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Adjoint twisted Reidemeister torsion and Gram matrices

Ka Ho Wong [a], Tian Yang [b,*]

[a] *Department of Mathematics, Yale University, New Haven, CT 06520, USA*
[b] *Department of Mathematics, Texas A&M University, College Station, TX 77843, USA*

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A B S T R A C T

We compute the adjoint twisted Reidemeister torsion for closed oriented hyperbolic 3-manifolds and for hyperbolic 3-manifolds with toroidal boundary. In our formula, we consider the manifold as obtained by doing a Dehn-filling along suitable boundary components of a fundamental shadow link complement, and the formula is in terms of the logarithmic holonomy of the meridians of the boundary components. As an important special case, we also write down a formula of the adjoint twisted Reidemeister torsion for the double of a hyperbolic 3-manifold with totally geodesic boundary in terms of the edge lengths of a geometric ideal triangulation of the manifold. These unexpected formulas were inspired by, and played an important role in, the study of the asymptotic expansion of quantum invariants [25].

1. Introduction

We compute the *adjoint twisted Reidemeister torsion* (see Section 2.1) for closed orientable hyperbolic 3-manifolds and for orientable hyperbolic 3-manifolds with toroidal

* Corresponding author.
E-mail addresses: kaho.wong@yale.edu (K.H. Wong), tianyang@math.tamu.edu (T. Yang).

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boundary with a representation of the fundamental group into $\mathrm{PSL}(2;\mathbb{C})$ for which the adjoint twisted Reidemeister torsion is defined.

To present the 3-manifolds, we use a 3-dimensional analogue of the pair-of-pants decompositions for surfaces, known as the *fundamental shadow link complements* ([6], see also Section 2.4). The fundamental shadow link complements form a universal family of 3-manifolds with toroidal boundary in the sense that all orientable 3-manifolds with empty or toroidal boundary can be obtained from one of them by doing a Dehn-filling along suitable boundary components [6]. Then in Theorem 1.1, we obtain an explicit formula of the adjoint twisted Reidemeister torsion of the fundamental shadow link complements, which turns out to be a product of the square root of the determinant of the values of the *Gram matrix function* (see Section 2.3) at the logarithmic holonomy of the meridians. As a consequence, in the main result of this paper, Theorem 1.4, we obtain an explicit formula of the adjoint twisted Reidemeister torsion of hyperbolic 3-manifolds obtained by doing a Dehn-filling along suitable boundary components of a fundamental shadow link complement. By [6,15], these manifolds contain most closed and cusped orientable hyperbolic 3-manifolds in the sense explained in Remark 1.5

To the best of our knowledge, this is by far the only explicit formula of the adjoint twisted Reidemeister torsion for most hyperbolic 3-manifolds. It is worth mentioning that the 1-loop Conjecture [8] suggests another formula of this quantity for cusped hyperbolic 3-manifolds in terms of the shape parameters.

In a setting dual to that in Theorem 1.1 and Theorem 1.4, we in Theorem 1.6 compute the adjoint twisted Reidemeister torsion of the double of a geometrically ideally triangulated hyperbolic 3-manifold with totally geodesic boundary, in terms of the edge lengths of the triangulation.

The relationship between the two intensively studied geometric quantities in our formulas, the adjoint twisted Reidemeister torsion and the Gram matrix, is completely unexpected, and is suggested by the asymptotic expansion of various quantum invariants of 3-manifolds proposed by the authors in [25]. This is one of the few examples where ideas from the study of quantum invariants shed light on a solution of purely geometric problems. In return, these formulas also play an essential role in the study of the asymptotic expansion of quantum invariants [25].

1.1. Fundamental shadow link complements

Theorem 1.1. *Let $M = \#^{d+1}(S^2\times S^1)\smallsetminus L_{FSL}$ be the complement of a fundamental shadow link L_{FSL} with n components $L_1,\dots,L_n$, which is the orientable double of the union of truncated tetrahedra $\Delta_1,\dots,\Delta_d$ along pairs of the triangles of truncation (see Section 2.4).*

(1) Let $\mathbf{m} = (m_1,\dots,m_n)$ be the system of the meridians of a tubular neighborhood of the components of L_{FSL}. For an $\mathbf{m}$-regular $\mathrm{PSL}(2;\mathbb{C})$*-character $[\rho]$ of M (see Definition 2.5), let $(u_1,\dots,u_n)$ be the logarithmic holonomies of $\mathbf{m}$ in ρ. For each*

$k \in \{1, \dots, d\}$, let $L_{k_1}, \dots, L_{k_6}$ be the components of L_{FSL} intersecting Δ_k, and let $\mathbb{G}_k = \mathbb{G}\left(\frac{u_{k_1}}{2}, \dots, \frac{u_{k_6}}{2}\right)$ be the value of the Gram matrix function at $\left(\frac{u_{k_1}}{2}, \dots, \frac{u_{k_6}}{2}\right)$. Then the adjoint twisted Reidemeister torsion of M with respect to $\boldsymbol{m}$ (see Definition 2.7) at $[\rho]$ is

$$\mathbb{T}_{(M,\boldsymbol{m})}([\rho]) = \pm 2^{3d} \prod_{k=1}^{d} \sqrt{\det \mathbb{G}_k}.$$

(2) In addition to the conditions of (1), let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ be a system of simple closed curves on ∂M, and let $(u_{\mu_1}, \dots, u_{\mu_n})$ be their logarithmic holonomies which are functions of $(u_1, \dots, u_n)$ near $[\rho]$. If $[\rho]$ is $\boldsymbol{\mu}$-regular, then the adjoint twisted Reidemeister torsion of M with respect to $\boldsymbol{\mu}$ at $[\rho]$ is

$$\mathbb{T}_{(M,\boldsymbol{\mu})}([\rho]) = \pm 2^{3d} \det\left(\frac{\partial(u_{\mu_1}, \dots, u_{\mu_n})}{\partial(u_1, \dots, u_n)}\right) \prod_{k=1}^{d} \sqrt{\det \mathbb{G}_k},$$

where $\frac{\partial(u_{\mu_1}, \dots, u_{\mu_n})}{\partial(u_1, \dots, u_n)}$ is the Jacobian matrix of $(u_{\mu_1}, \dots, u_{\mu_n})$ with respect to $(u_1, \dots, u_n)$ evaluated at $[\rho]$.

Remark 1.2. By (2.6) and the analyticity of both sides, the logarithmic holonomies of the system of longitudes, and hence of any system of simple closed curves on ∂M, can be explicitly written in terms of the $(u_1, \dots, u_n)$. Therefore, the formula in (2) can be written explicitly in terms of $(u_1, \dots, u_n)$.

Remark 1.3. By [21,20], all the characters near that of the holonomy representation of the complete hyperbolic structure of M are $\boldsymbol{\mu}$-regular for any system of simple closed curves $\boldsymbol{\mu}$ on ∂M.

1.2. Hyperbolic 3-manifolds

As a consequence of Theorem 1.1, we obtain in Theorem 1.4 a formula of the adjoint twisted Reidemeister torsion for hyperbolic 3-manifolds with empty or toroidal boundary obtained by doing a Dehn-filling along suitable boundary components of a fundamental shadow link complement, with a technique assumption on the holonomy representation of the hyperbolic structure. Recall from [6] that every orientable hyperbolic 3-manifolds with empty or toroidal boundary can be obtained in this way, and from [15] and as explained in Remark 1.5 for most closed and cusped hyperbolic 3-manifolds the technical assumption is satisfied.

Let M be a fundamental shadow link complement as in Theorem 1.1. For m with $0 \leqslant m \leqslant n$, let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$ be a system of simple closed curves on ∂M such that $\mu_i \subset T_i$, and let $\boldsymbol{\nu} = (\nu_{m+1}, \dots, \nu_n)$ be a system of simple closed curves on ∂M such that

$\nu_j \subset T_j$. Let $M_{\boldsymbol{\mu}}$ be the 3-manifold obtained from M by doing the Dehn-filling along $\boldsymbol{\mu}$. Then $\boldsymbol{\nu}$ can be considered as a system of simple closed curves on $\partial M_{\boldsymbol{\mu}}$. If $m = n$, then $\boldsymbol{\nu} = \emptyset$ and $M_{\boldsymbol{\mu}}$ is a closed 3-manifold,

Theorem 1.4. *Suppose $M_{\boldsymbol{\mu}}$ is hyperbolic. Let $[\rho_{\boldsymbol{\mu}}]$ be a $\boldsymbol{\nu}$-regular character of $M_{\boldsymbol{\mu}}$ and let ρ be the restriction of $\rho_{\boldsymbol{\mu}}$ on M. Let $(u_1, \dots, u_n)$ be the logarithmic holonomies of the system of meridians $\boldsymbol{m}$ in $[\rho]$ and for each $k \in \{1, \dots, d\}$, let $L_{k_1}, \dots, L_{k_6}$ be the components of L_{FSL} intersecting Δ_k and let $\mathbb{G}_k = \mathbb{G}\left(\frac{u_{k_1}}{2}, \dots, \frac{u_{k_6}}{2}\right)$ be the value of the Gram matrix function at $\left(\frac{u_{k_1}}{2}, \dots, \frac{u_{k_6}}{2}\right)$. Let $(u_{\mu_1}, \dots, u_{\mu_m})$ and $(u_{\nu_{m+1}}, \dots, u_{\nu_n})$ respectively be the logarithmic holonomies of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ considered as functions of $(u_1, \dots, u_n)$ near $[\rho]$. Let $(\gamma_1, \dots, \gamma_m)$ be a system of simple closed curves on ∂M that are isotopic to the core curves of the solid tori filled in and let $(u_{\gamma_1}, \dots, u_{\gamma_m})$ be their logarithmic holonomies in $[\rho]$. If $[\rho]$ is in the distinguished component of the* $\mathrm{PSL}(2;\mathbb{C})$*-character variety of M, then the adjoint twisted Reidemeister torsion of $M_{\boldsymbol{\mu}}$ with respect to $\boldsymbol{\nu}$ at $[\rho_{\boldsymbol{\mu}}]$ is*

$$\begin{aligned}&\mathbb{T}_{(M_{\boldsymbol{\mu}},\boldsymbol{\nu})}([\rho_{\boldsymbol{\mu}}])\\&\quad= \pm 2^{3d-2m} \det\left(\frac{\partial(u_{\mu_1}, \dots, u_{\mu_m}, u_{\nu_{m+1}}, \dots, u_{\nu_n})}{\partial(u_1, \dots, u_n)}\right) \prod_{k=1}^{d} \sqrt{\det \mathbb{G}_k} \prod_{i=1}^{m} \frac{1}{\sinh^2 \frac{u_{\gamma_i}}{2}},\end{aligned}$$

where $\frac{\partial(u_{\mu_1}, \dots, u_{\mu_m}, u_{\nu_{m+1}}, \dots, u_{\nu_n})}{\partial(u_1, \dots, u_n)}$ is the Jacobian matrix of $(u_{\mu_1}, \dots, u_{\mu_m}, u_{\nu_{m+1}}, \dots, u_{\nu_n})$ with respect to $(u_1, \dots, u_n)$ evaluated at $[\rho]$.

In particular, if $M_{\boldsymbol{\mu}}$ is closed, $\rho_{\boldsymbol{\mu}}$ is the holonomy representation of the hyperbolic structure and $[\rho]$ is in the distinguished component of the $\mathrm{PSL}(2;\mathbb{C})$*-character variety of M, then the adjoint twisted Reidemeister torsion of $M_{\boldsymbol{\mu}}$ is*

$$\mathrm{Tor}(M_{\boldsymbol{\mu}}; \mathrm{Ad}_{\rho_{\boldsymbol{\mu}}}) = \pm 2^{3d-2n} \det\left(\frac{\partial(u_{\mu_1}, \dots, u_{\mu_n})}{\partial(u_1, \dots, u_n)}\right) \prod_{k=1}^{d} \sqrt{\det \mathbb{G}_k} \prod_{i=1}^{n} \frac{1}{\sinh^2 \frac{u_{\gamma_i}}{2}}.$$

Remark 1.5. From [6,15], the manifolds $M_{\boldsymbol{\mu}}$ in Theorem 1.4 cover most closed and cusped orientable hyperbolic 3-manifolds in the sense that for each boundary component T_i of M, except for at most 114 simple closed curves μ_i, the complete hyperbolic metric on $M_{\boldsymbol{\mu}}$ can be connected to the complete hyperbolic metric on M by a one-parameter family of hyperbolic cone metrics on M. As a consequence, $[\rho]$ lies in the distinguished component of the $\mathrm{PSL}(2;\mathbb{C})$-character variety of M satisfying the condition in Theorem 1.4. We believe that this condition could be removed and the formula holds for all the closed hyperbolic 3-manifolds and hyperbolic 3-manifolds with toroidal boundary with a $\mathrm{PSL}(2;\mathbb{C})$-representation for which the adjoint twisted Reidemeister torsion is defined.

1.3. Double of hyperbolic polyhedral 3-manifolds

Theorem 1.6. *Let N be a hyperbolic polyhedral 3-manifold which is the union of truncated tetrahedra $\Delta_1, \dots, \Delta_d$ along pairs of hexagonal faces, and let M be the double of N with the double of the edges $e_1, \dots, e_n$ removed (see Section 2.5).*

(1) *For $i \in \{1, \dots, n\}$, let l_i be the lengths of e_i. Let $\boldsymbol{l}$ be the system of the preferred longitudes of M with the logarithmic holonomies $(2l_1, \dots, 2l_n)$. For each $k \in \{1, \dots, d\}$, let $e_{k_1}, \dots, e_{k_6}$ be the edges intersecting Δ_k, and let $\mathbb{G}_k = \mathbb{G}(l_{k_1}, \dots, l_{k_6})$ be the value of the Gram matrix function at $(l_{k_1}, \dots, l_{k_6})$. Let ρ be the holonomy representation of the hyperbolic cone metric on M obtained by doubling the hyperbolic polyhedral metric of N. Then*

$$\mathbb{T}_{(M,\boldsymbol{l})}([\rho]) = \pm 2^{3d} \prod_{k=1}^{d} \sqrt{\det \mathbb{G}_k}.$$

(2) *Let $\boldsymbol{m}$ be the system of meridians of a tubular neighborhood of the double of the edges, and let $(\theta_1, \dots, \theta_n)$ be the cone angles at the edges which are functions of the lengths $(l_1, \dots, l_n)$ of the edges of N. Then*

$$\mathbb{T}_{(M,\boldsymbol{m})}([\rho]) = \pm \mathbf{i}^n 2^{3d-n} \det\left(\frac{\partial(\theta_1, \dots, \theta_n)}{\partial(l_1, \dots, l_n)}\right) \prod_{k=1}^{d} \sqrt{\det \mathbb{G}_k},$$

where $\frac{\partial(\theta_1,\dots,\theta_n)}{\partial(l_1,\dots,l_n)}$ is the Jacobian matrix of $(\theta_1, \dots, \theta_n)$ with respect to $(l_1, \dots, l_n)$ evaluated at $[\rho]$.

(3) *Suppose $\overline{M}$ is the double of a geometrically ideally triangulated hyperbolic 3-manifold N with totally geodesic boundary (which is M with the removed double of edges filled back). Let ρ and $\overline{\rho}$ respectively be the holomony representations of M and $\overline{M}$. Let $(l_1, \dots, l_n)$ be the lengths of the edges of N and let $(\theta_1, \dots, \theta_n)$ be the cone angles considered as functions of $(l_1, \dots, l_n)$. For each $k \in \{1, \dots, d\}$, let $e_{k_1}, \dots, e_{k_6}$ be the edges intersecting Δ_k and let $\mathbb{G}_k = \mathbb{G}(l_{k_1}, \dots, l_{k_6})$ be the value of the Gram matrix function at $(l_{k_1}, \dots, l_{k_6})$. Then*

$$\mathrm{Tor}(\overline{M}; \mathrm{Ad}_{\overline{\rho}}) = \pm \mathbf{i}^n 2^{3d-3n} \det\left(\frac{\partial(\theta_1, \dots, \theta_n)}{\partial(l_1, \dots, l_n)}\right) \prod_{k=1}^{d} \sqrt{\det \mathbb{G}_k} \prod_{i=1}^{n} \frac{1}{\sinh^2 l_i},$$

where $\frac{\partial(\theta_1,\dots,\theta_n)}{\partial(l_1,\dots,l_n)}$ is the Jacobian matrix of $(\theta_1, \dots, \theta_n)$ with respect to $(l_1, \dots, l_n)$ evaluated at $[\rho]$.

Remark 1.7. Since the cone angles $(\theta_1, \dots, \theta_n)$ are the sums of the dihedral angles which by (2.7) can be explicitly written as functions of $(l_1, \dots, l_n)$, both of the formulas in (2) and (3) can be written explicitly in terms of the edge lengths $(l_1, \dots, l_n)$.

Remark 1.8. We believe that a similar formula of the adjoint twisted Reidemeister torsion of a geometrically ideally triangulated cusped hyperbolic 3-manifold and of a geometrically triangulated closed hyperbolic 3-manifold should also exist, respectively in terms of the decorated edge lengths and the edge lengths.

1.4. Outline of the proof

The main tool in the computation is the Mayer-Vietoris formula stated in Theorem 2.2. To use this formula, we in Sections 3 and 4 respectively compute the adjoint twisted Reidemeister torsion of the pairs of pants and of the D-blocks, and in Section 5 compute the Reidemeister torsion of the Mayer-Vietoris sequence. Then the results follow from Theorem 2.2.

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2. Preliminaries

2.1. Reidemeister torsions

Let C_* be a finite chain complex

$$0 \to \mathrm{C}_d \xrightarrow{\partial} \mathrm{C}_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathrm{C}_1 \xrightarrow{\partial} \mathrm{C}_0 \to 0$$

of $\mathbb{C}$-vector spaces, and for each C_k choose a basis $\mathbf{c}_k$. Let H_* be the homology of C_*, and for each H_k choose a basis $\mathbf{h}_k$ and a lift $\widetilde{\mathbf{h}}_k \subset \mathrm{C}_k$ of $\mathbf{h}_k$. We also choose a basis $\mathbf{b}_k$ for each image $\partial(\mathrm{C}_{k+1})$ and a lift $\widetilde{\mathbf{b}}_k \subset \mathrm{C}_{k+1}$ of $\mathbf{b}_k$. Then $\mathbf{b}_k \sqcup \widetilde{\mathbf{b}}_{k-1} \sqcup \widetilde{\mathbf{h}}_k$ form a new basis of C_k. Let $[\mathbf{b}_k \sqcup \widetilde{\mathbf{b}}_{k-1} \sqcup \widetilde{\mathbf{h}}_k; \mathrm{c}_k]$ be the determinant of the transition matrix from the basis $\mathbf{c}_k$ to the new basis $\mathbf{b}_k \sqcup \widetilde{\mathbf{b}}_{k-1} \sqcup \widetilde{\mathbf{h}}_k$. Then the Reidemeister torsion of the chain complex C_* with the chosen bases $\{\mathbf{c}_k\}$ and $\{\mathbf{h}_k\}$ is defined by

$$\mathrm{Tor}(\mathrm{C}_*, \{\mathbf{c}_k\}, \{\mathbf{h}_k\}) = \pm \prod_{k=0}^{d} [\mathbf{b}_k \sqcup \widetilde{\mathbf{b}}_{k-1} \sqcup \widetilde{\mathbf{h}}_k; \mathbf{c}_k]^{(-1)^{k+1}} \in \mathbb{C}^*/\{\pm 1\}. \tag{2.1}$$

It is easy to check that $\mathrm{Tor}(\mathrm{C}_*, \{\mathbf{c}_k\}, \{\mathbf{h}_k\})$ depends only on the choices of $\{\mathbf{c}_k\}$ and $\{\mathbf{h}_k\}$, and does not depend on the choices of $\{\mathbf{b}_k\}$ and the lifts $\{\widetilde{\mathbf{b}}_k\}$ and $\{\widetilde{\mathbf{h}}_k\}$.

We recall the twisted Reidemeister torsion of a CW-complex following the conventions in [22]. Let K be a finite CW-complex and let $\rho : \pi_1(K) \to \mathrm{SL}(N;\mathbb{C})$ be a representation of its fundamental group. Consider the twisted chain complex

$$\mathrm{C}_*(K;\rho) = \mathbb{C}^N \otimes_\rho \mathrm{C}_*(\widetilde{K};\mathbb{Z})$$

where $\mathrm{C}_*(\widetilde{K};\mathbb{Z})$ is the simplicial complex of the universal covering of K and $\otimes_\rho$ means the tensor product over $\mathbb{Z}$ modulo the relation

$$\mathbf{v} \otimes (\gamma \cdot \mathbf{c}) = \Big(\rho(\gamma)^T \cdot \mathbf{v}\Big) \otimes \mathbf{c},$$

where T is the transpose, $\mathbf{v} \in \mathbb{C}^N$, $\gamma \in \pi_1(K)$ and $\mathbf{c} \in \mathrm{C}_*(\widetilde{K};\mathbb{Z})$. The boundary operator on $\mathrm{C}_*(K;\rho)$ is defined by

$$\partial(\mathbf{v} \otimes \mathbf{c}) = \mathbf{v} \otimes \partial(\mathbf{c})$$

for $\mathbf{v} \in \mathbb{C}^N$ and $\mathbf{c} \in \mathrm{C}_*(\widetilde{K};\mathbb{Z})$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ be the standard basis of $\mathbb{C}^N$, and let $\{c_1^k, \dots, c_{d^k}^k\}$ denote the set of k-cells of K. Then we call

$$\mathbf{c}_k = \big\{\mathbf{e}_i \otimes c_s^k \bigm| i \in \{1,\dots,N\}, s \in \{1,\dots,d^k\}\big\}$$

the standard basis of $\mathrm{C}_k(K;\rho)$. Let $\mathrm{H}_*(K;\rho)$ be the homology of the chain complex $\mathrm{C}_*(K;\rho)$ and let $\mathbf{h}_k$ be a basis of $\mathrm{H}_k(K;\rho)$. Then the Reidemeister torsion of K twisted by ρ with the basis $\{\mathbf{h}_k\}$ is

$$\mathrm{Tor}(K, \{\mathbf{h}_k\};\rho) = \mathrm{Tor}(\mathrm{C}_*(K;\rho), \{\mathbf{c}_k\}, \{\mathbf{h}_k\}).$$

By [21], $\mathrm{Tor}(K, \{\mathbf{h}_k\};\rho)$ depends only on the conjugacy class of ρ. By for example [23], the Reidemeister torsion is invariant under elementary expansions and elementary collapses of CW-complexes; and by [19] it is invariant under subdivisions, hence defines an invariant of PL-manifolds and of topological manifolds of dimension less than or equal to 3.

A useful tool to compute the twisted Reidemeister torsion is the Mayer-Vietoris sequence. Suppose K is a finite CW-complex and $K_1, K_2, \dots, K_n$ are its sub-complexes. For $\{i,j\} \subset \{1,2,\dots,n\}$, let $K_{ij} = K_i \cap K_j$ if it is non-empty. Assume

(1) $K = K_1 \cup K_2 \cup \cdots \cup K_n$, and
(2) $K_i \cap K_j \cap K_k = \emptyset$ for all $\{i,j,k\} \subset \{1,\dots,n\}$.

For a representation $\rho : \pi_1(K) \to \mathrm{SL}(N;\mathbb{C})$, let ρ_k and ρ_{ij} respectively be the restriction of ρ to $\pi_1(K_k)$ and $\pi_1(K_{ij})$.

Lemma 2.1. *The follow sequence of chain complexes*

$$0 \to \bigoplus_{\{i,j\}\subset\{1,\dots,n\}} \mathrm{C}_*(K_{ij};\rho_{ij}) \xrightarrow{\delta} \bigoplus_{k=1}^{n} \mathrm{C}_*(K_k;\rho_k) \xrightarrow{\epsilon} \mathrm{C}_*(K;\rho) \to 0 \tag{2.2}$$

is exact, where ϵ *is the sum defined by*

$$\epsilon(\mathbf{c}_1, \ldots, \mathbf{c}_n) = \sum_{k=1}^{n} \mathbf{c}_k$$

and δ *is the alternating sum defined by*

$$(\delta \mathbf{c})_k = -\sum_{j=1}^{k-1} \mathbf{c}_{jk} + \sum_{l=k+1}^{n} \mathbf{c}_{kl}.$$

This short exact sequence can be found in for example [4, Proposition 15.2] for untwisted complexes, and the proof for the twisted case is similar. The short exact sequence (2.2) induces the following long exact sequence $\mathcal{H}$:

$$\begin{aligned} \cdots \to \mathrm{H}_{m+1}(K;\rho) \xrightarrow{\partial_{m+1}} & \bigoplus_{\{i,j\}\subset\{1,\ldots,n\}} \mathrm{H}_m(K_{ij};\rho_{ij}) \\ & \xrightarrow{\delta_m} \bigoplus_{k=1}^{n} \mathrm{H}_m(K_k;\rho_k) \xrightarrow{\epsilon_m} \mathrm{H}_m(K;\rho) \to \ldots, \end{aligned} \tag{2.3}$$

and the twisted Reidemeister torsion of K can be computed by those of $\{K_k\}$, $\{K_{ij}\}$ and $\mathcal{H}$.

Theorem 2.2 *(Mayer-Vietoris). ([21, Proposition 0.11]) Let* $\mathbf{h}_*$, $\{\mathbf{h}_{k,*}\}$ *and* $\{\mathbf{h}_{ij,*}\}$ *respectively be bases of* $\mathrm{H}_*(K;\rho)$, $\mathrm{H}_*(K_k;\rho_k)$ *and* $\mathrm{H}_*(K_{ij};\rho_{ij})$, *and let* $\mathbf{h}_{**}$ *be the union of* $\mathbf{h}_*$, $\sqcup_k \mathbf{h}_{k,*}$ *and* $\sqcup_{\{i,j\}} \mathbf{h}_{ij,*}$ *which is a basis of* $\mathcal{H}$. *Then*

$$\mathrm{Tor}(K, \{\mathbf{h}_*\};\rho) = \pm \frac{\prod_{k=1}^{n} \mathrm{Tor}(K_k, \mathbf{h}_{k,*};\rho_k)}{\prod_{\{i,j\}\subset\{1,\ldots,n\}} \mathrm{Tor}(K_{ij}, \mathbf{h}_{ij,*};\rho_{ij}) \cdot \mathrm{Tor}(\mathcal{H}, \mathbf{h}_{**})}.$$

In [21, Proposition 0.11], Theorem 2.2 is proved for the union of two sub-complexes, and the proof of the current form carries out in essentially the same way.

2.2. Adjoint twisted Reidemeister torsions

In this section we recall results of Porti [21] for the Reidemeister torsions of hyperbolic 3-manifolds twisted by the adjoint action $\mathrm{Ad}_\rho = \mathrm{Ad} \circ \rho$ of an irreducible $\mathrm{PSL}(2;\mathbb{C})$-representation ρ. Here Ad is the adjoint action of $\mathrm{PSL}(2;\mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2;\mathbb{C}) \cong \mathbb{C}^3$.

For a closed orientable hyperbolic 3-manifold M with the holonomy representation ρ, by the Weil local rigidity theorem and the Mostow rigidity theorem, $\mathrm{H}_k(M;\mathrm{Ad}_\rho) = 0$ for all k. Then the adjoint twisted Reidemeister torsion

$$\mathrm{Tor}(M; \mathrm{Ad}_\rho) \in \mathbb{C}^*/\{\pm 1\}$$

is defined without making any additional choice.

Now suppose M is a compact, orientable 3-manifold with boundary consisting of n disjoint tori $T_1 \dots, T_n$ whose interior admits a complete hyperbolic structure with finite volume. Let $\mathrm{X}(M)$ be the $\mathrm{PSL}(2;\mathbb{C})$-character variety of M.

By [23,7,9], every irreducible component of $\mathrm{X}(M)$ has dimension greater than or equal to n; and we denote by $\mathrm{X}^n(M) = \cup \mathrm{X}_k(M)$ the union of the irreducible components $\{\mathrm{X}_k(M)\}$ of $\mathrm{X}(M)$ that have dimension exact equal to n. If M is hyperbolic, then $\mathrm{X}^n(M)$ is non-empty because it contains the distinguished component $\mathrm{X}_0(M)$ containing the character of the holomony representation of the complete hyperbolic structure of M [23,20]. The main reason that we consider the space $\mathrm{X}^n(M)$ in this article instead of $\mathrm{X}_0(M)$ is that: If $M_{\boldsymbol{\mu}}$ is a hyperbolic 3-manifold obtained by doing a Dehn-filling along a system of simple closed curves $\boldsymbol{\mu}$ on ∂M, then it is not clear whether the restriction of the character of the holonomy representation of the hyperbolic structure on $M_{\boldsymbol{\mu}}$ to M always lies in $\mathrm{X}_0(M)$; but it always lies in $\mathrm{X}^n(M)$ by a standard Mayer-Vietoris sequence argument. This fact will be used in the proof of Theorem 1.6.

Below we recall two fundamental results (Theorem 2.3 and Theorem 2.8) of Porti [21]. Theorem 2.8 was originally proved for characters in $\mathrm{X}_0(M)$, but by essentially the same argument can be generalized to characters in $\mathrm{X}^n(X)$.

We denote by $\mathrm{X}^{\mathrm{irr}}(M)$ the Zariski-open subset of $\mathrm{X}(M)$ consisting of the irreducible characters.

Theorem 2.3. *[21, Section 3.3.3] For a system of simple closed curves $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ on ∂M with $\alpha_i \subset T_i$, $i \in \{1, \dots, n\}$, and a character $[\rho]$ in a Zariski open subset of $\mathrm{X}_0(M) \cap \mathrm{X}^{irr}(M)$, we have:*

(i) For $k \neq 1, 2$, $\mathrm{H}_k(M; \mathrm{Ad}\rho) = 0$.

(ii) For $i \in \{1, \dots, n\}$, up to scalar $\mathrm{Ad}_\rho(\pi_1(T_i))^T$ has a unique invariant vector $\mathbf{I}_i \in \mathbb{C}^3$; and

$$\mathrm{H}_1(M; \mathrm{Ad}\rho) \cong \mathbb{C}^n$$

with a basis

$$\mathbf{h}^1_{(M,\alpha)} = \{\mathbf{I}_1 \otimes [\alpha_1], \dots, \mathbf{I}_n \otimes [\alpha_n]\}$$

where $([\alpha_1], \dots, [\alpha_n]) \in \mathrm{H}_1(\partial M; \mathbb{Z}) \cong \bigoplus_{i=1}^n \mathrm{H}_1(T_i; \mathbb{Z})$.

(iii) Let $([T_1], \dots, [T_n]) \in \bigoplus_{i=1}^n \mathrm{H}_2(T_i; \mathbb{Z})$ be the fundamental classes of $T_1, \dots, T_n$. Then

$$\mathrm{H}_2(M; \mathrm{Ad}\rho) \cong \bigoplus_{i=1}^n \mathrm{H}_2(T_i; \mathrm{Ad}\rho) \cong \mathbb{C}^n$$

with a basis

$$\mathbf{h}^2_M = \{\mathbf{I}_1 \otimes [T_1], \ldots, \mathbf{I}_n \otimes [T_n]\}.$$

Remark 2.4 *([21,20,14]).* Important examples of the characters in Theorem 2.3 include the character of the holonomy representation of the complete hyperbolic structure on the interior of M, the restriction of the holonomy representation of the closed 3-manifold $M_{\boldsymbol{\mu}}$ obtained from M by doing the hyperbolic Dehn-filling along the system of simple closed curves $\boldsymbol{\mu}$ on ∂M, and the holonomy representation of a hyperbolic structure on the interior of M whose completion is a conical manifold with cone angles less than 2π.

Definition 2.5. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a system of simple closed curves on ∂M with $\alpha_i \subset T_i$, $i \in \{1, \ldots, n\}$. A character $[\rho]$ in $\mathrm{X}^n(M) \cap \mathrm{X}^{irr}(M)$ is* $\boldsymbol{\alpha}$-regular *if condition (ii) in Theorem 2.3 is satisfied.*

Remark 2.6. We notice that Definition 2.5 does not only consider characters in the distinguished component $\mathrm{X}_0(M)$, but also considers characters in $\mathrm{X}^n(M)$. By [21, Proposition 3.22], for characters in $\mathrm{X}_0(M)$, our definition of the $\boldsymbol{\alpha}$-regularity is equivalent to [21, Définition 3.21].

It follows that for any system of simple closed curves $\boldsymbol{\alpha}$ on ∂M, the $\boldsymbol{\alpha}$-regular characters are smooth points of $\mathrm{X}(M)$; and the logarithmic holonomies of $\boldsymbol{\alpha}$ form a local parametrization of $\mathrm{X}(M)$ near each of the $\boldsymbol{\alpha}$-regular characters. Here for a $\mathrm{PSL}(2;\mathbb{C})$-character $[\rho]$, the *logarithmic holonomy* of α_i is defined up to sign as the logarithm of the ratio of the eigenvalues of $\rho([\alpha_i])$.

Definition 2.7. *The* adjoint twisted Reidemeister torsion *of M* with respect to $\boldsymbol{\alpha}$ *is the function*

$$\mathbb{T}_{(M,\boldsymbol{\alpha})} : \mathrm{X}^n(M) \cap \mathrm{X}^{irr}(M) \to \mathbb{C}/\{\pm 1\}$$

defined by

$$\mathbb{T}_{(M,\boldsymbol{\alpha})}([\rho]) = \mathrm{Tor}(M, \{\mathbf{h}^1_{(M,\alpha)}, \mathbf{h}^2_M\}; \mathrm{Ad}_\rho)$$

if ρ is $\boldsymbol{\alpha}$-regular, and by 0 if otherwise.

Theorem 2.8. *[21, Theorem 4.1] Let M be a compact, orientable 3-manifold with boundary consisting of n disjoint tori $T_1 \ldots, T_n$ whose interior admits a complete hyperbolic structure with finite volume. Let $\mathbb{C}(\mathrm{X}^n(M) \cap \mathrm{X}^{irr}(M))$ be the ring of rational functions over $\mathrm{X}^n(M) \cap \mathrm{X}^{irr}(M)$. Then*

$$\begin{aligned} \mathrm{H}_1(\partial M; \mathbb{Z}) &\to \mathbb{C}(\mathrm{X}^n(M) \cap \mathrm{X}^{irr}(M)) \\ \boldsymbol{\alpha} \quad &\mapsto \quad \mathbb{T}_{(M,\boldsymbol{\alpha})} \end{aligned}$$

up to sign defines a function which is a $\mathbb{Z}$*-multilinear homomorphism with respect to the direct sum* $\mathrm{H}_1(\partial M;\mathbb{Z}) \cong \bigoplus_{i=1}^n \mathrm{H}_1(T_i;\mathbb{Z})$ *satisfying the following properties:*

(i) *For a system of simple closed curves* $\boldsymbol{\alpha}$ *on* ∂M*, if the component* $\mathrm{X}_k(M)$ *contains an* $\boldsymbol{\alpha}$*-regular character, then the support of* $\mathbb{T}_{(M,\boldsymbol{\alpha})}$ *contains a Zariski-open subset of* $\mathrm{X}_k(M) \cap \mathrm{X}^{irr}(M)$ *consisting of all the* $\boldsymbol{\alpha}$*-regular characters in* $\mathrm{X}_k(M)$*.*

(ii) (Change of Curves Formula). *Let* $\boldsymbol{\beta} = \{\beta_1,\dots,\beta_n\}$ *and* $\boldsymbol{\gamma} = \{\gamma_1,\dots,\gamma_n\}$ *be two systems of simple closed curves on* ∂M*. Let* $(u_{\beta_1},\dots,u_{\beta_n})$ *and* $(u_{\gamma_1},\dots,u_{\gamma_n})$ *respectively be the logarithmic holonomies of the curves in* $\boldsymbol{\beta}$ *and* $\boldsymbol{\gamma}$*. Then we have the equality of rational functions*

$$\mathbb{T}_{(M,\boldsymbol{\beta})} = \pm \det\left(\frac{\partial(u_{\beta_1},\dots,u_{\beta_n})}{\partial(u_{\gamma_1},\dots,u_{\gamma_n})}\right)\mathbb{T}_{(M,\boldsymbol{\gamma})} \tag{2.4}$$

on $\mathrm{X}_k(M) \cap \mathrm{X}^{irr}(M)$ *for the component* $\mathrm{X}_k(M)$ *containing a* $\boldsymbol{\gamma}$*-regular character, where* $\frac{\partial(u_{\beta_1},\dots,u_{\beta_n})}{\partial(u_{\gamma_1},\dots,u_{\gamma_n})}$ *is the Jocobian matrix of* $(u_{\beta_1},\dots,u_{\beta_n})$ *with respect to* $(u_{\gamma_1},\dots,u_{\gamma_n})$*.*

(iii) (Surgery Formula). *For* m *with* $0 \leqslant m \leqslant n$*, let* $\boldsymbol{\mu} = (\mu_1,\dots,\mu_m)$ *be a system of simple closed curves on* ∂M *such that* $\mu_i \subset T_i$*, and let* $\boldsymbol{\nu} = (\nu_{m+1},\dots,\nu_n)$ *be a system of simple closed curves on* ∂M *such that* $\nu_j \subset T_j$*. Let* $M_{\boldsymbol{\mu}}$ *be a hyperbolic* 3*-manifold obtained from* M *be doing the Dehn-filling along* $\boldsymbol{\mu}$*. Then* $\boldsymbol{\nu}$ *can be considered as a system of simple closed curves on* $\partial M_{\boldsymbol{\mu}}$*. Let* $[\rho_{\boldsymbol{\mu}}] \in \mathrm{X}^{n-m}(M_{\boldsymbol{\mu}}) \cap \mathrm{X}^{irr}(M_{\boldsymbol{\mu}})$ *and let* $[\rho] \in \mathrm{X}^n(M) \cap \mathrm{X}^{irr}(M)$ *be the restriction of* $[\rho_{\boldsymbol{\mu}}]$ *on* M*. Let* $(u_{\gamma_1},\dots,u_{\gamma_m})$ *be the logarithmic holonomies in* ρ *of the core curves* $\gamma_1,\dots,\gamma_m$ *of the solid tori filled in. If* $\rho_{\boldsymbol{\mu}}$ *is* $\boldsymbol{\nu}$*-regular, then* ρ *is* $\boldsymbol{\mu}\cup\boldsymbol{\nu}$*-regular, and*

$$\mathbb{T}_{(M_{\boldsymbol{\mu}},\boldsymbol{\nu})}([\rho_{\boldsymbol{\mu}}]) = \pm\mathbb{T}_{(M,\boldsymbol{\mu}\cup\boldsymbol{\nu})}([\rho])\prod_{i=1}^m \frac{1}{4\sinh^2\frac{u_{\gamma_i}}{2}}. \tag{2.5}$$

2.3. Gram matrix function and truncated hyperideal tetrahedra

Definition 2.9. *Let* $\mathrm{M}_{4\times4}(\mathbb{C})$ *be the space of* 4×4 *matrices with complex entries. The* Gram matrix function

$$\mathbb{G} : \mathbb{C}^6 \to \mathrm{M}_{4\times4}(\mathbb{C})$$

is defined for $\boldsymbol{z} = (z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34})$ *by*

$$\mathbb{G}(\boldsymbol{z}) = \begin{bmatrix} 1 & -\cosh z_{12} & -\cosh z_{13} & -\cosh z_{14} \\ -\cosh z_{12} & 1 & -\cosh z_{23} & -\cosh z_{24} \\ -\cosh z_{13} & -\cosh z_{23} & 1 & -\cosh z_{34} \\ -\cosh z_{14} & -\cosh z_{24} & -\cosh z_{34} & 1 \end{bmatrix}.$$

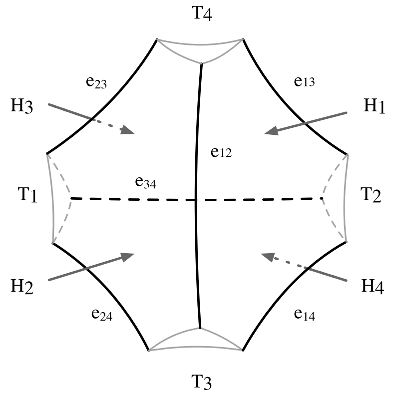

Fig. 1. Gram matrix in the dihedral angles.

The values of $\mathbb{G}$ at different $\boldsymbol{z}$ recover the Gram matrices of a truncated hyperideal tetrahedron in the dihedral angles and in the edge lengths. Recall from [1,11] that a truncated hyperideal tetrahedron Δ in $\mathbb{H}^3$ is a compact convex polyhedron that is diffeomorphic to a truncated tetrahedron in $\mathbb{E}^3$ with four hexagonal faces $\{H_1, H_2, H_3, H_4\}$ isometric to right-angled hyperbolic hexagons and four triangular faces $\{T_1, T_2, T_3, T_4\}$ isometric to hyperbolic triangles. We call the four triangular faces the *triangles of truncation*, and call the intersection of two hexagonal faces an *edge* and the angle between these two hexagonal faces the *dihedral angle* at this edge.

For $\{i, j\} \subset \{1, 2, 3, 4\}$, as in Fig. 1, if we let e_{ij} be the edge adjacent to the hexagonal faces H_i and H_j, and let α_{ij} and l_{ij} respectively be the dihedral angle at and the length of e_{ij}, then the *Gram matrix in the dihedral angles* of Δ is the matrix

$$\mathrm{G}_{\boldsymbol{\alpha}} = \begin{bmatrix} 1 & -\cos\alpha_{12} & -\cos\alpha_{13} & -\cos\alpha_{14} \\ -\cos\alpha_{12} & 1 & -\cos\alpha_{23} & -\cos\alpha_{24} \\ -\cos\alpha_{13} & -\cos\alpha_{23} & 1 & -\cos\alpha_{34} \\ -\cos\alpha_{14} & -\cos\alpha_{24} & -\cos\alpha_{34} & 1 \end{bmatrix}.$$

For $k, l \in \{1, 2, 3, 4\}$, let $G^{kl}_{\boldsymbol{\alpha}}$ be the kl-th cofactor of $\mathrm{G}_{\boldsymbol{\alpha}}$. Then by the hyperbolic Law of Cosine, we have

$$\cosh l_{ij} = \frac{G^{kl}_{\boldsymbol{\alpha}}}{\sqrt{G^{kk}_{\boldsymbol{\alpha}} G^{ll}_{\boldsymbol{\alpha}}}}, \tag{2.6}$$

where $\{k, l\} = \{1, 2, 3, 4\} \smallsetminus \{i, j\}$.

For $\{i, j\} \subset \{1, 2, 3, 4\}$, as in Fig. 2, if we let e_{ij} be the edge connecting the triangles of truncation T_i and T_j, and let l_{ij} and α_{ij} respectively be the length of and the dihedral angle at e_{ij}, then the *Gram matrix in the edge lengths* of Δ is the matrix

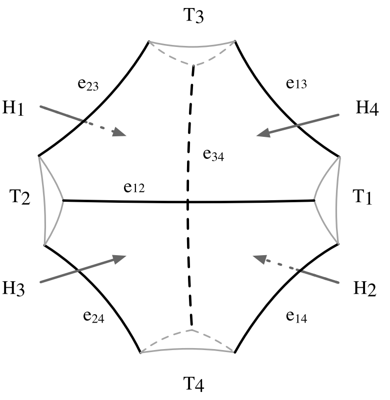

Fig. 2. Gram matrix in the edge lengths.

$$\mathrm{G}_{\boldsymbol{l}} = \begin{bmatrix} 1 & -\cosh l_{12} & -\cosh l_{13} & -\cosh l_{14} \\ -\cosh l_{12} & 1 & -\cosh l_{23} & -\cosh l_{24} \\ -\cosh l_{13} & -\cosh l_{23} & 1 & -\cosh l_{34} \\ -\cosh l_{14} & -\cosh l_{24} & -\cosh l_{34} & 1 \end{bmatrix}.$$

For $k, l \in \{1,2,3,4\}$, let $G_{\boldsymbol{l}}^{kl}$ be the kl-th cofactor of $\mathrm{G}_{\boldsymbol{l}}$. Then by the hyperbolic Law of Cosine, we have

$$\cos \alpha_{ij} = \frac{G_{\boldsymbol{l}}^{kl}}{\sqrt{G_{\boldsymbol{l}}^{kk} G_{\boldsymbol{l}}^{ll}}}, \tag{2.7}$$

where $\{k, l\} = \{1,2,3,4\} \smallsetminus \{i, j\}$.

We observe that, if $\boldsymbol{z} = (\mathbf{i}\alpha_{12}, \mathbf{i}\alpha_{13}, \mathbf{i}\alpha_{14}, \mathbf{i}\alpha_{23}, \mathbf{i}\alpha_{24}, \mathbf{i}\alpha_{34})$, then for Δ in Fig. 1,

$$\mathbb{G}(\boldsymbol{z}) = \mathrm{G}_{\boldsymbol{\alpha}};$$

and if $\boldsymbol{z} = (l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34})$, then for Δ in Fig. 2,

$$\mathbb{G}(\boldsymbol{z}) = \mathrm{G}_{\boldsymbol{l}}.$$

Remark 2.10. The way of assigning the edges $\{e_{ij}\}$ in the latter case is to consider Δ as a deeply truncated tetrahedron [16] that $T_1, \ldots, T_4$ are the faces and $H_1, \ldots, H_4$ are the faces of truncation. In this way, e_{ij} is the edge adjacent to or connecting the i-th and the j-th faces. For a general deeply truncated tetrahedron Δ, when two faces intersect we let $z_{ij} = \pm \mathbf{i}\alpha_{ij}$ and when two faces are disjoint we let $z_{ij} = \pm l_{ij}$, then $\mathbb{G}(\boldsymbol{z})$ coincides with the Gram matrix of the deeply truncated tetrahedron Δ. See [2, Section 2.1] for more details.

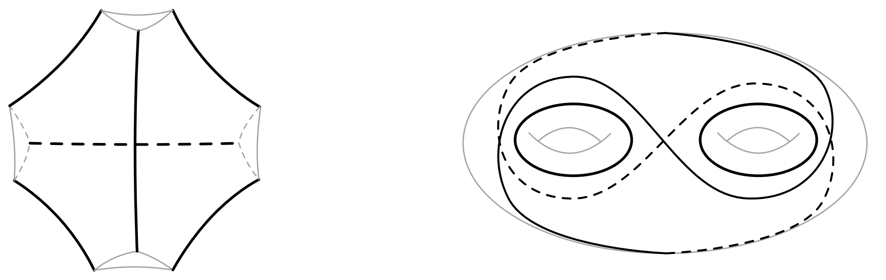

Fig. 3. The handlebody on the right is obtained from the truncated tetrahedron on the left by identifying the triangles on the top and the bottom by a horizontal reflection and the triangles on the left and the right by a vertical reflection.

2.4. Fundamental shadow link complements

In this section we recall the construction and basic properties of the fundamental shadow link complements. The building blocks for a fundamental shadow link complement are truncated tetrahedra as on the left of Fig. 3. If we take d building blocks $\Delta_1, \ldots, \Delta_d$ and glue them together along the triangles of truncation, then we obtain a (possibly non-orientable) handlebody of genus $d+1$ with a link on its boundary consisting of the edges of the building blocks, such as the right of Fig. 3. By taking the orientable double (the orientable double covering with the boundary quotient out by the deck involution) of this handlebody, we obtain a link L_{FSL} inside $\#^{d+1}(S^2 \times S^1)$. We call a link obtained this way a fundamental shadow link, and its complement $M = \#^{d+1}(S^2 \times S^1) \smallsetminus L_{\mathrm{FSL}}$ a *fundamental shadow link complement.* The fundamental shadow link complements form a universal family of 3-manifolds in the following sense.

Theorem 2.11 *([6]). Any compact orientable* 3*-manifold with empty or toroidal boundary can be obtained from a fundamental shadow link complement by doing an integral Dehn-filling along suitable boundary components.*

A hyperbolic cone metric on $\#^{d+1}(S^2 \times S^1)$ with singular locus L_{FSL} and with cone angles $2\alpha_1, \ldots, 2\alpha_n$ can be constructed as follows. For each $k \in \{1, \ldots, d\}$, let $e_{k_1}, \ldots, e_{k_6}$ be the edges of the building block Δ_k, and let $2\alpha_{k_i}$ be the cone angle of the component of L containing e_{k_i}. Suppose $\{\alpha_{k_1}, \ldots, \alpha_{k_6}\}$ form the set of dihedral angles of a truncated hyperideal tetrahedron, by abuse of notation still denoted by Δ_k. Then the hyperbolic metric on M whose completion has singular locus L_{FSL} with cone angles $2\alpha_1, \ldots, 2\alpha_n$ at the components is obtained by glueing the truncated hyperideal tetrahedra Δ_k's together along isometries between pairs of the triangles of truncation, then taking the orientable double. In this metric, the logarithmic holonomy of the meridian of a tubular neighborhood of the i-th component of L_{FSL} equals $2\mathbf{i}\alpha_i$. We also notice that when all the truncated hyperideal tetrahedra have edge lengths equal to zero, i.e., are the regular ideal hyperbolic octahedra, we obtain the complete hyperbolic structure on M.

For the purpose of computing the adjoint twisted Reidemeister torsion, we need the following alternative construction of the fundamental shadow link complements. The idea is that, instead of gluing the truncated tetrahedra together along the triangles of truncation first and then taking the orientable double, we take the double of each tetrahedron first along the hexagonal faces and then glue the resulting pieces together along the pairs of the double of the triangles of truncation. To be precise, for each Δ_k, $k \in \{1, \dots, d\}$, we let D_k be the union of Δ_k with its mirror image via the identity map between the four hexagonal faces and with the six edges removed. In the language of [5], D_k is a *D-block*. The boundary of each D_k is a union of four 3-puncture spheres (coming from the double of the four triangles of truncation) and six cylinders (coming from the boundary of a tubular neighborhood of the edges). We glue these D-blocks together via orientation reversing homeomorphisms between pairs of 3-puncture spheres part of the boundary, which send a triangle of truncation in one 3-puncture sphere to a triangle of truncation in the other 3-puncture sphere. The quotient space is a fundamental shadow link complement, and this construction could be considered as a 3-dimensional analog of the pair of pants decomposition of surfaces.

We call the double of a truncated hyperideal tetrahedra a *hyperbolic D-block*. Then a hyperbolic cone metric on M can alternatively be constructed as by gluing the hyperbolic D-blocks together via orientation reversing isometries between the hyperbolic 3-puncture spheres (double of the hyperbolic triangles of truncation with the three cone singularities removed) which preserve the hyperbolic triangles.

2.5. Double of hyperbolic polyhedral 3-manifolds

Dual to the construction of a fundamental shadow link complement is the construction of the double of a hyperbolic polyhedral 3-manifold. As defined in [17,18], a *hyperbolic polyhedral* 3-*manifold* N is obtained from d truncated hyperideal tetrahedra $\Delta_1, \dots, \Delta_d$ glued together via isometries between pairs of the hexagonal faces. The *cone angle* at an edge is the sum of the dihedral angles of the truncated hyperideal tetrahedra around the edge. If all the cone angles are equal to 2π, then N admits a hyperbolic metric with totally geodesic boundary and a geometric triangulation given by $\Delta_1, \dots, \Delta_d$. It is proved in [18, Theorem 1.2 (b)] that hyperbolic polyhedral 3-manifolds are rigid in the sense that they are up to isometry determined and infinitesimally determined by their cone angles.

To construct the double of N, we can also take the double of each tetrahedron first along the triangles of truncation and then glue the resulting pieces together. To be precise, for each truncated tetrahedron Δ_k, $k \in \{1, \dots, d\}$, we let D_k be the union of Δ_k with its mirror image via the identity map between the four triangles of truncation and with the double of the six edges removed. This is dual to the D-block in Section 2.4, hence we call it a *dual D-block*. The boundary of each D_k is a union of four 3-hole spheres (coming from the double of the four hexagonal faces) and six cylinders (coming from the double of the boundary of a tubular neighborhood of the edges). We then glue these dual D-blocks together via orientation reversing homeomorphisms between pairs of 3-hole spheres, and the quotient space M is the double of N with the double of the edges removed. If we fill the double of edges back in, topologically we get the double $\overline{M}$ of N.

Geometrically, if we let each truncated tetrahedron Δ_k be a truncated hyperideal tetrahedron, then the four 3-hole spheres are hyperbolic 3-hole spheres with geodesic boundary. If we require the gluing map between these hyperbolic 3-hole spheres to be isometries, then the quotient space is the double $\overline{M}$ of the hyperbolic polyhedral 3-manifold N, and M is obtained from $\overline{M}$ by removing all the double of the edges.

For $i \in \{1, \dots, n\}$, let l_i be the length of the edge e_i of the hyperbolic polyhedral manifold N. Since M comes from doubling, we can choose a *preferred longitude* on the boundary of a tubular neighborhood of the double of e_i (by isotoping e_i into Δ_k and then doubling) whose logarithmic holonomy equals $2l_i$.

3. Adjoint twisted Reidemeister torsion of the pairs of pants

Let P be a pair of pants with oriented boundary components γ_1, γ_2 and γ_3. Then $\pi_1(P)$ is a free group of rank 2 generated by $[\gamma_1]$ and $[\gamma_2]$. By [10,12], the $\mathrm{SL}(2;\mathbb{C})$-character variety of P is homeomorphic to $\mathbb{C}^3$ parametrized by the traces of the image of $[\gamma_1]$, $[\gamma_2]$ and $[\gamma_3]$; and a representation $\widetilde{\rho}: \pi_1(P) \to \mathrm{SL}(2;\mathbb{C})$ is irreducible if and only if

$$f_P\big(\mathrm{Tr}\widetilde{\rho}([\gamma_1]), \mathrm{Tr}\widetilde{\rho}([\gamma_2]), \mathrm{Tr}\widetilde{\rho}([\gamma_3])\big) \neq 0,$$

where f_P is the polynomial

$$f_P(x,y,z) = x^2 + y^2 + z^2 - xyz - 4.$$

The *logarithmic holonomies* of $(\gamma_1, \gamma_2, \gamma_3)$ in a representation $\widetilde{\rho}: \pi_1(P) \to \mathrm{SL}(2;\mathbb{C})$ are up to sign the complex numbers (u_1, u_2, u_3) satisfying

$$\big(\mathrm{Tr}\widetilde{\rho}([\gamma_1]), \mathrm{Tr}\widetilde{\rho}([\gamma_2]), \mathrm{Tr}\widetilde{\rho}([\gamma_3])\big) = \Big(-2\cosh\frac{u_1}{2}, -2\cosh\frac{u_2}{2}, -2\cosh\frac{u_3}{2}\Big).$$

In this way, if $\rho_0 : \pi_1(P) \to \mathrm{PSL}(2;\mathbb{C})$ is the holonomy representation of the complete hyperbolic structure on P and $\widetilde{\rho}_0 : \pi_1(P) \to \mathrm{SL}(2;\mathbb{C})$ is the lifting of ρ_0 with

$$\big(\mathrm{Tr}\widetilde{\rho}_0([\gamma_1]), \mathrm{Tr}\widetilde{\rho}_0([\gamma_2]), \mathrm{Tr}\widetilde{\rho}_0([\gamma_3])\big) = (-2,-2,-2),$$

then the logarithmic holonomies of $(\gamma_1, \gamma_2, \gamma_3)$ in $\widetilde{\rho}_0$ are $(0,0,0)$. The *Gram matrix* of $\widetilde{\rho}$ is defined as

$$\mathbb{G} = \begin{bmatrix} 1 & -\cosh\frac{u_3}{2} & -\cosh\frac{u_2}{2} \\ -\cosh\frac{u_3}{2} & 1 & -\cosh\frac{u_1}{2} \\ -\cosh\frac{u_2}{2} & -\cosh\frac{u_1}{2} & 1 \end{bmatrix}.$$

Then

$$f_P\big(\mathrm{Tr}\widetilde{\rho}([\gamma_1]),\mathrm{Tr}\widetilde{\rho}([\gamma_2]),\mathrm{Tr}\widetilde{\rho}([\gamma_3])\big)=-4\det\mathbb{G},$$

and $\widetilde{\rho}$ is irreducible if and only if $\det\mathbb{G}\neq 0$.

Since $\pi_1(P)$ is a free group, every $\mathrm{PSL}(2;\mathbb{C})$-representation of it lifts to an $\mathrm{SL}(2;\mathbb{C})$-representation. Hence the $\mathrm{SL}(2;\mathbb{C})$-character variety of P is a branched cover of the $\mathrm{PSL}(2;\mathbb{C})$-character variety of P, and the latter is an irreducible algebraic variety. For a representation $\rho:\pi_1(P)\to\mathrm{PSL}(2;\mathbb{C})$, we defined the logarithmic holonomies (u_1,u_2,u_3) and the Gram matrix $\mathbb{G}$ of ρ as those of a lifting $\widetilde{\rho}:\pi_1(P)\to\mathrm{SL}(2;\mathbb{C})$ of ρ. Notice that the logarithmic holonomies depend on the choice of the liftings of ρ, and a different lifting will change $\mathbb{G}$ by multiplying some rows and the corresponding columns by -1 at the same time, which does not change its determinant. Therefore, the *determinant of the Gram matrix* $\det\mathbb{G}$ is independent of the choice of the liftings, and is a well defined quantity of ρ.

For a representation $\rho:\pi_1(P)\to\mathrm{PSL}(2;\mathbb{C})$, let $\mathrm{Ad}_\rho=\mathrm{Ad}\circ\rho:\pi_1(P)\to\mathrm{SL}(3;\mathbb{C})$ be its adjoint representation. Since both Ad and Sym^2 are 3-dimensional irreducible representations of $\mathrm{SL}(2;\mathbb{C})$, they are equivalent by the Classification Theorem of finite dimensional irreducible representations of $\mathrm{SL}(2;\mathbb{C})$. In the rest of this paper, we will use the representation $\mathrm{Sym}^2\circ\widetilde{\rho}$ to do all the computations where $\widetilde{\rho}$ is a lifting of ρ to a representation into $\mathrm{SL}(2;\mathbb{C})$; and to simplify the notation still denote it by Ad_ρ. Notice that composing with Sym^2, the signs $\pm$ in front of the matrices will disappear and hence $\mathrm{Sym}^2\circ\widetilde{\rho}$ is independent of the choice of the lifting $\widetilde{\rho}$.

In addition, assume for each $i\in\{1,2,3\}$ that $\rho([\gamma_i])\neq\pm I$. Then up to sign we can canonically choose an invariant vector $\mathbf{I}_i$ of $\mathrm{Ad}_\rho([\gamma_i])^T$ as follows. Since $\rho([\gamma_i])$ is not the identity element in $\mathrm{PSL}(2;\mathbb{C})$, there is up to scalar a unique invariant vector of $\mathrm{Ad}_\rho([\gamma_i])^T$. To determine the scalar we consider the following Killing bilinear form κ on the Lie algebra $\mathfrak{sl}(2;\mathbb{C})\cong\mathbb{C}^3$ defined by

$$\kappa\left(\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix},\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}\right)=-2a_1b_3+a_2b_2-2a_3b_1,\tag{3.1}$$

which is up to scalar the unique Ad-invariant bilinear form on $\mathfrak{sl}(2;\mathbb{C})$. Then in the case that $\rho([\gamma_i])$ is not a parabolic element, we let $\mathbf{I}_i$ be up to sign the unique $\mathrm{Ad}_\rho([\gamma_i])^T$-invariant vector satisfying $\kappa(\mathbf{I}_i,\mathbf{I}_i)=1$.

Definition 3.1. *Let $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\gamma_3)$. An irreducible representation $\rho:\pi_1(P)\to\mathrm{PSL}(2;\mathbb{C})$ is $\boldsymbol{\gamma}$-*regular *if*

$$\mathbf{h}_P=\{\mathbf{I}_1\otimes[\gamma_1],\mathbf{I}_2\otimes[\gamma_2],\mathbf{I}_3\otimes[\gamma_3]\}$$

is a basis of $\mathrm{H}_1(P;\mathrm{Ad}_\rho)$, where $[\gamma_i]$ is the homology class of γ_i in $\mathrm{H}_1(P;\mathbb{Z})$, $i\in\{1,2,3\}$.

Let $\mathrm{X}(P)$ be the $\mathrm{PSL}(2;\mathbb{C})$-character variety of P. A character $[\rho] \in \mathrm{X}(P)$ is *$\boldsymbol{\gamma}$-regular* if ρ is a $\boldsymbol{\gamma}$-regular representation. Since $\pi_1(P)$ is a free group, an Euler characteristic counting argument shows that if $[\rho]$ is $\boldsymbol{\gamma}$-regular, then $\mathrm{H}_k(P;\mathrm{Ad}_\rho) = 0$ for $k \neq 1$.

The main result of this section is the following

Proposition 3.2. *Let $\rho : \pi_1(P) \to \mathrm{PSL}(2;\mathbb{C})$ be a $\boldsymbol{\gamma}$-regular representation, and for $i \in \{1,2,3\}$ let u_i be up to sign the logarithmic holonomy of γ_i in ρ. Then*

$$\mathrm{Tor}(P,\mathbf{h}_P;\mathrm{Ad}_\rho) = \pm\frac{1}{16\sinh\frac{u_1}{2}\sinh\frac{u_2}{2}\sinh\frac{u_3}{2}}.$$

Remark 3.3. Proposition 3.2 could be proved from other results in the literature, which is related to volume forms on the character variety, see [24,3,13]. We include a different proof here for the readers' convenience and as a warm up for the computations in the next section.

To prove Proposition 3.2, we need the following lemma, where the explicit computation of (1) and (2) will be used later.

Lemma 3.4. *The set of $\boldsymbol{\gamma}$-regular characters contains a Zariski-open subset $\mathrm{Z}(P)$ of $\mathrm{X}(P)$ consisting of the characters $[\rho]$ satisfying the following two conditions:*
(1)

$$\det[\mathbf{I}_1,\mathbf{I}_2,\mathbf{I}_3] \neq 0,$$

and
(2)

$$\det\Big[\mathbf{I}_1 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T\cdot\mathbf{I}_1,\ \mathbf{I}_2 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T\cdot\mathbf{I}_2,\ \mathbf{I}_3 - \mathrm{Ad}_\rho([\gamma_2])^T\cdot\mathbf{I}_3\Big] \neq 0.$$

Proof. Let us compute the determinants in conditions (1) and (2) first. We will do the computations for the holonomy representation of a hyperbolic metric on P with cone singularities around γ_1, γ_2 and γ_3 first. Then by analyticity the computation extends to the other representations. We would like to mention here that these computations will also hold the key to the proof of Proposition 3.2.

Let P be a hyperbolic 2-sphere with three cone singularities p_1, p_2 and p_3 removed. We let the cone angles at p_1, p_2 and p_3 respectively be $2\alpha_1$, $2\alpha_2$ and $2\alpha_3$ all of which are less than 2π, and let γ_1, γ_2 and γ_3 respectively be the simple loops around p_1, p_2 and p_3. In this case we have $u_i = \pm 2\mathbf{i}\alpha_i$.

Now $\rho([\gamma_i])^T$ is a rotation, hence has an eigenvector $\mathbf{v}_i^+$ of eigenvalue $e^{\mathbf{i}\alpha_i}$ and an eigenvector $\mathbf{v}_i^-$ of eigenvalue $e^{-\mathbf{i}\alpha_i}$. If

$$\mathbf{v}_i^+ = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{v}_i^- = \begin{bmatrix} c \\ d \end{bmatrix},$$

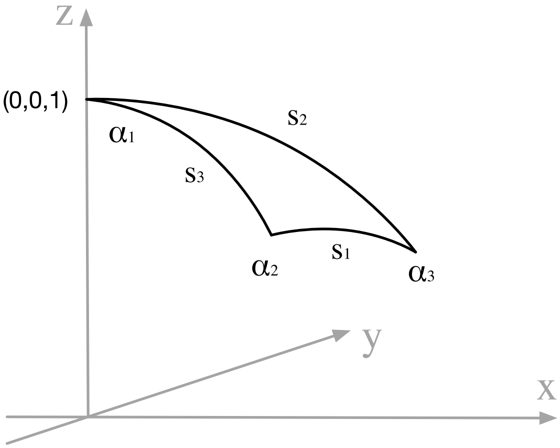

Fig. 4. T in $\mathbb{H}^3$.

then an invariant vector of $\mathrm{Ad}_\rho([\gamma_i])^T$ has the form

$$\begin{bmatrix} ac \\ ad+bc \\ bd \end{bmatrix}. \tag{3.2}$$

Indeed, if we identify $[a,b]^T$ with the polynomial $aX+bY$ and $[c,d]^T$ with $cX+dY$, then the polynomial $(aX+bY)(cX+dY) = acX^2 + (ad+bc)XY + bdY^2$ is invariant under $\left(\mathrm{Sym}^2 \circ \widetilde{\rho}([\gamma_i])\right)^T$.

Now P is the double of a hyperbolic triangle T with cone angles α_1, α_2 and α_3. For $i = 1,2,3$, let e_i be the edge of T opposite to p_i and let s_i be its lengths. To compute its holonomy representation ρ, we isometrically embedded T into $\mathbb{H}^3$ as follows. As in Fig. 4, we place p_1 at $(0,0,1)$, the edge e_2 in the xz-plane and T in the unit hemisphere centered at $(0,0,0)$ such that the y-coordinate of p_2 is negative. This could always be done by replacing T by its mirror image if necessary.

To simplify the notation, we for any $z \in \mathbb{C}$ let

$$D_z = \begin{bmatrix} e^{\frac{z}{2}} & 0 \\ 0 & e^{-\frac{z}{2}} \end{bmatrix}$$

and for $i = 1,2,3$, let

$$S_i = \begin{bmatrix} \cosh\frac{s_i}{2} & \sinh\frac{s_i}{2} \\ \sinh\frac{s_i}{2} & \cosh\frac{s_i}{2} \end{bmatrix}.$$

Suppose for each i, γ_i goes counterclockwise around p_i. Then by conjugating the tangent framings at p_2 and p_3 back to $p_1 = (0,0,1)$ and conjugating the tangent vectors of the axes of the rotations to $\frac{\partial}{\partial z}$, we have

$$\rho([\gamma_1]) = \pm D_{2\mathbf{i}\alpha_1},$$
$$\rho([\gamma_2]) = \pm D^{-1}_{\mathbf{i}\alpha_1} S_3 D_{2\mathbf{i}\alpha_2} S_3^{-1} D_{i\alpha_1} = \pm S_2 D^{-1}_{-\mathbf{i}\alpha_3} S_1^{-1} D_{2\mathbf{i}\alpha_2} S_1 D_{-\mathbf{i}\alpha_3} S_2^{-1},$$
$$\rho([\gamma_3]) = \pm S_2 D_{2\mathbf{i}\alpha_3} S_2^{-1} = \pm D^{-1}_{\mathbf{i}\alpha_1} S_3 D^{-1}_{\mathbf{i}\alpha_2} S_1^{-1} D_{2\mathbf{i}\alpha_3} S_1 D_{\mathbf{i}\alpha_2} S_3^{-1} D_{\mathbf{i}\alpha_1}.$$

Here we compute $\rho([\gamma_2])$ and $\rho([\gamma_3])$ in two ways for the purpose of computing different things later. Since both D_z and S_i are symmetric matrices, we have

$$\begin{aligned}
\rho([\gamma_1])^T &= \pm D_{2\mathbf{i}\alpha_1},\\
\rho([\gamma_2])^T &= \pm D_{\mathbf{i}\alpha_1} S_3^{-1} D_{2\mathbf{i}\alpha_2} S_3 D^{-1}_{\mathbf{i}\alpha_1} = \pm S_2^{-1} D_{-\mathbf{i}\alpha_3} S_1 D_{2\mathbf{i}\alpha_2} S_1^{-1} D^{-1}_{-\mathbf{i}\alpha_3} S_2,\\
\rho([\gamma_3])^T &= \pm S_2^{-1} D_{2\mathbf{i}\alpha_3} S_2 = \pm D_{\mathbf{i}\alpha_1} S_3^{-1} D_{\mathbf{i}\alpha_2} S_1 D_{2\mathbf{i}\alpha_3} S_1^{-1} D^{-1}_{\mathbf{i}\alpha_2} S_3 D^{-1}_{\mathbf{i}\alpha_1}.
\end{aligned} \tag{3.3}$$

Then

$$\begin{aligned}
[\mathbf{v}_1^+, \mathbf{v}_1^-] &= I,\\
[\mathbf{v}_2^+, \mathbf{v}_2^-] &= D_{\mathbf{i}\alpha_1} S_3^{-1} = S_2^{-1} D_{-\mathbf{i}\alpha_3} S_1,\\
[\mathbf{v}_3^+, \mathbf{v}_3^-] &= S_2^{-1} = D_{\mathbf{i}\alpha_1} S_3^{-1} D_{\mathbf{i}\alpha_2} S_1.
\end{aligned} \tag{3.4}$$

Using the first half of the second and third equations of (3.4), (3.2) and a direct computation, we have

$$\mathbf{I}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{I}_2 = \begin{bmatrix} -\frac{1}{2} e^{\mathbf{i}\alpha_1} \sinh s_3 \\ \cosh s_3 \\ -\frac{1}{2} e^{-\mathbf{i}\alpha_1} \sinh s_3 \end{bmatrix} \quad \text{and} \quad \mathbf{I}_3 = \begin{bmatrix} -\frac{1}{2}\sinh s_2 \\ \cosh s_2 \\ -\frac{1}{2}\sinh s_2 \end{bmatrix}. \tag{3.5}$$

Since $\kappa(\mathbf{I}_i, \mathbf{I}_i) = 1$ for $i \in \{1, 2, 3\}$, they are the canonical invariant vectors. Therefore

$$\det[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3] = -\frac{\mathbf{i}}{2} \sin\alpha_1 \sinh s_2 \sinh s_3. \tag{3.6}$$

This computes the determinant in (1) for the holonomy representation of a hyperbolic structure with cone singularities.

To compute the determinant in (2), we need the following auxiliary computations. For real numbers x and y, we let

$$X = \begin{bmatrix} \cosh\frac{x}{2} & \sinh\frac{x}{2} \\ \sinh\frac{x}{2} & \cosh\frac{x}{2} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} \cosh\frac{y}{2} & \sinh\frac{y}{2} \\ \sinh\frac{y}{2} & \cosh\frac{y}{2} \end{bmatrix},$$

and for a complex number z let D_z be as before. We let

$$\mathbf{I}^z_{xy} = \begin{bmatrix} ac \\ ad + bc \\ bd \end{bmatrix} \quad \text{if} \quad X^{-1} D_z Y = \pm \begin{bmatrix} a & c \\ b & d \end{bmatrix};$$

and let

$$\mathbf{I}^z_{wxy} = \begin{bmatrix} ac \\ ad+bc \\ bd \end{bmatrix} \quad \text{if} \quad D_w X^{-1} D_z Y = \pm \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

We notice that $\mathbf{I}^z_{xy}$ and $\mathbf{I}^z_{wxy}$ are independent of the signs $\pm$ in front of the 2×2 matrices. Then using the hyperbolic trigonometric identities $\cosh z - \cosh z' = 2\sinh\frac{z+z'}{2}\sinh\frac{z-z'}{2}$ and $\sinh z - \sinh z' = 2\cosh\frac{z+z'}{2}\sinh\frac{z-z'}{2}$ for any complex numbers z and z' and a direct computation, we have

$$\mathbf{I}^z_{xy} - \mathbf{I}^{z'}_{xy} = \sinh y \sinh\frac{z-z'}{2} \begin{bmatrix} \sinh\frac{z+z'}{2}\cosh x + \cosh\frac{z+z'}{2} \\ -2\sinh\frac{z+z'}{2}\sinh x \\ \sinh\frac{z+z'}{2}\cosh x - \cosh\frac{z+z'}{2} \end{bmatrix}, \tag{3.7}$$

and

$$\mathbf{I}^z_{wxy} - \mathbf{I}^{z'}_{wxy} = \sinh y \sinh\frac{z-z'}{2} \begin{bmatrix} e^w\big(\sinh\frac{z+z'}{2}\cosh x + \cosh\frac{z+z'}{2}\big) \\ -2\sinh\frac{z+z'}{2}\sinh x \\ e^{-w}\big(\sinh\frac{z+z'}{2}\cosh x - \cosh\frac{z+z'}{2}\big) \end{bmatrix}. \tag{3.8}$$

By the first half of the third equation of (3.3), we have

$$\rho([\gamma_3]^{-1})^T = \pm S_2^{-1} D_{-2\mathbf{i}\alpha_3} S_2. \tag{3.9}$$

By (3.9) and the first equation of (3.4), we have

$$[\mathbf{v}_1^+, \mathbf{v}_1^-] = I = S_2^{-1} D_0 S_2$$

and

$$\rho([\gamma_3]^{-1})^T \cdot [\mathbf{v}_1^+, \mathbf{v}_1^-] = \pm S_2^{-1} D_{-2\mathbf{i}\alpha_3} S_2.$$

Therefore, by (3.7)

$$\begin{aligned} \mathbf{I}_1 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_1 &= \mathbf{I}^0_{s_2 s_2} - \mathbf{I}^{-2\mathbf{i}\alpha_3}_{s_2 s_2} \\ &= \mathbf{i}\sinh s_2 \sin\alpha_3 \begin{bmatrix} -\mathbf{i}\sin\alpha_3\cosh s_2 + \cos\alpha_3 \\ 2\mathbf{i}\sin\alpha_3\sinh s_2 \\ -\mathbf{i}\sin\alpha_3\cosh s_2 - \cos\alpha_3 \end{bmatrix}. \end{aligned}$$

By (3.9) and the second half of the second equation of (3.4), we have

$$[\mathbf{v}_2^+, \mathbf{v}_2^-] = S_2^{-1} D_{-\mathbf{i}\alpha_3} S_1$$

and

$$\rho([\gamma_3]^{-1})^T \cdot [\mathbf{v}_2^+, \mathbf{v}_2^-] = \pm S_2^{-1} D_{-3\mathbf{i}\alpha_3} S_1.$$

Therefore, by (3.7)

$$\begin{aligned}\mathbf{I}_2 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_2 &= \mathbf{I}_{s_2 s_1}^{-\mathbf{i}\alpha_3} - \mathbf{I}_{s_2 s_1}^{-3\mathbf{i}\alpha_3} \\ &= \mathbf{i} \sinh s_1 \sin \alpha_3 \begin{bmatrix} -\mathbf{i}\sin(2\alpha_3)\cosh s_2 + \cos(2\alpha_3) \\ 2\mathbf{i}\sin(2\alpha_3)\sinh s_2 \\ -\mathbf{i}\sin(2\alpha_3)\cosh s_2 - \cos(2\alpha_3) \end{bmatrix}.\end{aligned}$$

By the first half of the second equation of (3.3) and the second half of the third equation of (3.4), we have

$$[\mathbf{v}_3^+, \mathbf{v}_3^-] = D_{\mathbf{i}\alpha_1} S_3^{-1} D_{\mathbf{i}\alpha_2} S_1$$

and

$$\rho([\gamma_2])^T \cdot [\mathbf{v}_3^+, \mathbf{v}_3^-] = \pm D_{\mathbf{i}\alpha_1} S_3^{-1} D_{3\mathbf{i}\alpha_2} S_1.$$

Therefore, by (3.8)

$$\begin{aligned}\mathbf{I}_3 - \mathrm{Ad}_\rho([\gamma_2])^T \cdot \mathbf{I}_3 &= \mathbf{I}_{(\mathbf{i}\alpha_1)s_3 s_1}^{\mathbf{i}\alpha_2} - \mathbf{I}_{(\mathbf{i}\alpha_1)s_3 s_1}^{3\mathbf{i}\alpha_2} \\ &= -\mathbf{i} \sinh s_1 \sin \alpha_2 \begin{bmatrix} e^{\mathbf{i}\alpha_1}\big(\mathbf{i}\sin(2\alpha_2)\cosh s_3 + \cos(2\alpha_2)\big) \\ -2\mathbf{i}\sin(2\alpha_2)\sinh s_3 \\ e^{-\mathbf{i}\alpha_1}\big(\mathbf{i}\sin(2\alpha_2)\cosh s_3 - \cos(2\alpha_2)\big) \end{bmatrix}.\end{aligned}$$

We observe that the matrix

$$\begin{aligned}&\begin{bmatrix} -\mathbf{i}\sin\alpha_3\cosh s_2 + \cos\alpha_3 & -\mathbf{i}\sin(2\alpha_3)\cosh s_2 + \cos(2\alpha_3) & e^{\mathbf{i}\alpha_1}\big(\mathbf{i}\sin(2\alpha_2)\cosh s_3 + \cos(2\alpha_2)\big) \\ 2\mathbf{i}\sin\alpha_3\sinh s_2 & 2\mathbf{i}\sin(2\alpha_3)\sinh s_2 & -2\mathbf{i}\sin(2\alpha_2)\sinh s_3 \\ -\mathbf{i}\sin\alpha_3\cosh s_2 - \cos\alpha_3 & -\mathbf{i}\sin(2\alpha_3)\cosh s_2 - \cos(2\alpha_3) & e^{-\mathbf{i}\alpha_1}\big(\mathbf{i}\sin(2\alpha_2)\cosh s_3 - \cos(2\alpha_2)\big) \end{bmatrix} \\ = &\begin{bmatrix} -\mathbf{i} & 0 & 1 \\ 0 & 2\mathbf{i} & 0 \\ -\mathbf{i} & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \sin\alpha_3\cosh s_2 & \sin(2\alpha_3)\cosh s_2 & -\cos\alpha_1\sin(2\alpha_2)\cosh s_3 - \sin\alpha_1\cos(2\alpha_2) \\ \sin\alpha_3\sinh s_2 & \sin(2\alpha_3)\sinh s_2 & -\sin(2\alpha_2)\sinh s_3 \\ \cos\alpha_3 & \cos(2\alpha_3) & -\sin\alpha_1\sin(2\alpha_2)\cosh s_3 + \cos\alpha_1\cos(2\alpha_2) \end{bmatrix}.\end{aligned}$$

Denoting the second matrix above by M, we have

$$\begin{aligned}&\det\Big[\mathbf{I}_1 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_1,\ \mathbf{I}_2 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_2,\ \mathbf{I}_3 - \mathrm{Ad}_\rho([\gamma_2])^T \cdot \mathbf{I}_3\Big] \\ &= -4\mathbf{i}\sinh^2 s_1 \sinh s_2 \sin\alpha_2 \sin^2\alpha_3 \det M.\end{aligned}$$

Computing the cofactors of M, we have $M_{13} = -\sin\alpha_3\sinh s_2$, $M_{23} = \sin\alpha_3\cosh s_2$ and $M_{33} = 0$. Then

$$\begin{aligned}
&\det M\\
=&-\sin\alpha_3\sinh s_2\big(-\cos\alpha_1\sin(2\alpha_2)\cosh s_3-\sin\alpha_1\cos(2\alpha_2)\big)\\
&-\sin\alpha_3\cosh s_2\sin(2\alpha_2)\sinh s_3\\
=&-\sinh s_2\sin\alpha_1\sin\alpha_3,
\end{aligned}$$

where the last equality comes from the use of the hyperbolic Law of Sine that $\sinh s_3 = \frac{\sinh s_2 \sin\alpha_3}{\sin\alpha_2}$ to get a common factor $\sinh s_2$, then the use of the hyperbolic Law of Cosine that $\cosh s_2 = \frac{\cos\alpha_2+\cos\alpha_1\cos\alpha_3}{\sin\alpha_1\sin\alpha_3}$ and $\cosh s_3 = \frac{\cos\alpha_3+\cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}$ to change the quantity into a function of the angles α_1, α_2 and α_3 only, finally the use of the double angle formulas to $\sin(2\alpha_2)$, $\cos(2\alpha_2)$ and $\sin(2\alpha_3)$ to get a function of $\{\sin\alpha_k\}$ and $\{\cos\alpha_k\}$ only then followed by a simplification.

Therefore,

$$\begin{aligned}
&\det\Big[\mathbf{I}_1-\mathrm{Ad}_\rho([\gamma_3]^{-1})^T\cdot\mathbf{I}_1,\ \mathbf{I}_2-\mathrm{Ad}_\rho([\gamma_3]^{-1})^T\cdot\mathbf{I}_2,\ \mathbf{I}_3-\mathrm{Ad}_\rho([\gamma_2])^T\cdot\mathbf{I}_3\Big]\\
=&4\mathbf{i}\sin\alpha_1\sin\alpha_2\sin^3\alpha_3\sinh^2 s_1\sinh^2 s_2.
\end{aligned}\tag{3.10}$$

This computes the determinant in (2) for the holonomy representation of a hyperbolic structure.

For the other characters in $\mathrm{X}(P)$, we observe that for the holonomy representation ρ of a hyperbolic structure with cone angles $(2\alpha_1, 2\alpha_2, 2\alpha_3)$, for any lifting $\widetilde{\rho}:\pi_1(P)\to \mathrm{SL}(2;\mathbb{C})$ of ρ, we have

$$\mathrm{Tr}\widetilde{\rho}([\gamma_i]) = \pm 2\cos\alpha_i$$

for $i\in\{1,2,3\}$. Then by the trigonometry identity and the hyperbolic Law of Cosine, we have

$$\sinh s_i = \pm\sqrt{\frac{-\det \mathrm{G}_{\boldsymbol{\alpha}}}{(1-\cos^2\alpha_j)(1-\cos^2\alpha_k)}}\tag{3.11}$$

for $\{i,j,k\}=\{1,2,3\}$, where

$$\mathrm{G}_{\boldsymbol{\alpha}} = \begin{bmatrix} 1 & -\cos\alpha_3 & -\cos\alpha_2\\ -\cos\alpha_3 & 1 & -\cos\alpha_1\\ -\cos\alpha_2 & -\cos\alpha_1 & 1\end{bmatrix}$$

is the Gram matrix in the angles of the hyperbolic triangle with angles $(\alpha_1,\alpha_2,\alpha_3)$. Therefore, the square of $\sinh s_i$ is a rational functions in $(\mathrm{Tr}\widetilde{\rho}([\gamma_1]), \mathrm{Tr}\widetilde{\rho}([\gamma_2]), \mathrm{Tr}\widetilde{\rho}([\gamma_3]))$. Since $\mathrm{X}(P)$ is an irreducible algebraic variety, by the analyticity of the functions on the right hand sides, (3.6) and (3.10) hold for the other characters $[\rho]$ in $\mathrm{X}(P)$.

Since the square of the determinants in conditions (1) and (2) is rational functions in the coordinates $(\mathrm{Tr}\widetilde{\rho}([\gamma_1]), \mathrm{Tr}\widetilde{\rho}([\gamma_2]), \mathrm{Tr}\widetilde{\rho}([\gamma_3]))$, the lifting of those characters form a

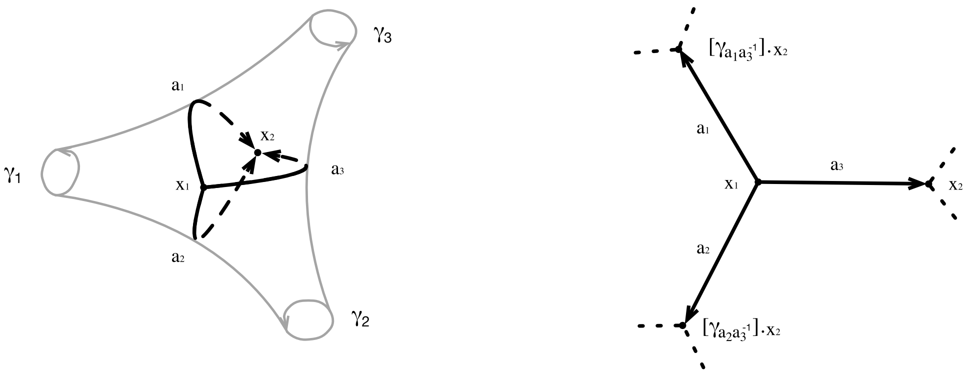

Fig. 5. The 1-dimensional CW complexes.

Zariski-open subset of $\mathbb{C}^3$, and hence those characters themselves form a Zariski-open subset of $\mathrm{X}(P)$.

Next we show that the representations satisfying (1) and (2) are $\boldsymbol{\gamma}$-regular. We will compute the homologies of P using its spine Γ, which is the 1-dimensional CW complex on the left of Fig. 5 consisting of two 0-cells x_1 and x_1 and three 1-cells a_1, a_2 and a_3 all of which are oriented from x_1 to x_2.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{C}^3$ and let the choice of representatives x_1, x_2, a_1, a_2 and a_3 in the universal covering of Γ be as drawn on the right of Fig. 5. Then $\mathrm{C}_0(P; \mathrm{Ad}_\rho) \cong \mathbb{C}^6$ with a natural basis $\{\mathbf{e}_i \otimes x_k\}$ for $i \in \{1,2,3\}$ and $k \in \{1,2\}$; $\mathrm{C}_1(P; \mathrm{Ad}_\rho) \cong \mathbb{C}^9$ with a natural basis $\{\mathbf{e}_i \otimes a_k\}$ for $i, k \in \{1,2,3\}$; and $\mathrm{C}_k(P; \mathrm{Ad}_\rho) = 0$ for $k \neq 0$ or 1.

We choose x_1 to be the base point of the fundamental group; and for $\{j,k\} \subset \{1,2,3\}$, let $\gamma_{a_j a_k^{-1}}$ be the curve starting from x_1 traveling along a_j to x_2 then along $-a_k$ back to x_1. In this way, we have $[\gamma_{a_1a_2^{-1}}] = [\gamma_1]$, $[\gamma_{a_2a_3^{-1}}] = [\gamma_2]$ and $[\gamma_{a_1a_3^{-1}}] = [\gamma_3]^{-1}$.

By condition (1), the vectors $\mathbf{I}_1 \otimes (a_1 - a_2)$, $\mathbf{I}_2 \otimes (a_2 - a_3)$ and $\mathbf{I}_3 \otimes (a_1 - a_3)$ are linearly independent in $\mathrm{C}_1(P; \mathrm{Ad}_\rho)$. Next we show that they lie in the kernel of $\partial : \mathrm{C}_1(P; \mathrm{Ad}_\rho) \to \mathrm{C}_0(P; \mathrm{Ad}_\rho)$. Indeed, for the image of $\mathbf{I}_1 \otimes (a_1 - a_2)$, we have

$$\begin{aligned}
\partial(\mathbf{I}_1 \otimes (a_1 - a_2)) &= \mathbf{I}_1 \otimes \partial(a_1 - a_2) \\
&= \mathbf{I}_1 \otimes \Big((x_1 - [\gamma_{a_1a_3^{-1}}] \cdot x_2) - (x_1 - [\gamma_{a_2a_3^{-1}}] \cdot x_2) \Big) \\
&= \mathbf{I}_1 \otimes \Big([\gamma_2] \cdot x_2 - [\gamma_3]^{-1} \cdot x_2 \Big) \\
&= \Big(\mathrm{Ad}_\rho([\gamma_2])^T \cdot \mathbf{I}_1 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_1 \Big) \otimes x_2 \\
&= \Big(\mathrm{Ad}_\rho([\gamma_2])^T \mathrm{Ad}_\rho([\gamma_1])^T \cdot \mathbf{I}_1 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_1 \Big) \otimes x_2 = 0,
\end{aligned}$$

where the penultimate equality comes from $\mathrm{Ad}_\rho([\gamma_1])^T \cdot \mathbf{I}_1 = \mathbf{I}_1$ and the last equation comes from $\gamma_1 \cdot \gamma_2 = \gamma_3^{-1}$. For the image of the other two vectors, we have

$$\begin{aligned}\partial(\mathbf{I}_2\otimes(a_2-a_3)) &= \mathbf{I}_2\otimes\partial(a_2-a_3)\\ &= \mathbf{I}_2\otimes\Big((x_1-[\gamma_{a_2a_3^{-1}}]\cdot x_2)-(x_1-x_2)\Big)\\ &= \mathbf{I}_2\otimes\Big(x_2-[\gamma_2]\cdot x_2\Big)\\ &= \Big(\mathbf{I}_2-\mathrm{Ad}_\rho([\gamma_2])^T\cdot\mathbf{I}_2\Big)\otimes x_2=0,\end{aligned}$$

and

$$\begin{aligned}\partial(\mathbf{I}_3\otimes(a_1-a_3)) &= \mathbf{I}_3\otimes\partial(a_1-a_3)\\ &= \mathbf{I}_3\otimes\Big((x_1-[\gamma_{a_1a_3^{-1}}]\cdot x_2)-(x_1-x_2)\Big)\\ &= \mathbf{I}_3\otimes\Big(x_2-[\gamma_3]^{-1}\cdot x_2\Big)\\ &= \Big(\mathbf{I}_3-\mathrm{Ad}_\rho([\gamma_3]^{-1})^T\cdot\mathbf{I}_3\Big)\otimes x_2=0,\end{aligned}$$

where the last equalities respectively come from $\mathrm{Ad}_\rho([\gamma_2])^T\cdot\mathbf{I}_2=\mathbf{I}_2$ and $\mathrm{Ad}_\rho([\gamma_3]^{-1})^T\cdot\mathbf{I}_3=\mathbf{I}_3$. Therefore, $\mathbf{I}_1\otimes(a_1-a_2)$, $\mathbf{I}_2\otimes(a_2-a_3)$ and $\mathbf{I}_3\otimes(a_1-a_3)$ represent three linearly independent elements $\mathbf{I}_1\otimes[\gamma_1]$, $\mathbf{I}_2\otimes[\gamma_2]$ and $\mathbf{I}_3\otimes[\gamma_3]$ in $\mathrm{H}_1(P;\mathrm{Ad}_\rho)$. Later we will prove that they also span, and hence form a basis of $\mathrm{H}_1(P;\mathrm{Ad}_\rho)$.

Now we claim that $\{\mathbf{I}_1\otimes(a_1-a_2),\mathbf{I}_2\otimes(a_2-a_3),\mathbf{I}_3\otimes(a_1-a_3)\}$ joint with six vectors $\{\mathbf{I}_1\otimes a_3,\mathbf{I}_2\otimes a_3,\mathbf{I}_3\otimes a_3,\mathbf{I}_1\otimes a_1,\mathbf{I}_2\otimes a_1,\mathbf{I}_3\otimes a_2\}$ form a basis of $\mathrm{C}_1(P;\mathrm{Ad}_\rho)$. Indeed, in the natural basis $\{\mathbf{e}_i\otimes a_k\}$, $i,k\in\{1,2,3\}$, the 9×9 matrix consisting of these vectors as the columns is obtained from the one consisting of $\{\mathbf{I}_j\otimes a_k\}$, $j,k\in\{1,2,3\}$, as the columns by a sequence of elementary column operations of type I, III, and II with a factor -1. The latter matrix is a block matrix with three 3×3 blocks $[\mathbf{I}_1,\mathbf{I}_2,\mathbf{I}_3]$ on the diagonal and $0's$ elsewhere, hence by condition (1) is non-singular and has determinant $\det[\mathbf{I}_1,\mathbf{I}_2,\mathbf{I}_3]^3$. As a consequence, the former matrix is also non-singular and up to sign has determinant $\det[\mathbf{I}_1,\mathbf{I}_2,\mathbf{I}_3]^3$.

In the next step, we will study the image of the six vectors $\{\mathbf{I}_1\otimes a_3,\mathbf{I}_2\otimes a_3,\mathbf{I}_3\otimes a_3,\mathbf{I}_1\otimes a_1,\mathbf{I}_2\otimes a_1,\mathbf{I}_3\otimes a_2\}$ under the boundary map ∂, and show that they span $\mathrm{C}_0(P;\mathrm{Ad}_\rho)$. We have for $j=1,2,3$,

$$\partial(\mathbf{I}_j\otimes a_3)=\mathbf{I}_j\otimes\partial a_3=\mathbf{I}_j\otimes(x_1-x_2)=\mathbf{I}_j\otimes x_1-\mathbf{I}_j\otimes x_2;$$

for $k=1,2$,

$$\partial(\mathbf{I}_k\otimes a_1)=\mathbf{I}_k\otimes\partial a_1=\mathbf{I}_k\otimes(x_1-[\gamma_{a_1a_3^{-1}}]\cdot x_2)=\mathbf{I}_k\otimes x_1-\Big(\mathrm{Ad}_\rho([\gamma_3]^{-1})^T\cdot\mathbf{I}_k\Big)\otimes x_2;$$

and

$$\partial(\mathbf{I}_3\otimes a_2)=\mathbf{I}_3\otimes\partial a_2=\mathbf{I}_3\otimes(x_1-[\gamma_{a_2a_3^{-1}}]\cdot x_2)=\mathbf{I}_3\otimes x_1-\Big(\mathrm{Ad}_\rho([\gamma_2])^T\cdot\mathbf{I}_3\Big)\otimes x_2.$$

Therefore, in the natural basis $\{\mathbf{e}_i \otimes x_k\}$, $i \in \{1,2,3\}$, $k \in \{1,2\}$, the 6×6 matrix consisting of $\{\partial(\mathbf{I}_1 \otimes a_3), \partial(\mathbf{I}_2 \otimes a_3), \partial(\mathbf{I}_3 \otimes a_3), \partial(\mathbf{I}_1 \otimes a_1), \partial(\mathbf{I}_2 \otimes a_1), \partial(\mathbf{I}_3 \otimes a_2)\}$ as the columns has four 3×3 blocks, where on the top it has two copies of $[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3]$, on the bottom left is has $[-\mathbf{I}_1, -\mathbf{I}_2, -\mathbf{I}_3]$ and on the bottom right

$$\Big[-\mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_1,\ -\mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_2,\ -\mathrm{Ad}_\rho([\gamma_2])^T \cdot \mathbf{I}_3\Big].$$

This matrix is row equivalent to (by adding the top blocks to the bottom) the one with two copies of $[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3]$ on the top, $0's$ on the bottom left and

$$\Big[\mathbf{I}_1 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_1,\ \mathbf{I}_2 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_2,\ \mathbf{I}_3 - \mathrm{Ad}_\rho([\gamma_2])^T \cdot \mathbf{I}_3\Big]$$

on the bottom right. The determinant of both of the 6×6 matrices is

$$\det[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3] \cdot \det\Big[\mathbf{I}_1 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_1,\ \mathbf{I}_2 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_2,\ \mathbf{I}_3 - \mathrm{Ad}_\rho([\gamma_2])^T \cdot \mathbf{I}_3\Big].$$

By conditions (1) and (2), the product above is nonzero and hence $\{\partial(\mathbf{I}_1 \otimes a_3), \partial(\mathbf{I}_2 \otimes a_3), \partial(\mathbf{I}_3 \otimes a_3), \partial(\mathbf{I}_1 \otimes a_1), \partial(\mathbf{I}_2 \otimes a_1), \partial(\mathbf{I}_3 \otimes a_2)\}$ span $\mathrm{C}_0(P; \mathrm{Ad}_\rho)$. This implies that $\mathrm{H}_0(P; \mathrm{Ad}_\rho) = 0$. Since there are no cells of dimension higher than or equal to 2, $\mathrm{H}_k(P; \mathrm{Ad}_\rho) = 0$ for $k \geqslant 2$.

Finally, since $\{\partial(\mathbf{I}_1 \otimes a_3), \partial(\mathbf{I}_2 \otimes a_3), \partial(\mathbf{I}_3 \otimes a_3), \partial(\mathbf{I}_1 \otimes a_1), \partial(\mathbf{I}_2 \otimes a_1), \partial(\mathbf{I}_3 \otimes a_2)\}$ span $\mathrm{C}_0(P; \mathrm{Ad}_\rho) \cong \mathbb{C}^6$, by dimension counting the kernel of $\partial : \mathrm{C}_1(P; \mathrm{Ad}_\rho) \to \mathrm{C}_0(P; \mathrm{Ad}_\rho)$ has dimension at most 3. Hence $\mathbf{I}_1 \otimes (a_1 - a_2)$, $\mathbf{I}_2 \otimes (a_2 - a_3)$ and $\mathbf{I}_3 \otimes (a_1 - a_3)$ span the kernel of ∂. This shows that the elements they represent $\mathbf{h}_P = \{\mathbf{I}_1 \otimes [\gamma_1], \mathbf{I}_2 \otimes [\gamma_2], \mathbf{I}_3 \otimes [\gamma_3]\}$ form a basis of $\mathrm{H}_1(P; \mathrm{Ad}_\rho)$, and $\mathrm{H}_1(P; \mathrm{Ad}_\rho) \cong \mathbb{C}^3$. This completes the proof. □

Proof of Proposition 3.2. Since the adjoint twisted Reidemeister torsion is invariant under subdivisions, elementary expansions and elementary collapses of CW-complexes by [19,23], we can do the computation using the spine Γ of P as on the left of Fig. 5.

The adjoint twisted Reidemeister torsion equals, up to sign, the determinant of the 9×9 matrix consisting of $\{\mathbf{I}_1 \otimes (a_1 - a_2), \mathbf{I}_2 \otimes (a_2 - a_3), \mathbf{I}_3 \otimes (a_1 - a_3), \mathbf{I}_1 \otimes a_3, \mathbf{I}_2 \otimes a_3, \mathbf{I}_3 \otimes a_3, \mathbf{I}_1 \otimes a_1, \mathbf{I}_2 \otimes a_1, \mathbf{I}_3 \otimes a_2\}$ as the columns divided by the determinant of the 6×6 matrix consisting of $\{\partial(\mathbf{I}_1 \otimes a_3), \partial(\mathbf{I}_2 \otimes a_3), \partial(\mathbf{I}_3 \otimes a_3), \partial(\mathbf{I}_1 \otimes a_1), \partial(\mathbf{I}_2 \otimes a_1), \partial(\mathbf{I}_3 \otimes a_2)\}$ as the columns.

By (3.6) and (3.10), for the holonomy representation of a hyperbolic structure we have

$$\begin{aligned}
&\mathrm{Tor}(P, \mathbf{h}_P; \mathrm{Ad}_\rho) \\
&= \pm \frac{\det[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3] \cdot \det[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3] \cdot \det[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3]}{\det[\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3] \cdot \det\Big[\mathbf{I}_1 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_1,\ \mathbf{I}_2 - \mathrm{Ad}_\rho([\gamma_3]^{-1})^T \cdot \mathbf{I}_2,\ \mathbf{I}_3 - \mathrm{Ad}_\rho([\gamma_2])^T \cdot \mathbf{I}_3\Big]} \\
&= \pm \frac{\mathbf{i}}{16 \sin\alpha_1 \sin\alpha_2 \sin\alpha_3} \\
&= \pm \frac{1}{16 \sinh\frac{u_1}{2} \sinh\frac{u_2}{2} \sinh\frac{u_3}{2}},
\end{aligned}$$

where the second equality comes from the hyperbolic Law of Sine that $\frac{\sinh s_3}{\sinh s_1} = \frac{\sin \alpha_3}{\sin \alpha_1}$.

Finally, by Lemma 3.4 and the analyticity, the result holds for all $\boldsymbol{\gamma}$-regular characters in $\mathrm{X}(P)$. $\square$

4. Adjoint twisted Reidemeister torsion of the D-blocks

Let Δ be a truncated tetrahedron with triangles of truncation T_1, T_2, T_3, T_4 and hexagonal faces H_1, H_2, H_3, H_4 such that T_k is opposite to H_k. Recall that an *edge* is the intersection of two hexagonal faces; and we call the intersection of a triangle of truncation and a hexagonal face a *short edge*. Let D be the union of Δ with its mirror image via the identity map between the four hexagonal faces $H_1, \dots, H_4$ and with the six edges removed. This is a *D-block* as defined in [5] and recalled in Section 2.4. For $\{j,k\} \subset \{1,2,3,4\}$ we let e_{jk} be the edge adjacent to H_j and H_k. For $\{j,k\} \subset \{1,2,3,4\}$, let γ_{jk} be a simple loop around e_{jk}.

The fundamental group $\pi_1(D)$ is a free group of rank 3 generated by $[\gamma_{12}]$, $[\gamma_{13}]$ and $[\gamma_{14}]$. By [12], the $\mathrm{SL}(2;\mathbb{C})$-character variety of D is homeomorphic to a hypersurface in $\mathbb{C}^7$ parametrized by the traces of the image of $[\gamma_{12}]$, $[\gamma_{13}]$, $[\gamma_{14}]$, $[\gamma_{12}\cdot\gamma_{13}]$, $[\gamma_{12}\cdot\gamma_{14}]$, $[\gamma_{13}\cdot\gamma_{14}]$ and $[\gamma_{12}\cdot\gamma_{13}\cdot\gamma_{14}]$, which is a double branched cover of $\mathbb{C}^6$ parametrized by the first six components. A representation $\widetilde{\rho}:\pi_1(D)\to\mathrm{SL}(2;\mathbb{C})$ is not in the branch locus if and only if

$$f_D\big(\mathrm{Tr}\widetilde{\rho}([\gamma_{12}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{13}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{14}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{12}\cdot\gamma_{13}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{12}\cdot\gamma_{14}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{13}\cdot\gamma_{14}])\big)\neq 0,$$

where f_D is the polynomial

$$\begin{aligned} f_D(t_1,t_2,t_3,t_{12},t_{13},t_{23}) &= \big(t_{12}t_3+t_{13}t_2+t_{23}t_1-t_1t_2t_3\big)^2 \\ &\quad -4\big(t_1^2+t_2^2+t_3^2+t_{12}^2+t_{13}^2+t_{23}^2-t_1t_2t_{12}-t_1t_3t_{13}-t_2t_3t_{23}+t_{12}t_{13}t_{23}-4\big). \end{aligned}$$

The *logarithmic holonomies* of $(\gamma_{12},\gamma_{13},\gamma_{14},\gamma_{23},\gamma_{24},\gamma_{34})$ in $\widetilde{\rho}$ are up to sign the complex numbers $(u_{12},u_{13},u_{14},u_{23},u_{24},u_{34})$ satisfying

$$\begin{aligned} &\big(\mathrm{Tr}\widetilde{\rho}([\gamma_{12}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{13}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{14}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{23}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{24}]),\mathrm{Tr}\widetilde{\rho}([\gamma_{34}])\big) \\ &\quad= \Big(-2\cosh\frac{u_{12}}{2},-2\cosh\frac{u_{13}}{2},-2\cosh\frac{u_{14}}{2},-2\cosh\frac{u_{23}}{2},-2\cosh\frac{u_{24}}{2},-2\cosh\frac{u_{34}}{2}\Big). \end{aligned}$$

In this way, if D is with the hyperbolic structure obtained by doubling the regular ideal octahedron, $\rho_0:\pi_1(D)\to\mathrm{PSL}(2;\mathbb{C})$ is the holonomy representation of this hyperbolic structure on D and $\widetilde{\rho}_0:\pi_1(D)\to\mathrm{SL}(2;\mathbb{C})$ is the lifting of ρ_0 with

$$\begin{aligned} &\big(\mathrm{Tr}\widetilde{\rho}_0([\gamma_{12}]),\mathrm{Tr}\widetilde{\rho}_0([\gamma_{13}]),\mathrm{Tr}\widetilde{\rho}_0([\gamma_{14}]),\mathrm{Tr}\widetilde{\rho}_0([\gamma_{23}]),\mathrm{Tr}\widetilde{\rho}_0([\gamma_{24}]),\mathrm{Tr}\widetilde{\rho}_0([\gamma_{34}])\big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = (-2,-2,-2,-2,-2,-2), \end{aligned}$$

then the logarithmic holonomies of $(\gamma_{12},\dots,\gamma_{34})$ in $\widetilde{\rho}_0$ are $(0,\dots,0)$. We notice that the complete hyperbolic structure on a fundamental shadow link complement is obtained by gluing such D-blocks together by isometries along the faces. Therefore, this hyperbolic structure can be considered as "the complete hyperbolic structure" on D.

The *Gram matrix* of a representation $\widetilde{\rho} : \pi_1(D) \to \mathrm{SL}(2;\mathbb{C})$ is the value of the Gram matrix function $\mathbb{G}$ defined in Definition 2.9 at $\big(\frac{u_{12}}{2}, \dots, \frac{u_{34}}{2}\big)$, i.e.,

$$\begin{aligned}\mathbb{G} &= \mathbb{G}\Big(\frac{u_{12}}{2}, \frac{u_{13}}{2}, \frac{u_{14}}{2}, \frac{u_{23}}{2}, \frac{u_{24}}{2}, \frac{u_{34}}{2}\Big)\\ &= \begin{bmatrix} 1 & -\cosh\frac{u_{12}}{2} & -\cosh\frac{u_{13}}{2} & -\cosh\frac{u_{14}}{2} \\ -\cosh\frac{u_{12}}{2} & 1 & -\cosh\frac{u_{23}}{2} & -\cosh\frac{u_{24}}{2} \\ -\cosh\frac{u_{13}}{2} & -\cosh\frac{u_{23}}{2} & 1 & -\cosh\frac{u_{34}}{2} \\ -\cosh\frac{u_{14}}{2} & -\cosh\frac{u_{24}}{2} & -\cosh\frac{u_{34}}{2} & 1 \end{bmatrix}.\end{aligned}$$

By the trace identity of the matrices in $\mathrm{SL}(2;\mathbb{C})$, for $\{j,k\} \subset \{2,3,4\}$,

$$\mathrm{Tr}\widetilde{\rho}([\gamma_{jk}]) = \mathrm{Tr}\widetilde{\rho}([\gamma_{1j}\cdot\gamma_{1k}^{-1})] = \mathrm{Tr}\widetilde{\rho}([\gamma_{1j}])\mathrm{Tr}\widetilde{\rho}([\gamma_{1k}]) - \mathrm{Tr}\widetilde{\rho}([\gamma_{1j}\cdot\gamma_{1k}]).$$

Then by a direct computation, we have

$$\begin{aligned}&f_D\big(\mathrm{Tr}\widetilde{\rho}([\gamma_{12}]), \mathrm{Tr}\widetilde{\rho}([\gamma_{13}]), \mathrm{Tr}\widetilde{\rho}([\gamma_{14}]), \mathrm{Tr}\widetilde{\rho}([\gamma_{12}\cdot\gamma_{13}]), \mathrm{Tr}\widetilde{\rho}([\gamma_{12}\cdot\gamma_{14}]), \mathrm{Tr}\widetilde{\rho}([\gamma_{13}\cdot\gamma_{14}])\big)\\ &\quad = 16\det\mathbb{G},\end{aligned}$$

and $\widetilde{\rho}$ is not in the branch locus of the double branched cover of the $\mathrm{SL}(2;\mathbb{C})$-character variety of D over $\mathbb{C}^6$ if and only if $\det\mathbb{G} \neq 0$.

Since $\pi_1(D)$ is a free group, every $\mathrm{PSL}(2;\mathbb{C})$-representation of it lifts to $\mathrm{SL}(2;\mathbb{C})$-representation Hence the $\mathrm{SL}(2;\mathbb{C})$-character variety of D is a branched cover of the $\mathrm{PSL}(2;\mathbb{C})$-character variety of D, and the latter is an irreducible algebraic variety. For a representation $\rho : \pi_1(D) \to \mathrm{PSL}(2;\mathbb{C})$, we defined the logarithmic holonomies $(u_{12}, \dots, u_{34})$ and the Gram matrix $\mathbb{G}$ of ρ as those of a lifting $\widetilde{\rho} : \pi_1(D) \to \mathrm{SL}(2;\mathbb{C})$ of ρ. Notice that the logarithmic holonomies depend on the choice of the liftings of ρ, and a different lifting will change $\mathbb{G}$ by multiplying some rows and the corresponding columns by -1 at the same time, which does not change its determinant. Therefore, the *determinant of the Gram matrix* $\det\mathbb{G}$ is independent of the choice of the liftings, and is a well defined quantity of ρ.

Let $\rho : \pi_1(D) \to \mathrm{PSL}(2;\mathbb{C})$ be a representation, and let $\mathrm{Ad}_\rho : \pi_1(D) \to \mathrm{SL}(3;\mathbb{C})$ be its adjoint representation. In addition, we assume for each $\{j,k\} \subset \{1,2,3,4\}$ that $\rho([\gamma_{jk}]) \neq \pm I$. Then in the case that $\rho([\gamma_{jk}])$ is not a parabolic element, we let $\mathbf{I}_{jk}$ be up to sign the unique invariant vector of $\mathrm{Ad}_\rho([\gamma_{jk}])^T$ with $\kappa(\mathbf{I}_{jk}, \mathbf{I}_{jk}) = 1$, where κ is the Killing bilinear form on $\mathfrak{sl}(2;\mathbb{C})$ defined in (3.1).

Definition 4.1. *Let* $\boldsymbol{\gamma} = (\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34})$. *An irreducible representation* $\rho : \pi_1(D) \to \mathrm{PSL}(2;\mathbb{C})$ *is* $\boldsymbol{\gamma}$-regular *if*

$$\mathbf{h}_D = \big\{\mathbf{I}_{jk} \otimes [\gamma_{jk}]\big\} \;\big|\; \{j,k\} \subset \{1,2,3,4\}\big\}$$

is a basis of $\mathrm{H}_1(D;\mathrm{Ad}_\rho)$*, where* $[\gamma_{jk}]$ *is the homology class of* γ_{jk} *in* $\mathrm{H}_1(D;\mathbb{Z})$*.*

Let $\mathrm{X}(D)$ be the $\mathrm{PSL}(2;\mathbb{C})$-character variety of D. A character $[\rho]\in\mathrm{X}(D)$ is $\boldsymbol{\gamma}$*-regular* if ρ is a $\boldsymbol{\gamma}$-regular representation. Since $\pi_1(D)$ is a free group, an Euler characteristic counting argument shows that if $[\rho]$ is $\boldsymbol{\gamma}$-regular, then $\mathrm{H}_k(D;\mathrm{Ad}_\rho)=0$ for $k\neq 1$.

The main result of this section is the following Proposition 4.2.

Proposition 4.2. *Let* $\rho:\pi_1(D)\to\mathrm{PSL}(2;\mathbb{C})$ *be a* $\boldsymbol{\gamma}$*-regular representation, and for* $\{j,k\}\subset\{1,2,3,4\}$ *let* u_{jk} *be up to sign the* logarithmic holonomy *of* γ_{jk} *in* ρ*. Then*

$$\mathrm{Tor}(D,\mathbf{h}_D;\mathrm{Ad}_\rho)=\pm\frac{\sqrt{\det\mathbb{G}\left(\frac{u_{12}}{2},\frac{u_{13}}{2},\frac{u_{14}}{2},\frac{u_{23}}{2},\frac{u_{24}}{2},\frac{u_{34}}{2}\right)}}{32\sinh\frac{u_{12}}{2}\sinh\frac{u_{13}}{2}\sinh\frac{u_{14}}{2}\sinh\frac{u_{23}}{2}\sinh\frac{u_{24}}{2}\sinh\frac{u_{34}}{2}}.$$

To prove Proposition 4.2, we need the following Lemma.

Lemma 4.3. *The set of* $\boldsymbol{\gamma}$*-regular characters contains a Zariski-open subset* $\mathrm{Z}(D)$ *of* $\mathrm{X}(D)$ *consisting of the characters* $[\rho]$ *satisfying the following two conditions:*

(1)

$$\begin{aligned}&\det[\mathbf{I}_{12},\mathbf{I}_{13},\mathbf{I}_{14}]\neq 0,\\&\det[\mathbf{I}_{12},\mathbf{I}_{23},\mathbf{I}_{24}]\neq 0,\\&\det[\mathbf{I}_{13},\mathbf{I}_{23},\mathbf{I}_{34}]\neq 0,\\&\det[\mathbf{I}_{14},\mathbf{I}_{24},\mathbf{I}_{34}]\neq 0,\end{aligned}$$

and

(2)

$$\det\Big[\mathbf{I}_{12}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot\mathbf{I}_{12},\ \mathbf{I}_{14}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot\mathbf{I}_{14},\mathbf{I}_{24}-\mathrm{Ad}_\rho([\gamma_{23}])^T\cdot\mathbf{I}_{24}\Big]\neq 0.$$

Proof. Let us compute the determinants in conditions (1) and (2) first. Similar to the proof of Lemma 3.4 we will do the computations for the holonomy representation of a hyperbolic metric on D with cone singularities around the edges e_{jk}'s first. Then by analyticity the computation extends to the other representations.

Now let Δ be a truncated hyperideal tetrahedron and let D be the union of Δ with its mirror image via the identity map between the four hexagonal faces $H_1,\dots,H_4$ and with the six edges $e_{12},\dots,e_{34}$ removed. This is a hyperbolic D-block defined in Section 2.5. For $\{j,k\}\subset\{1,2,3,4\}$ we let l_{jk} and α_{jk} respectively be the length of and the dihedral angle at the edge e_{jk}. We let s_{jk} be the length of the short edge adjacent to T_j and H_k, and notice that s_{jk} and s_{kj} are the lengths of different short edges.

Let $\rho:\pi_1(D)\to\mathrm{PSL}(2;\mathbb{C})$ be the holonomy representation of D and let $\mathrm{Ad}_\rho:\pi_1(D)\to\mathrm{SL}(3;\mathbb{C})$ be its adjoint representation. For $\{j,k\}\subset\{1,2,3,4\}$, let γ_{jk} be a

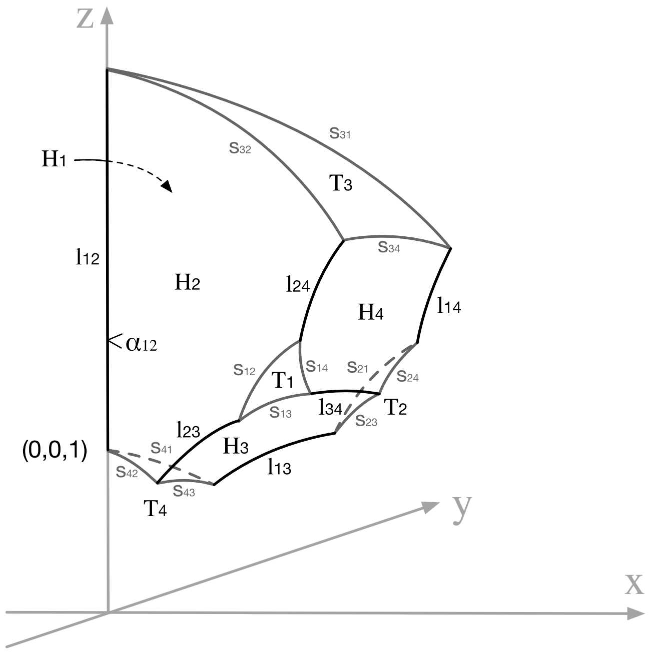

Fig. 6. Δ in $\mathbb{H}^3$.

simple loop around e_{jk}. Since $\rho(\gamma_{jk})$ is an elliptic element in $\mathrm{PSL}(2;\mathbb{C})$ which is not the identity matrix, $\mathrm{Ad}_\rho([\gamma_{jk}])^T$ has up to sign the canonical invariant vector $\mathbf{I}_{jk}$.

To compute the holonomy representation ρ of D, we isometrically embedded Δ into $\mathbb{H}^3$ as follows. As in Fig. 6, we place the intersection point of H_1, H_2 and T_4 at $(0,0,1)$, the edge e_{12} along the z-axis such that the intersection point of H_1, H_2 and T_3 is above $(0,0,1)$, the hexagonal face H_1 in the xz-plane and T_4 in the unit hemisphere centered at $(0,0,0)$ such that the y-coordinate of all the interior points of Δ are negative. This could always be done by using the mirror image of Δ if necessary.

For any complex number z let

$$D_z = \begin{bmatrix} e^{\frac{z}{2}} & 0 \\ 0 & e^{-\frac{z}{2}} \end{bmatrix},$$

and for $\{j,k\} \subset \{1,2,3,4\}$ let

$$S_{jk} = \begin{bmatrix} \cosh\frac{s_{jk}}{2} & \sinh\frac{s_{jk}}{2} \\ \sinh\frac{s_{jk}}{2} & \cosh\frac{s_{jk}}{2} \end{bmatrix}.$$

Suppose γ_{12}, γ_{14}, γ_{23} and γ_{24} go counterclockwise and γ_{13} goes clockwise around the corresponding edges observed from the perspective above T_3. By conjugating the tangent framings back to $p_1 = (0,0,1)$ and conjugating the tangent vectors of the axes of the rotations to $\frac{\partial}{\partial z}$, we have

$$\begin{aligned}
&\rho([\gamma_{12}]) = \pm D_{2\mathbf{i}\alpha_{12}},\\
&\rho([\gamma_{13}]) = \pm S_{41} D_{-2\mathbf{i}\alpha_{13}} S_{41}^{-1},\\
&\rho([\gamma_{14}]) = \pm D_{l_{12}} S_{31} D_{2\mathbf{i}\alpha_{14}} S_{31}^{-1} D_{l_{12}}^{-1} = \pm S_{41} D_{l_{13}} S_{21}^{-1} D_{-2\mathbf{i}\alpha_{14}} S_{21} D_{l_{13}}^{-1} S_{41}^{-1},\\
&\rho([\gamma_{23}]) = \pm D_{\mathbf{i}\alpha_{12}}^{-1} S_{42} D_{2\mathbf{i}\alpha_{23}} S_{42}^{-1} D_{\mathbf{i}\alpha_{12}},\\
&\rho([\gamma_{24}]) = \pm D_{\mathbf{i}\alpha_{12}}^{-1} S_{42} D_{l_{23}} S_{12}^{-1} D_{-2\mathbf{i}\alpha_{24}} S_{12} D_{l_{23}}^{-1} S_{42}^{-1} D_{\mathbf{i}\alpha_{12}}.
\end{aligned}$$

Here we write $\rho([\gamma_{14}])$ in two ways for the purpose of computing different things later. Since both D_z and S_{jk} are symmetric matrices, we have

$$\begin{aligned}
&\rho([\gamma_{12}])^T = \pm D_{2\mathbf{i}\alpha_{12}},\\
&\rho([\gamma_{13}])^T = \pm S_{41}^{-1} D_{-2\mathbf{i}\alpha_{13}} S_{41},\\
&\rho([\gamma_{14}])^T = \pm D_{l_{12}}^{-1} S_{31}^{-1} D_{2\mathbf{i}\alpha_{14}} S_{31} D_{l_{12}} = \pm S_{41}^{-1} D_{l_{13}}^{-1} S_{21} D_{-2\mathbf{i}\alpha_{14}} S_{21}^{-1} D_{l_{13}} S_{41},\\
&\rho([\gamma_{23}])^T = \pm D_{\mathbf{i}\alpha_{12}} S_{42}^{-1} D_{2\mathbf{i}\alpha_{23}} S_{42} D_{\mathbf{i}\alpha_{12}}^{-1},\\
&\rho([\gamma_{24}])^T = \pm D_{\mathbf{i}\alpha_{12}} S_{42}^{-1} D_{l_{23}}^{-1} S_{12} D_{-2\mathbf{i}\alpha_{24}} S_{12}^{-1} D_{l_{23}} S_{42} D_{\mathbf{i}\alpha_{12}}^{-1}.
\end{aligned} \tag{4.1}$$

Since $\rho([\gamma_{jk}])^T$ is a rotation of angle $2\alpha_{jk}$, it has an eigenvector $\mathbf{v}_{jk}^+$ with eigenvalue $e^{\mathbf{i}\alpha_{jk}}$ and an eigenvector $\mathbf{v}_{jk}^-$ with eigenvalue $e^{-\mathbf{i}\alpha_{jk}}$. By (4.1) we have

$$\begin{aligned}
&[\mathbf{v}_{12}^+, \mathbf{v}_{12}^-] = I,\\
&[\mathbf{v}_{13}^+, \mathbf{v}_{13}^-] = S_{41}^{-1},\\
&[\mathbf{v}_{14}^+, \mathbf{v}_{14}^-] = D_{l_{12}}^{-1} S_{31}^{-1} = S_{41}^{-1} D_{l_{13}}^{-1} S_{21},\\
&[\mathbf{v}_{24}^+, \mathbf{v}_{24}^-] = D_{\mathbf{i}\alpha_{12}} S_{42}^{-1} D_{l_{23}}^{-1} S_{12},
\end{aligned} \tag{4.2}$$

and by (3.2), the first half of the third equation of (4.2) and a direct computation we have

$$\mathbf{I}_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{I}_{13} = \begin{bmatrix} -\frac{1}{2}\sinh s_{41} \\ \cosh s_{41} \\ -\frac{1}{2}\sinh s_{41} \end{bmatrix} \quad \text{and} \quad \mathbf{I}_{14} = \begin{bmatrix} -\frac{1}{2}e^{-l_{12}}\sinh s_{31} \\ \cosh s_{31} \\ -\frac{1}{2}e^{l_{12}}\sinh s_{31} \end{bmatrix}. \tag{4.3}$$

Since $\kappa(\mathbf{I}_{12}, \mathbf{I}_{12}) = \kappa(\mathbf{I}_{13}, \mathbf{I}_{13}) = \kappa(\mathbf{I}_{14}, \mathbf{I}_{14}) = 1$, they are the canonical invariant vectors. Therefore,

$$\det[\mathbf{I}_{12}, \mathbf{I}_{13}, \mathbf{I}_{14}] = -\frac{1}{2}\sinh l_{12} \sinh s_{31} \sinh s_{41}. \tag{4.4}$$

Here we notice that by the hyperbolic Law of Sine for H_1, the quantity $\sinh l_{12} \sinh s_{31} \sinh s_{41}$ remains the same if we choose any edge and two adjacent short edges of H_1, hence is an intrinsic quantity of H_1.

For any $i \neq 1$, applying an orientation preserving isometry ϕ_i of $\mathbb{H}^3$ we can place H_i in $\mathbb{H}^3$ in the same way as H_1; and the invariant vector $\mathbf{I}_{ij}$, $i \neq j$, will be changed by

Ad_{ϕ_i}, which is a matrix in $\mathrm{SL}(3;\mathbb{C})$. Therefore, following the same computation as we did for (4.4), we have

$$\begin{aligned}
\det[\mathbf{I}_{12},\mathbf{I}_{23},\mathbf{I}_{24}] &= \frac{1}{2}\sinh l_{12}\sinh s_{32}\sinh s_{42},\\
\det[\mathbf{I}_{13},\mathbf{I}_{23},\mathbf{I}_{34}] &= -\frac{1}{2}\sinh l_{13}\sinh s_{23}\sinh s_{43},\\
\det[\mathbf{I}_{14},\mathbf{I}_{24},\mathbf{I}_{34}] &= \frac{1}{2}\sinh l_{14}\sinh s_{24}\sinh s_{34}.
\end{aligned} \tag{4.5}$$

This computes the determinants in (1) for the holonomy representation of a hyperbolic D-block.

To compute the determinant in (2), by the second equation of (4.1) and the first equation of (4.2), we have

$$[\mathbf{v}_{12}^{+},\mathbf{v}_{12}^{-}] = I = S_{41}^{-1}D_0S_{41}$$

and

$$\rho([\gamma_{13}])^T\cdot[\mathbf{v}_{12}^{+},\mathbf{v}_{12}^{-}] = \pm S_{41}^{-1}D_{-2\mathbf{i}\alpha_{13}}S_{41}.$$

Therefore, by (3.7) and the notation therein,

$$\begin{aligned}
\mathbf{I}_{12}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot\mathbf{I}_{12} &= \mathbf{I}^{0}_{s_{41}s_{41}}-\mathbf{I}^{-2\mathbf{i}\alpha_{13}}_{s_{41}s_{41}}\\
&= \mathbf{i}\sinh s_{41}\sin\alpha_{13}\begin{bmatrix}-\mathbf{i}\sin\alpha_{13}\cosh s_{41}+\cos\alpha_{13}\\ 2\mathbf{i}\sin\alpha_{13}\sinh s_{41}\\ -\mathbf{i}\sin\alpha_{13}\cosh s_{41}-\cos\alpha_{13}\end{bmatrix}.
\end{aligned}$$

By the second equation of (4.1) again and the second half of the third equation of (4.2), we have

$$[\mathbf{v}_{14}^{+},\mathbf{v}_{14}^{-}] = S_{41}^{-1}D_{-l_{13}}S_{21}$$

and

$$\rho([\gamma_{13}])^T\cdot[\mathbf{v}_{14}^{+},\mathbf{v}_{14}^{-}] = \pm S_{41}^{-1}D_{-l_{13}-2\mathbf{i}\alpha_{13}}S_{21}.$$

Therefore, by (3.7)

$$\begin{aligned}
&\mathbf{I}_{14}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot\mathbf{I}_{14}\\
&\quad= \mathbf{I}^{-l_{13}}_{s_{41}s_{21}}-\mathbf{I}^{-l_{13}-2\mathbf{i}\alpha_{13}}_{s_{41}s_{21}}\\
&\quad= \mathbf{i}\sinh s_{21}\sin\alpha_{13}\begin{bmatrix}-\sinh(l_{13}+\mathbf{i}\alpha_{13})\cosh s_{41}+\cosh(l_{13}+\mathbf{i}\alpha_{13})\\ 2\sinh(l_{13}+\mathbf{i}\alpha_{13})\sinh s_{41}\\ -\sinh(l_{13}+\mathbf{i}\alpha_{13})\cosh s_{41}-\cosh(l_{13}+\mathbf{i}\alpha_{13})\end{bmatrix}.
\end{aligned}$$

Finally, by the fourth equation of (4.1) and (4.2), we have

$$[\mathbf{v}_{24}^+, \mathbf{v}_{24}^-] = D_{\mathbf{i}\alpha_{12}} S_{42}^{-1} D_{-l_{23}} S_{12}$$

and

$$\rho([\gamma_{23}])^T \cdot [\mathbf{v}_{24}^+, \mathbf{v}_{24}^-] = \pm D_{\mathbf{i}\alpha_{12}} S_{42}^{-1} D_{-l_{23}+2\mathbf{i}\alpha_{23}} S_{12}.$$

Therefore, by (3.8)

$$\begin{aligned}
&\mathbf{I}_{24} - \mathrm{Ad}_\rho([\gamma_{23}])^T \cdot \mathbf{I}_{24}\\
&\quad = \mathbf{I}^{-l_{23}}_{(\mathbf{i}\alpha_{12})s_{42}s_{12}} - \mathbf{I}^{-l_{23}+2\mathbf{i}\alpha_{23}}_{(\mathbf{i}\alpha_{12})s_{42}s_{12}}\\
&\quad = -\mathbf{i}\sinh s_{12}\sin\alpha_{23} \begin{bmatrix} e^{\mathbf{i}\alpha_{12}}\big(-\sinh(l_{23}-\mathbf{i}\alpha_{23})\cosh s_{42} + \cosh(l_{23}-\mathbf{i}\alpha_{23})\big) \\ 2\sinh(l_{23}-\mathbf{i}\alpha_{23})\sinh s_{42} \\ e^{-\mathbf{i}\alpha_{12}}\big(-\sinh(l_{23}-\mathbf{i}\alpha_{23})\cosh s_{42} - \cosh(l_{23}-\mathbf{i}\alpha_{23})\big) \end{bmatrix}.
\end{aligned}$$

Putting all together, we have

$$\begin{aligned}
&\det\Big[\mathbf{I}_{12} - \mathrm{Ad}_\rho([\gamma_{13}])^T \cdot \mathbf{I}_{12},\ \mathbf{I}_{14} - \mathrm{Ad}_\rho([\gamma_{13}])^T \cdot \mathbf{I}_{14}, \mathbf{I}_{24} - \mathrm{Ad}_\rho([\gamma_{23}])^T \cdot \mathbf{I}_{24}\Big]\\
=&\mathbf{i}\sin^2\alpha_{13}\sin\alpha_{23}\sinh s_{12}\sinh s_{21}\sinh s_{41}\cdot\det\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix}\cdot\det M,
\end{aligned}$$

where M is the following matrix

$$\begin{bmatrix} \mathbf{i}\sin\alpha_{13}\cosh s_{41} & \sinh(l_{13}+\mathbf{i}\alpha_{13})\cosh s_{41} & \cos\alpha_{12}\sinh(l_{23}-\mathbf{i}\alpha_{23})\cosh s_{42} - \mathbf{i}\sin\alpha_{12}\cosh(l_{23}-\mathbf{i}\alpha_{23}) \\ \mathbf{i}\sin\alpha_{13}\sinh s_{41} & \sinh(l_{13}+\mathbf{i}\alpha_{13})\sinh s_{41} & \sinh(l_{23}-\mathbf{i}\alpha_{23})\sinh s_{42} \\ \cos\alpha_{13} & \cosh(l_{13}+\mathbf{i}\alpha_{13}) & -\mathbf{i}\sin\alpha_{12}\sinh(l_{23}-\mathbf{i}\alpha_{23})\cosh s_{42} + \cos\alpha_{12}\cosh(l_{23}-\mathbf{i}\alpha_{23}) \end{bmatrix}.$$

Computing the cofactors of Musing the hyperbolic angle sum formula, we have $M_{13} = -\sinh l_{13}\sinh s_{41}$, $M_{23} = \sinh l_{13}\cosh s_{41}$ and $M_{33} = 0$. Then

$$\begin{aligned}
\det M =& -\sinh l_{13}\sinh s_{41}\Big(\cos\alpha_{12}\sinh(l_{23}-\mathbf{i}\alpha_{23})\cosh s_{42} - \mathbf{i}\sin\alpha_{12}\cosh(l_{23}-\mathbf{i}\alpha_{23})\Big)\\
&+ \sinh l_{13}\cosh s_{41}\sinh(l_{23}-\mathbf{i}\alpha_{23})\sinh s_{42}\\
=&\frac{\sin\alpha_{12}\sinh l_{13}\sinh l_{23}\sinh s_{42}}{\sin\alpha_{13}},
\end{aligned}$$

where the last equality comes from the use of the hyperbolic Law of Sine that $\sinh s_{41} = \frac{\sinh s_{42}\sin\alpha_{23}}{\sin\alpha_{13}}$ to get a common factor $\sinh s_{41}$, the use of the hyperbolic Law of Cosine in T_4 to write $\cosh s_{41}$ and $\cosh s_{42}$ into trig-functions of the angles α_{12}, α_{13} and α_{23} and the use of the angle sum formula to expand $\sinh(l_{23} - i\alpha_{23})$ and $\cosh(l_{23} - i\alpha_{23})$ into trig- and hyperbolic trig-functions of α_{23} and l_{23}. Then after a final simplification, the imaginary part vanishes and the real part becomes the quantity above.

Therefore,

$$\begin{aligned}&\det\Big[\mathbf{I}_{12}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot\mathbf{I}_{12},\ \mathbf{I}_{14}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot\mathbf{I}_{14},\mathbf{I}_{24}-\mathrm{Ad}_\rho([\gamma_{23}])^T\cdot\mathbf{I}_{24}\Big]\\ =&4\mathbf{i}\sin\alpha_{12}\sin\alpha_{13}\sin\alpha_{23}(\sinh l_{13}\sinh s_{21}\sinh s_{41})(\sinh l_{23}\sinh s_{42}\sinh s_{12}).\end{aligned}\tag{4.6}$$

This computes the determinant in (2) for the holonomy representation of a hyperbolic D-block.

For the other characters in $\mathrm{X}(D)$, we observe that for the holonomy representation ρ of a hyperbolic D-block with cone angles $(2\alpha_{12},\dots,2\alpha_{34})$, for any lifting $\widetilde{\rho}:\pi_1(D)\to\mathrm{SL}(2;\mathbb{C})$ of ρ, we have

$$\mathrm{Tr}\widetilde{\rho}([\gamma_{jk}])=\pm 2\cos\alpha_{jk}$$

for $\{j,k\}\subset\{1,2,3,4\}$. Notice that l_{ij}, s_{ki} and s_{li} are the lengths of an edge and the two adjacent short edges around the face H_i, and all the determines in conditions (1) and (2) have factors products of the form $\sinh l_{ij}\sinh s_{ki}\sinh s_{li}$. We claim that

$$\sinh l_{ij}\sinh s_{ki}\sinh s_{li}=\pm\sqrt{\frac{-\det\mathrm{G}_{\boldsymbol{\alpha}}}{(1-\cos^2\alpha_{ij})(1-\cos^2\alpha_{ik})(1-\cos^2\alpha_{il})}}\tag{4.7}$$

for $\{i,j,k,l\}\subset\{1,2,3,4\}$, where $\mathrm{G}_{\boldsymbol{\alpha}}$ is the Gram matrix in the dihedral angles of the truncated hyperideal tetrahedron Δ recalled in Section 2.3. As a consequence, the square of $\cosh l_{ij}\sinh s_{ki}\sinh s_{li}$ is a rational function in $(\mathrm{Tr}\widetilde{\rho}([\gamma_{12}]),\dots,\mathrm{Tr}\widetilde{\rho}([\gamma_{34}]))$. Indeed, to see (4.7), using the hyperbolic Law of Cosine to the face H_i, we have

$$\begin{aligned}\sinh^2 l_{ij}\sinh^2 s_{ki}\sinh^2 s_{li}=&\bigg(\Big(\frac{\cosh s_{ji}+\cosh s_{ki}\cosh s_{li}}{\sinh s_{ki}\sinh s_{li}}\Big)^2-1\bigg)\sinh^2 s_{ki}\sinh^2 s_{li}\\ =&2\cosh s_{ji}\cosh s_{ki}\cosh s_{li}+\cosh^2 s_{ji}\\ &+\cosh^2 s_{ki}+\cosh^2 s_{li}-1;\end{aligned}$$

and using the hyperbolic Law of Cosine to the triangles of truncation T_j, T_k and T_l, we have

$$\begin{aligned}\cosh s_{ji}&=\frac{\cos\alpha_{kl}+\cos\alpha_{ik}\cos\alpha_{il}}{\sin\alpha_{ik}\sin\alpha_{il}},\\ \cosh s_{ki}&=\frac{\cos\alpha_{jl}+\cos\alpha_{ij}\cos\alpha_{il}}{\sin\alpha_{ij}\sin\alpha_{il}},\end{aligned}$$

and

$$\cosh s_{li}=\frac{\cos\alpha_{jk}+\cos\alpha_{ij}\cos\alpha_{ik}}{\sin\alpha_{ij}\sin\alpha_{ik}}.$$

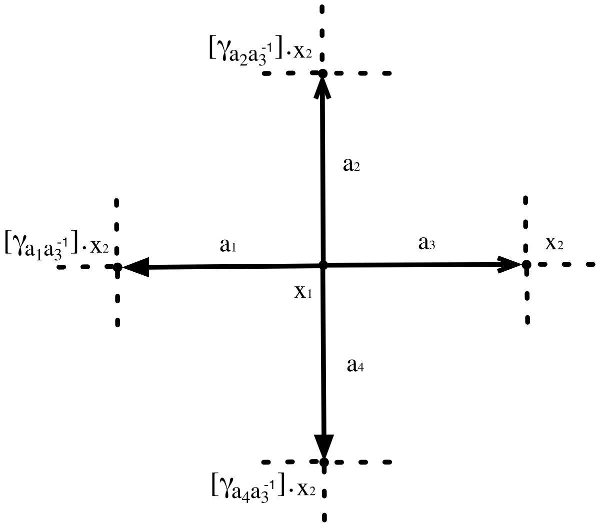

Fig. 7. The 1-dimensional CW complex.

Plugging these into the previous identity, we have (4.7). Since $\mathrm{X}(D)$ is an irreducible algebraic variety, by analyticity, (4.4), (4.5) and (4.6) hold for the other characters $[\rho]$ in $\mathrm{X}(D)$.

Since the square of the determinants in conditions (1) and (2) is rational functions in the coordinates $(\mathrm{Tr}\widetilde{\rho}([\gamma_{12}]), \dots, \mathrm{Tr}\widetilde{\rho}([\gamma_{34}]))$, the lifting of those characters form a Zariski-open subset of the $\mathrm{SL}(2;\mathbb{C})$ character variety of D, and hence those characters themselves form a Zariski-open subset of $\mathrm{X}(D)$.

Next we show that the representations satisfying (1) and (2) are $\boldsymbol{\gamma}$-regular. We will compute the homologies of D using its spine Γ, which is the 1-dimensional CW complex consisting of two 0-cells x_1 and x_2 (one dual to each copy of Δ) and four 1-cells a_1, a_2, a_3 and a_4 (one dual to each hexagonal face H_j) all of which are oriented from x_1 to x_2.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{C}^3$ and let the choice of representatives x_1, x_2, a_1, a_2, a_3 and a_4 in the universal covering of Γ as drawn in Fig. 7. Then $\mathrm{C}_0(D;\mathrm{Ad}_\rho) \cong \mathbb{C}^6$ with a natural basis $\{\mathbf{e}_i \otimes x_k\}$ for $i \in \{1,2,3\}$ and $k \in \{1,2\}$; $\mathrm{C}_1(D;\mathrm{Ad}_\rho) \cong \mathbb{C}^{12}$ with a natural basis $\{\mathbf{e}_i \otimes a_k\}$ for $i \in \{1,2,3\}$ and $k \in \{1,2,3,4\}$; and $\mathrm{C}_k(D;\mathrm{Ad}_\rho) = 0$ for $k \neq 0$ or 1.

We choose x_1 to be the base point of the fundamental group; and for $\{j,k\} \subset \{1,2,3\}$, let $\gamma_{a_j a_k^{-1}}$ be the curve starting from x_1 traveling along a_j to x_2 then along $-a_k$ back to x_1. In this way, we have $[\gamma_{a_k a_j^{-1}}] = [\gamma_{jk}]^{\pm 1}$. Checking the orientation carefully we have $[\gamma_{a_1 a_2^{-1}}] = [\gamma_{12}]$, $[\gamma_{a_2 a_3^{-1}}] = [\gamma_{23}]$ and $[\gamma_{a_1 a_3^{-1}}] = [\gamma_{13}]$.

By condition (1), we see that the vectors $\{\mathbf{I}_{jk} \otimes (a_j - a_k)\}$, $\{j,k\} \subset \{1,2,3,4\}$, are linearly independent in $\mathrm{C}_1(D;\mathrm{Ad}_\rho)$. To show that they lie in the kernel of $\partial : \mathrm{C}_1(D;\mathrm{Ad}_\rho) \to \mathrm{C}_0(D;\mathrm{Ad}_\rho)$, we have

$$
\begin{aligned}
\partial(\mathbf{I}_{jk}\otimes(a_j-a_k)) &= \mathbf{I}_{jk}\otimes\partial(a_j-a_k)\\
&= \mathbf{I}_{jk}\otimes\Big((x_1-[\gamma_{a_ja_3^{-1}}]\cdot x_2)-(x_1-[\gamma_{a_ka_3^{-1}}]\cdot x_2)\Big)\\
&= \mathbf{I}_{jk}\otimes\Big([\gamma_{a_ka_3^{-1}}]\cdot x_2-[\gamma_{a_ja_3^{-1}}]\cdot x_2\Big)\\
&= \Big(\mathrm{Ad}_\rho([\gamma_{a_ka_3^{-1}}])^T\cdot\mathbf{I}_{jk}-\mathrm{Ad}_\rho([\gamma_{a_ja_3^{-1}}])^T\cdot\mathbf{I}_{jk}\Big)\otimes x_2\\
&= \Big(\mathrm{Ad}_\rho([\gamma_{a_ka_3^{-1}}])^T\mathrm{Ad}_\rho([\gamma_{a_ja_k^{-1}}])^T\cdot\mathbf{I}_{jk}-\mathrm{Ad}_\rho([\gamma_{a_ja_3^{-1}}])^T\cdot\mathbf{I}_{jk}\Big)\otimes x_2\\
&= 0,
\end{aligned}
$$

where the penultimate equality comes from $\mathrm{Ad}_\rho([\gamma_{a_ja_k^{-1}}])^T\cdot\mathbf{I}_{jk}=\mathrm{Ad}_\rho([\gamma_{jk}]^{\pm1})^T\cdot\mathbf{I}_{jk}=\mathbf{I}_{jk}$ and the last equation comes from $\gamma_{a_ja_k^{-1}}\cdot\gamma_{a_ka_3^{-1}}=\gamma_{a_ja_3^{-1}}$. Therefore, $\{\mathbf{I}_{jk}\otimes(a_j-a_k)\}$, $\{j,k\}\subset\{1,2,3,4\}$, represent six linearly independent elements $\{\mathbf{I}_{jk}\otimes[\gamma_{jk}]\}$ in $\mathrm{H}_1(D;\mathrm{Ad}_\rho)$. Later we will prove that they also span, and hence form a basis of $\mathrm{H}_1(D;\mathrm{Ad}_\rho)$.

Now we claim that these six vectors $\{\mathbf{I}_{12}\otimes(a_1-a_2),\mathbf{I}_{13}\otimes(a_1-a_3),\mathbf{I}_{14}\otimes(a_1-a_4),\mathbf{I}_{23}\otimes(a_2-a_3),\mathbf{I}_{24}\otimes(a_2-a_4),\mathbf{I}_{34}\otimes(a_3-a_4)\}$ joint with the other six vectors $\{\mathbf{I}_{13}\otimes a_3,\mathbf{I}_{23}\otimes a_3,\mathbf{I}_{34}\otimes a_3,\mathbf{I}_{12}\otimes a_1,\mathbf{I}_{14}\otimes a_1,\mathbf{I}_{24}\otimes a_2\}$ form a basis of $\mathrm{C}_1(D;\mathrm{Ad}_\rho)$. Indeed, in the natural basis $\{\mathbf{e}_i\otimes a_k\}$ for $i\in\{1,2,3\}$ and $k\in\{1,2,3,4\}$, the 12×12 matrix consisting of these vectors as the columns is obtained from the one consisting of $\{\mathbf{I}_{jk}\otimes a_k\}$, $k\in\{1,2,3,4\}$ and $j\neq k$, as the columns by a sequence of elementary column operations of type I, III, and II with a factor -1. The latter matrix is a block matrix with four 3×3 blocks $[\mathbf{I}_{12},\mathbf{I}_{13},\mathbf{I}_{14}]$, $[\mathbf{I}_{12},\mathbf{I}_{23},\mathbf{I}_{24}]$, $[\mathbf{I}_{13},\mathbf{I}_{23},\mathbf{I}_{34}]$ and $[\mathbf{I}_{14},\mathbf{I}_{24},\mathbf{I}_{34}]$ on the diagonal and $0's$ elsewhere, hence has determinant

$$\det[\mathbf{I}_{12},\mathbf{I}_{13},\mathbf{I}_{14}]\cdot\det[\mathbf{I}_{12},\mathbf{I}_{23},\mathbf{I}_{24}]\cdot\det[\mathbf{I}_{13},\mathbf{I}_{23},\mathbf{I}_{34}]\cdot\det[\mathbf{I}_{14},\mathbf{I}_{24},\mathbf{I}_{34}]$$

and by condition (1) is non-singular. As a consequence, the former matrix is also non-singular and up to sign has the same determinant.

Next we will study the image of the six vectors $\{\mathbf{I}_{13}\otimes a_3,\mathbf{I}_{23}\otimes a_3,\mathbf{I}_{34}\otimes a_3,\mathbf{I}_{12}\otimes a_1,\mathbf{I}_{14}\otimes a_1,\mathbf{I}_{24}\otimes a_2\}$ under the boundary map ∂, and show that they span $\mathrm{C}_0(D;\mathrm{Ad}_\rho)$. We have for $j=1,2,4$,

$$\partial(\mathbf{I}_{j3}\otimes a_3)=\mathbf{I}_{j3}\otimes\partial a_3=\mathbf{I}_{j3}\otimes(x_1-x_2)=\mathbf{I}_{j3}\otimes x_1-\mathbf{I}_{j3}\otimes x_2;$$

for $k=2,4$,

$$\partial(\mathbf{I}_{1k}\otimes a_1)=\mathbf{I}_{1k}\otimes\partial a_1=\mathbf{I}_{1k}\otimes(x_1-[\gamma_{a_1a_3^{-1}}]\cdot x_2)=\mathbf{I}_{1k}\otimes x_1-\Big(\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot\mathbf{I}_{1k}\Big)\otimes x_2;$$

and

$$\partial(\mathbf{I}_{24}\otimes a_2)=\mathbf{I}_{24}\otimes\partial a_2=\mathbf{I}_{24}\otimes(x_1-[\gamma_{a_2a_3^{-1}}]\cdot x_2)=\mathbf{I}_{24}\otimes x_1-\Big(\mathrm{Ad}_\rho([\gamma_{23}])^T\cdot\mathbf{I}_{24}\Big)\otimes x_2.$$

Therefore, in the natural basis $\{\mathbf{e}_i \otimes x_k\}$, $i \in \{1,2,3\}$, $k \in \{1,2\}$, the 6×6 matrix consisting of $\{\partial(\mathbf{I}_{13}\otimes a_3), \partial(\mathbf{I}_{23}\otimes a_3), \partial(\mathbf{I}_{34}\otimes a_3), \partial(\mathbf{I}_{12}\otimes a_1), \partial(\mathbf{I}_{14}\otimes a_1), \partial(\mathbf{I}_{24}\otimes a_2)\}$ as the columns has four 3×3 blocks, where on the top left it has $[\mathbf{I}_{13}, \mathbf{I}_{23}, \mathbf{I}_{34}]$ and on the bottom left it has $[-\mathbf{I}_{13}, -\mathbf{I}_{23}, -\mathbf{I}_{34}]$; on the top right it has $[\mathbf{I}_{12}, \mathbf{I}_{14}, \mathbf{I}_{24}]$ and on the bottom right it has

$$\Big[-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot \mathbf{I}_{12},\ -\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot \mathbf{I}_{14},\ -\mathrm{Ad}_\rho([\gamma_{23}])^T\cdot \mathbf{I}_{24}\Big].$$

This matrix is row equivalent to (by adding the top blocks to the bottom) the one with $[\mathbf{I}_{13}, \mathbf{I}_{23}, \mathbf{I}_{34}]$ on the top left, $0's$ on the bottom left and

$$\Big[\mathbf{I}_{12}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot \mathbf{I}_{12},\ \mathbf{I}_{14}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot \mathbf{I}_{14}, \mathbf{I}_{24}-\mathrm{Ad}_\rho([\gamma_{23}])^T\cdot \mathbf{I}_{24}\Big]$$

on the bottom right. Hence the determinant of both of the 6×6 matrices are equal to

$$\det[\mathbf{I}_{13}, \mathbf{I}_{23}, \mathbf{I}_{34}]\cdot \det\Big[\mathbf{I}_{12}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot \mathbf{I}_{12},\ \mathbf{I}_{14}-\mathrm{Ad}_\rho([\gamma_{13}])^T\cdot \mathbf{I}_{14}, \mathbf{I}_{24}-\mathrm{Ad}_\rho([\gamma_{23}])^T\cdot \mathbf{I}_{24}\Big].$$

By conditions (1) and (2), the product above is nonzero, and hence $\{\partial(\mathbf{I}_{13}\otimes a_3), \partial(\mathbf{I}_{23}\otimes a_3), \partial(\mathbf{I}_{34}\otimes a_3), \partial(\mathbf{I}_{12}\otimes a_1), \partial(\mathbf{I}_{14}\otimes a_1), \partial(\mathbf{I}_{24}\otimes a_2)\}$ span $\mathrm{C}_0(D;\mathrm{Ad}_\rho)$. This implies that $\mathrm{H}_0(D;\mathrm{Ad}_\rho)=0$. Since there are no cells of dimension higher than or equal to 2, $\mathrm{H}_k(D;\mathrm{Ad}_\rho)=0$ for $k\geqslant 2$.

Now since $\{\partial(\mathbf{I}_{13}\otimes a_3), \partial(\mathbf{I}_{23}\otimes a_3), \partial(\mathbf{I}_{34}\otimes a_3), \partial(\mathbf{I}_{12}\otimes a_1), \partial(\mathbf{I}_{14}\otimes a_1), \partial(\mathbf{I}_{24}\otimes a_2)\}$ span $\mathrm{C}_0(D;\mathrm{Ad}_\rho)\cong\mathbb{C}^6$, by dimension counting the kernel of $\partial:\mathrm{C}_1(D;\mathrm{Ad}_\rho)\to\mathrm{C}_0(D;\mathrm{Ad}_\rho)$ has dimension at most 6. Hence $\{\mathbf{I}_{12}\otimes(a_1-a_2), \mathbf{I}_{13}\otimes(a_1-a_3), \mathbf{I}_{14}\otimes(a_1-a_4), \mathbf{I}_{23}\otimes(a_2-a_3), \mathbf{I}_{24}\otimes(a_2-a_4), \mathbf{I}_{34}\otimes(a_3-a_4)\}$ span the kernel of ∂. This shows that the elements they represent $\mathbf{h}_D=\{\mathbf{I}_{jk}\otimes[\gamma_{jk}]\}$, $\{j,k\}\subset\{1,2,3,4\}$, form a basis of $\mathrm{H}_1(D;\mathrm{Ad}_\rho)$, and $\mathrm{H}_1(D;\mathrm{Ad}_\rho)\cong\mathbb{C}^6$. This completes the proof. □

Proof of Proposition 4.2. Since the adjoint twisted Reidemeister torsion is invariant under subdivisions, elementary expansions and elementary collapses of CW-complexes by [19,23], we can do the computation using the spine Γ of D.

The adjoint twisted Reidemeistor torsion equals, up to sign, the determinant of the 12×12 matrix consisting of $\{\mathbf{I}_{12}\otimes(a_1-a_2), \mathbf{I}_{13}\otimes(a_1-a_3), \mathbf{I}_{14}\otimes(a_1-a_4), \mathbf{I}_{23}\otimes(a_2-a_3), \mathbf{I}_{24}\otimes(a_2-a_4), \mathbf{I}_{34}\otimes(a_3-a_4), \mathbf{I}_{13}\otimes a_3, \mathbf{I}_{23}\otimes a_3, \mathbf{I}_{34}\otimes a_3, \mathbf{I}_{12}\otimes a_1, \mathbf{I}_{14}\otimes a_1, \mathbf{I}_{24}\otimes a_2\}$ as the columns divided by the determinant of the 6×6 matrix consisting of $\{\partial(\mathbf{I}_{13}\otimes a_3), \partial(\mathbf{I}_{23}\otimes a_3), \partial(\mathbf{I}_{34}\otimes a_3), \partial(\mathbf{I}_{12}\otimes a_1), \partial(\mathbf{I}_{14}\otimes a_1), \partial(\mathbf{I}_{24}\otimes a_2)\}$ as the columns.

By (4.4), (4.5) and (4.6), we have for the holonomy representation of a hyperbolic D-block,

$$\begin{aligned}
&\mathrm{Tor}(D, \mathbf{h}_D; \mathrm{Ad}_\rho) \\
&= \pm \frac{\det[\mathbf{I}_{12}, \mathbf{I}_{13}, \mathbf{I}_{14}] \cdot \det[\mathbf{I}_{12}, \mathbf{I}_{23}, \mathbf{I}_{24}] \cdot \det[\mathbf{I}_{13}, \mathbf{I}_{23}, \mathbf{I}_{34}] \cdot \det[\mathbf{I}_{14}, \mathbf{I}_{24}, \mathbf{I}_{34}]}{\det[\mathbf{I}_{13}, \mathbf{I}_{23}, \mathbf{I}_{34}] \cdot \det\Big[\mathbf{I}_{12} - \mathrm{Ad}_\rho([\gamma_{13}])^T \cdot \mathbf{I}_{12},\ \mathbf{I}_{14} - \mathrm{Ad}_\rho([\gamma_{13}])^T \cdot \mathbf{I}_{14}, \mathbf{I}_{24} - \mathrm{Ad}_\rho([\gamma_{23}])^T \cdot \mathbf{I}_{24}\Big]} \\
&= \pm \frac{\mathbf{i} \sinh l_{14} \sinh s_{24} \sinh s_{34}}{32 \sin\alpha_{12} \sin\alpha_{13} \sin\alpha_{23}} \\
&= \pm \frac{\sqrt{\det \mathrm{G}_{\boldsymbol{\alpha}}}}{32 \sin\alpha_{12} \sin\alpha_{13} \sin\alpha_{14} \sin\alpha_{23} \sin\alpha_{24} \sin\alpha_{34}} \\
&= \pm \frac{\sqrt{\det \mathbb{G}\left(\frac{u_{12}}{2}, \frac{u_{13}}{2}, \frac{u_{14}}{2}, \frac{u_{23}}{2}, \frac{u_{24}}{2}, \frac{u_{34}}{2}\right)}}{32 \sinh\frac{u_{12}}{2} \sinh\frac{u_{13}}{2} \sinh\frac{u_{14}}{2} \sinh\frac{u_{23}}{2} \sinh\frac{u_{24}}{2} \sinh\frac{u_{34}}{2}},
\end{aligned}$$

where the last equality comes from (4.7).

Finally, by Lemma 4.3 and the analyticity of the involved functions, the result holds for all $\boldsymbol{\gamma}$-regular characters in $\mathrm{X}(D)$. □

5. Reidemeister torsion of the Mayer-Vietoris sequence

Let M be the complement of a fundamental shadow link with n components, and let $\rho : \pi_1(M) \to \mathrm{PSL}(2;\mathbb{C})$ be an irreducible representation. We insert a thickened pair of pants if necessary so that no D-block self-intersects. Suppose there are in total c thickened pairs of pants inserted, and the 3-dimensional objects (D-blocks and the thickened pairs of pants) intersect at p pairs of pants, then we have $p = c + 2d$. Order the c thickened pair of pants together with the d D-blocks by $D_1, \dots, D_{c+d}$, and order the p pairs of pants by $P_1, \dots, P_p$. Then by Lemma 2.1 there is the following short exact sequence of chain complexes

$$0 \to \bigoplus_{j=1}^{p} \mathrm{C}_*(P_j; \mathrm{Ad}_\rho) \xrightarrow{\delta} \bigoplus_{k=1}^{c+d} \mathrm{C}_*(D_k; \mathrm{Ad}_\rho) \xrightarrow{\epsilon} \mathrm{C}_*(M; \mathrm{Ad}_\rho) \to 0$$

with ϵ defined by the sum

$$\epsilon(\mathbf{c}_1, \dots, \mathbf{c}_{c+d}) = \sum_{k=1}^{c+d} \mathbf{c}_k \tag{5.1}$$

and δ defined by the alternating sum

$$(\delta\mathbf{c})_k = -\sum_j \mathbf{c}_j + \sum_l \mathbf{c}_l, \tag{5.2}$$

where j runs over the indices such that $P_j = D_{k'} \cap D_k$ for some $k' < k$ and l runs over the indices such that $P_l = D_k \cap D_{k''}$ for some $k < k''$.

For each $i \in \{1, \dots, n\}$, let $T_i = \partial N(L_i)$ be the boundary of a tubular neighborhood of the i-th component of L_{FSL}, m_i be the meridian of $N(L_i)$ and $\boldsymbol{m} = (m_1, \dots, m_n)$.

Suppose ρ is an $\boldsymbol{m}$-regular representation whose restriction to each pair of pants P_j is $\boldsymbol{\gamma}$-regular as defined in Definition 3.1, and to each D-block D_k is $\boldsymbol{\gamma}$-regular as defined in Definition 4.1, then the induced Mayer-Vietoris exact sequence $\mathcal{H}$ has four nonzero terms, i.e.,

$$0 \to \mathrm{H}_2(M;\mathrm{Ad}_\rho) \xrightarrow{\partial} \bigoplus_{j=1}^{p} \mathrm{H}_1(P_j;\mathrm{Ad}_\rho) \xrightarrow{\delta} \bigoplus_{k=1}^{c+d} \mathrm{H}_1(D_k;\mathrm{Ad}_\rho) \xrightarrow{\epsilon} \mathrm{H}_1(M;\mathrm{Ad}_\rho) \to 0. \tag{5.3}$$

Let $\mathbf{I}_i$ be up to sign the unique invariant vector of $\mathrm{Ad}_\rho([m_i])^T$ with $\kappa(\mathbf{I}_i,\mathbf{I}_i)=1$. Then by a diagram chasing, $\mathrm{H}_1(M;\mathrm{Ad}_\rho)$ has a basis $\mathbf{h}^1_{(M,\boldsymbol{m})}=\{\mathbf{I}_1\otimes[m_1],\dots,\mathbf{I}_n\otimes[m_n]\}$ and $\mathrm{H}_1(M;\mathrm{Ad}_\rho)$ has a basis $\mathbf{h}^2_M=\{\mathbf{I}_1\otimes[T_1],\dots,\mathbf{I}_n\otimes[T_n]\}$.

Proposition 5.1. *Let $\mathbf{h}_{P_j}$ be the basis of $\mathrm{H}_1(P_j;\mathrm{Ad}_\rho)$ in Definition 3.1 and let $\mathbf{h}_{D_k}$ be the basis of $\mathrm{H}_1(D_k;\mathrm{Ad}_\rho)$ in Definition 4.1. Let $\mathbf{h}_{**}$ be the union of $\mathbf{h}^1_{(M,\boldsymbol{m})}$, $\mathbf{h}^2_M$, $\sqcup_j\mathbf{h}_{P_j}$ and $\sqcup_k\mathbf{h}_{D_k}$. Then*

$$\mathrm{Tor}(\mathcal{H},\mathbf{h}_{**})=\pm 1. \tag{5.4}$$

Proof. By [21, Proposition 3.22, Corollary 3.23], Lemma 3.4 and Lemma 4.3 and the fact that a thickened pair of pants is simple homotopic to a pair of pants, with the chosen bases $\mathbf{h}^1_{(M,\boldsymbol{m})}$, $\mathbf{h}^2_M$, $\sqcup_j\mathbf{h}_{P_j}$ and $\sqcup_k\mathbf{h}_{D_k}$, we have

$$\mathrm{H}_2(M;\mathrm{Ad}_\rho)\cong\mathbb{C}^n,$$
$$\bigoplus_{j=1}^{p}\mathrm{H}_1(P_j;\mathrm{Ad}_\rho)\cong\mathbb{C}^{3p},$$
$$\bigoplus_{k=1}^{c+d}\mathrm{H}_1(D_k;\mathrm{Ad}_\rho)\cong\mathbb{C}^{3c+6d}$$

and

$$\mathrm{H}_1(M;\mathrm{Ad}_\rho)\cong\mathbb{C}^n.$$

In the rest of the proof, we will fix these isomorphisms and identify the linear maps ∂, δ and ϵ with the left multiplications of the corresponding matrices. In particular, ∂ corresponds to a $3p\times n$ matrix, δ corresponds to a $(3c+6d)\times 3p$ square matrix and ϵ corresponds to an $n\times(3c+6d)$ matrix.

For $C_3=\mathrm{H}_2(M;\mathrm{Ad}_\rho)$, we choose the lifting base $\widetilde{\mathbf{b}}_2$ to be $\mathbf{h}^2_M$. Then

$$[\widetilde{\mathbf{b}}_2;\mathbf{h}^2_M]=1. \tag{5.5}$$

For $C_2=\bigoplus_{j=1}^p\mathrm{H}_1(P_j;\mathrm{Ad}_\rho)$, we first order the vectors in $\widetilde{\mathbf{b}}_2=\mathbf{h}^2_M$ by $\{\mathbf{u}_1,\dots,\mathbf{u}_n\}$. Then $\mathbf{b}_2=\{\partial(\mathbf{u}_1),\dots,\partial(\mathbf{u}_n)\}$. We also order the vectors in $\sqcup_j\mathbf{h}_{P_j}$ by $\{\mathbf{v}_1,\dots,\mathbf{v}_{3p}\}$, and choose the lifting basis $\widetilde{\mathbf{b}}_1$ as follows. Since the sequence (5.3) is exact, δ has rank $3c+6d-n=3p-n$. Suppose a basis of the column space of δ consists of the columns $\{\mathbf{w}_{j_1},\dots,\mathbf{w}_{j_{3c+6d-n}}\}$ of δ, then we let $\widetilde{\mathbf{b}}_1=\{\mathbf{v}_{j_1},\dots,\mathbf{v}_{j_{3p-n}}\}$. Next we compute $\det[\mathbf{b}_2\sqcup\widetilde{\mathbf{b}}_1;\sqcup_j\mathbf{h}_{P_j}]$. Recall that there is a one-to-one correspondence between $\{\mathbf{u}_1,\dots,\mathbf{u}_n\}$ and the boundary components $\{T_1,\dots,T_k\}$ of M and a one-to-one correspondence between $\{\mathbf{v}_1,\dots,\mathbf{v}_{3p}\}$ and the boundary components of the disjoint union $\sqcup P_j$ of $\{P_j\}$. Then a diagram chasing shows that

$$\partial(\mathbf{u}_k)=\sum_{s=1}^{n_k}\pm\mathbf{v}_{i_s},$$

where n_k is the number of the boundary components of $\sqcup P_j$ intersecting T_k, $\mathbf{v}_{i_1},\dots,\mathbf{v}_{i_{n_k}}$ are the vectors corresponding to those boundary components of $\sqcup_j P_j$ and the signs $\pm$ are determined as follows. Fix an orientation of the longitude l_k of T_k, and suppose $P_{i_s}=D_r\cap D_t$ and D_r comes immediately before D_t along l_k in the chosen orientation. Then the sign in front of $\mathbf{v}_{i_s}$ is $+$ if $r>t$, and is $-$ if otherwise. Since each boundary component of $\sqcup_j P_j$ intersects exactly one boundary component of M, each row of the $n\times 3p$ matrix ∂ has exactly one nonzero entry, which equals either 1 or -1. Therefore, rows $j_1,\dots,j_{3p-n}$ of the matrix $\mathbf{b}_2\sqcup\widetilde{\mathbf{b}}_1$ have exactly two nonzero entries, one from $\mathbf{b}_2$ and one from $\widetilde{\mathbf{b}}_1$; and the other rows of $\mathbf{b}_2\sqcup\widetilde{\mathbf{b}}_1$ have exactly one nonzero entry. Let M be the $(3p-n)\times(3p-n)$ matrix consisting of the rows $j_1,\dots,j_{3p-n}$ of the columns $\mathbf{v}_{j_1},\dots,\mathbf{v}_{j_{3p-n}}$ of $\mathbf{b}_2\sqcup\widetilde{\mathbf{b}}_1$, and let N be the $n\times n$ matrix obtained from $\mathbf{b}_2\sqcup\widetilde{\mathbf{b}}_1$ by removing those rows and columns. Then each row of M and N contains exactly one nonzero entry, which equals 1 or -1, hence $\det M=\pm1$, $\det N=\pm1$ and $\det[\mathbf{b}_2\sqcup\widetilde{\mathbf{b}}_1]=\pm\det M\cdot\det N=\pm1$. Therefore,

$$[\mathbf{b}_2\sqcup\widetilde{\mathbf{b}}_1;\sqcup_j\mathbf{h}_{P_j}]=\pm1. \tag{5.6}$$

For $C_1=\bigoplus_{k=1}^{c+d}\mathrm{H}_1(D_k;\mathrm{Ad}_\rho)$, we have $\mathbf{b}_1=\{\delta(\mathbf{v}_{j_1}),\dots,\delta(\mathbf{v}_{j_{3p-n}})\}=\{\mathbf{w}_{j_1},\dots,\mathbf{w}_{j_{3c+6d-n}}\}$. We choose the lifting basis $\widetilde{\mathbf{b}}_0$ as follows. Since each P_j is adjacent to two of $\{D_1,\dots,D_{c+d}\}$ without redundancy and each edge of D_k connects two of $\{P_1,\dots,P_p\}$ without redundancy, by (5.2) each row of δ has exactly two nonzero entries each of which equals 1 or -1, and each column of δ has exactly two nonzero entries, one equals 1 and the other equals -1. For $t\notin\{j_1,\dots,j_{3c+6d-n}\}$, let $\mathbf{x}_t\in\mathbb{C}^{3c+6d}$ be the vector obtained from the column $\mathbf{w}_t$ of δ by replacing the entry -1 by 0. Then we let $\widetilde{\mathbf{b}}_0=\big\{\mathbf{x}_t\mid t\in\{1,\dots,3c+6d\}\smallsetminus\{j_1,\dots,j_{3c+6d-n}\}\big\}$. Now we claim that $\{\mathbf{x}_t\}$ are linearly independent and $\epsilon(\mathbf{x}_t)\neq0$ for each t so that $\mathbf{b}_1\sqcup\widetilde{\mathbf{b}}_0$ form a basis of C_1. Indeed, since each $\mathbf{x}_t$ contains only one nonzero component, to prove the linear independence it suffices to prove that no two nonzero entries of $\{\mathbf{x}_t\}$ are in the same row. Suppose otherwise that $\mathbf{x}_{t_1}$ and $\mathbf{x}_{t_2}$ have nonzero components in row k, then due to the fact that each row of δ has only two nonzero entries, the k-th component of all the columns $\mathbf{w}_{j_1},\dots,\mathbf{w}_{j_{3c+6d-n}}$ are 0. This contradicts the fact that $\{\mathbf{w}_{j_1},\dots,\mathbf{w}_{j_{3c+6d-n}}\}$ is a basis of the column space of δ since $\mathbf{w}_{t_1}$ and $\mathbf{w}_{t_2}$ have the k-th component equal to 1 and neither of them can be written as a linear combination of $\{\mathbf{w}_{j_1}, \dots, \mathbf{w}_{j_{3c+6d-n}}\}$. Also, since each edge of D_k belongs to exactly one boundary component of M, by (5.1) $\epsilon(\mathbf{x}_t)$ has exactly one nonzero component which equals 1, hence is nonzero. This finishes the proof of the claim. Next, we compute $\det[\mathbf{b}_1 \sqcup \widetilde{\mathbf{b}}_0]$. We observe that the matrix $[\mathbf{b}_1 \sqcup \widetilde{\mathbf{b}}_0]$ satisfies the following three properties:

(I) It is nonsingular.
(II) Each column has either exactly one nonzero component which equals ± 1; or has exactly two nonzero components, one equals 1 and the other equals -1.
(III) There is at least one column containing exactly one nonzero component.

We let $t_1, \dots, t_n$ be the rows where some $\mathbf{x}_t$ has nonzero components. Let M_1 be the $n \times n$ matrix consisting of the rows $t_1, \dots, t_n$ of the vectors $\{\mathbf{x}_t\}$, and let N_1 be the $(3c+6d-n) \times (3c+6d-n)$ matrix obtained from $\mathbf{b}_1$ by removing those rows. Since each column of M_1 contains exactly one 1 and no two $1'$s are in the same row, $\det M_1 = \pm 1$. As a consequence, we have $\det[\mathbf{b}_1 \sqcup \widetilde{\mathbf{b}}_0] = \pm \det M_1 \cdot \det N_1 = \pm \det N_1$. We claim that N_1 also satisfies the properties (I), (II) and (II). Indeed, (I) comes from the equality right above and (II) comes from the construction of N_1. For (III), suppose otherwise that all the columns of N_1 have one 1 and one -1, then all rows of N_1 add up to zero and N_1 is singular, which contradicts (I). Therefore, we can collect all the columns of N_1 containing only one nonzero components, and let M_2 be the square matrix consisting of the rows that contain those nonzero components, and let N_2 be the square matrix consisting of the other columns with those rows removed. Then $\det N_1 = \det M_2 \cdot \det N_2$. Since $\det N_1 \neq 0$, we have $\det M_2 \neq 0$. This implies that no two nonzero components of M_2 are in the same row. Together with the fact that all the columns of M_2 have only one nonzero entry ± 1, we have $\det M_2 = \pm 1$. This implies that $\det N_1 = \pm \det N_2$. By the same argument, we have that N_2 satisfies properties (I), (II) and (III), and we can recursively construct smaller square matrices $M_3, N_3, \dots, M_k, N_k, \dots$ that M_k consists of the rows containing those nonzero entries of the columns of N_{k-1} containing exactly one nonzero entry and N_k consists of the other columns of N_{k-1} with those rows removed, so that $\det M_k = \pm 1$, $\det N_{k-1} = \pm \det M_k \cdot \det N_k = \pm \det N_k$ and N_k satisfies (I), (II) and (III). This algorithm stops at some k when all columns of N_k contain exactly one nonzero entry ± 1, and we have $\det[\mathbf{b}_1 \sqcup \widetilde{\mathbf{b}}_0] = \pm \det N_1 = \dots = \pm \det N_k = \pm 1$. Therefore,

$$[\mathbf{b}_1 \sqcup \widetilde{\mathbf{b}}_0; \sqcup_k \mathbf{h}_{P_k}] = \pm 1. \tag{5.7}$$

For $C_0 = \mathrm{H}_1(M; \mathrm{Ad}_\rho)$, we have $\mathbf{b}_0 = \big\{\epsilon(\mathbf{x}_t) \mid t \in \{1, \dots, 3c+6d\} \setminus \{j_1, \dots, j_{3c+6d-n}\}\big\}$. Since $\mathbf{b}_1 \sqcup \widetilde{\mathbf{b}}_0$ form a basis of C_1 and $\mathbf{b}_1$ lies in the kernel of ϵ, $\mathbf{b}_0$ is a basis of C_0. In the previous paragraph, we show that each $\epsilon(\mathbf{x}_t)$ contains exactly nonzero entry 1, hence $\det[\mathbf{b}_0] = \pm 1$, which is the same as

$$[\mathbf{b}_0; \mathbf{h}^1_{(M,\boldsymbol{m})}] = \pm 1. \tag{5.8}$$

Therefore, by (5.5), (5.6), (5.7) and (5.8), we have

$$\mathrm{Tor}(\mathcal{H}; \mathbf{h}_{**}) = \frac{[\widetilde{\mathbf{b}}_2; \mathbf{h}^2_M] \cdot [\mathbf{b}_1 \sqcup \widetilde{\mathbf{b}}_0; \sqcup_k \mathbf{h}_{P_k}]}{[\mathbf{b}_2 \sqcup \widetilde{\mathbf{b}}_1; \sqcup_j \mathbf{h}_{P_j}] \cdot [\mathbf{b}_0; \mathbf{h}^1_{(M,\boldsymbol{m})}]} = \pm 1. \quad \square$$

6. Proof of Theorems 1.1, 1.4 and 1.6

Proof of Theorem 1.1. For (1), let M be a fundamental shadow link complement. Recall that M is the union of D-blocks by orientation reversing homeomorphisms between the 3-puncture spheres (which is homeomorphic to a pair of pants). For each pair of pants P and $i \in \{1,2,3\}$, let γ_i be the simple closed curve around the puncture p_i; and for each D-block D and $\{j,k\} \subset \{1,2,3,4\}$, let γ_{jk} be the simple closed curve around the edge e_{jk}. Then $(\gamma_1, \gamma_2, \gamma_3)$ is the restriction of the meridians $\boldsymbol{m}$ of M to P, and $(\gamma_{12}, \dots, \gamma_{34})$ is the restriction of $\boldsymbol{m}$ to D. Let $\rho : \pi_1(M) \to \mathrm{PSL}(2;\mathbb{C})$ be an $\boldsymbol{m}$-regular representation, and we will consider the following three cases:

- Case I. The restriction of $[\rho]$ to each pair of pants P_j is $\boldsymbol{\gamma}$-regular as defined in Definition 3.1, and to each D-block D_k is $\boldsymbol{\gamma}$-regular as defined in Definition 4.1.
- Case II. $[\rho]$ is not in Case I, and $\mathrm{Tr}\rho([m_i]) \neq \pm 2$ for all $i \in \{1,\dots,n\}$.
- Case III. Otherwise.

If $[\rho]$ is in Case I, then by Theorem 2.2, Propositions 3.2, 4.2 and 5.1, we have

$$\mathbb{T}_{(M,\boldsymbol{m})}([\rho]) = \mathrm{Tor}(M; \{\mathbf{h}^1_{(M,\boldsymbol{m})}, \mathbf{h}^2_M\}; \mathrm{Ad}_\rho) = \pm 2^{3d} \prod_{k=1}^{d} \sqrt{\det \mathbb{G}_k}.$$

This completes the proof of (1) for $[\rho]$ in Case I.

Next we show that each $[\rho]$ in Case II and Case III is in the closure of the set of characters in Case I in the classical (Hausdorff) topology, and the continuity of adjoint twisted Reidemeister torsion and the determinants of the Gram matrix functions will complete the proof.

For Case II, we first recall [22, Proposition 5.13] that, if ρ is $\boldsymbol{m}$-regular and $\mathrm{Tr}\rho([m_i]) \neq \pm 2$ for all $i \in \{1,\dots,n\}$, i.e., is in Case II, then the logarithmic holonomies $(u_1,\dots,u_n)$ form a local coordinates of $\mathrm{X}(M)$ near $[\rho]$. Since the restriction of $[\rho]$ to each P_j and D_k will possibly identity the traces of certain curves in $\boldsymbol{\gamma}$, we consider the following subsets of $\mathrm{X}(P_j)$ and $\mathrm{X}(D_k)$. For an equivalence relation $\sim$ on the index set $I_P = \{1,2,3\}$ with the set of equivalence classes $\overline{I_P}$, let

$$\mathrm{X}_{\overline{I_P}}(P) = \big\{[\rho] \in \mathrm{X}(P) \,\big|\, \text{for any lifting } \widetilde{\rho} \text{ of } \rho, \mathrm{Tr}\widetilde{\rho}([\gamma_a]) = \pm\mathrm{Tr}\widetilde{\rho}([\gamma_b]) \text{ for } a,b \in I_P \text{ with } a \sim b\big\};$$

and for an equivalence relation $\sim$ on the index set $I_D = \{12, \dots, 34\}$ with the set of equivalence classes $\overline{I_D}$, let

$$\begin{aligned}\mathrm{X}_{\overline{I_D}}(D) = \big\{[\rho] \in \mathrm{X}(D) \;\big|\; &\text{for any lifting } \widetilde{\rho} \text{ of } \rho, \mathrm{Tr}\widetilde{\rho}([\gamma_c]) \\ &= \pm \mathrm{Tr}\widetilde{\rho}([\gamma_d]) \text{ for } c, d \in I_D \text{ with } c \sim d\big\}.\end{aligned}$$

Then the restriction of $[\rho]$ to each P_j is in $\mathrm{X}_{\overline{I_P}}(P_j)$ for some $\overline{I_P}$; and the restriction of $[\rho]$ to each D_k is in $\mathrm{X}_{\overline{I_D}}(D_k)$ for some $\overline{I_D}$. Let

$$\mathrm{Z}_{\overline{I_P}}(P) = \mathrm{Z}(P) \cap \mathrm{X}_{\overline{I_P}}(P)$$

and let

$$\mathrm{Z}_{\overline{I_D}}(D) = \mathrm{Z}(D) \cap \mathrm{X}_{\overline{I_D}}(D).$$

Then by formulas (3.6), (3.10) and (3.11), for any quotient set $\overline{I_P}$, $\mathrm{Z}_{\overline{I_P}}(P)$ is dense in $\mathrm{X}_{\overline{I_P}}(P)$ in the classical topology; and by (4.4), (4.5), (4.6) and (4.7), for any quotient set $\overline{I_D}$, $\mathrm{Z}_{\overline{I_D}}(D)$ is dense in $\mathrm{X}_{\overline{I_D}}(D)$ in the classical topology. (Indeed, the numerators in the square root of the right hand side of both (3.11) and (4.7) have a constant term -1 which always stays under the identifications of the variables, hence the relevant analytic functions in the logarithmic holonomies never become the zero function.) As a consequence, any character in Case II is in the closure of the set of characters in Case I in the classical topology. This completes the proof of (1) for $[\rho]$ in Case II.

For a character $[\rho]$ in Case III, we show that it can be smoothly perturbed into Case I or Case II. Recall that the Killing form κ on $\mathfrak{sl}(2;\mathbb{C})$ defines a non-degenerate bilinear form $\langle\ ,\ \rangle : \mathrm{H}_1(M, \mathrm{Ad}_\rho) \times \mathrm{H}^1(M, \mathrm{Ad}_\rho) \to \mathbb{C}$, and the basis $\mathbf{h}^1_{(M,\boldsymbol{m})}$ of $\mathrm{H}_1(M, \mathrm{Ad}_\rho)$ gives an isomorphism between $\mathrm{H}_1(M, \mathrm{Ad}_\rho)$ and $\mathrm{H}^1(M, \mathrm{Ad}_\rho)$. For each $i \in \{1, \dots, n\}$, let $\mathbf{v}_i$ be the element in $\mathrm{H}^1(M, \mathrm{Ad}_\rho)$ dual to $\mathbf{I}_i \otimes [m_i]$ under this isomorphism, i.e., $\langle \mathbf{v}_i, \mathbf{I}_j \otimes [m_j] \rangle = \delta_{ij}$, the Kronecker symbol. Let $I \subset \{1, \dots, n\}$ be the subset of the indices i such that $\mathrm{Tr}\rho([m_i]) = \pm 2$, and let

$$\mathbf{v} = \sum_{i \in I} \mathbf{v}_i.$$

We consider $\mathbf{v}$ as a Zariski-tangent vector of $\mathrm{X}(M)$ at $[\rho]$. Since $[\rho]$ is $\boldsymbol{m}$-regular, it is a smooth point of $\mathrm{X}(M)$. As a consequence, $\mathbf{v}$ can be realized as the tangent vector of a deformation $[\rho_t]$, $t \in [0, \epsilon)$. Then $[\rho_t]$ is the desired perturbation of $[\rho]$, as for $t \neq 0$,

$$\mathrm{Tr}\rho_t([m_i]) \neq \mathrm{Tr}\rho([m_i]) = \pm 2$$

for $i \in I$, and

$$\mathrm{Tr}\rho_t([m_j]) = \mathrm{Tr}\rho([m_j]) \neq \pm 2$$

for $j \notin I$. This shows that any representation in Case III is in the closure of the set of the representations in Cases I and II in the classical topology, and completes the proof of (1) for $[\rho]$ in Case III.

(2) is a direct consequence of (1) and Theorem 2.8 (ii). $\square$

Proof of Theorem 1.4. Let $\boldsymbol{m}$ be the system of meridians of M. If the restriction $[\rho]$ of $[\rho_{\boldsymbol{\mu}}]$ to M is $\boldsymbol{m}$-regular, then the result follows directly from Theorem 1.1 and Theorem 2.8 (iii).

If $[\rho]$ is not $\boldsymbol{m}$-regular, then by Theorem 2.8 (i) that $\boldsymbol{m}$-regular characters are dense in the distinguished component of $\mathrm{X}(M)$, $[\rho]$ is a limit point of $\boldsymbol{m}$-regular characters. Then by the analyticity of the adjoint twisted Reidemeister torsion, the formula has a removable singularity at $[\rho]$ and hence can be evaluated by taking the limit of the values at the nearby $\boldsymbol{m}$-regular characters. $\square$

Proof of Theorem 1.6. From Section 2.5, we see that M is homeomorphic to a fundamental shadow link complement with the meridians (as of the fundamental shadow link complement) the preferred longitude $\boldsymbol{l}$. Let $\boldsymbol{m} = (m_1, \dots, m_n)$ be the simple closed curves around the edges, and let $(\gamma_1, \dots, \gamma_n)$ be the double of the edges. Then the holonomy representation ρ of the hyperbolic cone metric has the logarithmic holonomies $u_i = u_{\gamma_i} = 2l_i$ and $u_{m_i} = 2\mathbf{i}\theta_i$ for $i \in \{1, \dots, n\}$. Since a truncated hyperideal tetrahedron is determined and infinitesimally determined by its six edge lengths, ρ is $\boldsymbol{l}$-regular; and by [18, Theorem 1.2 (b)], ρ is determined and infinitesimally determined by its cone angles $(\theta_1, \dots, \theta_n)$, hence is $\boldsymbol{m}$-regular. Then (1) and (2) respectively follow from Theorem 1.1 (1) and (2), and (3) follows from Theorem 2.8 (iii). $\square$

References

[1] X. Bao, F. Bonahon, Hyperideal polyhedra in hyperbolic 3-space, Bull. Soc. Math. Fr. 130 (3) (2002) 457–491.
[2] G. Belletti, T. Yang, Discrete Fourier transform, quantum 6j-symbols and deeply truncated tetrahedra, preprint, arXiv:2009.03684.
[3] J. Bismut, F. Labourie, Symplectic Geometry and the Verlinde Formulas, Surveys in Differential Geometry: Differential Geometry Inspired by String Theory, Surv. Differ. Geom., vol. 5, Int. Press, Boston, MA, 1999, pp. 97–311.
[4] R. Bott, L. Tu, Differential Forms in Algebraic Topology, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, ISBN 0-387-90613-4, 1982, xiv+331 pp.
[5] F. Costantino, 6j-symbols, hyperbolic structures and the volume conjecture, Geom. Topol. 11 (2007) 1831–1854.
[6] F. Costantino, D. Thurston, 3-manifolds efficiently bound 4-manifolds, J. Topol. 1 (3) (2008) 703–745.
[7] M. Culler, P. Shalen, Varieties of group representations and splittings of 3-manifolds, Ann. Math. (2) 117 (1983) 109–146.
[8] T. Dimofte, S. Garoufalidis, The quantum content of the gluing equations, Geom. Topol. 17 (3) (2013) 1253–1315.
[9] E. Falbel, A. Guilloux, Dimension of character varieties for 3-manifolds, Proc. Am. Math. Soc. 145 (6) (2017) 2727–2737.
[10] R. Fricke, F. Klein, Vorlesungen über die Theorie der automorphen Funktionen, vol. 1, B.G. Teubner, Stuttgart, 1897.

[11] M. Fujii, Hyperbolic 3-manifolds with totally geodesic boundary, Osaka J. Math. 27 (3) (1990) 539–553.
[12] W. Goldman, Trace coordinates on Fricke spaces of some simple hyperbolic surfaces, in: Handbook of Teichmüller Theory, vol. II, in: IRMA Lect. Math. Theor. Phys., vol. 13, Eur. Math. Soc., Zürich, 2009, pp. 611–684.
[13] M. Heusener Michael, J. Porti, Holomorphic volume forms on representation varieties of surfaces with boundary, Ann. Henri Lebesgue 3 (2020) 341–380.
[14] C. Hodgson, S. Kerckhoff, Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery, J. Differ. Geom. 48 (1) (1998) 1–59.
[15] C. Hodgson, S. Kerckhoff, Universal bounds for hyperbolic Dehn surgery, Ann. Math. (2) 162 (1) (2005) 367–421.
[16] A. Kolpakov, J. Murakami, Volume of a doubly truncated hyperbolic tetrahedron, Aequ. Math. 85 (3) (2013) 449–463.
[17] F. Luo, A combinatorial curvature flow for compact 3-manifolds with boundary, Electron. Res. Announc. Am. Math. Soc. 11 (2005) 12–20.
[18] F. Luo, T. Yang, Volume and rigidity of hyperbolic polyhedral 3-manifolds, J. Topol. 11 (1) (2018) 1–29.
[19] J. Milnor, Whitehead torsion, Bull. Am. Math. Soc. 72 (1966) 358–426.
[20] W. Neumann, D. Zagier, Volumes of hyperbolic three-manifolds, Topology 24 (3) (1985) 307–332.
[21] J. Porti, Torsion de Reidemeister pour les variétés hyperboliques, Mem. Am. Math. Soc. 128 (612) (1997), x+139.
[22] J. Porti, Reidemeister torsion, hyperbolic three-manifolds, and character varieties, in: Handbook of Group Actions, vol. IV, in: Adv. Lect. Math. (ALM), vol. 41, Int. Press, Somerville, MA, 2018, pp. 447–507.
[23] W. Thurston, The Geometry and Topology of 3-Manifolds, Princeton Univ. Math. Dept., 1978, Available from, http://msri.org/publications/books/gt3m/.
[24] E. Witten, On quantum gauge theories in two dimensions, Commun. Math. Phys. 141 (1) (1991) 153–209.
[25] K.H. Wong, T. Yang, Asymptotic expansion of relative quantum invariants, arXiv:2103.15056.