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COMPLEX DYNAMICS INDUCED BY ADDITIVE ALLEE EFFECT IN A LESLIE-GOWER PREDATOR-PREY MODEL

Yue Yang[1,2], Fanwei Meng[1], Yancong Xu[*2] and Libin Rong[3]

[1]School of Mathematical Sciences, Qufu Normal University, Qufu, China

[2]Department of Mathematics, China Jiliang University, Hangzhou, China

[3]Department of Mathematics, University of Florida, FL, USA

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Abstract. In this paper, we investigate the complex dynamics of a predator-prey model, specifically the Leslie-Gower model, with additive Allee effect and simplified Holling III functional response. The model has been analyzed for various bifurcations, including the nilpotent cusp singularity of codimension 3, Bogdanov-Takens bifurcation of codimension 3, and Hopf bifurcation of codimension 2. Additionally, a codimension 2 cusp of limit cycles and the coexistent acute angle region of three limit cycles have been identified. Notably, the isola bifurcation of limit cycles, which indicates a new mechanism of sustained oscillation, has been observed for the first time in a Leslie-Gower predator-prey model with additive Allee effect. One-parameter and two-parameter bifurcation diagrams and corresponding phase portraits have been presented to verify the theoretical results. From a biological perspective, the additive Allee effect may result in system collapse, leading to the extinction of the predator population and survival of the prey. The finding of the isola bifurcation of limit cycles is of significant interest, highlighting a novel mechanism of sustained oscillation in this complex system.

1. **Introduction.** The predator-prey model as one of the typical ecosystems has played a significant role in the theoretical studies of biology and mathematics. Among them the Leslie-Gower predator-prey model taking the form

$$\begin{aligned}\frac{dx}{dt} &= rx(1-\frac{x}{k}) - yp(x),\\ \frac{dy}{dt} &= sy(1-\frac{y}{nx}),\end{aligned}$$

was introduced by Leslie [33, 34] to describe a scenario in which both growth rates of two species have upper limits, especially, the environment carrying capacity of the predator population represented as nx is proportional to the size of prey population.

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*Corresponding author: Yancong Xu, Yancongx@cjlu.edu.cn.

Given this, investigators have achieved fruitful results in investigating the Leslie-Gower predator-prey model [1, 2, 26, 29, 5, 42, 40, 50]. This paper deals with a Leslie-Gower predator-prey system incorporating the following two aspects: (i) the Allee effect in the growth function for prey, and (ii) simplified Holling III type functional response of predators.

Allee effect, started from the pioneer work of the well-known ecologist W.C. Allee [49], refers to any mechanism leading to a positive relationship between fitness and population density of any species [36, 25] and corresponds to a density-mediated reduction in the intrinsic growth rate of a population at low-density [19]. It originates from multiple factors, such as mate finding, inbreeding depression, foraging efficiency, anti-predator behavior, and environment conditioning etc. [44, 22, 23, 7, 8] and has attracted much attention in the last few decades [26, 40, 46, 27, 30, 56, 11, 41] because of its crucial potential effect on biological control and population dynamics. To describe Allee effect, diverse modeling methods have been considered, and the most typical form for a single population with Allee effect is performed as follows [15, 6]

$$\frac{dx}{dt} = r(1 - \frac{x}{K})(x - m)x,$$

here $x(t)$ is the population density at time t, r and K represent the intrinsic growth rate and the environmental carrying capacity, respectively. The term $x - m$ is deemed as Allee effect term, further, $0 < m \ll K$ corresponds to strong Allee effect, which indicates that there exists an extinction threshold and the population will go extinct if the initial size is below the threshold level [7, 46, 47], and $-K < m \leq 0$ describes weak Allee effect without such a threshold value, implying that the per capita growth rate is still positive despite decreases under low population density [47, 43].

Additive Allee effect has become the focus of public concern in recent years [1, 2, 19, 44, 11, 3, 48, 31, 54, 55, 37]. It was derived in [19, 44] with the following form

$$\frac{dx}{dt} = (r(1 - \frac{x}{K}) - \frac{m}{x + b})x, \tag{1}$$

here r, K occupy the same meaning as above, and the term $\frac{m}{x+b}$ is known as additive Affect effect. When $0 < m < br$, equation (1) describes the strong Allee effect and if $m > br$, it is said that the population is influenced by a weak Allee effect. Aguirre et al. [1] studied the existence of two limit cycles in a Leslie-Gower predator-prey model with additive Allee effect and simplified Holling IV functional response. Cai et al. [11] developed a Leslie-Gower predator-prey model with additive Allee effect and Holling II functional response, and presented the existence and stability of equilibria. Furthermore, they established the conditions for the existence of Hopf bifurcation. Lai et al. [31] investigated the stability and bifurcation in a predator-prey model combining the additive Allee effect with fear effect on prey species, and analyzed saddle-node bifurcation, transcritical bifurcation, and Hopf bifurcation. Molla et al. [37] proposed a predator-prey model with Holling type II response function, considering both variable prey refuge and additive Allee effect in the prey. They demonstrated that these two factors can lead to saddle-node bifurcation, Hopf bifurcation, or Bogdanov-Takens bifurcation.

Note that, isolas are isolated closed curves of solution branches while each isola always has two saddle-node bifurcation points, and the solution may be equilibrium, homoclinic cycle, heteroclinic cycle or limit cycle [4, 52, 17, 18]. This phenomenon has received significant attention in biological and chemical systems [24, 39]. The isola of equilibrium is relevant to the mushroom phenomenon [24], while the presence of an isola center of limit cycle indicates a new mechanism for sustained oscillations. Sandstede and Xu [39] derived the conditions that guarantee snaking or result in diagrams that either consist entirely of isolas. Aougab et al. [4] researched snaking branches of spatially localized stationary patterns that show localized rolls, which consist of isolas or of intertwined s-shaped curves. They promoted the results concerning orientable stable and unstable manifolds of rolls to the nonorientable case and further, discussed topological barriers that prevent snaking, thus allowing only isolas to occur. Xu et al. [52] firstly detected numerically isola bifurcation of periodic orbits in HIV model, which means that there is a parameter interval with the same oscillations.

The Leslie-Gower predator-prey model with Holling III functional response, but without an additive Allee effect, has received a lot of attention in recent decades [29, 28, 14, 53, 16, 51, 12]. Researchers, such as Hsu and Huang [28], have explored the global stability of the model and proposed the existence of one limit cycle. Others, like Collings [14], have studied the global stability and bifurcation behavior of a mite predator-prey system with a simplified Holling III functional response. They have shown the existence of Hopf bifurcation and saddle-node bifurcation of limit cycles and provided concrete stability regions, especially the bistability region. Huang et al. [29] have analyzed a Leslie-type predator-prey model with generalized Holling type III functional response and shown that the model can undergo Hopf bifurcation and degenerate focus type Bogdanov-Takens bifurcation of codimension 3 under suitable parameters, which illustrates the coexistence of two limit cycles. However, the dynamics of the Leslie-Gower predator-prey model with Holling III functional response and additive Allee effect has not been characterized yet. Inspired by the above results, in this paper, we will investigate a Leslie-Gower predator-prey system with additive Allee effect and simplified Holling III functional response. The system is represented by the following set of equations

$$\begin{aligned} \frac{dx}{dt} &= (r(1-\frac{x}{k}) - \frac{m}{x+b})x - \frac{qx^2y}{x^2+a}, \\ \frac{dy}{dt} &= sy(1-\frac{y}{nx}), \end{aligned} \tag{2}$$

where $x(t)$ and $y(t)$ denote the prey and predator populations at time t, respectively. The intrinsic growth rates of the prey and predator populations are denoted by r and s, respectively. The maximum rate of predation is represented by q, and a represents the half-saturation constant. The environmental carrying capacity of the prey is indicated by k, and the food quality that the prey provides for conversion into predator births is represented by n. The term $\frac{m}{x+b}$ represents the additive Allee effect, where m measures the degree of Allee effect, and b denotes the population size of the prey species at which fitness is half its maximum value. It is important to note that the Allee effect is considered weak if $0 < m < br$, and strong if $m > br$. The system is analyzed in the region $\Omega_1 = \{(x,y)|x > 0, y \geq 0\}$, and all parameters are assumed to be positive.

The article is organized as follows. Section 2 presents the asymptotic dynamics near $(0,0)$, the existence and types of equilibria. Bifurcation analysis, including

Bogdanov-Takens bifurcation of codimension 2 and 3, and Hopf bifurcation of codimension 2, is presented in Section 3. In Section 4, we carry out numerical simulations to verify our theoretical results. Biological interpretation is given in Section 5. Finally, we conclude this paper with conclusion and discussion.

2. The asymptotic dynamics near $(0, 0)$ and the analysis of equilibria.

2.1. The asymptotic dynamics near $(0, 0)$.

Noting that system (2) is not well defined at $x = 0$, it inspires us to think of that whether a blow-up transformation can be used to investigate the asymptotic dynamics near $(0, 0)$. Actually, our analysis is as follows.

Lemma 2.1. *The trajectories near* $(0, 0)$ *of system (2) with initial values located in the region* $\Omega_1 = \{(x, y)|x > 0, y \geq 0\}$ *will leave* $(0, 0)$ *when* $m + b(s - r) < 0$ *or when* $m + b(s - r) > 0, m - br < 0$*, and will be attracted to* $(0, 0)$ *when* $m - br > 0$.

Proof. Firstly, making transformation $dt = knx(x+b)(x^2+a)d\tau$ yields a polynomial system equivalent to system (2) in the region Ω_1 taking the form

$$\begin{aligned} \frac{dx}{d\tau} &= nx^2(x^2 + a)(r(x + b)(k - x) - mk) - knq(x + b)x^3y, \\ \frac{dy}{d\tau} &= ksy(x + b)(x^2 + a)(nx - y). \end{aligned} \tag{3}$$

It's worthy noting that $(0, 0)$ is an equilibrium of system (3) while the Jacobian matrix of system (3) at $(0, 0)$ is a null matrix. Taking the blow-up transformation $x = R\cos\theta, y = R\sin\theta, t = R\tau$, we can obtain that

$$\begin{aligned} \frac{dR}{dt} &= R(ka((br - m)n\cos^3\theta + bs\sin^2\theta(n\cos\theta - \sin\theta)) + O(R)), \\ \frac{d\theta}{dt} &= ka\sin\theta\cos\theta((m + b(s - r))n\cos\theta - bs\sin\theta) + O(R), \end{aligned} \tag{4}$$

where $(R, \theta) \in [0, +\infty) \times [0, \frac{\pi}{2}]$.

After a simple qualitative analysis, we know that when $m + b(s - r) < 0$, system (4) has two saddles $(0, 0)$ and $(0, \frac{\pi}{2})$. When $m + b(s - r) > 0$, system (4) has three equilibria $(0, 0)$, a saddle $(0, \frac{\pi}{2})$ and $(0, \arctan\frac{(m+b(s-r))n}{bs})$. Further, for $m - br < 0$, $(0, 0)$ is an unstable node and $(0, \arctan\frac{(m+b(s-r))n}{bs})$ is a saddle; for $m - br > 0$, $(0, 0)$ is a saddle and $(0, \arctan\frac{(m+b(s-r))n}{bs})$ is a stable node.

Hence, we can conclude that the trajectories near $(0, 0)$ of system (2) with initial values located in Ω_1 will leave $(0, 0)$ when $m + b(s - r) < 0$ (see Figure 1 (a) (b)) or when $m + b(s - r) > 0, m - br < 0$ (see Figure 1 (c) (d)) and will be attracted to $(0, 0)$ when $m - br > 0$ (see Figure 1 (e) (f)). □

2.2. The analysis of equilibria.

In this subsection, we study the existence and the type of equilibria in system (2). Firstly, we present the following result.

Lemma 2.2. *The positive invariant set of system (2) is the rectangular region* $\Omega_2 = \{(x, y)|0 < x \leq k, 0 \leq y \leq nk\}$.

Proof. By the first equation of system (2), we have $\frac{dx}{dt}|_{x>k} < 0$. Thus, we only focus on $0 < x \leq k$. On the other hand, we can easily get $\frac{dy}{dt} = sy(1 - \frac{y}{nx}) \leq sy(1 - \frac{y}{nk})$ for $0 < x \leq k$, which leads to $\frac{dy}{dt}|_{y>nk} < 0$. Therefore, all solutions of system (2) will ultimately move towards the region $\Omega_2 = \{(x, y)|0 < x \leq k, 0 \leq y \leq nk\}$ and this ends the proof. □

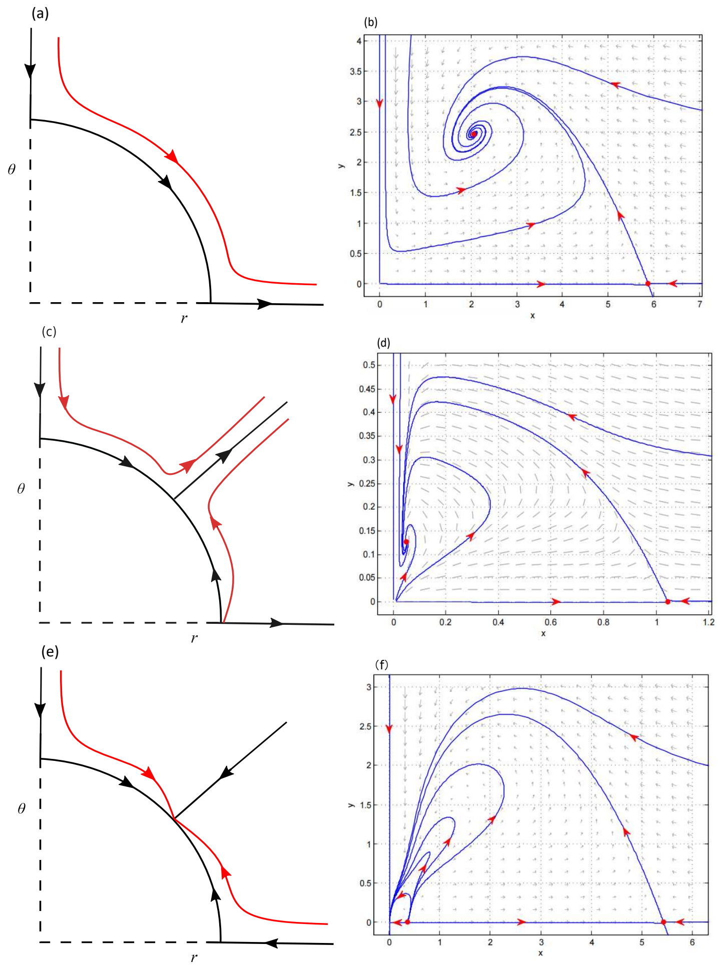

FIGURE 1. (a) Two equilibria of system (4) when $m+b(s-r)<0$. (b) The trajectories near $(0,0)$ of system (2) when $m+b(s-r)<0$. (c) Three equilibria of system (4) when $m+b(s-r)>0, m-br<0$. (d) The trajectories near $(0,0)$ of system (2) when $m+b(s-r)>0, m-br<0$. (e) Three equilibria of system (4) when $m-br>0$. (f) The trajectories near $(0,0)$ of system (2) when $m-br>0$.

For the boundary equilibria of system (2), by a simple qualitative analysis, we have the following Theorem.

Lemma 2.3. *System (2) has*
(I) no boundary equilibrium if $m>\frac{r(b+k)^2}{4k}$;
(II) a unique boundary equilibrium $B(\frac{k-b}{2},0)$ *if* $m=\frac{r(b+k)^2}{4k}$, *which is a degenerate equilibrium;*
(III) two boundary equilibria $B_1(\frac{1}{2}(k-b+\sqrt{\frac{r(b+k)^2-4km}{r}}),0)$ *and* $B_2(\frac{1}{2}(k-b-\sqrt{\frac{r(b+k)^2-4km}{r}}),0)$ *if* $m<\frac{r(b+k)^2}{4k}$.

For the existence of positive equilibria $E(x,y)$, a direct calculation indicates $y=nx$ and x being a positive root of

$$F(x)=\frac{-x(a_4x^4+a_3x^3+a_2x^2+a_1x+a_0)}{h(x)}, \tag{5}$$

where $a_4=r, a_3=knq+r(b-k), a_2=ar+k(bnq+m-br), a_1=ar(b-k), a_0=ak(m-br)$ and $h(x)=k(b+x)(a+x^2)$. Then we can easily obtain that $F'(x)=-\frac{nqx^2(3a+x^2)}{(a+x^2)^2}-\frac{bm}{(b+x)^2}-\frac{2rx}{k}+r$ and $F''(x)=\frac{2anqx(x^2-3a)}{(a+x^2)^3}+\frac{2bm}{(b+x)^3}-\frac{2r}{k}$.

Introduce the following notion and equation

$$\bar{F}_0(x):=\bar{F}(x)=a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0. \tag{6}$$

Because the denominator $h(x)$ of $F(x)$ is greater than zero, it is adequate to analyze the positive roots of equation $\bar{F}_0(x)$. According to Descartes's rule of signs, $\bar{F}_0(x)$ has at most four positive roots, which are expressed by size as $x_1<x_2<x_3<x_4$. The corresponding equilibria of system (2) are marked as $E_1(x_1,y_1), E_2(x_2,y_2)$, $E_3(x_3,y_3)$ and $E_4(x_4,y_4)$, respectively. For the specific conditions of the existence of positive equilibria in system (2), see Appendix A.

Now we turn to studying the type of positive equilibria in system (2). The Jacobian matrix of system (2) restricted to any positive equilibrium $E(x,nx)$ can be reduced to

$$J(E)=\begin{pmatrix} -\frac{2anqx^2}{(a+x^2)^2}-\frac{bm}{(b+x)^2}-\frac{2rx}{k}+r & -\frac{qx^2}{a+x^2} \\ ns & -s \end{pmatrix}.$$

It follows that

$$\begin{aligned} \det(J(E)) &= s\left(\frac{nqx^2(3a+x^2)}{(a+x^2)^2}+\frac{bm}{(b+x)^2}+r\left(\frac{2x}{k}-1\right)\right)=-sF'(x),\\ \operatorname{tr}(J(E)) &= -\frac{2anqx^2}{(a+x^2)^2}-\frac{bm}{(b+x)^2}-\frac{2rx}{k}+r-s \doteq p(x),\\ p'(x) &= \frac{4anqx(x^2-a)}{(a+x^2)^3}+\frac{2bm}{(b+x)^3}-\frac{2r}{k}. \end{aligned} \tag{7}$$

Through a simple qualitative analysis, we get the type of positive equilibria in system (2) as follows.

Theorem 2.4. *Supposing that $E_i(x_i,y_i)(i=1,2,3,4)$ are the simple positive equilibria of system (2), we have*
(I) if $F'(x_i)>0$, then $E_i(x_i,y_i)$ is a saddle;
(II) if $F'(x_i)<0$ and $p(x)<0$, then $E_i(x_i,y_i)$ is a stable node or focus;
(III) if $F'(x_i)<0$ and $p(x)>0$, then $E_i(x_i,y_i)$ is an unstable node or focus.
Here $F(x)$ and $p(x)$ are defined by (5) and (7), respectively.

Remark 2.5. Assuming that $E_i(x_i,y_i)(i=1,2,3,4)$ are the simple positive equilibria of system (2), we have $F'(x_i)=-\frac{\bar{F}(x_i)}{h(x_i)}-\frac{x_i(\bar{F}'(x_i)h(x_i)-\bar{F}(x_i)h'(x_i))}{h^2(x_i)}=-\frac{x_i\bar{F}'(x_i)}{h(x_i)}$. It is evident that $F'(x_1), F'(x_3)>0$ from the analysis in Appendix A, which implies $E_1(x_1,y_1)$ and $E_3(x_3,y_3)$ are saddles.

Based on the analysis of Appendix A, we can see that under suitable conditions, $\bar{F}_0(x)$ may possess positive roots $x_{i,i+1}(i=1,2,3)$ of multiplicity 2. This implies that two equilibria $E_i(x_i,y_i)$ and $E_{i+1}(x_{i+1},y_{i+1})$ may coincide at $E_{i,i+1}(x_{i,i+1},y_{i,i+1})$

for $i = 1, 2, 3$. Our attention will now shift to determining the type of these equilibria. For the sake of simplicity, we will express them uniformly as $E^*(x^*, nx^*)$. In fact, we can arrive at the following conclusion.

Theorem 2.6. *Suppose that $F(x^*) = F'(x^*) = 0$ and $F''(x^*) \neq 0$, then we have*
(I) if $p(x^*) \neq 0$, then $E^*(x^*, nx^*)$ is a saddle-node;
(II) if $p(x^*) = 0$ and $p'(x^*) \neq 0$, then $E^*(x^*, nx^*)$ is a cusp of codimension 2;
(III) if $p(x^*) = p'(x^*) = 0$ and $b \neq b_\pm$, then $E^*(x^*, nx^*)$ is a cusp of codimension 3, where $b_\pm = \frac{-A \pm x^*(a+x^{*2})^2\sqrt{B}}{4(a^3-2a^2x^{*2}-10ax^{*4}+5x^{*6})}$ and $A = 8a^3x^* + 11a^2x^{*3} - 38ax^{*5} + 7x^{*7}$, $B = 32a^2 - 8ax^{*2} + 9x^{*4}$.
Here $F(x)$ and $p(x)$ are defined by (5) and (7), respectively.

Proof. Firstly, we prove statement (I) of Theorem 2.6. It follows from $F(x^*) = F'(x^*) = 0$ that $m = \frac{r(b+x^*)^2(2ak-ax^*+x^{*3})}{k(2ab+3ax^*+x^{*3})}$ and $q = \frac{r(a+x^{*2})^2(k-b-2x^*)}{knx^*(2ab+3ax^*+x^{*3})}$. Substituting them to system (2) and making the transformation $X = x - x^*, Y = y - nx^*$ to shift system (2) around $E^*(x^*, nx^*)$ to the normal form system around the origin, we obtain (for convenience, in subsequent steps, we still denote X, Y and τ by x, y and t, respectively)

$$\begin{aligned}
\frac{dx}{dt} &= \hat{a}_{10}x + \hat{a}_{01}y + \hat{a}_{20}x^2 + \hat{a}_{11}xy + \hat{a}_{30}x^3 + \hat{a}_{21}x^2y + o(|x, y|^3),\\
\frac{dy}{dt} &= \hat{b}_{10}x + \hat{b}_{01}y + \hat{b}_{20}x^2 + \hat{b}_{11}xy + \hat{b}_{02}y^2 + \hat{b}_{30}x^3 + \hat{b}_{21}x^2y + \hat{b}_{12}xy^2\\
&\quad + o(|x, y|^3),
\end{aligned} \tag{8}$$

in which $\hat{a}_{ij}$ and $\hat{b}_{ij}$ are given in Appendix B.

Making the linear transformation of variables $X = \frac{\hat{b}_{01}x - \hat{a}_{01}y}{\hat{a}_{10}+\hat{b}_{01}}, Y = \frac{\hat{a}_{10}x + \hat{a}_{01}y}{\hat{a}_{10}+\hat{b}_{01}}$, system (8) is further transformed to

$$\begin{aligned}
\frac{dx}{dt} &= \hat{c}_{20}x^2 + \hat{c}_{11}xy + \hat{c}_{02}y^2 + \hat{c}_{30}x^3 + \hat{c}_{21}x^2y + \hat{c}_{12}xy^2 + \hat{c}_{03}y^3 + o(|x, y|^3),\\
\frac{dy}{dt} &= \hat{d}_{01}y + \hat{d}_{20}x^2 + \hat{d}_{11}xy + \hat{d}_{02}y^2 + \hat{d}_{30}x^3 + \hat{d}_{21}x^2y + \hat{d}_{12}xy^2 + \hat{d}_{03}y^3\\
&\quad + o(|x, y|^3),
\end{aligned} \tag{9}$$

in which $\hat{c}_{ij}$ and $\hat{d}_{ij}$ are given in Appendix B.

Note that $\hat{d}_{01} = p(x^*) \neq 0$, it leads to a center manifold

$$y = -\frac{\hat{d}_{20}}{\hat{d}_{01}}x^2 + o(x^2),$$

occuring in a small neighborhood of the origin. System (9) restricted to this manifold is expressed as

$$\frac{dx}{dt} = \hat{c}_{20}x^2 + o(x^2), \tag{10}$$

in which $\hat{c}_{20} = \frac{ks(2ab+3ax^*+x^{*3})}{2C}F''(x^*)$ with $C = ab(2ks + rx^*) + ax^*(-kr + 3ks + 2rx^*) + x^{*3}(br + k(s - r) + 2rx^*)$. In view of $F''(x^*) \neq 0$, we have that $\hat{c}_{20} \neq 0$ and (10) is topologically equivalent to

$$\frac{dx}{dt} = \pm x^2 + o(x^2).$$

Hence, equilibrium $E^*(x^*, nx^*)$ is a saddle-node, which finishes the proof of statement (I).

Next, we give the proof of statement (II). According to $F(x^*) = F'(x^*) = p(x^*) = 0$, we can easily obtain that $m = \frac{r(b+x^*)^2(2ak-ax^*+x^{*3})}{k(2ab+3ax^*+x^{*3})}$, $q = \frac{r(a+x^{*2})^2(-b+k-2x^*)}{knx^*(2ab+3ax^*+x^{*3})}$ and $s = \frac{rx^*(a+x^{*2})(-b+k-2x^*)}{k(2ab+3ax^*+x^{*3})}$, which are substituted to system (2). By the transformation $X = x - x^*, Y = y - nx^*$, we have (for convenience, in subsequent steps, we still denote X, Y and τ by x, y and t, respectively)

$$\begin{aligned}\frac{dx}{dt} &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + o(|x,y|^2),\\ \frac{dy}{dt} &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + o(|x,y|^2),\end{aligned}$$

where

$$\begin{aligned}a_{10} &= \frac{rx^*(a+x^{*2})(-b+k-2x^*)}{k(2ab+3ax^*+x^{*3})}, \quad a_{01} = \frac{rx^*(a+x^{*2})(b-k+2x^*)}{kn(2ab+3ax^*+x^{*3})},\\ a_{20} &= -\frac{r}{k(a+x^{*2})(b+x^*)(2ab+3ax^*+x^{*3})}(a^2(b^2-b(k-3x^*)+x^*(k\\ &\quad +x^*)) + ax^{*2}(5b^2-5b(k-3x^*)+x^*(10x^*-3k)) + x^{*6}),\\ a_{11} &= \frac{2ar(b-k+2x^*)}{kn(2ab+3ax^*+x^{*3})}, \quad b_{10} = \frac{nrx^*(a+x^{*2})(-b+k-2x^*)}{k(2ab+3ax^*+x^{*3})},\\ b_{01} &= \frac{rx^*(a+x^{*2})(b-k+2x^*)}{k(2ab+3ax^*+x^{*3})}, \quad b_{20} = \frac{nr(a+x^{*2})(b-k+2x^*)}{k(2ab+3ax^*+x^{*3})},\\ b_{11} &= \frac{2r(a+x^{*2})(-b+k-2x^*)}{k(2ab+3ax^*+x^{*3})}, \quad b_{02} = \frac{r(a+x^{*2})(b-k+2x^*)}{kn(2ab+3ax^*+x^{*3})}.\end{aligned}$$

Under the change of coordinates and time rescaling $X = x, Y = -\frac{b_{10}}{a_{10}}x + y, dt = \frac{1}{a_{01}}d\tau$, one has

$$\begin{aligned}\frac{dx}{dt} &= y + c_{20}x^2 + c_{11}xy + o(|x,y|^2),\\ \frac{dy}{dt} &= d_{20}x^2 + d_{11}xy + d_{02}y^2 + o(|x,y|^2),\end{aligned} \tag{11}$$

where

$$\begin{aligned}c_{20} &= \frac{a_{11}b_{10} + a_{10}a_{20}}{a_{01}a_{10}}, \quad c_{11} = \frac{a_{11}}{a_{01}},\\ d_{20} &= \frac{a_{10}^2b_{20} + a_{10}b_{10}(b_{11}-a_{20}) + b_{10}^2(b_{02}-a_{11})}{a_{01}a_{10}^2},\\ d_{11} &= \frac{a_{10}b_{11} - b_{10}(a_{11}-2b_{02})}{a_{01}a_{10}}, \quad d_{02} = \frac{b_{02}}{a_{01}}.\end{aligned}$$

Applying Lemma 3.1 in Perko [38] to system (11), we can get an equivalent system given by

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= e_{20}x^2 + e_{11}xy + o(|x,y|^2),\end{aligned}$$

where

$$e_{20} = d_{20} = -\frac{kn^2(2ab + 3ax^* + x^{*3})}{2rx^*(a + x^{*2})(b - k + 2x^*)}F''(x^*),$$
$$e_{11} = d_{11} + 2c_{20} = \frac{kn(2ab + 3ax^* + x^{*3})}{rx^*(a + x^{*2})(b - k + 2x^*)}p'(x^*).$$

Since $F''(x^*) \neq 0$ and $p'(x^*) \neq 0$, $E^*(x^*, nx^*)$ is a cusp of codimension 2 and this completes the proof of statement (II).

We now turn to the proof of the last statement. Solving $F(x^*) = F'(x^*) = p(x^*) = p'(x^*) = 0$ yields that $k = -\frac{x^*}{2a(a-x^*(2b+x^*))}(-a^2 + 4a(b + x^*)(b + 2x^*) + x^{*4})$, $s = -\frac{r}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}}(a + x^{*2})^2$, $m = -\frac{4ar(b+x^*)^3}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}}$ and $q = \frac{r(a+x^{*2})^3}{nx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})}$. Same as (II), we let $X = x - x^*, Y = y - y^*$ and obtain (for convenience, in subsequent steps, we still denote X, Y and τ by x, y and t, respectively)

$$\begin{aligned}\frac{dx}{dt} &= a^*_{10}x + a^*_{01}y + a^*_{20}x^2 + a^*_{11}xy + a^*_{30}x^3 + a^*_{21}x^2y + a^*_{40}x^4 + a^*_{31}x^3y \\ &\quad + o(|x, y|^4),\\ \frac{dy}{dt} &= b^*_{10}x + b^*_{01}y + b^*_{20}x^2 + b^*_{11}xy + b^*_{02}y^2 + b^*_{30}x^3 + b^*_{21}x^2y + b^*_{12}xy^2 + b^*_{40}x^4 \\ &\quad + b^*_{31}x^3y + b^*_{22}x^2y^2 + o(|x, y|^4),\end{aligned} \tag{12}$$

where a^*_{ij} and b^*_{ij} are given in Appendix B.

Making the transformation $X = x, Y = \frac{dx}{dt}$, system (12) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= c^*_{20}x^2 + c^*_{02}y^2 + c^*_{30}x^3 + c^*_{21}x^2y + c^*_{12}xy^2 + c^*_{40}x^4 + c^*_{31}x^3y + c^*_{22}x^2y^2 \\ &\quad + o(|x, y|^4),\end{aligned} \tag{13}$$

where c^*_{ij} are given in Appendix B.

By introducing $dt = (1 - c^*_{02}x)d\tau$ and $X = x, Y = (1 - c^*_{02}x)y$, system (13) can be converted into the form

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= c^*_{20}x^2 + (c^*_{30} - 2c^*_{02}c^*_{20})x^3 + c^*_{21}x^2y + (c^*_{12} - c^{*2}_{02})xy^2 + (c^{*2}_{02}c^*_{20} - 2c^*_{02}c^*_{30} \\ &\quad + c^*_{40})x^4 + (c^*_{31} - c^*_{02}c^*_{21})x^3y + (c^*_{22} - c^{*3}_{02})x^2y^2 + o(|x, y|^4).\end{aligned} \tag{14}$$

Since $c^*_{20} = \frac{r(a+x^{*2})^2F''(x^*)}{2(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})} \neq 0$, by changes of variables and scaling of time $X = x, Y = \frac{y}{\sqrt{c^*_{20}}}, \tau = \sqrt{c^*_{20}}t$ when $c^*_{20} > 0$ ($X = -x, Y = -\frac{y}{\sqrt{-c^*_{20}}}, \tau = \sqrt{-c^*_{20}}t$ when $c^*_{20} < 0$), system (14) can be further reexpressed as

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= x^2 \pm \frac{c^*_{30} - 2c^*_{02}c^*_{20}}{c^*_{20}}x^3 + \frac{c^{*2}_{02}c^*_{20} - 2c^*_{02}c^*_{30} + c^*_{40}}{c^*_{20}}x^4 + y(\frac{c^*_{21}}{\sqrt{\pm c^*_{20}}}x^2\end{aligned}$$

$$\pm\frac{c_{31}^*-c_{02}^*c_{21}^*}{\sqrt{\pm c_{20}^*}}x^3)+y^2((c_{12}^*-c_{02}^{*2})x\pm(c_{22}^*-c_{02}^{*3})x^2)+o(|x,y|^4). \quad (15)$$

By Proposition 5.3 in Lemontagne et al. [32], we can obtain an equivalent system of model (15) as follows

$$\begin{aligned}\frac{dx}{dt}&=y,\\ \frac{dy}{dt}&=x^2+Fx^3y+o(|x,y|^4),\end{aligned}$$

where

$$\begin{aligned}F&=\frac{c_{21}^*(c_{02}^*c_{20}^*-c_{30}^*)+c_{20}^*c_{31}^*}{(\pm c_{20}^*)^{\frac{3}{2}}}\\ &=\frac{-8a^2r^3D}{(\pm c_{20}^*)^{\frac{3}{2}}x^{*2}(b+x^*)^2(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})^3},\end{aligned}$$

with $D=2a^3(b^2+4bx^*+2x^{*2})-a^2x^{*2}(b+x^*)(4b-15x^*)-2ax^{*4}(5b+2x^*)(2b+3x^*)+x^{*6}(2b+x^*)(5b+x^*)$. It is evident that $D\neq 0$ when $b\neq b_\pm$, thus $E^*(x^*,y^*)$ is a cusp of codimension 3. We have accomplished the proof of Theorem 2.6. □

3. Bifurcation analysis.

3.1. Bogdanov-Takens bifurcation of codimension 2.

In this subsection, we study whether system (2) undergoes Bogdanov-Takens bifurcation of codimension 2 under small parameters perturbation if the bifurcation parameters are chosen suitably. Actually, we have the following result.

Theorem 3.1. *Suppose that $F(x^*)=F'(x^*)=p(x^*)=0$, $F''(x^*),p'(x^*)\neq 0$, then $E^*(x^*,nx^*)$ is a cusp of codimension 2. If we choose m and a as bifurcation parameters and assume that $k\neq\frac{a(b+x^*)(b+3x^*)+x^{*4}}{ab}$, then system (2) undergoes Bogdanov-Takens bifurcation of codimension 2. Here $F(x)$ and $p(x)$ are defined by (5) and (7), respectively.*

Proof. Choosing m and a as bifurcation parameters yields the following perturbed system

$$\begin{aligned}\frac{dx}{dt}&=(r(1-\frac{x}{k})-\frac{(m+\delta_1)}{x+b})x-\frac{qx^2y}{x^2+(a+\delta_2)},\\ \frac{dy}{dt}&=sy(1-\frac{y}{nx}),\end{aligned} \quad (16)$$

in which $r,s,k,m,a,b,q,n>0$ and $\delta=(\delta_1,\delta_2)\sim(0,0)$. We only focus on the dynamics of system (2) around the positive equilibrium $E^*(x^*,nx^*)$.

We can derive that $m=\frac{r(b+x^*)^2(2ak-ax^*+x^{*3})}{k(2ab+3ax^*+x^{*3})}$, $q=\frac{r(a+x^{*2})^2(-b+k-2x^*)}{knx^*(2ab+3ax^*+x^{*3})}$ and $s=\frac{rx^*(a+x^{*2})(-b+k-2x^*)}{k(2ab+3ax^*+x^{*3})}$ from $F(x^*)=F'(x^*)=p(x^*)=0$. Using $X=x-x^*,Y=y-nx^*$ to transform the positive equilibrium point $E^*(x^*,nx^*)$ of system (2) when $\delta=0$ into the origin and expanding the resulting system around the origin, we have (for convenience, in every subsequent transformation, we rename X,Y,τ by x,y,t, respectively)

$$\frac{dx}{dt}=\bar a_{00}+\bar a_{10}x+\bar a_{01}y+\bar a_{20}x^2+\bar a_{11}xy+o(|x,y|^2),$$

$$\frac{dy}{dt} = \bar{b}_{10}x + \bar{b}_{01}y + \bar{b}_{20}x^2 + \bar{b}_{11}xy + \bar{b}_{02}y^2 + o(|x,y|^2), \tag{17}$$

in which $\bar{a}_{ij}$ and $\bar{b}_{ij}$ are given in Appendix C and we also note that $\bar{a}_{00} = 0$ when $\delta = 0$.

Setting $X = x, Y = \frac{dx}{dt}$, we obtain

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \bar{c}_{00} + \bar{c}_{10}x + \bar{c}_{01}y + \bar{c}_{20}x^2 + \bar{c}_{11}xy + \bar{c}_{02}y^2 + P_1(x,y,\delta_1,\delta_2),\end{aligned} \tag{18}$$

in which $\bar{c}_{ij}$ are given in Appendix C and we also note that $\bar{c}_{00} = \bar{c}_{10} = \bar{c}_{01} = 0$ when $\delta = 0$. $P_1(x,y,\delta_1,\delta_2)$ is a C^∞ function at least of third order with respect to x, y, whose coefficients depend smoothly on δ_1 and δ_2.

Next, introducing time rescaling $dt = (1 - \bar{c}_{02}x)d\tau$ and changes of variables $X = x, Y = (1 - \bar{c}_{02}x)y$, system (18) becomes

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \bar{d}_{00} + \bar{d}_{10}x + \bar{d}_{01}y + \bar{d}_{20}x^2 + \bar{d}_{11}xy + P_2(x,y,\delta_1,\delta_2),\end{aligned} \tag{19}$$

in which $\bar{d}_{ij}$ are given in Appendix C and note that $\bar{d}_{00} = \bar{d}_{10} = \bar{d}_{01} = 0$ when $\delta = 0$. $P_2(x,y,\delta_1,\delta_2)$ is a C^∞ function at least of third order with respect to x, y, whose coefficients depend smoothly on δ_1 and δ_2.

Transformation $X = x + \frac{\bar{d}_{10}}{2\bar{d}_{20}}, Y = y$ brings the above model into an equivalent model

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \bar{e}_{00} + \bar{e}_{01}y + \bar{e}_{20}x^2 + \bar{e}_{11}xy + P_3(x,y,\delta_1,\delta_2),\end{aligned} \tag{20}$$

in which $\bar{e}_{ij}$ are given in Appendix C and note that $\bar{e}_{00} = \bar{e}_{01} = 0$ when $\delta = 0$. $P_3(x,y,\delta_1,\delta_2)$ is a C^∞ function at least of third order with respect to x, y, whose coefficients depend smoothly on δ_1 and δ_2.

Set $X = \frac{\bar{e}_{11}^2}{\bar{e}_{20}}x, Y = \frac{\bar{e}_{11}^3}{\bar{e}_{20}^2}y, \tau = \frac{\bar{e}_{20}}{\bar{e}_{11}}t$, it follows that

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \varphi_1 + \varphi_2 y + x^2 + xy + P_4(x,y,\delta_1,\delta_2),\end{aligned} \tag{21}$$

in which $P_4(x,y,\delta_1,\delta_2)$ is a C^∞ function at least of third order with respect to x, y, whose coefficients depend smoothly on δ_1 and δ_2. Moreover,

$$\varphi_1 = \frac{\bar{e}_{00}\bar{e}_{11}^4}{\bar{e}_{20}^3}, \quad \varphi_2 = \frac{\bar{e}_{01}\bar{e}_{11}}{\bar{e}_{20}}.$$

A direct calculation with the assistance of Mathematica leads to

$$\begin{aligned}&\left|\frac{\partial(\phi_1,\phi_2)}{\partial(\delta_1,\delta_2)}\right|_{\delta=0}\\ &= \frac{\bar{e}_{11}^5}{\bar{e}_{20}^4}\Big|_{\delta=0} \cdot \frac{r^2x^{*3}(a+x^{*2})^2(b-k+2x^*)^2(a(b^2-bk+4bx^*+3x^{*2})+x^{*4})}{k^2(b+x^*)(2ab+3ax^*+x^{*3})^2C},\end{aligned}$$

with $C = -a^2(b^2-b(k-3x^*)+3x^*(x^*-k))+ax^{*2}(3b^2-3bk+9bx^*-kx^*+6x^{*2})+x^{*6}$.

Obviously, $\left|\frac{\partial(\phi_1,\phi_2)}{\partial(\delta_1,\delta_2)}\right|_{\delta=0} \neq 0$ when $k \neq \frac{a(b+x^*)(b+3x^*)+x^{*4}}{ab}$. By the results of Bogdanov [9, 10] and Takens [45], we know that system (2) undergoes Bogdanov-Takens bifurcation of codimension 2 when (δ_1, δ_2) changes in a small neighborhood of $(0,0)$. This proves the Theorem. □

3.2. **Bogdanov-Takens bifurcation of codimension 3.** In this subsection, we will illustrate that system (2) may undergo Bogdanov-Takens bifurcation of codimension 3 under advisable parameters. Before the main result, we first state the relevant definition and property. Please see the references [13] and [21] for more details.

Definition 3.2. The bifurcation that results from unfolding the following normal form of a cusp of codimension 3,

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= x^2 \pm x^3 y, \end{aligned} \tag{22}$$

is called a cusp type degenerate Bogdanov-Takens bifurcation of codimension 3.

Proposition 3.3. *A universal unfolding of the normal form (22) is expressed by*

$$\left\{ \begin{array}{l} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = \zeta_1 + \zeta_2 y + \zeta_3 xy + x^2 \pm x^3 y + R(x,y,\rho), \end{array} \right. \tag{23}$$

where $\rho = (\rho_1, \rho_2, \rho_3) \sim (0,0,0)$, $\frac{D(\zeta_1,\zeta_2,\zeta_3)}{D(\rho_1,\rho_2,\rho_3)} \neq 0$ *for small* ρ *and*

$$\begin{aligned} R(x,y,\rho) = & \, y^2 O(|x,y|^2) + O(|x,y|^5) + O(\rho)(O(y^2) + O(|x,y|^3)) \\ & + O(\rho^2) O(|x,y|). \end{aligned} \tag{24}$$

Our main result is as follows.

Theorem 3.4. *Suppose that* $F(x^*) = F'(x^*) = p(x^*) = p'(x^*) = 0, F''(x^*) \neq 0$ *and* $b \neq b_\pm$, *in which* $b_\pm$ *are defined as shown in Theorem 2.6, then* $E^*(x^*, nx^*)$ *is a cusp of codimension 3. If we choose* m, a *and* n *as bifurcation parameters and assume that* $b \neq b_\pm^* \doteq \frac{-3a^2x^* + 7ax^{*3} - 6x^{*5} \pm \sqrt{-x^{*2}(a+x^{*2})^2(a^2+4ax^{*2}-6x^{*4})}}{2(a^2-2ax^{*2}+5x^{*4})}$, *then model (2) undergoes Bogdanov-Takens bifurcation of codimension 3 in a small neighborhood of* $E^*(x^*, nx^*)$. *Here* $F(x)$ *and* $p(x)$ *are defined by (5) and (7), respectively.*

Proof. Selecting m, a and n as bifurcation parameters, we have the following system

$$\begin{aligned} \frac{dx}{dt} &= (r(1-\frac{x}{k}) - \frac{(m+\kappa_1)}{x+b})x - \frac{qx^2y}{x^2+(a+\kappa_2)}, \\ \frac{dy}{dt} &= sy(1-\frac{y}{(n+\kappa_3)x}), \end{aligned} \tag{25}$$

where $\kappa = (\kappa_1, \kappa_2, \kappa_3) \sim (0,0,0)$.

By $F(x^*) = F'(x^*) = p(x^*) = p'(x^*) = 0$, we know that $k = \frac{-x^*}{2a(a-x^*(2b+x^*))}(-a^2 + 4a(b+x^*)(b+2x^*) + x^{*4})$, $s = \frac{-r(a+x^{*2})^2}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}}$, $m = \frac{-4ar(b+x^*)^3}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}}$, $n = \frac{r(a+x^{*2})^3}{qx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})}$. Next, we transform the positive equilibrium $E^*(x^*, nx^*)$ of system (25) when $\kappa = 0$ into the origin by setting $X = x - x^*, Y =$

$y - nx^*$. By the Taylor expansion, system (25) continues as (for convenience, in every subsequent transformation, we rename X, Y and τ as x, y and t, respectively)

$$\begin{aligned}\frac{dx}{dt} &= \bar{a}_{00}^* + \bar{a}_{10}^* x + \bar{a}_{01}^* y + \bar{a}_{20}^* x^2 + \bar{a}_{11}^* xy + \bar{a}_{30}^* x^3 + \bar{a}_{21}^* x^2 y + \bar{a}_{40}^* x^4 + \bar{a}_{31}^* x^3 y \\ &\quad + o(|x, y|^4), \\ \frac{dy}{dt} &= \bar{b}_{00}^* + \bar{b}_{10}^* x + \bar{b}_{01}^* y + \bar{b}_{20}^* x^2 + \bar{b}_{11}^* xy + \bar{b}_{02}^* y^2 + \bar{b}_{30}^* x^3 + \bar{b}_{21}^* x^2 y + \bar{b}_{12}^* xy^2 \\ &\quad + \bar{b}_{40}^* x^4 + \bar{b}_{31}^* x^3 y + \bar{b}_{22}^* x^2 y^2 + o(|x, y|^4), \end{aligned} \tag{26}$$

where $\bar{a}_{ij}^*$ and $\bar{b}_{ij}^*$ are given in Appendix C and it's worth noting that $\bar{a}_{00}^* = \bar{b}_{00}^* = 0$ when $\kappa = 0$.

Letting

$$X = x, \ Y = \frac{dx}{dt},$$

system (26) can be rewritten as follows

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{c}_{00}^* + \bar{c}_{10}^* x + \bar{c}_{01}^* y + \bar{c}_{20}^* x^2 + \bar{c}_{11}^* xy + \bar{c}_{02}^* y^2 + \bar{c}_{30}^* x^3 + \bar{c}_{21}^* x^2 y + \bar{c}_{12}^* xy^2 \\ &\quad + \bar{c}_{40}^* x^4 + \bar{c}_{31}^* x^3 y + \bar{c}_{22}^* x^2 y^2 + o(|x, y|^4), \end{aligned} \tag{27}$$

where $\bar{c}_{ij}^*$ are given in Appendix C and we also note that $\bar{c}_{00}^* = \bar{c}_{10}^* = \bar{c}_{01}^* = \bar{c}_{11}^* = 0$ when $\kappa = 0$.

To illustrate the existence of the Bogdanov-Takens bifurcation of codimension 3, we transform model (27) to the form of system (23) by following seven steps as shown in Li et al. [35] as follows.

Step I. First of all, remove the y^2-term from model (27) when $\kappa = 0$ by letting $x = X + \frac{\bar{c}_{02}^*}{2} X^2, y = Y + \bar{c}_{02}^* XY$. System (27) can be transformed into

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{d}_{00}^* + \bar{d}_{10}^* x + \bar{d}_{01}^* y + \bar{d}_{20}^* x^2 + \bar{d}_{11}^* xy + \bar{d}_{30}^* x^3 + \bar{d}_{21}^* x^2 y + \bar{d}_{12}^* xy^2 \\ &\quad + \bar{d}_{40}^* x^4 + \bar{d}_{31}^* x^3 y + \bar{d}_{22}^* x^2 y^2 + o(|x, y|^4), \end{aligned} \tag{28}$$

where $\bar{d}_{ij}^*$ are given in Appendix C and it's worth noting that $\bar{d}_{00}^* = \bar{d}_{10}^* = \bar{d}_{01}^* = \bar{d}_{11}^* = 0$ when $\kappa = 0$.

Step II. In this step, we remove the xy^2-term in model (28) when $\kappa = 0$. Making variables transformation $x = X + \frac{\bar{d}_{12}^*}{6} X^3, y = Y + \frac{\bar{d}_{12}^*}{2} X^2 Y$, we have

$$\begin{aligned}\frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{e}_{00}^* + \bar{e}_{10}^* x + \bar{e}_{01}^* y + \bar{e}_{20}^* x^2 + \bar{e}_{11}^* xy + \bar{e}_{30}^* x^3 + \bar{e}_{21}^* x^2 y + \bar{e}_{40}^* x^4 \\ &\quad + \bar{e}_{31}^* x^3 y + \bar{e}_{22}^* x^2 y^2 + o(|x, y|^4), \end{aligned} \tag{29}$$

where $\bar{e}_{ij}^*$ are given in Appendix C and it's worth noting that $\bar{e}_{00}^* = \bar{e}_{10}^* = \bar{e}_{01}^* = \bar{e}_{11}^* = 0$ when $\kappa = 0$.

Step III. We remove the x^2y^2-term in model (29) when $\kappa = 0$. Let $x = X + \frac{\bar{e}^*_{22}}{12}X^4, y = Y + \frac{\bar{e}^*_{22}}{3}X^3Y$, then system (29) becomes

$$\begin{aligned}
\frac{dx}{dt} &= y,\\
\frac{dy}{dt} &= \bar{f}^*_{00} + \bar{f}^*_{10}x + \bar{f}^*_{01}y + \bar{f}^*_{20}x^2 + \bar{f}^*_{11}xy + \bar{f}^*_{30}x^3 + \bar{f}^*_{21}x^2y + \bar{f}^*_{40}x^4\\
&\quad + \bar{f}^*_{31}x^3y + o(|x,y|^4),
\end{aligned} \tag{30}$$

where $\bar{f}^*_{ij}$ are given in Appendix C and it's worth noting that $\bar{f}^*_{00} = \bar{f}^*_{10} = \bar{f}^*_{01} = \bar{f}^*_{11} = 0$ when $\kappa = 0$.

Step IV. Removing the x^3 and x^4-terms in model (30) when $\kappa = 0$. We can easily get that $\bar{f}^*_{20} = -\frac{ar^2(a+x^{*2})^3}{x^*(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})^2} + O(\kappa)$, $\bar{f}^*_{20} \neq 0$ for small κ. With

$$\begin{aligned}
x &= X - \frac{\bar{f}^*_{30}}{4\bar{f}^*_{20}}X^2 + \frac{15\bar{f}^{*2}_{30} - 16\bar{f}^*_{20}\bar{f}^*_{40}}{80\bar{f}^{*2}_{20}}X^3, \quad y = Y,\\
dt &= (1 - \frac{\bar{f}^*_{30}}{2\bar{f}^*_{20}}X + \frac{45\bar{f}^{*2}_{30} - 48\bar{f}^*_{20}\bar{f}^*_{40}}{80\bar{f}^{*2}_{20}}X^2)d\tau,
\end{aligned}$$

it follows that

$$\begin{aligned}
\frac{dx}{dt} &= y,\\
\frac{dy}{dt} &= \bar{g}^*_{00} + \bar{g}^*_{10}x + \bar{g}^*_{01}y + \bar{g}^*_{20}x^2 + \bar{g}^*_{11}xy + \bar{g}^*_{30}x^3 + \bar{g}^*_{21}x^2y + \bar{g}^*_{40}x^4\\
&\quad + \bar{g}^*_{31}x^3y + o(|x,y|^4),
\end{aligned} \tag{31}$$

where $\bar{g}^*_{ij}$ are given in Appendix C and it's worth noting that $\bar{g}^*_{00} = \bar{g}^*_{10} = \bar{g}^*_{01} = \bar{g}^*_{11} = \bar{g}^*_{30} = \bar{g}^*_{40} = 0$ when $\kappa = 0$.

Step V. Removing the x^2y-term in model (31) when $\kappa = 0$. It's easy to know that $\bar{g}^*_{20} = -\frac{ar^2(a+x^{*2})^3}{x^*(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})^2} + O(\kappa)$, $g^*_{20} \neq 0$ for small κ. Setting

$$x = X, y = Y + \frac{\bar{g}^*_{21}}{3\bar{g}^*_{20}}Y^2 + \frac{\bar{g}^{*2}_{21}}{36\bar{g}^{*2}_{20}}Y^3, d\tau = (1 + \frac{\bar{g}^*_{21}}{3\bar{g}^*_{20}}Y + \frac{\bar{g}^{*2}_{21}}{36\bar{g}^{*2}_{20}}Y^2)dt,$$

we get an equivalent system to (31) as follows

$$\begin{aligned}
\frac{dx}{dt} &= y,\\
\frac{dy}{dt} &= \bar{h}^*_{00} + \bar{h}^*_{10}x + \bar{h}^*_{01}y + \bar{h}^*_{20}x^2 + \bar{h}^*_{11}xy + \bar{h}^*_{31}x^3y + R_1(x,y,\kappa),
\end{aligned} \tag{32}$$

where $\bar{h}^*_{ij}$ are given in Appendix C. It's worth noting that $\bar{h}^*_{00} = \bar{h}^*_{10} = \bar{h}^*_{01} = \bar{h}^*_{11} = 0$ when $\kappa = 0$ and $R_1(x,y,\kappa)$ possesses the property of (24).

Step VI. Changing $\bar{g}^*_{20}$ and $\bar{g}^*_{31}$ to 1 in model (32). A simple calculation shows that $\bar{h}^*_{20} = \bar{g}^*_{20} \neq 0$ and $\bar{h}^*_{31} \neq 0$ for small κ. Transformations

$$x = \bar{h}^{*\frac{1}{5}}_{20}\bar{h}^{*-\frac{2}{5}}_{31}X, \quad y = \bar{h}^{*\frac{4}{5}}_{20}\bar{h}^{*-\frac{3}{5}}_{31}Y, \quad t = \bar{h}^{*-\frac{3}{5}}_{20}\bar{h}^{*\frac{1}{5}}_{31}\tau,$$

bring the above system to

$$\begin{aligned}
\frac{dx}{dt} &= y,\\
\frac{dy}{dt} &= \bar{j}^*_{00} + \bar{j}^*_{10}x + \bar{j}^*_{01}y + \bar{j}^*_{11}xy + x^2 + x^3y + R_2(x,y,\kappa),
\end{aligned} \tag{33}$$

where $\bar{j}^*_{ij}$ are given in Appendix C. It's worth noting that $\bar{j}^*_{00} = \bar{j}^*_{10} = \bar{j}^*_{01} = \bar{j}^*_{11} = 0$ when $\kappa = 0$ and $R_2(x, y, \kappa)$ possesses the property of (24).

Step VII. Finally, we remove the $\bar{h}^*_{10}$-term in model (33) by introducing conversion of coordinates

$$x = X - \frac{\bar{j}^*_{10}}{2}, \quad y = Y,$$

then model (33) can be expressed as follows

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= \bar{\psi}_1 + \bar{\psi}_2 y + \bar{\psi}_3 xy + x^2 + x^3 y + R_3(x, y, \kappa), \end{aligned} \tag{34}$$

where

$$\bar{\psi}_1 = \bar{j}^*_{00} - \frac{1}{4}\bar{j}^{*2}_{10}, \quad \bar{\psi}_2 = \bar{j}^*_{01} - \frac{\bar{j}^{*3}_{10}}{8} - \frac{\bar{j}^*_{11}\bar{j}^*_{10}}{2}, \quad \bar{\psi}_3 = \bar{j}^*_{11} + \frac{3}{4}\bar{j}^{*2}_{10}.$$

It's worth noting that $\bar{\psi}_1 = \bar{\psi}_2 = \bar{\psi}_3 = 0$ when $\kappa = 0$ and $R_3(x, y, \kappa)$ possesses the property of (24). Moreover, with the help of Mathematica, we know that

$$\begin{aligned} &\left|\frac{\partial(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)}{\partial(\kappa_1, \kappa_2, \kappa_3)}\right|_{\kappa=0} \\ &= \bar{h}^{*\frac{4}{5}}_{31} \bar{h}^{*-\frac{12}{5}}_{20}|_{\kappa=0} \cdot \frac{4qr^2 x^{*4}\mathcal{F}}{(a + x^{*2})(b + x^*)^3(-a^2 + 4a(b + x^*)(b + 2x^*) + x^{*4})^2}, \end{aligned}$$

with $\mathcal{F} = a^2(2b^2 + 6bx^* + 5x^{*2}) - 2ax^{*2}(2b^2 + 7bx^* + 4x^{*2}) + x^{*4}(10b^2 + 12bx^* + 3x^{*2})$.

Obviously, $\left|\frac{\partial(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)}{\partial(\kappa_1, \kappa_2, \kappa_3)}\right|_{\kappa=0} \neq 0$ when $b \neq b^*_{\pm}$. Furthermore, system (34) has the same form as system (23). According to the conclusion of Li et al. [35], we can conclude that model (34) is the universal unfolding of the Bogdanov-Takens singularity (cusp case) of codimension 3. The remaining term $R_3(x, y, \kappa)$, satisfying the property of (24), has no influence on the bifurcation phenomena. The dynamics of system (2) in a small neighborhood of the positive equilibrium $E^*(x^*, nx^*)$, as (m, a, n) varies near $(m + \kappa_1, a + \kappa_2, n + \kappa_3)$, are equivalent to those of system (34) in a small neighborhood of $(0, 0, 0)$, as $(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)$ varies near $(0, 0, 0)$. This ends the proof. □

3.3. **Hopf bifurcation of codimension 2.** From the analysis of Appendix A, we see that Hopf bifurcation may occur at $E_2(x_2, y_2), E_4(x_4, y_4)$ (we denote them as $E_*(x_*, nx_*)$ for convenience) of system (2) because of $\det(J(E_*)) = -sF'(x_*) > 0$. Actually, we have the following Theorem.

Theorem 3.5. *Let $E_*(x_*, nx_*)$ be an equilibrium of system (2) accounting for either $E_2(x_2, y_2)$ or $E_4(x_4, y_4)$ and suppose that $F(x_*) = p(x_*) = 0$, then we have*
(I) if $\eta_{11} < 0$, then $E_*(x_*, nx_*)$ is a stable weak focus with multiplicity one and one stable limit cycle bifurcates from E_* by a supercritical Hopf bifurcation;
(II) if $\eta_{11} > 0$, then $E_*(x_*, nx_*)$ is an unstable weak focus with multiplicity one and one unstable limit cycle bifurcates from E_* by a subcritical Hopf bifurcation;
(III) if $\eta_{11} = 0$, then $E_*(x_*, nx_*)$ is a weak focus with multiplicity at least two and system (2) may exhibit a degenerate Hopf bifurcation of codimension at least 2, where

$$\begin{aligned} \eta_{11} = {} & bx^9_*(br + k(s - r))(k(s - r) + 2rx_*) + a^4 k(r + s)(8b^2 ks + 2x^2_*(3br + k(r \\ & +3s)) + 3bx_*(br - kr + 5ks)) + 2ax^5_*(-6b^3 ks(br + k(s - r)) - 2b^2 x_*(5r^2 \end{aligned}$$

$$\times(b-k)^2+2krs(11b-6k)+7k^2s^2)-x_*^3(109b^2r^2+52bkr(s-r)+3k^2 \times(r-s)^2)-b(60b^2r^2+k^2(10r-9s)(r-s)+bkr(-70r+83s))x_*^2-2r \times(33br+5k(-r+s))x_*^4-8r^2x_*^5)+2a^2x_*^3(4b^3ks(br-k(r+2s))+4b^2x_* \times(3r^2(b-k)^2+krs(3b-2k)-3k^2s^2)+x_*^3(179b^2r^2+2bkr(46s-73r) +k^2(r-s)(11r-s))+4rx_*^4(51br-15kr+11ks)+brx_*^2(r(b-k)(72b- 25k)+25ks(2b-k))+80r^2x_*^5)+2a^3(8b^4k^2s^2+6b^3ksx_*(3br-3kr+5k \times s)+x_*^4(81b^2r^2+12bkr(17s-3r)-k^2(r^2+22rs-15s^2))+2b^2x_*^2(3r^2 \times(b-k)^2+2krs(24b-13k)+27k^2s^2)+bx_*^3(2r^2(b-k)(18b-k)+9kr \times s(23b-7k)+49k^2s^2)+2rx_*^5(39br-3kr+35ks)+24r^2x_*^6). \tag{35}$$

Here $F(x)$ and $p(x)$ are defined by (5) and (7), respectively.

Proof. According to $F(x_*)=p(x_*)=0$, we know that $m=\frac{(b+x_*)^2}{k(a(b+2x_*)-bx_*^2)}(ak(r+s)+x_*^2(k(s-r)+2rx_*))$ and $q=\frac{(a+x_*^2)^2(b(ks+rx_*)+x_*(k(s-r)+2rx_*))}{knx_*^2(bx_*^2-a(b+2x_*))}$, which are substituted into system (2). Then by $X=x-x_*, Y=y-nx_*$ and the Taylor expansion, system (2) can be rewritten as (for convenience, we rename X, Y as x, y, respectively)

$$\begin{aligned} \frac{dx}{dt} &= \alpha_{10}x+\alpha_{01}y+\alpha_{20}x^2+\alpha_{11}xy+\alpha_{30}x^3+\alpha_{21}x^2y+\alpha_{40}x^4+\alpha_{31}x^3y \\ &\quad +o(|x,y|^4), \\ \frac{dy}{dt} &= \beta_{10}x+\beta_{01}y+\beta_{20}x^2+\beta_{11}xy+\beta_{02}y^2+\beta_{30}x^3+\beta_{21}x^2y+\beta_{12}xy^2 \\ &\quad +\beta_{40}x^4+\beta_{31}x^3y+\beta_{22}x^2y^2+o(|x,y|^4), \end{aligned} \tag{36}$$

in which α_{ij} and β_{ij} are defined in Appendix D and note that $\beta_{01}=-\alpha_{10}$.

A simple computation shows that $\alpha_{10}\beta_{01}-\alpha_{01}\beta_{10}=-sF'(x_*)>0$. Let $\lambda=\sqrt{\alpha_{10}\beta_{01}-\alpha_{01}\beta_{10}}$ and make a transformation of $x=-\alpha_{01}X, y=\alpha_{10}X-\lambda Y$ and $dt=\frac{1}{\lambda}d\tau$, then model (36) becomes (we still denote X, Y, τ by x, y, t, respectively)

$$\begin{aligned} \frac{dx}{dt} &= y+\delta_{20}x^2+\delta_{11}xy+\delta_{30}x^3+\delta_{21}x^2y+\delta_{40}x^4+\delta_{31}x^3y+o(|x,y|^4), \\ \frac{dy}{dt} &= -x+\gamma_{20}x^2+\gamma_{11}xy+\gamma_{02}y^2+\gamma_{30}x^3+\gamma_{21}x^2y+\gamma_{12}xy^2+\gamma_{40}x^4 \\ &\quad +\gamma_{31}x^3y+\gamma_{22}x^2y^2+o(|x,y|^4), \end{aligned} \tag{37}$$

in which δ_{ij} and γ_{ij} are defined in Appendix D.

Following the formula in Perko [38], we have the first Lyapunov coefficient as follows

$$\eta_1=\frac{s(b(ks+rx_*)+x_*(k(s-r)+2rx_*))^2\eta_{11}}{8k^4n^2(b+x_*)^2(bx_*^2-a(b+2x_*))^4\lambda^3},$$

in which η_{11} is defined as (35). Obviously, the sign of η_1 is the same as η_{11} and thus the proof is completed. □

Based on the case (III) of Theorem 3.5, we can conclude that system (2) may undergo degenerate Hopf bifurcation around $E_*(x_*, y_*)$ when $l_{11}=0$, i.e., $r=r_*^{\pm}\doteq\frac{ks(A\pm(bx_*^2-a(b+2x_*))\sqrt{C})}{2x_*B}$, in which A, B and C are given in Appendix D. A complicated calculation with the assistance of Maple and Mathematica yields the second Lyapunov coefficients as follows:

$$\eta_2 = \frac{s^2(b(ks + rx_*) + x_*(k(s-r) + 2rx_*))^3\eta_{22}}{288k^9n^4x_*^4(b+x_*)^4(bx_*^2 - a(b+2x_*))^9\lambda^7},$$

in which η_{22} is too long to be included here. More specifically, our conclusion is as follows.

Theorem 3.6. *Let $E_*(x_*, nx_*)$ be an equilibrium of system (2) accounting for either $E_2(x_2, y_2)$ or $E_4(x_4, y_4)$ and suppose that $F(x_*) = p(x_*) = 0$ and $r = r_*^{\pm}$, then we have*
(I) if $\eta_{22} < 0$, then $E_*(x_*, nx_*)$ is a stable weak focus with multiplicity 2. System (2) undergoes a degenerate Hopf bifurcation of codimension 2 and there can be up to two limit cycles bifurcating from E_*, the outermost being stable;
(II) if $\eta_{22} > 0$, then $E_*(x_*, nx_*)$ is an unstable weak focus with multiplicity 2. System (2) undergoes a degenerate Hopf bifurcation of codimension 2 and there can be up to two limit cycles bifurcating from E_*, the outermost being unstable;
(III) if $\eta_{22} = 0$, then $E_*(x_*, nx_*)$ is a weak focus with multiplicity at least 3 and system (2) may undergo a degenerate Hopf bifurcation of codimension at least 3. Here $F(x)$ and $p(x)$ are defined by (5) and (7), respectively.

4. Numerical simulations.

In this section, we will carry out numerical simulations to verify the theoretical results by using AUTO07P [20]. The corresponding parameter values are taken as follows:

$$r = 0.8,\ k = 6,\ q = 0.496,\ a = 0.65,\ s = 0.2,\ n = 1.2,\ m = 0.376,\ b = 0.2. \quad (38)$$

4.1. m and a as the primary bifurcation parameters, respectively.

We first consider the parameter m as the primary bifurcation parameter, while fixing the rest of parameter values as shown in (38). As a result, we observe one subcritical Hopf bifurcation point $HB(9.50285 \times 10^{-1}, 1.14034)$ at $m = 3.76381 \times 10^{-1}$, one saddle-node bifurcation point $SN(6.42520 \times 10^{-1}, 7.71024 \times 10^{-1})$ at $m = 4.07055 \times 10^{-1}$, a neutral saddle equilibrium $NS(0.075554626, 0.090665552)$ at $m = 0.21623993$ and one transcritical bifurcation point $TC(3.32272 \times 10^{-5}, 3.98726 \times 10^{-5})$ at $m = 1.60026 \times 10^{-1}$. Additionally, a family of unstable limit cycles approach a stable homoclinic cycle, as shown in Figure 2.

We observe an S-shaped limit cycle bifurcation branch originating from the subcritical Hopf bifurcation point HB, which has two saddle-node bifurcation points: $SNC_1(1.37206, 1.50088)$ at $m = 3.75330 \times 10^{-1}$ with a period of 3.72894×10^1 and $SNC_2(2.10132, 2.01884)$ at $m = 3.76744 \times 10^{-1}$ with a period of 4.30703×10^1. This indicates the coexistence of three limit cycles when $3.75330 \times 10^{-1} < m < 3.76382 \times 10^{-1}$, with the innermost and outermost cycles being unstable and the middle one being stable, as shown in Figure 3.

It is easy to see that there are two positive equilibria for $1.60026 \times 10^{-1} < m < 4.07055 \times 10^{-1}$, and only one boundary equilibrium for $m > 4.07055 \times 10^{-1}$. Biologically, m serves as a threshold indicating the coexistence of predator and prey when $1.60026 \times 10^{-1} < m \leq 4.07055 \times 10^{-1}$. However, system (2) will collapse when $m > 4.07055 \times 10^{-1}$ due to the extinction of the predator.

Secondly, we let

$$r = 0.65,\ k = 9,\ q = 0.25,\ a = 0.8,\ s = 0.03,\ n = 3.55,\ m = 0.2,\ b = 0.6, \quad (39)$$

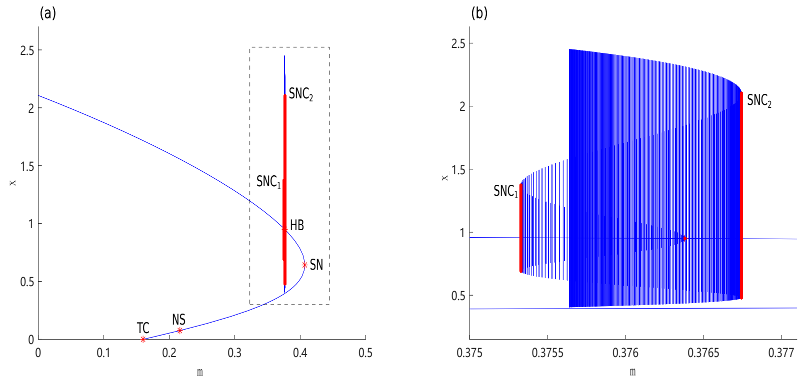

FIGURE 2. One-parameter bifurcation diagram of system (2) with respect to m. (a) m vs. x; (b) zoomed part in (a).

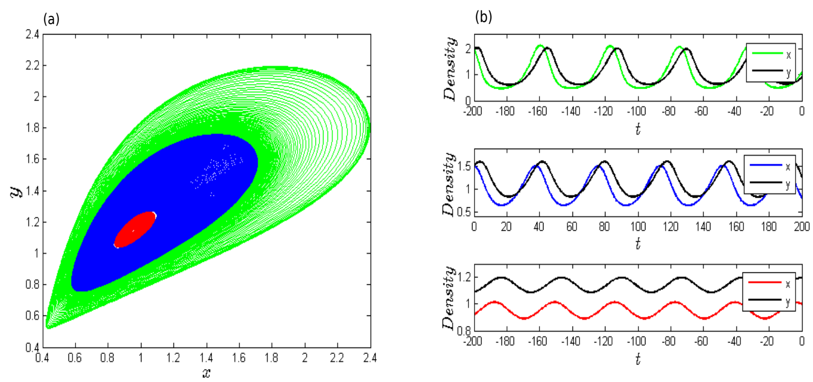

FIGURE 3. (a) The coexistent three limit cycles of system (2) with respect to m, where the outmost limit cycle (green) and the innermost limit cycle (red) are unstable, while the middle one is stable (blue); (b) The corresponding time evolutions of three limit cycles.

and take the parameter a as the primary bifurcation parameter. We observe two subcritical Hopf bifurcation points, denoted as HB_1 and HB_2, at $(4.04595 \times 10^{-1}, 1.43631)$ when $a = 1.80821 \times 10^{-1}$ and at $(7.91722 \times 10^{-1}, 2.81061)$ when $a = 6.11852 \times 10^{-1}$ respectively. Moreover, there is a saddle-node point SN at $(7.43039 \times 10^{-6}, 2.63779 \times 10^{-5})$ when $a = 0$, and two saddle-node bifurcation points of limit cycles, denoted as SNC_1 and SNC_2, at $(6.72268, 6.59548)$ when $a = 8.32992 \times 10^{-3}$ with period $= 2.31219 \times 10^2$, and at $(3.61649, 6.17642)$ when $a = 1.06304$ with period $= 7.24105 \times 10$ respectively. There are two intervals $8.32992 \times 10^{-3} < a < 1.80821 \times 10^{-1}$ and $6.11852 \times 10^{-1} < a < 1.06304$ where two limit cycles coexist, as shown in Figure 4 (a) (b). We note that the populations of predator and prey increase with the increasing half-saturation constant after some sustained oscillations.

4.2. m **and** a **as the primary bifurcation parameters.** Moving on, by taking m and a as primary bifurcation parameters and using the parameter values given

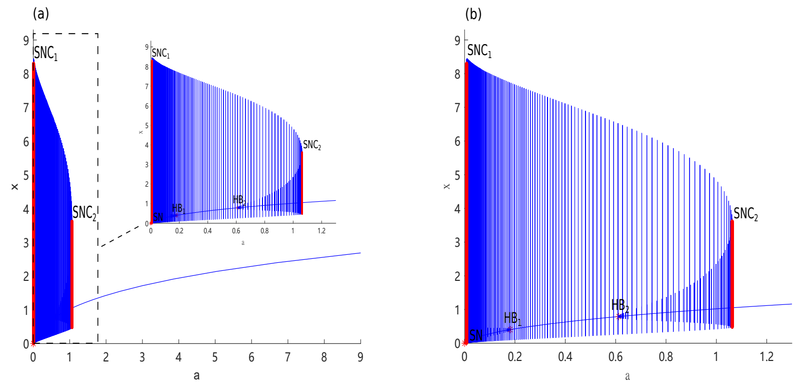

FIGURE 4. One-parameter bifurcation diagram of system (2) with respect to a, where $HB_1, HB_2, SN, SNC_1, SNC_2$ denote two subcritical Hopf bifurcation points, saddle-node point and two saddle-node bifurcation points of limit cycles, respectively. (a) a vs x; (b) zoomed part in (a).

in (38), we can obtain the bifurcation curves of saddle-node SN (blue), Hopf H (red), homoclinic Hom (green) and saddle-node of limit cycles SNL (black), as shown in Figure 5. Two Bogdanov-Takens (BT) bifurcation points $BT_1(9.42740 \times 10^{-1}, 1.13129)$ with $m = 5.42003 \times 10^{-1}, a = 1.75619$ and $BT_2(2.11331 \times 10^{-1}, 2.53597 \times 10^{-1})$ with $m = 2.35202 \times 10^{-1}, a = 8.82392 \times 10^{-2}$, are observed. Additionally, there are one generalized Hopf bifurcation point $GH(9.73781 \times 10^{-1}, 1.16854)$ with $m = 3.70134 \times 10^{-1}, a = 6.42377 \times 10^{-1}$, one cusp $CP(4.59146 \times 10^{-1}, 5.50975 \times 10^{-1})$ with $m = 1.16053 \times 10^{-1}, a = 1.21700 \times 10^{-2}$ and a codimension-2 cusp point of limit cycles $CPL(1.78649, 1.80917)$ (with double one Floquet multipliers) with $m = 3.79566 \times 10^{-1}, a = 6.59945 \times 10^{-1}$ and period $= 4.04041 \times 10^{1}$. The saddle-node bifurcation diagram of limit cycles is shown in Figure 5(a). There is an acute angle parameter region for the coexistence of three limit cycles. The entire bifurcation diagram is divided into eight regions: I-VIII and the corresponding phase portraits for each region are described as follows.

I: two saddles, an unstable node and a stable focus.

II: two saddles, an unstable node and a homoclinic cycle that contains a stable limit cycle enclosing an unstable focus.

III: two saddles, an unstable node and an unstable focus.

IV: a saddle and an unstable node.

V: two saddles, an unstable node and an unstable limit cycle that contains a stable focus.

VI: two saddles, an unstable node and a homoclinic cycle that contains two limit cycles (a big stable limit cycle contains a small unstable limit cycle) enclosing a stable hyperbolic positive equilibrium, bistability state.

VII: two saddles, an unstable node and three limit cycles (a big unstable limit cycle contains a medium stable limit cycle enclosing a small unstable limit cycle) that contains a stable focus, bistability state.

VIII: two saddles, an unstable node, and a big unstable limit cycle that contains a small stable limit cycle enclosing an unstable focus.

Please see Figure 6 for more details.

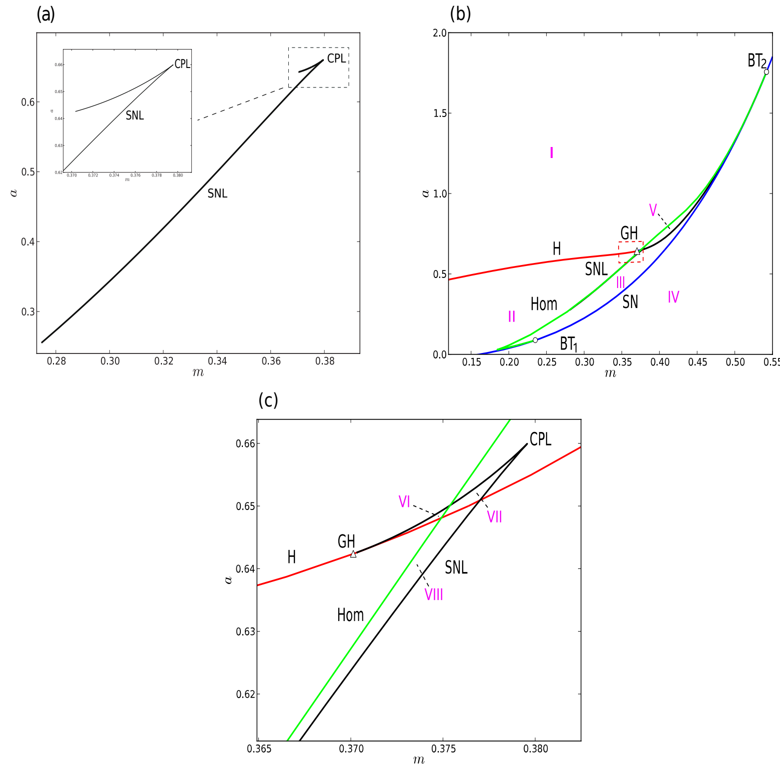

FIGURE 5. Two-parameter bifurcation diagram in system (2) with respect to m and a. (a) The saddle-node bifurcation curve of limit cycles SNL, where CPL denotes the codimension-2 cusp of limit cycles. (b) The locations of saddle-node bifurcation curve SN (blue), saddle-node bifurcation curve of limit cycles SNL (black), homoclinic bifurcation curve Hom (green) and Hopf bifurcation curve H (red). (c) Zoomed bifurcation diagram of (b). Here BT, GH and CPL represent the points of Bagdanov-Taken bifurcation point, generalized Hopf bifurcation point and cusp of limit cycles, respectively.

4.3. **m and n as the primary bifurcation parameters.** Now, considering m and n as the primary bifurcation parameters, and the first set of parameter values in (38), we can construct a two-parameter bifurcation diagram. This diagram includes the saddle-node bifurcation curve SN (blue), the Hopf bifurcation curve H (red), the homoclinic bifurcation curve Hom (green) and the saddle-node bifurcation curve of limit cycles SNL (black). The diagram is shown in Figure 7 (a, b, c). Notably, there are two BT bifurcation points: $BT_1(1.89611, 9.02782 \times 10^{-1})$ with $m = 7.27741 \times 10^{-1}, n = 4.76124 \times 10^{-1}$ and $BT_2(4.25581 \times 10^{-1}, 7.87462 \times 10^{-1})$ with $m = 3.39850 \times 10^{-1}, n = 1.85032$, respectively. There are also two generalized Hopf bifurcation points: $GH_1(9.89983 \times 10^{-1}, 1.17997)$ with $m = 3.71936 \times 10^{-1}, n = 1.19191$ and $GH_2(1.77236, 1.25389)$ with $m = 5.38342 \times 10^{-1}, n = 7.07466 \times 10^{-1}$, respectively.

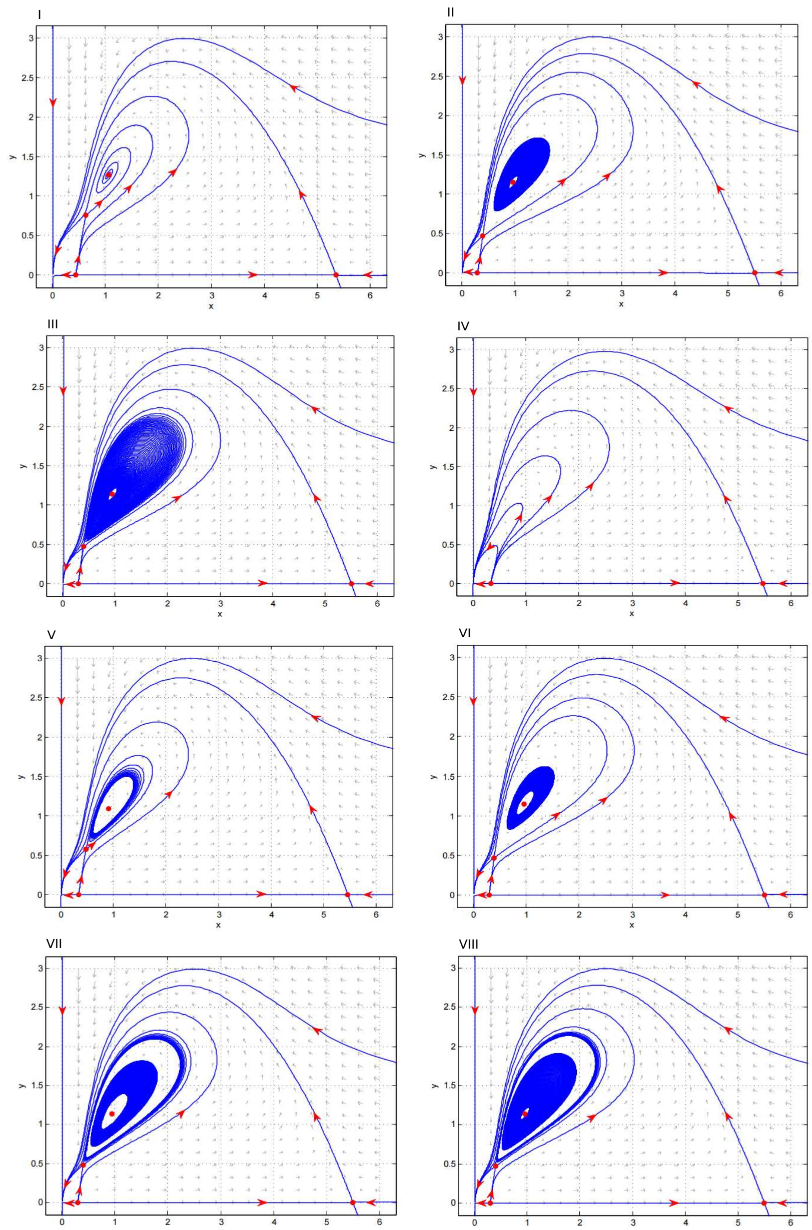

FIGURE 6. Phase portraits of regions: I-VIII in Figure 5.

Interestingly, the saddle-node bifurcation curve of limit cycles SNL connects the two generalized Hopf bifurcation points GH_1 and GH_2. There is a codimension-2 cusp of limit cycles CPL, which also indicates the existence of an acute angle parameter region for the coexistence of three limit cycles. The whole bifurcation diagram is divided into eight regions from I to VIII and the corresponding phase portraits are depicted as follows.

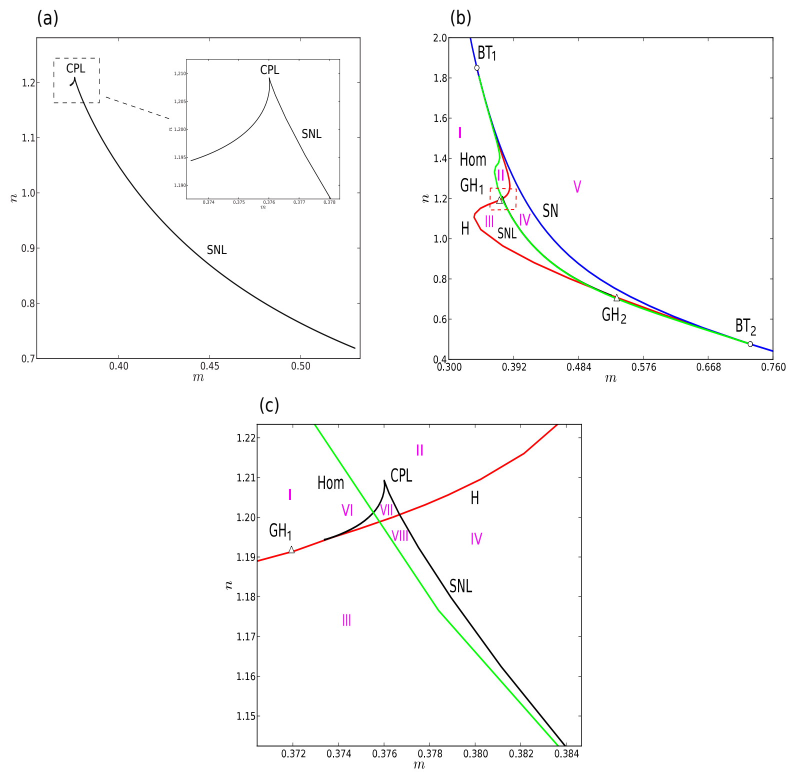

FIGURE 7. Two-parameter bifurcation diagram in system (2) with respect to m and n. (a) The saddle-node bifurcation curve of limit cycles SNL, where CPL denotes the codimension-2 cusp of limit cycles. (b) The location of saddle-node bifurcation curve SN (blue), saddle-node bifurcation curve of limit cycle SNL (black), homoclinic bifurcation curve Hom (green) and Hopf bifurcation curve H (red). (c) Zoomed bifurcation diagram of (b). Here BT, GH and CPL represent the points of Bagdanov-Taken bifurcation point, generalized Hopf bifurcation point and cusp of limit cycles, respectively.

I: two saddles, an unstable node and a stable focus.
II: two saddles, an unstable node and an unstable limit cycle that contains a stable focus.
III: two saddles, an unstable node and a stable limit cycle that contains an unstable focus.
IV: two saddles, an unstable node and an unstable focus.
V: a saddle and an unstable node.
VI: two saddles, an unstable node and a homoclinic cycle that contains two limit cycles (a big stable limit cycle contains a small unstable limit cycle) enclosing a stable hyperbolic positive equilibrium, bistability state.
VII: two saddles, an unstable node and a homoclinic cycle that contains three limit cycles (a big unstable limit cycle contains a medium stable limit cycle enclosing a small unstable limit cycle) enclosing a stable focus, bistability state.

VIII: two saddles, an unstable node and a homoclinic cycle that contains two limit cycles (a big unstable limit cycle contains a small stable limit cycle) enclosing an unstable focus.
Please see Figure 8 for more details.

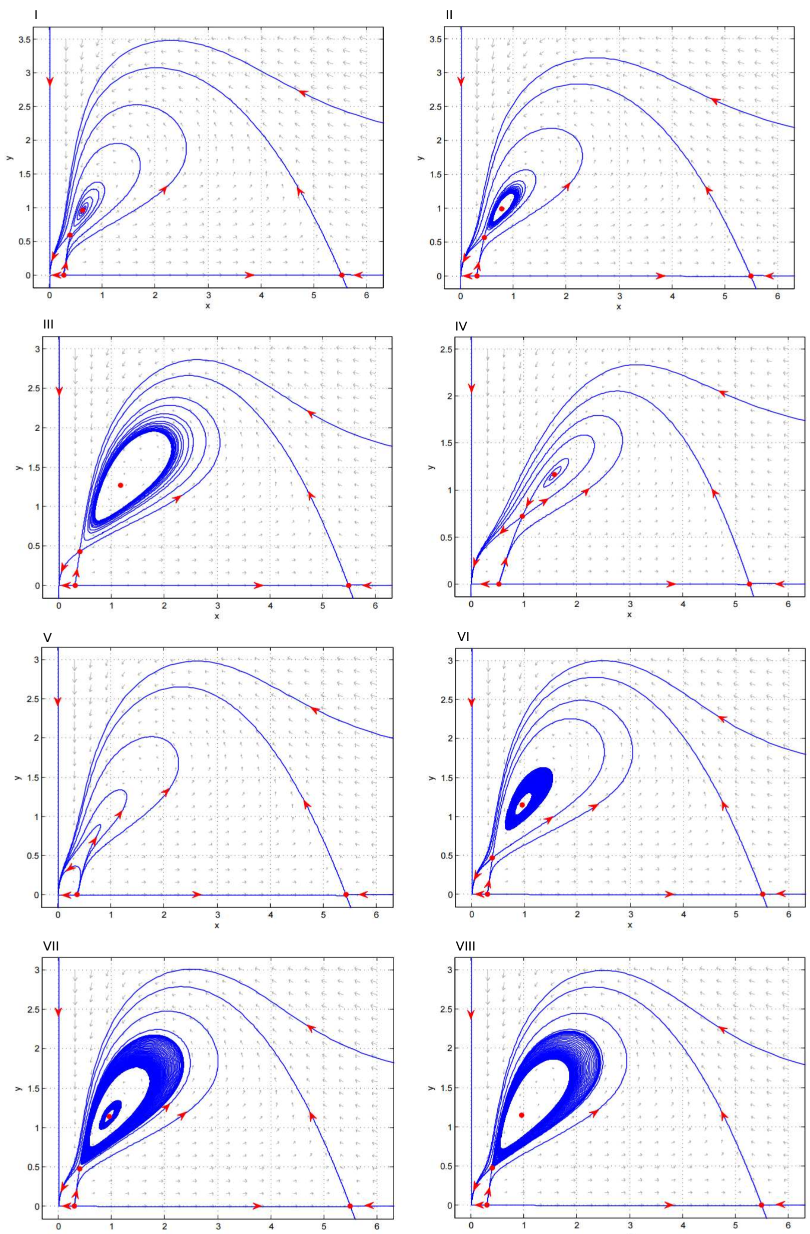

FIGURE 8. Phase portraits of regions: I-VIII in Figure 7.

4.4. a **and** n **as the primary bifurcation parameters.** If we consider a and n as the primary bifurcation parameters and use the parameter values in (39), we can create a two-parameter bifurcation diagram that includes the Hopf bifurcation curve H (red) and the saddle-node bifurcation curve of the limit cycle SNL (blue). This diagram is shown in Figure 9 (a). There are three generalized Hopf (GH) bifurcation points: $GH_1(2.93725 \times 10^{-1}, 1.00722)$ with $a = 1.34530 \times 10^{-5}$ and $n = 2.34217$, $GH_2(4.21597, 5.17619)$ with $a = 1.72751 \times 10^{-1}$ and $n = 1.22776$, and $GH_3(1.64766, 4.89813)$ with $a = 1.84974$ and $n = 2.97279$.

The saddle-node bifurcation curve of the limit cycles (SNL) bifurcating from the generalized Hopf bifurcation point GH_3 is evidently above the Hopf bifurcation curve (H), suggesting that there may be an isola bifurcation curve of the limit cycles. By setting n equal to $4, 4.05, 4.055$ and 4.115, we can obtain four isolas of the limit cycles [4, 52, 39], see Figure 9 (b, c) for details. Here the solid curve and the dotted curve represent the branches of stable limit cycle and unstable limit cycle, respectively.

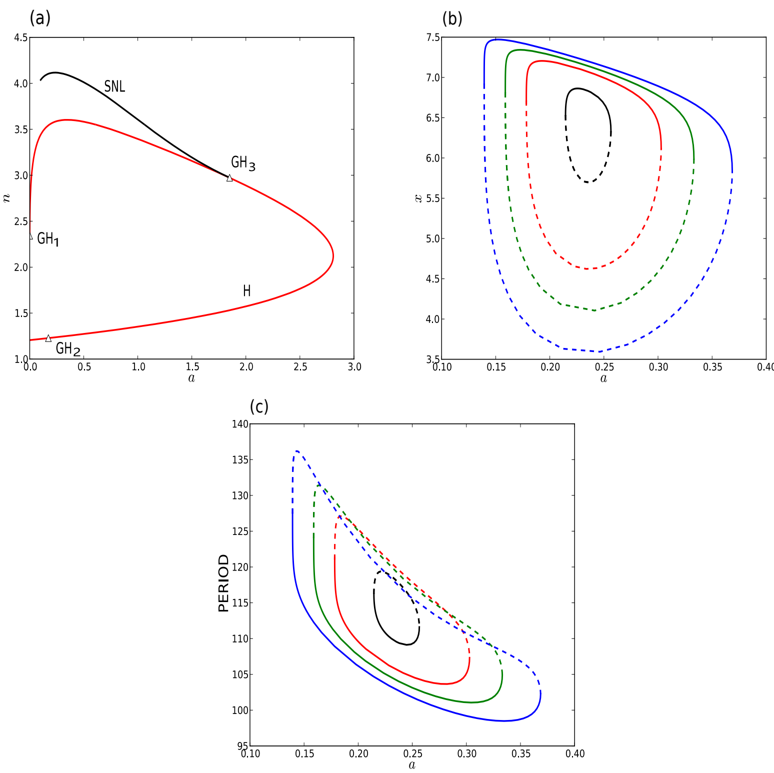

Figure 9. (a) Two-parameter bifurcation diagram with respect to a and n including the locations of Hopf bifurcation curve H (red) and saddle-node bifurcation curve of limit cycle SNL (blue). (b) A family of isolas of limit cycle in system (2) with respect to a; (c) Isolas about a and $Period$.

5. **Biological interpretations.** The inclusion of additive Allee effect has led to the emergence of rich dynamics including the codimension-2 cusp of limit cycles and the isola of limit cycles. This effect may cause the predator to go extinct while the prey survives after a long period of sustained oscillation. From the phase

portraits, it is observed that almost all bifurcation types occur near the boundary equilibrium located close to the origin. Two mechanisms have been identified as generators of these dynamics: one arises from the Hopf bifurcation point, while the other originates from an isola center of limit cycle. It is possible for the populations of predator and prey to increase as the half-saturation constant grows following sustained oscillations.

6. Conclusion and discussion.

This paper explores the intricate dynamics of a Leslie-Gower predator-prey model with additive Allee effect by using the dynamical systems approach. The study focuses on the existence and the type of boundary and positive equilibria, as well as saddle-node bifurcation, Hopf bifurcation, homoclinic bifurcation, saddle-node bifurcations of limit cycles and BT bifurcation of codimension 3. The research reveals the coexistence of two and three limit cycles, and highlights the significance of additive Allee effect in causing more complex dynamics, such as the codimension-3 BT bifurcation and codimension-2 cusp bifurcation of limit cycles. Of particular interest is the finding of isola bifurcation of limit cycles, which is reported for the first time in a Leslie-Gower predator-prey model through numerical analysis. The research establishes the existence of an acute angle parameter region where three coexistent limit cycles can be observed, indicating a codimension-2 cusp of limit cycles. Note that, the involvement of additive Allee effect may induce two kinds of oscillation mechanism for three coexistent limit cycles: one is from a Hopf bifurcation point; the other is: two limit cycles are bifurcating from a Hopf bifurcation point while the third one is from a homoclinic cycle mentioned in Aguirre et al. [2]. Numerical simulations and phase portraits are presented to illustrate the transition of global dynamics. It is suggested that future research should focus on analyzing the isola bifurcation of limit cycles, and identifying the boundary between two-limit-cycle region and three-limit-cycle region.

Acknowledgments. The authors are very grateful to Professor Pablo Aguirre for his helpful suggestions.

Appendix.

Appendix A. The Existence of Positive Equilibria in System (2).

To investigate the existence of positive equilibria $E_i(x_i, y_i)(i = 1, 2, 3, 4)$ of system (2) in the rectangular region $\Omega_2 = \{(x, y)|x \in (0, k], y \in [0, nk])\}$, we need to focus on the existence of positive roots $x_i(i = 1, 2, 3, 4)$ of equation (6)

$$\bar{F}_0(x) := \bar{F}(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0,$$

in the interval $(0, k)$, where $a_4 = r,\ a_3 = r(b - k) + knq,\ a_2 = ar + k(bnq + m - br),\ a_1 = ar(b-k)$ and $a_0 = ak(m-br)$. To achieve this, we introduce the following notions and equations

$$\begin{aligned}\bar{F}_1(x) &:= \bar{F}'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1 = 0,\\ \bar{F}_2(x) &:= \bar{F}''(x) = 2(6a_4x^2 + 3a_3x + a_2) = 0.\end{aligned}$$

Denote

$$\Delta_{\bar{F}_2} = 9a_3^2 - 24a_2a_4.$$

Then $\bar{F}_2(x)$ has no real root if $\Delta_{\bar{F}_2} < 0$, has one real root of multiplicity 2 if $\Delta_{\bar{F}_2} = 0$, which is denoted by $\bar{x}_{1,2} = -\frac{3a_3}{12a_4}$, and has two real roots if $\Delta_{\bar{F}_2} > 0$, which are marked as $\bar{x}_1$ and $\bar{x}_2$:

$$\bar{x}_1 = \frac{-3a_3 - \sqrt{9a_3^2 - 24a_2a_4}}{12a_4},$$
$$\bar{x}_2 = \frac{-3a_3 + \sqrt{9a_3^2 - 24a_2a_4}}{12a_4}.$$

Consequently, we analyze the existence of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ in three scenarios: $\Delta_{\bar{F}_2} < 0$, $\Delta_{\bar{F}_2} = 0$ and $\Delta_{\bar{F}_2} > 0$. Let $x_{i,i+1}$ be the coincidence point of x_i and $x_{i+1}(i = 1, 2, 3)$, and $x_{j,j+1,j+2}$ be the coincidence point of x_j, x_{j+1} and $x_{j+2}(j = 1, 2)$, then corresponding equilibria of system (2) are expressed as $E_{i,i+1}(x_{i,i+1}, y_{i,i+1})(i = 1, 2, 3)$ and $E_{j,j+1,j+2}(x_{j,j+1,j+2}, y_{j,j+1,j+2})(j = 1, 2)$, respectively.

Scenario 1: $\Delta_{\bar{F}_2} < 0$.
In this situation, $\bar{F}_2(x)$ has no real root and $\bar{F}''(x) > 0$ as $x \in (0, k]$, which demonstrates that $\bar{F}'(x)$ monotonically increases in the interval $(0, k]$. It should be noticed that $\bar{F}'(0) = a_1$ and $\bar{F}'(k) = r(a + k^2)(b + k) + k^2(nq(2b + 3k) + 2m) > 0$, then we have
(I) if $\bar{F}'(0) = a_1 \geq 0$, then $\bar{F}'(x) > 0, x \in (0, k]$, which indicates that $\bar{F}(x)$ is monotonically increasing in the interval $(0, k]$. Noting that $\bar{F}(0) = a_0$ and $\bar{F}(k) = akm + k^3(nq(b + k) + m) > 0$, we have
(i) for $\bar{F}(0) = a_0 < 0$, $\bar{F}_0(x)$ has a unique positive root in the interval $(0, k]$;
(ii) for $a_0 \geq 0$, $\bar{F}_0(x)$ has no positive root in the interval $(0, k]$;
(II) if $a_1 < 0$, then $\bar{F}_1(x)$ has a unique root $\hat{x}_1 \in (0, k]$, which indicates that $\bar{F}(x)$ monotonically decreases in $(0, \hat{x}_1)$ and monotonically increases in $(\hat{x}_1, k]$. Again noting that $\bar{F}(0) = a_0$ and $\bar{F}(k) > 0$, we have
(i) for $\bar{F}(0) = a_0 > 0$, $\bar{F}_0(x)$ has two positive roots in the interval $(0, k]$ if $\bar{F}(\hat{x}_1) < 0$, has one positive root of multiplicity 2 if $\bar{F}(\hat{x}_1) = 0$, has no positive root if $\bar{F}(\hat{x}_1) > 0$;
(ii) for $a_0 \leq 0$, $\bar{F}_0(x)$ has a unique positive root in the interval $(0, k]$.

Scenario 2: $\Delta_{\bar{F}_2} = 0$.
On this occasion, $\bar{F}_2(x)$ has one real root of multiplicity 2, which is denoted by $\bar{x}_{1,2} = -\frac{a_3}{4a_4}$ and $\bar{F}''(x) \geq 0, x \in (0, k]$, which illustrates that $\bar{F}'(x)$ is monotonically increasing in the interval $(0, k]$. It is noteworthy that $\bar{F}'(0) = a_1, \bar{F}'(\bar{x}_{1,2}) = \frac{a_3^3 - 4a_2a_3a_4 + 8a_1a_4^2}{8a_4^2}$ and $\bar{F}'(k) > 0$, then we have
(S2A) when $\bar{x}_{1,2} \leq 0$ or $\bar{x}_{1,2} \geq k$, we have $\bar{F}''(x) \geq 0, x \in (0, k]$, which illustrates that $\bar{F}'(x)$ is a monotonely increasing function in the interval $(0, k]$. Thus in this situation, the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1;
(S2B) when $0 < \bar{x}_{1,2} < k$, according to the signs of $\bar{F}'(0)$ and $\bar{F}'(\bar{x}_{1,2})$, we get
(I) if $\bar{F}'(0) = a_1 \geq 0$, then $\bar{F}'(x) > 0, x \in (0, k]$ and hence the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1 (I);
(II) if $a_1 < 0$ and $\bar{F}'(\bar{x}_{1,2}) \neq 0$, $\bar{F}_1(x)$ has a unique root in the interval $(0, k]$, and the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1 (II);
(III) if $a_1 < 0$ and $\bar{F}'(\bar{x}_{1,2}) = 0$, then $\bar{x}_{1,2}$ is a positive root of multiplicity 3 of

$\bar{F}_1(x)$. It shows that $\bar{F}(x)$ monotonically decreases in $(0, \bar{x}_{1,2})$ and monotonically increases in $(\bar{x}_{1,2}, k]$ and it should be mentioned that $\bar{F}(0) = a_0$ and $\bar{F}(k) > 0$. Therefore, we obtain that
(i) for $\bar{F}(0) = a_0 > 0$, $\bar{F}_0(x)$ has two positive roots in the interval $(0, k]$ if $\bar{F}(\bar{x}_{1,2}) < 0$, has one positive root of multiplicity 4 if $\bar{F}(\bar{x}_{1,2}) = 0$, has no positive root if $\bar{F}(\bar{x}_{1,2}) > 0$;
(ii) for $a_0 \leq 0$, $\bar{F}_0(x)$ has a unique positive root in the interval $(0, k]$.

Scenario 3: $\Delta_{\bar{F}_2} > 0$.
In this scenario, $\bar{F}_2(x)$ has two real roots, which are marked as $\bar{x}_1$ and $\bar{x}_2$. Obviously, $\bar{x}_1$ and $\bar{x}_2$ are the maximum and minimum value points of $\bar{F}'(x)$, respectively. By analyzing the positions of $\bar{x}_1$ and $\bar{x}_2$, we have
(S3A) when $\bar{x}_1 \geq k$ or $\bar{x}_2 \leq 0$, then $\bar{F}''(x) \geq 0, x \in (0, k]$ and the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1;
(S3B) when $0 < \bar{x}_1 < k \leq \bar{x}_2$, $\bar{F}'(x)$ monotonically increases in $(0, \bar{x}_1)$ and monotonically decreases in $(\bar{x}_1, k]$. Again noting that $\bar{F}'(0) = a_1, \bar{F}'(k) > 0$, we can obtain that
(I) if $\bar{F}'(0) = a_1 \geq 0$, then $\bar{F}'(x) > 0, x \in (0, k]$ and the distribution of positive roots of $\bar{F}_0(x)$ in the region $(0, k]$ is the same as Scenario 1 (I);
(II) if $a_1 < 0$, then $\bar{F}_1(x)$ has a unique root in the interval $(0, k]$. Consequently, the distribution of positive roots of $\bar{F}_0(x)$ in the region $(0, k]$ is the same as Scenario 1 (II);
(S3C) when $0 < \bar{x}_1 < \bar{x}_2 < k$, $\bar{F}'(x)$ firstly monotonically increases in $(0, \bar{x}_1)$, then monotonically decreases in $(\bar{x}_1, \bar{x}_2)$, and eventually monotonically increases in $(\bar{x}_2, k]$. Based on the signs of $\bar{F}'(0), \bar{F}'(\bar{x}_1)$ and $\bar{F}'(\bar{x}_2)$, we can obtain that
(I) if $\bar{F}'(0) = a_1 \geq 0$ and $\bar{F}'(\bar{x}_2) < 0$, then $\bar{F}_1(x)$ has two roots in the region $(0, k]$, which are denoted by $\hat{x}_2$ and $\hat{x}_3$. According to the signs of $\bar{F}(0), \bar{F}(\hat{x}_2)$ and $\bar{F}(\hat{x}_3)$, the distribution of the positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ can be summarized as follows
(i) for $\bar{F}(0) = a_0 \geq 0$, $\bar{F}_1(x)$ has two positive roots if $\bar{F}(\hat{x}_3) < 0$, denoted by $x_3 < x_4$, has one positive root of multiplicity 2 if $\bar{F}(\hat{x}_3) = 0$, and has no positive root if $\bar{F}(\hat{x}_3) > 0$;
(ii) for $a_0 < 0$, see Table 1;

signs of $\bar{F}(\hat{x}_2)$ and $\bar{F}(\hat{x}_3)$		Existence of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$
$\bar{F}(\hat{x}_2) > 0$	$\bar{F}(\hat{x}_3) < 0$	three positive roots, denoted by $x_2 < x_3 < x_4$
	$\bar{F}(\hat{x}_3) = 0$	two positive roots, one of them is a positive root of multiplicity 2, denoted by $x_2 < \hat{x}_3 = x_{3,4}$
	$\bar{F}(\hat{x}_3) > 0$	one positive root, denoted by x_2
$\bar{F}(\hat{x}_2) = 0$		two positive roots, one of them is a positive root of multiplicity 2, denoted by $\hat{x}_2 = x_{2,3} < x_4$
$\bar{F}(\hat{x}_2) < 0$		a unique positive root, denoted by x_4

Table 1. The distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ in (S3C) (I) (ii).

(II) if $a_1 \geq 0$ and $\bar{F}'(\bar{x}_2) = 0$, then $\bar{x}_2$ is a positive root of multiplicity 2 of equation $\bar{F}_1(x)$ and $\bar{F}'(x) \geq 0, x \in (0, k]$. We have that

(i) for $\bar{F}(0) = a_0 \geq 0$, $\bar{F}_0(x)$ has no positive root;
(ii) for $a_0 < 0$, $\bar{F}_0(x)$ has a unique positive root in the interval $(0, k]$ if $\bar{F}(\bar{x}_2) \neq 0$, and has one positive root of multiplicity 3 if $\bar{F}(\bar{x}_2) = 0$;
(III) if $a_1 \geq 0$ and $\bar{F}'(\bar{x}_2) > 0$, then $\bar{F}'(x) > 0, x \in (0, k]$. Thus the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1 (I);
(IV) if $a_1 < 0$ and $\bar{F}'(\bar{x}_1) < 0$, then $\bar{F}_1(x)$ has a unique positive root in the interval $(0, k]$. Therefore the distribution of the positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1 (II);
(V) if $a_1 < 0$ and $\bar{F}'(\bar{x}_1) = 0$, then $\bar{F}_1(x)$ has two positive roots $\bar{x}_1$ and $\hat{x}_2$ in the interval $(0, k]$, in which $\bar{x}_1$ is a root of multiplicity 2. With the assistance of signs of $\bar{F}(0), \bar{F}(\bar{x}_1)$ and $\bar{F}(\hat{x}_2)$, we have
(i) for $\bar{F}(0) = a_0 > 0$, see Table 2;
(ii) for $a_0 \leq 0$, $\bar{F}_0(x)$ has a unique positive root in the interval $(0, k]$, which is denoted by x_4;

<table>
<tr><th colspan="2">signs of $\bar{F}(\hat{x}_2)$ and $\bar{F}(\bar{x}_1)$</th><th>Existence of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$</th></tr>
<tr><td rowspan="3">$\bar{F}(\hat{x}_2) < 0$</td><td>$\bar{F}(\bar{x}_1) < 0$</td><td>two positive roots, denoted by $x_1 < x_4$</td></tr>
<tr><td>$\bar{F}(\bar{x}_1) = 0$</td><td>two positive roots, one of them is a positive root of multiplicity 3, denoted by $\bar{x}_1 = x_{1,2,3} < x_4$</td></tr>
<tr><td>$\bar{F}(\bar{x}_1) > 0$</td><td>two positive roots, denoted by $x_3 < x_4$</td></tr>
<tr><td colspan="2">$\bar{F}(\hat{x}_2) = 0$</td><td>one positive root of multiplicity 2, denoted by $\hat{x}_2 = x_{3,4}$</td></tr>
<tr><td colspan="2">$\bar{F}(\hat{x}_2) > 0$</td><td>no positive root</td></tr>
</table>

TABLE 2. The distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ in (S3C) (V) (i).

(VI) if $a_1 < 0, \bar{F}'(\bar{x}_1) > 0$ and $\bar{F}'(\bar{x}_2) > 0$, $\bar{F}_1(x)$ has a unique positive root in the interval $(0, k]$. Then we can get the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1 (II);
(VII) if $a_1 < 0, \bar{F}'(\bar{x}_1) > 0$ and $\bar{F}'(\bar{x}_2) = 0$, then $\bar{F}_1(x)$ has two roots $\hat{x}_2$ and $\bar{x}_2$ in the interval $(0, k]$, in which $\bar{x}_2$ is a root of multiplicity 2. Thus we have
(i) for $\bar{F}(0) = a_0 > 0$, see Table 3;
(ii) for $a_0 \leq 0$, $\bar{F}_0(x)$ has a unique positive root in the interval $(0, k]$ if $\bar{F}(\bar{x}_2) > 0$, which is denoted by x_3, has one positive root of multiplicity 3 if $\bar{F}(\bar{x}_2) = 0$, and has a unique positive root in the interval $(0, k]$ if $\bar{F}(\bar{x}_2) < 0$, which is denoted by x_4;

(VIII) if $a_1 < 0, \bar{F}'(\bar{x}_1) > 0$ and $\bar{F}'(\bar{x}_2) < 0$, $\bar{F}_1(x)$ has three positive roots in the region $(0, k]$, which are marked as $\hat{x}_2 < \hat{x}_3 < \hat{x}_4$. It means that $\bar{F}(x)$ firstly monotonically decreases in $(0, \hat{x}_2)$, then monotonically increases in $(\hat{x}_2, \hat{x}_3)$, monotonically decreases in $(\hat{x}_3, \hat{x}_4)$, and lastly monotonically increases in $(\hat{x}_4, k]$. Based on the signs of $\bar{F}(0), \bar{F}(\hat{x}_2), \bar{F}(\hat{x}_3)$ and $\bar{F}(\hat{x}_4)$, the distribution of the positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is concluded as follows
(i) for $\bar{F}(0) = a_0 > 0$, see Table 4;
(ii) for $a_0 \leq 0$, see Table 5;

(S3D) when $\bar{x}_1 \leq 0 < k \leq \bar{x}_2$, then $\bar{F}''(x) \leq 0, x \in (0, k]$, which suggests that $\bar{F}'(x)$ is a monotonically decreasing function in the interval $(0, k]$. With $\bar{F}'(k) > 0$,

signs of $\bar{F}(\hat{x}_2)$ and $\bar{F}(\bar{x}_2)$		Existence of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$
$\bar{F}(\hat{x}_2) < 0$	$\bar{F}(\bar{x}_2) > 0$	two positive roots, denoted by $x_1 < x_2$
	$\bar{F}(\bar{x}_2) = 0$	two positive roots, one of them is a positive root of multiplicity 3, denoted by $x_1 < \bar{x}_2 = x_{2,3,4}$
	$\bar{F}(\bar{x}_2) < 0$	two positive roots, denoted by $x_1 < x_4$
$\bar{F}(\hat{x}_2) = 0$		one positive root of multiplicity 2, denoted by $\hat{x}_2 = x_{1,2}$
$\bar{F}(\hat{x}_2) > 0$		no positive root

TABLE 3. The distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ in (S3C) (VII) (i).

signs of $\bar{F}(\hat{x}_2), \bar{F}(\hat{x}_3)$ and $\bar{F}(\hat{x}_4)$			Existence of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$
$\bar{F}(\hat{x}_2) < 0$	$\bar{F}(\hat{x}_3) > 0$	$\bar{F}(\hat{x}_4) < 0$	four positive roots, denoted by $x_1 < x_2 < x_3 < x_4$
		$\bar{F}(\hat{x}_4) = 0$	three positive roots, one of them is a positive root of multiplicity 2, denoted by $x_1 < x_2 < \hat{x}_4 = x_{3,4}$
		$\bar{F}(\hat{x}_4) > 0$	two positive roots, denoted by $x_1 < x_2$
	$\bar{F}(\hat{x}_3) = 0$		three positive roots, one of them is a positive root of multiplicity 2, denoted by $x_1 < \hat{x}_3 = x_{2,3} < x_4$
	$\bar{F}(\hat{x}_3) < 0$		two positive roots, denoted by $x_1 < x_4$
$\bar{F}(\hat{x}_2) = 0$		$\bar{F}(\hat{x}_4) < 0$	three positive roots, one of them is a positive root of multiplicity 2, denoted by $\hat{x}_2 = x_{1,2} < x_3 < x_4$
		$\bar{F}(\bar{x}_4) = 0$	two positive roots of multiplicity 2, denoted by $\hat{x}_2 = x_{1,2} < \hat{x}_4 = x_{3,4}$
		$\bar{F}(\hat{x}_4) > 0$	one positive root of multiplicity 2, denoted by $\hat{x}_2 = x_{1,2}$
$\bar{F}(\hat{x}_2) > 0$		$\bar{F}(\hat{x}_4) < 0$	two positive roots, denoted by $x_3 < x_4$
		$\bar{F}(\hat{x}_4) = 0$	one positive root of multiplicity 2, denoted by $\hat{x}_4 = x_{3,4}$
		$\bar{F}(\hat{x}_4) > 0$	no positive root

TABLE 4. The distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ in (S3C) (VIII) (i).

we have $\bar{F}'(x) > 0, x \in (0, k]$ and the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1 (I);
(S3E) when $\bar{x}_1 \leq 0 < \bar{x}_2 < k$, $\bar{F}_2(x)$ has a unique positive root $\bar{x}_2$. By this time $\bar{F}'(x)$ monotonically decreases in $(0, \bar{x}_2)$ and monotonically increases in $(\bar{x}_2, k]$. Judging the signs of $\bar{F}'(0)$ and $\bar{F}'(\bar{x}_2)$, we have that
(I) if $\bar{F}'(0) = a_1 > 0$ and $\bar{F}'(\bar{x}_2) > 0$, then $\bar{F}'(x) > 0, x \in (0, k]$ and the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0, k]$ is the same as Scenario 1 (I);
(II) if $a_1 > 0$ and $\bar{F}'(\bar{x}_2) = 0$, then $\bar{x}_2$ is a positive root of multiplicity 2 of $\bar{F}_1(x)$. Further, according to the signs of $\bar{F}(0)$ and $\bar{F}(\bar{x}_2)$, we can easily get
(i) for $\bar{F}(0) = a_0 \geq 0$, $\bar{F}_0(x)$ has no positive root;

signs of $\bar{F}(\hat{x}_3)$ and $\bar{F}(\hat{x}_4)$		Existence of positive roots of $\bar{F}_0(x)$ in the interval $(0,k]$
$\bar{F}(\hat{x}_3)>0$	$\bar{F}(\hat{x}_4)<0$	three positive roots, denoted by $x_2<x_3<x_4$
	$\bar{F}(\hat{x}_4)=0$	two positive roots, one of them is a positive root of multiplicity 2, denoted by $x_2<\hat{x}_4=x_{3,4}$
	$\bar{F}(\hat{x}_4)>0$	a unique positive root, denoted by x_2
$\bar{F}(\hat{x}_3)=0$		two positive roots, one of them is a positive root of multiplicity 2, denoted by $\hat{x}_3=x_{2,3}<x_4$
$\bar{F}(\hat{x}_3)<0$		a unique positive root, denoted by x_4

TABLE 5. The distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0,k]$ in (S3C) (VIII) (ii).

(ii) for $a_0<0$ and $\bar{F}'(\bar{x}_2)\neq 0$, $\bar{F}_0(x)$ has a unique positive root in the interval $(0,k)$;
(iii) for $a_0<0$ and $\bar{F}'(\bar{x}_2)=0$, $\bar{F}_0(x)$ has one positive root of multiplicity 3;
(III) if $a_1>0$ and $\bar{F}'(\bar{x}_2)<0$, then $\bar{F}_1(x)$ has two positive roots in the interval $(0,k]$, which are denoted by $\hat{x}_5$ and $\hat{x}_6$. Thus we have
(i) for $\bar{F}(0)=a_0\geq 0$, $\bar{F}_0(x)$ has two positive roots in the interval $(0,k]$ if $\bar{F}(\hat{x}_6)<0$, has one positive root of multiplicity 2 if $\bar{F}(\hat{x}_6)=0$, and has no positive root if $\bar{F}(\hat{x}_6)>0$;
(ii) for $a_0<0$, see Table 6;
(IV) if $a_1\leq 0$, then $\bar{F}_1(x)$ has a unique positive root in the interval $(0,k]$ and the distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0,k]$ is the same as Scenario 1 (II).

signs of $\bar{F}(\hat{x}_5)$ and $\bar{F}(\hat{x}_6)$		Existence of positive roots of $\bar{F}_0(x)$ in the interval $(0,k]$
$\bar{F}(\hat{x}_5)>0$	$\bar{F}(\hat{x}_6)<0$	three positive roots, denoted by $x_2<x_3<x_4$
	$\bar{F}(\hat{x}_6)=0$	two positive roots, one of them is a positive root of multiplicity 2, denoted by $x_2<\hat{x}_6=x_{3,4}$
	$\bar{F}(\hat{x}_6)>0$	a unique positive root, denoted by x_2
$\bar{F}(\hat{x}_5)=0$		two positive roots, one of them is a positive root of multiplicity 2, denoted by $\hat{x}_5=x_{2,3}<x_4$
$\bar{F}(\hat{x}_5)<0$		a unique positive root, denoted by x_4

TABLE 6. The distribution of positive roots of $\bar{F}_0(x)$ in the interval $(0,k]$ in (S3E) (III) (ii).

Appendix B. Coefficients in the proof of Theorem 2.6.

$$\hat{a}_{10}=\frac{rx^*(a+x^{*2})(-b+k-2x^*)}{k(2ab+3ax^*+x^{*3})},\quad \hat{a}_{01}=\frac{rx^*(a+x^{*2})(b-k+2x^*)}{kn(2ab+3ax^*+x^{*3})},$$

$$\hat{a}_{20}=-\frac{r}{k(a+x^{*2})(b+x^*)(2ab+3ax^*+x^{*3})}(a^2(b^2-b(k-3x^*)+x^*(k+x^*))$$
$$+ax^{*2}(5b^2-5b(k-3x^*)+x^*(10x^*-3k))+x^{*6}),$$

$$\hat{a}_{11}=\frac{2ar(b-k+2x^*)}{kn(2ab+3ax^*+x^{*3})},$$

$$\hat{a}_{30}=\frac{r}{k(2ab+3ax^*+x^{*3})}(\frac{4ax^*(x^{*2}-a)(b-k+2x^*)}{(a+x^{*2})^2}-\frac{b(2ak-ax^*+x^{*3})}{(b+x^*)^2}),$$

$$\hat{a}_{21} = \frac{ar(a-3x^{*2})(b-k+2x^*)}{knx^*(a+x^{*2})(2ab+3ax^*+x^{*3})}, \quad \hat{b}_{10} = ns, \quad \hat{b}_{01} = -s, \quad \hat{b}_{20} = -\frac{ns}{x^*},$$

$$\hat{b}_{11} = \frac{2s}{x^*}, \quad \hat{b}_{02} = -\frac{s}{nx^*}, \quad \hat{b}_{30} = \frac{ns}{x^{*2}}, \quad \hat{b}_{21} = -\frac{2s}{x^{*2}}, \quad \hat{b}_{12} = \frac{s}{nx^{*2}};$$

$$\hat{c}_{20} = \frac{\hat{a}_{10}(\hat{a}_{01}\hat{b}_{11} - \hat{a}_{11}\hat{b}_{01}) - \hat{a}_{10}^2\hat{b}_{02} + \hat{a}_{01}(\hat{a}_{20}\hat{b}_{01} - \hat{a}_{01}\hat{b}_{20})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{c}_{11} = \frac{-2\hat{a}_{01}^2\hat{b}_{20} + \hat{a}_{01}\hat{b}_{01}(2\hat{a}_{20} - \hat{b}_{11}) + \hat{a}_{11}\hat{b}_{01}^2 + \hat{a}_{10}(\hat{a}_{01}\hat{b}_{11} - \hat{b}_{01}(\hat{a}_{11} - 2\hat{b}_{02}))}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{c}_{02} = \frac{-\hat{a}_{01}^2\hat{b}_{20} + \hat{a}_{01}\hat{b}_{01}(\hat{a}_{20} - \hat{b}_{11}) + \hat{b}_{01}^2(\hat{a}_{11} - \hat{b}_{02})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{c}_{30} = \frac{\hat{a}_{10}(\hat{a}_{01}\hat{b}_{21} - \hat{a}_{21}\hat{b}_{01}) - \hat{a}_{10}^2\hat{b}_{12} + \hat{a}_{01}(\hat{a}_{30}\hat{b}_{01} - \hat{a}_{01}\hat{b}_{30})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{c}_{21} = \frac{1}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})}(-3\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{01}\hat{b}_{01}(3\hat{a}_{30} - \hat{b}_{21}) + \hat{a}_{21}\hat{b}_{01}^2 - \hat{a}_{10}^2\hat{b}_{12} + 2\hat{a}_{10}$$
$$\times(\hat{b}_{01}(\hat{b}_{12} - \hat{a}_{21}) + \hat{a}_{01}\hat{b}_{21})),$$

$$\hat{c}_{12} = \frac{1}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})}(-3\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{01}\hat{b}_{01}(3\hat{a}_{30} - 2\hat{b}_{21}) + \hat{b}_{01}^2(2\hat{a}_{21} - \hat{b}_{12}) + \hat{a}_{10}$$
$$\times(\hat{a}_{01}\hat{b}_{21} - \hat{b}_{01}(\hat{a}_{21} - 2\hat{b}_{12}))),$$

$$\hat{c}_{03} = \frac{-\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{01}\hat{b}_{01}(\hat{a}_{30} - \hat{b}_{21}) + \hat{b}_{01}^2(\hat{a}_{21} - \hat{b}_{12})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{01} = \frac{\hat{a}_{10}(\hat{a}_{10} + \hat{b}_{01}) + \hat{a}_{01}\hat{b}_{10} + \hat{b}_{01}^2}{\hat{a}_{10} + \hat{b}_{01}} = p(x^*),$$

$$\hat{d}_{20} = \frac{\hat{a}_{01}^2\hat{b}_{20} + \hat{a}_{10}\hat{a}_{01}(\hat{a}_{20} - \hat{b}_{11}) + \hat{a}_{10}^2(\hat{b}_{02} - \hat{a}_{11})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{11} = \frac{\hat{a}_{10}(\hat{b}_{01}(\hat{a}_{11} - 2\hat{b}_{02}) + \hat{a}_{01}(2\hat{a}_{20} - \hat{b}_{11})) + \hat{a}_{01}(2\hat{a}_{01}\hat{b}_{20} + \hat{b}_{01}\hat{b}_{11}) - \hat{a}_{10}^2\hat{a}_{11}}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{02} = \frac{\hat{a}_{01}^2\hat{b}_{20} + \hat{a}_{01}(\hat{a}_{10}\hat{a}_{20} + \hat{b}_{01}\hat{b}_{11}) + \hat{b}_{01}(\hat{a}_{10}\hat{a}_{11} + \hat{b}_{01}\hat{b}_{02})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{30} = \frac{\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{10}\hat{a}_{01}(\hat{a}_{30} - \hat{b}_{21}) + \hat{a}_{10}^2(\hat{b}_{12} - \hat{a}_{21})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})},$$

$$\hat{d}_{21} = \frac{1}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})}(\hat{a}_{10}^2(\hat{b}_{12} - 2\hat{a}_{21}) + \hat{a}_{10}(\hat{b}_{01}(\hat{a}_{21} - 2\hat{b}_{12}) + \hat{a}_{01}(3\hat{a}_{30} - 2\hat{b}_{21}))$$
$$+\hat{a}_{01}(3\hat{a}_{01}\hat{b}_{30} + \hat{b}_{01}\hat{b}_{21})),$$

$$\hat{d}_{12} = \frac{1}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})}(3\hat{a}_{01}^2\hat{b}_{30} + 2\hat{a}_{01}\hat{b}_{01}\hat{b}_{21} + \hat{a}_{10}(2\hat{b}_{01}(\hat{a}_{21} - \hat{b}_{12}) + \hat{a}_{01}(3\hat{a}_{30} -$$
$$\hat{b}_{21})) - \hat{a}_{10}^2\hat{a}_{21} + \hat{b}_{01}^2\hat{b}_{12}),$$

$$\hat{d}_{03} = \frac{\hat{a}_{01}^2\hat{b}_{30} + \hat{a}_{01}(\hat{a}_{10}\hat{a}_{30} + \hat{b}_{01}\hat{b}_{21}) + \hat{b}_{01}(\hat{a}_{10}\hat{a}_{21} + \hat{b}_{01}\hat{b}_{12})}{\hat{a}_{01}(\hat{a}_{10} + \hat{b}_{01})}.$$

$$a_{10}^* = -\frac{r(a+x^{*2})^2}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}},$$
$$a_{01}^* = \frac{r(a+x^{*2})^2}{n(a^2-4a(b+x^*)(b+2x^*)-x^{*4})},$$
$$a_{20}^* = \frac{ar(a+x^{*2})}{x^*(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$a_{11}^* = -\frac{2ar(a+x^{*2})}{nx^*(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$a_{30}^* = \frac{4arx^*(a-x^*(2b+x^*))}{(a+x^{*2})(b+x^*)(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$a_{21}^* = -\frac{ar(a-3x^{*2})}{nx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$a_{40}^* = \frac{ar}{-a^2+4a(b+x^*)(b+2x^*)+x^{*4}}(\frac{4x^*(x^{*2}-3a)}{(a+x^{*2})^2}+\frac{4b}{(b+x^*)^2}+\frac{1}{x^*}),$$
$$a_{31}^* = \frac{4ar(a-x^{*2})}{nx^*(a+x^{*2})(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$b_{10}^* = -\frac{nr(a+x^{*2})^2}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}},$$
$$b_{01}^* = \frac{r(a+x^{*2})^2}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}},$$
$$b_{20}^* = \frac{nr(a+x^{*2})^2}{x^*(a^2-4a(b+x^*)(b+2x^*)-x^{*4})},$$
$$b_{11}^* = \frac{2r(a+x^{*2})^2}{x^*(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$b_{02}^* = \frac{r(a+x^{*2})^2}{nx^*(a^2-4a(b+x^*)(b+2x^*)-x^{*4})},$$
$$b_{30}^* = \frac{nr(a+x^{*2})^2}{x^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$b_{21}^* = \frac{2r(a+x^{*2})^2}{x^{*2}(a^2-4a(b+x^*)(b+2x^*)-x^{*4})},$$
$$b_{12}^* = \frac{r(a+x^{*2})^2}{nx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$b_{40}^* = \frac{nr(a+x^{*2})^2}{x^{*3}(a^2-4a(b+x^*)(b+2x^*)-x^{*4})},$$
$$b_{31}^* = \frac{2r(a+x^{*2})^2}{x^{*3}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$
$$b_{22}^* = \frac{r(a+x^{*2})^2}{nx^{*3}(a^2-4a(b+x^*)(b+2x^*)-x^{*4})};$$
$$c_{20}^* = \frac{a_{10}^{*2}b_{02}^*}{a_{01}^*} - a_{10}^*b_{11}^* - a_{20}^*b_{01}^* + a_{11}^*b_{10}^* + a_{01}^*b_{20}^*,\ c_{02}^* = \frac{a_{11}^*+b_{02}^*}{a_{01}^*},$$
$$c_{30}^* = -a_{30}^*b_{01}^* + a_{21}^*b_{10}^* - a_{20}^*b_{11}^* + a_{11}^*b_{20}^* + a_{01}^*b_{30}^* + \frac{a_{10}^{*2}(a_{01}^*b_{12}^* - a_{11}^*b_{02}^*)}{a_{01}^{*2}} + a_{10}^*$$

$$\times(\frac{2a_{20}^*b_{02}^*}{a_{01}^*} - b_{21}^*),$$

$$c_{21}^* = \frac{1}{a_{01}^{*2}}(a_{10}^*(2a_{11}^*b_{02}^* - 2a_{01}^*(a_{21}^* + b_{12}^*) + a_{11}^{*2}) + a_{01}^*(a_{01}^*(3a_{30}^* + b_{21}^*) - a_{20}^*(a_{11}^* + 2b_{02}^*))),$$

$$c_{12}^* = \frac{1}{a_{01}^{*2}}(a_{01}^*(2a_{21}^* + b_{12}^*) - a_{11}^*(a_{11}^* + b_{02}^*)),$$

$$c_{40}^* = -a_{40}^*b_{01}^* + a_{31}^*b_{10}^* - a_{30}^*b_{11}^* + a_{21}^*b_{20}^* - a_{20}^*b_{21}^* + a_{11}^*b_{30}^* - a_{10}^*b_{31}^* + a_{01}^*b_{40}^* + \frac{a_{10}^{*2}a_{11}^{*2}b_{02}^*}{a_{01}^{*3}} - \frac{a_{10}^*(a_{10}^*a_{21}^*b_{02}^* + a_{11}^*(2a_{20}^*b_{02}^* + a_{10}^*b_{12}^*))}{a_{01}^{*2}} + \frac{1}{a_{01}^*}(a_{10}^{*2}b_{22}^* + 2 \times a_{30}^*a_{10}^*b_{02}^* + 2a_{20}^*a_{10}^*b_{12}^* + a_{20}^{*2}b_{02}^*),$$

$$c_{31}^* = \frac{1}{a_{01}^{*3}}(-a_{10}^*(2a_{11}^{*2}b_{02}^* - a_{01}^*a_{11}^*(3a_{21}^* + 2b_{12}^*) + a_{01}^*(a_{01}^*(3a_{31}^* + 2b_{22}^*) - 2a_{21}^*b_{02}^*) + a_{11}^{*3}) + a_{01}^*(a_{11}^*(2a_{20}^*b_{02}^* - a_{01}^*a_{30}^*) + a_{01}^*(-2a_{30}^*b_{02}^* - 2a_{20}^*(a_{21}^* + b_{12}^*) + a_{01}^*(4a_{40}^* + b_{31}^*)) + a_{20}^*a_{11}^{*2})),$$

$$c_{22}^* = \frac{a_{11}^{*2}b_{02}^* - a_{01}^*a_{11}^*(3a_{21}^* + b_{12}^*) + a_{01}^*(a_{01}^*(3a_{31}^* + b_{22}^*) - a_{21}^*b_{02}^*) + a_{11}^{*3}}{a_{01}^{*3}}.$$

Appendix C. Coefficients in the proof of Theorem 3.1 and Theorem 3.4.

$$\bar{a}_{00} = x^*(\frac{\delta_2 r x^*(a + x^{*2})(-b + k - 2x^*)}{k(2ab + 3ax^* + x^{*3})(a + \delta_2 + x^{*2})} - \frac{\delta_1}{b + x^*}),$$

$$\bar{a}_{10} = \frac{rx^*(-b + k - 2x^*)}{k(2ab + 3ax^* + x^{*3})(a + \delta_2 + x^{*2})^2}(4a\delta_2(a + x^{*2}) + \delta_2^2(3a + x^{*2}) + (a + x^{*2})^3) - \frac{b\delta_1}{(b + x^*)^2},$$

$$\bar{a}_{01} = \frac{rx^*(a + x^{*2})^2(b - k + 2x^*)}{kn(2ab + 3ax^* + x^{*3})(a + \delta_2 + x^{*2})},$$

$$\bar{a}_{20} = \frac{r(2ak - ax^* + x^{*3})}{k(2ab + 3ax^* + x^{*3})} - \frac{rx^*(2ak - ax^* + x^{*3})}{k(b + x^*)(2ab + 3ax^* + x^{*3})} + \frac{r(a + \delta_2)(a + x^{*2})^2(a + \delta_2 - 3x^{*2})(b - k + 2x^*)}{k(2ab + 3ax^* + x^{*3})(a + \delta_2 + x^{*2})^3} + \frac{\delta_1}{(b + x^*)^2} - \frac{\delta_1 x^*}{(b + x^*)^3} - \frac{r}{k},$$

$$\bar{a}_{11} = \frac{2r(a + \delta_2)(a + x^{*2})^2(b - k + 2x^*)}{kn(2ab + 3ax^* + x^{*3})(a + \delta_2 + x^{*2})^2},$$

$$\bar{b}_{10} = \frac{nrx^*(a + x^{*2})(-b + k - 2x^*)}{k(2ab + 3ax^* + x^{*3})},$$

$$\bar{b}_{01} = \frac{rx^*(a + x^{*2})(b - k + 2x^*)}{k(2ab + 3ax^* + x^{*3})},$$

$$\bar{b}_{20} = \frac{nr(a + x^{*2})(b - k + 2x^*)}{k(2ab + 3ax^* + x^{*3})},$$

$$\bar{b}_{11} = \frac{2r(a + x^{*2})(-b + k - 2x^*)}{k(2ab + 3ax^* + x^{*3})},$$

$$\bar{b}_{02} = \frac{r(a+x^{*2})(b-k+2x^*)}{kn(2ab+3ax^*+x^{*3})};$$

$$\bar{c}_{00} = \bar{a}_{00}(\frac{\bar{a}_{00}\bar{b}_{02}}{\bar{a}_{01}} - \bar{b}_{01}),$$

$$\bar{c}_{10} = -\frac{\bar{a}_{11}\bar{a}_{00}^2\bar{b}_{02}}{\bar{a}_{01}^2} - \bar{a}_{00}\bar{b}_{11} + \bar{a}_{10}(\frac{2\bar{a}_{00}\bar{b}_{02}}{\bar{a}_{01}} - \bar{b}_{01}) + \bar{a}_{01}\bar{b}_{10},$$

$$\bar{c}_{01} = -\frac{\bar{a}_{00}(\bar{a}_{11}+2\bar{b}_2)}{\bar{a}_{01}} + \bar{a}_{10} + \bar{b}_{01},$$

$$\bar{c}_{20} = -\bar{a}_{20}\bar{b}_{01} + \bar{a}_{11}\bar{b}_{10} - \bar{a}_{10}\bar{b}_{11} + \bar{a}_{01}\bar{b}_{20} + \frac{\bar{a}_{00}^2\bar{a}_{11}^2\bar{b}_{02}}{\bar{a}_{01}^3} - \frac{2\bar{a}_{00}\bar{a}_{10}\bar{a}_{11}\bar{b}_{02}}{\bar{a}_{01}^2} + \frac{(\bar{a}_{10}^2+2\bar{a}_{00}\bar{a}_{20})\bar{b}_{02}}{\bar{a}_{01}},$$

$$\bar{c}_{11} = -\frac{(\bar{a}_{01}\bar{a}_{10}-\bar{a}_{00}\bar{a}_{11})(\bar{a}_{11}+2\bar{b}_{02})}{\bar{a}_{01}^2} + 2\bar{a}_{20} + \bar{b}_{11},\ \bar{c}_{02} = \frac{\bar{a}_{11}+\bar{b}_{02}}{\bar{a}_{01}};$$

$$\bar{d}_{00} = \bar{c}_{00},\ \bar{d}_{10} = \bar{c}_{10} - 2\bar{c}_{00}\bar{c}_{02},\ \bar{d}_{01} = \bar{c}_{01},$$

$$\bar{d}_{20} = \bar{c}_{00}\bar{c}_{02}^2 - 2\bar{c}_{10}\bar{c}_{02} + \bar{c}_{20},\ \bar{d}_{11} = \bar{c}_{11} - \bar{c}_{01}\bar{c}_{02};$$

$$\bar{e}_{00} = \bar{d}_{00} - \frac{\bar{d}_{10}^2}{4\bar{d}_{20}},\ \bar{e}_{01} = \bar{d}_{01} - \frac{\bar{d}_{10}\bar{d}_{11}}{2\bar{d}_{20}},\ \bar{e}_{20} = \bar{d}_{20},\ \bar{e}_{11} = \bar{d}_{11}.$$

$$\bar{a}_{00}^* = \frac{1}{(b+x^*)(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})(a+\kappa_2+x^{*2})}(\kappa_1 x^*(a^2-4a \times(b+x^*)(b+2x^*)-x^{*4})(a+\kappa_2+x^{*2}) + \kappa_2 r x^*(a+x^{*2})^2(b+x^*)),$$

$$\bar{a}_{10}^* = \frac{1}{-a^2+4a(b+x^*)(b+2x^*)+x^{*4}}(-\frac{b\kappa_1}{(b+x^*)^2}(-a^2+4a(b+x^*)(b+2x^*) +x^{*4}) - \frac{2r}{(a+\kappa_2+x^{*2})^2}(a+\kappa_2)(a+x^{*2})^3 + r(3a+x^{*2})(a+x^{*2})),$$

$$\bar{a}_{01}^* = -\frac{qx^{*2}}{a+\kappa_2+x^{*2}},$$

$$\bar{a}_{20}^* = \frac{b\kappa_1}{(b+x^*)^3} + \frac{1}{x^*(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})(a+\kappa_2+x^{*2})^3}(r(\kappa_2 \times(4a^2-5ax^{*2}+3x^{*4})(a+x^{*2})^2 + \kappa_2^2(5a^2-8ax^{*2}-x^{*4})(a+x^{*2}) + 2a\kappa_2^3(a-x^{*2}) + a(a+x^{*2})^4)),$$

$$\bar{a}_{11}^* = -\frac{2qx^*(a+\kappa_2)}{(a+\kappa_2+x^{*2})^2},$$

$$\bar{a}_{30}^* = -\frac{4r}{(a^2-4a(b+x^*)(b+2x^*)-x^{*4})(a+\kappa_2+x^{*2})^4}(a+\kappa_2)(a+x^{*2})^3(a +\kappa_2-x^{*2}) + \frac{x^*}{(b+x^*)^4}(\kappa_1 - \frac{4ar}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}}(b+x^*)^3) - \frac{1}{(b+x^*)^3}(\kappa_1 - \frac{4ar}{a^2-4a(b+x^*)(b+2x^*)-x^{*4}}(b+x^*)^3),$$

$$\bar{a}_{21}^* = -\frac{q(a+\kappa_2)(a+\kappa_2-3x^{*2})}{(a+\kappa_2+x^{*2})^3},$$

$$\bar{a}_{40}^* = \frac{r(a+\kappa_2)(a+x^{*2})^3(a^2+2\kappa_2(a-5x^{*2})-10ax^{*2}+\kappa_2^2+5x^{*4})}{x^*(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})(a+\kappa_2+x^{*2})^5}$$

$$+\frac{b\kappa_1}{(b+x^*)^5}+\frac{4abr}{(b+x^*)^2(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})},$$

$$\bar{a}^*_{31}=-\frac{4qx^*(a+\kappa_2)(-a-\kappa_2+x^{*2})}{(a+\kappa_2+x^{*2})^4},$$

$$\bar{b}^*_{00}=(-\kappa_3 r^2x^*(a+x^{*2})^5)/(r(a+x^{*2})^3(a^2-4a(b+x^*)(b+2x^*)-x^{*4})-\kappa_3 \times qx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})^2),$$

$$\bar{b}^*_{10}=(r^3(a+x^{*2})^8)/(qx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})^2(\kappa_3qx^{*2}(-a^2+ 4a(b+x^*)(b+2x^*)+x^{*4})+r(a+x^{*2})^3)),$$

$$\bar{b}^*_{01}=(r(a+x^{*2})^2(r(a+x^{*2})^3-\kappa_3qx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})))/ (r(a+x^{*2})^3(a^2-4a(b+x^*)(b+2x^*)-x^{*4})-\kappa_3qx^{*2}(-a^2+4a(b+x^*) \times(b+2x^*)+x^{*4})^2),$$

$$\bar{b}^*_{20}=(-r^3(a+x^{*2})^8)/(qx^{*3}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})^2(\kappa_3qx^{*2}(-a^2+ 4a(b+x^*)(b+2x^*)+x^{*4})+r(a+x^{*2})^3)),$$

$$\bar{b}^*_{11}=(2r^2(a+x^{*2})^5)/(x^*(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})(\kappa_3qx^{*2}(-a^2+4a \times(b+x^*)(b+2x^*)+x^{*4})+r(a+x^{*2})^3)),$$

$$\bar{b}^*_{02}=(-qrx^*(a+x^{*2})^2)/(\kappa_3qx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})+r(a+ x^{*2})^3),$$

$$\bar{b}^*_{30}=(r^3(a+x^{*2})^8)/(qx^{*4}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})^2(\kappa_3qx^{*2}(-a^2+ 4a(b+x^*)(b+2x^*)+x^{*4})+r(a+x^{*2})^3)),$$

$$\bar{b}^*_{21}=-(2r^2(a+x^{*2})^5)/(x^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})(\kappa_3qx^{*2}(-a^2+ 4a(b+x^*)(b+2x^*)+x^{*4})+r(a+x^{*2})^3)),$$

$$\bar{b}^*_{12}=(qr(a+x^{*2})^2)(\kappa_3qx^{*2}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})+r(a+x^{*2})^3),$$

$$\bar{b}^*_{40}=-(r^3(a+x^{*2})^8)/(qx^{*5}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})^2(\kappa_3qx^{*2}(-a^2+ 4a(b+x^*)(b+2x^*)+x^{*4})+r(a+x^{*2})^3)),$$

$$\bar{b}^*_{31}=(2r^2(a+x^{*2})^5)/(x^{*3}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})(\kappa_3qx^{*2}(-a^2+ 4a(b+x^*)(b+2x^*)+x^{*4})+r(a+x^{*2})^3)),$$

$$\bar{b}^*_{22}=(-qr(a+x^{*2})^2)/(\kappa_3qx^{*3}(-a^2+4a(b+x^*)(b+2x^*)+x^{*4})+rx^*(a +x^{*2})^3);$$

$$\bar{c}^*_{00}=\frac{\bar{a}^{*2}_{00}\bar{b}^*_{02}}{\bar{a}^*_{01}}-\bar{a}^*_{00}\bar{b}^*_{01}+\bar{a}^*_{01}\bar{b}^*_{00},$$

$$\bar{c}^*_{10}=\frac{\bar{a}^{*2}_{00}\bar{b}^*_{12}}{\bar{a}^*_{01}}-\bar{a}^*_{00}\bar{b}^*_{11}+\bar{a}^*_{10}(\frac{2\bar{a}^*_{00}\bar{b}^*_{02}}{\bar{a}^*_{01}}-\bar{b}^*_{01})+\bar{a}^*_{11}(\bar{b}^*_{00}-\frac{\bar{a}^{*2}_{00}\bar{b}^*_{02}}{\bar{a}^{*2}_{01}})+\bar{a}^*_{01}\bar{b}^*_{10},$$

$$\bar{c}^*_{01}=-\frac{\bar{a}^*_{00}(\bar{a}^*_{11}+2\bar{b}^*_{02})}{\bar{a}^*_{01}}+\bar{a}^*_{10}+\bar{b}^*_{01},$$

$$\bar{c}^*_{20}=\frac{\bar{a}^{*2}_{00}\bar{a}^{*2}_{11}\bar{b}^*_{02}}{\bar{a}^{*3}_{01}}+\bar{a}^*_{11}\bar{b}^*_{10}+\bar{a}^*_{21}\bar{b}^*_{00}-\bar{a}^*_{20}\bar{b}^*_{01}-\bar{a}^*_{10}\bar{b}^*_{11}+\bar{a}^*_{01}\bar{b}^*_{20}-\bar{a}^*_{00}\bar{b}^*_{21}+\frac{1}{\bar{a}^*_{01}} \times(\bar{a}^{*2}_{00}\bar{b}^*_{22}+2\bar{a}^*_{20}\bar{a}^*_{00}\bar{b}^*_{02}+2\bar{a}^*_{10}\bar{a}^*_{00}\bar{b}^*_{12}+\bar{a}^{*2}_{10}\bar{b}^*_{02})-\frac{\bar{a}^*_{00}}{\bar{a}^{*2}_{01}}(2\bar{a}^*_{10}\bar{a}^*_{11}\bar{b}^*_{02}+\bar{a}^*_{00} \times(\bar{a}^*_{21}\bar{b}^*_{02}+\bar{a}^*_{11}\bar{b}^*_{12})),$$

$$\bar{c}_{11}^* = \frac{\bar{a}_{00}^*\bar{a}_{11}^*(\bar{a}_{11}^* + 2\bar{b}_{02}^*) - \bar{a}_{01}^*(\bar{a}_{10}^*(\bar{a}_{11}^* + 2\bar{b}_{02}^*) + 2\bar{a}_{00}^*(\bar{a}_{21}^* + \bar{b}_{12}^*))}{\bar{a}_{01}^{*2}} + 2\bar{a}_{20}^* + \bar{b}_{11}^*,$$

$$\bar{c}_{02}^* = \frac{\bar{a}_{11}^* + \bar{b}_{02}^*}{\bar{a}_{01}^*},$$

$$\bar{c}_{30}^* = -\frac{\bar{a}_{00}^{*2}\bar{a}_{11}^{*3}\bar{b}_{02}^*}{\bar{a}_{01}^{*4}} + \bar{a}_{11}^*\bar{b}_{20}^* + \bar{a}_{31}^*\bar{b}_{00}^* - \bar{a}_{30}^*\bar{b}_{01}^* + \bar{a}_{21}^*\bar{b}_{10}^* - \bar{a}_{20}^*\bar{b}_{11}^* - \bar{a}_{10}^*\bar{b}_{21}^* + \bar{a}_{01}^* \times\bar{b}_{30}^* - \bar{a}_{00}^*\bar{b}_{31}^* + \frac{1}{\bar{a}_{01}^{*3}}(\bar{a}_{00}^*\bar{a}_{11}^*(2(\bar{a}_{10}^*\bar{a}_{11}^* + \bar{a}_{00}^*\bar{a}_{21}^*)\bar{b}_{02}^* + \bar{a}_{00}^*\bar{a}_{11}^*\bar{b}_{12}^*)) + \frac{1}{\bar{a}_{01}^*} \times(\bar{a}_{10}^{*2}\bar{b}_{12}^* + 2\bar{a}_{10}^*(\bar{a}_{20}^*\bar{b}_{02}^* + \bar{a}_{00}^*\bar{b}_{22}^*) + 2\bar{a}_{00}^*(\bar{a}_{30}^*\bar{b}_{02}^* + \bar{a}_{20}^*\bar{b}_{12}^*)) - \frac{1}{\bar{a}_{01}^{*2}}(\bar{a}_{11}^*\bar{a}_{10}^{*2} \times\bar{b}_{02}^* + 2\bar{a}_{00}^*\bar{a}_{10}^*(\bar{a}_{21}^*\bar{b}_{02}^* + \bar{a}_{11}^*\bar{b}_{12}^*) + \bar{a}_{00}^*(\bar{a}_{00}^*(\bar{a}_{31}^*\bar{b}_{02}^* + \bar{a}_{21}^*\bar{b}_{12}^*) + \bar{a}_{11}^*(2\bar{a}_{20}^* \times\bar{b}_{02}^* + \bar{a}_{00}^*\bar{b}_{22}^*))),$$

$$\bar{c}_{21}^* = \frac{1}{\bar{a}_{01}^{*3}}(-\bar{a}_{01}^{*2}(\bar{a}_{20}^*(\bar{a}_{11}^* + 2\bar{b}_{02}^*) + 2\bar{a}_{10}^*(\bar{a}_{21}^* + \bar{b}_{12}^*) + \bar{a}_{00}^*(3\bar{a}_{31}^* + 2\bar{b}_{22}^*)) + \bar{a}_{01}^* \times(\bar{a}_{10}^*\bar{a}_{11}^*(\bar{a}_{11}^* + 2\bar{b}_{02}^*) + \bar{a}_{00}^*(\bar{a}_{21}^*(3\bar{a}_{11}^* + 2\bar{b}_{02}^*) + 2\bar{a}_{11}^*\bar{b}_{12}^*)) - \bar{a}_{00}^*\bar{a}_{11}^{*2}(\bar{a}_{11}^* + 2\bar{b}_{02}^*)) + 3\bar{a}_{30}^* + \bar{b}_{21}^*,$$

$$\bar{c}_{12}^* = \frac{\bar{a}_{01}^*(2\bar{a}_{21}^* + \bar{b}_{12}^*) - \bar{a}_{11}^*(\bar{a}_{11}^* + \bar{b}_{02}^*)}{\bar{a}_{01}^{*2}},$$

$$\bar{c}_{40}^* = -\bar{a}_{40}^*\bar{b}_{01}^* + \bar{a}_{31}^*\bar{b}_{10}^* - \bar{a}_{30}^*\bar{b}_{11}^* + \bar{a}_{21}^*\bar{b}_{20}^* - \bar{a}_{20}^*\bar{b}_{21}^* + \bar{a}_{11}^*\bar{b}_{30}^* - \bar{a}_{10}^*\bar{b}_{31}^* + \bar{a}_{01}^*\bar{b}_{40}^* + \frac{\bar{a}_{00}^{*2}\bar{a}_{11}^{*4}\bar{b}_{02}^*}{\bar{a}_{01}^{*5}} - \frac{\bar{a}_{00}^*\bar{a}_{11}^{*2}}{\bar{a}_{01}^{*4}}(2\bar{a}_{10}^*\bar{a}_{11}^*\bar{b}_{02}^* + \bar{a}_{00}^*(3\bar{a}_{21}^*\bar{b}_{02}^* + \bar{a}_{11}^*\bar{b}_{12}^*)) + \frac{1}{\bar{a}_{01}^*}(\bar{a}_{10}^{*2}\bar{b}_{22}^* + 2\bar{a}_{30}^*\bar{a}_{10}^*\bar{b}_{02}^* + \bar{a}_{20}^{*2}\bar{b}_{02}^* + 2\bar{a}_{00}^*(\bar{a}_{40}^*\bar{b}_{02}^* + \bar{a}_{30}^*\bar{b}_{12}^*) + 2\bar{a}_{20}^*(\bar{a}_{10}^*\bar{b}_{12}^* + \bar{a}_{00}^*\bar{b}_{22}^*)) + \frac{1}{\bar{a}_{01}^{*3}}(\bar{a}_{10}^{*2}\bar{a}_{11}^{*2}\bar{b}_{02}^* + 2\bar{a}_{00}^*\bar{a}_{10}^*\bar{a}_{11}^*(2\bar{a}_{21}^*\bar{b}_{02}^* + \bar{a}_{11}^*\bar{b}_{12}^*) + \bar{a}_{00}^*(\bar{a}_{11}^{*2}(2\bar{a}_{20}^*\bar{b}_{02}^* + \bar{a}_{00}^* \times\bar{b}_{22}^*) + 2\bar{a}_{00}^*\bar{a}_{11}^*(\bar{a}_{31}^*\bar{b}_{02}^* + \bar{a}_{21}^*\bar{b}_{12}^*) + \bar{a}_{00}^*\bar{a}_{21}^{*2}\bar{b}_{02}^*)) - \frac{1}{\bar{a}_{01}^{*2}}(\bar{a}_{10}^{*2}(\bar{a}_{21}^*\bar{b}_{02}^* + \bar{a}_{11}^* \times\bar{b}_{12}^*) + 2\bar{a}_{10}^*(\bar{a}_{00}^*(\bar{a}_{31}^*\bar{b}_{02}^* + \bar{a}_{21}^*\bar{b}_{12}^*) + \bar{a}_{11}^*(\bar{a}_{20}^*\bar{b}_{02}^* + \bar{a}_{00}^*\bar{b}_{22}^*)) + \bar{a}_{00}^*(2\bar{a}_{11}^*\bar{a}_{30}^* \times\bar{b}_{02}^* + 2\bar{a}_{20}^*(\bar{a}_{21}^*\bar{b}_{02}^* + \bar{a}_{11}^*\bar{b}_{12}^*) + \bar{a}_{00}^*(\bar{a}_{31}^*\bar{b}_{12}^* + \bar{a}_{21}^*\bar{b}_{22}^*))),$$

$$\bar{c}_{31}^* = \frac{1}{\bar{a}_{01}^{*4}}(-\bar{a}_{01}^{*3}(\bar{a}_{30}^*(\bar{a}_{11}^* + 2\bar{b}_{02}^*) + 2\bar{a}_{20}^*(\bar{a}_{21}^* + \bar{b}_{12}^*) + \bar{a}_{10}^*(3\bar{a}_{31}^* + 2\bar{b}_{22}^*)) + \bar{a}_{01}^{*2} \times(\bar{a}_{11}^*(2\bar{a}_{20}^*\bar{b}_{02}^* + \bar{a}_{10}^*(3\bar{a}_{21}^* + 2\bar{b}_{12}^*) + 2\bar{a}_{00}^*(2\bar{a}_{31}^* + \bar{b}_{22}^*)) + 2(\bar{a}_{10}^*\bar{a}_{21}^*\bar{b}_{02}^* + \bar{a}_{00}^* \times(\bar{a}_{31}^*\bar{b}_{02}^* + \bar{a}_{21}^*(\bar{a}_{21}^* + \bar{b}_{12}^*))) + \bar{a}_{20}^*\bar{a}_{11}^{*2}) - \bar{a}_{11}^*\bar{a}_{01}^*(\bar{a}_{10}^*\bar{a}_{11}^*(\bar{a}_{11}^* + 2\bar{b}_{02}^*) + 2\bar{a}_{00}^* \times(2\bar{a}_{21}^*(\bar{a}_{11}^* + \bar{b}_{02}^*) + \bar{a}_{11}^*\bar{b}_{12}^*)) + \bar{a}_{00}^*\bar{a}_{11}^{*3}(\bar{a}_{11}^* + 2\bar{b}_{02}^*)) + 4\bar{a}_{40}^* + \bar{b}_{31}^*,$$

$$\bar{c}_{22}^* = \frac{\bar{a}_{11}^{*2}\bar{b}_{02}^* - \bar{a}_{01}^*\bar{a}_{11}^*(3\bar{a}_{21}^* + \bar{b}_{12}^*) + \bar{a}_{01}^*(\bar{a}_{01}^*(3\bar{a}_{31}^* + \bar{b}_{22}^*) - \bar{a}_{21}^*\bar{b}_{02}^*) + \bar{a}_{11}^{*3}}{\bar{a}_{01}^{*3}};$$

$$\bar{d}_{00}^* = \bar{c}_{00}^*,\ \bar{d}_{10}^* = \bar{c}_{10}^* - \bar{c}_{00}^*\bar{c}_{02}^*,\ \bar{d}_{01}^* = \bar{c}_{01}^*,\ \bar{d}_{20}^* = \bar{c}_{00}^*\bar{c}_{02}^{*2} - \frac{\bar{c}_{10}^*\bar{c}_{02}^*}{2} + \bar{c}_{20}^*,$$

$$\bar{d}_{11}^* = \bar{c}_{11}^*,\ \bar{d}_{30}^* = \frac{1}{2}(\bar{c}_{10}^* - 2\bar{c}_{00}^*\bar{c}_{02}^*)\bar{c}_{02}^{*2} + \bar{c}_{30}^*,\ \bar{d}_{21}^* = \frac{\bar{c}_{02}^*\bar{c}_{11}^*}{2} + \bar{c}_{21}^*,$$

$$\bar{d}_{12}^* = 2\bar{c}_{02}^{*2} + \bar{c}_{12}^*,\ \bar{d}_{40}^* = \bar{c}_{00}^*\bar{c}_{02}^{*4} + \frac{1}{4}(\bar{c}_{02}^*(\bar{c}_{20}^* - 2\bar{c}_{02}^*\bar{c}_{10}^*) + 2\bar{c}_{30}^*)\bar{c}_{02}^* + \bar{c}_{40}^*,$$

$$\bar{d}_{31}^* = \bar{c}_{02}^*\bar{c}_{21}^* + \bar{c}_{31}^*,\ \bar{d}_{22}^* = -\bar{c}_{02}^{*3} + \frac{3\bar{c}_{12}^*\bar{c}_{02}^*}{2} + \bar{c}_{22}^*,\ \bar{e}_{00}^* = \bar{d}_{00}^*,\ \bar{e}_{10}^* = \bar{d}_{10}^*,$$

$$\bar{e}_{01}^* = \bar{d}_{01}^*,\ \bar{e}_{20}^* = \bar{d}_{20}^* - \frac{\bar{d}_{00}^*\bar{d}_{12}^*}{2},\ \bar{e}_{11}^* = \bar{d}_{11}^*,\ \bar{e}_{30}^* = \bar{d}_{30}^* - \frac{\bar{d}_{10}^*\bar{d}_{12}^*}{3},\ \bar{e}_{21}^* = \bar{d}_{21}^*,$$

$$\bar{e}_{40}^* = \frac{1}{4}\bar{d}_{00}^*\bar{d}_{12}^{*2} - \frac{\bar{d}_{20}^*\bar{d}_{12}^*}{6} + \bar{d}_{40}^*,\ \bar{e}_{31}^* = \frac{\bar{d}_{11}^*\bar{d}_{12}^*}{6} + \bar{d}_{31}^*,\ \bar{e}_{22}^* = \bar{d}_{22}^*,$$

$$\bar{f}_{00}^* = \bar{e}_{00}^*,\ \bar{f}_{10}^* = \bar{e}_{10}^*,\ \bar{f}_{01}^* = \bar{e}_{01}^*,\ \bar{f}_{20}^* = \bar{e}_{20}^*,\ \bar{f}_{11}^* = \bar{e}_{11}^*,$$

$$\bar{f}_{30}^* = \bar{e}_{30}^* - \frac{\bar{e}_{00}^*\bar{e}_{22}^*}{3},\ \bar{f}_{21}^* = \bar{e}_{21}^*,\ \bar{f}_{40}^* = \bar{e}_{40}^* - \frac{\bar{e}_{10}^*\bar{e}_{22}^*}{4},\ \bar{f}_{31}^* = \bar{e}_{31}^*,$$

$$\bar{g}_{00}^* = \bar{f}_{00}^*,\ \bar{g}_{10}^* = \bar{f}_{10}^* - \frac{\bar{f}_{00}^*\bar{f}_{30}^*}{2\bar{f}_{20}^*},\ \bar{g}_{01}^* = \bar{f}_{01}^*,$$

$$\bar{g}_{20}^* = \frac{9\bar{f}_{00}^*\bar{f}_{30}^{*2}}{16\bar{f}_{20}^{*2}} + \bar{f}_{20}^* - \frac{3(5\bar{f}_{10}^*\bar{f}_{30}^* + 4\bar{f}_{00}^*\bar{f}_{40}^*)}{20\bar{f}_{20}^*},\ \bar{g}_{11}^* = \bar{f}_{11}^* - \frac{\bar{f}_{01}^*\bar{f}_{30}^*}{2\bar{f}_{20}^*},$$

$$\bar{g}_{30}^* = \frac{\bar{f}_{10}^*(35\bar{f}_{30}^{*2} - 32\bar{f}_{20}^*\bar{f}_{40}^*)}{40\bar{f}_{20}^{*2}},$$

$$\bar{g}_{21}^* = \bar{f}_{21}^* - \frac{3(20\bar{f}_{11}^*\bar{f}_{20}^*\bar{f}_{30}^* + \bar{f}_{01}^*(16\bar{f}_{20}^*\bar{f}_{40}^* - 15\bar{f}_{30}^{*2}))}{80\bar{f}_{20}^{*2}},$$

$$\bar{g}_{40}^* = \frac{\bar{f}_{10}^*\bar{f}_{30}^*(16\bar{f}_{20}^*\bar{f}_{40}^* - 15\bar{f}_{30}^{*2})}{64\bar{f}_{20}^{*3}},\ \bar{g}_{31}^* = \frac{7\bar{f}_{11}^*\bar{f}_{30}^{*2}}{8\bar{f}_{20}^{*2}} + \bar{f}_{31}^* - \frac{5\bar{f}_{21}^*\bar{f}_{30}^* + 4\bar{f}_{11}^*\bar{f}_{40}^*}{5\bar{f}_{20}^*};$$

$$\bar{h}_{00}^* = \bar{g}_{00}^*,\ \bar{h}_{10}^* = \bar{g}_{10}^*,\ \bar{h}_{01}^* = \bar{g}_{01}^* - \frac{\bar{g}_{00}^*\bar{g}_{21}^*}{\bar{g}_{20}^*},\ \bar{h}_{20}^* = \bar{g}_{20}^*,\ \bar{h}_{11}^* = \bar{g}_{11}^* - \frac{\bar{g}_{10}^*\bar{g}_{21}^*}{\bar{g}_{20}^*},$$

$$\bar{h}_{31}^* = \bar{g}_{31}^* - \frac{\bar{g}_{21}^*\bar{g}_{30}^*}{\bar{g}_{20}^*};\bar{j}_{00}^* = \bar{h}_{00}^*\bar{h}_{31}^{*\frac{4}{5}}\bar{h}_{20}^{*-\frac{7}{5}},\ \bar{j}_{10}^* = \bar{h}_{10}^*\bar{h}_{31}^{*\frac{2}{5}}\bar{h}_{20}^{*-\frac{6}{5}},$$

$$\bar{j}_{01}^* = \bar{h}_{01}^*\bar{h}_{31}^{*\frac{1}{5}}\bar{h}_{20}^{*-\frac{3}{5}},\ \bar{j}_{11}^* = \bar{h}_{11}^*\bar{h}_{20}^{*-\frac{2}{5}}\bar{h}_{31}^{*-\frac{1}{5}}.$$

Appendix D. Coefficients in the proof of Theorem 3.5.

$$\alpha_{10} = s,\ \alpha_{01} = \frac{a + x_*^2}{kn(a(b + 2x_*) - bx_*^2)}(b(ks + rx_*) + x_*(k(s - r) + 2rx_*)),$$

$$\alpha_{20} = -\frac{1}{k}(\frac{a(a - 3x_*^2)}{x_*(a + x_*^2)(bx_*^2 - a(b + 2x_*))}(b(ks + rx_*) + x_*(k(s - r) + 2rx_*))$$
$$+\frac{b}{(b + x_*)(bx_*^2 - a(b + 2x_*))}(ak(r + s) + x_*^2(k(s - r) + 2rx_*)) + r),$$

$$\alpha_{11} = -\frac{2a}{knx_*(bx_*^2 - a(b + 2x_*))}(b(ks + rx_*) + x_*(k(s - r) + 2rx_*)),$$

$$\alpha_{30} = \frac{1}{k(a + x_*^2)^2(b + x_*)^2(bx_*^2 - a(b + 2x_*))}(4a(a - x_*^2)(b + x_*)^2(b(ks + rx_*)$$
$$+x_*(k(s - r) + 2rx_*)) + b(a + x_*^2)^2(ak(r + s) + x_*^2(k(s - r) + 2rx_*))),$$

$$\alpha_{21} = -\frac{a(a - 3x_*^2)(b(ks + rx_*) + x_*(k(s - r) + 2rx_*))}{knx_*^2(a + x_*^2)(bx_*^2 - a(b + 2x_*))},$$

$$\alpha_{40} = \frac{1}{kx_*(a + x_*^2)^3(b + x_*)^3(bx_*^2 - a(b + 2x_*))}(a(a^2 - 10ax_*^2 + 5x_*^4)(b + x_*)^3$$
$$\times(b(ks + rx_*) + x_*(k(s - r) + 2rx_*)) - bx_*(a + x_*^2)^3(ak(r + s) + x_*^2(k$$
$$\times(s - r) + 2rx_*))),$$

$$\alpha_{31} = \frac{4a(a - x_*^2)(b(ks + rx_*) + x_*(k(s - r) + 2rx_*))}{knx_*(a + x_*^2)^2(bx_*^2 - a(b + 2x_*))},$$

$$\beta_{10} = ns,\ \beta_{01} = -s,\ \beta_{20} = -\frac{ns}{x_*},\ \beta_{11} = \frac{2s}{x_*},\ \beta_{02} = -\frac{s}{nx_*},\ \beta_{30} = \frac{ns}{x_*^2},$$
$$\beta_{21} = -\frac{2s}{x_*^2},\ \beta_{12} = \frac{s}{nx_*^2},\ \beta_{40} = -\frac{ns}{x_*^3},\ \beta_{31} = \frac{2s}{x_*^3},\ \beta_{22} = -\frac{s}{nx_*^3};$$
$$\delta_{20} = \frac{\alpha_{10}\alpha_{11} - \alpha_{01}\alpha_{20}}{\lambda},\ \delta_{11} = -\alpha_{11},\ \delta_{30} = \frac{\alpha_{01}(\alpha_{01}\alpha_{30} - \alpha_{10}\alpha_{21})}{\lambda},$$
$$\delta_{21} = \alpha_{01}\alpha_{21}, \delta_{40} = \frac{\alpha_{01}^2(\alpha_{10}\alpha_{31} - \alpha_{01}\alpha_{40})}{\lambda},\ \delta_{31} = -\alpha_{01}^2\alpha_{31},$$
$$\gamma_{20} = \frac{-\alpha_{01}^2\beta_{20} + \alpha_{10}\alpha_{01}(\beta_{11} - \alpha_{20}) + \alpha_{10}^2(\alpha_{11} - \beta_{02})}{\lambda^2},$$
$$\gamma_{11} = -\frac{\alpha_{10}(\alpha_{11} - 2\beta_{02}) + \alpha_{01}\beta_{11}}{\lambda},$$
$$\gamma_{02} = -\beta_{02},\ \gamma_{30} = \frac{\alpha_{01}(\alpha_{01}^2\beta_{30} + \alpha_{10}\alpha_{01}(\alpha_{30} - \beta_{21}) + \alpha_{10}^2(\beta_{12} - \alpha_{21}))}{\lambda^2},$$
$$\gamma_{21} = \frac{\alpha_{01}(\alpha_{10}(\alpha_{21} - 2\beta_{12}) + \alpha_{01}\beta_{21})}{\lambda},\ \gamma_{12} = \alpha_{01}\beta_{12},$$
$$\gamma_{40} = -\frac{\alpha_{01}^2(\alpha_{01}^2\beta_{40} + \alpha_{10}\alpha_{01}(\alpha_{40} - \beta_{31}) + \alpha_{10}^2(\beta_{22} - \alpha_{31}))}{\lambda^2},$$
$$\gamma_{31} = -\frac{\alpha_{01}^2(\alpha_{10}(\alpha_{31} - 2\beta_{22}) + \alpha_{01}\beta_{31})}{\lambda},\ \gamma_{22} = -\alpha_{01}^2\beta_{22},$$
$$\begin{aligned}A = {} & -a^4(b^2(8k + 3x_*) + 6bx_*(2k + x_*) + 8kx_*^2) - 2a^3x_*(18b^3(b - k) + 4b^2x_* \\ & \times(24b - 13k) + 2x_*^3(102b - 11k) + 9bx_*^2(23b - 7k) + 70x_*^4) - 2a^2x_*^3(4b^3(b \\ & -k) + 4b^2x_*(3b - 2k) + 4x_*^3(23b - 3k) + 25bx_*^2(2b - k) + 44x_*^4) + 2ax_*^5(6b^4 \\ & +b^3(44x_* - 6k) + b^2x_*(83x_* - 24k) + bx_*^2(52x_* - 19k) + 2x_*^3(5x_* - 3k)) \\ & -bx_*^9(b - 2k + 2x_*),\end{aligned}$$
$$\begin{aligned}B = {} & a^4k(2x_*(3b + k) + 3b(b - k)) + 2a^3x_*(x_*^2(81b^2 - 36bk - k^2) + 6b^2(b - k)^2 \\ & +6x_*^3(13b - k) + 2bx_*(b - k)(18b - k) + 24x_*^4) + 2a^2x_*^3(x_*^2(179b^2 - 146bk \\ & +11k^2) + 12b^2(b - k)^2 + 12x_*^3(17b - 5k) + bx_*(b - k)(72b - 25k) + 80x_*^4) \\ & +2ax_*^5(x_*^2(-109b^2 + 52bk - 3k^2) - 10bx_*(6b^2 - 7bk + k^2) - 10b^2(b - k)^2 \\ & +2x_*^3(5k - 33b) - 8x_*^4) - bx_*^8(b - k)(k - 2x_*),\end{aligned}$$
$$\begin{aligned}C = {} & a^6(8bk - 3bx_* + 2kx_*)^2 - 2a^5x_*(192b^3k(k - b) + x_*^3(621b^2 - 1070bk + 112 \\ & \times k^2) + 8bx_*^2(48b^2 - 179bk + 40k^2) + 4b^2x_*(21b^2 - 231bk + 82k^2) + 158x_*^4 \\ & \times(3b - 2k) + 144x_*^5) + a^4x_*^2(528b^4(b - k)^2 + 192b^3x_*(b - k)(22b - 7k) + \\ & x_*^4(22879b^2 - 7448bk + 312k^2) + 16bx_*^3(1479b^2 - 811bk + 84k^2) + 48b^2x_*^2 \\ & \times(287b^2 - 242bk + 43k^2) + 4x_*^5(2951b - 412k) + 2500x_*^6) + 4a^3x_*^4(24b^4(b \\ & -k)^2 + 96b^3x_*(b - k)(2b + k) + x_*^4(1681b^2 - 42bk - 8k^2) + 4bx_*^3(339b^2 + \\ & 45bk - 16k^2) + 4b^2x_*^2(168b^2 + 27bk - 35k^2) + 2x_*^5(589b - 24k) + 340x_*^6) + \\ & a^2x_*^6(144b^4(b - k)^2 + x_*^4(6231b^2 - 860bk + 4k^2) + 16bx_*^3(393b^2 - 106bk + \\ & 2k^2) + 16b^2x_*^2(231b^2 - 138bk + 7k^2) + 192b^3x_*(6b^2 - 7bk + k^2) + 8x_*^5(453b \\ & -26k) + 984x_*^6) + 2ax_*^{10}(-12b^4 + 12b^3(k - 2x_*) + b^2x_*(64k + 11x_*) + 2b \\ & \times x_*^2(19k + 35x_*) + 4x_*^3(k + 14x_*)) + x_*^{14}(b - 2x_*)^2.\end{aligned}$$

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