# Young Students' Understandings of Function Graphs

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**Abstract:** This study involved 14-lesson classroom teaching experiments (CTE) with first and second grade students. We interviewed students at the beginning, middle, and end of the CTEs to assess their understandings of graphs of functional relationships. Here we report the levels of thinking that emerged from our analysis and the progress we observed with the second grade students. In summary, all students began the CTEs with at least an understanding of graphs in which they could coordinate quantities. By the end of the CTEs, one student reached a level of understanding in which she was able to extend beyond the representation at hand. Practical implications include guidance for developing instruction around learning to graph. Theoretical implications include contributing to answering open questions about students' developing understandings of mathematical representations.

## Algebraic Representations

Traditionally, elementary mathematics involves arithmetic, followed by an abrupt and largely superficial introduction to algebra in middle or early high school (Hiebert et al., 2005; Stigler et al., 1999; United States [US] Department of Education & National Center for Education Statistics, 1998a, 1998b, 1998c). Unfortunately, this approach does not fully prepare students to engage with more complex mathematical content in later grades. As a result, algebra serves as a detrimental "gatekeeper" course for access to advanced mathematics courses, particularly for students who are typically already marginalized from these courses (Kaput, 2008; Moses & Cobb, 2001).

In response, in the last couple of decades algebra has been identified as a central concern in mathematics education. It has been reconceptualized as a longitudinal strand of thinking across grades K–12 instead of as an isolated course for middle or high school (e.g., National Council of Teachers of Mathematics, 2000, 2006; National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010). Now it is argued that students need long-term, sustained algebra experiences beginning in Kindergarten, that take advantage of their natural intuitions about mathematical relationships, representations, and structure and build these ideas over time towards more sophisticated and formalized ways of mathematical thinking and representations. While progress has been made in this area, particularly with students in Grades 3-5 (e.g., Blanton et al., 2019), we lack a sufficient research base in the lower elementary grades (i.e., Grades K-2), especially around how students in these first grades of elementary school represent algebra ideas.

Recent work has shown that algebraic representations, such as variable notation (e.g., Blanton et al., 2017; Brizuela, Blanton, Gardiner et al., 2015; Brizuela, Blanton, Sawrey et al., 2015; Dougherty, 2008, 2010) and tables (Brizuela et al., 2021), are within the reach of young children and support them in engaging with algebraic reasoning. Yet, in the US, the use of graphs as a way to represent functional relationships is delayed until middle or high school grades. We argue this approach suffers similar pitfalls of the "algebra later" approach and aim to address this by studying young students' understandings of graphs. Specifically, using classroom teaching experiments (CTEs) and interviews we investigate the role of graphs in learning to represent and reason about functional relationships. Our study answers the following research question: *What are early elementary students' understandings of Cartesian graphs?*

Levels of understandings of graphs have been identified with late elementary school and middle school students (ages 10-14) in terms of bar graphs (e.g., García-Milá et al., 2014) and statistical graphs more broadly (e.g., Friel et al., 2001). Specifically, researchers have investigated students' use of tables versus raw data in the construction of bar graphs (e.g., García-Milá et al., 2014) and levels of understanding when interpreting bar graphs (e.g., Martí et al., 2010). However, in our review of the literature we have not identified prior studies on young students' use of Cartesian graphs to reason about functional relationships.

A parallel and closely related line of work has focused on students' use of tables to reason about functional relationships. For example, Brizuela et al. (2021) focus on Max, a five year old student who uses tables to organize data, which the authors refer to as *looking at* the table, and to reason about a functional relationship, which the authors refer to as *looking through* the table, using Kaput et al.'s (2008) framework. What is most surprising in Brizuela et al.'s (2021) study is the level of sophistication of Max's mathematical reasoning. It seems the table enables Max to see aspects of the functional relationship that otherwise would not be apparent to him. A key takeaway from this research and the aforementioned research on graphs is how these representations are used as tools which enabled students to reason in ways that they otherwise would not have been able to do (Hiebert, 1997), a fundamental motive in researching representations among young students.

# Method

## Participants

We conducted CTEs in Grade 1 and Grade 2 at an elementary school in the Northeastern US. We taught 14 lessons, eight of which were focused on graphing, in each of these grades and carried out individual interviews with four students in each one of these grades. Lessons were taught by a teacher-researcher and were about 30-40 minutes. All lessons were video recorded and transcribed. Given space restrictions, here we report only on the Grade 2 student interviews.

## Classroom Teaching Experiments

We used the CTE methodology because it involves sustained interactions with the same students, coupled with qualitative analysis of the lessons and interviews (Cobb, 2000). This method provides opportunities to identify levels of students' understandings and to explore how to support them in progressing to more advanced understandings. Moreover, embedded in the instructional sequence were cycles of revisiting the same representations with a new context and function. These cycles provided opportunities to test, review, revise, and retest our design conjectures (Sandoval, 2014) about how students would reason about algebraic representations and how we could support them in advancing their reasoning. For example, within each context (e.g., sandwiches) we revisited the table and graph. The lesson sequence was designed based on lessons we had developed in prior studies. Once one cycle of the lessons was developed we received expert input on the content and sequencing of the lessons. We also piloted two cycles of lessons with a different group of first grade students prior to conducting the CTE at the research site.

The features of students' understandings that we present are intimately connected with the curricular progression we implemented (Clements & Sarama, 2014). That is, the progression that we present is dependent on the instruction described in Table 1. Moreover, we understand this progression not as linear, but rather as a potential sequence of levels of understanding, or pieces of knowledge (i.e., knowledge in pieces; DiSessa, 1993) that students will develop and might demonstrate while learning to graph. Students may demonstrate an understanding at one level and then revert to another level in a new context (Clements & Sarama, 2014).

| **Grade 1 & 2 Lesson Sequence** | | | |
|---|---|---|---|
| **Lesson #** | **Task and related quantities explored** | **Equation of function** | **Representation(s)** |
| | *Interview 1* | | |
| 1<br>2 | Dogs' Noses: The relationship between the number of dogs and the number of noses on the dogs (Blanton et al., 2015). | y = x | table<br>human number line |
| 3<br>4 | Dogs' Eyes: The relationship between the number of dogs and the number of eyes on the dogs (LEAP GrK). | y = 2x | table<br>human number line |
| | *Interview 2* | | |
| 5 | Dogs' Eyes: The relationship between the number of dogs and the number of eyes on the dogs (LEAP GrK). | y = 2x | paper graph |
| 6<br>7<br>8 | Sandwiches: The relationship between the number of sandwiches and the number of pieces of bread needed to make those sandwiches. | y = 2x | table<br>human number line<br>paper graph |
| 9<br>10<br>11 | Bikes: The relationship between the number of bikes and the number of wheels on those bikes. | y = 2x | table<br>paper graph 1<br>paper graph 2 |
| 12<br>13<br>14 (Gr 1) | String task: The relationship between the number of pieces of string and the number of cuts. | y = x + 1 | table and paper graph<br>paper graph<br>human graph |
| 14 (Gr 2) | Find your representation: a game in which students matched a written description of a function relationship with a table and graph. | y = x, y = 2x, y = x + 1 | tables and graphs |
| | *Interview 3* | | |

Table 1. Summary of lesson sequence used in the Grades 1 and 2 CTEs.

## Individual Interviews

We individually interviewed four students from each grade before, during, and after the intervention. Students were selected for interviews based on teacher recommendations on which students would be most willing to speak with a researcher and would represent a variety of mathematical abilities. Here we report on the four interviews that were conducted with Grade 2 students.

In the interviews students were asked to reason about the relationship between number of birds and bird wings. While they did not use formal algebraic notation (i.e., $y = 2x$), the functional relationship they were asked to represent was a doubling relationship. They were given pictures of birds and asked to construct a table and to graph the relationship. If students were unable to construct the graph we provided them with a pre-constructed graph and asked them questions about it.

## Data Analysis

To develop levels of students' understandings of graphs we began by identifying relevant literature on trajectories and progressions in related contexts (e.g., Brizuela et al., 2021; Gabucio et al., 2010; García-Milá et al., 2014; Martí, 2009; Martí et al., 2010, 2011). First we described ways of thinking about graphs that might occur in the context of graphing functional relationships and then iteratively refined these levels by reviewing our data with the new levels in mind, revising, reviewing, and revising, until we encountered no new levels or situations that could not be described with one of our existing levels. This process resulted in a description of a six level trajectory that could be applied to the student interviews (i.e., the unit of analysis). To analyze one interview we watched the video and simultaneously read the interview transcript. When we observed an instance that indicated the student was reasoning at a specific level we noted this. We noted the level, evidence of thinking at that level, and the timestamp. The result was a table of each interview that had been conducted and a list of the levels that were observed in that interview along with evidence (i.e., transcript excerpts and a description of what happened) of that level. The three researchers who conducted the coding were the same three researchers who developed the trajectory, so we did not conduct coder training.

To ensure our analysis was reliable we had one primary coder (the first author of this paper) analyze each interview and identify evidence of students' understandings. This included providing a description of what the student was doing or saying and a timestamp of that moment. One of two secondary coders then reviewed each interview to check the primary coder's analysis. Differences were resolved using third-party resolution (Syed & Nelson, 2015).

# Findings

In the following we report on each level and share examples of student reasoning that illustrate each level. All the examples are in reference to the relationship between number of birds and number of bird wings because this is the problem context that students reasoned about in the interviews.

At *Level 0: Pre-Graph*, students can engage with the graph but not in a way that is unique to this type of representation or indicates that they understand graphs as different from another type of representation. In other words, they do not (necessarily) see the representation as a graph specifically. A student might read the labels "the number of birds" or "birds" and/or "the number of birds wings" or "wings," but they do not indicate (e.g., by gesturing) that the axes represent those quantities; they simply read the labels of the graph. They might also or instead read the numbers on the axes without indicating that those numbers represent quantities (such as "3 bird wings"). At this level students cannot reason about the representation as a graph. To a student at this level, the graph is no different from a table or any other type of representation.

At *Level 1: Basic*, a student might do or say something that indicates they understand the quantities being represented. For example, they might add labels to axes or say, "this is the number of birds" and gesture along the $x$-axis to indicate the axis shows the number of birds and they understand the label is a quantity that varies. At this level, students can start to construct a graph by adding labels to the axes, but they are not yet able to add points. Within this level we observed *Level 1: Basic-emergent,* which occurred when a student only attended to one axis. For example, the student might say "these are the number of birds" without mentioning bird wings. We also observed advanced understandings when students understood the quantities and could connect them but not yet coordinate them and plot a point. For example, some students would draw rounded lines connecting two corresponding quantities on each axis (see Figure 1) but never plot a point. Figure 1 shows a second-grade student's graph in which he represented the number of birds and bird wings without plotting points at the intersection of the two quantities. The key difference between *Level 1* and *0* is that at *Level 1* students recognize the axes and quantities they represent. For instance, they understand that the $x$-axis shows 1, 2, 3, and so on birds.

At *Level 2: Coordinate quantities*, a student might read a point by saying, “this means two birds and four wings,” or something similar. A key indicator that a student is reasoning at this level is when a student plots a point. The student might also draw straight guidelines between points to show the location of a point. Within this level, we observed variation. Specifically, we observed *Level 2: Coordinate quantities-emergent* when a student only interpreted part of a point but did not relate the two quantities (e.g., “this point shows two birds”). They simply indicated that they understood how the graph might work or what it could represent. As shown in Figure 1, a student might coordinate multiple quantities, but in this case, they did not identify the point, so we did not code this as fully coordinating quantities (i.e., advanced).

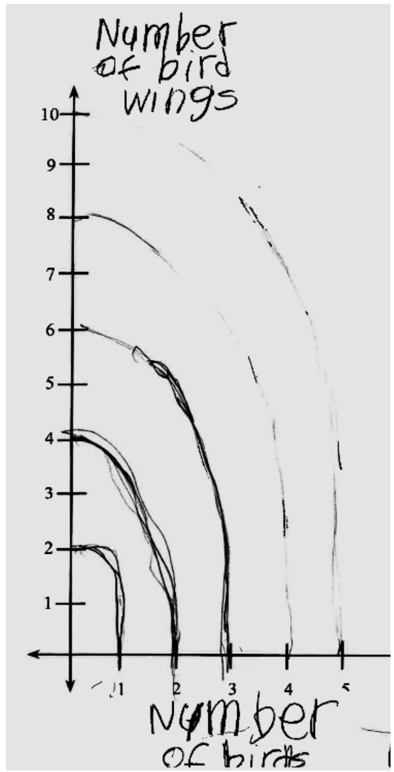


Figure 1. A second grader’s written work from his second interview.

At *Level 3: Infer relationships recursively*, students might relate two or more points on the graph. This would entail making a pre-functional observation, such as “after this point on the graph, there are always more than four wings,” or describing a recursive pattern; for example, they might describe the distance between two or more points as two. They may also interpret the points along a line as always going up by the same amount and say, “the bird wings go up by two every time.” This kind of thinking aligns with Blanton et al.’s (2015) level 3: recursive general for generalizing functional relationships. Within *Level 3: Infer relationships recursively-emergent*, we observed students describe patterns in terms of a list and say, “it goes 1, 2, 3, 4 and then 2, 4, 6, 8,” which aligns with Blanton et al.’s (2015) Level 2: recursive particular for generalizing functional relationships. A key difference between *Level 2* and *3* is that in *Level 3*, students relate two or more points and describe a relationship between the points.

At *Level 4: Infer relationships functionally* students can generalize a functional relationship through the graph. We describe this level as functional because it can be extended beyond consecutive points on the graph and describe the general relationship that is represented. That is, the distinction between level 3 and 4 is that in level 3, students focus recursively from point to point on a graph, whereas in level 4 they begin to reason functional by identifying and describing a relationship that can be generalized beyond consecutive points.

At *Level 5: Graph as object*, students understand the graph as something that extends beyond what is at hand. At this level students actually generalize about the representation, not just the relationship (*Level 4*). At an emergent *Level 5* understanding, a student might describe a far point on the graph by gesturing to indicate the location of a point that is not represented on the graph (e.g., 100 birds and 200 birds wings). A complete *Level 5* understanding involves seeing the graph as object, which aligns with Blanton et al.’s (2015) Level 8: Function as object and Sfard’s (1991) process-object model for concept formation. At this level, students might describe how the shape of a graph changes as a functional relationship changes, without having to replot points, treating the graph as an object. As we noted, they are able to generalize about the representation itself, not just the relationship represented.

In table 2 we share a summary of early elementary students' levels of understandings of graphs of functional relationships.

| **Level** | **Description** |
|---|---|
| **0: Pre-Graph** | The student does not understand the representation is a graph. |
| **1: Basic** | The student understands the graph as a mathematical representation, but they do not understand the parts of the graph, such as the meaning of the points, and thus cannot coordinate quantities or interpret points. |
| **2: Coordinate quantities** | The student understands what a point on the graph represents. They do not however understand how two or more points are related. |
| **3: Infer relationships recursively** | The student can relate two or more points on the graph by describing a recursive pattern. Students at this level cannot yet describe the general functional relationship represented in the graph. |
| **4: Infer relationships functionally** | The student can generalize a functional relationship in the graph. |
| **5: Graph as object** | At this level, the student understands the graph as something that extends beyond what is at hand. |

Table 2. Levels of early elementary students' understandings of graphs.

## Levels of understandings across the interviews

Next, we report on students' levels of understandings over the course of the CTE interviews. Table 3 shows that each Grade 2 student demonstrated at least a *Level 2* understanding and all but one student provided evidence of more sophisticated understandings in terms of our levels.

| **Tia** | | | | **Leo** | | | | **Nico** | | | | **Matt** | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **Interviews** | | | | **Interviews** | | | | **Interviews** | | | | **Interviews** | | |
| | **1** | **2** | **3** | | **1** | **2** | **3** | | **1** | **2** | **3** | | **1** | **2** | **3** |
| **5** | | | ■ | **5** | | | ■ | **5** | | | | **5** | | | |
| **4** | ■ | ■ | ■ | **4** | | | | **4** | | ■ | ■ | **4** | | | |
| **3** | ■ | ■ | ■ | **3** | ■ | | ■ | **3** | | ■ | | **3** | | | |
| **2** | ■ | ■ | ■ | **2** | ■ | ■ | ■ | **2** | ■ | ■ | ■ | **2** | ■ | ■ | ■ |
| **1** | ■ | ■ | ■ | **1** | ■ | ■ | ■ | **1** | ■ | ■ | ■ | **1** | ■ | ■ | ■ |
| **0** | | | | **0** | | | | **0** | | | | **0** | | | |

Table 3. Four second grade students' levels of understandings of graphs at the first, second, and third interview. A gray cell indicates that level was observed during that interview.

To provide context for our analysis we share brief details about Tia's third interview. We caveat the details about Tia's interview by stating that the questions the interviewer asked elicited a response that indicated certain understandings about the graph. However, this does not mean that this was Tia's highest understanding at this moment, rather it simply provides evidence around the different ways that students understand and talk about graphs.

Figure 2 shows the work Tia produced in her third interview. In the beginning of her interview, Tia added the correct labels to her graph (*Level 1*). Following that action, she demonstrated a *Level 2* understanding when she plotted several points and added guidelines connecting her points to the appropriate number on each axis.

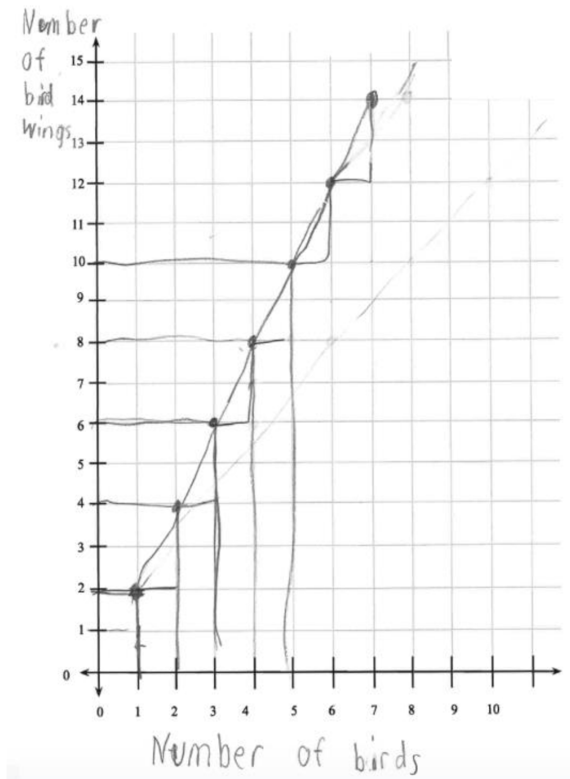


Figure 2. Tia's work from her third interview.

As she drew lines connecting (1, 2) to the axis, she described a recursive relationship (*Level 3*) when she said, "it's like a run rise line…because you run and then rise." She demonstrated with her pencil as she spoke. She continued to describe the relationship and began to describe a recursive relationship (*Level 3*). She said, "it gets higher by 2." She plotted the point (6, 12) and then justified it with the equation 6+6=12 (*Level 4*). When asked specifically about the relationship, she said, "we always have to double" (*Level 4*). Tia began to show that she was internalizing the graph (*Level 5*) when the interviewer covered up the numbers on the axes, and she accurately added a point to the graph. Similarly, she described the location of 100 birds and 200 wings, even though it was not on the graph at hand. She said, "it would be here, on that one, the white one," referring to a tile on the floor and gesturing to show that she "saw" the far point despite it not fitting on the graph.

## Discussion

In general, we saw less variation than we expected. One reason this might be the case is because currently our levels are quite broad. For instance, in Matt's interviews we observed an increase in the instances of evidence that demonstrated *Level 2* thinking. We think it would be useful to look specifically at what Matt was able to do at *Level 2* in his first, second and third interview. Comparing these instances of evidence might provide insight about variation within *Level 2*. We also note that we have data from Grade 1 students that will support us in this process. In general, the Grade 1 students did not demonstrate thinking as sophisticated thinking as that of the Grade 2 students. Thus, we anticipate these data will provide more insight about early sublevels, where there is not much prior research from which we can build.

Another important related point to our observation about variation is that these students started out with quite sophisticated understandings of graphs. Two of the students who were interviewed identified and described patterns in the graph (*Level 3* and *4*) prior to any instruction around graphs, and the two other students demonstrated a *Level 2* understanding prior to any instruction. We highlight this not only because it might explain why we did not observe as much variation as we may have expected but more importantly because it is remarkable what students were able to do with graphs before having any formal instruction around interpreting or constructing graphs. It is possible that students' interpretation of a table representing the same relationship that preceded their interpretation of the graph in the interview could have supported them in identifying and describing these relationships. Moreover, with exposure to just eight 30-minute lessons on graphs these students made progress, both in the number of times they demonstrated an understanding at a specific level but also as their understandings became more sophisticated and their levels increased. We argue this suggests that graphs and tables are within reach of early elementary students and potentially an important way for them to learn to represent functional relationships and other data.

In closing we highlight the value in this work and describe some ways that our work might be used by researchers and/or practitioners. Our research will result in levels of thinking demonstrated by students when using graphs to represent functional relationships. We imagine that these levels can serve as the foundation for designing instruction around teaching students to graph functional relationships. If, for instance, most students demonstrate the first step in learning to graph as interpreting the graph as a specific type of representation that is different from a table, then the first lesson in a sequence of lessons about graphing functions might be to simply learn to interpret a graph by reading the labels and understanding that those labels correspond to a number line representing quantities. The next step might be to learn to coordinate those quantities. Revealing these fine grain levels and sublevels is critical for designing instruction on this topic. Moreover, we hope that researchers who are investigating students' understandings about other types of representations will be able to build on our work in the same way that we leveraged Martí et al. (2009; 2010), García-Milá et al. (2014), Brizuela et al. (2021) and Blanton et al. (2017). We hope this future work challenges our findings and further develops our understandings about students' use of representations.

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