p-Adaptive Quadrature for the Chebyshev-based Boundary Integral Equation Method

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***Abstract*—The Chebyshev-based Boundary Integral (CBIE) method is a computationally efficient implementation of the Nyström method, which enables discretization and solution of boundary integral problems with high-order accuracy. The far source/target point interactions leverage the Nyström method, whereas self-singular and near-singular interactions are handled by expanding the unknown densities using Chebyshev polynomials and precomputing the interactions of the integral operators and the Chebyshev basis to high numerical accuracy. In this work, we present a p-adaptive quadrature scheme which enables these precomputation integrals to be evaluated efficiently to a desired level of accuracy without requiring any manual fine-tuning or knowledge of the geometry. This approach matches the same level of accuracy achieved by the previous implementations based on fixed-size refined integration grids, while achieving up to a 16.9X speed-up in computing time. The advantages of the p-adaptive approach are demonstrated by comparing it to the prior fixed-grid method for solving the Magnetic Field Integral Equation (MFIE) applied to an edge-refined metallic cube geometry.**

I. Introduction

Efficient and accurate numerical solution of Maxwell's equations is of significant importance, since most electromagnetic devices of engineering interest, such as antennas and nanophotonic devices, do not have analytical solutions. Boundary Integral Equation (BIE) methods are often used to solve such problems, since they require discretization of the problem on surfaces rather than in volumes, unlike other approaches such as Finite Difference and Finite Element methods. Typical BIE implementations use low-order Rao-Wilton-Glisson (RWG) basis functions with the Method of Moments (MoM) to discretize the system [1]. While this approach has been used successfully to solve many problems, its low-order nature is often not well suited to represent complex surfaces and may require very fine meshing to achieve acceptable error tolerances, leading to high memory requirements and long solution times. To overcome this potential drawback, a high-order polynomial representation for the surface current densities can be used instead; however, the implementation of these high-order representations in the context of MoM can be prohibitive due to Galerkin testing requiring expensive evaluation of four-dimensional integrals. On the other hand, Nyström methods, rely on point-matching instead of Galerkin for testing and can achieve similar accuracy with higher efficiency when compared to high-order MoM methods.

Recently, a Chebyshev-based Boundary Integral Equation (CBIE) implementation of the Nyström method has been introduced [2]– [4], which achieves both high accuracy and high computational efficiency. The CBIE method separates interactions between source and target points based on their distances into singular, near-singular, and far interactions. For the points which are far enough apart such that the kernels of the integral equations are not singular or near-singular, the method uses Fejér's first quadrature rule to compute the required integrals efficiently with high accuracy. For singular and near-singular interactions, however, custom quadrature weights are precomputed by integrating the kernels against tensor-products of Chebyshev polynomial basis functions on a highly refined grid. The action of the integral operator on an arbitrary density can then be evaluated by using a Chebyshev transform to expand the density in terms of its Chebyshev coefficients and multiplying and accumulating against the aforementioned precomputed weights. Previously published implementations of the CBIE method all used a single fixed-size refined grid, designated at compilation time, to evaluate these integration weights for all the singular and near-singular interactions. However, the fixed-grid approach requires manual, geometry-dependent tuning to determine a grid which is refined enough in order to not limit the accuracy of final solution, while also coarse enough so as to not incur an unnecessary computational penalty.

In this work we present a p-refinement-based adaptive integration method for dealing with the precomputations needed by the CBIE solver using a high-order nested Clenshaw-Curtis quadrature rule. The adaptive integration approach does not require a priori knowledge of the level of refinement required and can choose the coarsest refinement needed for each interaction to achieve the desired accuracy. The adaptive approach uses up to 5.6X fewer singular kernel evaluations than the static fixed-grid Fejér quadrature, which results in up to 16.9X improvement in wall-clock time for certain geometries. To showcase these improvements, we compare the accuracy and speed of the new p-adaptive and prior fixed-grid quadrature rules when used for solving the Magnetic Field Integral Equation(MFIE) on an edge refined perfect electrically conducting (PEC) cube with a side length of 2λ.

II. Adaptive Integration Approach

The p-adaptive integration approach leverages Fejér's second quadrature rule, which has the same nodes as Clenshaw-Curtis (CC) quadrature but does not include the end-points [5]. Similar to Fejér's first quadrature rule, this method is also based on integrating a Chebyshev expansion of the integrands; however, the arrangement of the qudrature nodes enables it to be used in a nested fashion (e.g., an order $2N$ rule shares

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half of its nodes with an order N rule). This allows efficient error estimation by evaluating an integral using an order N rule and comparing it against the result using an order $2N$ rule, while only requiring $2N$ total function evaluations of the integrand. Note that although the method is described here in one-dimension for simplicity, the actual implementation leverages a tensor-product rule to handle the 2D integrals. The inputs to the adaptive integrator are the order of the starting rule, N_β, and the number of times that it is allowed to double the refinement grid (p). If $p = 1$, the scheme is equivalent to a fixed quadrature method. However, for $p > 1$, the scheme first estimates the value of the integral (I_1) using an N_β-order rule, as well as its value (I_2) using a higher-order $2N_\beta$ refined rule and approximates the absolute integration error as $\text{abs}(I_2 - I_1)$. If the result is determined to be sufficiently accurate as defined by $\text{abs}(I_2 - I_1) < \max(tol_{abs}, tol_{rel} * \text{abs}(I_2))$, where tol_{abs} and tol_{res} are the prescribed absolute and relative tolerances respectively, the algorithm concludes and returns the result I_2. Otherwise, I_k for $k > 2$ is computed and compared against the previously computed I_{k-1} for increasing k until convergence is achieved or the maximum number of levels have been reached ($k = p$). The efficiency in the approach lies in the fact that all the expensive evaluations used to compute I_{k-1} are stored and reused to compute I_k, significantly reducing the number of additional function evaluations required.

III. Numerical Results

We use an 2λ side-length cube to compare the new p-adaptive approach with the previous fixed-grid method for a challenging object with sharp edges and corners in a realistic setting. An edge-refinement change of variables is used to properly resolve the singular field-enhancement at the edges [2]. Fig. 1(a) visualizes the solved current surface density on the cube, and Fig. 1(b) plots the the relative error of the solved scattered field at a point in the far-field (19λ above the surface of the cube) vs. the number of unknowns. A highly refined numerical result is used as the reference solution. Fig. 1(c) and (d) compare the total number of MFIE kernel evaluations and total time needed to evaluate all the precomputation integrals for the two approaches respectively. In Fig. 1, "Fejér: Best case" refers to fixed-grid Fejér quadrature that has been manually fine-tuned over multiple simulation runs to use the smallest possible singular refinement grid for each discretization, such that the refinement does not limit the accuracy of the solution. On the other hand, the curve labeled "Fejér" uses the same fixed-grid for every discretization, corresponding to the refinement needed to achieve the lowest error for the highest mesh discretization. All cases were run on a server with dual AMD Epyc 7763 CPUs using 128 cores. Although both methods can achieve similar solution accuracy for a given discretization, the new adaptive quadrature approach requires up to 5.6X fewer kernel evaluations in most cases than the fixed-grid method, which results in up to 16.9X savings in wall-clock time. Thus, the p-adaptive integration method introduced in this paper not only enables automatic error control in the accuracy of the singular and near-singular interactions (removing the requirement for specifying a geometry dependent fixed-grid refinement parameter), but also leads to significant improvement in computing times.

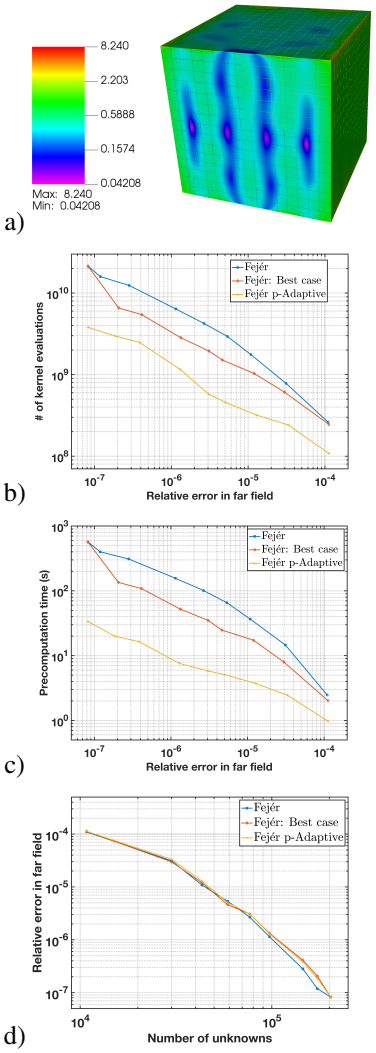

Fig. 1. Results using edge-refined PEC cube. a) Surface current density distribution ($\log(|\mathbf{J}|)$). b) Far-field error vs. number of unknowns. c) Number of kernel evaluations needed for precomputation integrals vs. solution accuracy. d) Total precomputation time vs. solution accuracy.

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