A Novel Approximation for the Spring Loaded Inverted Pendulum Model of Locomotion

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***Abstract*— The Spring-Loaded Inverted Pendulum (SLIP) is one of the simplest models of robot locomotion. SLIP is commonly used to predict the center of mass motion and derive simple control laws for stable locomotion. However, the SLIP model is not integrable, which means that no closed-form relation can be derived to understand how the design and control parameters of the SLIP model affect stable locomotion. There exist a number of different analytical approximations to the SLIP model when considering small step lengths and symmetric steps. In this paper, we present a novel approximation to the SLIP model without relying on the small step length and the symmetric step assumption. The model was found to accurately predict the stability of the SLIP model for large and asymmetric steps and was used to design a controller to stabilize the SLIP model in a couple of steps.**

I. Introduction

Modeling bipedal locomotion is a challenge because humans, animals, and robots have many degrees of freedom and complex dynamics [1], [2]. The complexity of legged dynamics motivates the development of heuristic models that simplify the dynamics while accurately capturing the motion of the center of mass. One such model is the Spring-Loaded Inverted Pendulum (SLIP) proposed by Blickhan [3].

The SLIP is a point mass representing the center of mass of the body and a spring representing the legs. The SLIP has been widely used to predict the mechanics and control aspects of not only human and animal locomotion [4]–[6], but also robot locomotion [7]–[13]. For example, predictions made using the SLIP model have been instrumental in designing controllers for both simple robots [14] and robots with many degrees of freedom and actuators [15]. Although the predictive power of the SLIP model was appreciable in these studies, the SLIP has non-integrable dynamics for which a closed-form relation between the design and control parameters cannot be analytically obtained.

The non-integrable nature of the SLIP model motivated a number of efforts for finding approximate analytical solutions for the SLIP model. Perhaps the most intuitive way of achieving a reduced complexity analytical solution is by neglecting the gravitational force and transforming the stance dynamics into a central force problem where angular momentum is conserved, see [16]. An alternative approach presented in [17] and later extended in [18], involves a small step-length approximation as well as assuming that the spring compression is negligible compared to the leg length. These assumptions result in both the angular momentum and the energy being conserved for symmetric steps and were used to model walking and walking-to-running transitions [19]. Recently, the authors in [20], [21] proposed a refined perturbation-based SLIP model that uses the small-angle approximation but obtains a more accurate albeit analytically more complex approximation.

Although the small-step approximation can be used to reasonably well approximate human walking with normal speeds, [19], situations where large and asymmetric steps are relevant exist, such as the fast walking of robots. It has been previously shown that, at least in walking with rigid legs, reducing the step length and increasing the stepping frequency reduces collision energy loss at heel strike and results in more efficient locomotion than increasing the step length and reducing the stepping frequency [22], [23]. However, by modeling the leg as a mass-less spring, the energy losses due to collision at heel strike can be minimized. Also, humans do not use small steps to walk fast, and robots may not need to use small and symmetric steps to walk fast. The latter has been recently shown in [24], where taking large steps, and using an optimal leg stiffness and precompression policy, was shown to be a desirable approach to walking at high speeds. Existing models that use a small step length approximation have limited prediction power when it comes to fast locomotion with large asymmetric steps [20]. Asymmetric stepping is used in human and robot locomotion and is crucial for acceleration and fast locomotion [25].

In this paper, we present a novel approximation to the SLIP model which preserves accuracy for large step lengths and asymmetric steps. The new model introduces a sine wave approximation of the horizontal ground reaction force and considers small vertical displacement during stance. The model includes the effect of gravity and does not invoke a small step length approximation. As a result, the model accurately captures the exact SLIP dynamics for a wide range of stepping conditions, which is important for studying both human and robot locomotion at high speeds. The approximate SLIP model was evaluated in terms of its accuracy in predicting the velocities of the body compared to the exact SLIP model. The approximate SLIP model was then used to derive a stiffness control strategy that stabilizes the SLIP model in a couple of steps. Finally, we show that the analytical controller design using the approximate SLIP model outperforms the controller that arises from using the

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linearized SLIP model when both controllers are applied to the exact SLIP model at race walking conditions.

The new model provides a template for legged locomotion that can be used to investigate the relation between design and control parameters that have been previously hidden in the non-integrable dynamics of the exact SLIP model. The new model can also be used to derive locomotion controllers for robots, similar to the controller presented in this paper.

II. A Novel Model of Locomotion

In this section, we describe the spring-loaded inverted pendulum model and derive a novel approximation of the planar SLIP model.

A. Planar SLIP Model

Let us consider a point mass representing the center of mass of the body and a mass-less spring representing the legs, shown in Fig. 1.

We assume that the mass-less foot acts as an anchor point during the stance phase and does not slip as the body vaults over it. The stance phase begins with the heel strike event, which is characterized by the foot coming to contact with the ground and ends with the toe-off event. Toe-off occurs when the spring leg releases all the energy it stores and reaches its fully-extended uncompressed length. If the vertical velocity of the body is positive at the toe-off point, the body takes off, and the locomotion gait changes from walking to running.

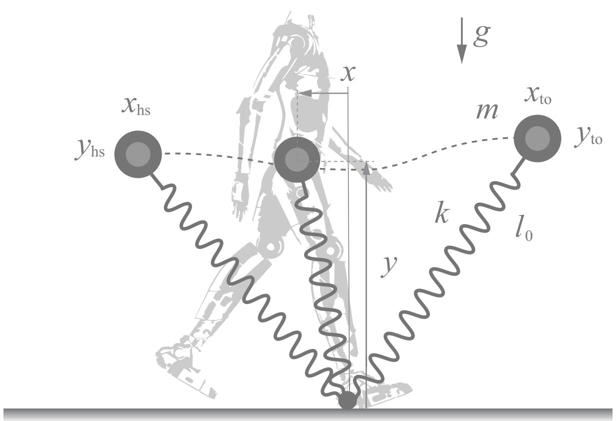

Fig. 1. SLIP model of robotic locomotion.

In the exact SLIP model, the spring leg is conservative and as such can be modeled by a potential energy function,

$$V(x,y) = \frac{1}{2}k(l_0 - \sqrt{x^2+y^2})^2, \tag{1}$$

where (x,y) is the position of the point mass, l_0 is the leg length, and k is the stiffness of the leg, see Fig. 1.

The stance dynamics of the exact SLIP model can be expressed by the following two equations,

$$m\ddot{x} = F_x(x,y) = \frac{kl_0x}{\sqrt{x^2+y^2}} - kx, \tag{2}$$

$$m\ddot{y} = F_y(x,y) - mg = \frac{kl_0y}{\sqrt{x^2+y^2}} - ky - mg, \tag{3}$$

where m denotes the mass while g denotes the gravitational acceleration.

These equations capture the nonlinear stance dynamics of locomotion for any step. These equations are non-integrable [1], [26]. In the remainder of this paper, we derive an approximation to these equations which retains the non-linear dynamics of the SLIP model at large steps and is integrable under some assumptions.

B. A Novel Approximation of the SLIP Model

In order to approximate the forces in the stance phase equations (2) and (3), we make two assumptions:

Assumption 1: The spring leg of the approximate SLIP model is conservative during stance, similar to the spring leg of the exact SLIP model. This assumption implies that there exists a scalar spring potential energy function for the approximate SLIP model,

$$\hat{V} = \hat{V}(x,y). \tag{4}$$

Assumption 2: The spring potential energy function of the approximate SLIP model is separable; it can be approximated by a product of two functions where one function depends only on the horizontal position of the center of mass, while the other function depends only on the vertical position of the center of mass,

$$\hat{V}(x,y) = V_x(x)V_y(y). \tag{5}$$

Based on these assumptions, we compute the horizontal and vertical forces in the approximate SLIP model,

$$\hat{F}_x(x,y) = -\frac{\mathrm{d}V_x(x)}{\mathrm{d}x}V_y(y), \tag{6}$$

$$\hat{F}_y(x,y) = -V_x(x)\frac{\mathrm{d}V_y(y)}{\mathrm{d}y}. \tag{7}$$

To approximate the horizontal force $\hat{F}_x(x,y)$ in (6), we note that the exact SLIP model has a near sinusoidal force profile for constant height center of mass motion $y = \bar{y}$, see Fig. 2.

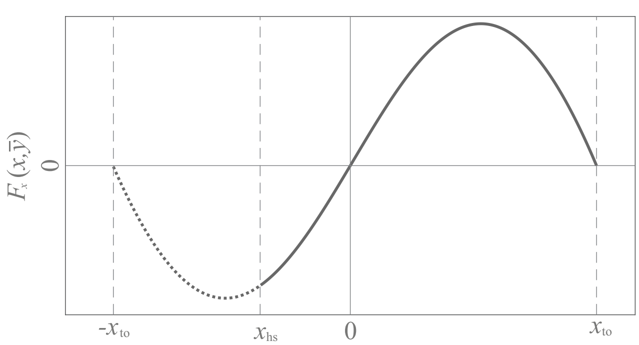

Fig. 2. Typical horizontal force of the exact SLIP model for constant height center of mass motion $y = \bar{y}$.

To approximate the vertical force $\hat{F}_y(x,y)$ in (7), we note that the exact SLIP model has a linear vertical force-deflection relation at mid-stance $F_y(0,y) = k(l_0 - y)$. Based on this linear relation, and the sinusoidal horizontal force profile in Fig. 2, we make the following assumptions,

$$\hat{F}_x(x,y) = \frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)V_y(y), \tag{8}$$

$$\hat{F}_y(x,y) = V_x(x)k(l_0-y), \tag{9}$$

where x_{to} is the horizontal position of the mass at toe-off, see Fig. 1.

Using (8) and (9), the computation of the potential energy function is reduced to the determination of two functions, $V_x(x)$ and $V_y(y)$. Based on *Assumption 1*, these two functions must satisfy the following equation,

$$\frac{\partial\hat{F}_x(x,y)}{\partial y} = \frac{\partial\hat{F}_y(x,y)}{\partial x}, \tag{10}$$

along with the boundary conditions that relate the approximate SLIP model with the exact SLIP model,

$$\hat{V}(x_{\text{hs}},y_{\text{hs}}) = V(x_{\text{hs}},y_{\text{hs}}), \tag{11}$$

$$\hat{V}(x_{\text{to}},y_{\text{to}}) = V(x_{\text{to}},y_{\text{to}}) = 0, \tag{12}$$

where $V(x_{\text{hs}},y_{\text{hs}})$ is the potential energy at heel strike while $V(x_{\text{to}},y_{\text{to}}) = 0$ is the potential energy at toe-off calculated using the exact SLIP model (1).

Using (8)–(12), we obtain the potential energy function of the approximate SLIP model,

$$\hat{V}(x,y) = V_x(x)V_y(y), \tag{13}$$

where

$$V_x(x) = \cos\left(\pi\frac{x}{x_{\text{to}}}\right)+1, \tag{14}$$

$$V_y(y) = \frac{V(x_{\text{hs}},y_{\text{hs}})}{\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)+1} - kl_0(y-y_{\text{hs}}) + \frac{k}{2}(y^2-y_{\text{hs}}^2). \tag{15}$$

The approximation presented here is somewhat similar to the one employed by the authors in [25], where a Fourier series approximation was used to achieve success in predicting the horizontal velocity at the toe-off for a set of conditions. However, the model in [25] does not provide a potential energy function $\hat{V}(x,y)$ and only considers vertical heel strike configurations $x_{\text{hs}} = 0$. Here, we consider arbitrary heel strike configurations and allow for the construction of a potential energy function, similar to the exact SLIP model.

III. Approximate Analytical Solution

In this section, an approximate solution of the SLIP dynamics is presented (Section III-A) based on the potential energy function derived in the previous section. Subsequently, the accuracy of the analytical solution is assessed with respect to the exact SLIP model (Section III-B).

A. Approximate Solution

In order to find a closed-form solution using the approximate SLIP dynamics, we make the following assumption:

Assumption 3: The position change of the center of mass in the vertical direction is negligible compared to the position change of the center of mass in the horizontal direction:

$$y(t) \approx y_{\text{hs}}. \tag{16}$$

This assumption is well suited to model fast locomotion.

Based on (16), we predict the horizontal velocity of the center of mass as a function of the horizontal position using the non-linear differential equation,

$$m\ddot{x} = -\frac{\mathrm{d}V_x(x)}{\mathrm{d}x}V_y(y_{\text{hs}}). \tag{17}$$

Using $\ddot{x} = \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}$, equation (17) separates variables and can be integrated into the following kinetic energy relation,

$$\frac{1}{2}m(\dot{x}^2-\dot{x}_{\text{hs}}^2) = V_y(y_{\text{hs}})\big(V_x(x_{\text{hs}})-V_x(x)\big). \tag{18}$$

Finally, we substitute (14) and (15) into (18) to obtain the horizontal velocity as a function of the horizontal position,

$$\dot{x}(x)^2 = \dot{x}_{\text{hs}}^2 + c_x(k)\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \cos\left(\pi\frac{x}{x_{\text{to}}}\right)\right), \tag{19}$$

where

$$c_x(k) = \frac{2}{m}\frac{V(x_{\text{hs}},y_{\text{hs}})}{\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)+1} = \frac{k}{m}\frac{\left(l_0-\sqrt{x_{\text{hs}}^2+y_{\text{hs}}^2}\right)^2}{\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)+1}. \tag{20}$$

Next, we proceed to find the vertical velocity as a function of the horizontal position of the center of mass. To do that, we consider the following equation,

$$m\ddot{y} = -V_x(x)\frac{\mathrm{d}V_y(y)}{\mathrm{d}y}\Big|_{y=y_{\text{hs}}} - mg. \tag{21}$$

Using $\ddot{y} = \frac{\mathrm{d}\dot{y}}{\mathrm{d}x}\dot{x}(x)$, equation (21) separates variables, but cannot be integrated to express the vertical velocity $\dot{y}(x)$ in terms of elementary functions. To express $\dot{y}(x)$ in terms of elementary functions we make the following assumption:

Assumption 4: The horizontal velocity of the center of mass does not change substantially during a single step, $\dot{x}(x)^2 \approx \dot{x}_{\text{hs}}^2$. This assumption was used to derive approximate formulas for fast walking [24]. According to (19), this assumption is well suited to locomotion where the velocity does not fluctuate much during the step, that is, when $c_x(k)$ (20) is small compared to $\dot{x}_{\text{hs}}^2$.

Using *Assumptions 3 and 4*, together with $\ddot{y} = \frac{\mathrm{d}\dot{y}}{\mathrm{d}x}\dot{x}_{\text{hs}}$, we integrate (21) to find the vertical velocity,

$$\begin{aligned}\dot{y}(x) = \dot{y}_{\text{hs}} + c_y(k)\Big(&\sin\left(\pi\frac{x}{x_{\text{to}}}\right) - \sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \frac{\pi}{x_{\text{to}}}(x-x_{\text{hs}})\Big)\\ &- \frac{g}{\dot{x}_{\text{hs}}}(x-x_{\text{hs}}),\end{aligned} \tag{22}$$

where

$$c_y(k) = \frac{x_{\text{to}}}{\pi}\frac{k}{m\dot{x}_{\text{hs}}}(l_0-y_{\text{hs}}). \tag{23}$$

Equations (19) and (22) can be further integrated, under the assumptions stated above, to obtain $x(t)$ and $y(x)$, but due to the analytical complexity, these functions are not presented here. Instead, in Section IV, we use (19) and (22) to analyze the stability of the approximate SLIP model and derive a controller for stable locomotion of the exact SLIP model.

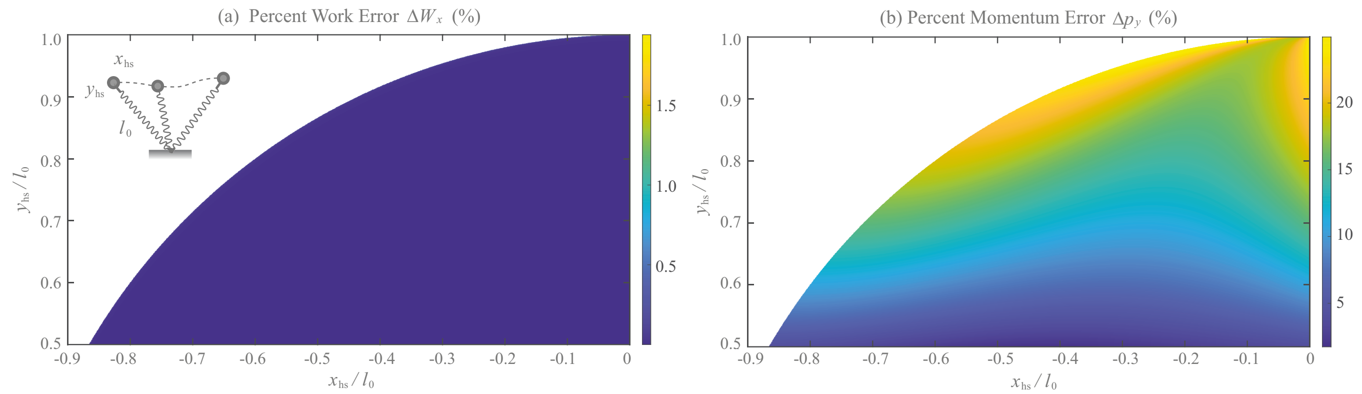

Fig. 3. Percent error in predicting the (a) work done by the horizontal force, and (b) momentum change due to the vertical force.

B. Approximation Error

In order to assess the accuracy of the predictions derived in the previous section, we compare the approximate SLIP model with the exact SLIP model.

We introduce two metrics to assess the error in our approximation. The first metric is the work done by the horizontal force since this quantity is computed when deriving the horizontal velocity (19),

$$\Delta W_x(x_{\text{hs}}, y_{\text{hs}}) = \left| 1 - \frac{\int_{x_{\text{hs}}}^{x_{\text{to}}} \hat{F}_x(x, y_{\text{hs}}) dx}{\int_{x_{\text{hs}}}^{x_{\text{to}}} F_x(x, y_{\text{hs}}) dx} \right| \times 100\%. \tag{24}$$

The second metric is the vertical momentum, since this quantity is computed when deriving the approximate expression for the vertical velocity (22),

$$\Delta p_y(x_{\text{hs}}, y_{\text{hs}}) = \left| 1 - \frac{\int_{x_{\text{hs}}}^{x_{\text{to}}} \frac{\hat{F}_y(x, y_{\text{hs}})}{\dot{x}_{\text{a}}(x)} dx}{\int_{x_{\text{hs}}}^{x_{\text{to}}} \frac{F_y(x, y_{\text{hs}})}{\dot{x}_{\text{e}}(x)} dx} \right| \times 100\%, \tag{25}$$

where $\dot{x}_{\text{a}}(x)$ is defined by (19) while $\dot{x}_{\text{e}}(x)$ is derived by integrating (2) with $y = y_{\text{hs}}$. These metrics give insight into the accuracy of the approximate SLIP model for predicting the horizontal and vertical velocity at toe-off for different heel strike positions, which is relevant to determining stability.

Figure 3 shows the errors defined by (24) and (25). It can be noticed that the model is 100% accurate in determining the heel strike velocity in the horizontal direction for the next step because it preserves the energy in the horizontal direction when $y = y_{\text{hs}}$. On the other hand, the error in momentum (25) goes up to about 25%. It is important to note, however, that the majority of high-error cases are in the region with short step lengths and little initial leg pre-compression, see Fig. 3b. In these cases, the horizontal forces of the approximate and the exact SLIP models are both small and while the relative error defined by (25) can be 25%, the difference between the predictions remains small.

We have derived an approximate model for the spring-loaded inverted pendulum template without relying on small-angle approximation, and demonstrated its potential in predicting toe-off velocity. Next, we use the approximate SLIP model to design a controller for stable locomotion.

IV. Analytical Control Design for Stable Locomotion

Stability of the approximate spring-loaded inverted pendulum model can be described by a two-dimensional return map, using the horizontal and vertical velocities. Although the two-dimensional map can be reduced to a one-dimensional map when there is no spring pre-compression at heel strike [16] and the vertical position of the center of mass at each heel strike is equal [1], here we consider the general case where pre-compression at heel strike is allowed.

In the remainder of this section, we (i) investigate the stability properties of the approximate SLIP model (Section IV-A), (ii) derive a controller using the approximate SLIP model (Section IV-B), and (iii) show that the approximate controller achieves steady-state locomotion when applied to the exact SLIP model. We also compare the performance of the proposed controller to the controller computed using the exact nonlinear SLIP dynamics, and the controller that uses the linearized SLIP dynamics. In the latter case, we assume a small leg angle and small relative leg compression since these assumptions are commonly employed in deriving approximate SLIP dynamics [17], [21], [27], [28].

A. Steady-State Locomotion

In order to find the relation between the horizontal position of the center of mass at heel strike x_{hs} and the horizontal velocity change from heel strike to toe-off, we use (19) to compute,

$$\dot{x}(x_{\text{to}})^2 - \dot{x}_{\text{hs}}^2 = \frac{k}{m}\left(l_0 - \sqrt{x_{\text{hs}}^2 + y_{\text{hs}}^2}\right)^2 \geq 0. \tag{26}$$

An important point here is to note that, because $l_0^2 = x_{\text{to}}^2 + y_{\text{hs}}^2$, symmetric steps defined by $x_{\text{hs}} = -x_{\text{to}}$ result in no horizontal velocity change between subsequent heel strikes.

By changing the heel strike position x_{hs} between a symmetric step $x_{\text{hs}} = -x_{\text{to}}$ and a fully asymmetric step $x_{\text{hs}} = 0$ (where the leg is vertical at heel strike), or by choosing the stiffness of the leg k, it is only possible to accelerate or keep constant horizontal velocity, but not possible to decelerate in absence of collision losses or air resistance. This is the case in the SLIP model if the vertical position of the center of mass does not change between heel strike and toe-off $y_{\text{hs}} = y_{\text{to}}$. This type of walking may emulate the near horizontal center of mass motion in cycling [24].

In the next section, we will introduce an additional control parameter to alleviate the aforementioned limitation.

B. Acceleration and Deceleration

In order to enable the approximate SLIP model to both accelerate and decelerate regardless of the position of the body at heel strike $x_{\text{hs}} \in [-x_{\text{to}}, 0]$, we assume that the stiffness of the leg can be changed at mid-stance similar to the strategy proposed by the authors in [29],

$$k \in \begin{cases} k_1 & if \quad x_{\text{hs}} \leq x \leq 0 \\ k_2 & if \quad 0 < x \leq x_{\text{to}} \end{cases}. \tag{27}$$

Next, we split the horizontal velocity prediction (19) into two separate parts and use (27) to compute $\dot{x}(x)^2$ as a function of both k_1 and k_2. Then by solving

$$\dot{x}(x_{\text{to}}) = \dot{x}(x_{\text{hs}}), \tag{28}$$

an analytical condition for mid-step stiffness change is derived to achieve steady-state horizontal velocity,

$$\frac{k_2}{k_1} = \frac{1}{2}\left(1 - \cos\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right). \tag{29}$$

Figure 4 shows the above relation. We observe that (29) (black line) separates the space of k_2/k_1 and $x_{\text{hs}}/x_{\text{to}}$, to two regions, one that results in horizontal acceleration (red) and the other that results in horizontal deceleration (blue). The result shows that mid-step stiffness change overcomes the limitation of the SLIP model, as it can not only accelerate but also decelerate, even if the vertical position of the center of mass remains the same at heel strike and toe-off $y_{\text{hs}} = y_{\text{to}}$.

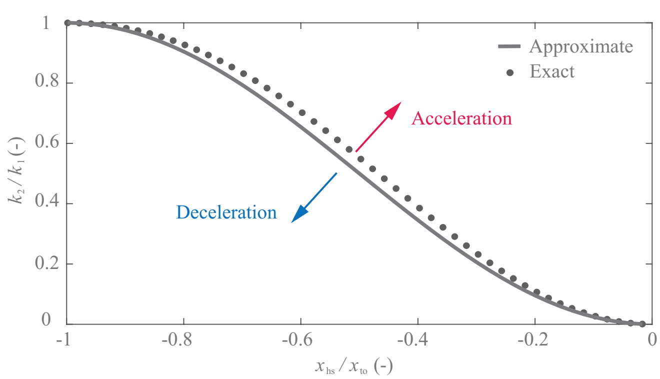

Fig. 4. Mid-step stiffness change for steady-state locomotion. The exact SLIP model was simulated at top race-walking speed $\dot{x}_{\text{hs}}/\sqrt{gl_0} = 1.2$, $\dot{y}_{\text{hs}} = 0$, $x_{\text{to}}/l_0 = 0.6$ [30] and using $y_{\text{hs}} = y_{\text{to}}$. The approximate relation (29) is 95% accurate when compared to the exact SLIP prediction.

Although the condition for steady-state horizontal velocity depends on the ratio of stiffness values before and after mid-step, this condition does not constrain the vertical velocity. However, at least in walking, the vertical velocity of the body cannot be arbitrary to prevent takeoff. Take-off will not accrue if the vertical velocity is non-positive when the leg is fully extended,

$$\dot{y}(x_{\text{to}}) \leq 0. \tag{30}$$

Similar to finding the horizontal velocity as a function of the mid-step stiffness change, the vertical velocity $\dot{y}(x)$ can also be found by incorporating the mid-step stiffness change (27) into (22). Then by using (29) and (30), we obtain the leg stiffness at heel strike to avoid take-off at steady-state walking,

$$\frac{k_1(l_0 - y_{\text{hs}})}{mg} \leq \frac{1 - \frac{x_{\text{hs}}}{x_{\text{to}}} - \frac{\dot{x}_{\text{hs}}\dot{y}_{\text{hs}}}{x_{\text{to}}g}}{\sin\left(\frac{\pi}{2}\frac{x_{\text{hs}}}{x_{\text{to}}}\right)^2 - \frac{1}{\pi}\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \frac{x_{\text{hs}}}{x_{\text{to}}}}. \tag{31}$$

The initial stiffness k_1 that results in zero vertical velocity at toe-off $\dot{y}(x_{\text{to}}) = 0$ is defined by the equality in (31).

In what follows, we introduce a controller for stable walking using the analytical relations defined in this section.

C. Controller

Using the analytical relations, (29) and (31), we define the following stiffness control policies,

$$k_{1n} = k_1 - G_{\text{y}}\dot{y}_{\text{hs}n} \tag{32}$$
$$k_{2n} = k_2 + G_{\text{x}}(\dot{x}_{\text{des}} - \dot{x}_{\text{hs}n}), \tag{33}$$

where k_{1n} and k_{2n} are the stiffness values for the current step (denoted with n), k_1 and k_2, are the stiffness values defined by (29) and (31) with $x_{\text{to}}^2 = l_0^2 - x_{\text{hs}}^2$, G_{x} and G_{y} are the feedback gains, while $\dot{x}_{\text{hs}n}$ and $\dot{y}_{\text{hs}n}$ are the horizontal and vertical velocities at the heel strike of the current step. In this controller, feedback is used to avoid take off $\dot{y}_{\text{hs}} = \dot{y}_{\text{des}} = 0$ and achieve a desired forward velocity $\dot{x}_{\text{hs}} = \dot{x}_{\text{des}}$.

In addition to (32) and (33), we use a stepping policy to define the desired foot placement of the SLIP model for the current step, based on the foot placement at the previous step,

$$x_{\text{hs}n} = x_{\text{hs}n-1} - \lambda(x_{\text{hs}n-1} + x_{\text{to}n-1}), \tag{34}$$

where $\lambda \in [0, 1]$ controls how fast the steps converge to the steady-state symmetric step at heel strike.

In the next section, we use this controller to generate stable locomotion of the exact SLIP model in a simulation study.

V. Evaluation

A simulation study was performed where the proposed approximate controller was tested on the exact SLIP dynamics. The task was to accelerate to race-walking velocities and achieve zero vertical velocity at toe-off without taking off. In this study, consistent with fast-walking, an instantaneous double-support phase was considered between consecutive steps similar to the model used in [24]. In addition, there are no collision losses at heel strike since foot mass and leg inertia were deemed negligible.

The controller, defined by the feedback control law for the leg stiffness (32), (33) and the stepping policy (34), was evaluated by comparing it to two different controllers (i) the exact SLIP controller derived using the nonlinear SLIP

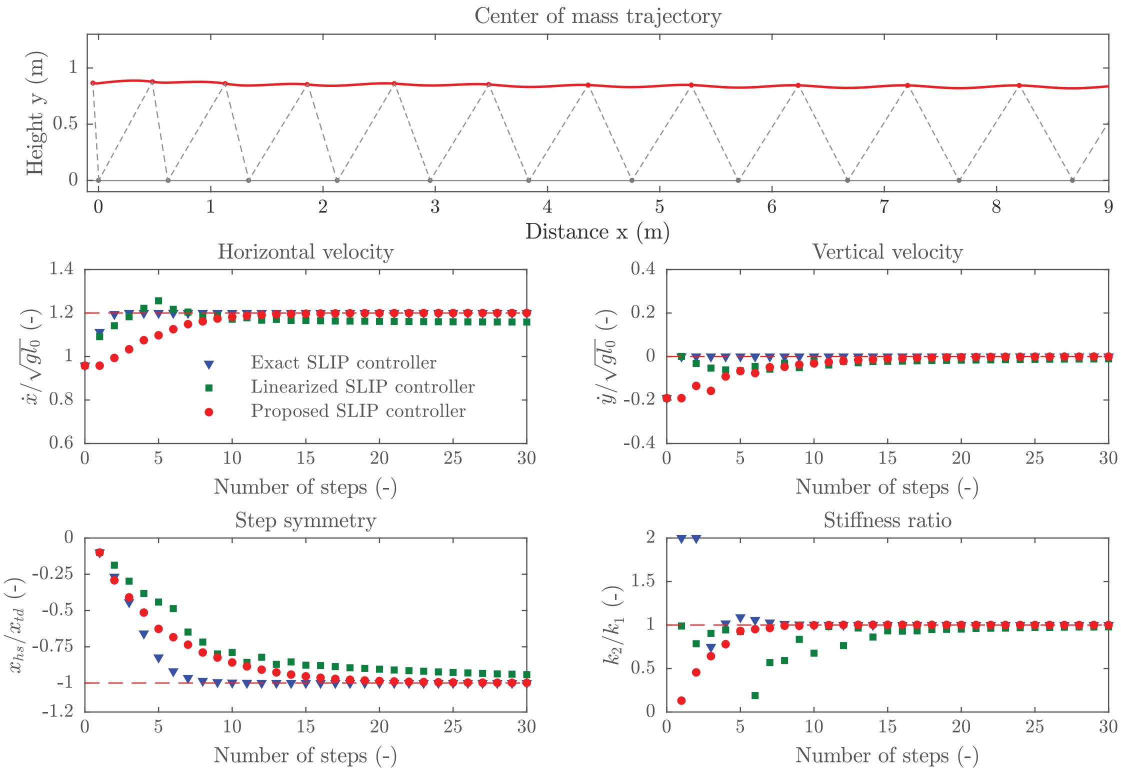

Fig. 5. Stable walking of the exact SLIP model at high speed and with large step length. (a) Center of mass trajectory for the proposed controller. (b) Dimensionless horizontal velocity at each heel strike. (c) Dimensionless vertical velocity at each heel strike. (d) Foot placement control. (e) Mid-step stiffness change ratio. (The stiffness change at mid-stance was limited by $k_2 \leq 2k_1$ for the exact SLIP and $k_2 \leq k_1$ for the other two controllers.)

dynamics and (ii) the approximate SLIP controller derived using the linearized SLIP dynamics. The linearized SLIP dynamics was derived assuming negligible leg compression and small step length.

(i) The exact SLIP controller was implemented using the exact SLIP dynamics and an optimization program that minimized the error between the desired and the actual horizontal velocities while constraining the takeoff vertical velocity to zero at every step. (ii) The linearized SLIP controller was implemented using the control laws defined by (32)–(34) where the steady-state stiffness values, k_1 and k_2, were computed using the linearized SLIP dynamics and an optimization program that minimized the error between the desired and the actual horizontal velocities while constraining the takeoff vertical velocity to zero at every step. (iii) The proposed SLIP controller was implemented using the control laws defined by (32)–(34).

The results are presented in Fig. 5 where the center of mass trajectory (Fig. 5a) is shown along with the heel strike horizontal and vertical velocities (Fig. 5bc), and the control parameters; the position of the center of mass at heel strike (Fig. 5d) and the ratio between the stiffness at heel strike and mid-stance (Fig. 5e). In all simulations, the model was started from a near-vertical configuration with velocities that results in the body taking off within one step without the controller. When the controllers were applied, no take-off accrued, and near-zero vertical velocity was achieved. The linearized SLIP controller took longer to converge, requiring high gains, and was challenging to tune (Fig. 5 green) compared to the proposed SLIP controller. The proposed SLIP controller resulted in steady-state walking at race-walking speeds ($\dot{x}/\sqrt{gl_0} = 1.2$) (Fig. 5 red) similar to the exact SLIP controller (Fig. 5 blue).

VI. Discussion and Conclusions

In this paper, we have defined a novel approximation for the SLIP model, which resulted in simplified dynamics that capture the non-linear behavior of the SLIP model. Based on the analytical solutions obtained, a closed-form mid-step stiffness change policy was derived for steady-state walking, see (29) and Fig. 4. Finally, numerical simulations were performed to demonstrate that the proposed control policy can be used for stable and fast locomotion with large steps when applied to the exact SLIP dynamics.

One shortcoming of the proposed approximation is the loss of accuracy for relatively high vertical velocities since the vertical position was assumed nearly constant during the motion, see *Assumption 3* in Section III-A. Another shortcoming of the proposed model is that it does not consider double support, and thus it is not well suited to model locomotion with finite double support [31], for example, slow walking, as opposed to locomotion with instantaneous or no double support such as fast walking and running. In future work, we aim to extend our model to 3D SLIP and derive controllers that can potentially compensate for large perturbations and transition smoothly between gaits.

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