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ACC for generalized log canonical thresholds for complex analytic spaces

In Memory of Professor Gang Xiao

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Abstract We show that generalized log canonical thresholds for complex analytic spaces satisfy the ACC (ascending chain condition) and we characterize the accumulation points.

Keywords log canonical threshold, ACC property, generalized pair, complex analytic space

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1 Introduction

Throughout this paper we work with pairs (X,B) where X is a normal complex analytic variety and B is an effective $\mathbb{R}$-divisor such that K_X+B is $\mathbb{R}$-Cartier. Understanding the singularities of such pairs plays a fundamental role in recent advances in the birational classification of algebraic varieties. One important measure of these singularities is the log canonical threshold. If (X,B) is log canonical and D is a non-zero effective $\mathbb{R}$-Cartier divisor, then the log canonical threshold is

$$\mathrm{lct}(X,B;D):=\sup\{t\mid (X,B+tD)\text{ is log canonical}\}.$$

Understanding the behaviour of log canonical thresholds is essential in a variety of contexts such as, for example, the termination of flips, moduli problems, and K-stability (see, for example, [1,11,13]). Perhaps the most important result in this context is the solution, by Hacon, McKernan and Xu, of Shokurov's "ACC for LCT's conjecture" [10] which we will now recall.

A set I of non-negative real numbers satisfies the ascending chain condition or ACC (resp. the descending chain condition or DCC) if any non-decreasing sequence $i_1\leqslant i_2\leqslant\cdots$ (resp. any non-increasing sequence $i_1\geqslant i_2\geqslant\cdots$) is eventually constant. Let I and J be two DCC sets of nonnegative real numbers and n a natural number. We define

$$\mathrm{LCT}_n(I,J):=\{\mathrm{lct}(X,B;D)\mid \dim X=n,\ \mathrm{coeff}(B)\in I,\ \mathrm{coeff}(D)\in J\}$$

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to be the set of all log canonical thresholds of n-dimensional lc pairs (X, B) with respect to divisors D such that the coefficients of B and D belong to I and J respectively. When X is quasi-projective, then by [10, Theorem 1.1], it follows that the set $\mathrm{LCT}_n(I, J)$ satisfies the ACC and that one can characterize the accumulation points under mild assumptions on the DCC sets I and J. By [10, Theorem 1.1], it follows that if $I \subset [0, 1]$, the only possible accumulation point of I is 1, and $I = I_+ := \{0\} \cup \{j = \sum_{k=1}^r i_k \in [0, 1] \mid i_k \in I\}$, then the only accumulation points of $\mathrm{LCT}_n(I, \mathbb{N})$ are $\mathrm{LCT}_{n-1}(I, \mathbb{N}) \setminus \{1\}$.

Recently, generalized pairs have begun playing an increasingly more prominent role in birational geometry (see [2] and the references therein). It has become apparent that it is important to also study singularities in this context. The analogs of the main results of [10], for generalized pairs were proven in [4, 9], and have already found several important applications.

Naturally, it is also expected that the ACC for LCT's will play an important role in many other contexts such as analytic varieties, foliated pairs, varieties in positive and mixed characteristics (see, for example, [5, 8, 12]). In view of recent progress in the minimal model program for analytic varieties (see [6, 7]), it is expected that the results of [10] should also hold for analytic varieties. Fujino has in fact shown that Shokurov's ACC for LCT's conjecture holds for analytic varieties [8]. One interesting phenomenon that occurs in the analytic case (which does not happen in the algebraic case or for compact analytic varieties) is that if $\lambda = \mathrm{lct}(X, B; D)$, then it is possible that $(X, B + \lambda D)$ is klt, i.e., there is no divisor E over X of log discrepancy $a(E; X, B + \lambda D) = 0$ (see [8, Example 1.3]). This is somewhat troubling as typically, many proofs by induction on the dimension involve studying the restriction of $K_X + B + \lambda D$ to an appropriate divisor E over X of log discrepancy $a(E; X, B + \lambda D) = 0$.

We will now give a more precise description of the main results of this paper. Let I, J be DCC sets and $n \in \mathbb{N}$, $f : X' \to X$ a proper bimeromorphic morphism of analytic varieties, $(X, B + M)$, M', P', P, D be as in Definition 2.3 so that

1. $(X, B + M)$ is glc of dimension n,
2. $M' = \sum_j \mu_j M'_j$, where M'_j are relatively nef Cartier divisors on X', and $\mu_j \in I$,
3. $P' = \sum_k \nu_k P'_k$, where P'_k are relatively nef Cartier divisors on X', and $\nu_k \in J$,
4. the coefficients of B belong to I and the coefficients of D belong to J.

Then $\mathrm{GLCT}_n(I, J) \subset \mathbb{R}$ is the set consisting of all the possible generalized log canonical thresholds $\mathrm{glct}(X, B + M; D + P)$ where $(X, B + M; D + P)$ are as above (see Definition 2.3).

Theorem 1.1. *The set* $\mathrm{GLCT}_n(I, J)$ *satisfies the ACC.*

Moreover, we give a precise description of the accumulation points of generalized log canonical thresholds as in [10, Theorem 1.11] and [9, Theorem 1.7]:

Theorem 1.2. *If* 1 *is the only accumulation point of the DCC set* $I \subset [0, 1]$ *and* $1 \in I = I_+$*, then the accumulation points of* $\mathrm{GLCT}_n(I) := \mathrm{GLCT}_n(I, \mathbb{N})$ *belong to* $\mathrm{GLCT}_{n-1}(I)$.

2 Preliminaries

Let X be a normal complex analytic space. A prime divisor P on X is an irreducible and reduced closed subspace of codimension one. An $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) D on X is a locally finite formal sum $D = \sum_i d_i D_i$ of distinct prime divisors D_i with coefficients $d_i \in \mathbb{R}$ (resp. $d_i \in \mathbb{Q}$). If for some point $x \in X$ there is a neighborhood $x \in U \subset X$ such that the restriction $D|_U$ of the $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) D is a finite $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of Cartier divisors, then we say that D is $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) at $x \in X$. If D is $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) at every $x \in X$ then we say that D is $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier).

Definition 2.1. We say that $(X, B + M)$ is a generalized pair if there is a proper bimeromorphic morphism $f : X' \to X$ and an f-nef $\mathbb{R}$-Cartier divisor M' on X' such that

1. X' and X are normal,
2. $M = f_* M'$ and $B \geqslant 0$,
3. $K_X + B + M$ is $\mathbb{R}$-Cartier.

We call B the boundary part and M the nef part of the generalized pair. We can always replace X' by a higher model that factors through f, and M' by its pullback.

We can write

$$K_{X'} + B' + M' = f^*(K_X + B + M)$$

and we say that the generalized pair $(X, B + M)$ is *generalized log canonical* (*glc*) at $x \in X$ if there is a neighborhood $x \in U \subset X$ such that $(X', B')|_U$ is sub-lc (see [7, Remark 3.2]) and is *generalized Kawamata log terminal* (*gklt*) at $x \in X$ if there is a neighborhood $x \in U \subset X$ such that $(X', B')|_U$ is sub-klt (see [7, Remark 3.2]). We can also define the *log discrepancy* $a(E, X, B + M) := a(E, X', B')$ for any divisor over X. We say that Z is a *log canonical center* (resp. *log canonical place*) of a glc pair $(X, B + M)$ if Z is the image of an lc center of (X', B') (resp. a log canonical place of (X', B')).

We say that a glc pair $(X, B+M)$ is *generalized divisorially log terminal* (*gdlt*) if we can choose $f : X' \to X$ (in the definition) to be a log resolution of (X, B) such that the log discrepancy is $a(E, X, B+M) > 0$ for every f-exceptional divisor E.

Definition 2.2. A set $I \subset \mathbb{R}$ satisfies the ACC (resp. DCC) if any non-decreasing (resp. non-increasing) sequence $I_k \subset I$ is eventually constant. We let ∂I be the set of accumulation points of I and $\bar{I} = I \cup \partial I$. If $I \subset [0, +\infty)$, then

$$I_+ = \{0\} \cup \left\{ \sum_{k=1}^{l} i_k \in [0, 1] \;\middle|\; i_k \in I \right\}$$

and

$$D(I) = \left\{ a \leqslant 1 \;\middle|\; a = \frac{m - 1 + f}{m},\ m \in \mathbb{N}^+,\ f \in I_+ \right\}.$$

If $I \subset [0, 1]$, then we let

$$\Phi(I) = \left\{ 1 - \frac{r}{m} \;\middle|\; r \in I,\ m \in \mathbb{N}^+ \right\}.$$

Definition 2.3 (Generalized log canonical thresholds for complex analytic spaces)**.** Let $(X, B+M)$ be a generalized log canonical pair and let D be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and $P = f_*P'$ where P' is a nef divisor on X'. Let c be the supremum of all real numbers such that $(X, B + M + t(D + P))$ is generalized log canonical, then c is called the generalized log canonical threshold of $D + P$ with respect to $(X, B + M)$ and is denoted by $\operatorname{lct}(X, B + M; D + P)$.

Lemma 2.4. *If $(X, B + M)$ and $D + P$ are as above, then $(X, B + M + c(D + P))$ is generalized log canonical.*

Proof. This follows directly from the definition. □

Remark 2.5. Note that the above definition differs from the one in [8] as there does not necessarily exist a non-Kawamata log terminal center of $(X, B + M + c(D + P))$. The issue is that X may not be compact. In this case, we may not have a log resolution, and the divisor D may have infinitely many components (see [8, Example 1.3]). If however X is (relatively) compact, then log resolutions exist, the two definitions agree, and we always have a log canonical center of $(X, B + M + c(D + P))$ (see [8, Remark 1.4]).

The next theorem is the analogue of dlt-blowups of generalized pairs in the complex analytic setting:

Theorem 2.6 (Dlt-blowup)**.** *Let X be a normal complex variety and $(X, B+M)$ a generalized pair as in Definition 2.1. Let U be any relatively compact Stein open subset of X and let V be any relatively compact open subset of U. Then we can take a Stein compact subset W of U such that $\Gamma(W, \mathcal{O}_X)$ is Noetherian, $V \subset W$, and after shrinking X around W suitably, we can construct a projective bimeromorphic morphism $g : Y \to X$ from a normal complex variety Y with the following properties*:

1. *Y is $\mathbb{Q}$-factorial over W;*
2. *$a(E, X, B + M) \leqslant 0$ for every g-exceptional divisor E on Y;*
3. *$(Y, B_Y^{\leqslant 1} + M_Y)$ is gdlt, where $K_Y + B_Y + M_Y = f^*(K_X + B + M)$.*

Proof. We will freely shrink X suitably without mentioning it explicitly. By taking a resolution of singularities, we can assume that $f: X' \to X$ is a projective bimeromorphic morphism such that $f^{-1}(U)$ is smooth and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f^{-1}B)$ is a simple normal crossing divisor on $f^{-1}(U)$. Let E be any f-exceptional divisor such that $f(E) \cap U \neq \emptyset$. Then, by enlarging V suitably, we may assume that $f(E) \cap V \neq \emptyset$. By [7, Lemma 2.16], we can take a Stein compact subset W of U such that $\Gamma(W, \mathcal{O}_X)$ is Noetherian and that $V \subset W$.

Write $K_{X'} + B' + M' = f^*(K_X + B + M)$ as in Definition 2.1 and let $B' = \sum_i a_i D_i$ be the irreducible decomposition, where each D_i is irreducible. Now we define a boundary

$$\Delta = \sum_{0<a_i<1} a_i D_i + \sum_{a_i \geqslant 1} D_i + \epsilon \sum_i E_i, \quad 1 \gg \epsilon > 0,$$

where E_i's are all the f-exceptional divisors such that $a(E_i, X, B+M) > 0$. Then we have

$$K_{X'} + \Delta + M' = f^*(K_X + B + M) + F$$

and we see that $-f_*F$ is effective. Let A be a general ample (over X) $\mathbb{Q}$-divisor such that $(X', \Delta + A + M')$ is gdlt and $K_{X'} + \Delta + A + M'$ is nef over W. Notice that for any $t > 0$, $tA + M'$ is f-ample. Therefore (over W) we can write

$$K_{X'} + \Delta + tA + M' \sim_{\mathbb{Q},f} K_{X'} + \Delta^t$$

for some klt pair (X', Δ^t). Then by [7, Theorem 1.7] we can run a $(K_{X'} + \Delta + M')$-MMP with scaling of A over X around W.

Let

$$(X_0, \Delta_0 + M') := (X', \Delta + M'), \quad F_0 := F, \quad M_0 := M' \quad \text{and} \quad A_0 := A.$$

Then we obtain a sequence of divisorial contractions and flips:

$$(X_0, \Delta_0 + M_0) \overset{\phi_0}{\dashrightarrow} (X_1, \Delta_1 + M_1) \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} (X_i, \Delta_i + M_i) \overset{\phi_i}{\dashrightarrow},$$

where Δ_i, M_i, F_i, A_i are the corresponding strict transforms. We also have the scaling numbers

$$1 \geqslant \lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_i \geqslant \cdots \geqslant 0$$

such that $K_{X_i} + \Delta_i + M_i + \lambda_i A_i$ is nef over W. Then by [7, Lemma 13.7] we know that

$$K_{X_k} + \Delta_k + M_k \in \overline{\mathrm{Mov}}(X_k/X; W)$$

for some $k \geqslant 0$. Thus by the negativity lemma (see [7, Lemma 4.6]) we have $-F_k \geqslant 0$ over W. Hence $-F_k$ is effective over some open neighborhood of W. Let

$$Y := X_k, \quad g: X_k \to X, \quad M_Y = M_k \quad \text{and} \quad K_Y + B_Y + M_Y = g^*(K_X + B + M).$$

Then $(Y, B_Y + M_Y)$ satisfies 1–3 above. □

3 Proof of the main theorems

Lemma 3.1. *We fix a positive integer n and a set $1 \in I \subset [0, \infty)$. Assume that $f: X' \to X$ is a projective morphism, and we have $\mathbb{R}$-divisors $B, D \geqslant 0$ on X and nef $\mathbb{R}$-divisors M', P' on X' such that*

1. *$(X, B+M)$ and $(X, B+D+M+P)$ are $(n+1)$-dimensional glc pairs with data given by M' and P', where $M = f_*M', P = f_*P'$;*
2. *$M' = \sum_j \mu_j M'_j$, where M'_j are relatively nef Cartier divisors and $\mu_j \in I$;*
3. *$P' = \sum_k \nu_k P'_k$, where P'_k are relatively nef Cartier divisors and $\nu_k \in I$;*
4. *the coefficients of $B, D, M' + P'$ belong to I.*

We further assume that there exists a non-gklt center V of $(X, B+D+M+P)$ such that V is not a non-gklt center of $(X, B+M)$ and $\dim V \leqslant \dim X - 2$. Then we can construct a generalized log canonical pair $(S, \Delta+N)$ with $S' \to S$ and N' as in Definition 2.1 such that

1. *S is a projective variety of dimension at most n;*
2. *the coefficients of Δ belong to $D(I)$;*
3. *$K_S + \Delta + N$ is numerically trivial;*

4 *$N' = \sum_i a_i N'_i$, where N'_i are relatively nef Cartier divisors and $a_i \in I$; and*

at least one of the following happens:

(i) *some component of Δ has coefficient of the form*

$$\frac{m-1+\alpha+c}{m},$$

where m is a positive integer, $\alpha \in I_+$, and $c \in I$ is the coefficient of some component of D or P';

(ii) *N'_i is not numerically trivial for some i and $a_i = g + \nu_k$, where $g \in I$ and $\nu_k > 0$ is a coefficient of P'.*

Proof. We can replace V with a maximal (with respect to inclusion) glc center of $(X, B+D+M+P)$ satisfying $\dim V \leqslant \dim X - 2$ and V is not a glc center of $(X, B+M)$. Let Q be an analytically sufficiently general point of V. Consider an open neighborhood U of Q and a Stein compact subset W of X such that $U \subset W$ and that $\Gamma(W, \mathcal{O}_X)$ is Noetherian. By Theorem 2.6, after shrinking X around W suitably, we can construct a projective bimeromorphic morphism $\pi : Y \to X$ with

$$K_Y + B_Y + D_Y + M_Y + P_Y = \pi^*(K_X + B + D + M + P)$$

such that

1. Y is $\mathbb{Q}$-factorial over W, and B_Y, D_Y, M_Y, P_Y are pushforwards of B', D', M', P' (after possibly replacing X' by a higher model);
2. $(Y, B_Y + D_Y + M_Y + P_Y)$ is gdlt, where $B_Y + D_Y$ is the boundary part and $M_Y + P_Y$ is the nef part;
3. $a(E, X, B+D+M+P) = 0$ holds for every π-exceptional divisor E; and
4. there exists a π-exceptional divisor F such that $\pi(F) = V$.

Let $\hat{D}$ be the strict transform of D on X' and $\hat{D}_Y$ be the pushforward of $\hat{D}$ on Y. We first claim that we can choose F in (4) such that $(\hat{D}_Y + P_Y)|_{F_v}$ is not numerically trivial, where $v \in V \cap U$ is an analytically sufficiently general point (see [7, Definition 2.50]). Let

$$E = \pi^*(D+P) - \hat{D}_Y - P_Y,$$

and then we can see $E \geqslant 0$ since $(X, B+M)$ is glc and every π-exceptional divisor E_i has log discrepancy $a(E_i, X, B+D+M+P) = 0$. Moreover, since V is not a log canonical center of $(X, B+M)$, E is non-trivial. Therefore by [3, Lemma 3.6.2] (see also [7, Section 11]) there is a component F of E with a covering family of curves C (contracted over X) such that $E \cdot C < 0$. So $(\hat{D}_Y + P_Y) \cdot C > 0$ for such curves and hence $(\hat{D}_Y + P_Y)|_F$ is not numerically trivial over sufficiently general points of V.

After replacing X' by a higher model, we may assume that $g : X' \to Y$ is a projective morphism. Let

$$K_{X'} + \Delta' + M' + P' = f^*(K_X + B + D + M + P)$$

and F' be the strict transform of F on X'. Let $\Delta_{F'}$ be the $\mathbb{R}$-divisor defined by the adjunction

$$K_{F'} + \Delta_{F'} = (K_{X'} + \Delta')|_{F'}$$

and Δ_F, M_F, P_F be the pushforwards of $\Delta_{F'}, M'|_{F'}, P'|_{F'}$, respectively, and then these data define a generalized pair $(F, \Delta_F + M_F + P_F)$ with nef part $M_F + P_F$ as in Definition 2.1 and we have

$$(K_Y + B_Y + D_Y + M_Y + P_Y)|_F \sim_{\mathbb{R}} K_F + \Delta_F + M_F + P_F.$$

Following [4, Definition 4.7 and Remark 4.8], $(F, \Delta_F + M_F + P_F)$ is generalized log canonical.

Next, we calculate the coefficients of Δ_F. Cutting by hyperplanes in Y we can assume that $\dim Y = 2$ (note that we are working over a Stein set W and Y is projective over W). Let $p \in F$ be a point and l_p be the Cartier index at $p \in Y$. In this case we may assume that F' is a normal curve isomorphic to F (since F is already normal), so we may also regard p as a point on F'. Then by classification of klt surface singularities we obtain that

$$\begin{aligned}\operatorname{mult}_p \Delta_F &= \frac{l_p - 1}{l_p} + \operatorname{mult}_p(B_Y + D_Y - F)|_F + \operatorname{mult}_p((g^*(M_Y + P_Y) - M' - P')|_{F'}) \\ &= \frac{l_p - 1 + \beta + \gamma}{l_p} \in D(I),\end{aligned}$$

where

$$\operatorname{mult}_p(B_Y + D_Y - F)|_F = \frac{\beta}{l_p}, \quad \operatorname{mult}_p((g^*(M_Y + P_Y) - M' - P')|_{F'}) = \frac{\gamma}{l_p}$$

and $\beta, \gamma, \beta + \gamma \in I_+$ by the assumptions on the coefficients of $B + D$ and $M' + P'$.

Let S (resp. S') be the general fiber of the Stein factorization of $F \to V$ (resp. $F' \to V$), $\Delta = \Delta_F|_S$, $N' = (M' + P')|_{S'}$ and $N = (M_F + P_F)|_S$. Then, these data define a generalized pair $(S, \Delta + N)$ with nef part N as in Definition 2.1 and we have

1. $(S, \Delta + N)$ is generalized log canonical;
2. $K_S + \Delta + N \sim_{\mathbb{R}} 0$;
3. the coefficients of N' belong to I;
4. $(\hat{D}_Y + P_Y)|_S$ is not numerically trivial.

If $P'|_{S'}$ is not numerically trivial, then (ii) in the statement is satisfied and we are done. So we can assume that $P'|_{S'} \equiv 0$, hence $P_F|_S \equiv 0$. If we write $g^*(P_Y) = P' + G$ and let G_S be the pushforward of $G|_{S'}$ on S, then we have

$$(\hat{D}_Y + P_Y)|_S = P_F|_S + G_S + \hat{D}_S \equiv G_S + \hat{D}_S \neq 0,$$

where $\hat{D}_S := \hat{D}_Y|_S$. Let R_F be the pushforward of $(g^*(M_Y + P_Y) - M' - P')|_{F'}$ on F and $R_S := R_F|_S$, and then $\operatorname{mult}_p(R_F) = \frac{\gamma}{l_p}$ in the previous computation. Notice that $\hat{D}_S \leqslant (B_Y + D_Y - F)|_S$ and $G_S \leqslant R_S$, therefore $G_S + \hat{D}_S \neq 0$ implies that (i) holds. □

Theorem 3.2. *Let Λ be a DCC set of non-negative real numbers and d a positive integer. Assume $X_i, B_i, M_i, M_i', D_i, P_i', P_i'$ are as in Definition 2.3 such that for any $i \geqslant 1$,*

1. *$(X_i, B_i + M_i)$ are glc pairs of dimension d;*
2. *$M_i' = \sum_{i,j} \mu_{i,j} M_{i,j}'$, where $M_{i,j}'$ are relatively nef Cartier divisors and $\mu_{i,j} \in \Lambda$;*
3. *$P_i' = \sum_{i,k} \nu_{i,k} P_{i,k}'$, where $P_{i,k}'$ are relatively nef Cartier divisors and $\nu_{i,k} \in \Lambda$;*
4. *the coefficients of B_i and D_i belong to Λ;*
5. *$(X_i, B_i + M_i + t_i D_i + t_i P_i)$ is glc and has a glc center V_i which is not a glc center of $(X_i, B_i + M_i)$ for some $t_i > 0$.*

Then $T = \{t_i\}_{i \geqslant 1}$ is an ACC set.

Proof. Assume that the sequence $\{t_i\}_{i \geqslant 1}$ is strictly increasing. If $\dim V_i = d - 1$ for infinitely many i, then we have $1 - t_i \lambda_i = \lambda_i' \in \Lambda \cap [0, 1]$ for some $0 < \lambda_i \in \Lambda$. Let

$$\Gamma_1 := \left\{ \frac{1}{\lambda} \;\middle|\; 0 < \lambda \in \Lambda \right\}, \quad \Gamma_2 := \{1 - \lambda' \mid \lambda' \in \Lambda \cap [0, 1]\},$$

and then Γ_1, Γ_2 and $\Gamma_1 \cdot \Gamma_2$ are ACC sets. Passing to a subsequence we have $\{t_i\}_{i \geqslant 1} \subset \Gamma_1 \cdot \Gamma_2$, which contradicts the assumption that $\{t_i\}_{i \geqslant 1}$ is strictly increasing. Therefore passing to a subsequence we can assume that $\dim V_i \leqslant d - 2$.

Let

$$I := \Lambda \cup (T \cdot \Lambda) \cup (\Lambda + T \cdot \Lambda),$$

and then I is also a DCC set. Possibly replacing $\{t_i\}_{i\geqslant 1}$ by a subsequence, by Lemma 3.1 and [4, Theorem 1.6], one of the following happens:

(i) $\frac{m_i-1+\alpha_i+t_ic_i}{m_i}$ belongs to a finite set Λ^0 for every $i\geqslant 1$, where $m_i\in\mathbb{N}^*$, $\alpha_i\in I_+$ and $0<c_i\in I$;

(ii) $g_i+t_i\nu_{i,k}$ belongs to a finite set Λ^0 for every $i\geqslant 1$, where $g_i\in I$ and $\nu_{i,k}>0$.

In either case we can conclude that t_i must belong to a finite set Λ^1, which is a contradiction and we are done. □

It is easy to see that Theorem 1.1 is equivalent to the following theorem (letting $\Lambda = I\cup J$), whose algebraic case is exactly [4, Theorem 1.5].

Theorem 3.3 (ACC for generalized lc thresholds). *Let Λ be a DCC set of nonnegative real numbers and d a natural number. Then there is an ACC set Θ depending only on Λ and d such that if $(X,B+M)$, M', P', P, D are as in Definition* 2.3*, and*

1. *$(X,B+M)$ is glc of dimension d;*
2. *$M'=\sum_j \mu_j M'_j$, where M'_j are relatively nef Cartier divisors and $\mu_j\in\Lambda$;*
3. *$P'=\sum_k \nu_k P'_k$, where P'_k are relatively nef Cartier divisors and $\nu_k\in\Lambda$; and*
4. *the coefficients of B and D belong to Λ,*

then the generalized lc threshold $\operatorname{glct}(X,B+M;D+P)$ *of $D+P$ with respect to $(X,B+M)$ belongs to Θ.*

Proof. Suppose that

$$c=\operatorname{glct}(X,B+M;D+P)>0,$$

and then there exists a non-increasing sequence $c_i\geqslant c_{i+1}\geqslant\cdots$ with $\lim_{i\to\infty}c_i=c$ and relatively compact open subsets $U_i\subset X$ such that

$$c_i=\operatorname{glct}(U_i,B_i+M_i;D_i+P_i),$$

where $B_i+M_i+D_i+P_i=(B+M+D+P)|_{U_i}$. By Remark 2.5 we know that $(U_i,B_i+M_i+c_iD_i+c_iP_i)$ has a glc center which is not a glc center of (U_i,B_i+M_i). Since the closure of an ACC set is also an ACC set, it suffices to consider relatively compact varieties X, and then the statement follows from Theorem 3.2. □

Proof of Theorem 1.2. Suppose c is an accumulation point of $\mathrm{GLCT}_n(I)$. Then again there exists a non-increasing sequence $c_i\geqslant c_{i+1}\geqslant\cdots$ with $\lim_{i\to\infty}c_i=c$ and relatively compact open subsets $U_i\subset X$ such that

$$c_i=\operatorname{glct}(U_i,B_i+M_i;D_i+P_i),$$

where $B_i+M_i+D_i+P_i=(B+M+D+P)|_{U_i}$. Therefore by Lemma 3.1 we know that $c_i\in\mathcal{N}_d(I,\mathbb{N},\mathbb{N})$, which is defined in [9, Definition 2.18]. Notice that since the generalized pair $(S,\Delta+N)$ constructed in Lemma 3.1 is projective, we are in the algebraic setting and so the proof follows from [9]. □

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References

1 Birkar C. Ascending chain condition for log canonical thresholds and termination of log flips. Duke Math J, 2007, 136: 173–180

2 Birkar C. Generalised pairs in birational geometry. EMS Surv Math Sci, 2021, 8: 5–24

3 Birkar C, Cascini P, Hacon C D, et al. Existence of minimal models for varieties of log general type. J Amer Math Soc, 2010, 23: 405–468

4 Birkar C, Zhang D Q. Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs. Publ Math Inst Hautes Études Sci, 2016, 123: 283–331

5 Chen Y A. ACC for foliated log canonical thresholds. arXiv:2202.11346, 2022

6 Das O, Hacon C D, Păun M. On the 4-dimensional minimal model program for Kähler varieties. arXiv:2205.12205, 2022

7 Fujino O. Minimal model program for projective morphisms between complex analytic spaces. arXiv:2201.11315, 2022

8 Fujino O. ACC for log canonical thresholds for complex analytic spaces. arXiv:2208.11872, 2022

9 Liu J H. Accumulation point theorem for generalized log canonical thresholds. arXiv:1810.12381, 2018

10 Hacon C D, M^{c}Kernan J, Xu C Y. ACC for log canonical thresholds. Ann of Math (2), 2014, 180: 523–571

11 Hacon C D, M^{c}Kernan J, Xu C Y. Boundedness of moduli of varieties of general type. J Eur Math Soc, 2018, 20: 865–901

12 Sato K. Ascending chain condition for F-pure thresholds with fixed embedding dimension. Int Math Res Not IMRS, 2021, 2021: 7205–7223

13 Xu C Y, Zhuang Z Q. Uniqueness of the minimizer of the normalized volume function. Camb J Math, 2021, 9: 149–176