)$.

A subset $H \subset \mathbf{P}(\mathbb{R}^d)$ is called a *projective hyperplane* if it is the image of some codimension-one linear subspace $W \subset \mathbb{R}^d \setminus \{0\} \rightarrow \mathbf{P}(\mathbb{R}^d)$. Given a properly convex domain

$\Omega \subset \mathbf{P}(\mathbb{R}^d)$ and $x \in \partial\Omega$, there always exists at least one projective hyperplane $H \subset \mathbf{P}(\mathbb{R}^d)$ with $x \in H$ and $H \cap \Omega = \emptyset$. In this case, H is called a *supporting hyperplane* of $\partial\Omega$ at x . When a boundary point $x \in \partial\Omega$ has a unique supporting hyperplane, we say that x is a C^1 -smooth point of $\partial\Omega$ and let $T_x\partial\Omega$ denote this unique supporting hyperplane.

Given a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ and $p, q \in \overline{\Omega}$, we will let $[p, q]_\Omega$ denote the closed projective line segment in $\overline{\Omega}$ that contains p and q . Then define $[p, q)_\Omega := [p, q]_\Omega \setminus \{q\}$, $(p, q]_\Omega := [p, q]_\Omega \setminus \{p\}$, and $(p, q)_\Omega := [p, q]_\Omega \setminus \{p, q\}$.

The *automorphism group* of a subset $S \subset \mathbf{P}(\mathbb{R}^d)$ is the group

$$\text{Aut}(S) := \{g \in \text{PGL}(d, \mathbb{R}) : g \cdot S = S\}.$$

Given a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ and a subgroup $\Gamma \leq \text{Aut}(\Omega)$, the *limit set* of Γ is

$$\Lambda_\Omega(\Gamma) := \partial\Omega \cap \bigcup_{p \in \Omega} \overline{\Gamma \cdot p},$$

where the closure is taken in $\mathbf{P}(\mathbb{R}^d)$. Equivalently, $\Lambda_\Omega(\Gamma)$ is the set of boundary points $x \in \partial\Omega$ where there exist $p \in \Omega$ and a sequence $(\gamma_n)_{n \geq 1}$ in Γ such that $\gamma_n(p) \rightarrow x$. The *convex hull* of Γ , denoted as $C_\Omega(\Gamma)$, is the closed convex hull of $\Lambda_\Omega(\Gamma)$ in Ω .

Given a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$, the *dual domain* is

$$\Omega^* := \{f \in \mathbf{P}(\mathbb{R}^{d*}) : f(x) \neq 0 \text{ for all } x \in \overline{\Omega}\}.$$

It is straightforward to show that Ω^* is a properly convex domain of $\mathbf{P}(\mathbb{R}^{d*})$ and under the natural identification $\text{PGL}(d, \mathbb{R}) = \text{PGL}(\mathbb{R}^{d*})$, we have $\text{Aut}(\Omega) = \text{Aut}(\Omega^*)$.

A properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ has a natural distance, called the *Hilbert distance*, which is defined by

$$d_\Omega(p, q) = \frac{1}{2} \log[a, p, q, b],$$

where L is a projective line containing p, q , $\{a, b\} = L \cap \partial\Omega$ with the ordering a, p, q, b along L , and $[a, p, q, b]$ is the standard projective cross ratio. Then (Ω, d_Ω) is a proper geodesic metric space and $\text{Aut}(\Omega)$ acts on (Ω, d_Ω) by isometries. Further, the line segment $[p, q]_\Omega$ joining $p, q \in \Omega$ can be parametrized to be a geodesic in (Ω, d_Ω) .

We recall that given two subsets $A, B \subset \Omega$, the *Hausdorff distance with respect to d_Ω* between A and B is defined as

$$d_\Omega^{\text{Haus}}(A, B) := \max \left\{ \sup_{a \in A} d_\Omega(a, B), \sup_{b \in B} d_\Omega(b, A) \right\}.$$

We will use the following well-known estimate on the Hausdorff distance between two line segments with respect to the Hilbert metric d_Ω .

Observation 8.1. Suppose that $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ is properly convex. If $p_1, p_2, q_1, q_2 \in \Omega$, then

$$d_\Omega^{\text{Haus}}([p_1, q_1]_\Omega, [p_2, q_2]_\Omega) \leq \max \{d_\Omega(p_1, p_2), d_\Omega(q_1, q_2)\}.$$

Proof. See, for instance, [25, Prop. 5.3]. □

8.2 | Convex hulls

A general subset of $\mathbf{P}(\mathbb{R}^d)$ has no well-defined convex hull, for instance, if $X = \{x_1, x_2\}$, then there is no natural way to choose between the two line projective line segments joining x_1 and x_2 . However, it was observed in [24] that for certain types of subsets, one can define a convex hull. We recall these observations here.

Given a subset $X \subset \mathbf{P}(\mathbb{R}^d)$ that is contained in some affine chart $\mathbb{A} \subset \mathbf{P}(\mathbb{R}^d)$, let $\text{ConvHull}_{\mathbb{A}}(X) \subset \mathbb{A}$ denote the convex hull of X in $\mathbb{A}$. For a general set (e.g., two points), this convex hull depends on the choice of $\mathbb{A}$ but when X is connected we have the following.

Observation 8.2 [24, Lem. 5.9]. Suppose that $X \subset \mathbf{P}(\mathbb{R}^d)$ is connected. If $\mathbb{A}_1$ and $\mathbb{A}_2$ are two affine charts that contain X , then

$$\text{ConvHull}_{\mathbb{A}_1}(X) = \text{ConvHull}_{\mathbb{A}_2}(X).$$

This leads to the following definition.

Definition 8.3. If $X \subset \mathbf{P}(\mathbb{R}^d)$ is connected and contained in some affine chart, then let $\text{ConvHull}(X)$ denote the convex hull of X in some (any) affine chart that contains X .

As a consequence of the definition, we have the following.

Observation 8.4. Suppose that $X \subset \mathbf{P}(\mathbb{R}^d)$ is connected and contained in some affine chart. If $g \in \text{PGL}(d, \mathbb{R})$, then

$$\text{ConvHull}(gX) = g \cdot \text{ConvHull}(X).$$

8.3 | Relatively Anosov representations into the projective linear group

In the context of convex real projective geometry, it is more natural to consider representations into $\text{PGL}(d, \mathbb{R})$. It is also helpful to identify $\text{Gr}_1(\mathbb{R}^d) = \mathbf{P}(\mathbb{R}^d)$ and $\text{Gr}_{d-1}(\mathbb{R}^d) = \mathbf{P}(\mathbb{R}^{d*})$ and assume that the boundary map of a relatively P_1 -Anosov representation has image in $\mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*})$. This leads to the following analogue of Definition 1.1.

Definition 8.5. Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic with Bowditch boundary $\partial(\Gamma, \mathcal{P})$. A representation $\rho: \Gamma \rightarrow \text{PGL}(d, \mathbb{R})$ is P_1 -Anosov relative to $\mathcal{P}$ if there exists a continuous map

$$\xi = (\xi^1, \xi^{d-1}): \partial(\Gamma, \mathcal{P}) \rightarrow \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*}),$$

which is

- (1) ρ -equivariant: if $\gamma \in \Gamma$, then $\rho(\gamma) \circ \xi = \xi \circ \gamma$,
- (2) transverse: if $x, y \in \partial(\Gamma, \mathcal{P})$ are distinct, then $\xi^1(x) \oplus \ker \xi^{d-1}(y) = \mathbb{R}^d$,

- (3) *strongly dynamics-preserving*: if $(\gamma_n)_{n \geq 1}$ is a sequence of elements in Γ where $\gamma_n \rightarrow x \in \partial(\Gamma, \mathcal{P})$ and $\gamma_n^{-1} \rightarrow y \in \partial(\Gamma, \mathcal{P})$, then

$$\lim_{n \rightarrow \infty} \rho(\gamma_n)v = \xi^1(x)$$

for all $v \in \mathbf{P}(\mathbb{R}^d) \setminus \mathbf{P}(\ker \xi^{d-1}(y))$.

8.4 | Relatively Anosov representations from visible subgroups

As mentioned in the introduction, a projectively visible subgroup (see Section 1.2.2 for the definition) acts as a convergence group on its limit set [17, Prop. 3.5]. Further, if the action on the limit set is geometrically finite, then the inclusion representation is relatively P_1 -Anosov.

Proposition 8.6. *Suppose that $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \text{Aut}(\Omega)$ is a projectively visible subgroup. If Γ acts on $\Lambda_\Omega(\Gamma)$ as a geometrically finite convergence group and $\mathcal{P}$ is a set of conjugacy representatives of the stabilizers of bounded parabolic points in $\Lambda_\Omega(\Gamma)$, then the inclusion representation $\Gamma \hookrightarrow \text{PGL}(d, \mathbb{R})$ is P_1 -Anosov relative to $\mathcal{P}$.*

Proof. By definition, there exists an equivariant homeomorphism $\xi^1 : \partial(\Gamma, \mathcal{P}) \rightarrow \Lambda_\Omega(\Gamma)$, see [42]. By the visibility property, each point in $\Lambda_\Omega(\Gamma)$ is a C^1 -smooth point of $\partial\Omega$. So, for every $x \in \partial(\Gamma, \mathcal{P})$, there exists a unique $\xi^{d-1}(x) \in \mathbf{P}(\mathbb{R}^{d*})$ such that

$$\mathbf{P}(\ker \xi^{d-1}(x)) = T_{\xi^1(x)}\partial\Omega.$$

Then let $\xi := (\xi^1, \xi^{d-1})$. Then ξ is continuous and equivariant. By the visibility property, if $x, y \in \partial(\Gamma, \mathcal{P})$ are distinct, then the open line segment in $\overline{\Omega}$ joining $\xi^1(x)$ to $\xi^1(y)$ is in Ω . Since $\mathbf{P}(\ker \xi^{d-1}(y)) \cap \Omega = \emptyset$, we must have $\xi^1(x) \notin \mathbf{P}(\ker \xi^{d-1}(y))$, and so,

$$\xi^1(x) \oplus \ker \xi^{d-1}(y) = \mathbb{R}^d.$$

Thus, ξ is transverse. Finally, by [17, Prop. 3.5], ξ is strongly dynamics-preserving. $\square$

9 | RELATIVELY ANOSOV REPRESENTATIONS WHOSE IMAGES PRESERVE A PROPERLY CONVEX DOMAIN

In this section, we prove a converse to Proposition 8.6 and characterize the relatively P_1 -Anosov representations that preserve a properly convex domain. This builds upon work in [17] and extends results in [18, 44] from the classical Anosov case to the relative one.

Let $\|\cdot\|_2$ denote both the Euclidean norm on $\mathbb{R}^d$ and the associated dual norm on $\mathbb{R}^{d*}$. Then let $\mathbb{S} \subset \mathbb{R}^d$ and $\mathbb{S}^* \subset \mathbb{R}^{d*}$ denote the unit balls relative to these norms. Also, let $\text{SL}^\pm(d, \mathbb{R}) = \{g \in \text{GL}(d, \mathbb{R}) : \det g = \pm 1\}$. The group $\text{SL}^\pm(d, \mathbb{R})$ acts on $\mathbb{S}$ and $\mathbb{S}^*$ by

$$g \cdot v = \frac{1}{\|g(v)\|_2} g(v) \quad \text{and} \quad g \cdot f = \frac{1}{\|f \circ g^{-1}\|_2} f \circ g^{-1}.$$

Definition 9.1. Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic, $\rho : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is P_1 -Anosov relative to $\mathcal{P}$, and $\xi : \partial(\Gamma, \mathcal{P}) \rightarrow \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*})$ is the Anosov boundary map, then we say that ρ has the *lifting property* if there exist lifts $\tilde{\xi} = (\tilde{\xi}^1, \tilde{\xi}^{d-1}) : \partial(\Gamma, \mathcal{P}) \rightarrow \mathbb{S} \times \mathbb{S}^*$ and $\tilde{\rho} : \Gamma \rightarrow \mathrm{SL}^\pm(d, \mathbb{R})$ of ξ and ρ with the following properties:

- (1) $\tilde{\xi}$ is continuous and $\tilde{\rho}$ -equivariant,
- (2) $\tilde{\xi}$ is *positive* in the following sense: if $x, y \in \partial(\Gamma, \mathcal{P})$ are distinct, then

$$\tilde{\xi}^{d-1}(y)(\tilde{\xi}^1(x)) > 0.$$

Proposition 9.2. Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is P_1 -Anosov relative to $\mathcal{P}$. Then the following are equivalent:

- (1) ρ has the lifting property,
- (2) there exists a properly convex domain $\Omega_0 \subset \mathbf{P}(\mathbb{R}^d)$ where $\rho(\Gamma) \leq \mathrm{Aut}(\Omega_0)$,
- (3) there exists a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ where $\rho(\Gamma) \leq \mathrm{Aut}(\Omega)$ is a projectively visible subgroup.

Remark 9.3. The equivalence (2) $\iff$ (3) follows from general results in [17] and the implication (2) $\Rightarrow$ (1) is elementary. So, the new content of Proposition 9.2 is the implication (1) $\Rightarrow$ (2).

The rest of the section is devoted to the proof of Proposition 9.2. So, fix $(\Gamma, \mathcal{P})$ and ρ as in the proposition, and let $\xi : \partial(\Gamma, \mathcal{P}) \rightarrow \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*})$ denote the Anosov boundary map of ρ .

Lemma 9.4. (2) $\iff$ (3).

Proof. Using the language in [17], [45, Prop. 4.4] implies that $\rho(\Gamma)$ is a $P_{k,d-k}$ -transverse group. Then the equivalence of (2) and (3) follows from [17, Prop. 4.4]. $\square$

Lemma 9.5 ((2) $\Rightarrow$ (1)). If there exists a properly convex domain $\Omega_0 \subset \mathbf{P}(\mathbb{R}^d)$ where $\rho(\Gamma) \leq \mathrm{Aut}(\Omega_0)$, then ρ has the lifting property.

Proof. We first observe that the strongly dynamics-preserving property implies that ξ^1 has image in $\partial\Omega_0$. Fix $x \in \partial(\Gamma, \mathcal{P})$ and a sequence $(\gamma_n)_{n \geq 1}$ in Γ with $\gamma_n \rightarrow x$. Passing to a subsequence, we can assume that $\gamma_n^{-1} \rightarrow y \in \partial(\Gamma, \mathcal{P})$. Then

$$\rho(\gamma_n)v \rightarrow \xi^1(x)$$

for all $v \in \mathbf{P}(\mathbb{R}^d) \setminus \mathbf{P}(\ker \xi^{d-1}(y))$. Since Ω_0 is open, there exists $v \in \Omega_0 \setminus \mathbf{P}(\ker \xi^{d-1}(y))$ and hence $\xi^1(x) \in \overline{\Omega_0}$. Since $\rho(\Gamma)$ acts properly on Ω_0 , we must have $\xi^1(x) \in \partial\Omega_0$. So, ξ^1 has image in $\partial\Omega_0$. The same argument shows that ξ^{d-1} has image in $\partial\Omega_0^*$.

The rest of the argument is identical to the proof of Case 1 in [44, Th. 3.1]. Let $\pi : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbf{P}(\mathbb{R}^d)$ denote the projection map. Since Ω_0 is properly convex, $\pi^{-1}(\Omega_0)$ has two connected components C_1 and C_2 . Moreover, both components are properly convex cones in $\mathbb{R}^d$ and $C_2 = -C_1$.

For $x \in \partial(\Gamma, \mathcal{P})$, let $\tilde{\xi}^1(x) \in \mathbb{S}$ denote the unique lift of $\xi^1(x)$ in $\overline{C_1} \cap \mathbb{S}$ and let $\tilde{\xi}^{d-1}(x)$ denote the unique lift of $\xi^{d-1}(x)$ such that $\tilde{\xi}^{d-1}(x) \in \mathbb{S}^*$ and $\tilde{\xi}^{d-1}(x)|_{C_1} > 0$. For $\gamma \in \Gamma$, let $\tilde{\rho}(\gamma) \in \mathrm{SL}^\pm(d, \mathbb{R})$

denote the unique lift of $\rho(\gamma)$ that preserves C_1 . Then $\tilde{\rho}$ is a homomorphism and $\tilde{\xi} = (\tilde{\xi}^1, \tilde{\xi}^{d-1})$ is continuous, $\tilde{\rho}$ -equivariant, and positive. So, ρ has the lifting property. $\square$

For the other direction, we closely follow the arguments in Section 5 of [24].

Lemma 9.6 ((1) $\Rightarrow$ (2)). *If ρ has the lifting property, then there exists a properly convex domain $\Omega_0 \subset \mathbf{P}(\mathbb{R}^d)$ where $\rho(\Gamma) \leq \text{Aut}(\Omega_0)$.*

Proof. Let $\tilde{\xi}, \tilde{\rho}$ denote lifts of ξ, ρ satisfying the lifting property. Then define

$$C_0 := \left\{ \left[\sum_{j=1}^N \lambda_j \tilde{\xi}^1(x_j) \right] : N \geq 2; \lambda_1, \dots, \lambda_N > 0; x_1, \dots, x_N \in \partial(\Gamma, \mathcal{P}) \text{ distinct} \right\}.$$

Since $\tilde{\xi}$ is $\tilde{\rho}$ -equivariant, $\tilde{\rho}(\gamma)C_0 = C_0$ for every $\gamma \in \Gamma$. Since $\tilde{\xi}$ is positive,

$$C_0 \cap \bigcup_{y \in \partial(\Gamma, \mathcal{P})} \mathbf{P}(\ker \xi^{d-1}(y)) = \emptyset. \quad (10)$$

Also, if we fix $x_1, x_2 \in \partial(\Gamma, \mathcal{P})$ distinct, then the positivity of $\tilde{\xi}$ implies that C_0 is bounded in the affine chart

$$\mathbb{A} := \{[v] \in \mathbf{P}(\mathbb{R}^d) : (\tilde{\xi}^{d-1}(x_1) + \tilde{\xi}^{d-1}(x_2))(v) \neq 0\}.$$

Fix $p \in C_0$. We claim that there exists a connected neighborhood U of p in $\mathbf{P}(\mathbb{R}^d)$ such that

$$\rho(\Gamma)U = \bigcup_{\gamma \in \Gamma} \rho(\gamma)U$$

is bounded in $\mathbb{A}$. Suppose not. Then there exist sequences $(p_n)_{n \geq 1}$ in $\mathbf{P}(\mathbb{R}^d)$ and $(\gamma_n)_{n \geq 1}$ in Γ such that $p_n \rightarrow p$ and $\rho(\gamma_n)p_n$ leaves every compact subset of $\mathbb{A}$. Passing to a subsequence, we can suppose that $\gamma_n \rightarrow x \in \partial(\Gamma, \mathcal{P})$ and $\gamma_n^{-1} \rightarrow y \in \partial(\Gamma, \mathcal{P})$. Then, by the strongly dynamics-preserving property,

$$\rho(\gamma_n)q \rightarrow \xi^1(x)$$

for all $q \in \mathbf{P}(\mathbb{R}^d) \setminus \mathbf{P}(\ker \xi^{d-1}(y))$ and the convergence is locally uniform. Equation (10) implies that $p \in \mathbf{P}(\mathbb{R}^d) \setminus \mathbf{P}(\ker \xi^{d-1}(y))$ and so $\rho(\gamma_n)p_n \rightarrow \xi^1(x)$. However, $\xi^1(x)$ lies in the closure of C_0 and C_0 is bounded in $\mathbb{A}$. This contradicts our assumption and hence such a set U exists.

Finally, the set

$$X := C_0 \cup \bigcup_{\gamma \in \Gamma} \rho(\gamma)U$$

is connected (since each of the sets in the union is path-connected, and $\rho(\gamma)U \cap C_0 \neq \emptyset$ for each $\gamma \in \Gamma$), bounded in $\mathbb{A}$, and preserved by $\rho(\Gamma)$. So, Observation 8.4 implies that

$$\Omega_0 := \text{ConvHull}(X)$$

is a properly convex domain where $\rho(\Gamma) \leq \text{Aut}(\Omega_0)$. $\square$

10 | STABILITY OF THE LIFTING PROPERTY

In this section, we prove Proposition 1.18 which we restate here.

Proposition 10.1. *Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho_0 : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is a representation. Let $\mathcal{A}_1(\rho_0)$ denote the set of representations in $\mathrm{Hom}_{\rho_0}(\Gamma, \mathrm{PGL}(d, \mathbb{R}))$ that are P_1 -Anosov relative to $\mathcal{P}$. Then the subset $\mathcal{A}_1^+(\rho_0) \subset \mathcal{A}_1(\rho_0)$ of representations with the lifting property is open and closed in $\mathcal{A}_1(\rho_0)$.*

10.1 | Lifting maps

In this subsection, we record some basic observations about lifting maps to covering spaces. Suppose that M is a compact Riemannian manifold and $\pi : \tilde{M} \rightarrow M$ is a Riemannian cover (i.e., $\tilde{M}$ is a Riemannian manifold and π is a covering map which is a local isometry). Fix $\epsilon > 0$ so that every metric ball of radius ϵ in M is normal.

Observation 10.2. If $p \in \tilde{M}$, then

- (1) π induces a diffeomorphism between metric balls $B_{\tilde{M}}(p, \epsilon) \rightarrow B_M(\pi(p), \epsilon)$,
- (2) $\pi^{-1}(q) \cap B_{\tilde{M}}(p, \epsilon)$ is a single point for any $q \in B_M(\pi(p), \epsilon)$.

Proof. For part (1), see, for instance, the proof of [11, Lem. 1.38]. Part (2) follows immediately from part (1). $\square$

Observation 10.3. Suppose that N is a compact topological space and $f, g : N \rightarrow M$ are continuous maps. If

$$\max_{x \in N} d_M(f(x), g(x)) < \epsilon$$

and f admits a continuous lift $\tilde{f} : N \rightarrow \tilde{M}$, then g admits a unique continuous lift $\tilde{g} : N \rightarrow \tilde{M}$ with

$$\max_{x \in N} d_{\tilde{M}}(\tilde{f}(x), \tilde{g}(x)) < \epsilon.$$

Proof. By Observation 10.2, for each $x \in N$, there is a unique $\tilde{g}(x) \in \pi^{-1}(g(x))$ such that $d_{\tilde{M}}(\tilde{f}(x), \tilde{g}(x)) < \epsilon$. By uniqueness, $\tilde{g}$ is continuous. $\square$

10.2 | Proof of Proposition 10.1

Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho_0 : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is a representation.

For $\rho \in \mathcal{A}_1(\rho_0)$, let ξ_ρ denote the Anosov boundary map. We will use the following stability result from [45].

Theorem 10.4 [45, Cor. 13.6]. *The map*

$$\mathcal{A}_1(\rho_0) \times \partial(\Gamma, \mathcal{P}) \ni (\rho, x) \mapsto \xi_\rho(x) \in \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*})$$

is continuous.

Fix Riemannian metrics on $\mathbb{S} \times \mathbb{S}^*$ and $\mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*})$ so that $\mathbb{S} \times \mathbb{S}^* \rightarrow \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*})$ is a Riemannian cover. We will let d denote the associated distance on both spaces. Then fix $\epsilon > 0$ satisfying Observation 10.3 with $M = \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*})$.

Lemma 10.5. $\mathcal{A}_1^+(\rho_0)$ is closed in $\mathcal{A}_1(\rho_0)$.

Proof. Suppose that $\rho_n \rightarrow \rho$ in $\mathcal{A}_1(\rho_0)$ where $\{\rho_n\} \subset \mathcal{A}_1^+(\rho_0)$. Let ξ_n (respectively, ξ) denote the Anosov boundary map of ρ_n (respectively, ρ) and let $\tilde{\xi}_n, \tilde{\rho}_n$ denote lifts of ξ_n, ρ_n satisfying the lifting property.

Theorem 10.4 implies that $\xi_n \rightarrow \xi$ uniformly. So, for n sufficiently large, we have

$$\max_{x \in \partial(\Gamma, \mathcal{P})} d(\xi_n(x), \xi(x)) < \epsilon.$$

So, by our choice of $\epsilon > 0$, there exists a unique continuous lift $\tilde{\xi}$ of ξ such that

$$\max_{x \in \partial(\Gamma, \mathcal{P})} d(\tilde{\xi}_n(x), \tilde{\xi}(x)) < \epsilon$$

for n sufficiently large. Further, $\tilde{\xi}_n$ converges pointwise to $\tilde{\xi}$. Then

$$\tilde{\xi}^{d-1}(y)(\tilde{\xi}^1(x)) = \lim_{n \rightarrow \infty} \tilde{\xi}_n^{d-1}(y)(\tilde{\xi}_n^1(x)) \geq 0$$

for all $x, y \in \partial(\Gamma, \mathcal{P})$. So, by transversality, we see that

$$\tilde{\xi}^{d-1}(y)(\tilde{\xi}^1(x)) > 0 \tag{11}$$

for all distinct $x, y \in \partial(\Gamma, \mathcal{P})$.

Finally, we construct the lift of ρ . Since Γ is finitely generated and $\mathrm{SL}^\pm(d, \mathbb{R}) \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is a finite cover, by passing to a further subsequence, we can suppose that

$$\tilde{\rho}(\gamma) := \lim_{n \rightarrow \infty} \tilde{\rho}_n(\gamma)$$

exists for all $\gamma \in \Gamma$. Since $\tilde{\xi}_n$ converges pointwise to $\tilde{\xi}$, we see that $\tilde{\xi}$ is $\tilde{\rho}$ -equivariant. Hence, ρ has the lifting property. $\square$

Lemma 10.6. $\mathcal{A}_1^+(\rho_0)$ is open in $\mathcal{A}_1(\rho_0)$.

Proof. It suffices to assume that $\rho_0 \in \mathcal{A}_1^+(\rho_0)$ and show that there exists an open neighborhood of ρ_0 in $\mathcal{A}_1(\rho_0)$ that is contained in $\mathcal{A}_1^+(\rho_0)$. Let ξ_{ρ_0} denote the Anosov boundary map of ρ_0 and let $\tilde{\xi}_{\rho_0}, \tilde{\rho}_0$ denote lifts of ξ_{ρ_0}, ρ_0 satisfying the lifting property.

By Theorem 10.4 and our choice of $\epsilon > 0$, we can find a neighborhood $\mathcal{O}$ of ρ_0 in $\mathcal{A}_1(\rho_0)$ such that if $\rho \in \mathcal{O}$, then the associated boundary map ξ_ρ admits a unique continuous lift $\tilde{\xi}_\rho : \partial(\Gamma, \mathcal{P}) \rightarrow \mathbb{S} \times \mathbb{S}^*$ with

$$d_{\max}(\tilde{\xi}_{\rho_0}, \tilde{\xi}_\rho) := \max_{x \in \partial(\Gamma, \mathcal{P})} d(\tilde{\xi}_{\rho_0}(x), \tilde{\xi}_\rho(x)) < \epsilon. \quad (12)$$

Fix a finite generating set $S \subset \Gamma$. Then we can find a subneighborhood $\mathcal{O}' \subset \mathcal{O}$ where for each $\gamma \in S$ and $\rho \in \mathcal{O}'$, there exists a lift $\tilde{\rho}(\gamma)$ of $\rho(\gamma)$ such that

$$\max_{v \in \mathbb{S}^* \times \mathbb{S}} d(\tilde{\rho}_0(\gamma)v, \tilde{\rho}(\gamma)v) < \epsilon/2.$$

By replacing $\mathcal{O}'$ with a relatively compact subset, we can also assume that there exists $C > 1$ such that: if $\gamma \in S$ and $\rho \in \mathcal{O}'$, then $\tilde{\rho}(\gamma)$ acts as a C -Lipschitz map on $\mathbb{S} \times \mathbb{S}^*$. Finally, by possibly replacing $\mathcal{O}'$ with a smaller neighborhood and using Theorem 10.4, we can assume that

$$d_{\max}(\tilde{\xi}_{\rho_0}, \tilde{\xi}_\rho) < \frac{\epsilon}{2C}$$

for all $\rho \in \mathcal{O}'$.

Now, if $\rho \in \mathcal{O}'$ and $\gamma \in S$, then (since $\tilde{\xi}_{\rho_0}$ is $\tilde{\rho}_0$ -equivariant)

$$\begin{aligned} d_{\max}(\tilde{\xi}_{\rho_0}, \tilde{\rho}(\gamma) \circ \tilde{\xi}_\rho \circ \gamma^{-1}) &= \max_{x \in \partial(\Gamma, \mathcal{P})} d(\tilde{\rho}_0(\gamma) \circ \tilde{\xi}_{\rho_0} \circ \gamma^{-1}(x), \tilde{\rho}(\gamma) \circ \tilde{\xi}_\rho \circ \gamma^{-1}(x)) \\ &< \epsilon/2 + \max_{x \in \partial(\Gamma, \mathcal{P})} d(\tilde{\rho}(\gamma) \circ \tilde{\xi}_{\rho_0} \circ \gamma^{-1}(x), \tilde{\rho}(\gamma) \circ \tilde{\xi}_\rho \circ \gamma^{-1}(x)) \\ &< \epsilon/2 + C d_{\max}(\tilde{\xi}_{\rho_0} \circ \gamma^{-1}, \tilde{\xi}_\rho \circ \gamma^{-1}) = \epsilon/2 + C d_{\max}(\tilde{\xi}_{\rho_0}, \tilde{\xi}_\rho) < \epsilon. \end{aligned}$$

So, by uniqueness of the lift $\tilde{\xi}_\rho$ satisfying Equation (12), we have $\tilde{\rho}(\gamma) \circ \tilde{\xi}_{\rho_0} \circ \gamma^{-1} = \tilde{\xi}_\rho$. Since at most one lift $\tilde{g} \in \text{SL}^\pm(d, \mathbb{R})$ of an element $\rho(\gamma) \in \rho(\Gamma)$ can satisfy the equation $\tilde{g} \circ \tilde{\xi}_{\rho_0} \circ \gamma^{-1} = \tilde{\xi}_\rho$, we then see that $\tilde{\rho}$ extends to a homomorphism of Γ and $\tilde{\xi}_\rho$ is $\tilde{\rho}$ -equivariant.

It remains to verify positivity. Fix a compact set $K \subset \{(x, y) \in \partial(\Gamma, \mathcal{P})^2 : x \neq y\}$ such that

$$\Gamma \cdot K = \{(x, y) \in \partial(\Gamma, \mathcal{P})^2 : x \neq y\} \quad (13)$$

(such a compact set exists by [39, Th. 1C]). By shrinking $\mathcal{O}'$, we may assume that

$$\tilde{\xi}_\rho^{d-1}(y) \left(\tilde{\xi}_\rho^1(x) \right) > 0 \quad (14)$$

for all $\rho \in \mathcal{O}'$ and $(x, y) \in K$. Fix $\rho \in \mathcal{O}'$. Since $\tilde{\xi}_\rho$ is $\tilde{\rho}$ -equivariant, Equation (13) implies that Equation (14) holds for all distinct $x, y \in \partial(\Gamma, \mathcal{P})$. Hence, we see that $\rho \in \mathcal{A}_1^+(\rho_0)$. $\square$

11 | REPRESENTATIONS OF RANK ONE GROUPS REVISITED

For the rest of the section, let G, K , and $X = G/K$ be as in Section 3. Then suppose that $\tau : G \rightarrow \text{PGL}(d, \mathbb{R})$ is a P_1 -proximal representation.

In this section, we prove three propositions. The first two characterize exactly when $\tau(G)$ preserves a properly convex domain and the third proposition establishes a structure theorem in the case it does. The first and third propositions imply Proposition 1.22.

Proposition 11.1. *If X is not isometric to real hyperbolic 2-space (equivalently, G is not locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$), then $\tau(G)$ preserves a properly convex domain.*

Proposition 11.2. *Suppose that X is isometric to real hyperbolic 2-space and*

$$\mathbb{R}^d = \bigoplus_{j=1}^m V_j$$

is a decomposition into $\tau(G)$ -irreducible subspaces. Then, $\tau(G)$ preserves a properly convex domain if and only if $\max_{1 \leq j \leq m} \dim V_j$ is odd.

Proposition 11.3. *Suppose that $\tau(G)$ preserves some properly convex domain. Then there exists a $\tau(G)$ -invariant properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ such that: if $\Gamma \leq G$ is a geometrically finite subgroup, then*

- (1) $\tau(\Gamma)$ is a projectively visible subgroup of $\mathrm{Aut}(\Omega)$ and acts geometrically finitely on its limit set.
- (2) If $C_\Gamma := C_\Omega(\tau(\Gamma))$, then (C_Γ, d_Ω) is Gromov-hyperbolic.

Arguing exactly as in the proof of Proposition 4.2, there exists a continuous τ -equivariant, transverse, strongly dynamics-preserving map

$$\zeta = (\zeta^1, \zeta^{d-1}) : \partial_\infty X \rightarrow \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*}).$$

Arguing as in the first step of the proof of Lemma 9.5, we obtain the following.

Observation 11.4. *If $\tau(G)$ preserves a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$, then*

$$\zeta^1(\partial_\infty X) \subset \partial\Omega \quad \text{and} \quad \zeta^{d-1}(\partial X) \subset \partial\Omega^*.$$

11.1 | Proof of Proposition 11.1

Suppose that X is not isometric to hyperbolic 2-space. Then $\partial_\infty X$ is a sphere with dimension at least two and in particular is simply connected.

As in Section 9, let $\mathbb{S} \subset \mathbb{R}^d$ and $\mathbb{S}^* \subset \mathbb{R}^{d*}$ denote the unit spheres relative to the Euclidean norms. Then, since $\mathbb{S} \rightarrow \mathbf{P}(\mathbb{R}^d)$ is a covering map and $\partial_\infty X$ is simply connected, we can lift ζ^1 to a continuous map $\tilde{\zeta}^1 : \partial_\infty X \rightarrow \mathbb{S}$. For the same reasons, we can lift ζ^{d-1} to a continuous map $\tilde{\zeta}^{d-1} : \partial_\infty X \rightarrow \mathbb{S}^*$. By transversality,

$$\tilde{\zeta}^{d-1}(x)(\tilde{\zeta}^1(y)) \neq 0 \tag{15}$$

for all distinct $x, y \in \partial_\infty X$. Since $\partial_\infty X$ minus any point is connected, Equation (15) has the same sign for all distinct $x, y \in \partial_\infty X$. So, by possibly replacing $\tilde{\zeta}^{d-1}$ by $-\tilde{\zeta}^{d-1}$, we may assume that Equation (15) is positive for all distinct $x, y \in \partial_\infty X$.

Since $\partial_\infty X$ is connected, ζ^1 has exactly two continuous lifts to $\mathbb{S}$. So, if $g \in G$ and $\tilde{h} \in \mathrm{SL}^\pm(d, \mathbb{R})$ is a lift of $\tau(g)$, then either $\tilde{h} \circ \zeta^1 \circ g^{-1} = \tilde{\zeta}^1$ or $\tilde{h} \circ \zeta^1 \circ g^{-1} = -\tilde{\zeta}^1$. So, for every $g \in G$, there exists a unique lift $\tilde{\tau}(g) \in \mathrm{SL}^\pm(d, \mathbb{R})$ of $\tau(g)$ such that $\tilde{\tau}(g) \circ \zeta^1 \circ g^{-1} = \tilde{\zeta}^1$. By uniqueness, $\tilde{\tau}$ is a representation.

Then arguing as in the proof of Lemma 9.6, we see that $\tau(G)$ preserves a properly convex domain.

11.2 | Proof of Proposition 11.2

Suppose that X is isometric to real hyperbolic 2-space and $\mathbb{R}^d = \bigoplus_{j=1}^m V_j$ is a decomposition into $\tau(G)$ -irreducible subspaces. Then G is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ and hence τ induces a Lie algebra representation $d\tau : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(d, \mathbb{R})$. Since every such Lie algebra representation integrates to a representation $\mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(d, \mathbb{R})$ and G is connected, there exists a representation $\hat{\tau} : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(d, \mathbb{R})$ with the same image as τ . So, by possibly replacing τ with $\hat{\tau}$, we can assume that $G = \mathrm{SL}(2, \mathbb{R})$.

Let $d_j := \dim V_j$ and let $\tau_j : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(V_j)$ be the restriction of τ to V_j . By possibly relabeling, we can assume $d_1 \geq d_2 \geq \dots \geq d_m$. Recall that V_j is isomorphic to the vector space of homogeneous polynomials in two variables with degree $d_j - 1$ where $\tau_j(g)f = f \circ g^{-1}$. Then one can check that

$$\lambda_k(\tau_j(g)) = \lambda_1(g)^{d_j+1-2k}$$

for all $g \in \mathrm{SL}(2, \mathbb{R})$. Then, since τ is P_1 -proximal, we must have $d_1 > d_2$.

Let $\iota_j : V_j \hookrightarrow \mathbb{R}^d$ be the inclusion map and let $\pi_j : \mathbb{R}^d \rightarrow V_j$ be the projection relative to the decomposition $\mathbb{R}^d = \bigoplus_{j=1}^m V_j$. Then the adjoint $\pi_j^* : V_j^* \rightarrow \mathbb{R}^{d*}$ of π_j , which is given by

$$\pi_j^*(f) = f \circ \pi_j,$$

defines an inclusion. Since τ_1 is P_1 -proximal, the proof of Proposition 4.2 implies that there exists a boundary map $\zeta_1 : \partial_\infty X \rightarrow \mathbf{P}(V_1) \times \mathbf{P}(V_1^*)$ associated to τ_1 . Then, by the strongly dynamics-preserving property

$$\zeta = (\iota_1, \pi_1^*) \circ \zeta_1. \quad (16)$$

Lemma 11.5. $\tau(\mathrm{SL}(2, \mathbb{R}))$ preserves a properly convex domain in $\mathbf{P}(\mathbb{R}^d)$ if and only if $\tau_1(\mathrm{SL}(2, \mathbb{R}))$ preserves a properly convex domain in $\mathbf{P}(V_1)$.

Proof. First, suppose that $\tau(\mathrm{SL}(2, \mathbb{R}))$ preserves a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$. By Observation 11.4 and Equation (16),

$$\zeta_1^1(\partial_\infty X) = \zeta^1(\partial_\infty X) \subset \partial\Omega.$$

Hence, $C := \overline{\Omega} \cap \mathbf{P}(V_1)$ is a nonempty $\tau_1(\mathrm{SL}(2, \mathbb{R}))$ -invariant properly convex closed set in $\mathbf{P}(V_1)$. Since τ_1 is irreducible, C must have nonempty interior in $\mathbf{P}(V_1)$. So, $\tau_1(\mathrm{SL}(2, \mathbb{R}))$ preserves a properly convex domain in $\mathbf{P}(V_1)$.

Next, suppose that $\tau_1(\mathrm{SL}(2, \mathbb{R}))$ preserves a properly convex domain in $\Omega_1 \subset \mathbf{P}(V_1)$. By Observation 11.4 applied to τ_1 ,

$$\zeta_1^{d-1}(\partial_\infty X) \subset \partial\Omega_1^*.$$

Then, Equation (16) implies that

$$\mathbf{P}(\ker \zeta^{d-1}(x)) \cap \Omega_1 = \emptyset$$

for all $x \in \partial_\infty X$.

Fix a point $p_0 \in \Omega_1$ and an affine chart $\mathbb{A} \subset \mathbf{P}(\mathbb{R}^d)$ that contains Ω_1 as a bounded set. Arguing as in the proof of Lemma 9.6, there exists a connected neighborhood U of p_0 in $\mathbf{P}(\mathbb{R}^d)$ such that

$$\tau(\mathrm{SL}(2, \mathbb{R}))U = \bigcup_{g \in \mathrm{SL}(2, \mathbb{R})} \tau(g)U$$

is bounded in $\mathbb{A}$. Then the set $X := \Omega_1 \cup \tau(\mathrm{SL}(2, \mathbb{R}))U$ is connected, bounded in $\mathbb{A}$, and preserved by $\tau(\mathrm{SL}(2, \mathbb{R}))$. So, by Observation 8.4

$$\Omega := \mathrm{ConvHull}(X)$$

is a properly convex domain where $\tau(\mathrm{SL}(2, \mathbb{R})) \leq \mathrm{Aut}(\Omega)$. □

Lemma 11.6. $\tau_1(\mathrm{SL}(2, \mathbb{R}))$ preserves a properly convex domain in $\mathbf{P}(V_1)$ if and only if d_1 is odd.

Proof. As described above, we can identify V_1 with the vector space of homogeneous polynomials in two variables x_1, x_2 with degree $d_1 - 1$. Under this identification, one can check that

$$\zeta_1^1([a : b]) = [(ax_2 + bx_1)^{d_1-1}],$$

where we identify $\partial_\infty X = \mathbf{P}(\mathbb{R}^2)$.

Case 1: Assume that d_1 is odd. Then

$$\Omega := \{[f] : f \in V_1 \text{ is convex and } f > 0 \text{ on } \mathbb{R}^2 \setminus \{0\}\}$$

is a properly convex domain in $\mathbf{P}(V_1)$ preserved by $\tau_1(\mathrm{SL}(2, \mathbb{R}))$. (Notice that this set is *properly* convex since any polynomial representing a point in Ω must have nonzero $x_1^{d_1-1}$ coefficient.).

Case 2: Assume that d_1 is even. Suppose for a contradiction that $\tau_1(\mathrm{SL}(2, \mathbb{R}))$ preserves a properly convex domain $\Omega \subset \mathbf{P}(V_1)$. Then, by Observation 11.4,

$$\zeta_1^1(\partial_\infty X) \subset \partial\Omega \quad \text{and} \quad \zeta_1^{d-1}(\partial_\infty X) \subset \partial\Omega^*.$$

However,

$$\zeta_1^1([1 : t]) = [(x_2 + tx_1)^{d_1-1}] = [x_2^{d_1-1} + tx_2^{d_1-2}x_1 + \cdots + t^{d_1-1}x_1^{d_1-1}]$$

and, since $d_1 - 1$ is odd, the curve $t \mapsto \zeta_1^1([1 : t])$ passes through the hyperplane

$$H := \mathbf{P}(\ker \zeta_1^{d_1-1}([1 : 0])) = \mathbf{P}\left(\left\langle x_2^{d_1-1}, x_2^{d_1-2}x_1, \dots, x_2x_1^{d_1-2} \right\rangle\right).$$

So, H cannot be a supporting hyperplane of Ω , but this contradicts Observation 11.4. $\square$

11.3 | Proof of Proposition 11.3

Now suppose that $\tau(G)$ preserves a properly convex domain $\Omega_0 \subset \mathbf{P}(\mathbb{R}^d)$.

Lemma 11.7. *There exists a properly convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^d)$ such that:*

- (1) $\Omega_0 \subset \Omega$,
- (2) $\tau(G) \leq \text{Aut}(\Omega)$,
- (3) $\zeta^1(\partial_\infty X) \subset \partial\Omega$ and $\zeta^{d-1}(\partial_\infty X) \subset \partial\Omega^*$,
- (4) if $x, y \in \zeta^1(\partial_\infty X)$, then $(x, y)_\Omega \subset \Omega$,
- (5) if $x \in \partial_\infty X$, then $\zeta^1(x)$ is a C^1 -smooth point of $\partial\Omega$ and $T_{\zeta^1(x)}\partial\Omega = \mathbf{P}(\ker \zeta^{d-1}(x))$,
- (6) if $(g_n)_{n \geq 1}$ is a sequence in G with $g_n \rightarrow x \in \partial_\infty X$ and $g_n^{-1} \rightarrow y \in \partial_\infty X$, then

$$\tau(\gamma_n)(p) \rightarrow \zeta^1(x)$$

for all $p \in \Omega$.

Proof. This is nearly identical to the proof of [17, Prop. 4.4]. We sketch the proof here for completeness.

Fix a compact subset $K \subset \Omega_0^*$ with nonempty interior. Then let D be the convex hull of $\overline{\tau(G) \cdot K}$ in Ω_0^* . Notice that D is a properly convex domain since $K \subset D \subset \Omega_0^*$ and K has nonempty interior. Then let $\Omega := D^*$. Then Ω is a properly convex domain, $\Omega_0 \subset \Omega$, and $\tau(G) \leq \text{Aut}(\Omega)$. Observation 11.4 implies that $\zeta^1(\partial_\infty X) \subset \partial\Omega$ and $\zeta^{d-1}(\partial_\infty X) \subset \partial\Omega^*$. It remains to verify (4), (5), and (6).

Let C be a connected component of the preimage of Ω in $\mathbb{R}^d$. Then C is a properly convex cone. Also, by the strongly dynamics-preserving property,

$$\overline{\tau(G) \cdot K} = \tau(G) \cdot K \cup \zeta^{d-1}(\partial_\infty X).$$

(4): Fix $x, y \in \zeta^1(\partial_\infty X)$ and $p \in (x, y)_\Omega$. Also fix lifts $\tilde{x}, \tilde{y} \in \overline{C}$ of x, y . Then $p = [\lambda\tilde{x} + (1-\lambda)\tilde{y}]$ for some $\lambda \in (0, 1)$. Suppose for a contradiction that $p \in \partial\Omega$. Then there exists $f \in \partial\Omega^* = \partial D$ such that $f(p) = 0$. We can write $f = [\sum_{j=1}^m f_j]$ where $f_j \in \mathbb{R}^{d*}$, $f_j|_C > 0$, and

$$[f_j] \in \overline{\tau(G) \cdot K}.$$

Case 1: Assume $[f_1] \in \tau(G) \cdot K$. Since $\tau(G) \cdot K \subset \Omega_0^*$ and $\zeta^1(\partial_\infty X) \subset \partial\Omega_0$, then $f_1(\tilde{x}) > 0$ and $f_1(\tilde{y}) > 0$. So,

$$\sum_{j=1}^m f_j(\lambda\tilde{x} + (1-\lambda)\tilde{y}) \geq f_1(\lambda\tilde{x} + (1-\lambda)\tilde{y}) > 0$$

and hence $f(p) \neq 0$. Contradiction.

Case 2: Assume $[f_1] \in \zeta^{d-1}(\partial_\infty X)$. Then by transversality, $f_1(x)$ and $f_1(y)$ cannot both be zero. Hence,

$$\sum_{j=1}^m f_j(\lambda \tilde{x} + (1-\lambda)\tilde{y}) \geq f_1(\lambda \tilde{x} + (1-\lambda)\tilde{y}) > 0$$

and hence, $f(p) \neq 0$. Contradiction.

(5): Fix $x \in \partial_\infty X$ and fix a supporting hyperplane H at $\zeta^1(x)$. Then $H = \mathbf{P}(\ker f)$ for some $f \in \partial\Omega^* = \partial D$. We can then write $f = [\sum_{j=1}^m f_j]$ where $f_j \in \mathbb{R}^{d*}$, $f_j|_C > 0$, and

$$[f_j] \in \overline{\tau(G) \cdot K}.$$

Arguing as in the proof of (4), we see that $m = 1$ and $[f_1] = \zeta^{d-1}(x)$. Hence, $H = \mathbf{P}(\ker \zeta^{d-1}(x))$. Since H was an arbitrary supporting hyperplane at $\zeta^1(x)$, we see that $\zeta^1(x)$ is a C^1 -smooth point of $\partial\Omega$ and $T_{\zeta^1(x)}\partial\Omega = \mathbf{P}(\ker \zeta^{d-1}(x))$.

(6): Suppose that $g_n \rightarrow x \in \partial_\infty X$ and $g_n^{-1} \rightarrow y \in \partial_\infty X$. By the strongly dynamics-preserving property

$$\tau(g_n)(v) \rightarrow \zeta^1(x)$$

for all $v \in \mathbf{P}(\mathbb{R}^d) \setminus \mathbf{P}(\ker \zeta^{d-1}(y))$. Part (5) of this lemma implies that

$$\mathbf{P}(\ker \zeta^{d-1}(y)) \cap \Omega = \emptyset$$

and so $\tau(g_n)(p) \rightarrow \zeta^1(x)$ for all $p \in \Omega$. □

Let C denote the convex hull of $\zeta^1(\partial_\infty \Gamma)$ in Ω . We will show that $\tau(G)$ acts cocompactly on C . To do this, we will use Lemma 8.7 in [18], which is based on a result and argument of Kapovich–Leeb–Porti (namely, Theorem 1.1 in [27] and Proposition 5.26 in [28]). Alternatively, it is possible to give an elementary, but longer, argument following the proof of [44, Prop. 3.6].

Lemma 11.8. $\tau(G)$ acts cocompactly on C .

Proof. Fix a cocompact lattice $\Gamma \leq G$. Then $\rho = \tau|_\Gamma$ is P_1 -Anosov and, if we identify $\partial_\infty \Gamma = \partial_\infty X$, then ρ has Anosov boundary map ζ .

Let C be a connected component of the preimage of Ω in $\mathbb{R}^d$. Then C is a properly convex cone. Following the notation in [18, Sec. 8], let

$$\tilde{\Lambda}_{\rho(\Gamma)}^* := \{f \in \mathbb{R}^{d*} : f|_C > 0 \text{ and } [f] \in \zeta^{d-1}(\partial_\infty X)\}$$

and

$$\Omega_{\max} := \left\{ [v] \in \mathbf{P}(\mathbb{R}^d) : f(v) > 0 \text{ for all } f \in \tilde{\Lambda}_{\rho(\Gamma)}^* \right\}.$$

Then $\Omega \subset \Omega_{\max}$ and so $\Omega_{\max} \neq \emptyset$. Further, Lemma 11.7(4) implies that C coincides with the convex hull of $\zeta^1(\partial_\infty \Gamma)$ in $\Omega_{\max}$. So, Lemma 8.7 in [18] implies that $\rho(\Gamma) = \tau(\Gamma)$ acts cocompactly on C . Thus, $\tau(\Gamma)$ acts cocompactly on C . $\square$

Lemma 11.9. (C, d_Ω) is Gromov-hyperbolic.

Proof. Since G contains uniform lattices, Lemma 11.8 and the fundamental lemma of geometric group theory imply that (C, d_Ω) is quasi-isometric to X . $\square$

We may now conclude the proof of our proposition.

Proof of Proposition 11.3. Suppose that $\Gamma \leq G$ is geometrically finite.

We first observe that $\zeta^1(\Lambda_X(\Gamma)) = \Lambda_\Omega(\tau(\Gamma))$. Fix $x \in \Lambda_\Omega(\tau(\Gamma))$. Then there exists $p \in \Omega$ and a sequence $(\gamma_n)_{n \geq 1}$ in Γ such that $\tau(\gamma_n)(p) \rightarrow x$. Passing to a subsequence, we can suppose that $\gamma_n \rightarrow x^+ \in \Lambda_X(\Gamma)$ and $\gamma_n^{-1} \rightarrow x^- \in \Lambda_X(\Gamma)$. Then Lemma 11.7 part (6) implies that $x = \zeta^1(x^+) \in \zeta^1(\Lambda_X(\Gamma))$. Conversely, fix $x \in \zeta^1(\Lambda_X(\Gamma))$. Then there exists a sequence $(\gamma_n)_{n \geq 1}$ in Γ such that $\gamma_n \rightarrow x$. Passing to a subsequence, we can suppose that $\gamma_n^{-1} \rightarrow y$. By Lemma 11.7 part (6),

$$\tau(\gamma_n)(p) \rightarrow \zeta^1(x)$$

for all $p \in \Omega$. So $x \in \Lambda_\Omega(\tau(\Gamma))$. Thus $\zeta^1(\Lambda_X(\Gamma)) = \Lambda_\Omega(\tau(\Gamma))$.

Then Lemma 11.7 parts (4) and (5) imply that $\tau(\Gamma)$ is a projectively visible subgroup of $\text{Aut}(\Omega)$. Since ζ^1 induces a homeomorphism $\Lambda_X(\Gamma) \rightarrow \Lambda_\Omega(\tau(\Gamma))$, we see that $\tau(\Gamma)$ acts geometrically finitely on its limit set.

The inclusion $(C_\Gamma, d_\Omega) \hookrightarrow (C, d_\Omega)$ is isometric and hence Lemma 11.9 implies that (C_Γ, d_Ω) is Gromov-hyperbolic. $\square$

PART 3. MISCELLANEOUS EXAMPLES

12 | PING-PONG WITH UNIPOTENTS IN PROJECTIVE SPACE

In this section, we show that certain free products are relatively P_1 -Anosov. Before stating the result, we need to introduce some terminology.

For $k \leq d/2$, let $\mathcal{F}_{k,d-k} = \mathcal{F}_{k,d-k}(\mathbb{K}^d)$ denote the space of partial flags of the form $F^k \subset F^{d-k} \subset \mathbb{K}^d$ where $\dim F^j = j$. A subgroup $\Gamma \leq \text{SL}(d, \mathbb{K})$ is P_k -divergent if $\lim_{n \rightarrow \infty} \frac{\mu_k}{\mu_{k+1}}(\gamma_n) = \infty$ for every escaping sequence $(\gamma_n)_{n \geq 1}$ in Γ . Such a group has well-defined limit set in $\mathcal{F}_{k,d-k}$ defined by

$$\Lambda_{k,d-k}(\Gamma) := \{F : \exists (\gamma_n)_{n \geq 1} \text{ in } \Gamma \text{ with } \gamma_n \rightarrow \infty \text{ and } F = \lim(U_k, U_{d-k})(\gamma_n)\}.$$

For relatively Anosov groups, the following holds.

Observation 12.1. If $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}$ with Anosov boundary map ξ , then $\rho(\Gamma)$ is P_k -divergent and ξ induces a homeomorphism

$$\partial(\Gamma, \mathcal{P}) \rightarrow \Lambda_{k,d-k}(\rho(\Gamma)).$$

In particular, if $P \in \mathcal{P}$, then $\Lambda_{k,d-k}(\rho(P))$ consists of a single point.

Proof. The strongly dynamics-preserving property and Observation 2.5 imply that $\rho(\Gamma)$ is P_k -divergent and ξ induces a homeomorphism $\partial(\Gamma, \mathcal{P}) \rightarrow \Lambda_{k,d-k}(\rho(\Gamma))$. $\square$

Recall that an element $g \in \mathrm{SL}(d, \mathbb{K})$ is P_1 -proximal if $\lambda_1(g) > \lambda_2(g)$. In this case, let $\ell_g^+ \in \mathbf{P}(\mathbb{K}^d)$ denote the eigenline corresponding to $\lambda_1(g)$. Then there exists a unique g -invariant codimension-one subspace $H_g^- \in \mathrm{Gr}_{d-1}(\mathbb{K}^d)$ such that $\ell_g^+ \oplus H_g^- = \mathbb{K}^d$.

An element $g \in \mathrm{SL}(d, \mathbb{K})$ is P_1 -biproximal if both g and g^{-1} are P_1 -proximal. In this case, we let $\ell_g^- := \ell_{g^{-1}}^+$ and $H_g^+ := H_{g^{-1}}^-$. Notice that in this case, $\ell_g^+ \subset H_g^+$ and $\ell_g^- \subset H_g^-$. Moreover, by writing a P_1 -biproximal element g in Jordan normal form, one can show that

$$g^n(F) \xrightarrow{n \rightarrow +\infty} (\ell_g^+, H_g^+)$$

for all $F \in \mathcal{F}_{1,d-1}$ transverse to (ℓ_g^-, H_g^-) .

Proposition 12.2. *Suppose that $\gamma \in \mathrm{SL}(d, \mathbb{K})$ is P_1 -biproximal, $U \leq \mathrm{SL}(d, \mathbb{K})$ is a P_1 -divergent discrete weakly unipotent group where $\Lambda_{1,d}(U) = \{F_U\}$ is a single element, and F_U is transverse to the flags $F_\gamma^+ := (\ell_\gamma^+, H_\gamma^+)$ and $F_\gamma^- := (\ell_\gamma^-, H_\gamma^-)$.*

*Then there exist $N \geq 1$ and a finite-index subgroup $U' \leq U$ such that the group Γ generated by γ^N and U' is naturally isomorphic to the free product $\langle \gamma^N \rangle * U'$ and the inclusion $\Gamma \hookrightarrow \mathrm{SL}(d, \mathbb{K})$ is P_1 -Anosov relative to $\{U'\}$.*

Remark 12.3. It is possible to add more P_1 -biproximal elements or weakly unipotent groups, as long as their limit flags are transverse. We skip this more general case as the proof is the same, just with more notation.

The rest of the section is devoted to the proof of Proposition 12.2, so fix γ and U as in the statement.

Let $\mathcal{F} := \mathcal{F}_{1,d-1}(\mathbb{K}^d)$ and let $d_{\mathcal{F}}$ be the distance on $\mathcal{F}$ defined by

$$d_{\mathcal{F}}(F_1, F_2) = d_{\mathbf{P}(\mathbb{K}^d)}(F_1^1, F_2^1) + d_{\mathrm{Gr}_{d-1}(\mathbb{K}^d)}(F_1^{d-1}, F_2^{d-1}).$$

Fix $\epsilon > 0$ such that the metric balls

$$B_{\mathcal{F}}(F_U, 2\epsilon), B_{\mathcal{F}}(F_\gamma^+, 2\epsilon), B_{\mathcal{F}}(F_\gamma^-, 2\epsilon)$$

are disjoint and any two flags in different balls are transverse. Let

$$\mathcal{Z}_U := \overline{\mathcal{N}_{\mathcal{F}}(\{F \in \mathcal{F} : F \text{ is not transverse to } F_U\}, \epsilon)}$$

and

$$\mathcal{Z}_\gamma^\pm := \overline{\mathcal{N}_{\mathcal{F}}(\{F \in \mathcal{F} : F \text{ is not transverse to } F_\gamma^\pm\}, \epsilon)}.$$

After possibly shrinking $\epsilon > 0$, we may also assume that

$$\mathcal{O} := \mathcal{F} \setminus \overline{\mathcal{Z}_U \cup \mathcal{Z}_\gamma^+ \cup \mathcal{Z}_\gamma^-}$$

is open and nonempty.

Lemma 12.4. *By replacing γ by a sufficiently large power, we may assume that $\gamma^{\pm 1}$ is ϵ -Lipschitz on $\mathcal{F} \setminus \mathcal{Z}_\gamma^\mp$ and $\gamma^{\pm 1}(\mathcal{F} \setminus \mathcal{Z}_\gamma^\mp) \subset \mathcal{B}_\mathcal{F}(F_\gamma^\pm, \epsilon)$.*

Proof. By conjugating, we can assume that

$$F_\gamma^+ = (\langle e_1 \rangle, \langle e_1, \dots, e_{d-1} \rangle) \quad \text{and} \quad F_\gamma^- = (\langle e_d \rangle, \langle e_2, \dots, e_d \rangle).$$

Then,

$$\gamma = \begin{pmatrix} \lambda_1 & & \\ & A & \\ & & \lambda_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{K}$, $A \in \text{GL}(d-2, \mathbb{K})$, and

$$|\lambda_1| > \lambda_1(A) \geq \lambda_{d-2}(A) > |\lambda_2|.$$

Since

$$\lambda_1(A) = \lim_{n \rightarrow \infty} \mu_1(A^n)^{1/n} \quad \text{and} \quad \frac{1}{\lambda_{d-2}(A)} = \lim_{n \rightarrow \infty} \mu_1(A^{-n})^{1/n},$$

the result follows from a straightforward calculation in affine charts. $\square$

Lemma 12.5. *By replacing U with a finite-index subgroup, we may assume that: if $u \in U \setminus \{\text{id}\}$, then u is ϵ -Lipschitz on $\mathcal{F} \setminus \mathcal{Z}_U$ and $u(\mathcal{F} \setminus \mathcal{Z}_U) \subset \mathcal{B}_\mathcal{F}(F_U, \epsilon)$.*

Proof. By conjugating, we can assume that

$$F_U = (\langle e_1 \rangle, \langle e_1, \dots, e_{d-1} \rangle).$$

Since U is P_1 -divergent and $\Lambda_{1,d-1}(U) = \{F_U\}$, for any escaping sequence $(u_n)_{n \geq 1}$ in U , we have

$$\lim_{n \rightarrow \infty} \frac{1}{\mu_1(u_n)} u_n = e_1 \langle \cdot, e_n \rangle \in \text{End}(\mathbb{R}^d),$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. Then a straightforward calculation in affine charts provides a finite subset $K \subset U$ such that: if $u \in U \setminus K$, then u is ϵ -Lipschitz on $\mathcal{F} \setminus \mathcal{Z}_U$ and $u(\mathcal{F} \setminus \mathcal{Z}_U) \subset \mathcal{B}_\mathcal{F}(F_U, \epsilon)$.

By [45, Th. 8.1(2)], U is finitely generated. Then, U is residually finite by a theorem of Malcev [32]. So there exists a finite-index subgroup $U' \leq U$ with $U' \cap K = \{\text{id}\}$. $\square$

Lemma 12.6. *The group Γ generated by γ and U is naturally isomorphic to the free product $\langle \gamma \rangle * U$.*

Proof. Let $\tau : \langle \gamma \rangle * U \rightarrow \Gamma$ be the obvious homomorphism. It is clearly onto and so we just have to show that it is one-to-one. Suppose that w is a nontrivial word in $\langle \gamma \rangle * U$. Fix $F \in \mathcal{O}$. Then

Lemmas 12.4 and 12.5 imply that

$$\tau(w)F \in \mathcal{B}_F(F_\gamma^+, \epsilon) \cup \mathcal{B}_F(F_\gamma^-, \epsilon) \cup \mathcal{B}_F(F_U, \epsilon).$$

So, $\tau(w)F \notin \mathcal{O}$ and hence $\tau(w) \neq \text{id}$. □

For the arguments that follow fix a finite symmetric generating set of U and let $|u|$ denote the associated word length of an element $u \in U$.

Next, we describe the Bowditch boundary of Γ . Let $S := \{\gamma, \gamma^{-1}\} \cup U \setminus \{\text{id}\}$ and let $\mathcal{W} := \{x = x_1 x_2 \cdots\}$ be the set of all *finite and infinite* reduced words in S (i.e., no letter is followed by its inverse) such that

- x has no consecutive elements in U , and
- x does not end in U .

We assume that the empty word $\emptyset$ is an element of $\mathcal{W}$. Also, let $\mathcal{W}_\infty \subset \mathcal{W}$ denote the subset of infinite-length words. Informally, finite-length words correspond to parabolic boundary points; this will be made more precise presently.

Since Γ is naturally isomorphic to the free product $\langle \gamma \rangle * U$, $\mathcal{W}$ admits a natural action of Γ , where Γ acts on nonempty words by left multiplication, $\gamma^{\pm 1} \cdot \emptyset = \gamma^{\pm 1}$, and $U \cdot \emptyset = \emptyset$. Notice that if $x = x_1 \cdots x_m \in \mathcal{W} \setminus \mathcal{W}_\infty$, then

$$\text{Stab}_\Gamma(x) = (x_1 \cdots x_m)U(x_1 \cdots x_m)^{-1}.$$

Further, $\mathcal{W}$ has a natural topology that can be described as follows. For $x = x_1 x_2 \cdots \in \mathcal{W}_\infty$ and $N \geq 1$, let

$$B_N(x) := \{y_1 y_2 \cdots \in \mathcal{W} : y_n = x_n \text{ for all } n \leq N\}.$$

For $x = x_1 \cdots x_m \in \mathcal{W} \setminus \mathcal{W}_\infty$ and $N \geq 1$, let

$$B_N(x) := \{x\} \cup \{y_1 y_2 \cdots \in \mathcal{W} : y_n = x_n \text{ for all } n \leq m, y_{m+1} \in U, \text{ and } |y_{m+1}| \geq N\}.$$

Then $\{B_N(x) : x \in \mathcal{W}, N \geq 1\}$ generates a topology on $\mathcal{W}$.

With this topology, one can check that Γ acts as a convergence group on $\mathcal{W}$, the points in $\mathcal{W}_\infty$ are conical limit points, and the points in $\mathcal{W} \setminus \mathcal{W}_\infty$ are bounded parabolic points. So, Γ is relatively hyperbolic with respect to $\mathcal{P} := \{U\}$ and we can identify $\partial(\Gamma, \mathcal{P}) = \mathcal{W}$.

Next, we define boundary maps for the inclusion $\Gamma \hookrightarrow \text{SL}(d, \mathbb{K})$.

Lemma 12.7. *If $x = x_1 x_2 \cdots \in \mathcal{W}_\infty$ and $F \in \mathcal{O}$, then the limit*

$$F_x := \lim_{n \rightarrow \infty} x_1 \cdots x_n(F)$$

exists and does not depend on $F \in \mathcal{O}$.

Proof. If $F \in \mathcal{O}$, then Lemmas 12.4 and 12.5 imply that

$$d_F(x_1 \cdots x_{n+1}(F_1), x_1 \cdots x_n(F_1)) \leq \epsilon^n \text{diam } F,$$

and so, $(x_1 \cdots x_n(F))_{n \geq 1}$ is a Cauchy sequence and hence the limit exists.

Further, if $F_1, F_2 \in \mathcal{O}$, then Lemmas 12.4 and 12.5 imply that

$$d_F(x_1 \cdots x_n(F_1), x_1 \cdots x_n(F_2)) \leq \epsilon^n \text{diam } F.$$

So, the limit does not depend on $F \in \mathcal{O}$. □

Define $\xi : \partial(\Gamma, \mathcal{P}) \rightarrow F$ by

$$\xi(x) = \begin{cases} F_x & \text{if } x \in \mathcal{W}_\infty \\ (x_1 \cdots x_m)F_U & \text{if } x = x_1 \cdots x_m \in \mathcal{W} \setminus \mathcal{W}_\infty \end{cases}.$$

Notice that

$$\xi(x) = (x_1 \cdots x_m)\xi(x_{m+1} \cdots)$$

for all $x = x_1 x_2 \cdots \in \mathcal{W}$ and hence ξ is ρ -equivariant.

Lemma 12.8. ξ is continuous.

Proof. Fix a converging sequence $y_n \rightarrow x$ in $\mathcal{W}$.

Case 1: Assume $x = x_1 x_2 \cdots \in \mathcal{W}_\infty$. Suppose $y_n = y_{n,1} y_{n,2} \cdots$. Then for any $j \geq 1$, $y_{n,j} = x_j$ for n sufficiently large (depending on j). So, for any $m \geq 1$, Lemmas 12.4 and 12.5 imply that

$$\begin{aligned} \limsup_{n \rightarrow \infty} d_F(\xi(x), \xi(y_n)) &= \limsup_{n \rightarrow \infty} d_F(x_1 \cdots x_m \xi(x_{m+1} \cdots), x_1 \cdots x_m \xi(y_{n,m+1} \cdots)) \\ &\leq \epsilon^m \text{diam } F. \end{aligned}$$

Since $m \geq 1$ was arbitrary and $\epsilon \in (0, 1)$, we have $\xi(y_n) \rightarrow \xi(x)$.

Case 2: Assume $x = x_1 \cdots x_m \in \mathcal{W} \setminus \mathcal{W}_\infty$. We may assume that $y_n \neq x$ for all n . Then passing to a tail of $(y_n)_{n \geq 1}$, we may assume that $y_n = x_1 \cdots x_m y_{n,m+1} \bar{y}_n$ where $\bar{y}_n \in \mathcal{W}$, $y_{n,m+1} \in U$, and $|y_{n,m+1}| \rightarrow \infty$. Then

$$\xi(y_n) = x_1 \cdots x_m y_{n,m+1} \xi(\bar{y}_n).$$

The word $\bar{y}_n$ has to start with either γ or γ^{-1} , hence Lemma 12.4 implies that $\xi(\bar{y}_n) \in B(F_\gamma^+, \epsilon)cf \cup B(F_\gamma^-, \epsilon)$. So, by our choice of $\epsilon > 0$, any accumulation point of $(\xi(\bar{y}_n))_{n \geq 1}$ is transverse to F_U . Thus, since $|y_{n,m+1}| \rightarrow \infty$ and $\Lambda_{1,d-1}(U) = \{F_U\}$, Observation 2.5 implies that

$$\lim_{n \rightarrow \infty} \xi(y_n) = x_1 \cdots x_m \lim_{n \rightarrow \infty} y_{n,m+1} \xi(\bar{y}_n) = x_1 \cdots x_m F_U = \xi(x).$$

So, ξ is continuous. □

Lemma 12.9. ξ is transverse.

Proof. Fix $x, y \in \mathcal{W}$ distinct. After possibly relabeling and translating by Γ , it is enough to consider the following cases.

Case 1: Assume $x \neq \emptyset$, $y \neq \emptyset$, and $x_1 \neq y_1$. Then

$$\xi(x) = x_1 \xi(x_2 \cdots) \quad \text{and} \quad \xi(y) = y_1 \xi(y_2 \cdots).$$

So Lemmas 12.4 and 12.5 imply that

$$\xi(x), \xi(y) \in B_F(F_U, \epsilon) \cup B_F(F_\gamma^+, \epsilon) \cup B_F(F_\gamma^-, \epsilon).$$

Since $x_1 \neq y_1$, they are contained in different balls, and so, by our choice of $\epsilon > 0$, $\xi(x)$ and $\xi(y)$ are transverse.

Case 2: Assume $x = \emptyset$ and $y \neq \emptyset$. After possibly translating by an element of U , we may also assume that $y_1 \notin U$. Then

$$\xi(y) \in B_F(F_\gamma^+, \epsilon) \cup B_F(F_\gamma^-, \epsilon),$$

and so, $\xi(y)$ is transverse to $\xi(x) = F_U$. □

Lemma 12.10. ξ is strongly dynamics-preserving.

Proof. Suppose that $(\gamma_n)_{n \geq 1}$ is an escaping sequence in Γ with $\gamma_n \rightarrow x \in \mathcal{W}$ and $\gamma_n^{-1} \rightarrow y \in \mathcal{W}$. We claim that

$$\lim_{n \rightarrow \infty} \gamma_n F = \xi(x)$$

for all $F \in \mathcal{O}$. To that end fix $F \in \mathcal{O}$.

By Lemma 12.6, we can write $\gamma_n = z_{n,1} z_{n,2} \cdots z_{n,m_n}$ as a reduced word in S that has no consecutive elements in U .

Case 1: Assume $x = x_1 x_2 \cdots \in \mathcal{W}_\infty$. Then $z_{n,j} = x_j$ for n sufficiently large (depending on j). For any $k \geq 1$ and n sufficiently large (depending on k), Lemmas 12.4 and 12.5 imply that

$$d_F(x_1 \cdots x_k F, \gamma_n F) = d_F(x_1 \cdots x_k F, x_1 \cdots x_k z_{n,k+1} \cdots z_{n,m_n} F) \leq \epsilon^k \text{diam } F.$$

So,

$$\lim_{n \rightarrow \infty} \gamma_n F = \lim_{k \rightarrow \infty} x_1 \cdots x_k F = F_x = \xi(x).$$

Case 2: Assume $x = x_1 \cdots x_m \in \mathcal{W} \setminus \mathcal{W}_\infty$. Then passing to a tail of γ_n , we can assume that $z_{n,j} = x_j$ for all $1 \leq j \leq m$, $z_{n,m+1} \in U$, and $\lim_{n \rightarrow \infty} |z_{n,m+1}| = \infty$. Let $\tilde{\gamma}_n := z_{n,m+2} \cdots z_{n,m_n}$. If $\tilde{\gamma}_n = \text{id}$, then by Observation 2.5

$$\lim_{n \rightarrow \infty} \gamma_n F = x_1 \cdots x_m \lim_{n \rightarrow \infty} z_{n,m+1} F = x_1 \cdots x_m F_U = \xi(x)$$

since $F \in \mathcal{O}$ is transverse to F_U and $\Lambda_{1,d-1}(U) = \{F_U\}$. Otherwise, if $\tilde{\gamma}_n \neq \text{id}$, then $z_{n,m+2} \in \{\gamma, \gamma^{-1}\}$. So,

$$\tilde{\gamma}_n F \in B_F(F_\gamma^+, \epsilon) \cup B_F(F_\gamma^-, \epsilon).$$

In particular, any accumulation point of $(\tilde{\gamma}_n F)_{n \geq 1}$ is transverse to F_U . Then, since $z_{n,m+1} \in U$, $\lim_{n \rightarrow \infty} |z_{n,m+1}| = \infty$, and $\Lambda_{1,d-1}(U) = \{F_U\}$, Observation 2.5 implies that

$$\lim_{n \rightarrow \infty} \gamma_n F = x_1 \cdots x_m \lim_{n \rightarrow \infty} z_{n,m+1} \tilde{\gamma}_n F = x_1 \cdots x_m F_U = \xi(x).$$

Similar reasoning shows that

$$\lim_{n \rightarrow \infty} \gamma_n^{-1} F = \xi(y)$$

for all $F \in \mathcal{O}$. Thus, by Observation 2.5,

$$\lim_{n \rightarrow \infty} \gamma_n V = \xi^k(x)$$

for all $V \in \text{Gr}_k(\mathbb{K}^d)$ transverse to $\xi^{d-k}(y)$. $\square$

Thus, the inclusion $\Gamma \hookrightarrow \text{SL}(d, \mathbb{K})$ is P_1 -Anosov relative to $\mathcal{P} = \{U\}$.

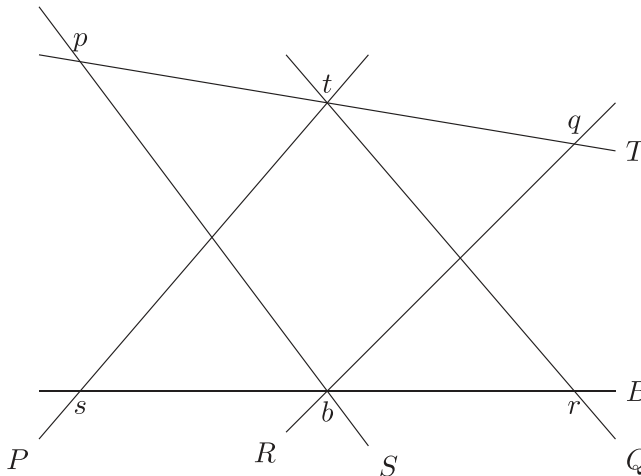
13 | PAPPUS-SCHWARTZ REPRESENTATIONS

In [36], certain representations of the (projectivized) modular group $\text{PSL}(2, \mathbb{Z})$ into $\text{PGL}(3, \mathbb{R})$ were obtained by considering the iterated application of Pappus's theorem, from projective geometry, on certain configurations of points and lines in the real projective plane.

Here, we establish that these representations are relatively Anosov. This mostly involves reformulating results in [36] in the language of (relatively) Anosov representations.

We first define the configurations of points and lines we consider. Given points $p, q \in \mathbf{P}(\mathbb{R}^3)$, write pq to denote the projective line containing p and q . Dually, given projective lines $P, Q \subset \mathbf{P}(\mathbb{R}^3)$, write PQ to denote the intersection of the lines P and Q . In the discussion that follows, we identify elements of $\text{Gr}_2(\mathbb{R}^3)$ with projective lines in $\mathbf{P}(\mathbb{R}^3)$.

- An *overmarked box* is a pair of 6-tuples $((p, q, r, s, t, b), (P, Q, R, S, T, B))$ in $(\mathbf{P}(\mathbb{R}^3))^6 \times (\text{Gr}_2(\mathbb{R}^3))^6$ satisfying the incidence relations required by Pappus's theorem (shown in the following figure)



- A *marked box* is an equivalence class of overmarked boxes under the involution

$$((p, q, r, s, t, b), (P, Q, R, S, T, B)) \mapsto ((q, p, s, r, t, b), (Q, P, S, R, T, B))$$

(corresponding to “flipping around the central axis tb ”).

- The *convex interior* of a marked box is the open quadrilateral with vertices p, q, r, s (in that order). We shall not make much direct use of convex interiors of marked boxes here, but they are useful mental tools for thinking of these objects geometrically.

Given a marked box $\mathfrak{B}$, let $\rho_{\mathfrak{B}} : \mathrm{PSL}(2, \mathbb{Z}) \rightarrow \mathrm{PGL}(3, \mathbb{R})$ be a Pappus–Schwartz representation as defined in [36, Th. 2.4] (see also [7]). This representation is defined as follows. First, let

$$a = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $\mathrm{PSL}(2, \mathbb{Z})$ has presentation $\mathrm{PSL}(2, \mathbb{Z}) = \langle a, d : a^3 = d^2 = 1 \rangle$. Then:

- $\rho_{\mathfrak{B}}(d)$ is the projective duality that sends

$$\mathfrak{B} = [((p, q, r, s, t, b), (P, Q, R, S, T, B))]$$

to its “dual”/“exterior” marked box

$$\iota(\mathfrak{B}) := [((s, r, p, q, b, t), (R, S, Q, P, B, T))],$$

and

- $\rho_{\mathfrak{B}}(a)$ is the 3-cycle that cycles between the original box, the dual to the “top” box produced by an application of Pappus’s theorem to $\mathfrak{B}$, and the dual to the “bottom” box (see [36, Fig. 2.3]). In symbols, $\rho_{\mathfrak{B}}(a)$ sends

$$\mathfrak{B} = [((p, q, r, s, t, b), (P, Q, R, S, T, B))]$$

to

$$[(PS, QR, p, q, (qs)(pr), t), (qs, pr, Q, P, (QR)(PS), T))]$$

to

$$[((s, r, PS, QR, b, (qs)(pr)), (S, R, qs, pr, B, (QR)(PS)))]$$

back to $\mathfrak{B}$.

Next, let $\mathbb{H}_{\mathbb{R}}^2$ denote real hyperbolic 2-space and identify $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}_0(\mathbb{H}_{\mathbb{R}}^2)$ via the Poincaré upper half-plane model. If we let $\mathcal{P}$ denote a set of representatives for the conjugacy classes of maximal parabolic subgroups in $\mathrm{PSL}(2, \mathbb{Z})$, then $\mathrm{PSL}(2, \mathbb{Z})$ is relatively hyperbolic with respect to $\mathcal{P}$ and the Bowditch boundary naturally identifies with the Gromov boundary $\partial_{\infty} \mathbb{H}_{\mathbb{R}}^2$ of $\mathbb{H}_{\mathbb{R}}^2$.

By [36, Sec. 3.2, 3.3] (see also [7, Sec. 5.3]), there is a continuous $\rho_{\mathfrak{B}}$ -equivariant map

$$\xi_{\mathfrak{B}} = (\xi_{\mathfrak{B}}^1, \xi_{\mathfrak{B}}^2) : \partial_{\infty} \mathbb{H}_{\mathbb{R}}^2 \rightarrow \mathbf{P}(\mathbb{R}^3) \times \mathrm{Gr}_2(\mathbb{R}^3).$$

Moreover, this map is transverse [36, Th. 3.3].

The strongly dynamics-preserving property follows from the proof of [36, Lem. 4.2.3]. For the reader’s convenience, we will derive the property directly from the statement of [36, Lem. 4.2.3].

Lemma 13.1 [36, Lem. 4.2.3]. *If $(\gamma_n)_{n \geq 1}$ is a sequence in $\mathrm{PSL}(2, \mathbb{Z})$ and $\epsilon > 0$, then there exist $N \geq 1$ and $x, y \in \partial_\infty \mathbb{H}_{\mathbb{R}}^2$ such that*

$$\rho_{\mathfrak{B}}(\gamma_N)(\mathbf{P}(\mathbb{R}^3) \setminus \mathcal{N}_{\mathbf{P}}(\xi_{\mathfrak{B}}^2(y), \epsilon)) \subset B_{\mathbf{P}}(\xi_{\mathfrak{B}}^1(x), \epsilon)$$

(where $\mathcal{N}_{\mathbf{P}}$ and $B_{\mathbf{P}}$ denote, respectively, an open neighborhood and an open ball with respect to the angle metric defined on $\mathbf{P}(\mathbb{R}^3)$ in Section 2.3).

Proposition 13.2. $\xi_{\mathfrak{B}}$ is strongly dynamics-preserving.

Proof. Suppose that $(\gamma_n)_{n \geq 1}$ is a sequence in $\mathrm{PSL}(2, \mathbb{Z})$ such that $\gamma_n \rightarrow x \in \partial_\infty \mathbb{H}_{\mathbb{R}}^2$ and $\gamma_n^{-1} \rightarrow y \in \partial_\infty \mathbb{H}_{\mathbb{R}}^2$. It is enough to verify that every subsequence of $(\gamma_n)_{n \geq 1}$ has a subsequence that verifies the strongly dynamics-preserving property.

So, fix a subsequence $(\gamma_{n_j})_{j \geq 1}$. Replacing $(\gamma_{n_j})_{j \geq 1}$ by a subsequence, we can suppose that for each $j \geq 1$, there exist $x_j, y_j \in \partial_\infty \mathbb{H}_{\mathbb{R}}^2$ such that

$$\rho_{\mathfrak{B}}(\gamma_{n_j})(\mathbf{P}(\mathbb{R}^3) \setminus \mathcal{N}(\xi_{\mathfrak{B}}^2(y_j), 2^{-j})) \subset B_{\mathbf{P}}(\xi_{\mathfrak{B}}^1(x_j), 2^{-j}).$$

Passing to a further subsequence, we can assume that $x_j \rightarrow x_\infty$ and $y_j \rightarrow y_\infty$. Then

$$\rho_{\mathfrak{B}}(\gamma_{n_j})v \rightarrow \xi_{\mathfrak{B}}^1(x_\infty)$$

for all $v \in \mathbf{P}(\mathbb{R}^3) \setminus \xi_{\mathfrak{B}}^2(y_\infty)$.

Fix $z \in \partial_\infty \mathbb{H}_{\mathbb{R}}^2 \setminus \{x, y, x_\infty, y_\infty\}$. Then by the transversality, equivariance, and continuity of the boundary map,

$$\xi_{\mathfrak{B}}^1(x_\infty) = \lim_{j \rightarrow \infty} \rho_{\mathfrak{B}}(\gamma_{n_j})\xi_{\mathfrak{B}}^1(z) = \lim_{j \rightarrow \infty} \xi_{\mathfrak{B}}^1(\gamma_{n_j}(z)) = \xi_{\mathfrak{B}}^1(x).$$

By Observation 2.5 and transversality,

$$\xi_{\mathfrak{B}}^2(y_\infty) = \lim_{j \rightarrow \infty} \rho_{\mathfrak{B}}(\gamma_{n_j})^{-1}\xi_{\mathfrak{B}}^2(z).$$

So, a similar argument also shows that $\xi_{\mathfrak{B}}^2(y_\infty) = \xi_{\mathfrak{B}}^2(y)$. Thus,

$$\rho_{\mathfrak{B}}(\gamma_{n_j})v \rightarrow \xi_{\mathfrak{B}}(x)$$

for all $v \in \mathbf{P}(\mathbb{R}^3) \setminus \xi_{\mathfrak{B}}^2(y)$. □

Hence $\rho_{\mathfrak{B}}$ is $\mathbf{P}_1$ -Anosov relative to $\mathcal{P}$.

14 | SEMISIMPLIFICATION

A representation into $\mathrm{SL}(d, \mathbb{K})$ is called *semisimple* if the Zariski closure of its image is a reductive group. Associated to a representation $\rho: \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$, there is a natural conjugacy class of

semisimple representations defined as follows. Let G be the Zariski closure of $\rho(\Gamma)$ in $\mathrm{SL}(d, \mathbb{K})$ and choose a Levi decomposition $G = L \ltimes U$, where U is the unipotent radical of G . Let ρ^{ss} denote the representation obtained by composing ρ with the projection onto L . We call any representation in the conjugacy class of ρ^{ss} a *semisimplification* of ρ . Since L is unique up to conjugation, this definition does not depend on the chosen Levi decomposition.

When Γ is a word-hyperbolic group, it is known that ρ is P_k -Anosov if and only if some (any) semisimplification of ρ is P_k -Anosov [20, Prop. 4.13]. This is quite useful, see, for instance, the proof of Theorem 1.2 in [15] or the proof of Proposition 1.2 in [30].

In this section, we observe that the forward direction of this statement is also true for relatively Anosov representations, while the backward direction is false.

Proposition 14.1. *There exists a representation $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ of a relatively hyperbolic group $(\Gamma, \mathcal{P})$ where every semisimplification of ρ is P_1 -Anosov relative to $\mathcal{P}$, but ρ is not P_1 -Anosov relative to $\mathcal{P}$.*

Proof. Let $\Gamma = \langle a, b \rangle \leq \mathrm{PSL}(2, \mathbb{R})$ be a discrete free group where a is hyperbolic and

$$b = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then, Γ is hyperbolic relative to $\mathcal{P} = \{\langle b \rangle\}$. Fix lifts $\tilde{a}, \tilde{b} \in \mathrm{SL}(2, \mathbb{R})$ of $a, b \in \mathrm{PSL}(2, \mathbb{R})$ and consider the representation $\rho : \Gamma \rightarrow \mathrm{SL}(4, \mathbb{R})$ defined by

$$\rho(a) = \mathrm{id}_2 \oplus \tilde{a} \quad \text{and} \quad \rho(b) = \tilde{b} \oplus \tilde{b}.$$

Notice that

$$\lim_{n \rightarrow \infty} \rho(b^n)[x_1 : x_2 : x_3 : x_4] = [x_2 : 0 : x_4 : 0]$$

for all $[x_1 : x_2 : x_3 : x_4] \in \mathbf{P}(\mathbb{R}^4)$ with $x_2 \neq 0$ or $x_4 \neq 0$. So, there cannot exist a ρ -equivariant strongly dynamics-preserving map into $\mathbf{P}(\mathbb{R}^d) \times \mathrm{Gr}_{d-1}(\mathbb{R}^d)$, and so, ρ is not P_1 -Anosov relative to $\mathcal{P}$. However, the representation $\rho^{ss} : \Gamma \rightarrow \mathrm{SL}(4, \mathbb{R})$ defined by

$$\rho^{ss}(a) = \mathrm{id}_2 \oplus \tilde{a} \quad \text{and} \quad \rho(b) = \mathrm{id}_2 \oplus \tilde{b}$$

is a semisimplification of ρ and is P_1 -Anosov relative to $\mathcal{P}$. □

Proposition 14.2. *Suppose that $(\Gamma, \mathcal{P})$ is relatively hyperbolic. If $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ is P_k -Anosov relative to $\mathcal{P}$, then so is every semisimplification of ρ .*

The rest of the section is devoted to the proof of Proposition 14.2. So fix a relatively hyperbolic group $(\Gamma, \mathcal{P})$ and a representation $\rho : \Gamma \rightarrow \mathrm{SL}(d, \mathbb{K})$ that is P_k -Anosov relative to $\mathcal{P}$. Then fix a semisimplification ρ^{ss} of ρ .

If $\gamma \in \Gamma$ is a loxodromic element (see [45, Sec. 3.2]), then let $\gamma^\pm \in \partial(\Gamma, \mathcal{P})$ denote the attracting/repelling fixed points of γ .

Following the proof of [20, Prop. 4.13], there exists a ρ^{ss} -equivariant, transverse, continuous map $\xi_{ss} : \partial(\Gamma, \mathcal{P}) \rightarrow \text{Gr}_k(\mathbb{K}^d) \times \text{Gr}_{d-k}(\mathbb{K}^d)$ with the following property (called *dynamics-preserving* in [20]): if $\gamma \in \Gamma$ is a loxodromic element, then $\rho^{ss}(\gamma)$ is P_k -proximal and $\xi_{ss}^k(\gamma^+)$, $\xi_{ss}^{d-k}(\gamma^-)$ are the attracting/repelling spaces of $\rho^{ss}(\gamma)$.

It remains to show that ξ_{ss} is strongly dynamics-preserving. We begin by showing ρ^{ss} is P_k -divergent.

Lemma 14.3. $\lim_{n \rightarrow \infty} \frac{\mu_k}{\mu_{k+1}}(\rho^{ss}(\gamma_n)) = \infty$ for any escaping sequence $(\gamma_n)_{n \geq 1}$ in Γ .

Proof. By Theorem 1.3, there exists a weak cusp space X for $(\Gamma, \mathcal{P})$ such that ρ is P_k -Anosov relative to X . Fix $x_0 \in X$. Then by [45, Th. 6.1] there exist $\alpha, \beta > 0$ such that

$$\log \frac{\mu_k}{\mu_{k+1}}(\rho(\gamma)) \geq \alpha d_X(x_0, \gamma(x_0)) - \beta$$

for all $\gamma \in \Gamma$. So, for $\gamma \in \Gamma$, we have

$$\log \frac{\lambda_k}{\lambda_{k+1}}(\rho^{ss}(\gamma)) = \log \frac{\lambda_k}{\lambda_{k+1}}(\rho(\gamma)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\mu_k}{\mu_{k+1}}(\rho(\gamma)^n) \geq \alpha \ell_X(\gamma), \quad (17)$$

where $\ell_X(\gamma) := \lim_{n \rightarrow \infty} \frac{1}{n} d_X(x_0, \gamma^n(x_0))$.

Since ρ^{ss} is semisimple, by [20, Th. 4.12], there exist $C_1 > 1$ and a finite set $F_1 \subset \Gamma$ with the following property: for every $\gamma \in \Gamma$, there is some $f \in F_1$ such that

$$\frac{1}{C_1} \mu_j(\rho^{ss}(\gamma)) \leq \lambda_j(\rho^{ss}(\gamma f)) \leq C_1 \mu_j(\rho^{ss}(\gamma)), \quad (18)$$

for all $1 \leq j \leq d$.

Now fix an escaping sequence $(\gamma_n)_{n \geq 1}$. It suffices to consider the case when

$$\lim_{n \rightarrow \infty} \frac{\mu_k}{\mu_{k+1}}(\rho^{ss}(\gamma_n))$$

exists in $\mathbb{R} \cup \{\infty\}$ and show that the limit is infinite. Passing to a subsequence, we can suppose that $\gamma_n \rightarrow x$ and $\gamma_n^{-1} \rightarrow y$. Pick $\alpha \in \Gamma$ such that $\alpha^{-1}(y) \notin F_1(x)$. For each n , fix $f_n \in F_1$ such that $\gamma_n \alpha f_n$ satisfies Equation (18). Passing to a further subsequence, we can suppose that $f := f_n$ for all n . Then $\gamma_n \alpha f \rightarrow x$ and $(\gamma_n \alpha f)^{-1} \rightarrow f^{-1} \alpha^{-1}(y)$. By our choice of α , we have $f^{-1} \alpha^{-1}(y) \neq x$ which implies that $\gamma_n \alpha f$ is a loxodromic element for n sufficiently large. Further, $(\gamma_n \alpha f)^+ \rightarrow x$ and $(\gamma_n \alpha f)^- \rightarrow y$. Then, since $(\gamma_n \alpha f)$ is escaping sequence, we must have $\lim_{n \rightarrow \infty} \ell_X(\gamma_n \alpha f) = \infty$.

Then, by Equations (18) and (17),

$$\lim_{n \rightarrow \infty} \frac{\mu_k}{\mu_{k+1}}(\rho^{ss}(\gamma_n)) \geq \lim_{n \rightarrow \infty} \frac{\mu_k}{\mu_{k+1}}(\rho^{ss}(\gamma_n \alpha f)) \geq \lim_{n \rightarrow \infty} \frac{\lambda_k}{\lambda_{k+1}}(\rho^{ss}(\gamma_n \alpha f)) = \infty. \quad \square$$

To complete the proof that ξ_{ss} is strongly dynamics-preserving, we recall a few results. First, since ρ^{ss} is semisimple and $\rho^{ss}(\Gamma)$ contains a P_k -proximal element, [1] and [3, Cor. 6.3] imply that there exist a finite set $F_2 \subset \Gamma$ and some $C_2 > 0$ with the following property: for every $\gamma \in \Gamma$, there

is some $f \in F$ such that $\rho^{ss}(\gamma f)$ is P_k -proximal and

$$\frac{\mu_1 \cdots \mu_k}{\lambda_1 \cdots \lambda_k}(\rho^{ss}(\gamma f)) \leq C_2. \quad (19)$$

Also, by [38, Prop. 2.5(i)], there exists $C_3 > 0$ such that: if $g \in \mathrm{SL}(d, \mathbb{K})$ is P_k -proximal and $V_g^+ \in \mathrm{Gr}_k(\mathbb{K}^d)$ is the attracting subspace, then

$$d_{\mathrm{Gr}_k(\mathbb{K}^d)}(V_g^+, U_k(g)) \leq C_3 \frac{\mu_{k+1}}{\mu_k}(g) \frac{\mu_1 \cdots \mu_k}{\lambda_1 \cdots \lambda_k}(g). \quad (20)$$

Finally, by [10, Lem. A.4], if $g, h \in \mathrm{GL}(d, \mathbb{K})$, $\mu_k(g) > \mu_{k+1}(g)$, and $\mu_k(gh) > \mu_{k+1}(gh)$, then

$$d_{\mathrm{Gr}_k(\mathbb{K}^d)}(U_k(gh), U_k(g)) \leq \frac{\mu_1}{\mu_d}(h) \frac{\mu_{k+1}}{\mu_k}(g). \quad (21)$$

Lemma 14.4. ξ_{ss} is strongly dynamics-preserving.

Proof. Fix an escaping sequence $(\gamma_n)_{n \geq 1}$ in Γ such that $\gamma_n \rightarrow x$ and $\gamma_n^{-1} \rightarrow y$. By Lemma 14.3 and Observation 2.5, it suffices to show that $U_k(\rho^{ss}(\gamma_n)) \rightarrow \xi_{ss}^k(x)$ and $U_{d-k}(\rho^{ss}(\gamma_n)^{-1}) \rightarrow \xi_{ss}^{d-k}(y)$.

For each n , fix $f_n \in F_2$ such that $\rho^{ss}(\gamma_n f_n)$ is P_k -proximal and satisfies Equation (19). Then $\frac{\lambda_k}{\lambda_{k+1}}(\rho(\gamma_n f_n)) = \frac{\lambda_k}{\lambda_{k+1}}(\rho^{ss}(\gamma_n f_n)) > 1$, and hence, by Proposition 2.6(1), each $\gamma_n f_n$ must be a non-peripheral element of $(\Gamma, \mathcal{P})$. So, by the dynamics-preserving property, $\xi_{ss}^k((\gamma_n f_n)^+)$ is the attracting k -plane of $\rho^{ss}(\gamma_n f_n)$.

Since F_2 is a finite set and $\gamma_n \rightarrow x$, we must have $\gamma_n f_n \rightarrow x$. This, in turn, implies that $(\gamma_n f_n)^+ \rightarrow x$. Then, by Equations (21), (20), and (19)

$$\begin{aligned} \limsup_{n \rightarrow \infty} d_{\mathrm{Gr}_k(\mathbb{K}^d)}(\xi_{ss}^k(x), U_k(\rho^{ss}(\gamma_n))) &= \limsup_{n \rightarrow \infty} d_{\mathrm{Gr}_k(\mathbb{K}^d)}(\xi_{ss}^k((\gamma_n f_n)^+), U_k(\rho^{ss}(\gamma_n f_n))) \\ &\lesssim \limsup_{n \rightarrow \infty} \frac{\mu_{k+1}}{\mu_k}(\rho^{ss}(\gamma_n f_n)) \frac{\mu_1 \cdots \mu_k}{\lambda_1 \cdots \lambda_k}(\rho^{ss}(\gamma_n f_n)) = 0. \end{aligned}$$

So, $U_k(\rho^{ss}(\gamma_n)) \rightarrow \xi_{ss}^k(x)$.

The proof that $U_{d-k}(\rho^{ss}(\gamma_n)^{-1}) \rightarrow \xi_{ss}^{d-k}(y)$ is nearly identical. $\square$

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