

K-stability for varieties with a big anticanonical class

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To Claire Voisin on the occasion of the conference in her honor

Abstract. We extend the algebraic K-stability theory to projective klt pairs with a big anticanonical class. While in general such a pair could behave pathologically, it is observed in this note that the K-semistability condition will force them to have a klt anticanonical model, whose stability property is the same as that of the original pair.

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Contents

1. Introduction	2
2. S-invariants	3
3. Finite generation	5

1. Introduction

There has been tremendous progress in algebraic K-stability theory of log Fano pairs (see [Xu21] for a survey of the topic). In the recent works [DZ22] and [DR22], the Kähler–Einstein problem is considered for a Kähler manifold (X, ω) such that $-K_X$ is big. More precisely, in [DZ22] the authors prove a transcendental Yau–Tian–Donaldson theorem for twisted big Kähler–Einstein metrics. As a consequence of their result, in the algebraic setting, uniform K-stability of X with a big anticanonical class implies the existence of a Kähler–Einstein metric. In this note we want to show that the K-stability theory in this setting, *i.e.* a projective klt pair with a big anticanonical class, essentially follows from the original (log) Fano case.

In general, there could be pathological examples in projective varieties X with a big anticanonical class $-K_X$; *e.g.* the anticanonical ring $R(X, -K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, -mK_X)$ is not necessarily finitely generated (see Example 3.8). However, we will show that the K-stability condition implies that X is of log Fano type.

Theorem 1.1. *Let (X, Δ) be a klt projective pair with $-K_X - \Delta$ big. Assume $\delta(X, \Delta) \geq 1$. Then there exists an effective $\mathbb{Q}$ -divisor Γ such that $(X, \Delta + \Gamma)$ is a log Fano pair, *i.e.* $(X, \Delta + \Gamma)$ is klt and $-K_X - \Delta - \Gamma$ is ample. In particular,*

$$R(X, -r(K_X + \Delta)) = \bigoplus_{m \in r \cdot \mathbb{N}} H^0(X, -m(K_X + \Delta))$$

is finitely generated for any r such that $r(K_X + \Delta)$ is Cartier.

Here $\delta(X, \Delta)$ is defined in the exactly same fashion as in the case when $-K_X - \Delta$ is ample (see [FO18, BJ20]), *i.e.*

$$\delta(X, \Delta) = \inf_E \frac{A_{X, \Delta}(E)}{S_{X, \Delta}(E)}.$$

For the stronger and more precise statement, see Theorem 3.4. We note that the above finite generation is asked in [DR22].

The above observation makes it possible to use existing birational geometry techniques to study K-stability questions for X with a big anticanonical class. In fact, without too much difficulty, it reduces K-stability questions for (X, Δ) to K-stability questions for its anticanonical model (Z, Δ_Z) , as we can see from the following statement.

Theorem 1.2. *Let (X, Δ) be a klt projective pair with $-K_X - \Delta$ big. Assume $R = \bigoplus_{m \in r \cdot \mathbb{N}} H^0(-m(K_X + \Delta))$ is finitely generated, and denote by (Z, Δ_Z) the anticanonical model. Then (X, Δ) is K-semistable (resp. K-stable, uniformly K-stable) if and only if (Z, Δ_Z) is K-semistable (resp. K-stable, uniformly K-stable). In particular, uniform K-stability of (X, Δ) is the same as K-stability of (X, Δ) .*

Remark 1.3. In [DR22], Ding stability notions for a projective klt pair (X, Δ) with big $-K_X - \Delta$ are developed. If one assumes $R = \bigoplus_{m \in r \cdot \mathbb{N}} H^0(-m(K_X + \Delta))$ is finitely generated and denotes by (Z, Δ_Z) the anticanonical model, then one can show a similar statement to Theorem 1.2; *i.e.* the Ding stability notions for (X, Δ) are equivalent to the notions for (Z, Δ_Z) .

Notation and Convention.— Throughout this paper, we work over an algebraically closed field $\mathbb{k}$ of characteristic 0. We follow the standard terminology from [KM98, Koll13].

For a normal log pair (X, Δ) such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier and a divisor E over X , we denote by $A_{X, \Delta}(E)$ the log discrepancy of E with respect to (X, Δ) .

We say a klt projective pair (X, Δ) is *log Fano* if (X, Δ) is klt and $-K_X - \Delta$ is ample, and a klt projective pair (X, Δ) is *of log Fano type* if there exists an effective $\mathbb{Q}$ -divisor D such that $(X, \Delta + D)$ is a log Fano pair.

We say an effective $\mathbb{Q}$ -divisor Γ on a projective log pair (X, Δ) is an N -*complement* for a positive integer N if $N(K_X + \Delta + \Gamma) \sim 0$ and $(X, \Delta + \Gamma)$ is log canonical. A $\mathbb{Q}$ -*complement* is an N -complement for some N .

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2. S-invariants

Let (X, Δ) be an n -dimensional projective normal pair such that $-K_X - \Delta$ is big. For any prime divisor E which appears on a birational model $\mu: Y \rightarrow X$, the S -invariant is defined as

$$S_{X, \Delta}(E) := \frac{1}{\text{vol}(-K_X - \Delta)} \int_0^\infty \text{vol}(-\mu^*(K_X + \Delta) - tE) dt.$$

Definition 2.1. If (X, Δ) is klt, we define

$$\delta(X, \Delta) := \inf_E \frac{A_{X, \Delta}(E)}{S_{X, \Delta}(E)},$$

where E runs through all valuations over (X, Δ) . We say (X, Δ) is uniformly K-stable (resp. K-semistable), if $\delta(X, \Delta) > 1$ (resp. $\delta(X, \Delta) \geq 1$). We say (X, Δ) is K-stable if $A_{X, \Delta}(E) > S_{X, \Delta}(E)$ for any E over X .

Remark 2.2. When (X, Δ) is log Fano, the equivalence between this way of defining K-stability notions using valuations and the original one using test configurations, called the Fujita–Li criterion, is proved in [Fuj19], [Li17] and [BX19]. For (X, Δ) with a big anticanonical class, the current definition is formulated in [DZ22].

Remark 2.3. Theorem 1.2 says K-stability is indeed the same as uniform K-stability. For a log Fano pair, this is proved in [LXZ22] (see [XZ22] for a different proof).

Fix $m \in r \cdot \mathbb{N}$, let $R_m = H^0(X, -m(K_X + \Delta))$, and assume $N_m := \dim H^0(X, -m(K_X + \Delta)) > 0$. Following [FO18], we say a $\mathbb{Q}$ -divisor D is an m -*basis type divisor* if

$$\frac{1}{m \cdot N_m} \text{ord}_E(\text{div}(s_1) + \cdots + \text{div}(s_{N_m}))$$

for a basis $\{s_1, \dots, s_{N_m}\}$ of R_m . In particular, $D \sim_{\mathbb{Q}} -K_X - \Delta$.

We define $S_{X, \Delta, m}(E)$ (or $S_m(E)$ if (X, Δ) is clear) for any E over X as follows: E yields a decreasing filtration $\mathcal{F}_E^\lambda$ ($\lambda \in \mathbb{R}$) on $R_m := H^0(X, -m(K_X + \Delta))$ by

$$\mathcal{F}_E^\lambda R_m = \left\{ s \in H^0(X, -m(K_X + \Delta)) \mid \text{ord}_E(s) \geq \lambda \right\},$$

and

$$S_m(E) = \frac{1}{m \cdot N_m} \text{ord}_E(\text{div}(s_1) + \cdots + \text{div}(s_{N_m}))$$

for any basis $\{s_1, \dots, s_{N_m}\}$ of R_m compatible with $\mathcal{F}_E^\lambda$ ($\lambda \in \mathbb{R}$). Here the basis $\{s_1, \dots, s_{N_m}\}$ is compatible with $\mathcal{F}_E^\lambda$ ($\lambda \in \mathbb{R}$) if for any λ , all the elements s_i contained in $\mathcal{F}_E^\lambda R_m$ span $\mathcal{F}_E^\lambda R_m$. Then

$$S_m(E) = \frac{1}{m \cdot N_m} \sum_{\lambda \in \mathbb{N}} \lambda \cdot \dim \text{Gr}_E^\lambda R_m,$$

where $\text{Gr}_E^\lambda R_m := \mathcal{F}_E^\lambda R_m / \mathcal{F}_E^{\lambda+1} R_m$.

We also define

$$\delta_m(X, \Delta) := \inf_E \frac{A_{X, \Delta}(E)}{S_m(E)}.$$

The following are basic properties proved in [BJ20].

Theorem 2.4. *Keep the notation as above.*

- (1) We have $\lim_{m \rightarrow \infty} S_m(E) = S(E)$.
- (2) For any $\varepsilon > 0$, there exists an m_0 such that for any E over X and $m \geq m_0$ with $m \in r \cdot \mathbb{N}$,

$$S_m(E) \leq (1 + \varepsilon) S(E).$$

- (3) We have $\delta_m(X, \Delta) = \inf_D \text{lct}(X, \Delta; D)$, where D runs through all m -basis type divisors.
- (4) We have $\lim_{m \rightarrow \infty} \delta_m(X, \Delta) = \delta(X, \Delta)$.

Proof. Statement (1) follows from [BJ20, Lemma 2.9] and (2) from [BJ20, Corollary 2.10]. Statement (3) is [BJ20, Proposition 4.3], and (4) is [BJ20, Theorem 4.4]. $\square$

We can consider more general filtrations.

Definition 2.5. By a (linearly bounded) filtration $\mathcal{F}$ on $R(X, -r(K_X + \Delta)) = \bigoplus_{m \in r \cdot \mathbb{N}} R_m$, we mean the data of a family $\mathcal{F}^\lambda R_m \subseteq R_m$ of $\mathbb{k}$ -vector subspaces of R_m for $m \in r \cdot \mathbb{N}$ and $\lambda \in \mathbb{R}$, satisfying

- (1) $\mathcal{F}^{\lambda'} R_m \subseteq \mathcal{F}^\lambda R_m$ when $\lambda \geq \lambda'$;
- (2) $\mathcal{F}^\lambda R_m = \bigcap_{\lambda' < \lambda} \mathcal{F}^{\lambda'} R_m$ for any λ ;
- (3) there exist $e_-, e_+ \in \mathbb{R}$ such that $\mathcal{F}^{me_-} R_m = R_m$ and $\mathcal{F}^{me_+} R_m = 0$ for any m ;
- (4) $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subseteq \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$.

For any filtration $\mathcal{F}$ on R , we can define $S_m(\mathcal{F})$ and $S(\mathcal{F})$ as in [BJ20, Sections 2.5 and 2.6, pp. 15–16], and we have

$$(2.1) \quad \lim_{m \rightarrow \infty} S_m(\mathcal{F}) \longrightarrow S(\mathcal{F});$$

see [BJ20, Lemma 2.9].

Lemma 2.6. *If A is an effective ample $\mathbb{Q}$ -divisor on X such that $-K_X - \Delta - A$ is pseudoeffective, then $S_{X, \Delta}(A) \geq \frac{1}{n+1}$.*

Proof. Since $-K_X - \Delta - A$ is pseudoeffective, for any $t \geq 0$, we have

$$\text{vol}(-K_X - \Delta - tA) \geq \text{vol}((1-t)(-K_X - \Delta)).$$

Thus

$$\begin{aligned} S(A) &= \frac{1}{\text{vol}(-K_X - \Delta)} \int_0^{+\infty} \text{vol}(-K_X - \Delta - tA) dt \\ &\geq \frac{1}{\text{vol}(-K_X - \Delta)} \int_0^1 \text{vol}((1-t)(-K_X - \Delta)) dt \\ &= \frac{1}{(n+1)}. \end{aligned}$$

$\square$

3. Finite generation

3.1. $\mathbb{Q}$ -complements and finite generation

For a $\mathbb{Q}$ -divisor D with $|rD| \neq \emptyset$ and any $m \in \mathbb{N}$, we denote by $\text{Bs}(|mrD|)$ the base ideal. We can define the log canonical threshold of the asymptotic linear series as follows:

$$\text{lct}(X, \Delta; \|\cdot\|) := \sup_{\ell} \text{lct}\left(X, \Delta; \frac{1}{\ell^r} \text{Bs}(|\ell^r(-K_X - \Delta)|)\right).$$

We can define a sequence of multiplier ideals

$$\mathcal{I}\left(X, \Delta; \frac{1}{r} \text{Bs}(|rD|)\right) \subseteq \mathcal{I}\left(X, \Delta; \frac{1}{2r} \text{Bs}(|2rD|)\right) \subseteq \cdots \subseteq \mathcal{I}\left(X, \Delta; \frac{1}{\ell!r} \text{Bs}(|\ell!rD|)\right) \subseteq \cdots.$$

By the ascending chain condition of ideals, this sequence will stabilize. We denote the maximal element by $\mathcal{I}(X, \Delta; \|\cdot\|)$ and call it the *asymptotic multiplier ideal sheaf* of D . For more background, see [Laz04, Section 11.1]. Recall that for any ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, we have $\text{lct}(X, \Delta; \mathfrak{a}) > 1$ if and only if $\mathcal{I}(X, \Delta; \mathfrak{a}) > 1$.

Lemma 3.1. *Assume (X, Δ) is a projective pair with $-K_X - \Delta$ big. If*

$$\text{lct}(X, \Delta; \|\cdot\|) > 1 \quad (\text{or equivalently } \mathcal{I}(X, \Delta; \|\cdot\|) = \mathcal{O}_X),$$

then (X, Δ) is of log Fano type.

Proof. From the assumption, there exists an effective $\mathbb{Q}$ -divisor $D \sim_{\mathbb{Q}} -(K_X + \Delta)$ such that $(X, \Delta + D)$ is klt. Since D is big, then $D \sim_{\mathbb{Q}} A + E$ for an ample $\mathbb{Q}$ -divisor A and an effective $\mathbb{Q}$ -divisor E . Set

$$\Gamma := (1 - \varepsilon)D + \varepsilon E$$

for $0 < \varepsilon \ll 1$; then $(X, \Delta + \Gamma)$ is klt, and $-K_X - \Delta - \Gamma \sim \varepsilon A$ is ample. Thus (X, Δ) is of log Fano type. $\square$

Definition 3.2. For any projective pair (X, Δ) , we define the constant $a(X, \Delta)$ by

$$(3.1) \quad a(X, \Delta) = \sup_{t \in \mathbb{R}} \left\{ \begin{array}{l} \text{there exists an ample divisor } A \text{ such that } A - t(K_X + \Delta) \\ \text{is ample and } -K_X - \Delta - A \text{ is pseudoeffective} \end{array} \right\}.$$

If $-K_X - \Delta$ is big, then $a(X, \Delta) > 0$; if $-K_X - \Delta$ is ample, then $a(X, \Delta) = +\infty$.

Assumption 3.3. Let (X, Δ) be an n -dimensional klt projective pair with $-K_X - \Delta$ big. Assume

$$(3.2) \quad \delta(X, \Delta) > \frac{n+1}{n+1+a_0}, \quad \text{where } a_0 = a(X, \Delta).$$

Now we can show the following.

Theorem 3.4. *Let (X, Δ) satisfy Assumption 3.3; then (X, Δ) is of log Fano type. In particular, any Cartier divisor E on X satisfies that $R(X, E) := \bigoplus_{m \in \mathbb{N}} H^0(X, mE)$ is finitely generated.*

Proof. Let us first prove this when $\delta(X, \Delta) > 1$ as it is quite straightforward. By Theorem 2.4, we know that for a sufficiently large m and any m -basis type divisor D ,

$$\text{lct}(X, \Delta; D) \geq \delta_m(X, \Delta) > 1.$$

Thus we can apply Lemma 3.1.

In the general case, we may assume $\delta(X, \Delta) \leq 1$, and we need some perturbation argument. By our definition of $a(X, \Delta)$, for any $t \in (0, a(X, \Delta))$, there exists an ample $\mathbb{Q}$ -divisor A such that

$$-K_X - \Delta - A \sim_{\mathbb{Q}} E_1 \quad \text{and} \quad A - t(K_X + \Delta) \sim_{\mathbb{Q}} A_0,$$

where E_1 is an effective $\mathbb{Q}$ -divisor and A_0 is an ample $\mathbb{Q}$ -divisor. Moreover, by (3.2) we may assume

$$(3.3) \quad 1 - \delta(X, \Delta) < \frac{t}{n+1} \delta(X, \Delta).$$

Fix $m_0 \in \mathbb{N}$ such that $|m_0 A|$ is base-point-free. Then for any prime divisor $H \in |m_0 A|$, by Lemma 2.6,

$$\begin{aligned} S(H) &= \frac{1}{\text{vol}(-K_X - \Delta)} \int \text{vol}(-K_X - \Delta - tH) dt \\ &= \frac{1}{m_0} S_{X, \Delta}(A) \\ &\geq \frac{1}{m_0(n+1)}. \end{aligned}$$

We can choose an m -basis type $\mathbb{Q}$ -divisor D_m compatible with H , so we can write $D_m = F_m + b_m H$, where

$$(3.4) \quad \lim_{m \rightarrow \infty} b_m = \lim_{m \rightarrow \infty} S_m(H) = S(H) \geq \frac{1}{m_0(n+1)}.$$

By (3.3), (3.4), and the equality $\lim_m \delta_m(X, \Delta) = \delta(X, \Delta)$, we can find a sufficiently large m and a positive δ' such that $\delta' < \min\{\delta_m(X, \Delta), 1\}$ and

$$(3.5) \quad 1 - \delta' < t m_0 b_m \delta'.$$

Then $(X, \Delta + \delta' F_m)$ is klt, as $(X, \Delta + \delta' D_m)$ is klt and $D_m = F_m + b_m H$. Moreover,

$$-K_X - \Delta - \delta' F_m \sim_{\mathbb{Q}} -(1 - \delta')(K_X + \Delta) + \delta' b_m H,$$

which implies $(X, \Delta + \delta' F_m)$ is a log Fano pair since

$$\begin{aligned} -(1 - \delta')(K_X + \Delta) + \delta' b_m H &\sim_{\mathbb{Q}} (1 - \delta') \left(-(K_X + \Delta) + \frac{1}{t} A \right) + \left(\delta' b_m m_0 - \frac{1 - \delta'}{t} \right) A \\ &\sim_{\mathbb{Q}} \frac{1 - \delta'}{t} A_0 + \left(\delta' b_m m_0 - \frac{1 - \delta'}{t} \right) A \end{aligned}$$

is ample by (3.5).

The last statement then follows from [BCH⁺10]. $\square$

Corollary 3.5. *Let (X, Δ) satisfy Assumption 3.3. Let $r(K_X + \Delta)$ be Cartier and $Z := \text{Proj } R(X, -r(K_X + \Delta))$. Denote by Δ_Z the birational transform of Δ on Z ; then (Z, Δ_Z) is a log Fano pair.*

Proof. We know $f: X \dashrightarrow Z$ is a birational contraction; i.e. $\text{Ex}(f^{-1})$ does not contain any divisor, and $f_*(K_X + \Delta) = K_Z + \Delta_Z$ is antiample.

It follows from Theorem 3.4 that there exists a $\mathbb{Q}$ -complement Γ for (X, Δ) such that $(X, \Delta + \Gamma)$ is klt. Then $(Z, \Delta_Z + f_*\Gamma)$ is klt as the pullbacks of $K_Z + \Delta_Z + f_*\Gamma$ and $K_X + \Delta + \Gamma$ on a common resolution are equal. So (Z, Δ_Z) is klt. $\square$

3.2. K-stability of the anticanonical model

Let (X, Δ) be a projective log pair with big $-K_X - \Delta$. Let (Z, Δ_Z) be its anticanonical model; i.e. $Z = \text{Proj } R(X, -r(K_X + \Delta))$, and Δ_Z is the birational transform of Δ on Z . Let Y be a common resolution.

$$(3.6) \quad \begin{array}{ccc} & Y & \\ \mu \swarrow & & \searrow \pi \\ (X, \Delta) & \xrightarrow{f} & (Z, \Delta_Z). \end{array}$$

Then

$$\pi^*(K_Z + \Delta_Z) - \mu^*(K_X + \Delta) = B \geq 0$$

Lemma 3.6. *Let (X, Δ) satisfy Assumption 3.3. Then for any prime divisor E over X ,*

$$A_{X,\Delta}(E) = A_{Z,\Delta_Z}(E) + \text{ord}_E(B) \text{ and } S_{X,\Delta}(E) = S_{Z,\Delta_Z}(E) + \text{ord}_E(B).$$

Proof. For the log discrepancy function, this follows directly from the definition. Since

$$|\mu^*(-m(K_X + \Delta))| = |\pi^*(-m(K_Z + \Delta_Z))| + mB,$$

we have $S_{X,\Delta,m}(E) = S_{Z,\Delta_Z,m}(E) + \text{ord}_E(B)$. Therefore, the same is true for the S -function. $\square$

Lemma 3.7. *If (Z, Δ_Z) is klt, there exists a $t > 0$ depending on Z (but not E) such that for any divisor E over X*

$$A_{Z,\Delta_Z}(E) \geq t \cdot \text{ord}_E(B).$$

Proof. Since (Z, Δ_Z) is klt, we know that there exists a $t > 0$ such that if we write $\pi^*(K_Z + \Delta_Z) = K_Y + \Delta_1$, then $(K_Y + \Delta_1 + tB)$ is sub-lc for some $t > 0$; i.e. for any E ,

$$A_{Z,\Delta_Z}(E) \geq t \cdot \text{ord}_E(B). \quad \square$$

Proof of Theorem 1.2. Since

$$\delta(X, \Delta) = \inf_E \frac{A_{Z,\Delta_Z}(E) + \text{ord}_E(B)}{S_{Z,\Delta_Z}(E) + \text{ord}_E(B)},$$

it is clear that $\delta(X, \Delta) \geq 1$ if and only if $A_{Z,\Delta_Z}(E) \geq S_{Z,\Delta_Z}(E)$, i.e. (Z, Δ) is klt and $\delta(Z, \Delta_Z) \geq 1$. Moreover,

$$A_{X,\Delta}(E) = A_{Z,\Delta_Z}(E) + \text{ord}_E(B) > S_{Z,\Delta_Z}(E) + \text{ord}_E(B) = S_{X,\Delta}(E)$$

if and only if $A_{Z,\Delta_Z}(E) > S_{Z,\Delta_Z}(E)$.

Assume $\delta(X, \Delta) > 1$; then $\delta(Z, \Delta_Z) \geq \delta(X, \Delta)$. Conversely, if $\delta(Z, \Delta_Z) > 1$, an easy calculation shows that

$$\delta(X, \Delta) \geq \frac{\delta(Z, \Delta_Z)(t+1)}{\delta(Z, \Delta_Z) + t} > 1,$$

where t is the constant from Lemma 3.7. $\square$

Example 3.8. This example has appeared in several works to present pathological phenomena, see e.g. [Gon12]: Let S be the blowup of $\mathbb{P}^2$ at nine very general points. Then $-K_S$ is known to be nef but not semiample. In fact, there will be a unique cubic curve passing through these nine points, and if we denote by E its birational transform on S , then for any $m \in \mathbb{N}$, $|-mK_S|$ has one element mE .

Let H be an ample Cartier divisor on S and $X = \mathbb{P}_S(E)$, where $E := \mathcal{O}_S + \mathcal{O}_S(H)$. Denote by $\pi: X \rightarrow S$ the natural morphism. We claim $-K_X$ is big. In fact, since

$$\omega_{X/S} = \wedge^2 \mathcal{O}_{\mathbb{P}(E)}(-2),$$

we have

$$\begin{aligned} H^0(\mathcal{O}_X(-mK_X)) &= H^0(S, \pi_*(\mathcal{O}_X(-mK_X))) \\ &= H^0(S, \text{Sym}^{2m}(E) \otimes (\wedge^2 E)^{\otimes -m} \otimes \omega_S^{\otimes m}) \\ &= H^0\left(S, \left(\bigoplus_{i=0}^{2m} \mathcal{O}_S(iH)\right) \otimes \mathcal{O}_S(-mH - mK_S)\right) \\ &= H^0\left(S, \bigoplus_{i=0}^m \mathcal{O}_S(iH - mK_S)\right), \end{aligned}$$

and since $-K_S \sim E$ is nef, we have

$$\begin{aligned} \text{vol}_X(-K_X) &= 6 \int_0^1 \frac{1}{2} (tH - K_S)^2 = 3 \int_0^1 (t^2 H^2 - 2tH(-K_S)) \\ &= H^2 + 3H \cdot (-K_S) > 0. \end{aligned}$$

However, the algebra $\bigoplus_{i \leq m} H^0(iH - mK_S)$ is not finitely generated, since

$$\sum_{1 \leq j \leq m-1} H^0(\mathcal{O}_S(-jK_S)) \otimes H^0(\mathcal{O}_S(H - (m-j)K_S)) \longrightarrow H^0(\mathcal{O}_S(H - mK_S))$$

is not surjective for any m . Thus we need generators from $H^0(\mathcal{O}_S(H - mK_S))$ for every m .

By Theorem 1.1, we know $\delta(X) < 1$. Here we give a direct verification of this. We denote by $Y \subseteq X$ the section given by

$$E = \mathcal{O}_S \oplus \mathcal{O}_S(H) \longrightarrow \mathcal{O}_S.$$

Then similarly to before, we have

$$H^0(\mathcal{O}_X(-mK_X - m_0Y)) = H^0\left(S, \left(\bigoplus_{i=m_0}^{2m} \mathcal{O}_S(iH)\right) \otimes \mathcal{O}_S(-mH - mK_S)\right),$$

where we follow the convention that if $m_0 > 2m$, then the direct sum is 0. Hence a direct calculation implies

$$\text{vol}(-K_X - tY) = \begin{cases} H^2 + 3H \cdot (-K_S) & \text{if } t \leq 1, \\ (2-t)((t^2 - t + 1)H^2 + 3tH \cdot (-K_S)) & \text{if } 1 \leq t \leq 2. \end{cases}$$

By an elementary calculation,

$$S_X(Y) = \frac{\frac{7}{4}H^2 + 5H \cdot (-K_S)}{H^2 + 3H \cdot (-K_S)} > \frac{5}{3} > 1 = A_X(Y),$$

which implies $\delta(X) < \frac{3}{5}$.

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