

ABSTRACT. We compute the  $v_1$ -periodic  $\mathbb{R}$ -motivic stable homotopy groups. The main tool is the effective slice spectral sequence. Along the way, we also analyze  $\mathbb{C}$ -motivic and  $\eta$ -periodic  $v_1$ -periodic homotopy from the same perspective.

## 1. INTRODUCTION

4 The computation of the stable homotopy groups of spheres is a difficult but  
 5 central problem of stable homotopy theory. There is much that we do not know  
 6 about stable homotopy. However, the  $v_1$ -periodic stable homotopy groups (also  
 7 known as the homotopy groups of the spectrum  $J$ ) are completely understood,  
 8 and they have interesting number-theoretic properties.

9 The goal of this article is to explore  $v_1$ -periodic stable homotopy in the  $\mathbb{R}$ -  
 10 motivic context. This choice of ground field represents a middle ground between  
 11 the well-understood  $\mathbb{C}$ -motivic situation and the much more difficult situation of  
 12 an arbitrary field, in which arithmetic necessarily enters into the picture.

13 From our perspective, the field  $\mathbb{R}$  introduces just one piece of arithmetic: the  
 14 failure of  $-1$  to have a square root. This leads to complications in  $\mathbb{R}$ -motivic  
 15 homotopical computations, but they can be managed with care and attention to  
 16 detail.

17 Classically,  $v_1$ -periodic homotopy is detected by the connective spectrum  $j^{\text{top}}$ ,  
 18 which is defined to be the fiber of a map  $\text{ko}^{\text{top}} \xrightarrow{\psi^3 - 1} \Sigma^4 \text{ksp}^{\text{top}}$ , where  $\text{ko}^{\text{top}}$  is the  
 19 connective real  $K$ -theory spectrum,  $\text{ksp}^{\text{top}}$  is the connective symplectic  $K$ -theory  
 20 spectrum, and  $\psi^3$  is an Adams operation. (The “top” superscripts indicate that  
 21 we are discussing the classical context here, rather than the motivic context.)

22 In fact,  $\text{ko}^{\text{top}}$  itself is the more natural target for the map  $\psi^3 - 1$ . However,  
 23 the fiber of  $\text{ko}^{\text{top}} \xrightarrow{\psi^3 - 1} \text{ko}^{\text{top}}$  has a minor defect. It has some additional homotopy  
 24 classes in stems  $-1$ ,  $0$ , and  $1$  that do not correspond to homotopy classes for the

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1 sphere spectrum. In other words, the map from  $S^0$  to this fiber is not surjective in  
2 homotopy. If we change the target of  $\psi^3 - 1$  from  $\text{ko}^{\text{top}}$  to its 3-connective cover  
3  $\Sigma^4 \text{ksp}^{\text{top}}$ , then this problem disappears, and the map from  $S^0$  to the fiber is onto  
4 in homotopy.

5 It is possible to mimic these constructions in motivic stable homotopy theory  
6 [BH20]. At the prime 2, one can define the motivic connective spectrum  $j$  to  
7 be the fiber of a map  $\text{ko} \xrightarrow{\psi^3 - 1} \Sigma^{4,2} \text{ksp}$ , where  $\text{ko}$  is the very effective connective  
8 Hermitian  $K$ -theory spectrum,  $\text{ksp}$  is defined in terms of very effective covers of  
9  $\text{ko}$ , and  $\psi^3$  is a motivic lift of an Adams operation.

10 However, from a computational perspective, this definition of  $j$  introduces  
11 apparently unnecessary complications. It is possible to compute the homotopy of  
12  $\mathbb{R}$ -motivic  $j$  using the techniques that appear later in this manuscript. However,  
13 the computation is slightly messy, involving some exceptional differentials and  
14 exceptional hidden extensions in low dimensions. In any case, the homotopy of  
15 the  $\mathbb{R}$ -motivic sphere does not surject onto the homotopy of  $\mathbb{R}$ -motivic  $j$ . In other  
16 words, the main rationale for using  $\text{ksp}$  in the first place does not apply in the  
17 motivic situation.

18 On the other hand, the computation of the homotopy of the  $\mathbb{R}$ -motivic fiber of  
19  $\text{ko} \xrightarrow{\psi^3 - 1} \text{ko}$  is much cleaner. Moreover, it tells us just as much about  $v_1$ -periodic  $\mathbb{R}$ -  
20 motivic homotopy as  $j$ . In other words, it has all of the computational advantages  
21 of  $j$ , while avoiding some unfortunate complications.

22 Consequently, in this manuscript, we will be solely concerned with the fiber of  
23  $\text{ko} \xrightarrow{\psi^3 - 1} \text{ko}$ . We use the notation  $L$  for this fiber in order to avoid confusion with  
24 the traditional meaning of  $j$ . The symbol  $L$  is meant to draw a connection to the  
25 classical  $K(1)$ -local sphere  $L_{K(1)} S^0$ , which is the fiber of  $\text{KO}^{\text{top}} \xrightarrow{\psi^3 - 1} \text{KO}^{\text{top}}$ . Our  
26 main result is a computation of the homotopy of  $L$ .

27 **Theorem 1.1.** *The homotopy of the  $\mathbb{R}$ -motivic spectrum  $L$  is depicted in Figures  
28 13–19 via the  $E_\infty$ -page of the effective spectral sequence, including all hidden  
29 extensions by  $\rho$ ,  $\mathbf{h}$ , and  $\eta$ .*

30 The proof of Theorem 1.1 appears in Section 5. See especially Theorem 5.12  
31 and Proposition 5.13.

32 Beware that the homotopy of the  $\mathbb{R}$ -motivic spheres does not surject onto the  
33 homotopy of  $\mathbb{R}$ -motivic  $L$ . It is possible that we may have not yet constructed  
34 the “correct” motivic version of the classical connective spectrum  $j^{\text{top}}$ . These  
35 considerations raise questions about vector bundles and the motivic Adams  
36 conjecture. We make no attempt to study these more geometric issues.<sup>1</sup>

37 We claim to compute the  $v_1$ -periodic  $\mathbb{R}$ -motivic stable homotopy groups, but  
38 this claim deserves some clarification. We do not use an intrinsic definition of  
39  $v_1$ -periodic  $\mathbb{R}$ -motivic homotopy, although such a definition could probably be

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<sup>1</sup>After the first version of this manuscript appeared, some of these issues have been addressed in [AERY23].

1 formulated in terms of the motivic  $K(1)$ -local sphere. See [BOQ23] for some  
2 progress on motivic  $K(1)$ -localization.

3 Rather, we merely compute the homotopy of  $L$ , and we observe that it detects  
4 large-scale structure in the stable homotopy of the  $\mathbb{R}$ -motivic sphere, which was  
5 described in a range in [BI22]. In other words, we have a practical description of  
6  $\mathbb{R}$ -motivic  $v_1$ -periodic homotopy, not a theoretical one.

7 The careful reader may object that our approach with effective spectral sequences  
8 is long-winded and unnecessarily complicated. In fact, the homotopy of  $L$  could  
9 be determined by direct analysis of the long exact sequence associated to the  
10 defining fiber sequence for  $L$ . However, there is a disadvantage in this direct  
11 approach. We find that the effective filtration is useful additional information  
12 about the homotopy of  $L$  that helps us understand the computation. The effective  
13 filtration is part of the “higher structure” of the homotopy of  $L$ . For example,  
14 some subtle phenomena, such as hidden multiplicative extensions, can only shift  
15 into higher effective filtration, so detailed knowledge of effective filtrations of  
16 homotopy classes can rule out possibilities that may otherwise be difficult to  
17 analyze. Another example occurs with Toda brackets, which may be computable  
18 using effective differentials. While we have no immediate uses for this higher  
19 structure, we know from experience that it inevitably becomes important in deeper  
20 homotopical analyses.

21 **1.1. Charts.** We provide a series of charts that display the effective spectral  
22 sequences for  $\text{ko}$  and  $L$ , as well as their  $\mathbb{C}$ -motivic counterparts. We consider these  
23 charts to be the central achievement of this manuscript. We encourage the reader  
24 to rely heavily on the charts. In a sense, they provide an illustrated guide to our  
25 computations.

26 Caution must be exercised in the comparison to [BI22] since the Adams fil-  
27 trations and effective filtrations are different. As in [BI22], our charts consider  
28 each coweight separately; we have found that this is a practical way of studying  
29  $\mathbb{R}$ -motivic homotopy groups. Periodicity by  $\tau^4$  (which is not a permanent cycle,  
30 but should be thought of as a periodicity operator in coweight 4) allows us to give  
31 a fairly compact depiction of the homotopy of  $L$  in coweights congruent to 0, 1,  
32 and 2 modulo 4; see Figures 13, 14, and 15.

33 The homotopy of  $L$  in coweights congruent to 3 modulo 4 is much more  
34 interesting but harder to describe. See Figures 17 and 18.

35 **1.2. Completions.** We are computing exclusively in the 2-complete context. This  
36 simplifies all questions surrounding convergence of spectral sequences. Also, the  
37 final computational 2-complete answers are easier to state than their 2-localized  
38 or integral counterparts.

39 We generally omit completions from our notation for brevity. For example, we  
40 write  $\mathbb{Z}$  for the 2-adic integers, and we write  $\text{KO}$  for the 2-completed  $\mathbb{R}$ -motivic  
41 Hermitian  $K$ -theory spectrum.

42 Section 2.3 discusses these topics in slightly more detail.

1 1.3. **Regarding the element 2.** When passing from the effective  $E_\infty$ -page to  
2 stable homotopy groups, one must choose homotopy elements that are represented  
3 by each element of the  $E_\infty$ -page. For the element 2 in the  $E_\infty$ -page, there is more  
4 than one choice in  $\pi_{0,0}$  because of the presence of elements in the  $E_\infty$ -page in  
5 higher effective filtration.

6 From the perspective of abelian groups, the element  $2 = 1 + 1$  is the obvious  
7 choice of homotopy element. However, there is another element  $h$ , also detected by  
8 2 in the effective spectral sequence, that turns out to be a much more convenient  
9 choice. The difference between  $h$  and 2 in homotopy is detected by the element  
10  $\rho h_1$  in higher filtration (to be discussed later). Experience has shown that the  
11 motivic stable homotopy groups are easier to describe in terms of  $h$  than in terms  
12 of 2. For example, we have the relations  $h\rho = 0$  and  $h\eta = 0$ , where  $\rho$  and  $\eta$  are  
13 homotopy elements detected by  $\rho$  and  $h_1$  respectively. However, neither  $2\rho$  nor  $2\eta$   
14 are zero. Because of the presence of elements in higher filtration, the homotopy  
15 elements  $\rho$  and  $\eta$  are not uniquely defined by the effective  $E_\infty$ -page elements  
16 that detect them. However, the mentioned relations hold for all choices. In this  
17 discussion, the exact definitions of  $\rho$  and  $\eta$  are less important than the observation  
18 that they satisfy nicer relations with respect to  $h$  than with respect to 2.

19 There are two additional reasons why the element  $h$  plays a central role. First,  
20 it corresponds to the hyperbolic plane under the isomorphism between motivic  $\pi_{0,0}$   
21 and the Grothendieck–Witt group of symmetric bilinear forms [Mor04]. Second,  
22 it plays the role of the zeroth Hopf map, in the sense that the Steenrod operations  
23 on its cofiber are simpler than the Steenrod operations for the cofiber of 2.

24 Consequently, instead of describing motivic stable homotopy groups as a module  
25 over the 2-adic integers  $\mathbb{Z}$  (i.e., in terms of the action of 2), it is easier to describe  
26 the homotopy groups in terms of the action of  $h$ .

27 1.4. **Future directions.** Our work points toward several open problems.

28 **Problem 1.2.** Compute motivic  $v_1$ -periodic homotopy over an arbitrary base  
29 field. Using [BH20], one can define  $L$  as the fiber of the map  $\psi^3 - 1$ , and it is  
30 conceivable that one could carry out the effective spectral sequence for  $L$  in this  
31 level of generality, similar to the kind of computations that appear in [RSØ19] and  
32 [RSØ21]. See Section 1.5 for further discussion. For prime fields of characteristic  
33 not two, some explicit computations were carried out in [KQ23].

34 **Problem 1.3.** Recompute the homotopy of  $L$  using the  $\mathbb{R}$ -motivic Adams spectral  
35 sequence. This would be a useful comparison object for further computations  
36 with the Adams spectral sequence for the  $\mathbb{R}$ -motivic sphere. The classical Adams  
37 spectral sequence for  $j^{\text{top}}$  was studied by Davis [Dav75], but it was only recently  
38 computed completely by Bruner and Rognes [BR22]. We are proposing a motivic  
39 analogue of their results.

1 **Problem 1.4.** Carry out the effective spectral sequence for the  $\mathbb{R}$ -motivic sphere  
2 in a range. These computations would serve as a useful companion to  $\mathbb{R}$ -motivic  
3 Adams spectral sequence computations [BI22]. The idea is to build on the  
4 techniques that are developed in this manuscript.

5 **Problem 1.5.** Compute the  $v_1$ -periodic  $C_2$ -equivariant stable homotopy groups.  
6 More precisely, carry out the  $C_2$ -effective spectral sequence for a  $C_2$ -equivariant  
7 version of  $L$ . The details will be similar to but more complicated than the  
8 computations in this manuscript. See [Kon23] for the effective approach to the  
9  $C_2$ -equivariant version of  $\text{ko}$ . Alternatively, one might compute the  $v_1$ -periodic  
10  $C_2$ -equivariant stable homotopy groups by periodicizing the  $v_1$ -periodic  $\mathbb{R}$ -motivic  
11 groups with respect to  $\tau$ , as considered by Behrens and Shah [BS20].

12 Recall that the  $\mathbb{R}$ -motivic and  $C_2$ -equivariant stable homotopy groups are  
13 isomorphic in a range [BG121]. Consequently, we anticipate that some version of  
14 the structure described in this manuscript appears in the  $C_2$ -equivariant context  
15 as well.

16 In the equivariant context, we mention Balderrama's [Bal21] computation of the  
17 homotopy groups of the Borel  $C_2$ -equivariant  $K(1)$ -local sphere, using techniques  
18 that are entirely different from ours. Roughly speaking, Balderrama computes  
19 the  $\tau^4 v_1^4$ -periodicization of our result. The effective  $E_\infty$  charts in Figures 13–19  
20 possess an obvious regularity every 8 stems, and Balderrama's computation sees  
21 that regular pattern.

22 **Problem 1.6.** Study  $K(1)$ -localization in the motivic context, which ought to be  
23 something like localization with respect to  $KGL/2$ .<sup>2</sup> Compute  $K(1)$ -local motivic  
24 homotopy. This would provide an intrinsic definition of  $v_1$ -periodic homotopy that  
25 would improve upon the practical computational perspective of this manuscript.

26 A guide to the motivic situation could lie in the work of Balderrama [Bal21]  
27 and Carrick [Car22] on equivariant localizations.

28 **1.5. Towards  $v_1$ -periodic homotopy over general base fields.** Our explicit  
29 computations point the way towards a complete computation of the  $v_1$ -periodic  
30 motivic stable homotopy groups over arbitrary fields. The situation here is  
31 analogous to the  $\eta$ -periodic  $\mathbb{R}$ -motivic computations of [GI16], which foreshadowed  
32 the more general  $\eta$ -periodic computations of [Wil18], [OR20], and [BH20].

33 **Problem 1.7.** Let  $k$  be an arbitrary field of characteristic different from 2.  
34 Let  $GW(k)$  be the Grothendieck–Witt ring of symmetric bilinear forms over  $k$ .  
35 Describe the 2-primary homotopy groups of the  $k$ -motivic spectrum  $L$  in terms  
36 of the cokernels and kernels of multiplication by various powers of 2 and of  $\mathbf{h}$  on  
37  $GW(k)$ .

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<sup>2</sup>After the first version of this manuscript appeared, some progress has occurred in [BOQ23].

1 Problem 1.7 is stated only in terms of 2-primary computations because that  
2 is the most interesting part. We expect that the generalization to odd primes is  
3 straightforward.

4 The exact powers of 2 and  $h$  that are required in Problem 1.7 depend not only  
5 on the coweight but also on the stem. Figures 17 and 18 show that  $2^{v(j)+3}$  is the  
6 relevant power of 2 in most stems in coweight  $4j - 1$ . Here  $v(j)$  is the 2-adic  
7 valuation of  $j$ , i.e., largest number  $v$  such that  $2^v$  divides  $j$ . In coweight  $4j - 1$   
8 and stem  $4i - 1$ , we see larger powers of 2, as well as powers of  $h$ .

9 Similar observations apply to the kernels that contribute to coweight  $4i$ .

10 **1.6. Outline.** Section 2 contains some background information that we will need  
11 to get started on our computations. We briefly discuss convergence of the effective  
12 spectral sequences that we will use. We recall some results of Bachmann–Hopkins  
13 [BH20] about motivic Adams operations and of Ananyevskiy–Röndigs–Østvær  
14 [ARØ20] about the slices of  $ko$ .

15 In Section 2, we have taken some care to eliminate details that we do not use.  
16 In other words, Section 2 describes the minimal hypotheses necessary in order to  
17 carry out our computations.

18 Section 3 considers  $\mathbb{C}$ -motivic computations, which play two roles in our work.  
19 First, they serve as a warmup to the more intricate  $\mathbb{R}$ -motivic computations.  
20 Second, the comparison between  $\mathbb{R}$ -motivic and  $\mathbb{C}$ -motivic homotopy is a necessary  
21 ingredient for our computations. In this section, we describe the effective spectral  
22 sequence for  $ko^{\mathbb{C}}$ . This material is well-known, since it is the same (up to regrading)  
23 as the  $\mathbb{C}$ -motivic Adams–Novikov spectral sequence for  $ko^{\mathbb{C}}$ , which is nearly the  
24 same as the classical Adams–Novikov spectral sequence for  $ko^{\text{top}}$ . We then use  
25 the fiber sequence

$$L^{\mathbb{C}} \longrightarrow ko^{\mathbb{C}} \xrightarrow{\psi^3 - 1} ko^{\mathbb{C}}$$

26 in order to determine the  $E_1$ -page of the effective spectral sequence for  $L^{\mathbb{C}}$ .

27 We next completely analyze the effective spectral sequence for the  $\eta$ -period-  
28 icization  $L^{\mathbb{C}}[\eta^{-1}]$ . The  $\eta$ -periodic spectral sequence is significantly simpler than  
29 the unperiodicized spectral sequence. We note the close similarity between the  
30 homotopy of  $L^{\mathbb{C}}[\eta^{-1}]$  and the computations of Andrew–Miller [AM17].

31 The  $\eta$ -periodic effective differentials completely determine the unperiodicized  
32 effective differentials for  $L^{\mathbb{C}}$ . Finally, we determine hidden extensions in the  
33 effective  $E_{\infty}$ -page for  $L^{\mathbb{C}}$ .

34 Section 3 completely computes the homotopy of  $L^{\mathbb{C}}$ , but the effective spectral  
35 sequence is not necessarily the simplest way of obtaining the computation. Nev-  
36 ertheless, we have chosen this approach because of its relationship to our later  
37  $\mathbb{R}$ -motivic computations.

38 Section 4 analyzes the effective spectral sequence for  $\mathbb{R}$ -motivic  $ko$ , including  
39 all differentials and hidden extensions. The  $E_1$ -page is readily determined from  
40 the work of Ananyevskiy–Röndigs–Østvær [ARØ20] on the slices of  $ko$ . We draw  
41 particular attention to the formula

$$(1.1) \quad (\tau h_1)^2 = \tau^2 \cdot h_1^2 + \rho^2 \cdot v_1^2.$$

1 This formula has a major impact on the shape of the answers that we obtain. In  
2 a sense, our work merely draws algebraic conclusions from Equation (1.1) and  
3  $\eta$ -periodic information. The hidden extensions in the effective  $E_\infty$ -page for  $\text{ko}$  are  
4 easily determined by comparison to the  $\mathbb{C}$ -motivic case, using the relationship  
5 between  $\mathbb{C}$ -motivic and  $\mathbb{R}$ -motivic homotopy that is described in [BS20, Corollary  
6 1.9].

7 Our computation of the homotopy of  $\mathbb{R}$ -motivic  $\text{ko}$  is not original. See [Kon23]  
8 for a  $C_2$ -equivariant analogue of the effective spectral sequence for  $\text{ko}$ . The  $\mathbb{R}$ -  
9 motivic computation can be extracted from the  $C_2$ -equivariant computation by  
10 dropping the “negative cone” elements. Also, Hill [Hil11] computed the Adams  
11 spectral sequence for  $\text{ko}$ , although the  $\mathbb{R}$ -motivic spectrum  $\text{ko}$  had not yet been  
12 constructed at the time.

13 The next step, undertaken in Section 4.2, is to analyze the effect of  $\psi^3$  on the  
14 effective spectral sequence of  $\text{ko}$ . This follows from a straightforward comparison  
15 to the classical case, together with careful bookkeeping. In turn, this leads to  
16 a complete understanding of the effective  $E_1$ -page of  $L$ , which is described in  
17 Section 5.1. Again, this is mostly a matter of careful bookkeeping.

18 Section 5.2 completely analyzes the effective spectral sequence for  $\eta$ -periodic  
19  $L[\eta^{-1}]$ . This information is essentially already well-known, either from [GI16]  
20 or from Ormsby–Röndigs [OR20], although those references do not specifically  
21 mention  $L$ .

22 As in the  $\mathbb{C}$ -motivic situation of Section 3,  $\eta$ -periodic information yields ev-  
23 erything that we need to know about the unperiodic situation, including all  
24 multiplicative relations in the effective  $E_1$ -page for  $L$  (see Section 5.3) and all  
25 differentials (see Sections 5.4 and 5.5). We again emphasize the significance of  
26 Equation (1.1) in carrying out the details. Finally, Section 5.6 studies hidden  
27 extensions in the effective  $E_\infty$ -page for  $L$ . As for  $\text{ko}$ , these hidden extensions  
28 follow by comparison to the  $\mathbb{C}$ -motivic case.

29 **1.7. Notation.** We use the following notation conventions.

- 30 •  $v(n)$  is the 2-adic valuation of  $n$ , i.e., the largest integer  $v$  such that  $2^v$  divides  
31  $n$ .
- 32 • Except in Section 2, everything is implicitly 2-completed. For example,  $S$  is  
33 actually the 2-complete  $\mathbb{R}$ -motivic sphere spectrum. Similarly,  $\mathbb{Z}$  is the 2-adic  
34 integers.
- 35 •  $s_*(X)$  are the slices of a motivic spectrum  $X$ .
- 36 •  $E_r(X)$  is the  $E_r$ -page of the effective spectral sequence for a motivic spectrum  
37  $X$ .
- 38 • We find the effective slice filtration to be slightly inconvenient for our purposes.  
39 We prefer to use the “Adams–Novikov filtration”, which equals twice the  
40 effective filtration minus the stem.
- 41 • Coweight equals the stem minus the motivic weight.

- Elements in  $E_r(X)$  are tri-graded. We write  $E_r^{s,f,w}(X)$  to denote the part with topological dimension  $s$ , Adams–Novikov filtration  $f$ , and motivic weight  $w$ .
  - We use unadorned symbols for  $\mathbb{R}$ -motivic spectra. For example,  $\text{ko}$  is the very effective cover of the  $\mathbb{R}$ -motivic Hermitian  $K$ -theory spectrum.
  - $X^{\mathbb{C}}$  is the  $\mathbb{C}$ -motivic extension-of-scalars spectrum of an  $\mathbb{R}$ -motivic spectrum  $X$ .
  - $X^{\text{top}}$  is the Betti realization of an  $\mathbb{R}$ -motivic spectrum  $X$ .
  - $S$  is the  $\mathbb{R}$ -motivic sphere spectrum.
  - $\text{KO}$  is the  $\mathbb{R}$ -motivic spectrum that represents Hermitian  $K$ -theory (also known as  $\text{KQ}$ ).
  - $\text{ko}$  is the very effective connective cover of  $\text{KO}$ .
  - $HA$  is the  $\mathbb{R}$ -motivic Eilenberg–Mac Lane spectrum on the group  $A$ .
  - $\psi^3$  is an Adams operation. We use the same symbol in the  $\mathbb{R}$ -motivic,  $\mathbb{C}$ -motivic, and classical situations.
  - $L$  is the fiber of  $\text{ko} \xrightarrow{\psi^3-1} \text{ko}$ .
  - $\Sigma^{s,w} X$  is a (bigraded) suspension of a motivic spectrum  $X$ .
  - $\pi_{*,*}(X)$  are the bigraded stable homotopy groups of an  $\mathbb{R}$ -motivic or  $\mathbb{C}$ -motivic spectrum.
  - Recall that  $\varepsilon$  is the motivic homotopy class that is represented by the twist map  $S \wedge S \rightarrow S \wedge S$ , where  $S$  is the motivic sphere spectrum. Let  $\mathbf{h}$  be the element  $1 - \varepsilon$ , which corresponds to the hyperbolic plane under the isomorphism between  $\pi_{0,0}(S)$  and the Grothendieck–Witt ring  $GW(\mathbb{R})$  [Mor04].
  - The element  $\rho$  belongs to the  $\mathbb{R}$ -motivic homology of a point. It is the class represented by  $-1$  in the Milnor  $K$ -theory of  $\mathbb{R}$ . Since  $\rho$  survives all of the spectral sequences under consideration, we use the same symbol for the corresponding homotopy class. However, there is a choice of homotopy class represented by  $\rho$  because of the presence of elements in higher filtration. There is an inconsistency in the literature about this choice. Following [Bac18], we define  $\rho$  such that  $\varepsilon = \rho\eta - 1$ , or equivalently  $2 = \rho\eta + \mathbf{h}$ .
- We frequently use names for indecomposables that consist of more than one symbol. For example, Theorem 2.1 discusses the indecomposable element  $v_1^2$  of the effective  $E_1$ -page for  $\text{ko}^{\mathbb{C}}$ . These longer names are slightly more cumbersome. This is especially the case when we consider products. We will use expressions of the form  $x \cdot y$  for clarity.
- On the other hand, our names are particularly convenient because they reflect the origins of the elements in terms of the spectral sequences that we use. For example, consider the indecomposable element  $2v_1^2$  of the effective  $E_{\infty}$ -page for  $\text{ko}^{\mathbb{C}}$ , as discussed in Theorem 3.3 (see also Figure 2). This name reflects the

1 element's origin in the effective  $E_1$ -page. It also illuminates relations such as

$$2v_1^2 \cdot 2v_1^2 = 4 \cdot v_1^4$$

2 However, one must be careful about possible error terms in such formulas; see  
3 especially Equation (1.1).

4

## 2. BACKGROUND

5 In this section only, we write  $\text{ko}$  for the integral version of the very effective  
6 cover of the Hermitian  $K$ -theory spectrum, and we use the usual decorations to  
7 indicate localizations and completions of  $\text{ko}$ . In the rest of the manuscript,  $\text{ko}$  is  
8 assumed to be 2-completed.

9 **2.1. The effective slices of  $\text{ko}$ .** We recall the structure of the effective slices of  
10  $\text{ko}$ .

**Theorem 2.1** ([ARØ20, Theorem 17]). *The slices of  $\text{ko}$  are*

$$s_*(\text{ko}) = H\mathbb{Z}[h_1, v_1^2]/(2h_1),$$

11 where  $v_1^2$  and  $h_1$  have degrees  $(4, 0, 2)$  and  $(1, 1, 1)$  respectively.

12 We explain the expression in Theorem 2.1. Each monomial of degree  $(s, f, w)$   
13 contributes a summand of  $\Sigma^{s,w} HA$  in the  $(\frac{s+f}{2})$ th slice. Here  $HA$  is the motivic  
14 Eilenberg–Mac Lane spectrum associated to  $A$ . The abelian group  $A$  is  $\mathbb{F}_2$  when  
15 the monomial is 2-torsion, and is  $\mathbb{Z}$  when the monomial is torsion free. We list  
16 the first three slices as examples:

$$\begin{aligned} s_0(\text{ko}) &= H\mathbb{Z}\{1\}, \\ s_1(\text{ko}) &= \Sigma^{1,1} H\mathbb{F}_2\{h_1\}, \\ s_2(\text{ko}) &= \Sigma^{2,2} H\mathbb{F}_2\{h_1^2\} \vee \Sigma^{4,2} H\mathbb{Z}\{v_1^2\}. \end{aligned}$$

17 Beware that the multiplicative structure of  $s_*(\text{ko})$  is not completely captured  
18 by the notation in Theorem 2.1. The essential multiplicative relation is Equation  
19 (1.1), which follows immediately from the general formulas in [ARØ20].

**Remark 2.2.** The calculation of the slices of the motivic sphere spectrum, due  
to Röndigs, Spitzweck, and Østvær [RSØ19], is commonly expressed at the prime  
2 as

$$s_*(S) = H\mathbb{Z} \otimes \text{Ext}_{BP_*BP}^{*,*}(BP_*, BP_*).$$

20 Analogously, Theorem 2.1 says that

$$s_*(\text{ko}) = H\mathbb{Z} \otimes \text{Ext}_{BP_*BP}^{*,*}(BP_*, BP_*(\text{ko}^{\text{top}})).$$

21 However, we do not know of a general theorem relating the slices of a motivic  
22 spectrum with the Adams–Novikov  $E_2$ -page for its topological counterpart.

1 2.2. **The Adams operation  $\psi^3$  and the spectrum  $L$ .** Bachmann and Hopkins  
2 [BH20] constructed a motivic analogue of the classical Adams operation  $\psi^3$ . We  
3 summarize the results that we need.

4 **Theorem 2.3** ([BH20]). *There is a unital ring map  $\psi^3 : \mathrm{ko} \left[ \frac{1}{3} \right] \rightarrow \mathrm{ko} \left[ \frac{1}{3} \right]$  whose  
5 Betti realization is the classical Adams operation  $\psi^3$ .*

6 *Proof.* There is a unital ring map  $\psi^3 : \mathrm{KO} \left[ \frac{1}{3} \right] \rightarrow \mathrm{KO} \left[ \frac{1}{3} \right]$  [BH20, Theorem 3.1],  
7 which is an  $E_\infty$ -map. Its Betti realization is also an  $E_\infty$ -map whose action on  
8 the classical Bott element is multiplication by 81. These properties uniquely  
9 characterize the classical Adams operation.

10 Now apply very effective covers, and the result about  $\mathrm{ko}$  follows formally.  $\square$

11 The original result is more general in more than one sense. First, it works over  
12 general base schemes in which 2 is invertible, while we only use the construction  
13 over  $\mathbb{R}$ . Second, its values are computed more precisely than just compatibility  
14 with the classical values.

15 **Corollary 2.4.**

16 (1)  $\psi^3 : \pi_{*,*}(\mathrm{ko}_2^\wedge) \rightarrow \pi_{*,*}(\mathrm{ko}_2^\wedge)$  is a ring map.

17 (2) If  $x$  is in the image of the unit map  $\pi_{*,*}(S_2^\wedge) \rightarrow \pi_{*,*}(\mathrm{ko}_2^\wedge)$ , then  $\psi^3(x) = x$ .

18 (3) There is a commutative diagram

$$\begin{array}{ccc} \pi_{*,*}(\mathrm{ko}_2^\wedge) & \xrightarrow{\psi^3} & \pi_{*,*}(\mathrm{ko}_2^\wedge) \\ \downarrow & & \downarrow \\ \pi_*((\mathrm{ko}^{\mathrm{top}})_2^\wedge) & \xrightarrow{\psi^3} & \pi_*((\mathrm{ko}^{\mathrm{top}})_2^\wedge), \end{array}$$

19 where the vertical maps are Betti realization homomorphisms.

20 *Proof.* These are computational consequences of Theorem 2.3. Part (1) follows  
21 from the fact that  $\psi^3$  is a ring map. Part (2) follows from the fact that  $\psi^3$  is  
22 unital. Part (3) follows from the fact that the Betti realization of the motivic  
23 Adams operation is the classical Adams operation.  $\square$

24 **Remark 2.5.** Corollary 2.4 can also be stated in a localized sense rather than  
25 completed sense, but we will not need that.

26 **Definition 2.6.** Let  $L$  be the fiber of the map  $\mathrm{ko} \left[ \frac{1}{3} \right] \xrightarrow{\psi^3-1} \mathrm{ko} \left[ \frac{1}{3} \right]$ .

27 Note that our definition of  $L$  is already localized; we do not consider an integral  
28 version. Except for this section,  $L$  is assumed to be 2-completed.

1 The most important point for us is that there is a fiber sequence

$$L_2^\wedge \longrightarrow \mathrm{ko}_2^\wedge \xrightarrow{\psi^3-1} \mathrm{ko}_2^\wedge$$

2 of completed spectra since completion preserves fiber sequences.

3 **2.3. Convergence of the effective spectral sequence.** The *effective spectral*  
4 *sequence* for a motivic spectrum  $X$  denotes the spectral sequence associated to  
5 the effective slice filtration of  $X$ . We refer to [Lev13, RSØ19] for details on the  
6 construction and properties of this spectral sequence.

7 The effective slice filtration [Voe02] has truncations  $f^q(X)$  and quotients (i.e.,  
8 slices)  $s_q(X)$ . The  $E_1$ -page of the effective spectral sequence is  $\pi_{*,*}(s_*(X))$ . In  
9 good cases, it converges to the homotopy groups of a completion of  $X$ . We also  
10 use the very effective slice filtration [SØ12], but only to define  $\mathrm{ko}$ .

11 The slice functors do not necessarily commute with completions, i.e.,  $s_*(X)_2^\wedge$   
12 and  $s_*(X_2^\wedge)$  are not always equivalent. Consequently, we must carefully define the  
13 spectral sequences that we use to study completed spectra. On the other hand,  
14 the effective slices do interact nicely with localizations [Spi08, Corollary 4.6].

**Theorem 2.7.** *There are strongly convergent spectral sequences*

$$E_1^{s,f,w}(\mathrm{ko}) = \pi_{s,w} \left( s_{\frac{s+f}{2}}(\mathrm{ko})_2^\wedge \right) \implies \pi_{s,w}(\mathrm{ko}_2^\wedge)$$

and

$$E_1^{s,f,w}(L) = \pi_{s,w} \left( s_{\frac{s+f}{2}}(L)_2^\wedge \right) \implies \pi_{s,w}(L_2^\wedge)$$

15 with differentials  $d_r : E_r^{s,f,w} \rightarrow E_r^{s-1,f+2r+1,w}$ .

16 We remind the reader that our grading of the effective spectral sequence is  
17 different than the standard grading in the literature. Briefly,  $s$  represents the  
18 topological stem,  $f$  represents the Adams–Novikov filtration (not the effective  
19 filtration), and  $w$  represents the motivic weight. See Section 1.7 for more discussion.

20 *Proof.* We discuss the spectral sequence for  $\mathrm{ko}$  in detail; most of the argument for  
21  $L$  is the same.

22 Consider the effective slice tower

$$f^0(\mathrm{ko}) \leftarrow f^1(\mathrm{ko}) \leftarrow f^2(\mathrm{ko}) \leftarrow \dots$$

23 Now take the 2-completion of this tower to obtain

$$f^0(\mathrm{ko})_2^\wedge \leftarrow f^1(\mathrm{ko})_2^\wedge \leftarrow f^2(\mathrm{ko})_2^\wedge \leftarrow \dots$$

24 The resulting layers are the same as  $s_*(\mathrm{ko})_2^\wedge$  since completion respects cofiber  
25 sequences. Beware that this is not necessarily the same as the slice tower of  
26 the completion  $\mathrm{ko}_2^\wedge$ , since slices do not interact nicely with completions. The  
27 associated spectral sequence of this tower is the one described in the statement of  
28 the theorem.

It remains to determine the target of the completed spectral sequence. The limit  
of the uncompleted slice tower of  $\mathrm{ko}$  is equivalent to its  $\eta$ -completion [RSØ19],

[ARØ20], i.e.,

$$\text{holim } f^n(\text{ko}) \simeq \text{ko}_\eta^\wedge.$$

1 Completion respects limits, so the limit  $\text{holim}(f^n(\text{ko})_2^\wedge)$  of the completed slice  
2 tower is equivalent to  $(\text{ko}_\eta^\wedge)_2^\wedge$ , which is equivalent to  $\text{ko}_2^\wedge$  by [HKO11, Theorem 1].  
3 Consequently, the completed effective spectral sequence of  $\text{ko}$  converges to the  
4 homotopy of  $\text{ko}_2^\wedge$ , as desired.

5 Strong convergence follows from [Boa99, Theorem 7.1], which has a technical  
6 hypothesis involving derived  $E_\infty$ -pages. For  $\text{ko}$ , this technical hypothesis follows  
7 directly from the computations of Section 4. For  $L$ , the technical hypothesis  
8 follows directly from the computations in Sections 5.4 and 5.5.  $\square$

9 **Remark 2.8.** By construction, we have a fiber sequence

$$s_*(L)_2^\wedge \longrightarrow s_*(\text{ko})_2^\wedge \xrightarrow{\psi^3-1} s_*(\text{ko})_2^\wedge,$$

10 which yields a long exact sequence

$$\cdots \longrightarrow E_1^{s,f,w}(L) \longrightarrow E_1^{s,f,w}(\text{ko}) \xrightarrow{\psi^3-1} E_1^{s,f,w}(\text{ko}) \longrightarrow \cdots.$$

11 This long exact sequence will be our main tool for computing  $E_1(L)$  in Section  
12 5.1.

### 13 3. $\mathbb{C}$ -MOTIVIC COMPUTATIONS

14 In this section, we carry out a preliminary computation of the effective spectral  
15 sequences for  $\text{ko}^\mathbb{C}$  and  $L^\mathbb{C}$ . We also consider the  $\eta$ -periodic spectral sequences. We  
16 are primarily interested in  $\mathbb{R}$ -motivic computations, but we will need to compare  
17 our  $\mathbb{R}$ -motivic computations to their  $\mathbb{C}$ -motivic counterparts.

18 **3.1. The effective spectral sequence for  $\text{ko}^\mathbb{C}$ .** We review the effective spectral  
19 sequence for  $\text{ko}^\mathbb{C}$ .

20 **Proposition 3.1.** *The effective spectral sequence for  $\text{ko}^\mathbb{C}$  takes the form*

$$E_1(\text{ko}^\mathbb{C}) = \mathbb{Z}[\tau, h_1, v_1^2]/2h_1.$$

21 *Proof.* This follows from Theorem 2.1 by taking stable homotopy groups. There  
22 are no possible error terms to complicate the multiplicative structure.  $\square$

23 Table 1 lists the generators of  $E_1(\text{ko}^\mathbb{C})$ . Figure 1 depicts  $E_1(\text{ko}^\mathbb{C})$  in graphical  
24 form.

Table 1: Multiplicative generators for  $E_1(\text{ko}^\mathbb{C})$

coweight	$(s, f, w)$	$x$	$d_1(x)$	$\psi^3(x)$
0	$(1, 1, 1)$	$h_1$		$h_1$
1	$(0, 0, -1)$	$\tau$		$\tau$

Table 1: Multiplicative generators for  $E_1(\text{ko}^{\mathbb{C}})$

coweight	$(s, f, w)$	$x$	$d_1(x)$	$\psi^3(x)$
2	$(4, 0, 2)$	$v_1^2$	$\tau h_1^3$	$9v_1^2$

1 **Proposition 3.2.** *Table 1 gives the values of the effective  $d_1$  differential on the  
2 multiplicative generators of  $E_1(\text{ko}^{\mathbb{C}})$ .*

3 *Proof.* The  $\mathbb{C}$ -motivic effective spectral sequence is identical to the  $\mathbb{C}$ -motivic  
4 Adams–Novikov spectral sequence up to reindexing. This claim does not appear  
5 to be cleanly stated in the literature, but it is a computational consequence of the  
6 weight 0 result of [Lev15, Theorem 1]. Alternatively, there is only one pattern of  
7 effective differentials that computes the motivic stable homotopy groups of  $\text{ko}^{\mathbb{C}}$ ,  
8 which were previously described using the  $\mathbb{C}$ -motivic Adams spectral sequence  
9 [IS11].  $\square$

10 **Theorem 3.3.** *The  $E_{\infty}$ -page of the effective spectral sequence for  $\text{ko}^{\mathbb{C}}$  takes the  
11 form*

$$E_{\infty}(\text{ko}^{\mathbb{C}}) = \frac{\mathbb{Z}[\tau, h_1, 2v_1^2, v_1^4]}{2h_1, \tau h_1^3, (2v_1^2)^2 = 4 \cdot v_1^4}.$$

12 *Proof.* For degree reasons, there can be no higher differentials in the effective  
13 spectral sequence for  $\text{ko}^{\mathbb{C}}$ .  $\square$

14 Table 2 lists the multiplicative generators of  $E_{\infty}(\text{ko}^{\mathbb{C}})$ . Figure 2 depicts  $E_{\infty}(\text{ko}^{\mathbb{C}})$   
15 in graphical form.

Table 2: Multiplicative generators for  $E_{\infty}(\text{ko}^{\mathbb{C}})$

coweight	$(s, f, w)$	$x$	$\psi^3(x)$
0	$(1, 1, 1)$	$h_1$	$h_1$
1	$(0, 0, -1)$	$\tau$	$\tau$
2	$(4, 0, 2)$	$2v_1^2$	$9 \cdot 2v_1^2$
4	$(8, 0, 4)$	$v_1^4$	$81v_1^4$

16 **Remark 3.4.** There are no possible hidden extensions in  $E_{\infty}(\text{ko}^{\mathbb{C}})$  for degree  
17 reasons. Therefore, Theorem 3.3 describes  $\pi_{*,*}(\text{ko}^{\mathbb{C}})$  as a ring.

18 **3.2. The effective  $E_1$ -page for  $L^{\mathbb{C}}$ .** Our next goal is to describe the effective  
19  $E_1$ -page  $E_1(L^{\mathbb{C}})$ . First we must study the values of  $\psi^3$  on  $\text{ko}^{\mathbb{C}}$ .

1 **Lemma 3.5.** *The map  $E_\infty(\text{ko}^\mathbb{C}) \rightarrow E_\infty(\text{ko}^\mathbb{C})$  induced by  $\psi^3$  on effective  $E_\infty$ -pages*  
 2 *takes the values shown in Table 2.*

3 *Proof.* All values follow immediately by comparison along Betti realization to the  
 4 values of classical  $\psi^3$ .  $\square$

5 **Lemma 3.6.** *The map  $E_1(\text{ko}^\mathbb{C}) \rightarrow E_1(\text{ko}^\mathbb{C})$  induced by  $\psi^3$  on effective  $E_1$ -pages*  
 6 *takes the values shown in Table 1.*

7 *Proof.* The values of  $\psi^3$  on  $E_1(\text{ko}^\mathbb{C})$  are compatible with the values of  $\psi^3$  on  
 8  $E_\infty(\text{ko}^\mathbb{C})$ , as shown in Table 2 (see also Lemma 3.5). This immediately yields all  
 9 values.  $\square$

10 In order to describe  $E_1(L^\mathbb{C})$ , we need some elementary number theory.

11 **Definition 3.7.** Let  $v(n)$  be the 2-adic valuation of  $n$ , i.e., the exponent of the  
 12 largest power of 2 that divides  $n$ .

**Lemma 3.8.**

$$v(3^n - 1) = \begin{cases} 1 & \text{if } v(n) = 0 \\ 2 + v(n) & \text{if } v(n) > 0 \end{cases}$$

13 *Proof.* Let  $n = 2^a \cdot b$ , where  $b$  is an odd number, so  $v(n) = a$ . Then

$$3^n - 1 = (1 + 3^{2^a} + (3^{2^a})^2 + \cdots + (3^{2^a})^{b-1}) (3 - 1) \prod_{i=0}^{a-1} (1 + 3^{2^i}).$$

14 The first factor is odd, so it does not contribute to the 2-adic valuation. The  
 15 factor  $(1 + 3^{2^i})$  has valuation 1 if  $i > 0$ , and it has valuation 2 if  $i = 0$ .  $\square$

16 **Proposition 3.9.** *The chart in Figure 3 depicts the effective  $E_1$ -page of  $L^\mathbb{C}$ .*

17 *Proof.* The long exact sequence

$$\cdots \rightarrow E_1(L^\mathbb{C}) \rightarrow E_1(\text{ko}^\mathbb{C}) \xrightarrow{\psi^3-1} E_1(\text{ko}^\mathbb{C}) \rightarrow \cdots$$

18 induces a short exact sequence

$$0 \rightarrow \Sigma^{-1}C \rightarrow E_1(L^\mathbb{C}) \rightarrow K \rightarrow 0,$$

19 where  $C$  and  $K$  are the cokernel and kernel of  $E_1(\text{ko}^\mathbb{C}) \xrightarrow{\psi^3-1} E_1(\text{ko}^\mathbb{C})$  respectively.  
 20 The cokernel and kernel can be computed directly from the information given in  
 21 Table 1 (see also Lemma 3.6).

22 The kernel is additively generated by all multiples of  $h_1$  in  $E_1(\text{ko}^\mathbb{C})$ , together  
 23 with the elements  $\tau^k$  for  $k \geq 0$ .

24 The cokernel  $C$  is nearly the same as  $E_1(\text{ko}^\mathbb{C})$  itself. We must impose the  
 25 relations  $(3^{2k} - 1)v_1^{2k} = 0$  for all  $k > 0$ . Lemma 3.8 says that  $3^{2k} - 1$  equals

- 1  $2^{v(2k)+2} \cdot u$ , where  $u$  is an odd number, i.e., a unit in our 2-adic context. Therefore,  
2 the relation  $(3^{2k} - 1)v_1^{2k} = 0$  is equivalent to the relation  $2^{v(2k)+2}v_1^{2k} = 0$ .  $\square$

3 Table 3 lists some elements of the effective  $E_1$ -page of  $L^{\mathbb{C}}$ . In fact, these elements  
4 are multiplicative generators for  $E_1(L^{\mathbb{C}})$ . By inspection, all elements of  $E_1(L^{\mathbb{C}})$   
5 are of the form  $\tau^a h_1^b x$ , for some  $x$  in the table.

6 We use the same notation for elements of  $E_1(L^{\mathbb{C}})$  and their images in  $E_1(\text{ko}^{\mathbb{C}})$ .  
7 On the other hand, we define the elements  $\iota x$  of  $E_1(L^{\mathbb{C}})$  by the property that  
8 they are the image of  $x$  under the map  $\iota : \Sigma^{-1}E_1(\text{ko}) \rightarrow E_1(L)$ . For example, the  
9 element 1 of  $E_1(\text{ko})$  maps to  $\iota$ .

Table 3: Multiplicative generators for  $E_1(L^{\mathbb{C}})$ :  $k \geq 0$

coweight	$(s, f, w)$	generator
1	$(0, 0, -1)$	$\tau$
$2k$	$(4k + 1, 1, 2k + 1)$	$h_1 v_1^{2k}$
$2k - 1$	$(4k - 1, 1, 2k)$	$\iota v_1^{2k}$

10 **Remark 3.10.** Our choice of notation for elements of  $E_1(L^{\mathbb{C}})$  is helpful for the  
11 particular analysis at hand. The generators of  $E_1(L^{\mathbb{C}})$  also have traditional names  
12 from the perspective of the Adams–Novikov spectral sequence. Namely,  $h_1 v_1^{2k}$   
13 and  $\iota v_1^{2k}$  correspond to  $\alpha_{2k+1}$  and  $\alpha_{2k/v(8k)}$  respectively. However, the  $\alpha$ -family  
14 perspective is not so helpful for us.

15 **3.3. The effective spectral sequence of  $L^{\mathbb{C}}[\eta^{-1}]$ .** Next, we describe the effective  
16 spectral sequence of  $L^{\mathbb{C}}[\eta^{-1}]$ .

17 In the  $\eta$ -periodic context, the element  $h_1$  is a unit. Therefore, powers of  $h_1$   
18 are inconsequential for computational purposes. Consequently, we have removed  
19 these powers from all  $\eta$ -periodic formulas. The appropriate powers of  $h_1$  can be  
20 easily reconstructed from the degrees of elements (although this reconstruction is  
21 typically not necessary).

22 **Proposition 3.11.** *The effective  $E_1$ -page for  $L^{\mathbb{C}}[\eta^{-1}]$  is given by*

$$E_1(L^{\mathbb{C}}[\eta^{-1}]) = \mathbb{F}_2[h_1^{\pm 1}, \tau, v_1^2, \iota]/\iota^2.$$

23 *Proof.* The functors  $s_*$  commute with homotopy colimits [Spi08, Corollary 4.6].  
24 Therefore, we can just invert  $h_1$  in  $E_1(\text{ko}^{\mathbb{C}})$  to obtain

$$E_1(\text{ko}^{\mathbb{C}}[\eta^{-1}]) = \mathbb{F}_2[h_1^{\pm 1}, \tau, v_1^2].$$

25 See Proposition 3.1 (and Figure 1) for the description of  $E_1(\text{ko}^{\mathbb{C}})$ .

26 The map  $E_1(\text{ko}^{\mathbb{C}}[\eta^{-1}]) \xrightarrow{\psi^3 - 1} E_1(\text{ko}^{\mathbb{C}}[\eta^{-1}])$  is trivial because  $(\psi^3 - 1)(v_1^{2k})$  is a  
27 multiple of 2, as shown in Table 1 (see also Lemma 3.6). Therefore, the long exact

1 sequence

$$\cdots \longrightarrow E_1(L^{\mathbb{C}}[\eta^{-1}]) \longrightarrow E_1(\mathrm{ko}^{\mathbb{C}}[\eta^{-1}]) \xrightarrow{\psi^3-1} E_1(\mathrm{ko}^{\mathbb{C}}[\eta^{-1}]) \longrightarrow \cdots$$

2 implies that  $E_1(L^{\mathbb{C}}[\eta^{-1}])$  splits as

$$E_1(\mathrm{ko}^{\mathbb{C}}[\eta^{-1}]) \oplus \Sigma^{-1} E_1(\mathrm{ko}^{\mathbb{C}}[\eta^{-1}]).$$

3 This establishes the additive structure of  $E_1(L[\eta^{-1}])$ , as well as most of the  
4 multiplicative structure.

5 The relation  $\iota^2 = 0$  is immediate because there are no possible non-zero values  
6 for  $\iota^2$ .  $\square$

7 **Proposition 3.12.** *In the effective spectral sequence for  $L^{\mathbb{C}}[\eta^{-1}]$ , we have  $d_1(v_1^2) =$   
8  $\tau$ . The effective differentials are zero on all other multiplicative generators on all  
9 pages.*

10 *Proof.* The value of  $d_1(v_1^2)$  in  $E_1(L^{\mathbb{C}}[\eta^{-1}])$  follows by comparison of effective  
11 spectral sequences along the maps  $L^{\mathbb{C}} \rightarrow L^{\mathbb{C}}[\eta^{-1}]$  and  $L^{\mathbb{C}} \rightarrow \mathrm{ko}^{\mathbb{C}}$ . Table 1 (see  
12 also Proposition 3.2) gives the value of  $d_1(v_1^2)$  in  $E_1(\mathrm{ko}^{\mathbb{C}})$ .  $\square$

13 **Remark 3.13.** The effective spectral sequence for  $L^{\mathbb{C}}[\eta^{-1}]$  is very close to the  
14 effective spectral sequence for the  $\eta$ -periodic sphere  $S^{\mathbb{C}}[\eta^{-1}]$ . The effective spectral  
15 sequence for  $S^{\mathbb{C}}[\eta^{-1}]$  is the same (up to reindexing) as the motivic Adams–Novikov  
16 spectral sequence for  $S^{\mathbb{C}}[\eta^{-1}]$ . This motivic Adams–Novikov spectral sequence is  
17 analyzed in [AM17]. The element  $\iota$  is not present in  $E_1(S^{\mathbb{C}}[\eta^{-1}])$ , but its multiples  
18  $\iota(v_1^2)^k$  are present.

### 19 3.4. Effective differentials for $L^{\mathbb{C}}$ .

20 **Proposition 3.14.** *Table 4 gives the values of the effective  $d_1$  differentials on  
21 the multiplicative generators of  $E_1(L^{\mathbb{C}})$ . There are no higher differentials in the  
22 effective spectral sequence for  $L^{\mathbb{C}}$ .*

23 *Proof.* All of these differentials follow immediately from the effective  $d_1$  differentials  
24 for  $L^{\mathbb{C}}[\eta^{-1}]$ , which are determined by Proposition 3.12.

25 For degree reasons, there are no possible higher differentials.  $\square$

Table 4: Effective  $d_1$  differentials for  $L^{\mathbb{C}}$ :  $k \geq 0$

coweight	$(s, f, w)$	$x$	$d_1(x)$
1	$(0, 0, -1)$	$\tau$	
$4k$	$(8k+1, 1, 4k+1)$	$h_1 v_1^{4k}$	
$4k+2$	$(8k+5, 1, 4k+3)$	$h_1 v_1^{4k+2}$	$\tau h_1^3 \cdot h_1 v_1^{4k}$

Table 4: Effective  $d_1$  differentials for  $L^{\mathbb{C}}$ :  $k \geq 0$

coweight	$(s, f, w)$	$x$	$d_1(x)$
$4k - 1$	$(8k - 1, 1, 4k)$	$\iota v_1^{4k}$	
$4k + 1$	$(8k + 3, 1, 4k + 2)$	$\iota v_1^{4k+2}$	$\tau h_1^3 \cdot \iota v_1^{4k}$

1 **Theorem 3.15.** *The  $E_{\infty}$ -page of the effective spectral sequence for  $L^{\mathbb{C}}$  is depicted  
2 in Figure 4.*

3 *Proof.* Because there are no higher effective differentials for  $L^{\mathbb{C}}$ , we obtain the  
4 effective  $E_{\infty}$ -page immediately from the effective  $d_1$  differentials in Table 4 (see  
5 also Proposition 3.14).  $\square$

6 3.5. **Hidden extensions in  $E_{\infty}(L^{\mathbb{C}})$ .**

7 **Proposition 3.16.** *In the effective spectral sequence for  $L^{\mathbb{C}}$ , the elements  $h_1 v_1^{4k}$   
8 do not support hidden  $\mathbf{h}$  extensions for all  $k \geq 0$ .*

9 *Proof.* The elements  $h_1 v_1^{4k}$  detect elements in  $\pi_{*,*} L^{\mathbb{C}}$  that are in the image of  
10 the homotopy  $\pi_{*,*} S^{\mathbb{C}}$  of the  $\mathbb{C}$ -motivic sphere. In the  $\mathbb{C}$ -motivic sphere, these  
11  $v_1$ -periodic elements are annihilated by  $\mathbf{h}$ .  $\square$

12 **Remark 3.17.** The proof of Proposition 3.16 appeals to knowledge of the homotopy  
13 of the  $\mathbb{C}$ -motivic sphere. In fact, one can avoid this by use of Toda brackets  
14 in the homotopy of  $L^{\mathbb{C}}$ . Namely, in the homotopy of  $L^{\mathbb{C}}$ , the  $E_{\infty}$ -page element  
15  $h_1 v_1^{4k+4}$  detects an element in the bracket  $\langle \mathbf{h}^3 \sigma, \mathbf{h}, \alpha \rangle$ , where  $\alpha$  is detected by  $h_1 v_1^{4k}$   
16 and  $\sigma$  is detected by  $\iota v_1^4$ . By induction,

$$\langle \mathbf{h}^3 \sigma, \mathbf{h}, \alpha \rangle \mathbf{h} = \mathbf{h}^3 \cdot \sigma \langle \mathbf{h}, \alpha, \mathbf{h} \rangle = \mathbf{h}^3 \cdot \sigma \cdot \tau \eta \cdot \alpha = 0.$$

17 **Proposition 3.18.** *In the effective spectral sequence for  $L^{\mathbb{C}}$ , there are hidden  $\mathbf{h}$   
18 extensions from  $\iota v_1^{4k+2}$  to  $\tau h_1^2 \cdot h_1 v_1^{4k}$  for all  $k \geq 0$ .*

19 *Proof.* Recall that  $\tau \eta^2 = \langle \mathbf{h}, \eta, \mathbf{h} \rangle$  in the homotopy of the  $\mathbb{C}$ -motivic sphere [Isa19,  
20 Table 7.23]. If  $\alpha$  is a homotopy element of  $L^{\mathbb{C}}$  such that  $\mathbf{h}\alpha$  is zero, then

$$\alpha \cdot \tau \eta^2 = \alpha \langle \mathbf{h}, \eta, \mathbf{h} \rangle = \langle \alpha, \mathbf{h}, \eta \rangle \mathbf{h}.$$

21 In particular, let  $\alpha$  be detected by  $h_1 v_1^{4k}$ . Note that  $\mathbf{h}\alpha = 0$  by Proposition 3.16.  
22 Then  $\tau h_1^2 \cdot h_1 v_1^{4k}$  detects a homotopy element that is divisible by  $\mathbf{h}$ , so  $\tau h_1^2 \cdot h_1 v_1^{4k}$   
23 must be the target of a hidden  $\mathbf{h}$  extension. There is only one possible source for  
24 this extension.  $\square$

#### 4. THE EFFECTIVE SPECTRAL SEQUENCE FOR $k\Omega$

We now study the effective spectral sequence for  $\mathbb{R}$ -motivic  $ko$ .

**3 Proposition 4.1.** *The effective spectral sequence for  $ko$  takes the form*

$$E_1(\text{ko}) = \frac{\mathbb{Z}[\rho, \tau^2, h_1, \tau h_1, v_1^2]}{2\rho, 2h_1, 2 \cdot \tau h_1, (\tau h_1)^2 = \tau^2 \cdot h_1^2 + \rho^2 \cdot v_1^2}$$

4 *Proof.* The additive structure follows from Theorem 2.1 by taking stable homotopy  
 5 groups. We need that the homotopy groups of  $\mathbb{R}$ -motivic  $H\mathbb{Z}$  are

$$H\mathbb{Z}_{*,*} = \mathbb{Z}[\tau^2, \rho]/2\rho,$$

6 and the homotopy groups of  $\mathbb{R}$ -motivic  $H\mathbb{F}_2$  are

$$(H\mathbb{F}_2)_{*,*} = \mathbb{F}_2[\tau, \rho].$$

<sup>7</sup> The multiplicative structure is mostly also immediate from Theorem 2.1. As explained in [Kon23], our formula for  $(\tau h_1)^2$  is equivalent to the formula  $\eta^2 \xrightarrow{\delta} \sqrt{\alpha}$  given in [ARØ20, p. 1029].  $\square$

Table 5 lists the generators of  $E_1(ko)$ . Figure 5 depicts  $E_1(ko)$  in graphical form.

Table 5: Multiplicative generators for  $E_1(\text{ko})$

coweight	$(s, f, w)$	$x$	$d_1(x)$	$\psi^3(x)$	image in $E_1(\mathrm{ko}[\eta^{-1}])$
0	$(-1, 1, -1)$	$\rho$		$\rho$	$\rho$
0	$(1, 1, 1)$	$h_1$		$h_1$	1
1	$(1, 1, 0)$	$\tau h_1$		$\tau h_1$	$\tau \cdot h_1$
2	$(0, 0, -2)$	$\tau^2$	$\rho^2 \cdot \tau h_1$	$\tau^2$	$\tau^2 + \rho^2 \cdot v_1^2 \cdot h_1^{-2}$
2	$(4, 0, 2)$	$v_1^2$	$\tau h_1 \cdot h_1^2$	$9v_1^2$	$v_1^2$

**12 Proposition 4.2.** Table 5 gives the values of the effective  $d_1$  differential on the  
 13 multiplicative generators of  $E_1(k_0)$ .

**14** *Proof.* The value of  $d_1(\tau^2)$  follows from [ARØ20, Theorem 20] and  $\mathbb{R}$ -motivic  
**15** Steenrod algebra actions. Then the value of  $d_1(v_1^2)$  follows from Equation (1.1).

16 Alternatively, there is only one pattern of effective differentials that computes  
 17 the motivic stable homotopy groups of  $ko$ , which were previously computed with  
 18 the  $\mathbb{R}$ -motivic Adams spectral sequence [Hil11].  $\square$

19 The entire  $d_1$  differential in the effective spectral sequence for  $\text{ko}$  can easily be  
 20 deduced from Proposition 4.2 and the Leibniz rule.

1 **Theorem 4.3.** *The  $E_\infty$ -page of the effective spectral sequence for  $\text{ko}$  is depicted  
2 in Figures 6, 7, and 8.*

3 *Proof.* The Leibniz rule, together with the values in Table 5 (see also Proposition  
4 4.2), completely determines the effective  $d_1$  differential on  $E_1(\text{ko})$ . The  $E_2$ -page  
5 can then be determined directly. However, the computation is not entirely  
6 straightforward. Of particular note is the differential

$$d_1(\tau^2 \cdot \tau h_1 \cdot v_1^2) = \tau^4 \cdot h_1^4 + \rho^4 \cdot v_1^4,$$

7 which yields the relation

$$(4.1) \quad \tau^4 \cdot h_1^4 = \rho^4 \cdot v_1^4$$

8 in  $E_2(\text{ko})$ .

9 For degree reasons, there can be no higher differentials in the effective spectral  
10 sequence for  $\text{ko}$ .  $\square$

11 For legibility, Figures 6, 7, and 8 display  $E_\infty(\text{ko})$  in three different charts  
12 separated by coweight modulo 4. There is no chart for coweights 3 mod 4 because  
13  $E_\infty(\text{ko})$  is zero in those coweights.

14 Figure 9 illustrates part of the analysis of the  $d_1$  differentials and the determi-  
15 nation of  $E_2(\text{ko})$ ; it is meant to be representative, not thorough. The chart shows  
16 some of the elements in coweights 1 and 2 mod 4, together with the  $d_1$  differentials  
17 that relate these elements. In this chart, one can see that  $\tau^2 \cdot h_1^2 + \rho^2 \cdot v_1^2$  survives  
18 to  $E_2(\text{ko})$ . This element survives to  $E_\infty(\text{ko})$ . It is labelled  $(\tau h_1)^2$  in Figure 8, in  
19 accordance with Equation (1.1).

20 **Remark 4.4.** There is an alternative, slightly more structured, method for  
21 obtaining  $E_\infty(\text{ko})$ . One can filter  $E_1(\text{ko})$  by powers of  $\tau h_1$  and obtain a spectral  
22 sequence that converges to  $E_2(\text{ko})$ . In this spectral sequence, we have the relation  
23  $\tau^2 \cdot h_1^2 = \rho^2 \cdot v_1^2$ . There are differentials  $d_1(\tau^2) = \rho^2 \cdot \tau h_1$  and  $d_1(v_1^2) = h_1^2 \cdot \tau h_1$ .  
24 Then there is a higher differential  $d_3(\tau^2 \cdot v_1^2) = (\tau h_1)^3$ . None of this is essential to  
25 our study, but the interested reader may wish to carry out the details.

26

27 Table 6 lists the multiplicative generators of  $E_\infty(\text{ko})$ . It is possible to give a  
28 complete list of relations. However, the long list is not so helpful for understanding  
29 the structure of  $E_\infty(\text{ko})$ . The charts in Figures 6, 7, and 8 are more useful for  
30 this purpose.

Table 6: Multiplicative generators for  $E_\infty(\text{ko})$

coweight	$(s, f, w)$	$x$	$\psi^3(x)$
0	$(-1, 1, -1)$	$\rho$	$\rho$
0	$(1, 1, 1)$	$h_1$	$h_1$
1	$(1, 1, 0)$	$\tau h_1$	$\tau h_1$

Table 6: Multiplicative generators for  $E_\infty(\text{ko})$ 

coweight	$(s, f, w)$	$x$	$\psi^3(x)$
2	$(0, 0, -2)$	$2\tau^2$	$2\tau^2$
2	$(4, 0, 2)$	$2v_1^2$	$9 \cdot 2v_1^2$
4	$(0, 0, -4)$	$\tau^4$	$\tau^4$
4	$(4, 0, 0)$	$2\tau^2v_1^2$	$9 \cdot 2\tau^2v_1^2$
4	$(8, 0, 4)$	$v_1^4$	$81v_1^4$

1 **Proposition 4.5.** *Table 7 lists some hidden extensions by  $\rho$ ,  $\mathbf{h}$ , and  $\eta$  in the  
 2 effective spectral sequence for  $\text{ko}$ . All other hidden extensions by  $\rho$ ,  $\mathbf{h}$ , and  $\eta$  are  
 3  $v_1^4$ -multiples and  $\tau^4$ -multiples of these.*

4 *Proof.* Recall from [BS20, Corollary 1.9] that the homotopy of  $\text{ko}/\rho$  is isomorphic  
 5 to the homotopy of  $\text{ko}^{\mathbb{C}}$ . Therefore, we completely understand the homotopy of  
 6  $\text{ko}/\rho$  from Theorem 3.3 and Figure 2.

7 The hidden  $\rho$  extensions follow from inspection of the long exact sequence  
 8 associated to the cofiber sequence

$$\Sigma^{-1, -1}\text{ko} \xrightarrow{\rho} \text{ko} \rightarrow \text{ko}/\rho.$$

9 The map  $\text{ko} \rightarrow \text{ko}/\rho$  takes the elements  $\tau^4 \cdot h_1^3$  and  $(\tau h_1)^2 h_1$  to zero because there  
 10 are no possible targets in the homotopy of  $\text{ko}/\rho$ . Therefore, those two elements  
 11 must receive hidden  $\rho$  extensions, and there is only one possibility in both cases.  
 12 The relation  $\tau^4 \cdot h_1^4 = \rho^4 \cdot v_1^4$  (see Equation (4.1)) then implies that  $2\tau^2v_1^2$  also  
 13 supports an  $h_1$  extension.

14 The map  $\text{ko}/\rho \rightarrow \Sigma^{0, -1}\text{ko}$  takes  $\tau^3$  and  $\tau^3 h_1$  to  $2\tau^2$  and  $\rho(\tau h_1)^2$  respectively.  
 15 There is an  $h_1$  extension connecting  $\tau^3$  and  $\tau^3 h_1$  in  $\text{ko}/\rho$ , so there must be a  
 16 hidden  $\eta$  extension from  $2\tau^2$  to  $\rho(\tau h_1)^2$ .

17 The hidden  $\mathbf{h}$  extension on  $\tau h_1$  follows from the analogous hidden extension in  
 18 the homotopy groups of the  $\mathbb{R}$ -motivic sphere [DI17] [BI22], using the unit map  
 19  $S \rightarrow \text{ko}$ . Alternatively, this hidden extension is computed in [Hil11, Proposition  
 20 4.3] in the context of the  $\mathbb{R}$ -motivic Adams spectral sequence for  $\text{ko}$ .

21 Finally, multiply by  $\tau h_1$  to obtain the hidden  $\mathbf{h}$  extension on  $(\tau h_1)^2$ .

22 For degree reasons, there are no other possible hidden extensions to consider.  $\square$

 Table 7: Hidden extensions in  $E_\infty(\text{ko})$ 

coweight	source	type	target	$(s, f, w)$
2	$2v_1^2$	$\rho$	$(\tau h_1)^2 h_1$	$(3, 3, 1)$

Table 7: Hidden extensions in  $E_\infty(\text{ko})$ 

coweight	source	type	target	$(s, f, w)$
4	$2\tau^2 v_1^2$	$\rho$	$\tau^4 \cdot h_1^3$	(3, 3, -1)
4	$2\tau^2 v_1^2$	$\eta$	$\rho^3 \cdot v_1^4$	(5, 3, 1)
2	$2\tau^2$	$\eta$	$\rho(\tau h_1)^2$	(1, 3, -1)
1	$\tau h_1$	$\mathbf{h}$	$\rho \cdot \tau h_1 \cdot h_1$	(1, 3, 0)
2	$(\tau h_1)^2$	$\mathbf{h}$	$\rho(\tau h_1)^2 h_1$	(2, 4, 0)

1 **Remark 4.6.** We have completely analyzed the  $E_\infty$ -page of the effective spectral  
 2 sequence for  $\text{ko}$ , but this is not quite the same as completely describing the  
 3 homotopy of  $\text{ko}$ . In particular, one must choose an element of  $\pi_{*,*}\text{ko}$  that is  
 4 represented by each multiplicative generator of  $E_\infty(\text{ko})$  (see Table 6). In some  
 5 cases, there is more than one choice because of the presence of elements in  
 6 higher filtration in the  $E_\infty$ -page. The choices of  $\rho$ ,  $h_1$ ,  $\tau h_1$ , and  $\tau^4$  can be made  
 7 arbitrarily; the ring structure is unaffected by these choices. The elements  $2\tau^2$   
 8 and  $2v_1^2$  are already well-defined because there are no elements in higher filtration.  
 9 Finally, the choices of  $2\tau^2 v_1^2$  and  $v_1^4$  can then be uniquely specified by the relations  
 10  $\rho \cdot 2\tau^2 v_1^2 = \tau^4 \cdot h_1^3$  and  $\rho^4 \cdot v_1^4 = \tau^4 \cdot h_1^4$ .

11 4.1.  **$\eta$ -periodic  $\text{ko}$ .** Later we will need some information about the  $\eta$ -periodic  
 12 spectrum  $\text{ko}[\eta^{-1}]$ . As in Section 3.3, powers of  $h_1$  are inconsequential for compu-  
 13 tational purposes in the  $\eta$ -periodic context. Consequently, we have removed these  
 14 powers from all  $\eta$ -periodic formulas.

15 **Proposition 4.7.** *The effective  $E_1$ -page for  $\text{ko}$  is given by*

$$E_1(\text{ko}[\eta^{-1}]) = \mathbb{F}_2[h_1^{\pm 1}, \tau, \rho, v_1^2].$$

16 *Moreover, the periodicization map  $\text{ko} \rightarrow \text{ko}[\eta^{-1}]$  induces the map on effective  
 17  $E_1$ -pages whose values are given in Table 5.*

18 The first part of Proposition 4.7 was first proved in [ARØ20, Theorem 19],  
 19 although the notation is different.

20 *Proof.* The functors  $s_*$  commute with homotopy colimits [Spi08, Corollary 4.6].  
 21 Therefore, we can just invert  $h_1$  in the description of  $E_1(\text{ko})$  given in Proposition  
 22 4.1 (see also Figure 5).

23 After inverting  $h_1$ , the relation  $2h_1$  in  $E_1(\text{ko})$  implies that  $2 = 0$  in  $E_1(\text{ko}[\eta^{-1}])$ .  
 24 This gives that

$$E_1(\text{ko}[\eta^{-1}]) = \frac{\mathbb{F}_2[h_1^{\pm 1}, \rho, \tau^2, \tau h_1, v_1^2]}{\tau^2 = h_1^{-2}(\tau h_1)^2 + h_1^{-2} \cdot \rho^2 \cdot v_1^2}.$$

- 1 Because of the relation, the generator  $\tau^2$  is redundant.  
2 The values of the periodicization map given in Table 5 are immediate from the  
3 algebraic analysis of the previous paragraph.  $\square$

- 4 **Remark 4.8.** Table 5 gives an unexpected value for  $\tau^2$ . Recall that  $\tau^2$  is inde-  
5 composable in  $E_1(ko)$ , so there is no inconsistency. The unexpected value arises  
6 from Equation (1.1).

- 7 **4.2. The Adams operation  $\psi^3$  in effective spectral sequences.** Our goal  
8 in this section is to study  $\psi^3$  as a map of effective spectral sequences. This will  
9 allow us to compute the  $E_1$ -page of the effective spectral sequence for  $L$ .

- 10 **Lemma 4.9.** *The map  $E_\infty(ko) \rightarrow E_\infty(ko)$  induced by  $\psi^3$  on effective  $E_\infty$ -pages*  
11 *takes the values shown in Table 6.*

- 12 *Proof.* Corollary 2.4(2) gives the values of  $\psi^3$  on  $\rho$ ,  $h_1$ , and  $\tau h_1$ .

- 13 The value of  $\psi^3$  on  $\tau^4$  is determined immediately by comparison along Betti  
14 realization to the classical value  $\psi^3(1) = 1$ . The computation is greatly simplified  
15 by ignoring terms in higher effective filtration. Similarly, the value of  $\psi^3$  on  $2\tau^2$  is  
16 determined by the classical value  $\psi^3(2) = 2$ .

- 17 The remaining values in Table 6 are also determined by comparison along Betti  
18 realization to the classical values  $\psi^3(2v_1^2) = 9 \cdot 2v_1^2$  and  $\psi^3(v_1^4) = 81v_1^4$ .  $\square$

- 19 **Lemma 4.10.** *The map  $E_1(ko) \rightarrow E_1(ko)$  induced by  $\psi^3$  on effective  $E_1$ -pages*  
20 *takes the values shown in Table 5.*

- 21 *Proof.* The values of  $\psi^3$  on  $E_1(ko)$  are compatible with the values of  $\psi^3$  on  $E_\infty(ko)$ ,  
22 as shown in Table 6. This immediately yields the value of  $\psi^3$  on  $\rho$ ,  $h_1$ , and  $\tau h_1$ .

- 23 The value of  $\psi^3((\tau^2)^2)$  must be  $(\tau^2)^2$  by compatibility with the value of  $\psi^3(\tau^4)$   
24 in  $E_\infty(ko)$ . Then the relation  $\psi^3((\tau^2)^2) = (\psi^3(\tau^2))^2$  implies that  $\psi^3(\tau^2) = \tau^2$ .

- 25 Similarly, the value of  $\psi^3((v_1^2)^2)$  must be  $81(v_1^2)^2$  by compatibility with the  
26 value of  $\psi^3(v_1^4)$  in  $E_\infty(ko)$ . Then the relation  $\psi^3((v_1^2)^2) = (\psi^3(v_1^2))^2$  implies that  
27  $\psi^3(v_1^2) = 9v_1^2$ .  $\square$

- 28 **Remark 4.11.** Since  $\psi^3$  is a ring homomorphism, all values of  $\psi^3$  on  $E_1(ko)$  are  
29 readily determined by the values on multiplicative generators given in Table 5. In  
30 particular, for all  $k \geq 0$ ,

$$\psi^3(v_1^{2k}) = 9^k v_1^{2k}.$$

- 31 **Remark 4.12.** Table 5 implies that  $\psi^3(v_1^4) = 81v_1^4$ . The careful reader will notice  
32 that this expression appears to be simpler than the analogous formula in [BH20,  
33 Theorem 3.1(2)]. The difference is explained by the fact that we are working  
34 only up to higher effective filtration. In particular, our formulas do not reflect  
35 the difference between the homotopy elements 2 and  $h$ , since their difference is

1 detected by  $\rho h_1$  in higher effective filtration. This also means that our formulas  
2 are less precise, but that has no consequence for our computational results.

3                   5. THE EFFECTIVE SPECTRAL SEQUENCE FOR  $L$

4   5.1. **The effective  $E_1$ -page of  $L$ .** In this section we compute the  $E_1$ -page of  
5 the effective spectral sequence for  $L$ .

6   The fiber sequence  $L \rightarrow \text{ko} \xrightarrow{\psi^3-1} \text{ko}$  induces a fiber sequence

$$s_*L \longrightarrow s_*\text{ko} \xrightarrow{\psi^3-1} s_*\text{ko}$$

7   on slices. Upon taking homotopy groups, we obtain a long exact sequence

$$\cdots \longrightarrow E_1(L) \longrightarrow E_1(\text{ko}) \xrightarrow{\psi^3-1} E_1(\text{ko}) \longrightarrow \cdots.$$

8   Table 5 (see also Lemma 4.10) gives us complete computational knowledge of the  
9   map  $E_1(\text{ko}) \rightarrow E_1(\text{ko})$ . This allows us to compute  $E_1(L)$ .

10   **Proposition 5.1.** *The chart in Figure 10 depicts the effective  $E_1$ -page of  $L$ .*

11   *Proof.* The long exact sequence

$$\cdots \longrightarrow E_1(L) \longrightarrow E_1(\text{ko}) \xrightarrow{\psi^3-1} E_1(\text{ko}) \longrightarrow \cdots$$

12   induces a short exact sequence

$$0 \longrightarrow \Sigma^{-1}C \longrightarrow E_1(L) \longrightarrow K \longrightarrow 0,$$

13   where  $C$  and  $K$  are the cokernel and kernel of  $E_1(\text{ko}) \xrightarrow{\psi^3-1} E_1(\text{ko})$ . The cokernel  
14   and kernel can be computed directly from the information given in Lemma 4.10.  
15   See also Remark 4.11.

16   The kernel consists of all elements in  $E_1(\text{ko})$  with the exception of the integer  
17   multiples of  $\tau^{2j} \cdot v_1^{2k}$  for  $j \geq 0$  and  $k > 0$ .

18   The cokernel  $C$  is nearly the same as  $E_1(\text{ko})$  itself. We must impose the relations  
19    $(3^{2k} - 1)v_1^{2k} = 0$  for all  $k > 0$ . Lemma 3.8 says that  $3^{2k} - 1$  equals  $2^{v(2k)+2} \cdot u$ , where  
20    $u$  is an odd number, i.e., a unit in our 2-adic context. Therefore, the relation  
21    $(3^{2k} - 1)v_1^{2k} = 0$  is equivalent to the relation  $2^{v(2k)+2}v_1^{2k} = 0$ .  $\square$

22   Table 8 lists some elements of the effective  $E_1$ -page of  $L$ . In fact, by inspection  
23   these elements are multiplicative generators for  $E_1(L)$ .

24   We use the same notation for elements of  $E_1(L)$  and their images in  $E_1(\text{ko})$ .  
25   On the other hand, we define the element  $\iota x$  of  $E_1(L)$  to be the image of  $x$  under  
26   the map  $\iota : \Sigma^{-1}E_1(\text{ko}) \rightarrow E_1(L)$ . For example, the element 1 of  $E_1(\text{ko})$  maps to  $\iota$   
27   in  $E_1(L)$ .

Table 8: Multiplicative generators for  $E_1(L)$ :  $k \geq 0$ 

coweight	$(s, f, w)$	generator	image in $E_1(L[\eta^{-1}])$
2	$(0, 0, -2)$	$\tau^2$	$\tau^2 + \rho^2 \cdot v_1^2$
$2k + 1$	$(4k + 1, 1, 2k)$	$\tau h_1 v_1^{2k}$	$\tau(v_1^2)^k$
$2k$	$(4k - 1, 1, 2k - 1)$	$\rho v_1^{2k}$	$\rho(v_1^2)^k$
$2k$	$(4k + 1, 1, 2k + 1)$	$h_1 v_1^{2k}$	$(v_1^2)^k$
$2k - 1$	$(4k - 1, 1, 2k)$	$\iota v_1^{2k}$	$\iota(v_1^2)^k$

1 5.2. **The effective spectral sequence for  $L[\eta^{-1}]$ .** In Section 5.1, we determined  
 2 the effective  $E_1$ -page of  $L$ . The next steps in the analysis of the effective spectral  
 3 sequence for  $L$  are to determine the multiplicative structure of  $E_1(L)$  (see Section  
 4 5.3) and to determine the effective differentials (see Sections 5.4 and 5.5).

5 Before doing so, we collect some information on the  $\eta$ -periodicization  $L[\eta^{-1}]$ .  
 6 We will study  $L[\eta^{-1}]$  by comparing to the more easily understood  $\text{ko}[\eta^{-1}]$ .

7 As in Sections 3.3 and 4.1, powers of  $h_1$  are inconsequential for computational  
 8 purposes in the  $\eta$ -periodic context. Consequently, we have removed these powers  
 9 from all  $\eta$ -periodic formulas.

10 **Proposition 5.2.** *The effective  $E_1$ -page for  $L[\eta^{-1}]$  is given by*

$$E_1(L[\eta^{-1}]) = \mathbb{F}_2[h_1^{\pm 1}, \tau, \rho, v_1^2, \iota]/\iota^2.$$

11 *Moreover, the periodicization map  $L \rightarrow L[\eta^{-1}]$  induces the map  $E_1(L) \rightarrow E_1(L[\eta^{-1}])$   
 12 whose values are given in Table 8.*

13 *Proof.* As in Proposition 4.7, we can just invert  $h_1$  in the additive description of  
 14  $E_1(L)$  given in Proposition 5.1.

15 The map  $E_1(\text{ko}[\eta^{-1}]) \xrightarrow{\psi^3 - 1} E_1(\text{ko}[\eta^{-1}])$  is trivial because  $(\psi^3 - 1)(h_1) = 0$ , as  
 16 shown in Table 5 (see also Lemma 4.10). Therefore, the long exact sequence

$$\cdots \rightarrow E_1(L[\eta^{-1}]) \rightarrow E_1(\text{ko}[\eta^{-1}]) \xrightarrow{\psi^3 - 1} E_1(\text{ko}[\eta^{-1}]) \rightarrow \cdots$$

17 splits as

$$E_1(L[\eta^{-1}]) \cong E_1(\text{ko}[\eta^{-1}]) \oplus \Sigma^{-1} E_1(\text{ko}[\eta^{-1}]).$$

18 With Proposition 4.7, this establishes the additive structure of  $E_1(L[\eta^{-1}])$ , as well  
 19 as most of the multiplicative structure.

20 The relation  $\iota^2 = 0$  is immediate because there are no possible non-zero values  
 21 for  $\iota^2$ .  $\square$

1 **Remark 5.3.** As in Remark 4.8, Table 8 gives an unexpected value for  $\tau^2$ , which  
2 arises from Equation (1.1). Also, the last column of Table 8 leaves out of  $h_1$  for  
3 readability.

4 **Remark 5.4.** Note that  $E_1(L[\eta^{-1}])$  is very close to the effective  $E_1$ -page for  
5 the  $\eta$ -periodic sphere  $S[\eta^{-1}]$  [RSØ19, Theorem 2.32] [OR20, Theorem 2.3]. The  
6 element  $\iota$  is not present in  $E_1(S[\eta^{-1}])$ , but the elements  $\iota v_1^{2k}$  are present.

7 **Proposition 5.5.** *Some values of the differentials in the effective spectral sequence  
8 of  $L[\eta^{-1}]$  are:*  
9 (1)  $d_1(v_1^2) = \tau$ .

10 (2)  $d_{n+1}(v_1^{2^n}) = \rho^{n+1} \cdot \iota v_1^{2^n}$  for  $n \geq 2$ .

11 *The effective differentials are zero on all other multiplicative generators on all  
12 pages.*

13 Following our convention throughout this section, we have omitted the powers  
14 of  $h_1$  from the formulas in Proposition 5.5.

15 *Proof.* The  $d_1$  differential follows from [RSØ19, Lemma 4.2] or [OR20, Theorem  
16 2.6].

17 To study the higher differentials, consider the map  $S[\eta^{-1}] \rightarrow L[\eta^{-1}]$ . This map  
18 induces an isomorphism on stable homotopy groups, except in coweight  $-1$ . This  
19 follows from a minor adjustment to [BH20, Theorem 1.1]. The adjustment arises  
20 from the fact that our  $L[\eta^{-1}]$  is the fiber of  $\text{ko}[\eta^{-1}] \xrightarrow{\psi^3-1} \text{ko}[\eta^{-1}]$ , while [BH20,  
21 Theorem 1.1] refers to the fiber of  $\text{ko}[\eta^{-1}] \xrightarrow{\psi^3-1} \Sigma^{8,4} \text{ko}[\eta^{-1}]$ .

22 The homotopy of  $S[\eta^{-1}]$  is completely computed in [GI16], so the homotopy of  
23  $L[\eta^{-1}]$  is known (except in coweight  $-1$ ). There is only one pattern of differentials  
24 that is compatible with the known values for  $L[\eta^{-1}]$ .  $\square$

25 **Remark 5.6.** In the language of [OR20, Section 4], Proposition 5.5 establishes  
26 the profile of the  $\eta$ -periodic effective spectral sequence over  $\mathbb{R}$ .

27 **5.3. Multiplicative relations for  $E_1(L)$ .** In this section, we will completely  
28 describe the product structure on  $E_1(L)$ . We do not need all of this structure for  
29 our later computations, but we include it for completeness.

30 **Proposition 5.7.** *Table 9 lists some products in  $E_1(L)$ .*

Table 9: Products in  $E_1(L)$ :  $j \geq 0$  and  $k \geq 0$

	$\rho v_1^{2j}$	$h_1 v_1^{2j}$	$\tau h_1 v_1^{2j}$	$\iota v_1^{2j}$
$\rho v_1^{2k}$	$\rho \cdot \rho v_1^{2j+2k}$			

Table 9: Products in  $E_1(L)$ :  $j \geq 0$  and  $k \geq 0$

	$\rho v_1^{2j}$	$h_1 v_1^{2j}$	$\tau h_1 v_1^{2j}$	$\iota v_1^{2j}$
$h_1 v_1^{2k}$	$\rho \cdot h_1 v_1^{2j+2k}$	$h_1 \cdot h_1 v_1^{2j+2k}$		
$\tau h_1 v_1^{2k}$	$\rho \cdot \tau h_1 v_1^{2j+2k}$	$h_1 \cdot \tau h_1 v_1^{2j+2k}$	$\tau^2 \cdot h_1 \cdot h_1 v_1^{2j+2k} +$ $+ \rho \cdot \rho v_1^{2j+2k+2}$	
$\iota v_1^{2k}$	$\rho \cdot \iota v_1^{2j+2k}$	$h_1 \cdot \iota v_1^{2j+2k}$	$\tau h_1 \cdot \iota v_1^{2j+2k}$	0

1 *Proof.* All of these products are detected by  $E_1(L[\eta^{-1}])$ , which is described  
2 in Proposition 5.2. We need the values of the periodicization map  $E_1(L) \rightarrow$   
3  $E_1(L[\eta^{-1}])$  given in Table 8.  $\square$

4 **5.4. The effective  $d_1$  differential for  $L$ .** Our next task is to compute the  
5 differentials in the effective spectral sequence for  $L$ .

6 **Proposition 5.8.** *Table 10 gives the values of the effective  $d_1$  differential on the*  
7 *multiplicative generators of  $E_1(L)$ .*

Table 10: Effective  $d_1$  differentials for  $L$ :  $k \geq 0$

coweight	$(s, f, w)$	$x$	$d_1(x)$
2	$(0, 0, -2)$	$\tau^2$	$\rho^2 \cdot \tau h_1$
$4k$	$(8k-1, 1, 4k-1)$	$\rho v_1^{4k}$	
$4k+2$	$(8k+3, 1, 4k+1)$	$\rho v_1^{4k+2}$	$\rho h_1^2 \cdot \tau h_1 v_1^{4k}$
$4k$	$(8k+1, 1, 4k+1)$	$h_1 v_1^{4k}$	
$4k+2$	$(8k+5, 1, 4k+3)$	$h_1 v_1^{4k+2}$	$h_1^3 \cdot \tau h_1 v_1^{4k}$
$4k+3$	$(8k+5, 1, 4k+2)$	$\tau h_1 v_1^{4k+2}$	$\tau^2 \cdot h_1^3 \cdot h_1 v_1^{4k} + \rho^2 h_1 \cdot h_1 v_1^{4k+2}$
$4k+1$	$(8k+1, 1, 4k)$	$\tau h_1 v_1^{4k}$	
$4k+1$	$(8k+3, 1, 4k+2)$	$\iota v_1^{4k+2}$	$\tau h_1 \cdot h_1^2 \cdot \iota v_1^{4k}$
$4k-1$	$(8k-1, 1, 4k)$	$\iota v_1^{4k}$	

8 *Proof.* All of these differentials follow immediately from the effective  $d_1$  differentials  
9 for  $L[\eta^{-1}]$ , which are all determined by Proposition 5.5(1). Beware that the exact  
10 values of the map  $E_1(L) \rightarrow E_1(L[\eta^{-1}])$ , as shown in Table 8, are important.

11 For example, consider the differential on the element  $\tau h_1 v_1^{4k+2}$ . It maps to  
12  $\tau(v_1^2)^{2k+1}$  in  $E_1(L[\eta^{-1}])$  (up to  $h_1$  multiples, which as usual we ignore in the  
13  $\eta$ -periodic situation). The  $\eta$ -periodic differential on this latter element is  $\tau^2(v_1^2)^{2k}$ .

1 Finally, we need to find an element of  $E_1(L)$  in the correct degree whose  $\eta$ -period-  
2 icization is  $\tau^2(v_1^2)^{2k}$ , The only possibility is  $\tau^2 \cdot h_1^3 \cdot h_1 v_1^{4k} + \rho^2 h_1 \cdot h_1 v_1^{4k+2}$ .  $\square$

3 **Remark 5.9.** All  $d_1$  differentials in  $E_1(L)$  can be deduced from the information  
4 in Table 10 and the Leibniz rule, but the computations can be complicated by  
5 the multiplicative relations of Table 9. For example,

$$d_1(\tau^2 \cdot \tau h_1 v_1^2) = \rho^2 \cdot \tau h_1 \cdot \tau h_1 v_1^2 + \tau^2(\tau^2 \cdot h_1^4 + \rho^2 h_1 \cdot h_1 v_1^2) = \tau^4 \cdot h_1^4 + \rho^4 \cdot v_1^4.$$

6 Having completely analyzed the slice  $d_1$  differentials for  $E_1(L)$ , it is now possible  
7 to compute the  $E_2$ -page of the slice spectral sequence for  $L$ .

8 **Proposition 5.10.** *The  $E_2$ -page of the effective spectral sequence for  $L$  is depicted  
9 in Figures 11, 12, 14, and 15.*

10 For legibility, Figures 11, 12, 14, and 15 display  $E_2(L)$  in four different charts  
11 separated by coweight modulo 4. Note that Figures 14 and 15 also serve as  $E_\infty$ -  
12 page charts in coweights 1 and 2 modulo 4 because there are no higher differentials  
13 that affect these coweights.

14 *Proof.* The Leibniz rule, together with the values in Table 10, completely deter-  
15 mines the effective  $d_1$  differential on  $E_1(L)$ . The  $E_2$ -page can then be determined  
16 directly. However, as in the proof of Theorem 4.3, the computation is not entirely  
17 straightforward.

18 It turns out that the  $d_1$  differential preserves the image of the map  $\Sigma^{-1}E_1(\text{ko}) \rightarrow$   
19  $E_1(L)$ . Moreover, it turns out that all  $d_1$  differentials with values in the image of  
20  $\Sigma^{-1}E_1(\text{ko}) \rightarrow E_1(L)$  also have source in this image. (This is not for formal reasons;  
21 in fact, the higher effective differentials do not have this property.) Consequently,  
22 the determination of the  $E_2$ -page splits into two separate computations: one for  
23 the image of  $\Sigma^{-1}E_1(\text{ko}) \rightarrow E_1(L)$ , and one for the cokernel of the same map.

24 In more concrete terms, we can determine  $E_2(L)$  by first considering only  
25 elements of the form  $\iota x$ , and then separately considering only elements that are  
26 not of this form.

27 The  $d_1$  differential on the image of  $\Sigma^{-1}E_1(\text{ko}) \rightarrow E_1(L)$  is identical to the  
28  $d_1$  differential for  $\text{ko}$  discussed in Section 4. The  $d_1$  differential on the cokernel  
29 of  $\Sigma^{-1}E_1(\text{ko}) \rightarrow E_1(L)$  is similar to the  $d_1$  differential on  $E_1(\text{ko})$ , but slightly  
30 different. The difference is created by the absence of the elements  $v_1^{2k}$  in  $E_1(L)$ .  $\square$

31 **5.5. Higher differentials.** We now consider the higher differentials in the effec-  
32 tive spectral sequence for  $L$ .

33 By inspection of the charts for  $E_2(L)$ , the only possible higher differentials have  
34 source in coweight congruent to 0 modulo 4 and value in coweight congruent to 3  
35 modulo 4. In other words, in coweights congruent to 1 and 2 modulo 4, we have  
36 that  $E_2(L)$  equals  $E_\infty(L)$ .

37 It turns out that there are many higher differentials. In fact, nearly all of  
38 the elements in  $E_2(L)$  in coweight congruent to 0 modulo 4 support differentials.  
39 While it is possible to write down explicit formulas for all of these differentials,

1 the formulas would be cumbersome and not so helpful. Rather, we give a more  
2 qualitative description of the differentials because it is more useful for computation.

3 **Proposition 5.11.** *Consider the elements of  $E_2(L)$  in coweights congruent to 0  
4 modulo 4 that belong to the cokernel of the map  $\Sigma^{-1}E_2(\text{ko}) \rightarrow E_2(L)$ .*

5 (1) *The only permanent cycles are the multiples of 1, the multiples of  $2\tau^{4k}$  for  
6  $k \geq 0$ , and  $\rho^a h_1^b$  for all  $a \geq 0$  and  $b \geq 0$ .*

7 (2) *Excluding the elements listed in (1), if an element has coweight congruent  
8 to  $2^{r-1}$  modulo  $2^r$ , then it supports a  $d_r$  differential.*

9 Proposition 5.11 may seem imprecise because it does not give the values of the  
10 differentials. However, there is only one non-zero possible value in every case, so  
11 there is no ambiguity.

12 *Proof.* These differentials follow immediately from the  $\eta$ -periodic differentials of  
13 Proposition 5.5, together with multiplicative relations in  $E_2(L)$ .

14 For example, consider the element  $\tau^8 \cdot \rho v_1^{12}$  in coweight 20, which is congruent  
15 to  $2^2$  modulo  $2^3$ . Using Table 8, we find that this element maps to  $\rho^9(v_1^2)^{10}$  in  
16  $E_2(L[\eta^{-1}])$ . Here we are using that  $\tau^2$  is zero in  $E_2(L[\eta^{-1}])$  since it is hit by  
17 an  $\eta$ -periodic  $d_1$  differential. Proposition 5.5 says that this element supports an  
18  $\eta$ -periodic  $d_3$  differential. It follows that  $\tau^8 \cdot \rho v_1^{12}$  also supports a  $d_3$  differential.  $\square$

19 **Theorem 5.12.** *The  $E_\infty$ -page of the effective spectral sequence for  $L$  is depicted  
20 in Figures 13, 14, 15, 16, 17, 18, and 19.*

21 *Proof.* The  $E_\infty$ -page can be deduced directly from the higher differentials described  
22 in Proposition 5.11.  $\square$

23 The  $E_\infty$ -page in coweights congruent to 3 modulo 4 is by far the most com-  
24 plicated case. Figures 17, 18, and 19 display  $E_\infty(L)$  in coweights congruent to 3  
25 modulo 8, 7 modulo 16, and 15 modulo 32 respectively.

26 In each case (and more generally in coweights congruent to  $2^{n-1} - 1$  modulo  $2^n$ ),  
27 we see similar patterns with minor variations. The lower boundary of each chart  
28 takes the same shape. The upper boundary of the  $\tau$ -periodic portion of each chart  
29 also takes the same shape. However, the filtration jump between the lower and  
30 upper boundaries increases linearly with  $n$ .

31 In addition to the  $\tau$ -periodic portion of each chart, there are also  $\tau$ -torsion,  
32  $\eta$ -periodic regions. These consist of bands of infinite  $h_1$ -towers of width  $n$  that  
33 repeat every  $2^{n+1}$  stems. The first such band starts at  $\iota v_1^{2^{n-1}}$ .

34 **5.6. Hidden extensions.** Our last goal is to compute hidden extensions by  $\rho$ ,  
35  $h$ , and  $\eta$ . See [Isa19, Section 4.1] for a precise definition of a hidden extension.  
36 Fortunately, none of the complications associated with crossing extensions occur  
37 in this manuscript.

1 **Proposition 5.13.** *Table 11 lists some hidden extensions by  $\rho$ ,  $\mathbf{h}$ , and  $\eta$  in the  
2 effective spectral sequence for  $L$ .*

3 *Proof.* The last column of Table 11 indicates the reason for each hidden extension.  
4 Some of the hidden extensions follow from the analogous extensions for  $\mathrm{ko}$  given  
5 in Table 7, using the maps  $\Sigma^{-1}\mathrm{ko} \rightarrow L$  and  $L \rightarrow \mathrm{ko}$ .  
6 Other extensions follow from the long exact sequence associated to the cofiber  
7 sequence

$$\Sigma^{-1,-1}L \xrightarrow{\rho} L \longrightarrow L/\rho.$$

8 Here we need that the homotopy of  $L/\rho$  is isomorphic to the homotopy of  $L^{\mathbb{C}}$ ,  
9 as shown in [BS20, Corollary 1.9]. For example, the hidden  $\mathbf{h}$  extensions of  
10 Proposition 3.18 give hidden  $\mathbf{h}$  extensions in  $L/\rho$ , which then imply the hidden  
11 extension from  $\iota 4v_1^2$  to  $h_1^2 \cdot \tau h_1$ .

12  $\square$

Table 11: Hidden extensions in  $E_{\infty}(L)$

coweight	source	type	target	$(s, f, w)$	proof
0	$\iota \cdot \tau h_1$	$\mathbf{h}$	$\iota \cdot \rho h_1 \cdot \tau h_1$	(0, 2, 0)	$\Sigma^{-1}\mathrm{ko} \rightarrow L$
1	$\tau h_1$	$\mathbf{h}$	$\rho h_1 \cdot \tau h_1$	(1, 1, 0)	$L \rightarrow \mathrm{ko}$
1	$\iota(\tau h_1)^2$	$\mathbf{h}$	$\iota \cdot \rho h_1(\tau h_1)^2$	(1, 3, 0)	$\Sigma^{-1}\mathrm{ko} \rightarrow L$
1	$\iota \cdot 2\tau^2$	$\eta$	$\iota \cdot \rho(\tau h_1)^2$	(-1, 1, -2)	$\Sigma^{-1}\mathrm{ko} \rightarrow L$
1	$\iota 2v_1^2$	$\rho$	$\iota \cdot h_1(\tau h_1)^2$	(3, 1, 2)	$\Sigma^{-1}\mathrm{ko} \rightarrow L$
1	$\iota 4v_1^2$	$\mathbf{h}$	$h_1^2 \cdot \tau h_1$	(3, 1, 2)	$L/\rho$
2	$(\tau h_1)^2$	$\mathbf{h}$	$\rho h_1(\tau h_1)^2$	(2, 2, 0)	$L \rightarrow \mathrm{ko}$
3	$\iota 4\tau^2 v_1^2$	$\mathbf{h}$	$(\tau h_1)^3$	(3, 1, 0)	$L/\rho$
3	$\iota 2\tau^2 v_1^2$	$\rho$	$\iota \tau^4 \cdot h_1^3$	(3, 1, 0)	$\Sigma^{-1}\mathrm{ko} \rightarrow L$
3	$\iota 2\tau^2 v_1^2$	$\eta$	$\rho^3 \cdot \iota v_1^4$	(3, 1, 0)	$\Sigma^{-1}\mathrm{ko} \rightarrow L$
2	$2\tau^2$	$\eta$	$\rho(\tau h_1)^2$	(0, 0, -2)	$L \rightarrow \mathrm{ko}$
3	$(\tau h_1)^3$	$\mathbf{h}$	$\iota \tau^4 \cdot \rho^2 h_1^6$	(3, 3, 0)	$L/\rho$
5	$\iota v_1^4 \cdot 8\tau^2$	$\mathbf{h}$	$\rho^2 \cdot \tau h_1 v_1^4$	(7, 1, 2)	$L/\rho$

13 **Remark 5.14.** The hidden extensions in Table 11 are  $\tau^4$ -periodic in the following  
14 sense. If we take the source and target of each extension in  $E_1(L)$  and multiply  
15 by  $\tau^4$ , then we obtain permanent cycles that are related by a hidden extension.  
16 For example, the hidden  $\mathbf{h}$  extension from  $\tau h_1$  to  $\rho h_1 \cdot \tau h_1$  generalizes to a family  
17 of hidden extensions from  $\tau^{4k+1} h_1$  to  $\rho h_1 \cdot \tau^{4k+1} h_1$  for all  $k \geq 0$ .

1 **Remark 5.15.** Similarly to the  $\tau^4$ -periodicity discussed in Remark 5.14, most of  
2 the hidden extensions in Table 11 are  $v_1^4$ -periodic as well. For example, the hidden  
3  $\mathbf{h}$  extension from  $\tau h_1$  to  $\rho h_1 \cdot \tau h_1$  generalizes to a family of hidden extensions from  
4  $\tau h_1 v_1^{4k}$  to  $\rho h_1 \cdot \tau h_1 v_1^{4k}$  for all  $k \geq 0$ . There are three exceptions, which appear  
5 below the horizontal divider at the bottom of the table. These exceptions are  
6 discussed in more detail in Remarks 5.16, 5.17, and 5.18.

7 **Remark 5.16.** The hidden  $\eta$  extension from  $2\tau^2$  to  $\rho(\tau h_1)^2$  is  $\tau^4$ -periodic as in  
8 Remark 5.14, but it is not  $v_1^4$ -periodic. The elements  $2\tau^2 v_1^{4k}$  are not permanent  
9 cycles for  $k \geq 1$ .

10 **Remark 5.17.** The hidden  $\mathbf{h}$  extension from  $\iota v_1^4 \cdot 8\tau^2$  to  $\rho^2 \cdot \tau h_1 v_1^4$  is  $v_1^4$ -periodic,  
11 but the situation is slightly more complicated than in Remark 5.15. For all  $k$ ,  
12  $\rho^2 \cdot \tau h_1 v_1^{4k}$  receives a hidden  $\mathbf{h}$  extension from an appropriate multiple of  $\iota v_1^{4k} \cdot 2\tau^2$ .  
13 For example, as shown in Figure 14, there is a hidden  $\mathbf{h}$  extension from  $\iota v_1^{4k} \cdot 16\tau^2$   
14 to  $\rho^2 \cdot \tau h_1 v_1^8$ .

15 **Remark 5.18.** The hidden  $\mathbf{h}$  extension from  $(\tau h_1)^3$  to  $\iota \tau^4 \cdot \rho^2 h_1^6$  is  $v_1^4$ -periodic,  
16 but the situation is more complicated than in Remarks 5.15 and 5.17. For all  
17  $k \geq 0$ , the element  $(\tau h_1)^2 \tau h_1 v_1^{4k}$  supports a hidden  $\mathbf{h}$  extension to the element of  
18  $E_\infty(L)$  of highest filtration in the appropriate degree. For example, as shown in  
19 Figure 18, there is a hidden  $\mathbf{h}$  extension from  $(\tau h_1)^2 \cdot \tau^5 h_1$  to  $\iota \tau^8 \cdot \rho^3 h_1^7$ . Figures  
20 17, 18, and 19 show several extensions of this type.

## 21 6. CHARTS

22 We explain the notation used in the charts.

- 23 • The horizontal coordinate is the stem  $s$ . The vertical coordinate is the  
24 Adams-Novikov filtration  $f$  (see Section 1.7 for further discussion).
- 25 • Black or green circles represent copies of  $\mathbb{F}_2$ , periodicized by some power of  $\tau$ .  
26 The relevant power of  $\tau$  varies from chart to chart.
- 27 • Black or green unfilled boxes represent copies of  $\mathbb{Z}$  (the 2-adic integers),  
28 periodicized by some power of  $\tau$ . The relevant power of  $\tau$  varies from chart  
29 to chart.
- 30 • Black or green boxes containing a number  $n$  represent copies of  $\mathbb{Z}/2^n$ , peri-  
31 odicized by some power of  $\tau$ . The relevant power of  $\tau$  varies from chart to  
32 chart.
- 33 • Red unfilled boxes represent copies of  $\mathbb{Z}$  (the 2-adic integers) that are not  
34  $\tau^k$ -periodic for any  $k$ .
- 35 • Green objects represent elements in the image of the map  $E_1(\Sigma^{-1}ko) \rightarrow E_1(L)$   
36 (or  $E_1(\Sigma^{-1}ko^\mathbb{C}) \rightarrow E_1(L^\mathbb{C})$ ). Beware that the color refers to the  $E_1$ -page origin  
37 of the element, not the properties of the homotopical element that it detects.

- 1 For example, in Figure 4, the element  $\tau h_1^3$  detects an element in  $\pi_{3,2}L^{\mathbb{C}}$  that  
 2 maps to zero in  $\pi_{3,2}ko^{\mathbb{C}}$ , so it is in the image of  $\pi_{4,2}ko^{\mathbb{C}}$ . Nevertheless, the  
 3 element is colored black because it is not in the image on  $E_1$ -pages.
- 4 • Black objects represent elements in the cokernel of the map  $E_1(\Sigma^{-1}ko) \rightarrow$   
 5  $E_1(L)$  (or  $E_1(\Sigma^{-1}ko^{\mathbb{C}}) \rightarrow E_1(L^{\mathbb{C}})$ ). In other words, they are detected by the  
 6 map  $L \rightarrow ko$  (or  $L^{\mathbb{C}} \rightarrow ko^{\mathbb{C}}$ ). As in the previous paragraph, beware of the  
 7 distinction between  $E_1$ -page origins and homotopical properties.
- 8 • Lines of slope 1 represent  $h_1$ -multiplications.
- 9 • Black or green arrows of slope 1 represent infinite sequences of elements that  
 10 are  $\tau^k$ -periodic for some  $k > 0$  and are connected by  $h_1$ -multiplications.
- 11 • Red arrows of slope 1 represent infinite sequences of elements that are con-  
 12 nected by  $h_1$ -multiplications and are not  $\tau^k$ -periodic for any  $k$ .
- 13 • Lines of slope  $-1$  represent  $\rho$ -multiplications.
- 14 • Dashed lines of slope  $-1$  represent  $\rho$ -multiplications whose values are multiples  
 15 of  $\tau^k$  for some  $k > 0$ . For example, in Figure 6, we have  $\rho \cdot \rho^3 v_1^4$  equals  $\tau^4 \cdot h_1^4$ .
- 16 • Black or green arrows of slope  $-1$  represent infinite sequences of elements  
 17 that are  $\tau^k$ -periodic for some  $k > 0$  and are connected by  $\rho$ -multiplications.
- 18 • Light blue lines of slope  $-3$  represent effective  $d_1$  differentials.
- 19 • Dashed light blue lines of slope  $-3$  represent effective  $d_1$  differentials that hit  
 20 multiples of  $\tau^k$ , for some  $k > 0$ . For example, the dashed line in Figure 1  
 21 indicates that  $d_1(v_1^2)$  equals  $\tau h_1^3$ .
- 22 • Dark blue lines indicate hidden extensions by  $h$ ,  $\rho$ , or  $h_1$ .
- 23 • Dashed dark blue lines indicate hidden extensions whose value is a multiple  
 24 of  $\tau^k$  for some  $k > 0$ . For example, in Figure 4, there is a hidden  $h$  extension  
 25 from  $\iota 4v_1^2$  to  $\tau h_1^3$ .

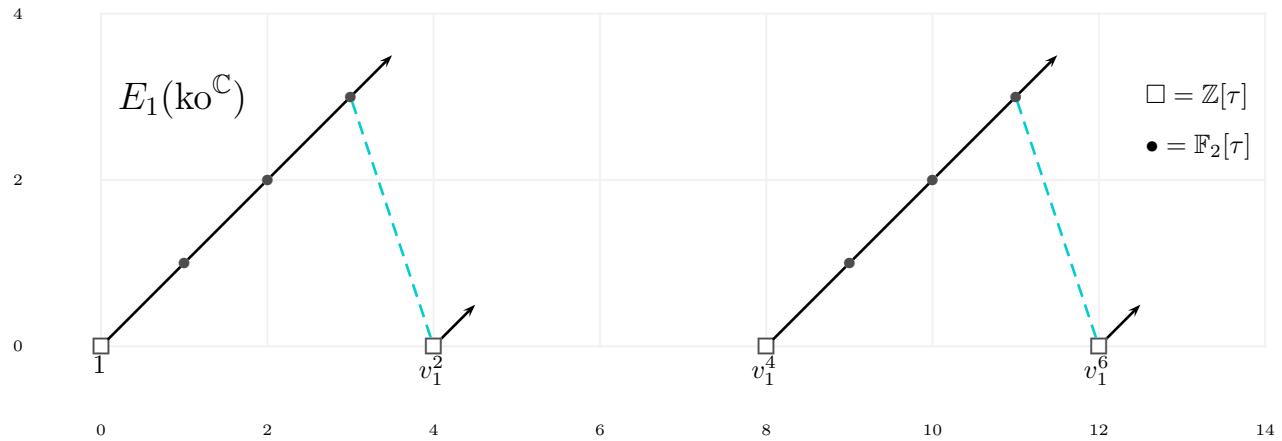


FIGURE 1. The  $E_1$ -page of the effective spectral sequence for  $\text{ko}^{\mathbb{C}}$

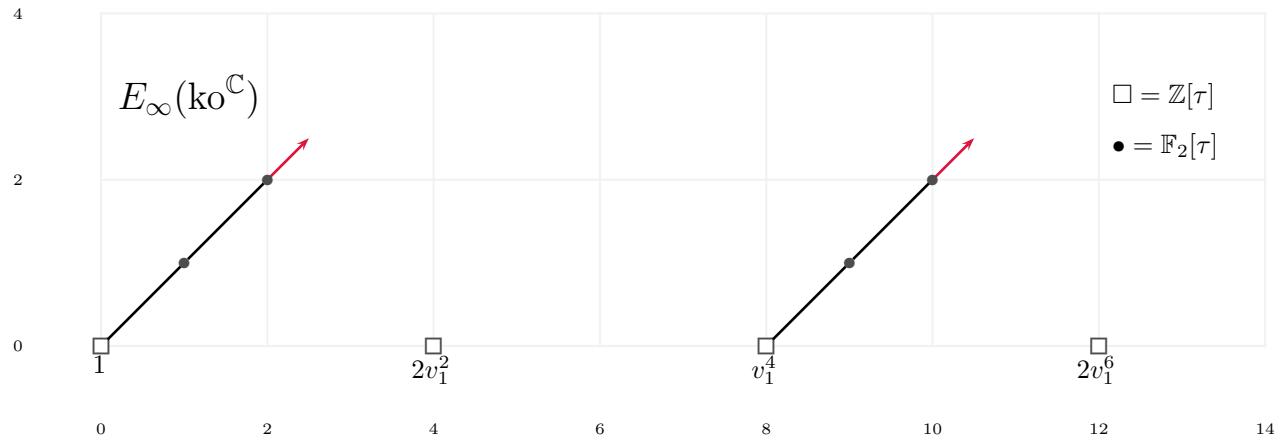


FIGURE 2. The  $E_{\infty}$ -page of the effective spectral sequence for  $\text{ko}^{\mathbb{C}}$

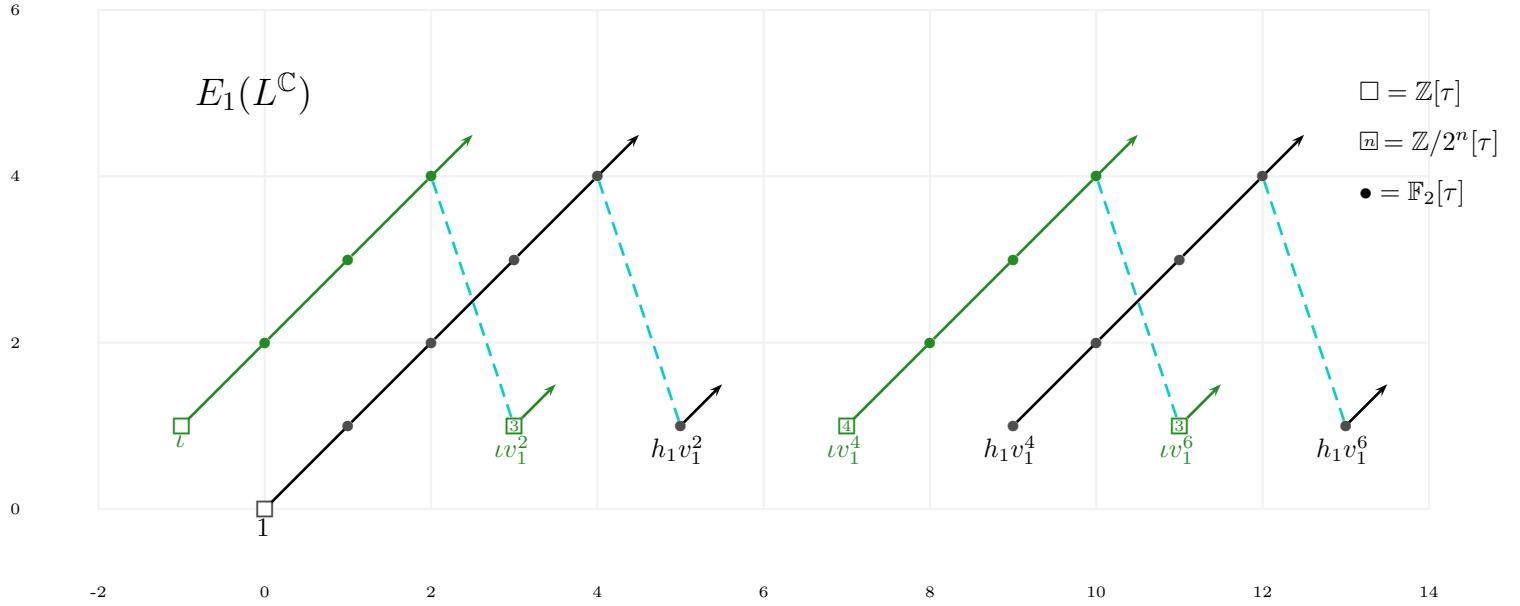


FIGURE 3. The  $E_1$ -page of the effective spectral sequence for  $L^C$

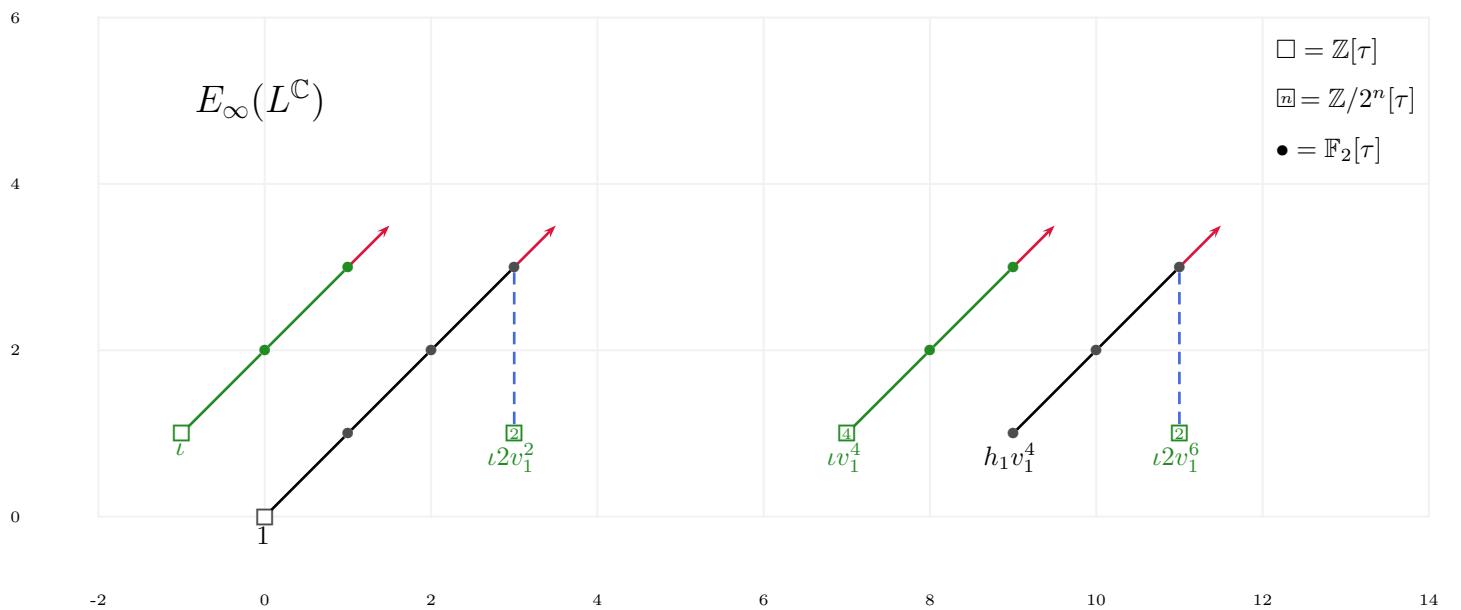


FIGURE 4. The  $E_\infty$ -page of the effective spectral sequence for  $L^C$

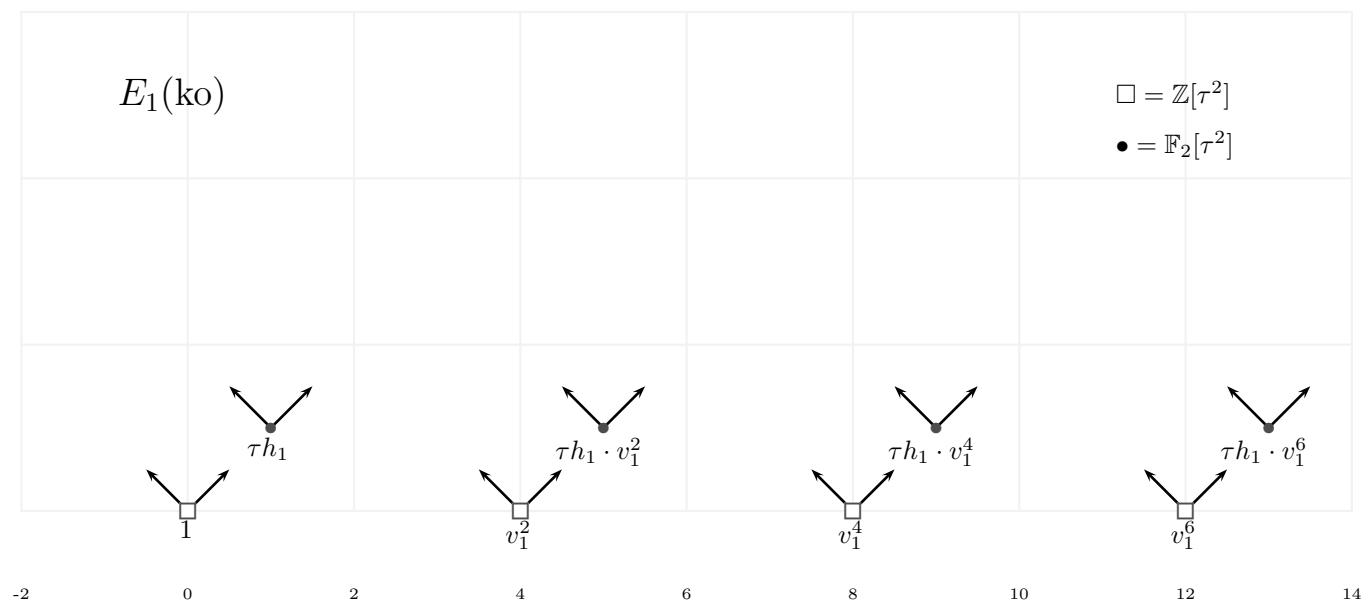
6

 $E_1(\text{ko})$ 

4

2

0

 $\square = \mathbb{Z}[\tau^2]$  $\bullet = \mathbb{F}_2[\tau^2]$ FIGURE 5. The  $E_1$ -page of the effective spectral sequence for  $\text{ko}$

10

 $E_\infty(\text{ko})$ , COWEIGHTS 0 MOD 4

8

 $\square = \mathbb{Z}[\tau^4]$ 

6

 $\bullet = \mathbb{F}_2[\tau^4]$ 

4

2

0

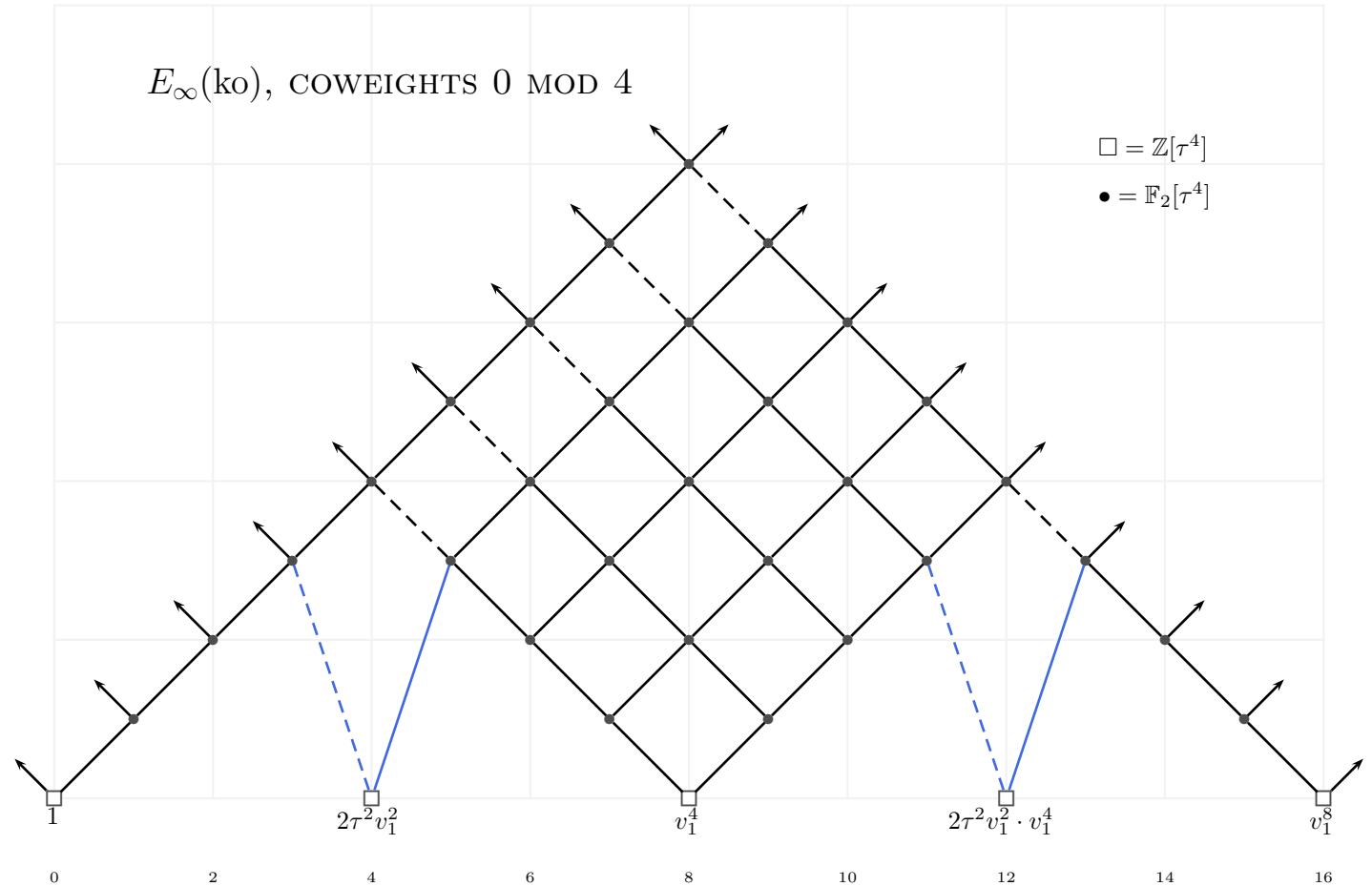


FIGURE 6. The  $E_\infty$ -page of the effective spectral sequence for  $\text{ko}$  in coweights 0 mod 4

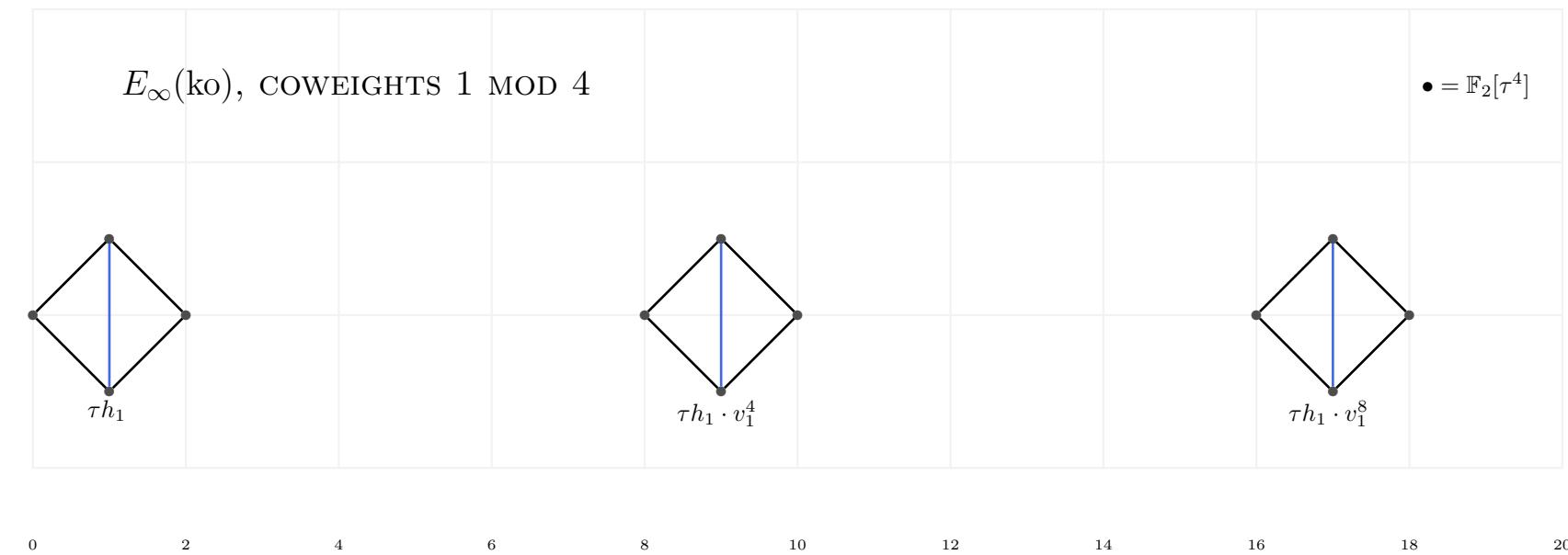


FIGURE 7. The  $E_\infty$ -page of the effective spectral sequence for  $\text{ko}$  in coweights  $1 \bmod 4$

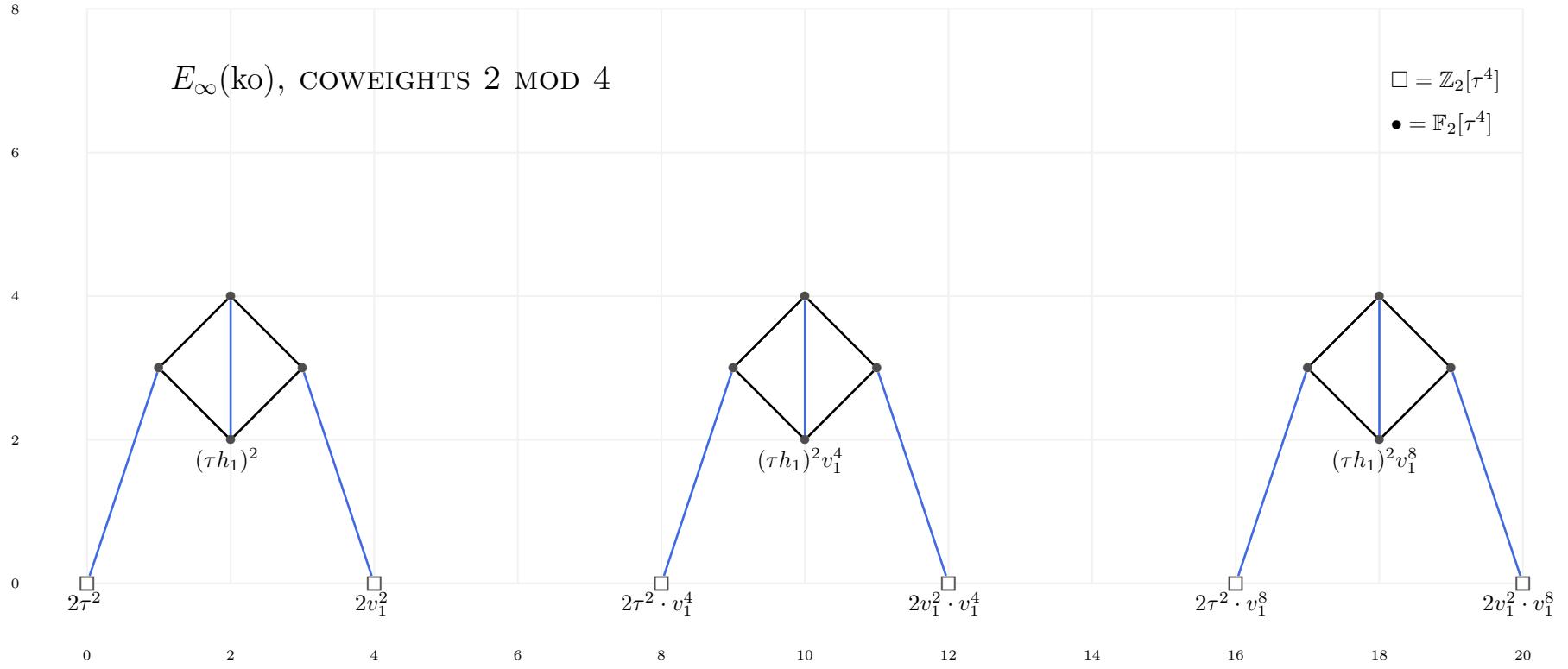


FIGURE 8. The  $E_\infty$ -page of the effective spectral sequence for  $\text{ko}$  in coweights  $2 \bmod 4$

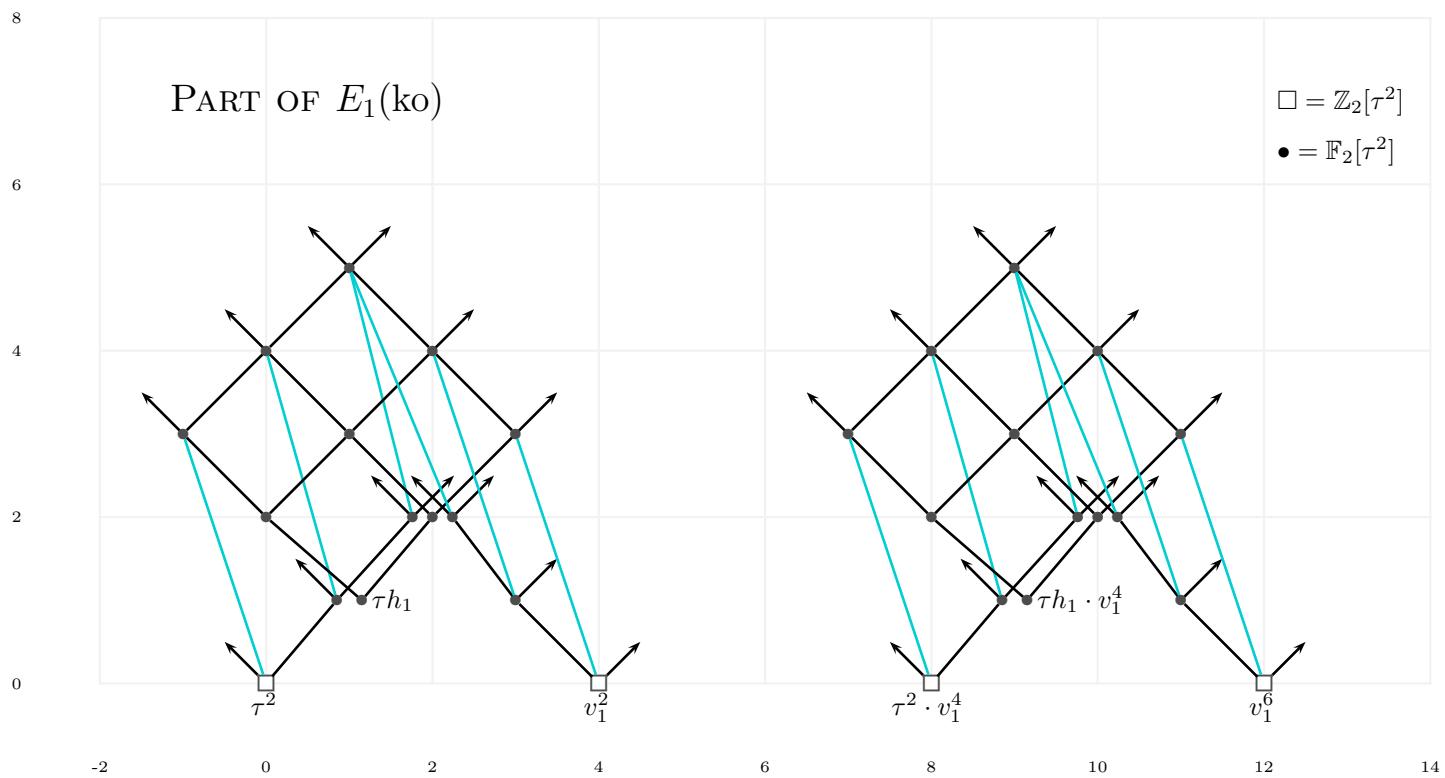


FIGURE 9. Some differentials in the effective spectral sequence for  $ko$

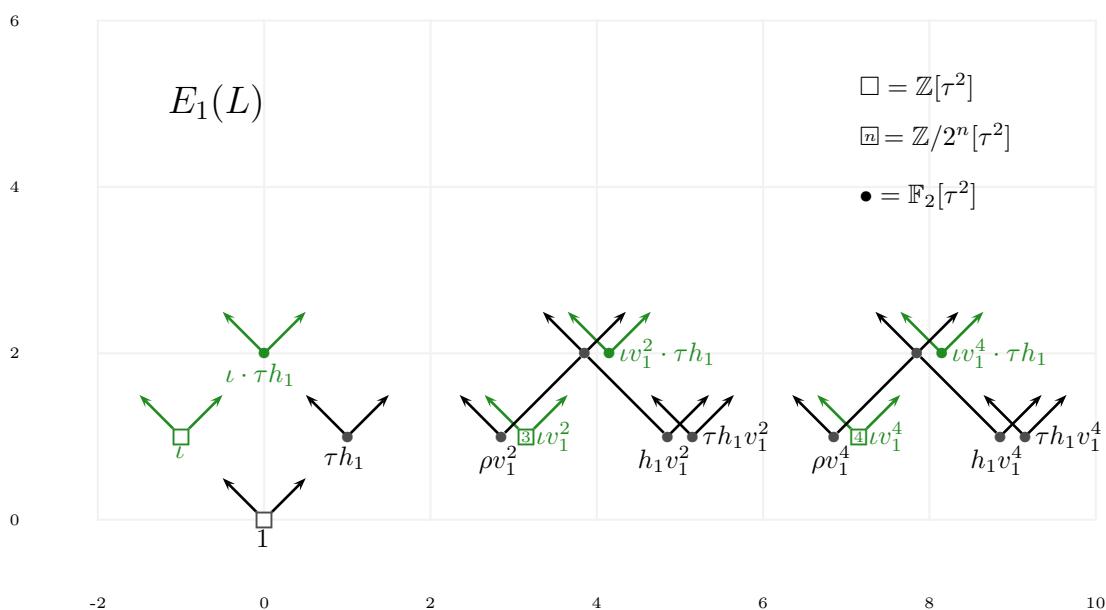


FIGURE 10. The  $E_1$ -page of the effective spectral sequence for  $L$

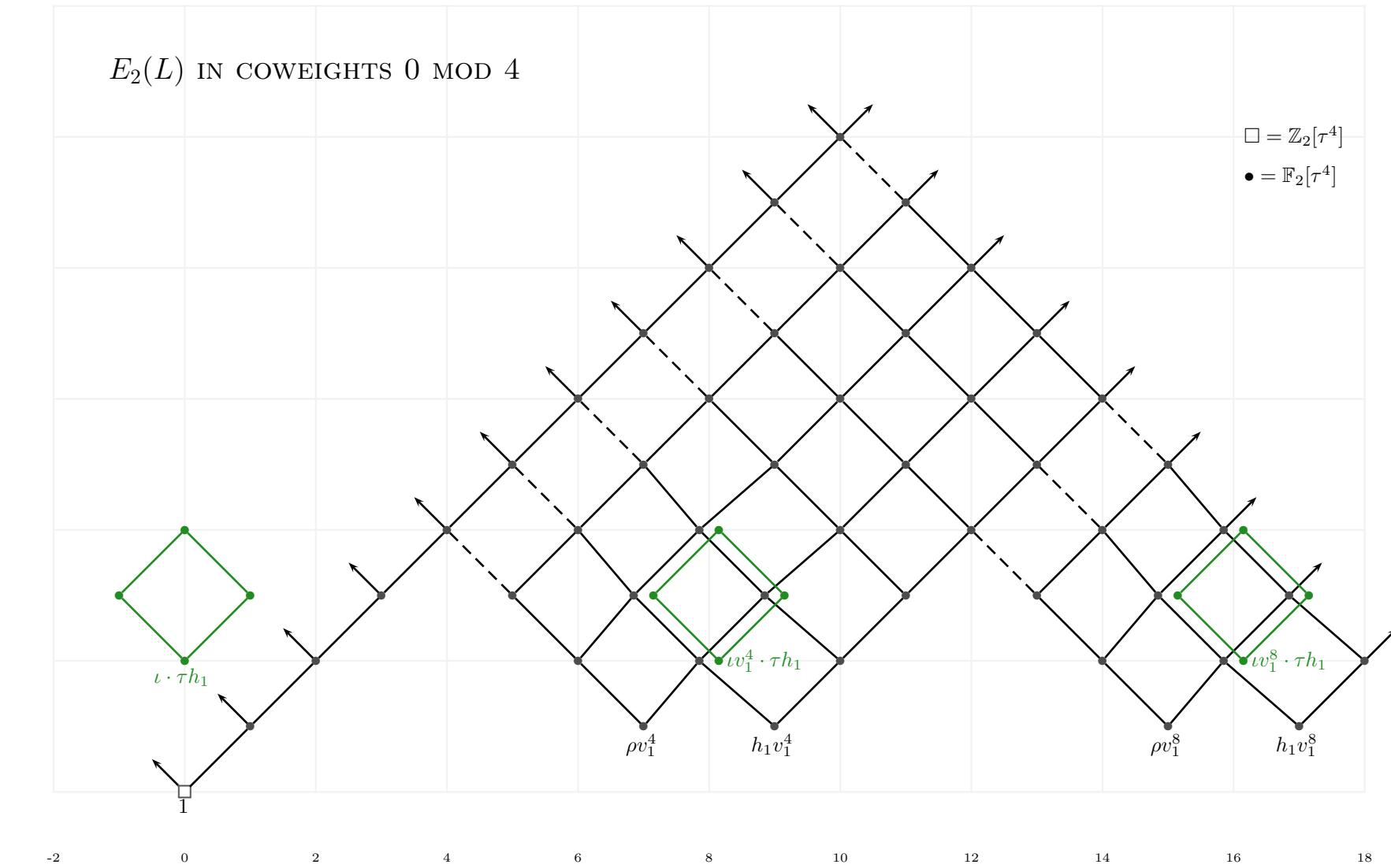


FIGURE 11. The  $E_2$ -page of the effective spectral sequence for  $L$  in coweights 0 mod 4

$E_2(L)$  IN COWEIGHTS 3 MOD 4

$\square = \mathbb{Z}_2[\tau^4]$   
 $\blacksquare = \mathbb{Z}/2^n[\tau^4]$   
 $\bullet = \mathbb{F}_2[\tau^4]$

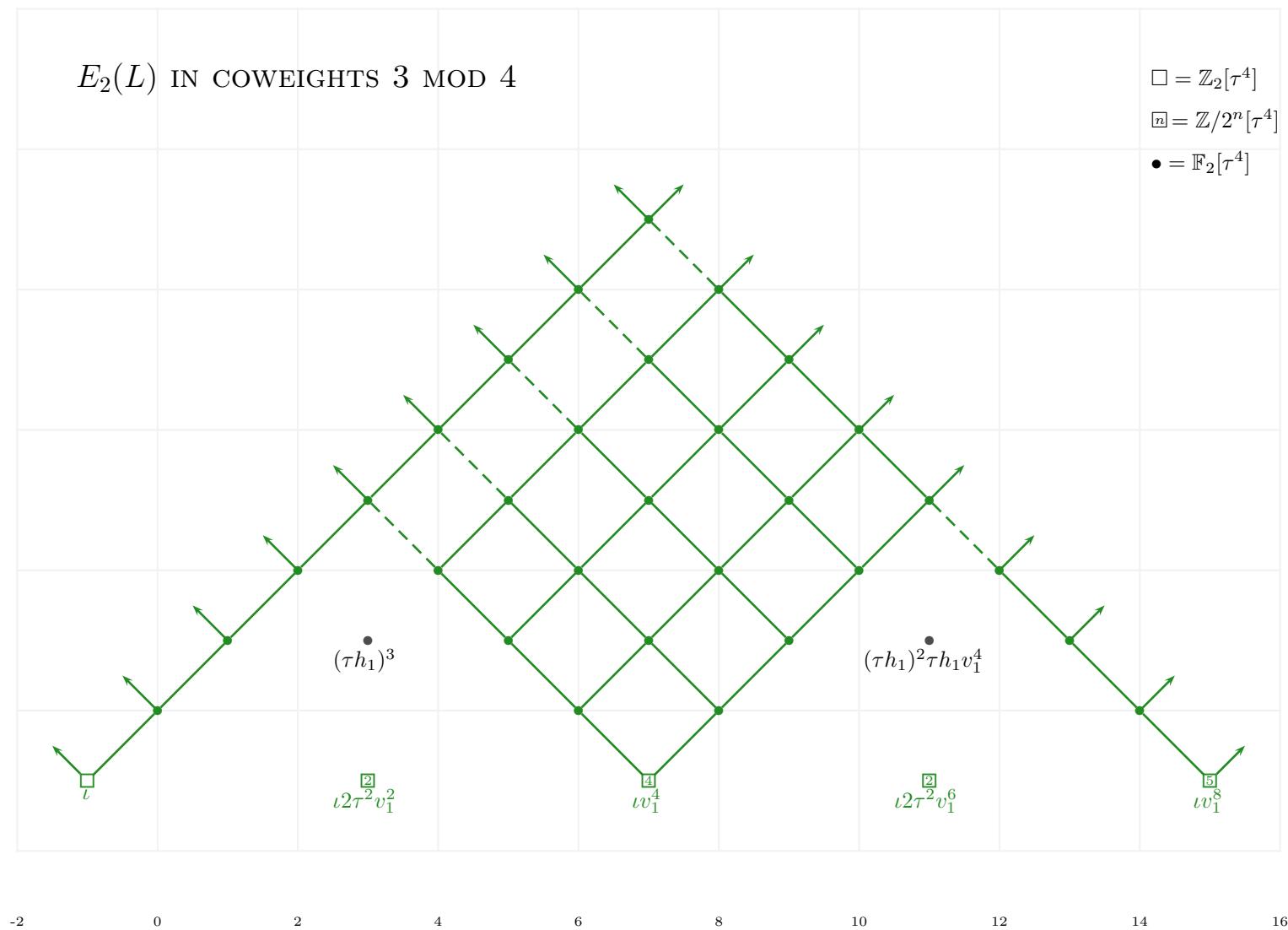


FIGURE 12. The  $E_2$ -page of the effective spectral sequence for  $L$  in coweights 3 mod 4

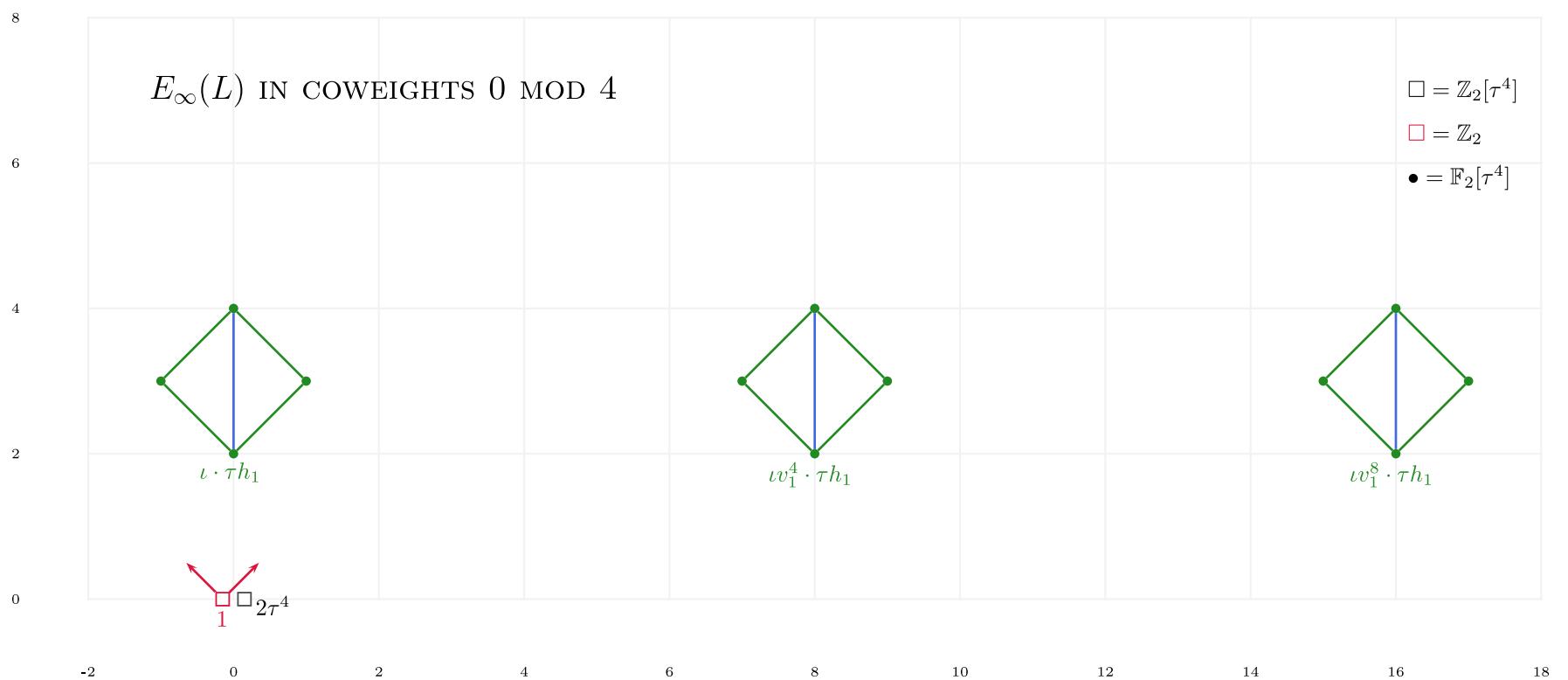


FIGURE 13. The  $E_\infty$ -page of the effective spectral sequence for  $L$  in coweights 0 mod 4

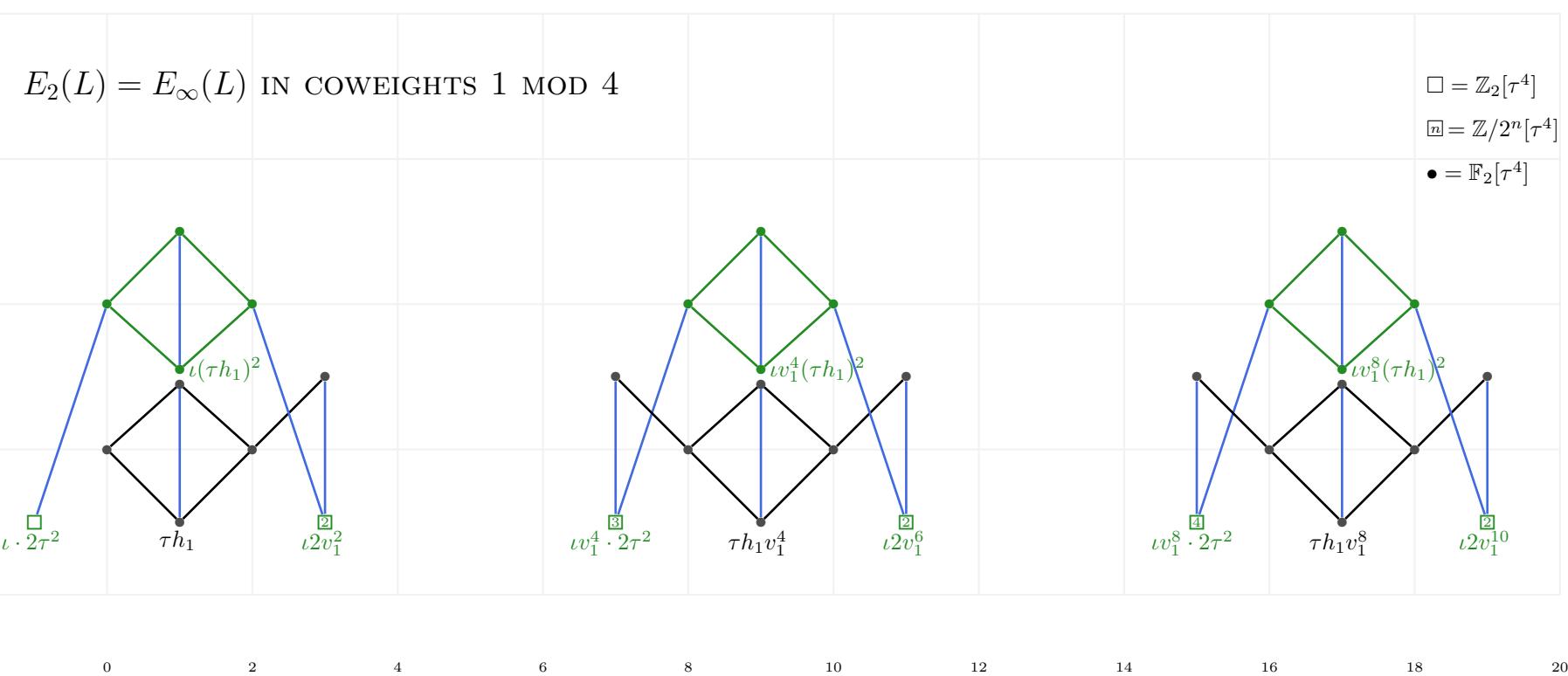


FIGURE 14. The  $E_\infty$ -page of the effective spectral sequence for  $L$  in coweights 1 mod 4

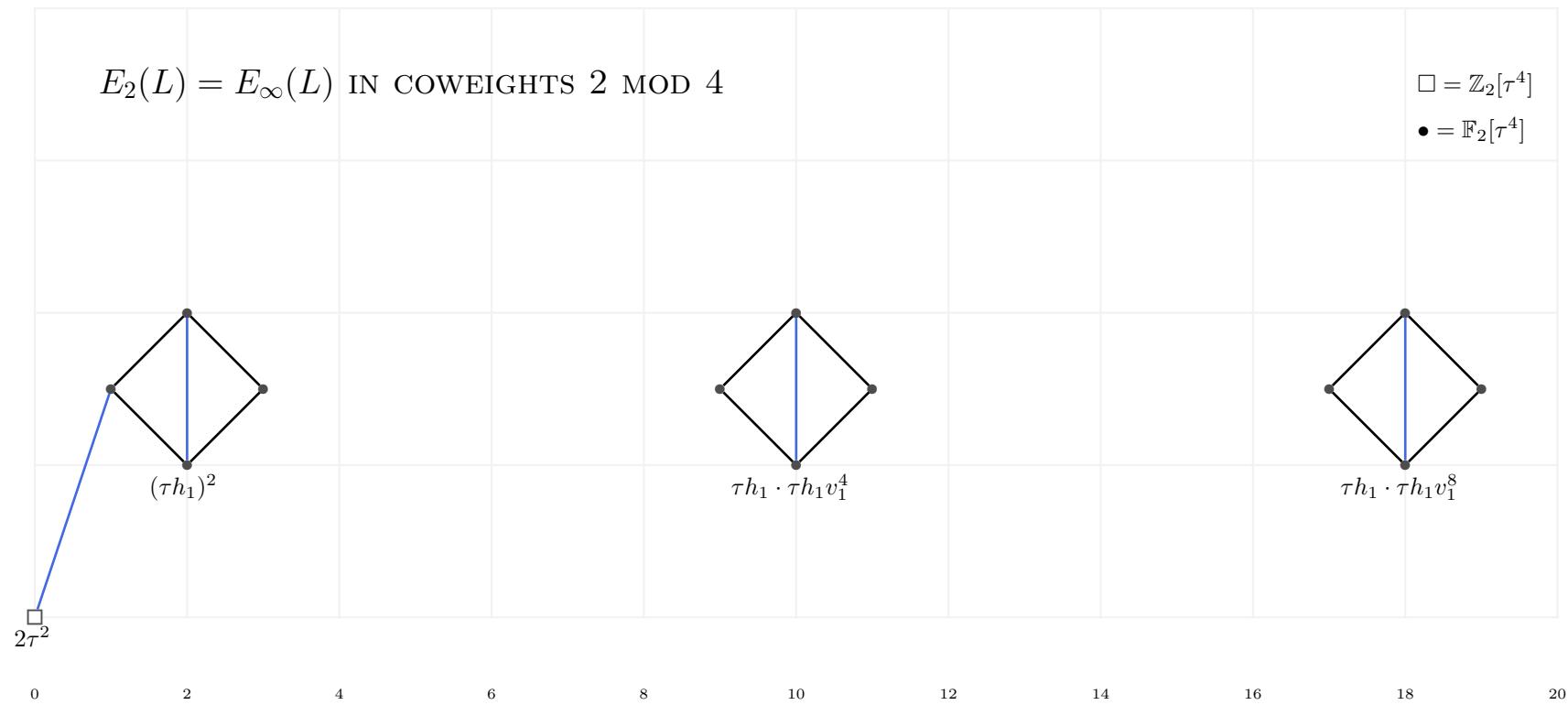


FIGURE 15. The  $E_\infty$ -page of the effective spectral sequence for  $L$  in coweights  $2 \bmod 4$

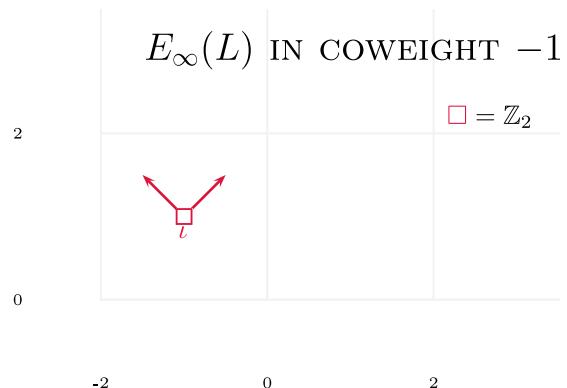


FIGURE 16. The  $E_\infty$ -page of the effective spectral sequence for  $L$  in coweight  $-1$

$E_\infty(L)$  IN COWEIGHTS 3 MOD 8

12

$\square = \mathbb{Z}_2[\tau^8]$

$\blacksquare = \mathbb{Z}/2^n[\tau^8]$

$\bullet = \mathbb{F}_2[\tau^8]$

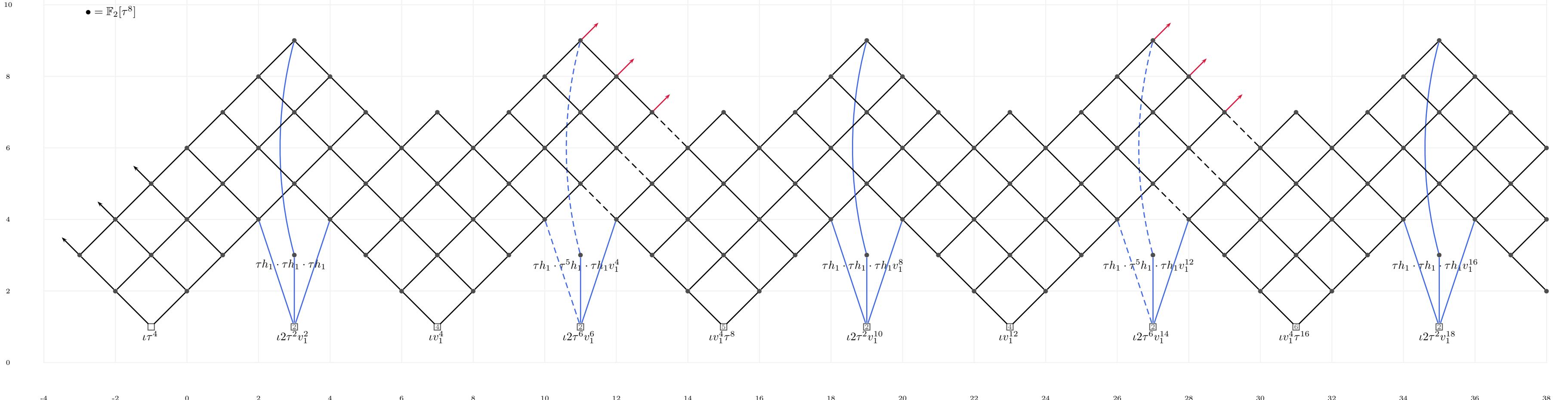
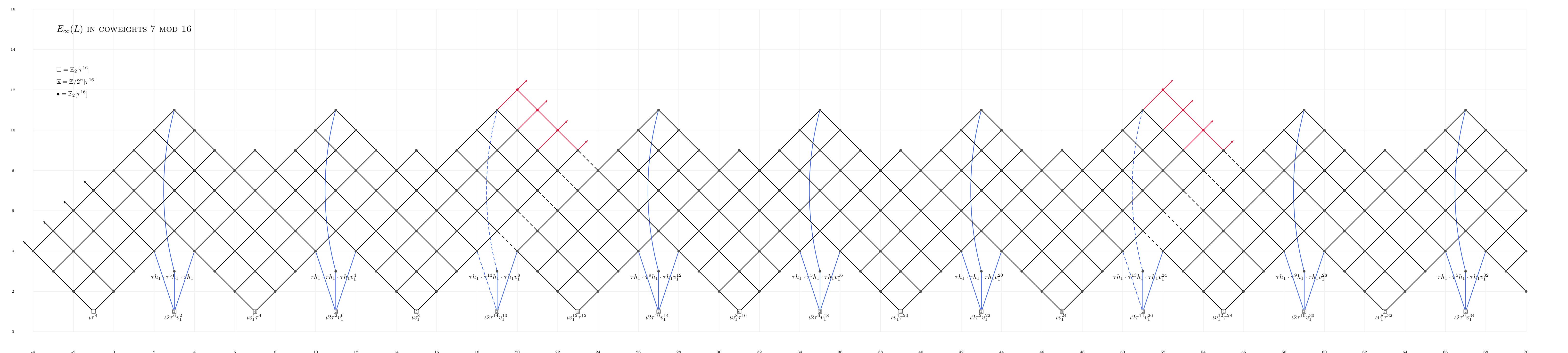


FIGURE 17. The  $E_\infty$ -page of the effective spectral sequence for  $L$  in coweights 3 mod 8



spectral sequence for  $L$  in coweights  $7 \bmod 16$

$E_\infty(L)$  IN COWEIGHTS 15 MOD 32

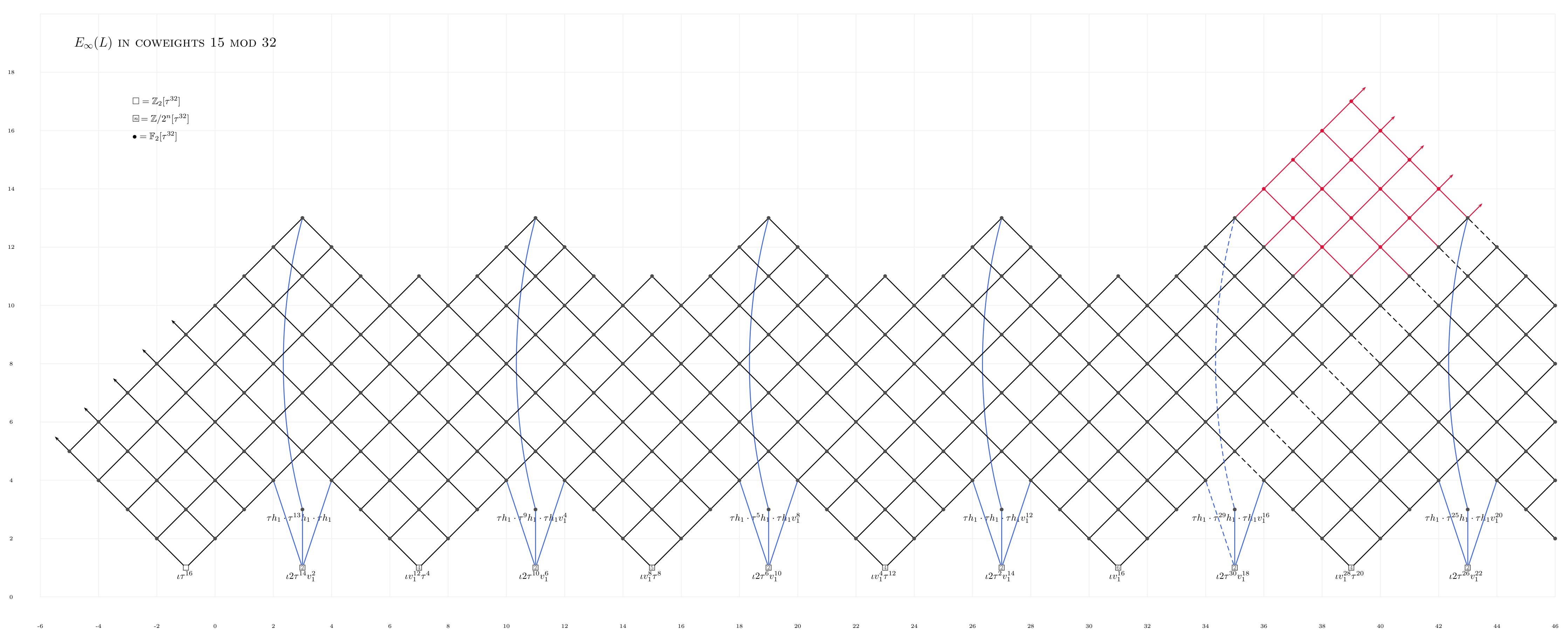


FIGURE 19. The  $E_\infty$ -page of the effective spectral sequence for  $L$  in coweights 15 mod 32

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