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A New Expression for the Passivity Bound for a Class of Sampled-Data Systems

Rodney G. Roberts, *Senior Member, IEEE,* Carl A. Moore, Jr., and J. Edward Colgate, *Fellow, IEEE*

***Abstract*—In this article, we characterize the passivity of a class of haptic systems modeled as a simple sampled-data system. We guarantee passivity by ensuring that there is sufficient damping in the haptic interface. Previous work established a necessary and sufficient bound on damping, but the corresponding mathematical expressions were complicated, and the derivation was not completely rigorous. After providing a rigorous proof, we derive a more tractable expression. Using this improved expression, we establish passivity conditions for several classes of transfer functions representing virtual environments, including some special cases with time delay. The original results assumed that the operator can be modeled by a passive but otherwise arbitrary transfer function. This assumption is weakened to allow the operator model to have a shortage of passivity. This requires only a slight modification of the passivity bound.**

***Index Terms*—Haptics and Haptic Interfaces, Telerobotics and Teleoperation**

I. Introduction

PASSIVITY is an important approach to ensuring stability of a haptic device [1], [2], [3], [4], [5], [6]. Colgate and Schenkel [7] used a passivity-based approach to establish a necessary and sufficient condition for the passivity of a sampled data system corresponding to a 1-DOF haptic interface. Their passivity condition was given by a lower bound on the viscous damping of the actuator, with necessity proved using Nyquist plane methods and sufficiency proved using an energy-based argument. Other research involving viscous or Coulomb friction include [8], [9], [10], [11]. Hulin *et al.* [12] derived stability boundaries while taking into account human operator dynamics. Nonlinear virtual environments [13], which can also be analyzed using energy-based arguments to prove passivity, and multiple degree-of-freedom haptic systems [6], [14] have also been the subject of study. Mashayekhi *et al.* [15] presented a stability analysis of a haptic device experiencing time delay using a frequency response function analysis via a continuous time model of the system. Other works taking into account time delay include [16], [17], [18]. Pecly and Hashtrudi-Zaad [19] analyzed the passivity and uncoupled stability of haptic systems when a viscoelastic virtual environment is implemented using a two-parameter Tustin-like general discretization method. A Routh-Hurwitz approach was described in [20].

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R. G. Roberts is with the Department of Electrical and Computer Engineering, Florida A&M University-Florida State University College of Engineering, Tallahassee, FL, 32310 USA e-mail: rroberts@eng.famu.fsu.edu.

C. A. Moore, Jr., is with the Department of Mechanical Engineering, Florida A&M University-Florida State University College of Engineering, Tallahassee, FL, 32310 USA e-mail: camoore@eng.famu.fsu.edu.

J. Edward Colgate, is with the Department of Mechanical Engineering, Northwestern University, Evanston, IL, 60208 USA e-mail: colgate@northwestern.edu.

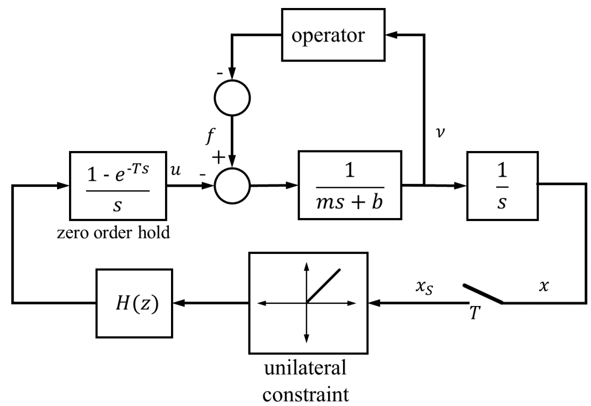

Fig. 1: Model of a 1-DOF haptic system. The system consists of a haptic interface modeled as a combination of a rigid body of mass m with viscous damping $b > 0$, an ideal sampler operating at a sampling rate of $1/T$, a virtual environment (feedback controller) represented by a stable linear, shift-invariant transfer function $H(z)$, a zero order hold, a unilateral constraint, and a human operator modeled by a transfer function $Z_O(s)$.

In this article, we reexamine the approach in [7]. In particular, we clarify some of the proofs, introduce an improved expression for the passivity bound, and analyze a broader class of virtual environments. The haptic system that we will study is shown in Fig. 1. It consists of a haptic interface modeled as a combination of a rigid body of mass m with viscous damping $b > 0$, an ideal sampler operating at a sampling rate of $1/T$, a virtual environment (feedback controller) represented by a stable linear, shift-invariant transfer function $H(z)$, a zero order hold, a unilateral constraint, and a human operator modeled by a passive but otherwise arbitrary transfer function $Z_O(s)$. Amplifier and sensor dynamics, nonlinearity, and noise are ignored.

A necessary and sufficient condition for the passivity of the sampled data system in Fig. 1 without the unilateral constraint is presented in [7] in terms of a lower bound for the damping constant b. In particular, it was shown that the system without the unilateral constraint is passive if and only if

$$b > \frac{T}{2}\frac{\mathrm{Re}\{(1-e^{-j\omega T})H(e^{j\omega T})\}}{1-\cos\omega T} \text{ for } 0 \le \omega \le \omega_N \quad (1)$$

where $\omega_N = \pi/T$ is the Nyquist frequency. Although the analysis was performed without the unilateral constraint, (1) still implies passivity for the system with the unilateral constraint [7].

In this paper, we reexamine the haptic system shown in Fig. 1. In particular, we more rigorously justify the passivity analysis in [7] and introduce a simpler expression for the passivity bound for b that leads to the analysis of virtual environments that would not have been tractable with the previous passivity bound expression.

The remainder of the paper is outlined as follows. In the next section, we review relevant background and formulate the problem. Earlier work [7] used an intermediate result that was not properly justified. In Section III, we provide a correct proof of this important result used to prove that (1) is a necessary condition for passivity provided that the operator can be considered a passive system. In Section IV, we demonstrate that (1) is also a sufficient condition. In Section V, we introduce an equivalent but much simpler expression for (1) and employ it in Section VI to derive closed form expressions for several classes of H(z) transfer functions including cases where time delay is present. These bounds would have been difficult to determine using the prior passivity bound expression. In Section VII, we weaken the assumption that the operator model is passive to the assumption that the operator model may have a shortage of passivity. Lastly, conclusions appear in Section VIII.

II. Problem Formulation

The passivity problem for the system shown in Fig. 1 is formulated in the following way. We initially assume that the unilateral constraint is not present. If the transfer function corresponding to the operator block is denoted $Z_O(s)$, then the upper portion of the block diagram excluding the sampler, unilateral constraint, and the transfer function $H(z)$, can be replaced with the single transfer function

$$G(s) = \frac{1-e^{-sT}}{s^2}\frac{1}{ms+b+Z_O(s)}. \quad (2)$$

The transfer function $G(s)$ incorporates the zero order hold, the operator, the haptic device, and the integrator. The system in Fig. 1 contains an ideal sampler that samples the signal $x(t)$ with a sampling period T. The necessary sampling theory background for discrete-time control systems is based on modeling the ideal sampler using a pulse train $\delta_T(t) = \sum_{n=0}^{\infty}\delta(t-nT)$ where $\delta(t)$ is the unit impulse function and T is the sampling period [21]. The pulse train captures the sampled values of the signal $x(t)$ shown in Fig. 1:

$$x_S(t) = x(t)\delta_T(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT).$$

The Laplace transform of $x_S(t)$ is then given by

$$X_S(s) = \sum_{k=0}^{\infty} x(kT)e^{-ksT} = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(s+jn\omega_s) \quad (3)$$

where $\omega_s = 2\pi/T$ and where $X(s)$ is the Laplace transform of $x(t)$. The right hand side of (3) follows from sampling theory [21]. In a similar way, the ideal sampler requires a modification of the transfer function $G(s)$ given by (2):

$$G^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jn\omega_s). \quad (4)$$

Equation (4) is called the starred transform of $G(s)$ [21]. Without the unilateral constraint, the closed loop characteristic equation of the sampled data system is

$$1 + G^*(s)H(e^{sT}) = 0. \quad (5)$$

Equation (5) determines the stability of the system and will be used in the next section to derive a necessary condition for passivity. The fact that (5) is also a sufficient condition will be proved using an energy argument in Section IV.

III. Necessary Condition

Several steps are required to show that (1) is a necessary condition. These steps are illustrated in Fig. 2. First, observe that the passivity of $Z_O(s)$ means that the values of $Z_O(j\omega)$ lie in the closed right half plane. Since passivity is the only requirement specified for $Z_O(s)$, we consider the whole closed right half plane when studying the possible range of the human operator model $Z_O(j\omega)$ as indicated in Fig. 2(a). Adding $mj\omega + b$ to $Z_O(j\omega)$ has the effect of shifting the area of interest, i.e., the closed right half plane, to the right by b. Taking the reciprocal, we find that the image of

$$G_0(s) = \frac{1}{ms+b+Z_O(s)}$$

as $s = j\omega$ varies over all real ω is equal to $R_1 = \bar{D}_{\frac{1}{2b}}(\frac{1}{2b})$ where $\bar{D}_r(z_0)$ denotes the closed disk in the complex plane of radius r centered at z_0. This follows from the fact that the function $1/z$ maps $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \ge b\}$ to R_1. These two plane operations are shown in Fig. 2(b) and (c), respectively.

We next determine the range of possible values of $G^*(j\omega)$. Since $\omega_s = 2\pi/T$ and $e^{j(\omega+n\omega_s)T} = e^{j\omega T}$, we can write

$$G^*(j\omega) = \sum_{n=-\infty}^{\infty} a_n(\omega)b_n(\omega)$$

where

$$a_n(\omega) = \frac{1}{T}\frac{e^{-j\omega T}-1}{(\omega+n\omega_s)^2} = T\frac{e^{-j\omega T}-1}{(\omega T+2n\pi)^2} \quad (6)$$

and

$$b_n(\omega) = G_0(j\omega + jn\omega_s). \quad (7)$$

The $a_n(\omega)$ terms are fixed while the $b_n(\omega)$ terms depend on $Z_O(s)$ but can take on any values in R_1. Using this notation, we can write $R_{G^*}(j\omega)$, the range of possible values of $G^*(j\omega)$, as

$$R_{G^*}(\omega) = \left\{ \sum_{n=-\infty}^{\infty} a_n(\omega)b_n(\omega) \;\middle|\; b_n(\omega) \in R_1 \right\}.$$

It was pointed out in [7] that $R_{G^*}(\omega) = r(j\omega)R_1$ where

$$r(j\omega) = \sum_{n=-\infty}^{\infty} a_n(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\frac{e^{-j\omega T}-1}{(\omega+n\omega_s)^2},$$

but a rigorous derivation was not provided and some of the underlying assumptions need to be modified for the argument to hold. We now present a formal statement and proof of this result.

Theorem 1. $R_{G^*}(\omega) = r(j\omega)R_1$.

Proof. Let $a_n(\omega)$ and $b_n(\omega)$ be given by (6) and (7), respectively. We follow the standard approach of proving the equality of two sets by demonstrating that each contains the other. First, we show that $r(j\omega)R_1$ is contained in $R_{G^*}(\omega)$. Let $c(\omega) \in r(j\omega)R_1$. Then there is a $d(\omega) \in R_1$ such that

$$c(\omega) = r(j\omega)d(\omega) = \sum_{n=-\infty}^{\infty} a_n(\omega)d(\omega). \tag{8}$$

Since $d(\omega) \in R_1$, we can choose $Z_O(s)$ so that $b_n(\omega)$ in (7) equals $d(\omega)$ for each integer n. Hence, the term on the right-hand side of (8) is clearly in $R_{G^*}(\omega)$. This proves the first inclusion, i.e., $r(j\omega)R_1 \subset R_{G^*}(\omega)$.

To prove the other set inclusion, we define $\hat{r}(\omega)$ as

$$\hat{r}(\omega) = \frac{r(j\omega)}{T(e^{-j\omega T}-1)} = \sum_{n=-\infty}^{\infty} \frac{1}{(\omega T+2n\pi)^2}. \tag{9}$$

Assume that $\hat{r}(\omega)$ is finite. Since $\hat{r}(\omega) \geq 0$, it follows that

$$\hat{r}(\omega)R_1 = \hat{r}(\omega)\bar{D}_{\frac{1}{2b}}\left(\frac{1}{2b}\right) = \bar{D}_{\frac{\hat{r}(\omega)}{2b}}\left(\frac{\hat{r}(\omega)}{2b}\right). \tag{10}$$

Next, note that for $b_n(\omega) \in R_1$, $n = 0, \pm 1, \pm 2, \ldots$, we have

$$\begin{aligned}
\left|\sum_{n=-\infty}^{\infty} \frac{b_n(\omega)}{(\omega T+2n\pi)^2} - \frac{\hat{r}(\omega)}{2b}\right| &= \left|\sum_{n=-\infty}^{\infty} \frac{b_n(\omega)-\frac{1}{2b}}{(\omega T+2n\pi)^2}\right| \\
&\leq \sum_{n=-\infty}^{\infty} \frac{|b_n(\omega)-\frac{1}{2b}|}{(\omega T+2n\pi)^2} \\
&\leq \sum_{n=-\infty}^{\infty} \frac{\frac{1}{2b}}{(\omega T+2n\pi)^2} \\
&= \frac{\hat{r}(\omega)}{2b}
\end{aligned} \tag{11}$$

where we have used the fact that $b_n(\omega) \in R_1 = \bar{D}_{\frac{1}{2b}}(\frac{1}{2b})$ to obtain the second inequality. Equations (11) and (10) imply that

$$\sum_{n=-\infty}^{\infty} \frac{b_n(\omega)}{(\omega T+2n\pi)^2} \in \bar{D}_{\frac{\hat{r}(\omega)}{2b}}\left(\frac{\hat{r}(\omega)}{2b}\right) = \hat{r}(\omega)R_1. \tag{12}$$

On the other hand, if $\hat{r}(\omega)$ is infinite, then it can be shown that $\hat{r}(\omega)R_1$ is the open right half plane along with the origin and (12) still holds. Lastly, we note that

$$\begin{aligned}
G^*(j\omega) &= \sum_{n=-\infty}^{\infty} a_n(\omega)b_n(\omega) \\
&= T(e^{-j\omega T}-1)\sum_{n=-\infty}^{\infty} \frac{b_n(\omega)}{(\omega T+2n\pi)^2} \\
&\in T(e^{-j\omega T}-1)[\hat{r}(\omega)R_1] = r(j\omega)R_1,
\end{aligned}$$

proving that $R_{G^*}(\omega) \subset r(j\omega)R_1$. Together with the other inclusion result, this implies that $R_{G^*}(\omega) = r(j\omega)R_1$. This completes the proof. □

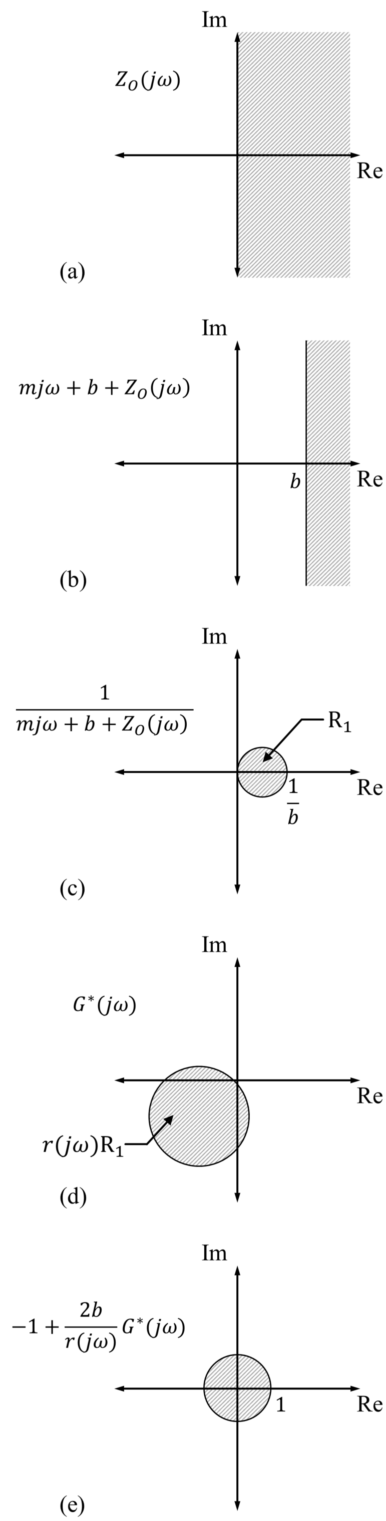

Fig. 2: Sequence of Nyquist plane transformations mapping the possible range of the passive operator transfer function $Z_O(j\omega)$ to the closed unit disk.

A careful examination of the above proof shows that the property that the $a_n(\omega)$ terms all have the same phase for a fixed ω is critical. Because of this, $\hat{r}(\omega)$ was real and positive, so that (10) holds. If, for example, a first order hold were used instead of a zero order hold, the corresponding $a_n(\omega)$ terms would not have this property and the situation would be more complicated. This observation implies that, while the theorem is true, the approach described in [7] needs further clarification.

By Theorem 1, the range of $G^*(j\omega)$ is a rotated and scaled version $r(j\omega)R_1$ for R_1 as shown in Fig. 2(d). It then follows by scaling $G^*(j\omega)$ by $2b/r(j\omega)$ and shifting by -1 that the range of $-1+\frac{2b}{r(s)}G^*(s)$ is the closed unit disk as shown in Fig. 2(e). The authors of [7] use this fact to derive the bound in (1) by noting that one can write

$$\frac{1+G^*(s)H(e^{sT})}{1+\frac{r(s)}{2b}H(e^{sT})}=1+Z_1(s)Z_2(s) \tag{13}$$

where

$$Z_1(s)=-1+\frac{2b}{r(s)}G^*(s)$$

and

$$Z_2(s)=\frac{r(s)H(e^{sT})}{2b+r(s)H(e^{sT})}.$$

The authors of [7] apply the small gain theorem [22] to obtain a necessary and sufficient condition for stability. The key point is that (13) and the closed loop characteristic equation (5) have the same unstable roots provided that b is properly bounded. First, note that the poles of $H(e^{sT})$ are also roots of (13) but are assumed to be stable. The function $r(s)$ introduces imaginary poles at integer multiples of the sampling frequency. The only situation that we need to address is when there is a value of ω for which $|Z_2(j\omega)|\geq 1$. In that case, there would exist a passive operator transfer function $Z_O(s)$ such that $Z_1(j\omega)=-1/Z_2(j\omega)$, i.e., $1+Z_1(s)Z_2(s)=0$. Hence, we must have $|Z_2(j\omega)|<1$, i.e.,

$$\left|\frac{r(j\omega)H(e^{j\omega T})}{2b+r(j\omega)H(e^{j\omega T})}\right|<1. \tag{14}$$

This expression can be rewritten as $|r(j\omega)H(e^{j\omega T})|<|2b+r(j\omega)H(e^{j\omega T})|$, and since the imaginary parts of $r(s)H(e^{sT})$ and $2b+r(s)H(e^{sT})$ are equal, it follows that (14) is equivalent to

$$b>\mathrm{Re}\Big\{-r(j\omega)H(e^{j\omega T})\Big\}=-\Phi(\omega)\sum_{n=-\infty}^{\infty}\frac{1}{(\omega+n\omega_s)^2} \tag{15}$$

where

$$\Phi(\omega)=\mathrm{Re}\{(e^{-j\omega T}-1)H(e^{j\omega T})/T\}.$$

We thus conclude that satisfying (15) for $0\leq\omega\leq\omega_N$ is a necessary requirement for passivity. It will be shown in Section V that (15) is equivalent to (1).

IV. Sufficient Condition

The fact that (1) is a sufficient condition for passivity was proved in [7] using an energy argument. This approach was extended in [23] to determine a passivity criterion for a class of sampled-data bilateral teleoperation systems. However, the proofs provided in [7] and [23] are lengthy and require several non-obvious steps that are difficult to motivate. While such an analysis is necessarily involved, we introduce a more concise and straightforward approach. First, we describe an energy-based approach to passivity to determine a sufficient criterion. We then reformulate the proposed passivity bound into an equivalent form that is more amenable to the energy arguments of [7] and [23]. Once this is done, we derive the necessary inequalities to guarantee passivity.

A. An Energy Based Argument to Prove Passivity

It is well known that a passive system connected to a strictly passive system is passive. Since we assume that the operator transfer function $Z_O(s)$ is passive, we only need to show that the remaining part of Fig. 1 without the operator branch is a strictly passive system. In that case, the force f is considered to be the system input. The intuitive statement for passivity is that the kinetic energy of the mass is never as great as the total energy input by the source $f(t)$:

$$\frac{1}{2}mv^2(t)<\int_0^t f(\tau)v(\tau)\,d\tau. \tag{16}$$

The equation corresponding to the output of the differencer in Fig. 1 is

$$m\dot{v}+bv=f-u.$$

Multiplying by v and integrating yields

$$\frac{1}{2}mv^2(t)=\int_0^t f(\tau)v(\tau)\,d\tau-\int_0^t u(\tau)v(\tau)\,d\tau-\int_0^t bv^2(\tau)\,d\tau,$$

which, combined with equation (16), gives

$$\int_0^t u(\tau)v(\tau)\,d\tau+b\int_0^t v^2(\tau)\,d\tau>0.$$

The limits of integration can be extended to all positive and negative values of t by introducing a function $v_\theta(\tau)$ which is equal to $v(\tau)$ for $0\leq\tau\leq\theta$ and equal to zero otherwise:

$$\int_{-\infty}^{\infty}u(\tau)v_t(\tau)\,d\tau+b\int_{-\infty}^{\infty}v_t^2(\tau)\,d\tau>0.$$

By Plancherel's theorem, this is equivalent to

$$\int_{-\infty}^{\infty}U(j\omega)\overline{V(j\omega)}\,d\omega+b\int_{-\infty}^{\infty}V(j\omega)\overline{V(j\omega)}\,d\omega>0 \tag{17}$$

for admissible $V(j\omega)$ where $U(j\omega)$ and $V(j\omega)$ are the Fourier transforms of $u(\tau)$ and $v_t(\tau)$, respectively. Note that $\overline{V(j\omega)}$ denotes the complex conjugate of $V(j\omega)$. From Fig. 1 and equation (3), we observe that $X(s)=V(s)/s$,

$$X_S(j\omega)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{V(j\omega+jn\omega_s)}{j\omega+jn\omega_s},$$

and

$$U(j\omega) = \frac{F(\omega)}{-j\omega} \sum_{n=-\infty}^{\infty} \frac{V(j\omega + jn\omega_s)}{j\omega + jn\omega_s}$$

where we have introduced

$$F(\omega) = (e^{-j\omega T} - 1)H(e^{j\omega T})/T. \tag{18}$$

For convenience, we will use the standard inner product notation

$$\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt$$

and corresponding norm

$$\|f(t)\| = \sqrt{\langle f(t), f(t)\rangle},$$

which we will use for both the time and frequency domains. It should be clear which domain the inner product and norm refer to from the context of the discussion, but it should be noted that the inner products in the time and frequency domains are related by Plancherel's theorem as $\langle f(t), g(t)\rangle = \frac{1}{2\pi}\langle F(j\omega), G(j\omega)\rangle$. The passivity condition (17) can then be written as

$$\langle U(j\omega), V(j\omega)\rangle + b\|V(j\omega)\|^2 > 0. \tag{19}$$

B. Reformulating the Passivity Bound (15)

Before reformulating the proposed passivity condition, we make some observations concerning the maximum and minimum operators of two numbers: $\max\{x, y\}$ and $\min\{x, y\}$. Observe that any real number x satisfies

(i) $x = \max\{x, 0\} + \min\{x, 0\}$
(ii) $\max\{x, 0\} \geq 0$ and $\min\{x, 0\} \leq 0$
(iii) $\max\{-x, 0\} = -\min\{x, 0\}$
(iv) $\min\{ax, 0\} = a\min\{x, 0\}$ for $a > 0$.

More generally, any real-valued function $f(t)$ can be written as $f(t) = f^+(t) + f^-(t)$ where $f^+(t) = \max\{f(t), 0\} \geq 0$ and $f^-(t) = \min\{f(t), 0\} \leq 0$. Furthermore, $\max\{-f(t), 0\} = -\min\{f(t), 0\}$ and for any positive number a, $\min\{af(t), 0\} = a\min\{f(t), 0\}$.

Since the damping parameter b of the interface is positive, the bound (15) is equivalent to

$$b > \max\left\{ -\Phi(\omega) \sum_{n=-\infty}^{\infty} \frac{1}{(\omega + n\omega_s)^2}, 0\right\} \tag{20}$$

for $0 \leq \omega \leq \omega_N$. Applying the above observations about the maximum and minimum operators to (20), we can rewrite the proposed passivity bound as

$$b > -\Phi^-(\omega) \sum_{n=-\infty}^{\infty} \frac{1}{(\omega + n\omega_s)^2} \text{ for } 0 \leq \omega \leq \omega_N$$

where $\Phi^-(\omega)$ denotes $\min\{\Phi(\omega), 0\} \leq 0$, $\Phi^+(\omega)$ denotes $\max\{\Phi(\omega), 0\} \geq 0$, and $\Phi(\omega) = \Phi^+(\omega) + \Phi^-(\omega)$. In particular, we have that the second term in (19) satisfies

$$b\|V(j\omega)\|^2 > -\int_{-\infty}^{\infty} \Phi^-(\omega) \sum_{n=-\infty}^{\infty} \frac{|V(j\omega)|^2}{(\omega + n\omega_s)^2}\, d\omega. \tag{21}$$

C. An Inequality for $\langle U(j\omega), V(j\omega)\rangle$

The proposed bound (1) is given in terms of the real part of a function of ω. Taking a cue from this observation, we show that part of the integrand of $\langle U(j\omega), V(j\omega)\rangle$ can be replaced by its real part. Consider

$$\langle U(j\omega), V(j\omega)\rangle = \sum_{n=-\infty}^{\infty} \mathcal{L}_n[F(\omega)] \tag{22}$$

where we have interchanged integration and summation and introduced a sequence of linear operators

$$\mathcal{L}_n[F(\omega)] = \int_{-\infty}^{\infty} F(\omega)\frac{V(j\omega + jn\omega_s)}{j\omega + jn\omega_s}\overline{\left[\frac{V(j\omega)}{j\omega}\right]}\, d\omega.$$

Now $F(\omega)$, which was defined in (18), has two important properties, viz., $F(\omega + n\omega_s) = F(\omega)$ and $F(-\omega) = \overline{F(\omega)}$, which together imply that $F(-\omega - n\omega_s) = \overline{F(\omega)}$. Using the change of variables $-\omega - n\omega_s$ for ω, we find that $\mathcal{L}_n[F(\omega)] = \mathcal{L}_n[\overline{F(\omega)}]$:

$$\begin{aligned}
&\mathcal{L}_n[F(\omega)] \\
&= \int_{-\infty}^{\infty} F(-\omega - n\omega_s)\frac{V(-j\omega)}{-j\omega}\overline{\left[\frac{V(-j\omega - jn\omega_s)}{-j\omega - jn\omega_s}\right]}\, d\omega \\
&= \int_{-\infty}^{\infty} \overline{F}(\omega)\frac{V(j\omega + jn\omega_s)}{j\omega + jn\omega_s}\overline{\left[\frac{V(j\omega)}{j\omega}\right]}\, d\omega \\
&= \mathcal{L}_n[\overline{F(\omega)}],
\end{aligned}$$

which implies that

$$\mathcal{L}_n[\mathrm{Re}\{F(\omega)\}] = \mathcal{L}_n[(F(\omega) + \overline{F(\omega)})/2] = \mathcal{L}_n[F(\omega)].$$

Consequently, we can replace $F(\omega)$ by $\mathrm{Re}\{F(\omega)\} = \Phi(\omega)$ in (22) to obtain

$$\begin{aligned}
&\langle U(j\omega), V(j\omega)\rangle \\
&= \int_{-\infty}^{\infty} \Phi(\omega) \sum_{n=-\infty}^{\infty} \frac{V(j\omega + jn\omega_s)}{j\omega + jn\omega_s}\overline{\left[\frac{V(j\omega)}{j\omega}\right]}\, d\omega.
\end{aligned}$$

We next write $\langle U(j\omega), V(j\omega)\rangle$ in terms of $\Phi^+(\omega)$ and $\Phi^-(\omega)$ as

$$\langle U(j\omega), V(j\omega)\rangle = I_1 + I_2 \tag{23}$$

where

$$I_1 = \int_{-\infty}^{\infty} \Phi^+(\omega) \sum_{n=-\infty}^{\infty} \frac{V(j\omega + jn\omega_s)}{j\omega + jn\omega_s}\overline{\left[\frac{V(j\omega)}{j\omega}\right]}\, d\omega \tag{24}$$

and

$$I_2 = \int_{-\infty}^{\infty} \Phi^-(\omega) \sum_{n=-\infty}^{\infty} \frac{V(j\omega + jn\omega_s)}{j\omega + jn\omega_s}\overline{\left[\frac{V(j\omega)}{j\omega}\right]}\, d\omega. \tag{25}$$

Since $\Phi^+(\omega)$ is real, non-negative, and periodic with period ω_s, we can write

$$I_1 = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} Z(\omega + n\omega_s)\overline{Z(\omega)}\, d\omega \tag{26}$$

where $Z(\omega) = \sqrt{\Phi^+(\omega)}V(j\omega)/(j\omega)$. Furthermore, by Plancherel's theorem,

$$I_1 = \langle TZ_T(\omega), Z(\omega)\rangle = 2\pi T\langle z_T(t), z(t)\rangle \tag{27}$$

where $Z_T(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Z(\omega + n\omega_s)$ is the sampled version of $Z(\omega)$ in the frequency domain and $z(t)$ and $z_T(t) = \sum_{k=-\infty}^{\infty} z(t)\delta(t-kT)$ are the time domain expressions for $Z(\omega)$ and $Z_T(\omega)$, respectively. Applying the sifting property to $\langle z_T(t), z(t)\rangle$, we find that

$$\begin{aligned} I_1 &= 2\pi T \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z(t)\delta(t-kT)\overline{z(t)}\,dt \\ &= 2\pi T \sum_{k=-\infty}^{\infty} |z(kT)|^2 \geq 0. \end{aligned} \tag{28}$$

In other words, as one would expect from a signal correlation interpretation, the inner product of a sampled signal with its original signal is non-negative.

Turning our attention to I_2, we observe that

$$-I_2 = \sum_{n=-\infty}^{\infty} \langle F_n(\omega), G_n(\omega)\rangle \tag{29}$$

where

$$F_n(\omega) = \sqrt{-\Phi^-(\omega)}\frac{V(j\omega + jn\omega_s)}{j\omega},$$

and

$$G_n(\omega) = \sqrt{-\Phi^-(\omega)}\frac{V(j\omega)}{j\omega + jn\omega_s},$$

and where $\sqrt{-\Phi^-(\omega)}$ is the positive square root of the non-negative real-valued function $-\Phi^-(\omega)$. Applying the integral version of the Cauchy-Schwarz inequality gives

$$-I_2 \leq \sum_{n=-\infty}^{\infty} A_n B_n \tag{30}$$

where

$$A_n = \sqrt{\int_{-\infty}^{\infty} -\Phi^-(\omega)\frac{|V(j\omega + jn\omega_s)|^2}{\omega^2}\,d\omega} \tag{31}$$

and

$$B_n = \sqrt{\int_{-\infty}^{\infty} -\Phi^-(\omega)\frac{|V(j\omega)|^2}{(\omega + n\omega_s)^2}\,d\omega}.$$

Applying the summation version of the Cauchy-Schwarz inequality to (30) gives

$$-I_2 \leq \sqrt{\sum_{n=-\infty}^{\infty} A_n^2}\sqrt{\sum_{n=-\infty}^{\infty} B_n^2}. \tag{32}$$

Lastly, using the change of variables of $\omega - n\omega_s$ for ω in (31) and using the fact that $\Phi(\omega) = \Phi(\omega - n\omega_s)$, we find that $A_n = B_{-n}$, and equation (32) becomes

$$-I_2 \leq \sum_{n=-\infty}^{\infty} B_n^2 = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} -\Phi^-(\omega)\frac{|V(j\omega)|^2}{(\omega + n\omega_s)^2}\,d\omega.$$

Since $I_1 \geq 0$ and

$$I_2 \geq \int_{-\infty}^{\infty} \Phi^-(\omega)\sum_{n=-\infty}^{\infty}\frac{|V(j\omega)|^2}{(\omega + n\omega_s)^2}\,d\omega,$$

it follows that

$$\langle U(j\omega), V(j\omega)\rangle \geq \int_{-\infty}^{\infty} \Phi^-(\omega)\sum_{n=-\infty}^{\infty}\frac{|V(j\omega)|^2}{(\omega + n\omega_s)^2}\,d\omega. \tag{33}$$

D. Proving Sufficiency

Combining (21) and (33) gives the passivity condition (19), which implies that (20), and equivalently, (1), is a sufficient condition for passivity.

V. New Expressions for $r(j\omega)$ and the Passivity Bound for b

A. Expressions for $r(j\omega)$

Now that it has been established that the bound $b > \mathrm{Re}\{-r(j\omega)H(e^{j\omega T})\}$ for $0 \leq \omega \leq \omega_N$ is a necessary and sufficient condition for the haptic system described in [7] to be passive, we turn our attention to finding a particularly nice closed form expression for

$$r(j\omega) = \sum_{n=-\infty}^{\infty} T\frac{e^{-j\omega T} - 1}{(\omega T + 2n\pi)^2}. \tag{34}$$

Expressions for $r(j\omega)$ were presented in [7] and [24] as

$$r(j\omega) = \frac{T}{2}\frac{e^{-j\omega T} - 1}{1 - \cos\omega T} \tag{35}$$

and

$$r(j\omega) = (e^{-j\omega T} - 1)\frac{T}{4}\csc^2(\omega T/2), \tag{36}$$

respectively. Before presenting what we propose is the best expression $r(j\omega)$, we introduce simplifications of (35) and (36).

Using the fact that $e^{-j\omega T} - 1 = 2\sin(\omega T/2)e^{-j(\omega T+\pi)/2}$, it is easy to show that (36) can be written as

$$r(j\omega) = \frac{T}{2}\csc(\omega T/2)e^{-j(\frac{\omega T+\pi}{2})}. \tag{37}$$

Similarly, applying suitable trigonometric identities to (35), we find that $r(j\omega)$ can be written as

$$r(j\omega) = \frac{-T}{2}\left(1 + j\cot\frac{\omega T}{2}\right). \tag{38}$$

Not only are these last two expressions simpler, but they also provide better geometric insight into the s-domain representation of the haptic system. It is pointed out in [7] and [24] that one can view $r(j\omega)R_1$ as a scaled, rotated version of R_1. The new expressions derived in (37) and (38) make this especially obvious. Since $0 \leq \omega T \leq \omega_N T = \pi$, it follows that $\csc(\omega T/2) > 0$ so that the polar form for (37) is

$$r(j\omega) = \frac{T}{2}\csc(\omega T/2)\angle\frac{-\omega T - \pi}{2}. \tag{39}$$

Hence $r(j\omega)$ rotates R_1 by 90° to 180° clockwise as ω varies from 0 to π/T while at the same time scaling by $\frac{T}{2}\csc(\omega T/2)$. Expression (38) provides similar insight since the real part of $r(j\omega)$ is the fixed negative number $-T/2$ and the imaginary part goes from negative infinity to zero as ω varies from 0 to π/T. From this observation we can conclude the same rotational properties of $r(j\omega)$ as we did from (39).

While expressions (38) and (39) are insightful for visualizing how R_1 is rotated and, in the case of (39), scaled, we are more interested in the role that $r(j\omega)$ plays in the passivity bound (15) for $0 \leq \omega \leq \omega_N = \pi/T$. This motivates our next goal: finding simpler expressions for $r(j\omega)$ and the passivity bound.

B. New Expressions for $r(j\omega)$ and the Passivity Bound for b

Previous formulas for $r(j\omega)$ have relied on bringing out the $T(e^{j\omega T}-1)$ term from the summation in (34) and finding a closed form expression for the resulting summation term, i.e., by finding a closed form expression for $\hat{r}(j\omega)$ in (9). We will take a different approach and derive a much simpler expression for $r(j\omega)$ using a sampling based argument that takes advantage of the time delay interpretation of e^{-sT}. Let

$$k(s) = (1-e^{-sT})r(s). \tag{40}$$

Using the fact that $e^{-(s+jn\omega_s)T} = e^{-sT}$, we have that

$$\begin{aligned} k(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{(1-e^{-sT})^2}{(s+jn\omega_s)^2} \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{(1-e^{-(s+jn\omega_s)T})^2}{(s+jn\omega_s)^2}. \end{aligned} \tag{41}$$

From sampling theory [21], we can write (41) as

$$k(s) = C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s+jn\omega_s) = \sum_{k=0}^{\infty} c(kT)e^{-ksT} \tag{42}$$

where

$$C(s) = \frac{(1-e^{-sT})^2}{s^2} = \left(\frac{1-e^{-sT}}{s}\right)^2$$

and

$$c(t) = \mathcal{L}^{-1}[F(s)] = \begin{cases} t & 0 \le t < T \\ 2T - t & T \le t \le 2T \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

Equation (43) follows from the observation that $C(s)$ can be interpreted as the Laplace transform of the convolution of two unit rectangular pulses with support $[0, T]$ so that $c(t)$ is a triangular function with support $[0, 2T]$ and height T. Since $c(kT)$ equals T for $k = 1$ and equals 0 for all other integer values of k, there is only one nonzero term in the last infinite sum in (42). Consequently, we avoid the need for evaluating an infinite sum and simply obtain $k(s) = Te^{-sT}$. This fact along with (40) implies that

$$r(j\omega) = \frac{T}{e^{j\omega T}-1}. \tag{44}$$

It is easy to verify that (44) and the earlier expressions for $r(j\omega)$ are all equal to one another. However, equation (44) is significantly simpler and gives a more tractable version of the passivity bound (1):

$$b > \mathrm{Re}\left\{\frac{TH(e^{j\omega T})}{1-e^{j\omega T}}\right\} \text{ for } 0 \le \omega \le \omega_N = \pi/T. \tag{45}$$

In the next section, we take advantage of the greater simplicity of (45) to determine passivity bounds for a larger family of digital controllers $H(z)$ than previously studied.

VI. Closed Form Passivity Bounds for Certain Classes of Transfer Functions $H(z)$

In [7], the passivity bound is investigated in detail for a digital controller $H(z)$ corresponding to a virtual spring and damper modeled using a backward difference approach. Together with a unilateral constraint operator, this setup can be used to implement a virtual wall. In this section, we will use (45) to find closed form expressions for the lower bound of b for some other special cases of transfer functions $H(z)$.

Example 1: Transfer Functions Based on Backward Difference Differentiation

We can model a difference equation version of an n-th order differentiation using backward difference differentiation by

$$H_n(z) = \left(\frac{z-1}{Tz}\right)^n.$$

For $H_0(z) = 1$ with $z = e^{j\omega T}$, we have

$$\begin{aligned} \mathrm{Re}\left\{\frac{TH_0(z)}{1-z}\right\} &= \frac{1}{2}\left[\frac{T}{1-z}+\frac{T}{1-z^{-1}}\right] \\ &= \frac{1}{2}\left[\frac{T(1-z)}{1-z}\right] = \frac{T}{2} \end{aligned}$$

where we have used the fact that the complex conjugate $\overline{z}$ of z is $\overline{z} = z^{-1}$ for $z = e^{j\omega T}$. Next consider $H_1(z)$, which corresponds to a velocity estimate obtained via backward difference differentiation of position. In this case, we have

$$\mathrm{Re}\left\{\left.\frac{TH_1(z)}{1-z}\right|_{z=e^{j\omega T}}\right\} = \mathrm{Re}\{-e^{-j\omega T}\} = -\cos\omega T \le 1$$

where equality is achieved for $\omega T = \pi$. More generally, for higher order derivative estimates $H_n(z)$,

$$\begin{aligned} \frac{TH_n(z)}{1-z} &= \frac{1}{T^{n-1}}\left[-z^{-1}(1-z^{-1})^{n-1}\right] \\ &= \frac{1}{T^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{k+1}z^{-k-1}. \end{aligned}$$

For $z = e^{j\omega T}$ and $n \ge 2$, we have that

$$\mathrm{Re}\left\{\frac{TH_n(z)}{1-z}\right\} = \frac{1}{T^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{k+1}\cos(k+1)\omega T$$

is also maximized by $\omega T = \pi$, giving the tight bound

$$\mathrm{Re}\left\{\frac{TH_n(z)}{1-z}\right\} \le \left(\frac{2}{T}\right)^{n-1}.$$

Hence,

$$b > \sum_{n=0}^{N}\alpha_n\left(\frac{2}{T}\right)^{n-1} \tag{46}$$

is a necessary and sufficient condition for passivity when

$$H(z) = \sum_{n=0}^{N}\alpha_n H_n(z)$$

where $\alpha_n \ge 0$ for $n = 0, 1, 2, \ldots, N$ and at least one of the α_n coefficients is positive.

We have as an important special case the virtual wall example considered in [7]:

$$H(z) = K + B\frac{z-1}{Tz}.$$

In this case, the bound given by (46) becomes

$$b > \frac{KT}{2} + B.$$

Before proceeding to our next example, we note that due to the $1-z$ in the denominator of (45), one would expect that except for the addition of a constant term B, $H(z)$ would typically need a $z-1$ or $1-z$ term in its numerator to prevent (45) from going to infinity as z approaches 1. To avoid such difficulties, $H(z)$ will have a zero at $z=1$ in the remaining examples except of course when $H(z)$ contains an isolated constant term. Interestingly, in the case of $H_0(z)=1$, while the term $T/(1-z)$ approaches infinity as $z=e^{j\omega T}$ approaches 1, it can be shown that the real part of $T/(1-z)$ remains fixed at $T/2$ as can be seen from (38).

Example 2: Special Cases of FIR Filters

We next consider finite impulse response transfer functions of the form

$$H(z) = (1-z)\sum_{n=0}^{N}\alpha_n z^{-n}. \tag{47}$$

In this case,

$$\mathrm{Re}\left\{\frac{TH(z)}{1-z}\bigg|_{z=e^{j\omega T}}\right\} = T\sum_{n=0}^{N}\alpha_n\cos n\omega T \le T\sum_{n=0}^{N}|\alpha_n|, \tag{48}$$

and we obtain a passivity bound

$$b > T\sum_{n=0}^{N}|\alpha_n|, \tag{49}$$

which is not generally tight unless the α_n coefficients satisfy a certain sign structure. If the α_n's are non-negative with at least one being positive, then each summand in the first summation in (48) is maximized when $\omega T = 0$, in which case $b > T\sum_{n=0}^{N}\alpha_n$ is a tight bound. For example, if $H(z) = (1-z)\sum_{n=0}^{N}z^{-k}$, then the passivity requirement is $b > (N+1)T$.

The situation is not as straightforward if one or more α_k coefficients are negative. However, if the coefficients satisfy the alternating sign structure $\alpha_0 \ge 0, \alpha_1 \le 0, \alpha_2 \ge 0, ...$ with at least one α_i being nonzero, the maximum would occur when $\omega T = \pi$, i.e., $z=-1$, in which case $T\sum_{n=0}^{N}\alpha_n\cos n\omega T = T\sum_{n=0}^{N}(-1)^n\alpha_n = T\sum_{n=0}^{N}|\alpha_n|$ and (49) is a tight bound. An interesting special case with all negative α_k's that still has a relatively tight bound is

$$H(z) = (z-1)\sum_{n=0}^{N}z^{-n}.$$

In this case,

$$\mathrm{Re}\left\{\frac{TH(z)}{1-z}\bigg|_{z=e^{j\omega T}}\right\} = -T\sum_{n=0}^{N}\cos n\omega T, \tag{50}$$

requiring us to find a lower bound for $\sum_{n=0}^{N}\cos n\omega T$ due to the minus sign in (50). Fortunately, this sum is related to the Dirichlet kernel, $D_N(x) = 1+2\sum_{k=1}^{N}\cos kx$, which has been studied extensively in the theory of Fourier series. It is known that for relatively large N, $\min_x D_N(x) \sim -C(2N+1)$ where $C = 0.2172336282$ [25]. We thus conclude that

$$\min_x \sum_{n=0}^{N}\cos nx \sim \frac{1-C(2N+1)}{2}$$

and that $b > (CN-D)T$ is an approximate lower bound for b for sufficiently large N where $D = (1-C)/2 = 0.3913831859$. The Dirichlet kernel will also play a role in Example 4 when we consider time delay.

Example 3: Special Cases of IIR Filters

We next apply the improved passivity bound expression to some special cases of infinite impulse response filters. Suppose that

$$H(z) = \frac{z-1}{z+a}$$

where $|a|<1$. With some straightforward algebra, it can be shown that for $z = e^{j\omega T}$,

$$\mathrm{Re}\left\{\frac{T}{1-z}\left(\frac{z-1}{z+a}\right)\right\} = -T\frac{\cos\omega T + a}{a^2+2a\cos\omega T+1}. \tag{51}$$

Since $|a|<1$, the denominator in (51) is strictly positive and it can be shown that (51) decreases as $\cos\omega T$ increases so that (51) is maximized by $\cos\omega T = -1$, i.e., $\omega T = \pi$. Hence the passivity bound becomes

$$b > \frac{T}{1-a}.$$

The situation is similar for

$$H(z) = \frac{1-z}{z+a}$$

with $|a|<1$ except now $\mathrm{Re}\{H(z)T/(1-z)\}$ increases as $\cos\omega T$ increases and is maximized for $\omega T = 0$ giving a passivity bound of

$$b > \frac{T}{1+a}.$$

The solution for $a=1$, when the pole is on the unit circle rather than inside the open unit disk, has a different form. In this case, equation (51) becomes $\mathrm{Re}\{-T/(1+z)\} = -T/2$ and is hence independent of ω. This is further confirmed from the expression

$$\frac{T}{1-z}\left(\frac{z-1}{z+1}\right)\bigg|_{z=e^{j\omega T}} = \frac{-T}{2}[1-j\tan(\omega T/2)]. \tag{52}$$

While (52) does approach infinity as ωT approaches π, its real part remains fixed at $-T/2$, once again implying that the passivity bound in this case is $b > -T/2$. This is similar to the situation discussed at the end of Example 1 for the constant transfer function $H_0(z)$.

Equation (52) is related to Tustin's approximation

$$s = \frac{2}{T}\frac{z-1}{z+1},$$

also known as the bilinear transformation. Applying Tustin's approximation to $H(s) = K + Bs$ gives an approximate discrete transfer function

$$H(z) = K + B\frac{2}{T}\frac{z-1}{z+1}$$

that can be used to model a virtual wall. In this case, the passivity bound (45) becomes [19], [26]

$$b > \frac{KT}{2} - B.$$

Although the above bound implies that no damping b is needed if $B > KT/2$, one should remember that Tustin's approximation is an ideal case and that the above result ignores effects such as the inherent computational time in an actual implementation of Tustin's approximation. Indeed, if we introduce a time delay t_d in the analysis of Section III, we obtain for the Tustin case

$$\begin{aligned}&\mathrm{Re}\{\tfrac{T}{1-z}(\tfrac{2}{T}\tfrac{z-1}{z+1})e^{-j\omega t_d}\}\\ &\quad= \mathrm{Re}\Big\{\Big[-1 + j\tan\frac{\omega T}{2}\Big]\Big[\cos\omega t_d - j\sin\omega t_d\Big]\Big\}\\ &\quad= -\cos\omega t_d + \sin\omega t_d \tan\frac{\omega T}{2}.\end{aligned}$$

For $0 < t_d < T$, this expression approaches positive infinity as ωT approaches π due to the $\sin\omega t_d \tan\frac{\omega T}{2}$ term. Hence, the necessary condition of Section III is violated and the system is not passive for a small time delay when using the Tustin approximation. A similar analysis for $H(z) = 1$ and $H_m(z) = (\frac{1-z}{Tz})^m$ shows that they do not suffer the same problem as the Tustin approximation. Other authors have also expressed concern about using Tustin's approximation for this application [26]. It is interesting to note that, as shown in Example 4 below, the above analysis is not a problem for the Tustin case when t_d is a multiple of the sampling period T.

Example 4: Special Cases with Time Delay

In this example we will continue to consider the effect that time delay has on the passivity bound [17], [18]. Once again note that a time delay of t_d is modeled by introducing a block with transfer function $e^{-t_d s}$ into the lower portion of Fig. 1. If we restrict ourselves to time delays that are multiples of the sampling period, i.e., $t_d = nT$, then the delay would be modeled by $z^{-n} = e^{-jn\omega T}$. Consequently, the passivity bound would become $b > \mathrm{Re}\{TH(z)z^{-n}/(1-z)\}$ where $z = e^{j\omega T}$ for $0 \le \omega \le \omega_N$. The following are some examples illustrating the effect that a time delay $t_d = nT$ has on the passivity bound.

We first note that introducing a time delay $t_d = nT$ does not fundamentally change the form of the general solution to the FIR examples described in Example 2. Equation (47) becomes $H(z)z^{-n} = (1-z)\sum_{k=0}^{N}\alpha_n z^{-k-n}$ and the passivity bound analysis is similar to the case when there was no delay.

We next consider virtual systems modeled by backward difference differentiation. Using the partial fraction expansion

$$\frac{1}{(1-z)z^n} = \frac{1}{1-z} + \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^n},$$

we can show that for

$$H_m(z) = \left(\frac{z-1}{Tz}\right)^m,$$

we have

$$\mathrm{Re}\left\{\frac{TH_0(z)}{(1-z)z^n}\right\} = \frac{T}{2}D_n(\omega T),$$

where $D_n(u) = 1 + 2\cos u + 2\cos 2u + \cdots + 2\cos nu$ is the Dirichlet kernel,

$$\mathrm{Re}\left\{\frac{TH_1(z)}{(1-z)z^n}\right\} = -\cos(n+1)\omega T,$$

and more generally,

$$\begin{aligned}&\mathrm{Re}\left\{\tfrac{TH_m(z)}{(1-z)z^n}\right\}\\ &\quad= \frac{1}{T^{m-1}}\sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^{k+1}\cos(n+k+1)\omega T\\ &\quad\le \left(\frac{2}{T}\right)^{m-1}.\end{aligned}$$

For the important case $H(z) = K + B\frac{z-1}{Tz}$, we have

$$\mathrm{Re}\left\{\frac{TH(z)}{(1-z)z^n}\right\} = \frac{KT}{2}D_n(\omega T) - B\cos(n+1)\omega T. \quad (53)$$

A conservative bound for (53) is $b > (2n+1)kT/2 + B$. The maximum value of (53) depends on KT, B, and n. For example, if $n = 1$,

$$\begin{aligned}\mathrm{Re}\left\{\frac{TH(z)}{(1-z)z}\right\} &= \mathrm{Re}\left\{\frac{KT}{1-z} + \frac{KT}{z} - \frac{B}{z^2}\right\}\\ &= \frac{KT}{2} + KT\cos\omega T - B\cos 2\omega T\\ &= \frac{KT}{2} + B + KT\cos\omega T - 2B\cos^2\omega T\end{aligned}$$

where we have used the identity $\cos 2\omega T = 2\cos^2\omega T - 1$ to obtain an expression that is a quadratic in $\cos\omega T$. Finding the passivity bound for b is simply a constrained quadratic optimization problem that is maximized by $\cos\omega T = \frac{KT}{4B}$ if $KT < 4B$ and by $\cos\omega T = 1$ if $KT \ge 4B$. Hence, the passivity bound when there is a delay of $t_d = T$ is

$$b > \begin{cases}\frac{KT}{2} + B + (KT - 2B) & \text{if } KT \ge 4B\\ \frac{KT}{2} + B + \frac{K^2T^2}{8B} & \text{if } KT < 4B.\end{cases}$$

Tustin's approximation model for differentiation is handled in a similar manner. For $H(z) = K + B\frac{2}{T}\frac{z-1}{z+1}$,

$$\mathrm{Re}\left\{\frac{TH(z)}{(1-z)z^n}\right\} = \frac{KT}{2}D_n(\omega T) - B(-1)^n D_n(\omega T + \pi).$$

For the special case when $n = 1$,

$$\mathrm{Re}\left\{\frac{TH(z)}{(1-z)z}\right\} = \frac{KT}{2} + B + (KT - 2B)\cos\omega T,$$

and the passivity bound becomes

$$b > \frac{KT}{2} + B + |KT - 2B|.$$

It should once again be noted that the Tustin approximation can be sensitive to arbitrary time delays including the small time delay case analyzed in Example 3.

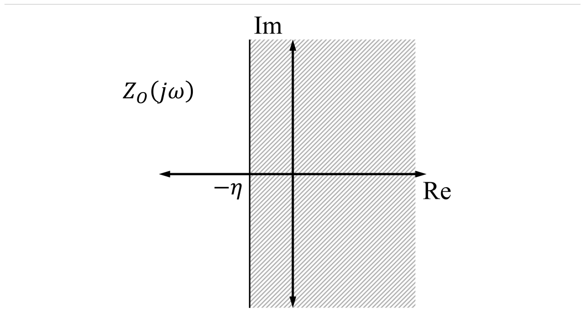

Fig. 3: The possible range of $Z_O(j\omega)$ for the case when the operator corresponds to a non-passive system with a shortage of passivity η.

VII. Non-Passive Operators

In the previous sections, we only considered passive transfer functions $Z_O(s)$. However, it has been shown in the literature that this assumption does not always hold [27], [28]. We now consider the case when the operator transfer function has a shortage of passivity $\eta > 0$, i.e., when the range of $Z_O(j\omega)$ is to the right of $-\eta$ as shown in Fig. 3.

One can address the necessary condition for passivity using the same approach described in Section III except that $Z_O(s)$ needs to be shifted an additional η units to the right and the passivity bound becomes

$$b > \mathrm{Re}\left\{\frac{TH(e^{j\omega T})}{1-e^{j\omega T}}\right\} + \eta \text{ for } 0 \leq \omega \leq \omega_N = \pi/T. \quad (54)$$

This demonstrates that (54) is a necessary condition. Modifying the analysis in Section IV to show that this is also a sufficient condition requires more creativity.

To address sufficiency, we write $Z_O(s) = -\eta + Z_P(s)$ where $Z_P(s) = Z_O(s) + \eta$ corresponds to a passive transfer function. We then replace the operator block $Z_O(s)$ with a parallel connection of blocks with transfer functions $Z_P(s)$ and $-\eta$ as shown on the left hand side of Fig. 4. The $-\eta$ block can then be connected in a negative feedback configuration with the $1/(ms+b)$ block so that these two blocks together are equivalent to the single block $1/(ms+b-\eta)$. Consequently, the two block diagrams shown in Fig. 4 are equivalent. Substituting the second block diagram of Fig. 4 into the original block diagram of Fig. 1, one obtains Fig. 5. Hence, the previous analysis for both the necessary and sufficient conditions would still hold with b replaced by $b - \eta$, and we obtain the result that (54) is a necessary and sufficient condition for passivity when the shortage of passivity of $Z_O(s)$ is η.

VIII. Conclusions

In this article, we provided a rigorous justification of a well-known passivity bound for a class of sampled-data systems. Such systems can serve as a simple model of a haptic system. We also presented a concise derivation of the passivity bound using a sample-based approach that resulted in a significantly simpler expression. Previous studies based on the earlier

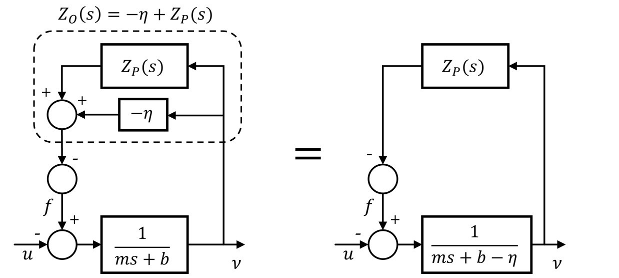

Fig. 4: The two block diagrams shown have the same transfer functions. The block diagram on the left shows the shortage of passivity η present in the operator block. The parameter η is redistributed to the haptic interface block in the block diagram on the right.

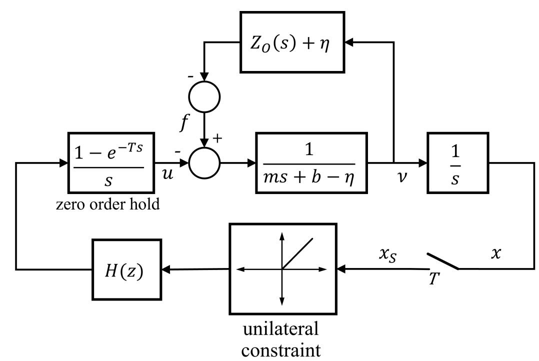

Fig. 5: An equivalent block diagram of the 1-DOF haptic device shown in Fig. 1 for the shortage of passivity case using the block on the right in Fig. 4.

bound were limited to simple virtual spring/damper systems. Using the more tractable expression established in the paper, we derived closed form passivity bounds for three classes of transfer functions corresponding to virtual environments. We also presented bounds for simple virtual spring/damper systems experiencing a time delay. Lastly, we considered the possibility that the operator may correspond to a non-passive system and modified the passivity bound for the case when the operator transfer function has a shortage of passivity.

IX. Acknowledgments

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References

[1] R. J. Adams and B. Hannaford, "Stable haptic interaction with virtual environments," *IEEE Trans. Robot. Automat.*, vol. 15, no. 3, pp. 465-474, Jun. 1999.

[2] B. Hannaford and Jee-Hwan Ryu, "Time-domain passivity control of haptic interfaces," *IEEE Trans. Robot. Automat.*, vol. 18, no. 1, pp. 1-10, Feb. 2002.

[3] J.-H. Ryu, D.-S. Kwon, and B. Hannaford, "Stable teleoperation with time-domain passivity control," *IEEE Trans. Robot. Automat.*, vol. 20, no. 2, pp. 365–373, Apr. 2004.

[4] J. H. Ryu, Y. S. Kim, and B. Hannaford, "Sampled- and continuous-time passivity and stability of virtual environments," *IEEE Trans. Robot.*, vol. 20, no. 4, pp. 772–776, Aug. 2004.

[5] M. Tavakoli, A. Aziminejad, R. Patel, and M. Moallem, "Discrete-time bilateral teleoperation: Modelling and stability analysis," *IET Contr. Theory Appl.*, vol. 2, no. 6, pp. 496–512, Jun. 2008.

[6] J. -P. Kim, S. -Y. Baek and J. Ryu, "A force bounding approach for multi-degree-of-freedom haptic interaction," *IEEE/ASME Trans. Mechatron.*, vol. 20, no. 3, pp. 1193-1203, Jun. 2015.

[7] J. E. Colgate and G. G. Schenkel, "Passivity of a class of sampled-data systems: Application to haptic interfaces," *J. Robot. Syst.*, 14(1), pp. 37-47, Jan. 1997.

[8] B. E. Miller, J. E. Colgate and R. A. Freeman, "On the role of dissipation in haptic systems" *IEEE Trans. Robot.*, vol. 20, no. 4, pp. 768-771, Aug. 2004.

[9] N. Diolaiti, G. Niemeyer, F. Barbagli and J. K. Salisbury, "Stability of haptic rendering: Discretization, quantization, time delay, and Coulomb effects," *IEEE Trans. Robot.*, vol. 22, no. 2, pp. 256-268, Apr. 2006.

[10] J.J. Gil, E. Sanchez, T. Hulin, C. Preusche, and G. Hirzinger, "Stability boundary for haptic rendering: Influence of damping and delay," *ASME. J. Comput. Inf. Sci. Eng.*, vol. 9, no. 4, Mar. 2009.

[11] A. Mashayekhi, R. B. Boozarjomehry, A. Nahvi, A. Meghdari, and P. Asgari, "Improved passivity criterion in haptic rendering: Influence of Coulomb and viscous friction," *Adv. Robot.*, 28:10, 695-706, 2014.

[12] T. Hulin, C. Preusche, and G. Hirzinger, "Stability boundary for haptic rendering: Influence of human operator," *Proc. IEEE/RSJ Int. Conf. Intell. Robots Syst.*, Nice, France, pp. 3483-3488, Sep. 2008.

[13] B. E. Miller, J. E. Colgate and R. A. Freeman, "Guaranteed stability of haptic systems with nonlinear virtual environments," *IEEE Trans. Robot. Automat.*, vol. 16, no. 6, pp. 712-719, Dec. 2000

[14] S.-Y. Baek, S. Park and J. Ryu, "A force bounding approach in joint space for interacting with dynamic multi-degrees of freedom virtual objects," *IEEE Trans. Haptics*, vol. 12, no. 2, pp. 217-223, Apr.-Jun. 2019.

[15] A. Mashayekhi, S. Behbahani, F. Ficuciello, and B. Siciliano, "Analytical stability criterion in haptic rendering: The role of damping," *IEEE/ASME Trans. Mechatron.*, vol. 23, no. 2, pp. 596-603, Apr. 2018.

[16] B. E. Miller, J. E. Colgate and R. A. Freeman, "Environment delay in haptic systems," *Proc. IEEE Int. Conf. Robot. Automat.*, San Francisco, CA, USA, pp. 2434-2439, Apr. 2000.

[17] T. Hulin, A. Albu-Schäffer and G. Hirzinger, "Passivity and stability boundaries for haptic systems with time delay," *IEEE Trans. Control Syst. Technol.*, vol. 22, no. 4, pp. 1297-1309, Jul. 2014.

[18] N. Miandashti and M. Tavakoli, "Stability of sampled-data, delayed haptic interaction and teleoperation," *Proc. IEEE Haptics Symp.*, Houston, TX, USA, pp. 215-220, Jul. 2014.

[19] L. Pecly and K. Hashtrudi-Zaad, "Enhanced haptic passivity and uncoupled stability with Tustin-like general discretization method," *Proc. IEEE World Haptics Conf.*, Montreal, QC, Canada, pp. 109-114, Jul. 2021.

[20] J.J. Gil, A. Avello, A. Rubio and J. Florez, "Stability analysis of a 1 DOF haptic interface using the Routh-Hurwitz criterion," *IEEE Trans. Control Syst. Technol.*, vol. 12, no. 4, pp. 583-588, Jul. 2004.

[21] C. L. Phillips and H. T. Nagle, *Digital Control System Analysis and Design, Third Edition*, New Jersey, Prentice-Hall, 1995.

[22] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, New York, Academic Press, 1975.

[23] A. Jazayeri and M. Tavakoli, "A passivity criterion for sampled-data bilateral teleoperation systems," *IEEE Trans. Haptics*, vol. 6, no. 03, pp. 363-369, Jul.-Sep. 2013.

[24] J.E. Colgate, "Coordinate transformations and logical operations for minimizing conservativeness in coupled stability criteria," *ASME J. Dynam. Syst. Measurement Contr.*, 116(4), pp. 643-649, Dec. 1994.

[25] A. Wiggins, "The minimum of the Dirichlet kernel," http://www-personal.umd.umich.edu/ adwiggin/TeachingFiles/FourierSeries/Resources/DirichletKernel.pdf, accessed March 31, 2023.

[26] A. Haddadi and K. Hashtrudi-Zaad, "Stability analysis of haptic interfaces for different types of sampled signals and virtual environment implementations," *Proc. IEEE Haptics Symp.*, Waltham, MA, USA, pp. 293-299, Mar. 2010.

[27] M. Dyck, A. Jazayeri and M. Tavakoli, "Is the human operator in a teleoperation system passive?," *Proc. IEEE World Haptics Conf.*, Daejeon, South Korea, pp. 683-688, Apr. 2013.

[28] N. Miandashti and M. Tavakoli, "Stability of sampled-data, delayed haptic interaction under passive or active operator," *IET Contr. Theory Appl.*, vol. 8, issue 17, pp. 1769-1780, 2014.

Rodney G. Roberts is a professor of electrical and computer engineering at the Florida A&M University-Florida State University College of Engineering, Tallahassee, Florida, USA. His research interests include robot kinematics and dynamics, teleoperation, and haptics. He earned his B.S. degrees in electrical engineering and mathematics from the Rose-Hulman Institute of Technology and his M.S. and Ph.D. degrees in electrical engineering from Purdue University. He was a National Research Council Fellow at Wright Patterson Air Force Base and a Fulbright U.S. Scholar, Faculty of Science and Technology, University of Macau, in 2016.

Carl A. Moore Jr. is an Associate Professor of Mechanical Engineering at the FAMU-FSU College of Engineering. He was instrumental in developing cobots, a human-robot collaborative technology for applications that require humans to work in physical contact with robots. Today, he is using cobots to develop remote manipulation systems to connect people across the barrier of distance. Through grants from NASA and NSF, Moore develops success strategies for underrepresented graduate and undergraduate STEM majors.

J. Edward Colgate is the Walter P. Murphy Professor of Mechanical Engineering at Northwestern University. His research interests include haptic interfaces, human-robot interaction, electroadhesion, and robot dexterity. He is the founding Editor-in-Chief of the IEEE Transactions on Haptics, a member of the National Academy of Engineering, and a Fellow of the IEEE and the National Academy of Inventors. Colgate and collaborator Michael Peshkin are the inventors of cobots and a variety of surface haptic technologies.