Equivariant Hopf Bifurcation in a Class of Partial Functional Differential Equations on a Circular Domain

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Abstract

Circular domains frequently appear in mathematical modeling in the fields of ecology, biology and chemistry. In this paper, we investigate the equivariant Hopf bifurcation of partial functional differential equations with Neumann boundary condition on a two-dimensional disk. The properties of these bifurcations at equilibriums are analyzed rigorously by studying the equivariant normal forms. Two reaction-diffusion systems with discrete time delays are selected as numerical examples to verify the theoretical results, in which spatially inhomogeneous periodic solutions including standing waves and rotating waves, and spatially homogeneous periodic solutions are found near the bifurcation points.

Keywords: Circular domain, partial functional differential equation, equivariant Hopf bifurcation, standing wave, rotating wave.

I. INTRODUCTION

The research on the reaction-diffusion equation plays an important role in physics, chemistry, medicine, biology, and ecology. Many mathematical problems, such as the existence, boundedness, regularity and stability of solutions and traveling waves have been raised in [1–4]. Recently, the effect of time delay has drawn a lot of attention and Hopf bifurcation analysis has become an effective tool to explain complex phenomena in reaction-diffusion systems, and even in more general partial functional differential equations (PFDEs). [5] gave a general Hopf bifurcation theorem for PFDEs by restricting the system to an eigenspace of the Laplacian. [6] gave a framework for directly calculating the normal form of PFDEs with parameters. Based on these theories, many achievements have been made in the study of local Hopf bifurcation [7–11] or other codimension-two bifurcations [12–14].

The phenomena of symmetry appear a lot in real-world models, which usually leads to multiple eigenvalues, and the standard Hopf bifurcation theory of functional differential equations cannot be applied to solve such problems. [15] used group theory to characterize transitions in symmetric systems and worked out the bifurcation theory for a number of symmetry groups. Based on these theories, there have been many subsequent studies on symmetry. First, some researchers were concerned about nonlinear optical systems, which can effectively characterize optical problems such as circular diffraction [16–18]. Besides, a Hopfield-Cohen-Grossberg network consisting of n identical elements [19–21] also has a certain symmetry. Furthermore, there have been many studies on the regions with $O(2)$ symmetry where the models are established. For example, in [22], the authors considered Hopf bifurcation in the presence of $O(2)$ symmetry and distinguished the phase portraits of the normal form into six cases. [23] studied a delay parabolic equation in a disk with the Neumann boundary conditions and proved the existence of rotating waves with methods of eigenfunction.

In recent years, a sequence of results about equivariant Hopf bifurcation in neutral functional differential equations [24, 25] and functional differential equations of mixed type [26] have been established. In particular, [27] applied the equivariant Hopf bifurcation theorem to study the Hopf bifurcation of a delayed Ginzburg-Landau equation on a two-dimensional disk with the homogeneous Dirichlet boundary condition. More recently, Qu and Guo applied Lyapunov-Schmidt reduction to study the existence of inhomogeneous steady-state solutions on a unit disk [28], whereas different kinds of spatial-temporal solutions with symmetry have been detected by investigating isotropy subgroups of these equations [15, 21, 29, 30].

In fact, the disk, a typical region with $O(2)$ symmetry, is usually used to describe many real-world problems. For example in physics, rotating waves were often observed in the case of a circular aperture [31–33]. In chemical experiments, one usually studies chemical reactions in circular Petri dishes, whose size may affect the existence and pattern of spiral waves [34, 35], and the most suitable boundary condition for chemical species in a disk is the homogeneous Neumann boundary condition [35]. Besides, in the field of ecology, some lakes could be abstracted as circular domains to study the interaction between predator and prey, and the mathematical modeling of predator-prey systems on the circular domain has been summarized in [36, 37]. We found that a complete derivation of normal forms and bifurcation analysis in general PFDEs on two-dimensional circular domains remains lacking. Therefore, in this paper, we aim to consider general PFDEs with homogeneous Neumann boundary conditions defined on a disk and to improve the center manifold reduction technique established in [6, 38] to the normal form derivation for PFDEs on circular domains to fill the gap.

Compared to the results in [5], due to the $O(2)$ symmetry leading to multiple pure imaginary eigenvalues, the eigenspace of the Laplacian is sometimes two-dimensional, which gives rise to higher dimensional center subspace of the equilibrium at the bifurcation point. By introducing similar operators as in [6], we derive the normal form of the equivariant Hopf bifurcation of general PFDEs on a disk in explicit formulas, which can be directly applied to some models with practical significance or the model on other kinds of circular domains, for example, an annulus or a circular sector. In addition to the results in [27], there are clearer classification and quantitative criteria for the existence, stability, and approximate expressions of various solutions. With the aid of normal forms, we find standing wave solutions and rotating wave solutions in a delayed predator-prey model. This is done after all the coefficients in the normal forms are explicitly computed.

The structure of the article is as follows. In Sec. II, the eigenvalue problem of the Laplace operator on a circular domain is reviewed and the existence of Hopf bifurcation is explored. In Sec. III, we study the properties of equivariant Hopf bifurcation on the center manifold. The normal forms are also rigorously derived in this section. In Sec. IV, two types of reaction-diffusion equations with discrete time delay are selected and numerically solved to verify the theoretical results.

II. PRELIMINARIES

A. The eigenvalue problem of the Laplace operator on a circular domain

The eigenvalue problem associated with Laplace operators on a circular domain could be given in a standard way, see [39, 40]. For the convenience of our research, we use a similar method to treat the eigenvalue problem and state the main results here. Consider a disk as follows:

$$\mathbb{D} = \{(r,\theta) : 0 \leq r \leq R, 0 \leq \theta \leq 2\pi\}.$$

The Laplace operator defined in the Cartesian coordinates is $\Delta\varphi = \frac{\partial^2}{\partial x^2}\varphi + \frac{\partial^2}{\partial y^2}\varphi$. Letting $x = r\cos(\theta), y = r\sin(\theta)$, it can be converted into the polar coordinates as $\Delta_{r\theta}\varphi = \frac{\partial^2}{\partial r^2}\varphi + \frac{1}{r}\cdot\frac{\partial}{\partial r}\varphi + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\varphi$.

One needs to consider the following eigenvalue problem and calculate the eigenvectors on the disk:

$$\begin{cases} \Delta_{r\theta}\phi = -\lambda\phi, \\ \phi_r'(R,\theta) = 0, \quad \theta \in [0, 2\pi]. \end{cases} \tag{1}$$

Using the method of separation of variables and letting $\phi(r,\theta) = P(r)\Phi(\theta)$, we get that the eigenfunction corresponding to λ_{nm} is

$$\phi_{nm}(r,\theta) = J_n\left(\sqrt{\lambda_{nm}}r\right)\Phi_n(\theta), \tag{2}$$

with

$$\Phi_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \tag{3}$$

and

$$J_n(\rho) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\Gamma(n+m+1)}\left(\frac{\rho}{2}\right)^{n+2m}. \tag{4}$$

λ_{nm} is chosen such that the boundary condition $P'(R) = 0$ is satisfied.

Remark II.1. *Considering the Neumann boundary conditions, we have*

$$J_n'(\sqrt{\lambda_{nm}}R) = 0,$$

which indicates that at $r = R$, λ_{nm} are roots of $J_n'(\sqrt{\lambda}r)$. We use α_{nm} to represent these nonzero roots and assume that they are indexed in increasing order, i.e. $J_n'(\alpha_{nm}) = 0, \alpha_{n1} < \alpha_{n2} < \alpha_{n3} < \cdots$, where $n \geq 0$ is the indices of the Bessel function (4) and $m \geq 1$ are the indices for these roots. So $\lambda_{nm} = (\alpha_{nm}/R)^2$. For convience, we use $\alpha_{00} = 0, \lambda_{00} = 0$.

Remark II.2. *From the standard Sturm-Liouville theorem, we know that for any given non-negative integer n, $J_n\left(\frac{\alpha_{nm}}{R}r\right)$ are the orthogonal sets with weight r on the interval $[0,R]$, where $J_n'(\alpha_{nm}) = 0$. That is, for any given m,k, we have*

$$\int_0^R rJ_n\left(\frac{\alpha_{nm}}{R}r\right)J_n\left(\frac{\alpha_{nk}}{R}r\right)\mathrm{d}r = \begin{cases} 0, & m \neq k, \\ \frac{R^2}{2}\left[1-\left(\frac{n}{\alpha_{nm}}\right)^2\right]J_n^2(\alpha_{nm}), & m = k. \end{cases}$$

Furthermore, for any given non-negative integer n, the $\mathbb{L}^2$ norm with weight r of function systems $\left\{J_n\left(\frac{\alpha_{nm}}{R}r\right)\right\}$ that include α_{00} are complete in the space $\mathbb{L}^2[0,R]$. Besides, the trigonometric function systems are orthogonal in the interval $[0,2\pi]$ and complete in the space $\mathbb{L}^2[0,2\pi]$. Therefore, for $n = 0,1,2,\cdots$, $m = 1,2,\cdots$, we use a complexification of the space and the system of functions

$$\phi_0,\ \phi_{nm}^c,\ \phi_{nm}^s,$$

constitutes an orthogonal basis with weight r in the space $\mathbb{L}^2\{0 \leq \theta \leq 2\pi, 0 \leq r \leq R\}$ with

$$\phi_0 = J_0\left(\frac{\alpha_{00}}{R}r\right) = 1,\ \phi_{nm}^c = J_n\left(\frac{\alpha_{nm}}{R}r\right)\mathrm{e}^{\mathrm{i}n\theta},\ \phi_{nm}^s = \overline{\phi_{nm}^c} = J_n\left(\frac{\alpha_{nm}}{R}r\right)\mathrm{e}^{-\mathrm{i}n\theta}.$$

From the above analysis, we can draw the following conclusions.

Theorem II.3. *The solution of the eigenvalue problem (1) can be written as*

$$\varphi(r,\theta) = \phi_0(r,\theta) + A_{nm}\sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty}\phi_{nm}^c(r,\theta) + B_{nm}\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\phi_{nm}^s(r,\theta),$$

where

$$A_{nm} = \frac{\delta_n}{R^2\pi\left[1-\left(\frac{n}{\alpha_{nm}}\right)^2\right]J_n^2(\alpha_{nm})}\int_0^R\int_0^{2\pi} r\varphi(r,\theta)J_n\left(\frac{\alpha_{nm}}{R}r\right)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}r\mathrm{d}\theta,$$

$$B_{nm} = \frac{2}{R^2\pi\left[1-\left(\frac{n}{\alpha_{nm}}\right)^2\right]J_n^2(\alpha_{nm})}\int_0^R\int_0^{2\pi} r\varphi(r,\theta)J_n\left(\frac{\alpha_{nm}}{R}r\right)\mathrm{e}^{\mathrm{i}n\theta}\mathrm{d}r\mathrm{d}\theta,$$

$$\delta_n = \begin{cases} 1, n = 0, \\ 2, n \neq 0. \end{cases}$$

This means, for $n = 0$, the eigenspace corresponding to the eigenvalue λ_{0m} is spanned by ϕ_{0m}^c, $m = 0,1,\cdots$. For $n > 0$, the eigenspace corresponding to the eigenvalue λ_{nm} is spanned by ϕ_{nm}^c and ϕ_{nm}^s, $m = 1,2,\cdots$.

Remark II.4. *The above eigenvalue problems can be directly applied to an annular domain*

$$\tilde{\mathbb{D}} = \{(r,\theta):\ R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}.$$

The difference is that the homogeneous Neumann condition is given at $r = R_1$ and $r = R_2$. Besides, $P(r)$ becomes

$$P_n(\sigma_{nm}, r) = A_n J_n(\frac{\sigma_{nm}}{R_2} r) + B_n N_n(\frac{\sigma_{nm}}{R_2} r),$$

with $P_n'(\sigma_{nm}, R_1) = P_n'(\sigma_{nm}, R_2) = 0$, where

$$N_n(\rho) = \frac{J_n(\rho)\cos n\pi - J_{-n}(\rho)}{\sin n\pi}.$$

Then $\frac{P_n(\sigma_{nm},r)}{\|P_n(\sigma_{nm},r)\|_{2,2}}$ forms a standard orthogonal basis of the space $\mathbb{L}^2\{R_1 \le r \le R_2\}$. In what follows, there will be no significant difference in the subsequent process of bifurcation analysis.

In fact, the above eigenvalue analysis in a circular sector domain

$$\hat{\mathbb{D}} = \{(r,\theta):\ 0 \le r \le R,\ 0 \le \theta \le \Theta,\ \Theta < \pi\},$$

is also similar. The difference is that the homogeneous Neumann conditions make $\Phi(\theta)$ become $\Phi_n(\theta) = \cos\frac{n\pi}{\Theta}\theta$. The subsequent calculation process is even simpler, which is a direct extension of the case in one-dimensional intervals.

B. The existence of the Hopf bifurcation

We consider a general PFDEs with homogeneous Neumann boundary conditions defined on a disk as follows:

$$\frac{\partial U(t,x,y)}{\partial t} = D(\nu)\Delta U(t,x,y) + L(\nu)U_t(x,y) + F\left(U_t(x,y),\nu\right), \tag{5}$$

where $t \in [0,+\infty)$, $\Omega = \left\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < R^2\right\}$, $D(\nu) = diag\{d_1(\nu), d_2(\nu), \cdots, d_n(\nu)\}$, $d_i(\nu) > 0$, $i = 1,2,\cdots,n$, $\nu \in \mathbb{R}$, $U(t,x,y) = (u_1(t,x,y), u_2(t,x,y), \cdots, u_n(t,x,y))^{\mathrm{T}}$, $U_t(\vartheta)(x,y) = U(t+\vartheta,x,y)$, $\vartheta \in [-1,0]$, $U_t(\vartheta)(x,y) = (u_t^1(\vartheta)(x,y), u_t^2(\vartheta)(x,y), \cdots, u_t^n(\vartheta)(x,y))^{\mathrm{T}}$, $U_t(x,y) \in \tilde{\mathscr{C}} := C([-1,0],\ \tilde{\mathscr{X}}_{\mathbb{C}})$, $L: \mathbb{R}\times\tilde{\mathscr{C}} \to \mathscr{X}_{\mathbb{C}}$ is a bounded linear operator, and $F: \tilde{\mathscr{C}}\times\mathbb{R} \to \mathscr{X}_{\tilde{\mathscr{C}}}$ is a C^k $(k \ge 3)$ function such that $F(0,\nu) = 0$, $D_\varphi F(0,\nu) = 0$ that stands for the Fréchet derivative of $F(\varphi,\nu)$ with respect to φ at $\varphi = 0$.

$$\tilde{\mathscr{X}}_{\mathbb{C}} = \left\{\tilde{U}(x,y) \in W^{2,2}(\Omega) : \nabla\tilde{U}(x,y)\cdot\eta = 0, (x,y) \in \partial\Omega\right\},$$

where η is the out-of-unit normal vector. Here we use the complexification space $\tilde{\mathscr{X}}_{\mathbb{C}}$, because a complex form of eigenvector is more suitable to shorten the expressions in the normal form derivation.

Let us now explore the existing conditions of Hopf bifurcation based on system (5). Again, we use $x = r\cos\theta$, $y = r\sin\theta$ and the domain Ω is transformed into $\mathbb{D} = \{(r,\theta) : 0 \le r < R, 0 \le \theta < 2\pi\}$. For simplicity, we still use symbols in (5). System (5) can be converted in polar coordinates to

$$\frac{\partial U(t,r,\theta)}{\partial t} = D(\nu)\Delta_{r\theta}U(t,r,\theta) + L(\nu)U_t(r,\theta) + F\left(U_t(r,\theta),\nu\right), \tag{6}$$

and analogously, define the phase space

$$\mathscr{C} := C([-1,0],\mathscr{X}_{\mathbb{C}}),$$

where

$$\mathscr{X}_{\mathbb{C}} = \left\{\tilde{U}(r,\theta) \in W^{2,2}(\mathbb{D}) : \partial_r\tilde{U}(R,\theta) = 0,\ \theta \in [0,2\pi)\right\},$$

with inner product

$$\langle u(r,\theta), v(r,\theta)\rangle = \iint_{\mathbb{D}} ru(r,\theta)\bar{v}(r,\theta)\mathrm{d}r\mathrm{d}\theta$$

weighted r for $u(r,\theta),\ v(r,\theta) \in \mathscr{X}_{\mathbb{C}}$. Then, $U_t(r,\theta) \in \mathscr{C}$.

Linearizing system (6) at the origin, we have

$$\frac{\partial U(t,r,\theta)}{\partial t} = D(\nu)\Delta_{r\theta}U(t,r,\theta) + L(\nu)U_t(r,\theta). \tag{7}$$

The characteristic equations of (7) are

$$\gamma\varphi - D(\nu)\Delta_{r\theta}\varphi - L(\nu)(\mathrm{e}^{\gamma\cdot}\varphi) = 0, \tag{8}$$

where $\mathrm{e}^{\gamma\cdot}(\vartheta)\varphi = \mathrm{e}^{\gamma\vartheta}\varphi$, for $\vartheta \in [-1,0]$, and γ is an eigenvalue of Eq. (7). By Theorem II.3, we find that solving (8) is equivalent to solving the following two groups of characteristic equations. The first group is given as

$$\det\left[\gamma I + \lambda_{0m}D(\nu) - L(\nu)(\mathrm{e}^{\gamma\cdot}I)\right] = 0,\ m = 0,1,2,\cdots. \tag{9}$$

The second groups of equations have multiple roots as the eigenspace of $\lambda_{nm},\ n = 1,2,\cdots,\ m = 1,2,\cdots$ is of two-dimensional. They are

$$\det\left[\gamma I + \lambda_{nm}D(\nu) - L(\nu)(\mathrm{e}^{\gamma\cdot}I)\right]^2 = 0,\ n = 1,2,\cdots,\ m = 1,2,\cdots. \tag{10}$$

In order to consider the Hopf bifurcation, we assume that the following conditions hold for some $\nu_{\hat{\lambda}},\ \hat{\lambda} = \lambda_{0m}$ or λ_{nm}.

(H_1) There exists a neighborhood $\mathscr{U}_1$ of $\nu_{\hat{\lambda}}$, $\hat{\lambda} = \lambda_{0m}$ such that for $\nu \in \mathscr{U}_1$, system (7) has a pair of complex simple conjugate eigenvalues $\alpha_{\hat{\lambda}}(\nu) \pm \mathrm{i}\omega_{\hat{\lambda}}(\nu)$ and the remaining eigenvalues of (7) have non-zero real part for $\nu \in \mathscr{U}_1$.

(H_2) There exists a neighborhood $\mathscr{U}_2$ of $\nu_{\hat{\lambda}}$, $\hat{\lambda} = \lambda_{nm}$ such that for $\nu \in \mathscr{U}_2$, system (7) has a pairs of complex repeated conjugate eigenvalues $\alpha_{\hat{\lambda}}(\nu) \pm \mathrm{i}\omega_{\hat{\lambda}}(\nu)$ (both geometric multiplicity and algebraic multiplicity are two) and the remaining eigenvalues of (7) have non-zero real part for $\nu \in \mathscr{U}_2$.

(H_3) $\alpha_{\hat{\lambda}}(\nu) \pm \mathrm{i}\omega_{\hat{\lambda}}(\nu)$ are continuously differential in ν with $\alpha_{\hat{\lambda}}(\nu_{\hat{\lambda}}) = 0$, $\omega_{\hat{\lambda}}(\nu_{\hat{\lambda}}) = \omega_{\hat{\lambda}} > 0$.

Remark II.5. *According to [15], problem (6) is Γ equivariant, with $\Gamma = O(2) \times S^1$. For instance, write the right hand of system (6) as $\mathscr{F}(U(t,r,\theta))$, we have*

$$\mathscr{F}(\kappa U(t,r,\theta)) = \kappa \mathscr{F}(U(t,r,\theta)), \forall \kappa \in \Gamma.$$

Thus, in what follows, any solution after the action of this group is still a solution of the equation.

By [15, 20, 38, 41] and Remark II.5, if (H_1) and (H_3) or (H_2) and (H_3) hold, noting $\hat{\nu} = \min\{\nu_{\hat{\lambda}}\}$, we know Hopf bifurcations occur at the critical values $\nu = \hat{\nu}$. When $\hat{\lambda} = \lambda_{0m}$, $m = 0,1,2,\cdots$, the center subspace of the equilibrium is two-dimensional, so we call this a standard Hopf bifurcation. When $\hat{\lambda} = \lambda_{nm}$, $n = 1,2,\cdots$, $m = 1,2,\cdots$, the center subspace of the equilibrium is four-dimensional, we say this is a (real) equivariant Hopf bifurcation. In the coming section, we will calculate the equivariant Hopf bifurcation at the origin.

III. HOPF BIFURCATION ANALYSIS

A. Normal form for PFDEs

In this section, we will investigate the properties of the equivariant Hopf bifurcation at the origin, using the theory in [5, 6, 22, 30, 38].

Letting $\nu = \hat{\nu} + \mu$, where $\hat{\nu}$ is given in Sec. II B and $\mu \in \mathbb{R}$, following the method proposed in [6], and using μ as a new variable, the Taylor expansions of $L(\hat{\nu} + \mu)$ and $D(\hat{\nu} + \mu)$ are as follows:

$$L(\hat{\nu} + \mu) = \tilde{L}_0 + \mu \tilde{L}_1 + \frac{1}{2}\mu^2 \tilde{L}_2 + \cdots,$$

$$D(\hat{\nu} + \mu) = \tilde{D}_0 + \mu \tilde{D}_1 + \frac{1}{2}\mu^2 \tilde{D}_2 + \cdots,$$

where $\tilde{D}_0 = D(\hat{v}),\ \tilde{L}_0(\cdot) = L(\hat{v})(\cdot)$ is a linear operator from $\mathscr{C}$ to $\mathscr{X}_{\mathbb{C}}$. Now, in the space $\mathscr{C}$, system (6) is equivalent to

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \tilde{D}_0 \Delta_{r\theta} U(t) + \tilde{L}_0 U_t + \tilde{F}\left(U_t, \mu\right), \tag{11}$$

where $\tilde{F}(\varphi,\mu) = [D(\hat{v}+\mu) - \tilde{D}_0]\Delta_{r\theta}\varphi(0) + [L(\hat{v}+\mu) - \tilde{L}_0](\varphi) + F(\varphi,\hat{v}+\mu)$, and the linearization system of (11) is obtained as follows

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \tilde{D}_0 \Delta_{r\theta} U(t) + \tilde{L}_0 U_t. \tag{12}$$

1. Decomposition of $\mathscr{C}$

Let $A : \mathscr{C} \to \mathscr{X}_{\mathbb{C}}$ represent the infinitesimal generators of the semigroup induced by the solutions of (12), and A^* is the adjoint operator of A, which satisfy

$$A\varphi(\vartheta) = \begin{cases} \varphi'(\vartheta), & \vartheta \in [-1,0), \\ \int_{-\tau}^{0} \mathrm{d}\eta(\mu,\vartheta)\varphi(\vartheta), & \vartheta = 0, \end{cases} \tag{13}$$

$$A^*\psi(\rho) = \begin{cases} -\psi'(\rho), & \rho \in (0,1], \\ \int_{-\tau}^{0} \psi(-\rho)\mathrm{d}\eta(\mu,\vartheta), & \rho = 0. \end{cases} \tag{14}$$

In addition, define a bilinear pairing

$$(\psi,\varphi) = \int_0^R \int_0^{2\pi} r\left[\overline{\psi(0)}\varphi(0) - \int_{-\tau}^{0}\int_{\xi=0}^{\vartheta} \overline{\psi(\xi-\vartheta)}\mathrm{d}\eta(\hat{v},\vartheta)\varphi(\xi)\mathrm{d}\xi\right]\mathrm{d}r\mathrm{d}\theta. \tag{15}$$

From the discussion in Sec. II B, we know that A has a pair of repeated purely imaginary eigenvalues $\pm \mathrm{i}\omega_{\hat{\lambda}}$ which are also eigenvalues of A^*. Let the central subspace P and P^* be the generalized eigenspace of A and A^* about $\Lambda_0 = \{\pm \mathrm{i}\omega_{\hat{\lambda}}\}$, respectively, where $\pm \mathrm{i}\omega_{\hat{\lambda}}$ are repeated. P^* is the adjoint space of P. An important task is to decompose the space $\mathscr{C}$ through the relationship of bases in P and P^*, and we write $\mathscr{C} = P_{CN} \oplus Q_S$, where P_{CN} is the central subspace and Q_S is its complementary space.

Define

$$\hat{\phi}^c_{nm} = \frac{\phi^c_{nm}}{\|\phi^c_{nm}\|_{2,2}},\ \hat{\phi}^s_{nm} = \frac{\phi^s_{nm}}{\|\phi^s_{nm}\|_{2,2}}.$$

Lemma III.1. *Let the basis of P is*

$$\Phi_{r\theta}(\vartheta) = \left(\Phi^1_{r\theta}(\vartheta), \Phi^2_{r\theta}(\vartheta)\right) = \left(\Phi^1(\vartheta)\cdot\hat{\phi}^c_{nm}, \Phi^2(\vartheta)\cdot\hat{\phi}^s_{nm}\right),\ \vartheta \in [-1,0], \tag{16}$$

with

$$\Phi^1_{r\theta}(\vartheta)=\left(\Phi_1(\vartheta)\cdot\hat{\phi}^c_{nm},\Phi_2(\vartheta)\cdot\hat{\phi}^c_{nm}\right)=\left(\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\xi\hat{\phi}^c_{nm},\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\bar{\xi}\hat{\phi}^c_{nm}\right),$$
$$\Phi^2_{r\theta}(\vartheta)=\left(\Phi_3(\vartheta)\cdot\hat{\phi}^s_{nm},\Phi_4(\vartheta)\cdot\hat{\phi}^s_{nm}\right)=\left(\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\xi\hat{\phi}^s_{nm},\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\bar{\xi}\hat{\phi}^s_{nm}\right),$$

where ξ can be noted as $\xi=(p_{11},p_{12},\cdots,p_{1n})^{\mathrm{T}}$. A basis for the adjoint space P^* is

$$\Psi_{r\theta}(\rho)=\left(\Psi^1_{r\theta}(\rho),\Psi^2_{r\theta}(\rho)\right)^{\mathrm{T}}=\left(\Psi^1(\rho)\cdot\hat{\phi}^c_{nm},\Psi^2(\rho)\cdot\hat{\phi}^s_{nm}\right)^{\mathrm{T}},\ \rho\in[0,1],\tag{17}$$

with

$$\Psi^1_{r\theta}(\rho)=\left(\Psi_1(\rho)\cdot\hat{\phi}^c_{nm},\Psi_2(\rho)\cdot\hat{\phi}^c_{nm}\right)^{\mathrm{T}}=\left(q^{-1}\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\rho}\eta\hat{\phi}^c_{nm},\bar{q}^{-1}\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\rho}\bar{\eta}\hat{\phi}^c_{nm}\right)^{\mathrm{T}},$$
$$\Psi^2_{r\theta}(\rho)=\left(\Psi_3(\rho)\cdot\hat{\phi}^s_{nm},\Psi_4(\rho)\cdot\hat{\phi}^s_{nm}\right)^{\mathrm{T}}=\left(q^{-1}\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\rho}\eta\hat{\phi}^s_{nm},\bar{q}^{-1}\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\rho}\bar{\eta}\hat{\phi}^s_{nm}\right)^{\mathrm{T}},$$

where $\eta=(q_{11},q_{12},\cdots,q_{1n})$ and q in the expressions of $\Psi^1_{r\theta}(\rho)$ and $\Psi^2_{r\theta}(\rho)$ can be obtained by $(\Psi_{r\theta},\Phi_{r\theta})=I$, according to the adjoint bilinear form defined in (15).

One can decompose U_t into two parts:

$$U_t=U_t^P+U_t^Q=\sum_{k=1}^{2}\Phi^k_{r\theta}\left(\Psi^k_{r\theta},U_t\right)+U_t^Q=\sum_{k=1}^{2}\Phi^k_{r\theta}z^k_{r\theta}+y_t,\tag{18}$$

where $z^k_{r\theta}=\left(\Psi^k_{r\theta},U_t\right),\ y_t\in Q_s$. Define $z=(z^1_{r\theta},z^2_{r\theta})\equiv(z_1,z_2,z_3,z_4)^{\mathrm{T}}$ as the local coordinate system on the four-dimensional center manifold, which is induced by the basis $\Phi_{r\theta}$.

Then, we get that

$$\begin{aligned}\dot{z}(t)&=\tilde{B}z(t)+\begin{pmatrix}\left\langle\tilde{F}(\sum_{k=1}^{2}\Phi^k_{r\theta}z^k_{r\theta}+y,\mu),\Psi^1_{r\theta}(0)\right\rangle\\ \left\langle\tilde{F}(\sum_{k=1}^{2}\Phi^k_{r\theta}z^k_{r\theta}+y,\mu),\Psi^2_{r\theta}(0)\right\rangle\end{pmatrix},\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=A_Qy+(I-\pi)X_0\tilde{F}\left(\sum_{k=1}^{2}\Phi^k_{r\theta}z^k_{r\theta}+y,\mu\right),\end{aligned}\tag{19}$$

where A_Q is the restriction of A on Q_s, $A_Q\varphi=A\varphi$ for $\varphi\in Q_s$, $\pi:\mathscr{C}\to P_{CN}$ is the projection, and

$$\tilde{B}=\begin{pmatrix}\mathrm{i}\omega_{\hat{\lambda}}&0&0&0\\0&-\mathrm{i}\omega_{\hat{\lambda}}&0&0\\0&0&\mathrm{i}\omega_{\hat{\lambda}}&0\\0&0&0&-\mathrm{i}\omega_{\hat{\lambda}}\end{pmatrix}.$$

According to the formal Taylor expansion $\tilde{F}(\varphi,\mu)=\sum_{j\geq2}\frac{1}{j!}\tilde{F}_j(\varphi,\mu)$, (19) can be written as

$$\begin{aligned}\dot{z}(t)&=\tilde{B}z(t)+\sum_{j\geq2}\frac{1}{j!}f^1_j(z,y,\mu),\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=A_Qy+\sum_{j\geq2}\frac{1}{j!}f^2_j(z,y,\mu),\end{aligned}\tag{20}$$

where $f_j = (f_j^1, f_j^2),\ j \geq 2$ are defined by

$$
\begin{aligned}
f_j^1(z,y,\mu) &= \begin{pmatrix} \left\langle \tilde{F}(\sum_{k=1}^{2} \Phi_{r\theta}^k z_{r\theta}^k + y, \mu), \Psi_{r\theta}^1(0) \right\rangle \\ \left\langle \tilde{F}(\sum_{k=1}^{2} \Phi_{r\theta}^k z_{r\theta}^k + y, \mu), \Psi_{r\theta}^2(0) \right\rangle \end{pmatrix}, \\
f_j^2(z,y,\mu) &= (I-\pi) X_0 \tilde{F}_j \left(\sum_{k=1}^{2} \Phi_{r\theta}^k z_{r\theta}^k + y, \mu \right).
\end{aligned}
\tag{21}
$$

Referring to [6], the normal form is

$$
\begin{aligned}
\dot{z}(t) &= \tilde{B}z(t) + \frac{1}{2} g_2^1(z,y,\mu) + \frac{1}{6} g_3^1(z,y,\mu) + h.o.t., \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= A_Q y + \frac{1}{2} g_2^2(z,y,\mu) + \frac{1}{6} g_3^2(z,y,\mu) + h.o.t.,
\end{aligned}
\tag{22}
$$

where $g = (g_j^1, g_j^2),\ j \geq 2$ is given by

$$
g_j(z,y,\mu) = \bar{f}_j(z,y,\mu) - M_j U_j(z,\mu),
$$

where $\bar{f}_j^1$ is the terms of order j in (z,y) obtained after the computation of normal forms up to order $j-1$, $U_j = (U_j^1, U_j^2)$ denotes the change of variables about the transformation from f_j to g_j, and the operator $M_j = (M_j^1, M_j^2)$ is defined by

$$
\begin{aligned}
M_j^1 &: \mathbb{V}_j^5(\mathscr{C}^4) \to \mathbb{V}_j^5(\mathscr{C}^4), \\
M_j^1 U_j^1 &= D_z U_j^1(z,\mu) \tilde{B} z - \tilde{B} U_j^1(z,\mu), \\
M_j^2 &: \mathbb{V}_j^5(Q_s) \to \mathbb{V}_j^5(\mathrm{Ker}\pi), \\
M_j^2 U_j^2 &= D_z U_j^2(z,\mu) \tilde{B} z - A_Q U_j^2(z,\mu),
\end{aligned}
\tag{23}
$$

where $\mathbb{V}_j^5(Y)$ denotes the space homogeneous polynomials of $z = (z_1, z_2, z_3, z_4)^{\mathrm{T}}$ and μ with coefficients in $\mathscr{C}^4$.

It is easy to verify that

$$
M_j^1(\mu z^p e_k) = \mathrm{i}\omega_{\hat{\lambda}} \mu \left(p_1 - p_2 + p_3 - p_4 + (-1)^k \right) z^p e_k,\ |p| = j-1, \tag{24}
$$

where $j \geq 2,\ k = 1,2,3,4$, and $\{e_1, e_2, e_3, e_4\}$ is the canonical basis for $\mathscr{C}^4$. Applying the center manifold theory [5, 42], the existence of an invariant local center manifold of the origin can be obtained, and referring to [6], the flow on it is given by the four-dimensional ODEs

$$
\dot{z}(t) = \tilde{B}z(t) + \frac{1}{2} g_2^1(z,0,\mu) + \frac{1}{6} g_3^1(z,0,\mu) + \cdots. \tag{25}
$$

2. The normal form

By calculating $g_2^1(z,0,\mu)$ and $g_3^1(z,0,\mu)$, the normal form truncated to the third order on the center manifold can be summarized as follows:

$$
\begin{aligned}
\dot{z}_1 =& \mathrm{i}\omega_{\hat{\lambda}} z_1 + B_{11} z_1 \mu + B_{2100} z_1^2 z_2 + B_{2001} z_1^2 z_4 + B_{0120} z_3^2 z_2 + B_{0021} z_3^2 z_4 + B_{1110} z_1 z_2 z_3 + B_{1011} z_1 z_3 z_4,\\
\dot{z}_2 =& -\mathrm{i}\omega_{\hat{\lambda}} z_2 + \overline{B_{11}} z_2 \mu + \overline{B_{2100}} z_1 z_2^2 + \overline{B_{2001}} z_2^2 z_3 + \overline{B_{0120}} z_4^2 z_1 + \overline{B_{0021}} z_4^2 z_3 + \overline{B_{1110}} z_1 z_2 z_4 + \overline{B_{1011}} z_2 z_3 z_4,\\
\dot{z}_3 =& \mathrm{i}\omega_{\hat{\lambda}} z_3 + B_{11} z_3 \mu + B_{2100} z_3^2 z_4 + B_{2001} z_3^2 z_2 + B_{0120} z_1^2 z_4 + B_{0021} z_1^2 z_2 + B_{1110} z_1 z_3 z_4 + B_{1011} z_1 z_2 z_3,\\
\dot{z}_4 =& -\mathrm{i}\omega_{\hat{\lambda}} z_4 + \overline{B_{11}} z_4 \mu + \overline{B_{2100}} z_3 z_4^2 + \overline{B_{2001}} z_4^2 z_1 + \overline{B_{0120}} z_2^2 z_3 + \overline{B_{0021}} z_1^2 z_2 + \overline{B_{1110}} z_2 z_3 z_4 + \overline{B_{1011}} z_1 z_2 z_4.
\end{aligned}
\tag{26}
$$

Please refer to Sec. 1 in supplementary materials for the specific calculation process.

Lemma III.2. *By [22], the normal form truncated to the third order can be reduced to*

$$
\begin{aligned}
\dot{z}_1 &= \mathrm{i}\omega_{\hat{\lambda}} z_1 + B_{11} z_1 \mu + B_{2001} z_1^2 z_4 + B_{1110} z_1 z_2 z_3,\\
\dot{z}_2 &= -\mathrm{i}\omega_{\hat{\lambda}} z_2 + \overline{B_{11}} z_2 \mu + \overline{B_{2001}} z_3 z_2^2 + \overline{B_{1110}} z_1 z_2 z_4,\\
\dot{z}_3 &= \mathrm{i}\omega_{\hat{\lambda}} z_3 + B_{11} z_3 \mu + B_{2001} z_3^2 z_2 + B_{1110} z_1 z_3 z_4,\\
\dot{z}_4 &= -\mathrm{i}\omega_{\hat{\lambda}} z_4 + \overline{B_{11}} z_4 \mu + \overline{B_{2001}} z_1 z_4^2 + \overline{B_{1110}} z_2 z_3 z_4.
\end{aligned}
\tag{27}
$$

and the symmetry properties are preserved.

The proof is given in Sec. 2 in supplementary materials.

Introducing double sets of polar coordinates

$$
\begin{aligned}
z_1 &= \rho_1 \mathrm{e}^{\mathrm{i}\chi_1},\ z_4 = \rho_1 \mathrm{e}^{-\mathrm{i}\chi_1},\\
z_3 &= \rho_2 \mathrm{e}^{\mathrm{i}\chi_2},\ z_2 = \rho_2 \mathrm{e}^{-\mathrm{i}\chi_2},
\end{aligned}
\tag{28}
$$

we can obtain that

$$
\begin{aligned}
\dot{\rho}_1 &= (a_1\mu + a_2\rho_1^2 + a_3\rho_2^2)\rho_1,\\
\dot{\chi}_1 &= \omega_{\hat{\lambda}},\\
\dot{\rho}_2 &= (a_1\mu + a_2\rho_2^2 + a_3\rho_1^2)\rho_2,\\
\dot{\chi}_2 &= \omega_{\hat{\lambda}},
\end{aligned}
\tag{29}
$$

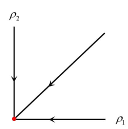
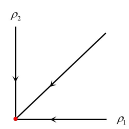
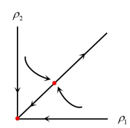
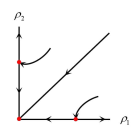
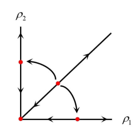
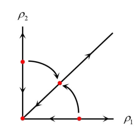
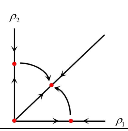
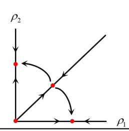
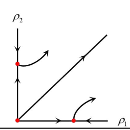
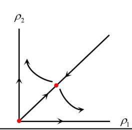
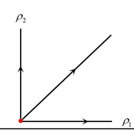
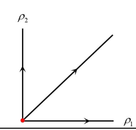
with

$$a_1 = \mathrm{Re}\{B_{11}\},\ a_2 = \mathrm{Re}\{B_{2001}\},\ a_3 = \mathrm{Re}\{B_{1110}\}.$$

TABLE I. The six unfoldings of system (29).

Case	1	2	3	4	5	6
a_2	−	−	−	+	+	+
a_2+a_3	−	−	+	−	+	+
a_2-a_3	−	+	−	+	+	−

TABLE II. The dynamical classifications of system (29) in each case.

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
$a_1\mu<0$						
$a_1\mu>0$						

Based on the above analysis, by [22, 43], we get, when $a_1\mu<0(>0)$, system (29) has six unfoldings(see Table I) and their dynamical classifications are shown in Table II.

Therefore, we can draw the following conclusions.

Theorem III.3. *We are mainly concerned with the properties corresponding to the following four equilibrium points of (29).*

(i) $(\rho_1,\rho_2)=(0,0)$ *corresponds to the origin in the four-dimensional phase space and undergoes a stationary solution, which is spatially homogeneous.*

(ii) *For* $a_1\mu\cdot a_2<0$, $(\rho_1,\rho_2)=(0,\sqrt{\frac{-a_1\mu}{a_2}})$ *corresponds to a periodic solution in the plane of* (z_2,z_3)*, which is spatially inhomogeneous. At this point, the periodic solution restricted to the center subspace has the following approximate form:*

$$U_t(\vartheta)(r,\theta)\approx\sum_{i=1}^{n}2|p_{1i}|\sqrt{\frac{-a_1\mu}{a_2}}J_n(\sqrt{\lambda_{nm}}r)\cos(\mathrm{Arg}(p_{1i})+\omega_{\hat{\lambda}}\vartheta+\omega_{\hat{\lambda}}t+n\theta)\mathrm{e}_i,$$

where e_i *is the ith unit coordinate vector of* $\mathbb{R}^n$*. When* $a_1\mu>0(<0)$, $a_2<0(>0)$*, system undergoes an anticlockwise rotating wave. Only when* $a_1\mu>0$, $a_2+a_3<0$, $a_2-a_3>0$*, the periodic* *solution of the equivariant Hopf bifurcation is orbitally asymptotically stable.*

(iii) *For $a_1\mu \cdot a_2 < 0$, $(\rho_1,\rho_2) = (\sqrt{\frac{-a_1\mu}{a_2}},0)$ corresponds to a a periodic solution in the plane of (z_1,z_4), which is spatially inhomogeneous. At this point, the periodic solution restricted to the center subspace has the following approximate form:*

$$U_t(\vartheta)(r,\theta) \approx \sum_{i=1}^{n} 2|p_{1i}|\sqrt{\frac{-a_1\mu}{a_2}} J_n(\sqrt{\lambda_{nm}}r)\cos(\mathrm{Arg}(p_{1i})+\omega_{\hat{\lambda}}\vartheta+\omega_{\hat{\lambda}}t-n\theta)\mathrm{e}_i.$$

When $a_1\mu > 0(< 0)$, $a_2 < 0(> 0)$, system undergoes a clockwise rotating wave. Its stability conditions are as same as (ii).

(iv) *For $a_1\mu \cdot (a_2+a_3) < 0$, $(\rho_1,\rho_2) = (\sqrt{\frac{-a_1\mu}{a_2+a_3}},\sqrt{\frac{-a_1\mu}{a_2+a_3}})$ corresponds to a periodic solution, which is spatially inhomogeneous. At this point, the periodic solution restricted to the center subspace has the following approximate form:*

$$U_t(\vartheta)(r,\theta) \approx \sum_{i=1}^{n} 4|p_{1i}|\sqrt{\frac{-a_1\mu}{a_2+a_3}} J_n(\sqrt{\lambda_{nm}}r)\cos(\mathrm{Arg}(p_{1i})+\omega_{\hat{\lambda}}\vartheta+\omega_{\hat{\lambda}}t)\cos(n\theta)\mathrm{e}_i.$$

When $a_1\mu > 0(< 0)$, $a_2+a_3 < 0(> 0)$, system undergoes a standing wave. Only when $a_1\mu >$ 0, $a_2+a_3 < 0$, $a_2-a_3 < 0$, the periodic solution of the equivariant Hopf bifurcation is orbitally asymptotically stable.

The proof is given in Sec. 3 in supplementary materials.

Corollary III.4. *For $\lambda_{0m}, m = 0,1,2,\cdots$, the corresponding characteristic function is ϕ^c_{0m}. Define $\hat{\phi}^c_{0m} = \frac{\phi^c_{0m}}{\|\phi^c_{0m}\|_{2,2}}$. According to [6], after a similar calculation process shown above, the following normal form on the center manifold is obtained,*

$$\begin{aligned}\dot{z}_1 &= \mathrm{i}\omega_{\hat{\lambda}} z_1 + B^*_{11} z_1\mu + B^*_{2100} z_1^2 z_2 + \cdots,\\ \dot{z}_2 &= -\mathrm{i}\omega_{\hat{\lambda}} z_2 + \overline{B^*_{11}} z_2\mu + \overline{B^*_{2100}} z_1 z_2^2 + \cdots.\end{aligned}$$

Introducing a set of polar coordinates, we can get

$$\dot{\rho} = (a^*_1\mu + a^*_2\rho^2)\rho + o(\mu^2\rho + |(\rho,\mu)|^4),$$

with

$$a^*_1 = \mathrm{Re}\{B^*_{11}\},\ a^*_2 = \mathrm{Re}\{B^*_{2100}\},$$

where the specific representation of B^*_{11} and B^*_{2100} is shown in [6].

Besides, we get that

(i) *When $a^*_2 < 0(> 0)$, the periodic solution is orbitally asymptotically stable(unstable).*

(ii) *When $a^*_1 a^*_2 < 0(> 0)$, the bifurcation is supercritical(subcritical).*

B. Application to a class of reaction-diffusion model with discrete time delay

In a specific model, it is necessary to calculate $A_{p_1p_2p_3p_4}$, $S_{y(0)z_k}$, $S_{y(-1)z_k}$, $k=1,2,3,4$ to determine the explicit expression of $B_{p_1p_2p_3p_4}$ in the normal form, and the time delay is often selected as a bifurcation parameter to study the Hopf bifurcation. Therefore, in order to provide a more general symbolic expression, we will consider a class of reaction-diffusion system with discrete time delay defined on a disk as follows:

$$\begin{cases} \frac{\partial u(t,r,\theta)}{\partial t} = d_1\Delta_{r\theta}u(t,r,\theta)+F^{(1)}(u(t,r,\theta),v(t,r,\theta)),\ (r,\theta)\in\mathbb{D},\ t>0, \\ \frac{\partial v(t,r,\theta)}{\partial t} = d_2\Delta_{r\theta}v(t,r,\theta)+F^{(2)}(u(t,r,\theta),v(t,r,\theta),u(t-\tau,r,\theta),v(t-\tau,r,\theta)),\ (r,\theta)\in\mathbb{D},\ t>0, \\ \partial_r u(\cdot,R,\theta)=\partial_r v(\cdot,R,\theta)=0,\ \theta\in[0,2\pi). \end{cases} \tag{30}$$

This type of model covers some predator-prey systems and chemical reaction models, etc. While in practice there are many ways to introduce the time delay τ, we demonstrate the critical method of analysis by including τ in the second equation for simplicity. Other types of systems can also refer to this process for calculation.

Assume that the model has a positive equilibrium point $E^*(u^*,v^*)$ and select the time delay τ as the bifurcation parameter. Letting $\bar{u}(t,r,\theta)=u(\tau t,r,\theta)-u^*, \bar{v}(t,r,\theta)=v(\tau t,r,\theta)-v^*$, we drop the bar for simplicity. Then system (30) can be transformed into

$$\begin{cases} \frac{\partial u(t,r,\theta)}{\partial t} =&\tau d_1\Delta_{r\theta}u(t,r,\theta)+\tau\left[a_{11}\left(u(t,r,\theta)+u^*\right)+a_{12}\left(v(t,r,\theta)+v^*\right)\right] \\ & +\tau\sum_{i+j\geq 2}\frac{1}{i!j!}F^{(1)}_{ij}(0,0)u^i(t,r,\theta)v^j(t,r,\theta), \\ \frac{\partial v(t,r,\theta)}{\partial t} =&\tau d_2\Delta_{r\theta}v(t,r,\theta)+\tau\left[a_{21}\left(u(t,r,\theta)+u^*\right)+a_{22}\left(v(t,r,\theta)+v^*\right)\right] \\ & +\tau\left[b_{21}\left(u(t-1,r,\theta)+u^*\right)+b_{22}\left(v(t-1,r,\theta)+v^*\right)\right] \\ & +\tau\sum_{i+j+l+k\geq 2}\frac{1}{i!j!k!l!}F^{(2)}_{ijkl}(0,0,0,0)u^i(t,r,\theta)v^j(t,r,\theta)u^k(t-1,r,\theta)v^l(t-1,r,\theta), \end{cases} \tag{31}$$

with

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial F^{(1)}(u^*,v^*)}{\partial u(t)} & \frac{\partial F^{(1)}(u^*,v^*)}{\partial v(t)} \\ \frac{\partial F^{(2)}(u^*,v^*,u^*,v^*)}{\partial u(t)} & \frac{\partial F^{(2)}(u^*,v^*,u^*,v^*)}{\partial v(t)} \end{pmatrix},$$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{\partial F^{(2)}(u^*,v^*,u^*,v^*)}{\partial u(t-\tau)} & \frac{\partial F^{(2)}(u^*,v^*,u^*,v^*)}{\partial v(t-\tau)} \end{pmatrix},$$

$$F_{ij}^{(1)} = \frac{\partial^{i+j} F^{(1)}}{\partial u^i \partial v^j}(0,0),$$
$$F_{ijkl}^{(2)} = \frac{\partial^{i+j+k+l} F^{(2)}}{\partial u^i \partial v^j \partial u^k(t-\tau) \partial v^l(t-\tau)}(0,0,0,0).$$

Letting $\tau = \hat{\tau} + \mu$, where $\mu \in \mathbb{R}$ and $\hat{\tau}$ is the critical values at which Hopf bifurcations occur, then system (31) can be written in an abstract form like (11), where operators L_0 and $\tilde{F}$ are given, respectively, by

$$L_0(\varphi) = (\hat{\tau} + \mu) \begin{pmatrix} a_{11}\varphi_1(0) + a_{12}\varphi_2(0) \\ b_{21}\varphi_1(-1) + b_{22}\varphi_2(-1) + a_{21}\varphi_1(0) + a_{22}\varphi_2(0) \end{pmatrix},$$

$$\tilde{F}(\varphi, \mu) = (\hat{\tau} + \mu) \begin{pmatrix} \sum_{i+j\geq 2} \frac{1}{i!j!} F_{ij}^{(1)}(0,0)\varphi_1^i(0)\varphi_2^j(0) \\ \sum_{i+j+k+l\geq 2} \frac{1}{i!j!k!l!} F_{ijkl}^{(2)}(0,0,0,0)\varphi_1^i(0)\varphi_2^j(0)\varphi_1^k(-1)\varphi_2^l(-1) \end{pmatrix},$$
$$\varphi = (\varphi_1, \varphi_2)^{\mathrm{T}} \in \mathscr{C}.$$

Choosing $\xi = (1, p_0)^T$, with $p_0 = \frac{\mathrm{i}\omega_{\hat{\lambda}} + d_1\lambda_{nm} - a_{11}}{a_{12}}$, we get that the bases of P is

$$\Phi_{r\theta}(\vartheta) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}\vartheta}\hat{\phi}_{nm}^c & \mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}\vartheta}\hat{\phi}_{nm}^c & \mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}\vartheta}\hat{\phi}_{nm}^s & \mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}\vartheta}\hat{\phi}_{nm}^s \\ p_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}\vartheta}\hat{\phi}_{nm}^c & \bar{p}_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}\vartheta}\hat{\phi}_{nm}^c & p_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}\vartheta}\hat{\phi}_{nm}^s & \bar{p}_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}\vartheta}\hat{\phi}_{nm}^s \end{pmatrix},$$

and the basis of P^* can also be obtained. The explicit formulas of $A_{p_1p_2p_3p_4}$, $S_{y(0)z_k}$, $S_{y(-1)z_k}$, $k = 1,2,3,4$, and $h_{jp_1p_2p_3p_4}$ in the calculation of normal form are shown in Secs. 4-6 in supplementary materials.

IV. NUMERICAL SIMULATIONS

In this section, we present two reaction-diffusion systems with discrete time delays as numerical examples. The time delay is chosen as a bifurcation parameter in both two models to simulate spatially homogeneous periodic solutions and spatially inhomogeneous periodic solutions, including standing and rotating waves.

A. Numerical example 1: A diffusive Brusselator model with delayed feedback

[44] studied a diffusive Brusselator model with delayed feedback. We put this model on a disk with Neumann boundary conditions and perform some numerical simulations. The model in polar

form now turns to be

$$\begin{cases} \frac{\partial u(t,r,\theta)}{\partial t} = d_1\Delta_{r\theta}u(t,r,\theta)+a-(b+1)u(t,r,\theta)+u^2(t,r,\theta)v(t,r,\theta),\ (r,\theta)\in\mathbb{D},\ t>0, \\ \frac{\partial v(t,r,\theta)}{\partial t} = d_2\Delta_{r\theta}v(t,r,\theta)+bu(t,r,\theta)-u^2(t,r,\theta)v(t,r,\theta)+g(v(t-\tau,r,\theta)-v(t,r,\theta)),\ (r,\theta)\in\mathbb{D},\ t>0, \\ \partial_r u(\cdot,R,\theta)=\partial_r v(\cdot,R,\theta)=0,\ \theta\in[0,2\pi). \end{cases} \tag{32}$$

Fixing $a=1,\ b=1.5,\ g=2,\ d_1=2,\ d_2=5,\ R=10$, we get that the unique positive equilibrium solution of the model is $(1,1.5)$. When $\hat{\lambda}=\lambda_{00}$, $\omega\approx 0.6166$ and $\hat{\tau}\approx 0.7128$. According to the common analysis of the standard Hopf bifurcation, we get that when $\tau\in[0,0.7128)$, E^* is locally asymptotically stable. When τ increases from zero and crosses the critical value $\hat{\tau}\approx 0.7128$, a family of periodic solutions are bifurcated from E^*, which is spatially homogeneous. By Corollary III.4, the Hopf bifurcation is supercritical and the periodic solutions are stable since $a_1^*a_2^*\approx -0.8264,\ a_2^*\approx -0.6920$. This is actually the periodic solution of the corresponding delay differential equations without diffusion, so the figure is not given here.

B. Numerical example 2: A delayed predator-prey model with group defense and nonlocal competition

In [45], the authors investigated a predator-prey model with group defence and nonlocal competition. Here, we use the method established above to investigate the dynamics of such a model on a disk.

$$\begin{cases} \frac{\partial u(t,r,\theta)}{\partial t} = d_1\Delta_{r\theta}u(t,r,\theta)+bu(t,r,\theta)\left(1-\frac{\hat{u}(t,r,\theta)}{K}\right)-au^{\alpha}(t,r,\theta)v(t,r,\theta),\ (r,\theta)\in\mathbb{D},\ t>0, \\ \frac{\partial v(t,r,\theta)}{\partial t} = d_2\Delta_{r\theta}v(t,r,\theta)-dv(t,r,\theta)+aeu^{\alpha}(t-\tau,r,\theta)v(t,r,\theta),\ (r,\theta)\in\mathbb{D},\ t>0, \\ \partial_r u(\cdot,R,\theta)=\partial_r v(\cdot,R,\theta)=0,\ \theta\in[0,2\pi), \end{cases} \tag{33}$$

where the state u, v and parameters are defined in [45]. In particular, $\hat{u}(r,\theta,t)$ depicts the nonlocal competition with the form of

$$\hat{u}(r,\theta,t)=\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}\bar{r}u\left(\bar{r},\bar{\theta},t\right)\mathrm{d}\bar{\theta}\mathrm{d}\bar{r}.$$

According to subsection II B, characteristic equations of the linearization equation at the positive equilibrium point E^* for system (33) are

$$\begin{cases} \gamma^2+P_{0m}\gamma+Q_{0m}-a_{12}b_{21}\mathrm{e}^{-\gamma\tau}=0,\ m=0,1,2,\cdots, \\ \left(\gamma^2+\bar{P}_{nm}\gamma+\bar{Q}_{nm}-a_{12}b_{21}\mathrm{e}^{-\gamma\tau}\right)^2=0,\ n=1,2,\cdots,\ m=1,2,\cdots, \end{cases} \tag{34}$$

with

$$P_{0m} = (d_1 + d_2)\lambda_{0m} - a_{11} - c_{11},\ Q_{0m} = (d_1\lambda_{0m} - a_{11} - c_{11})d_2\lambda_{0m},\ m = 0,1,2,\cdots;$$
$$\bar{P}_{nm} = (d_1 + d_2)\lambda_{nm} - a_{11},\ \bar{Q}_{nm} = (d_1\lambda_{nm} - a_{11})d_2\lambda_{nm},\ n = 1,2,\cdots,\ m = 1,2,\cdots,$$

where the expressions of $a_{11},\ a_{12},\ b_{21},\ c_{11}$ are shown in [45].

Fixing $b = 0.25,\ K = 20,\ a = 0.3,\ d = 0.7,\ e = 0.5,\ d_1 = 0.3,\ d_2 = 0.75,\ R = 6$, applying the same mathematical analysis method mentioned in [41, 45], at the unique positive constant steady solution, the first two bifurcation curves on the $\alpha - \tau$ plane are shown in Figure 1. We select $\alpha = 0.6$ on the plane of $\alpha - \tau$. When $\hat{\lambda} = \lambda_{11}$, Hopf bifurcation occurs at $\hat{\tau} = \tau^0_{\lambda_{11}} \approx 1.7825$. We know that when $\tau < 1.7825$, E^* is locally asymptotically stable, and when $\tau > 1.7825$, E^* is unstable. The bifurcation generated at this time is an equivariant hopf bifurcation. It can be obtained through numerical calculation that $\mu = 1.2175,\ B_{11} \approx 0.0021 - 0.0911\mathrm{i},\ B_{2001} \approx -0.1075 + 0.0745\mathrm{i},\ B_{1110} \approx -0.1813 + 0.1620\mathrm{i}$. Thus, $a_1\mu \approx 0.0026,\ a_2 \approx -0.1075,\ a_2 + a_3 \approx -0.2888,\ a_2 - a_3 \approx 0.0738$, which corresponds to Case 2 when $a_1\mu > 0$ in Table II. By Theorem III.3, we know that system possesses an unstable standing wave (see Figure 2-4) and two orbitally asymptotically stable rotating waves (see Figure 5-6).

Remark IV.1. *We can see from Fig. 1 that as α changes, a double Hopf point HH appears. Below the lower line is the stable region of the system where $\tau < min\{\tau_{\lambda^0_{00}}, \tau_{\lambda^0_{11}}\}$. Above the lower line where $\tau > \{\tau_{\lambda^0_{00}}, \tau_{\lambda^0_{11}}\}$ the system may produce spatially homogeneous or inhomogeneous period solutions. Investigating the detailed bifurcation sets might require studying a center manifold that is at least six-dimensional.*

Remark IV.2. *We can find that only when the initial value restricted to the center subspace satisfies $\rho_1 = \rho_2$, the spatially inhomogeneous periodic solution is in the form of standing waves. For example, we select the initial value as $u(t,r,\theta) = u^* + \varsigma_1(t,r)\cdot\cos(\theta + \hat{\theta}),\ t \in [-\tau, 0);\ v(t,r,\theta) = v^* + \varsigma_2(t,r)\cdot\cos(\theta + \hat{\theta}),\ t \in [-\tau, 0)$, which has the following approximate form restricted to the center manifold*

$$U_0(\vartheta)(r,\theta) \approx 4(\mathrm{Re}\{\varsigma_1(\vartheta,r)\}, \mathrm{Re}\{p_0 \cdot \varsigma_2(\vartheta,r)\})^{\mathrm{T}}\cos(\theta + \hat{\theta}),$$

with $z_1 = z_2 = z_3 = z_4 = 1$. Thus, the spatially inhomogeneous periodic solutions in Figs. 2-4 are in the form of standing waves. No matter what value of $\hat{\theta}$ is taken, the simulation is a standing wave solution, which reflects the effect of $O(2)$ equivariance. However, when the initial values of *u and v are chosen with other forms, solutions of the system are attracted by one of two coexisting stable rotating waves (see Figs. 5 and 6), which may be clockwise (Fig. 6) or counterclockwise (Fig. 5).*

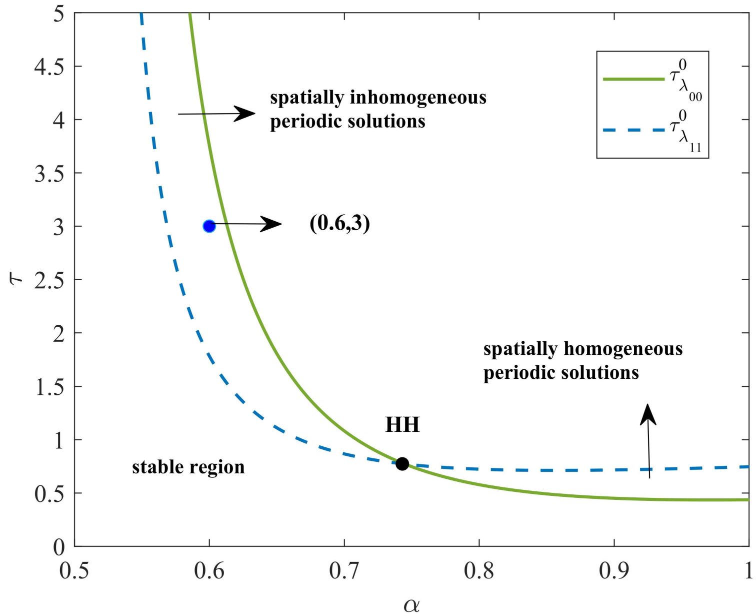


FIG. 1. Partial bifurcation curves on the $\alpha - \tau$ plane.

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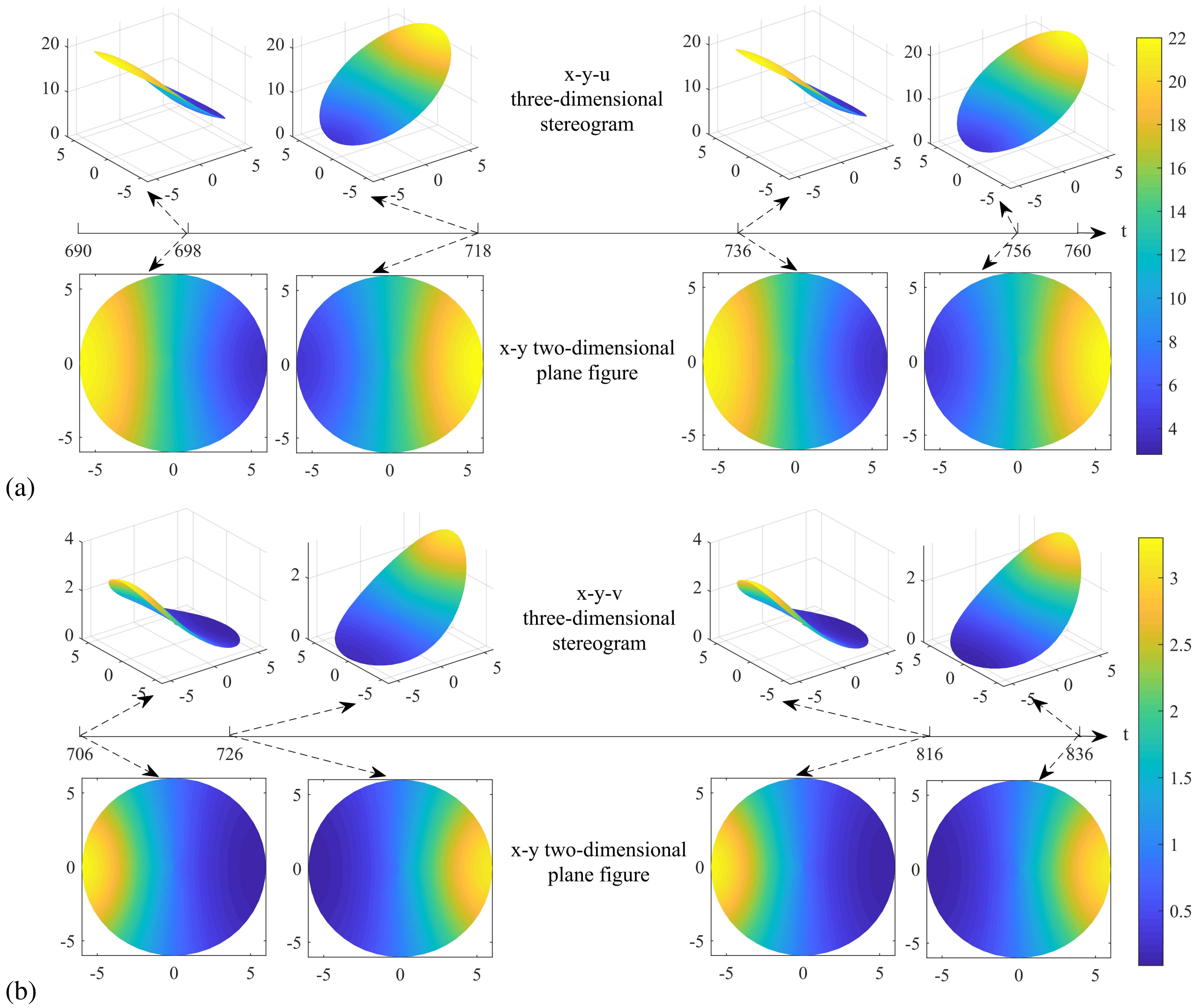


FIG. 2. The system produces standing waves with parameters: $b = 0.25,\ K = 20,\ a = 0.3,\ d = 0.7,\ e = 0.5, \alpha = 0.6,\ d_1 = 0.3,\ d_2 = 0.75,\ R = 6,\ \tau = 3$. Initial values are $u(t,r,\theta) = 13.0320 + 0.01 \cdot \cos t \cdot \cos r \cdot \cos\theta,\ v(t,r,\theta) = 0.8108 + 0.01 \cdot \cos t \cdot \cos r \cdot \cos\theta,\ t \in [-\tau, 0)$. $(a) : u,\ (b) : v$.

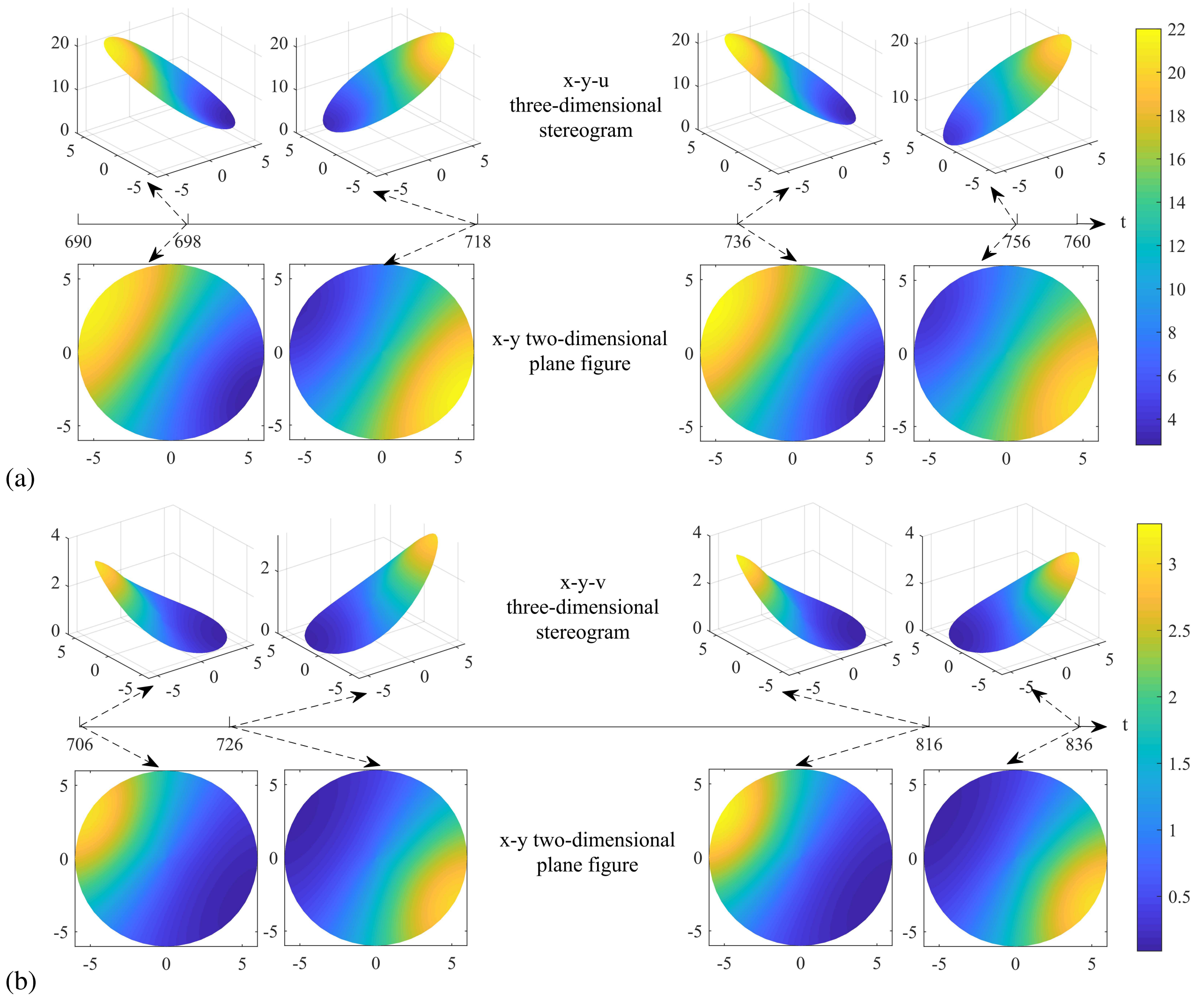


FIG. 3. The system produces standing waves with parameters: $b = 0.25$, $K = 20$, $a = 0.3$, $d = 0.7$, $e = 0.5$, $\alpha = 0.6$, $d_1 = 0.3$, $d_2 = 0.75$, $R = 6$, $\tau = 3$. Initial values are $u(t,r,\theta) = 13.0320 + 0.01 \cdot \cos t \cdot \cos r \cdot \cos(\theta + \frac{\pi}{6})$, $v(t,r,\theta) = 0.8108 + 0.01 \cdot \cos t \cdot \cos r \cdot \cos(\theta + \frac{\pi}{6})$, $t \in [-\tau, 0)$. $(a) : u$, $(b) : v$.

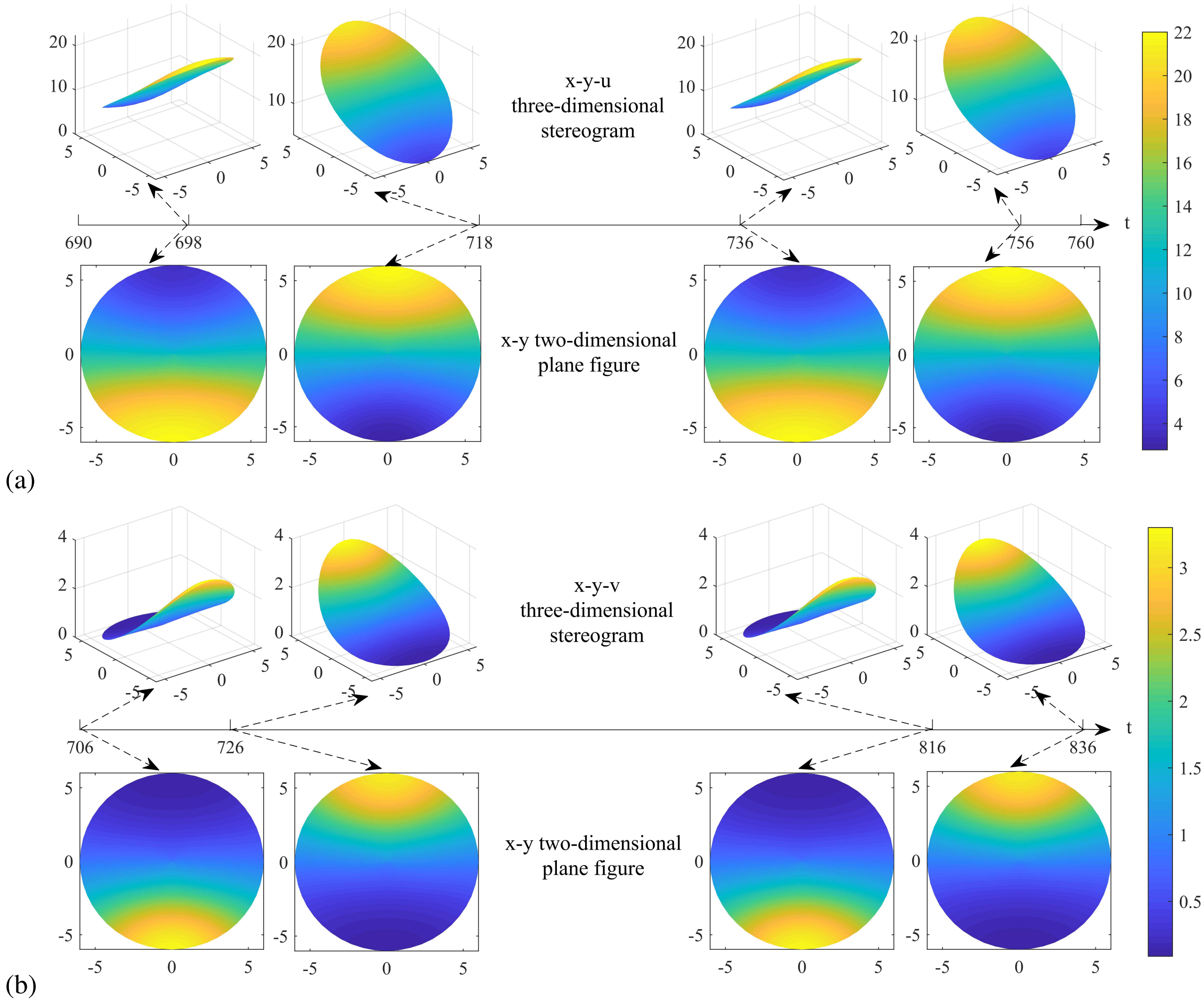
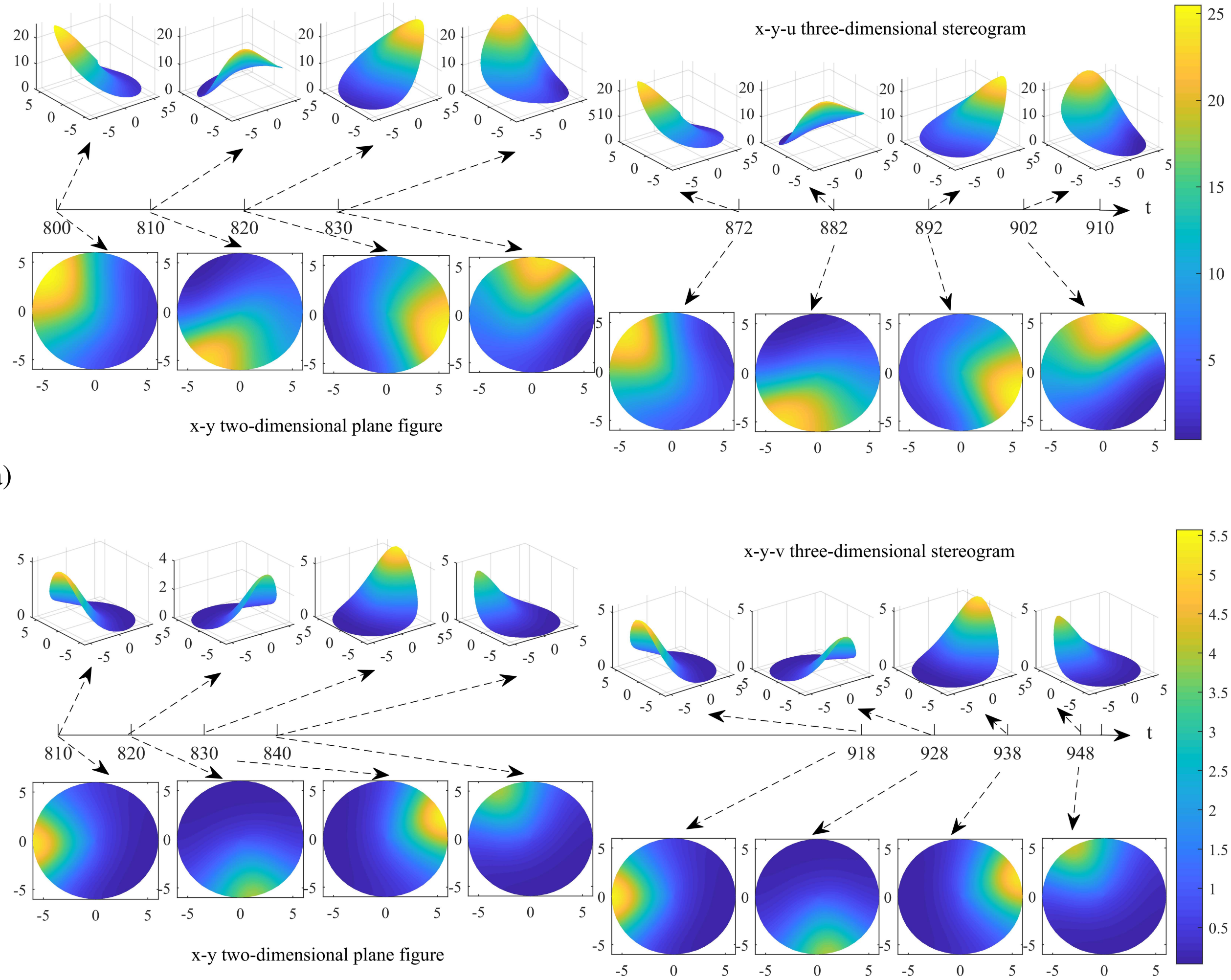


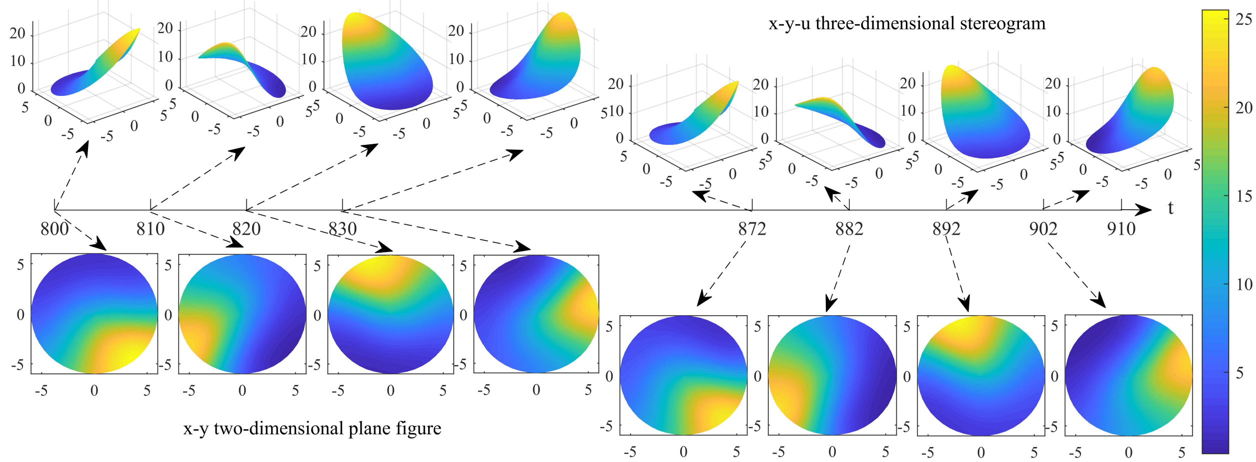
FIG. 4. The system produces standing waves with parameters: $b = 0.25,\ K = 20,\ a = 0.3,\ d = 0.7,\ e = 0.5, \alpha = 0.6,\ d_1 = 0.3,\ d_2 = 0.75,\ R = 6,\ \tau = 3$. Initial values are $u(t,r,\theta) = 13.0320 + 0.01 \cdot \cos t \cdot \cos r \cdot \cos(\theta - \frac{\pi}{2}),\ v(t,r,\theta) = 0.8108 + 0.01 \cdot \cos t \cdot \cos r \cdot \cos(\theta - \frac{\pi}{2}),\ t \in [-\tau, 0).\ (a) : u,\ (b) : v.$



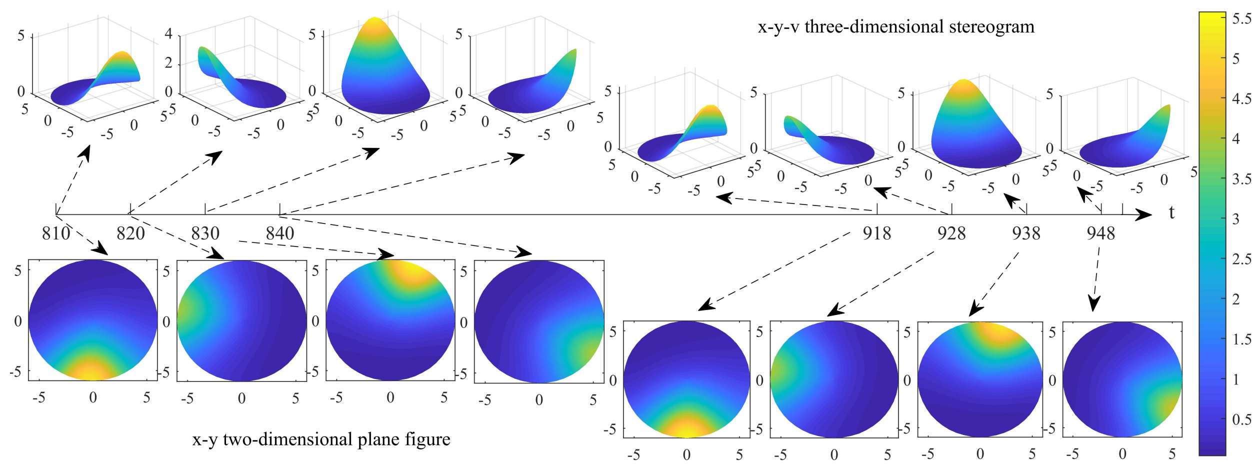
(a)

(b)

FIG. 5. The system produces rotating waves with parameters: $b = 0.25,\ K = 20,\ a = 0.3,\ d = 0.7,\ e = 0.5, \alpha = 0.6,\ d_1 = 0.3,\ d_2 = 0.75,\ R = 6,\ \tau = 3$. Initial values are $u(t,r,\theta) = 13.0320 + 0.01 \cdot \cos t \cdot \cos r \cdot \sin\theta,\ v(t,r,\theta) = 0.8108 + 0.01 \cdot \cos t \cdot \cos r \cdot \cos\theta,\ t \in [-\tau, 0)$. $(a) : u,\ (b) : v$.



(a)



(b)

FIG. 6. The system produces counterpropagating waves with parameters: $b = 0.25,\ K = 20,\ a = 0.3,\ d = 0.7,\ e = 0.5, \alpha = 0.6,\ d_1 = 0.3,\ d_2 = 0.75,\ R = 6,\ \tau = 3$. Initial values are $u(t,r,\theta) = 13.0320 + 0.01 \cdot \cos t \cdot \cos r \cdot \cos\theta,\ v(t,r,\theta) = 0.8108 + 0.01 \cdot \cos t \cdot \cos r \cdot \sin\theta,\ t \in [-\tau, 0)$. $(a) : u,\ (b) : v$.

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Supplementary Materials of "Equivariant Hopf Bifurcation in a Class of Partial Functional Differential Equations on a Circular Domain"

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I. THE SPECIFIC CALCULATION OF THE NORMAL FORM

A. The calculation of $g_2^1(z,0,\mu)$

For $j=2$, similar to the results in [1, 2], we have

$$\mathrm{Im}(M_2^1)^c = \mathrm{span}\left\{\mu z_1 e_1,\ \mu z_3 e_1,\ \mu z_2 e_2,\ \mu z_4 e_2,\ \mu z_3 e_3,\ \mu z_1 e_3,\ \mu z_4 e_4,\ \mu z_2 e_4\right\},$$

then

$$\mathrm{Im}(M_2^1)^c \cap \mathrm{span}\left\{\mu z^p e_k; |p|=1, k=1,2,3,4\right\} = \mathrm{span}\left\{\mu z_1 e_1, \mu z_3 e_1, \mu z_2 e_2, \mu z_4 e_2, \mu z_1 e_3, \mu z_3 e_3, \mu z_2 e_4, \mu z_4 e_4\right\}.$$

Therefore, the second order term of $\tilde{F}(U_t,\mu)$ is

$$\tilde{F}_2(U_t,\mu) = \mu \tilde{D}_1 \Delta U_t(0) + \mu \tilde{L}_1 U_t + F_2(U_t,\mu), \tag{1}$$

and

$$\tilde{F}_2(z,y,\mu) = \tilde{F}(\Phi_{r\theta} z + y,\mu) = \mu \tilde{D}_1 \Delta(\Phi_{r\theta}(0) z + y(0)) + \mu \tilde{L}_1(\Phi_{r\theta} z + y) + F_2(\Phi_{r\theta} z + y,\mu). \tag{2}$$

Since $F(0,\mu)=0,\ D_\varphi F(0,\mu)=0$, $F_2(\Phi_{r\theta} z + y,\mu)$ can be written as follows

$$\begin{aligned} F_2(\Phi_{r\theta} z + y,\mu) =& F_2(\Phi_{r\theta} z + y, 0) \\ =& \sum_{p_1+p_2+p_3+p_4=2} A_{p_1 p_2 p_3 p_4} \left(\hat{\phi}_{nm}^c\right)^{p_1+p_2} \left(\hat{\phi}_{nm}^s\right)^{p_3+p_4} z_1^{p_1} z_2^{p_2} z_3^{p_3} z_4^{p_4} + S_2(\Phi_{r\theta} z, y) + o(|y|^2), \end{aligned} \tag{3}$$

where S_2 represents the linear terms of y, which can be calculated by $DF_2(\Phi_{r\theta} z + y, 0)|_{y=0}(y)$.

By (1)-(3), noticing the fact $\int_0^R \int_0^{2\pi} r \hat{\phi}_{nm}^c \hat{\phi}_{nm}^s \mathrm{d}\theta \mathrm{d}r = 1$, and the relationship of $\Phi_{r\theta}$ and $\Psi_{r\theta}$, we obtain

$$\frac{1}{2} g_2^1(z,0,\mu) = \frac{1}{2} \mathrm{Proj}_{\mathrm{Ker}(M_2^1)} f_2^1(z,0,\mu) = \begin{pmatrix} B_{11} \mu z_1 \\ \overline{B_{11}} \mu z_2 \\ B_{11} \mu z_3 \\ \overline{B_{11}} \mu z_4 \end{pmatrix}, \tag{4}$$

with

$$B_{11} = \frac{1}{2} \overline{\Psi_1(0)} (-\lambda_{nm} \tilde{D}_1 \Phi_1(0) + \tilde{L}_1 \Phi_1). \tag{5}$$

B. The calculation of $g_3^1(z,0,\mu)$

For $j=3$, we have

$$\begin{aligned}\mathrm{Im}(M_3^1)^c = \mathrm{span}\{&z_1^2z_2e_1,\ z_1^2z_2e_3,\ z_1^2z_4e_1,\ z_1^2z_4e_3,\ z_3^2z_2e_1,\ z_3^2z_2e_3,\ z_3^2z_4e_1,\ z_3^2z_4e_3,\ z_2^2z_1e_2,\ z_2^2z_1e_4,\\ &z_2^2z_3e_2,\ z_2^2z_4e_4,\ z_4^2z_1e_2,\ z_4^2z_1e_4,\ z_4^2z_3e_2,\ z_4^2z_3e_4,\ z_1z_2z_3e_1,\ z_1z_2z_3e_3,\\ &z_1z_3z_4e_1,\ z_1z_3z_4e_3,\ z_1z_2z_4e_2,\ z_1z_2z_4e_4,\ z_2z_3z_4e_2,\ z_2z_3z_4e_4\},\end{aligned}$$

see [1, 2] again. Then

$$\mathrm{Im}(M_3^1)^c \cap \mathrm{span}\{\mu z^p e_k; |p|=2, k=1,2,3,4\} = \emptyset.$$

We define

$$\bar{f}_3^1(z,0,\mu) = \frac{3}{2}\left[D_z f_2^1(z,0,\mu)U_2^1(z,\mu) + D_y f_2^1(z,0,\mu)U_2^2(z,\mu) - D_z U_2^1(z,\mu) g_2^1(z,0,\mu)\right] + f_3^1(z,0,\mu). \tag{6}$$

According to [3], the normal form up to the third order is

$$g_3^1(z,0,\mu) = \mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\bar{f}_3^1(z,0,\mu) = \mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\bar{f}_3^1(z,0,0) + o(\mu^2|x|).$$

Since $g_2^1(z,0,0)=0$, we only need to calcalate three parts $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)} f_3^1(z,0,0)$, $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\left(D_z f_2^1(z,0,0)U_2^1(z,0)\right)$ and $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\left(D_y f_2^1(z,0,0)U_2^2(z,0)\right)$.

1. *The calculation of* $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)} f_3^1(z,0,0)$.

Writing $\tilde{F}_3(\Phi_{r\theta}z,\mu)$ as $\tilde{F}_3(\Phi_{r\theta}z,0) = \sum_{p_1+p_2+p_3+p_4=3} A_{p_1p_2p_3p_4}\left(\hat{\phi}_{nm}^c\right)^{p_1+p_2}\left(\hat{\phi}_{nm}^s\right)^{p_3+p_4} z_1^{p_1}z_2^{p_2}z_3^{p_3}z_4^{p_4}$, then we have

$$\begin{aligned} f_3^1(z,0,0) &= \begin{pmatrix} \left\langle \tilde{F}_3(\Phi_{r\theta}z,0), \Psi_{r\theta}^1(0)\right\rangle \\ \left\langle \tilde{F}_3(\Phi_{r\theta}z,0), \Psi_{r\theta}^2(0)\right\rangle \end{pmatrix} \\ &= \overline{\Psi(0)} \begin{pmatrix} \sum_{p_1+p_2+p_3+p_4=3} A_{p_1p_2p_3p_4}\int_0^R\int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^{p_1+p_2}\left(\hat{\phi}_{nm}^s\right)^{p_3+p_4+1}\mathrm{d}\theta\mathrm{d}r z_1^{p_1}z_2^{p_2}z_3^{p_3}z_4^{p_4} \\ \sum_{p_1+p_2+p_3+p_4=3} A_{p_1p_2p_3p_4}\int_0^R\int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^{p_1+p_2+1}\left(\hat{\phi}_{nm}^s\right)^{p_3+p_4}\mathrm{d}\theta\mathrm{d}r z_1^{p_1}z_2^{p_2}z_3^{p_3}z_4^{p_4} \end{pmatrix}. \end{aligned}$$

Noticing the fact

$$\int_0^R\int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^4\mathrm{d}\theta\mathrm{d}r = \int_0^R\int_0^{2\pi} r\left(\hat{\phi}_{nm}^s\right)^4\mathrm{d}\theta\mathrm{d}r = 0,$$

$$\int_0^R\int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^3\hat{\phi}_{nm}^s\mathrm{d}\theta\mathrm{d}r = \int_0^R\int_0^{2\pi} r\hat{\phi}_{nm}^c\left(\hat{\phi}_{nm}^s\right)^3\mathrm{d}\theta\mathrm{d}r = 0,$$

$$\int_0^R \int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^2 \left(\hat{\phi}_{nm}^s\right)^2 \mathrm{d}\theta\mathrm{d}r \triangleq \mathrm{M}_{22},$$

and the relationship of $\Phi_{r\theta}$ and $\Psi_{r\theta}$, we get

$$\frac{1}{3!}\mathrm{Proj}_{\mathrm{Ker}(M_3^1)} f_3^1(z,0,0) = \begin{pmatrix} C_{2001}z_1^2z_4 + C_{1110}z_1z_2z_3 \\ \overline{C_{2001}}z_2^2z_3 + \overline{C_{1110}}z_1z_2z_4 \\ C_{2001}z_3^2z_2 + C_{1110}z_1z_3z_4 \\ \overline{C_{2001}}z_4^2z_1 + \overline{C_{1110}}z_2z_3z_4 \end{pmatrix}, \tag{7}$$

where

$$C_{2001} = \frac{1}{6}\overline{\Psi_1(0)}A_{2001}\mathrm{M}_{22},\ C_{1110} = \frac{1}{6}\overline{\Psi_1(0)}A_{1110}\mathrm{M}_{22}, \tag{8}$$

2. *The calculation of* $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\left(D_z f_2^1(z,0,0)U_2^1(z,0)\right)$.

We have

$$\begin{aligned} f_2^1(z,0,0) &= \begin{pmatrix} \left\langle \tilde{F}_2(\Phi_{r\theta}z,0), \Psi_{r\theta}^1(0)\right\rangle \\ \left\langle \tilde{F}_2(\Phi_{r\theta}z,0), \Psi_{r\theta}^2(0)\right\rangle \end{pmatrix} \\ &= \overline{\Psi(0)}\begin{pmatrix} \sum_{p_1+p_2+p_3+p_4=2} A_{p_1p_2p_3p_4}\int_0^R\int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^{p_1+p_2}\left(\hat{\phi}_{nm}^s\right)^{p_3+p_4+1}\mathrm{d}\theta\mathrm{d}r z_1^{p_1}z_2^{p_2}z_3^{p_3}z_4^{p_4} \\ \sum_{p_1+p_2+p_3+p_4=2} A_{p_1p_2p_3p_4}\int_0^R\int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^{p_1+p_2+1}\left(\hat{\phi}_{nm}^s\right)^{p_3+p_4}\mathrm{d}\theta\mathrm{d}r z_1^{p_1}z_2^{p_2}z_3^{p_3}z_4^{p_4} \end{pmatrix}. \end{aligned} \tag{9}$$

Noticing the fact that

$$\int_0^R \int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^3 \mathrm{d}\theta\mathrm{d}r = \int_0^R \int_0^{2\pi} r\left(\hat{\phi}_{nm}^s\right)^3 \mathrm{d}\theta\mathrm{d}r = 0,$$

$$\int_0^R \int_0^{2\pi} r\left(\hat{\phi}_{nm}^c\right)^2 \hat{\phi}_{nm}^s \mathrm{d}\theta\mathrm{d}r = \int_0^R \int_0^{2\pi} r\hat{\phi}_{nm}^c \left(\hat{\phi}_{nm}^s\right)^2 \mathrm{d}\theta\mathrm{d}r = 0,$$

then we get $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\left(D_z f_2^1(z,0,0)U_2^1(z,0)\right) = 0$.

3. *The calculation of* $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\left(D_y f_2^1(z,0,0)U_2^2(z,0)\right)$.

Firstly, we calculate the Fréchet derivative $D_y f_2^1(z,0,0) : Q_s \to \mathscr{X}_{\mathbb{C}}$. By (2) and (3), $\tilde{F}_2(z,y,0)$ can be written as

$$\begin{aligned} \tilde{F}_2(z,y,0) &= S_2(\Phi_{r\theta}z,y) + o(z^2,y^2) \\ &= S_{yz_1}(y)z_1\hat{\phi}_{nm}^c + S_{yz_2}(y)z_2\hat{\phi}_{nm}^c + S_{yz_3}(y)z_3\hat{\phi}_{nm}^s + S_{yz_4}(y)z_4\hat{\phi}_{nm}^s + o(z^2,y^2), \end{aligned} \tag{10}$$

where $S_{yz_k}(k=1,2,3,4):Q_s\to\mathscr{X}_{\mathbb{C}}$ are linear operators, and $S_{yz_k}(\varphi)=S_{y(0)z_k}(\varphi(0))+S_{y(-1)z_k}(\varphi(-1))$.

Let $U_2^2(z,0)\triangleq h(z)=\sum_{j\geq 0}h_j(z)\hat{\phi}_{jk}(r,\theta)$, with

$$h_{jk}(z)=\begin{pmatrix} h_{jk}^{(1)}(z)\\ h_{jk}^{(2)}(z)\end{pmatrix}=\sum_{p_1+p_2+p_3+p_4=2}\begin{pmatrix} h_{jkp_1p_2p_3p_4}^{(1)}(z)\\ h_{jkp_1p_2p_3p_4}^{(2)}(z)\end{pmatrix}z_1^{p_1}z_2^{p_2}z_3^{p_3}z_4^{p_4}.$$

Therefore,

$$\begin{aligned}D_y\tilde{F}_2(z,0,0)\left(U_2^2(z,0)\right)&=\begin{pmatrix}\langle D_y\tilde{F}_2(z,0,0)\left(U_2^2(z,0)\right),\Psi_{r\theta}^1(0)\rangle\\ \langle D_y\tilde{F}_2(z,0,0)\left(U_2^2(z,0)\right),\Psi_{r\theta}^2(0)\rangle\end{pmatrix}\\ &=\overline{\Psi(0)}\begin{pmatrix}\sum_{j\geq 0}\left[\mathrm{M}_{jkcs}S_{yz_1}(h_{jk})z_1+\mathrm{M}_{jkcs}S_{yz_2}(h_{jk})z_2\right.\\ \left.+\mathrm{M}_{jkss}S_{yz_3}(h_{jk})z_3+\mathrm{M}_{jkss}S_{yz_4}(h_{jk})z_4\right]\\ \sum_{j\geq 0}[\mathrm{M}_{jkcc}S_{yz_1}(h_{jk})z_1+\mathrm{M}_{jkcc}S_{yz_2}(h_{jk})z_2\\ +\mathrm{M}_{jksc}S_{yz_3}(h_{jk})z_3+\mathrm{M}_{jksc}S_{yz_4}(h_{jk})z_4]\end{pmatrix},\end{aligned}$$

where

$$\mathrm{M}_{jksc}=\mathrm{M}_{jkcs}=\int_0^R\int_0^{2\pi}r\hat{\phi}_{jk}\hat{\phi}_{nm}^c\hat{\phi}_{nm}^s\mathrm{d}\theta\mathrm{d}r=\begin{cases}\mathrm{M}_{0kcs}^c & \hat{\phi}_0 \ or\ \hat{\phi}_{0k}^c,\\ 0, & otherwise,\end{cases}$$

$$\mathrm{M}_{jkss}=\int_0^R\int_0^{2\pi}r\hat{\phi}_{jk}\hat{\phi}_{nm}^s\hat{\phi}_{nm}^s\mathrm{d}\theta\mathrm{d}r=\begin{cases}\mathrm{M}_{2nkss}^c, & \hat{\phi}_{jk}=\hat{\phi}_{2nk}^c,\\ 0, & otherwise,\end{cases}$$

$$\mathrm{M}_{jkcc}=\int_0^R\int_0^{2\pi}r\hat{\phi}_{jk}\hat{\phi}_{nm}^c\hat{\phi}_{nm}^c\mathrm{d}\theta\mathrm{d}r=\begin{cases}\mathrm{M}_{2nkcc}^s, & \hat{\phi}_{jk}=\hat{\phi}_{2nk}^s,\\ 0, & otherwise.\end{cases}$$

Moreover, we have $D_y\tilde{F}_2(z,0,0)\left(U_2^2(z,0)\right)=\overline{\Psi(0)}\begin{pmatrix}N_1\\ N_2\end{pmatrix}$, with

$$\begin{aligned}N_1=&\mathrm{M}_{0kcs}^c\left(S_{yz_1}(h_{0k}^{ccs})z_1+S_{yz_2}(h_{0k}^{ccs})z_2\right)+\mathrm{M}_{2nkss}^c\left(S_{yz_3}(h_{2nk}^{css})z_3+S_{yz_4}(h_{2nk}^{css})z_4\right),\\ N_2=&\mathrm{M}_{2nkcc}^s\left(S_{yz_1}(h_{2nk}^{ccs})z_1+S_{yz_2}(h_{2nk}^{ccs})z_2\right)+\mathrm{M}_{0kcs}^c\left(S_{yz_3}(h_{0k}^{ccs})z_3+S_{yz_4}(h_{0k}^{ccs})z_4\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{1}{3!}\mathrm{Proj}_{\mathrm{Ker}(\mathrm{M}_3^1)}\left(D_yf_2^1(z,0,0)U_2^2(z,0)\right)\\ =&\begin{pmatrix}E_{2100}^1z_1^2z_2+E_{2001}^1z_1^2z_4+E_{0120}^1z_3^2z_2+E_{0021}^1z_3^2z_4+E_{1110}^1z_1z_2z_3+E_{1011}^1z_1z_3z_4\\ \overline{E_{2100}^1}z_1z_2^2+\overline{E_{2001}^1}z_2^2z_3+\overline{E_{0120}^1}z_4^2z_1+\overline{E_{0021}^1}z_3z_4^2+\overline{E_{1110}^1}z_1z_2z_4+\overline{E_{1011}^1}z_2z_3z_4\\ E_{2100}^2z_1^2z_2+E_{2001}^2z_1^2z_4+E_{0120}^2z_3^2z_2+E_{0021}^2z_3^2z_4+E_{1110}^2z_1z_2z_3+E_{1011}^2z_1z_3z_4\\ \overline{E_{2100}^2}z_1z_2^2+\overline{E_{2001}^2}z_2^2z_3+\overline{E_{0120}^2}z_4^2z_1+\overline{E_{0021}^2}z_3z_4^2+\overline{E_{1110}^2}z_1z_2z_4+\overline{E_{1011}^2}z_2z_3z_4\end{pmatrix},\end{aligned}\tag{11}$$

where

$$
\begin{aligned}
E^1_{2100} &= \frac{1}{6}\overline{\Psi_1(0)}\left[\mathrm{M}^c_{0kcs}\left(S_{yz_1}(h^{ccs}_{0k1100})+S_{yz_2}(h^{ccs}_{0k2000})\right)\right],\\
E^1_{2001} &= \frac{1}{6}\overline{\Psi_1(0)}\left[\mathrm{M}^c_{0kcs}S_{yz_1}(h^{ccs}_{0k1001})+\mathrm{M}^c_{2nkss}S_{yz_4}(h^{css}_{2nk2000})\right],\\
E^1_{0120} &= \frac{1}{6}\overline{\Psi_1(0)}\left[\mathrm{M}^c_{0kcs}S_{yz_2}(h^{ccs}_{0k0020}))+\mathrm{M}^c_{2nkss}S_{yz_3}(h^{css}_{2nk0110})\right],\\
E^1_{0021} &= \frac{1}{6}\overline{\Psi_1(0)}\left[\mathrm{M}^c_{2nkss}\left(S_{yz_3}(h^{css}_{2nk0011})+S_{yz_4}(h^{css}_{2nk0020})\right)\right],\\
E^1_{1110} &= \frac{1}{6}\overline{\Psi_1(0)}\left[\mathrm{M}^c_{0kcs}\left(S_{yz_1}(h^{ccs}_{0k0110})+S_{yz_2}(h^{ccs}_{0k1010})\right)+\mathrm{M}^c_{2nkss}S_{yz_3}(h^{css}_{2nk1100})\right],\\
E^1_{1011} &= \frac{1}{6}\overline{\Psi_1(0)}\left[\mathrm{M}^c_{0kcs}S_{yz_1}(h^{ccs}_{0k0011})+\mathrm{M}^c_{2nkss}\left(S_{yz_3}(h^{css}_{2nk1001})+S_{yz_4}(h^{css}_{2nk1010})\right)\right],\\
E^2_{2100} &= \frac{1}{6}\overline{\Psi_3(0)}\left[\mathrm{M}^s_{2nkcc}\left(S_{yz_1}(h^{ccs}_{2nk1100})+S_{yz_2}(h^{ccs}_{2nk2000})\right)\right],\\
E^2_{2001} &= \frac{1}{6}\overline{\Psi_3(0)}\left[\mathrm{M}^s_{2nkcc}S_{yz_1}(h^{ccs}_{2nk1001})+\mathrm{M}^c_{0kcs}S_{yz_2}(h^{ccs}_{0k2000})\right],\\
E^2_{0120} &= \frac{1}{6}\overline{\Psi_3(0)}\left[\mathrm{M}^s_{2nkcc}S_{yz_2}(h^{ccs}_{2nk0020})+\mathrm{M}^c_{0kcs}S_{yz_3}(h^{ccs}_{0k0110})\right],\\
E^2_{0021} &= \frac{1}{6}\overline{\Psi_3(0)}\left[\mathrm{M}^c_{0kcs}\left(S_{yz_3}(h^{ccs}_{0k0011})+S_{yz_4}(h^{ccs}_{0k0020})\right)\right],\\
E^2_{1110} &= \frac{1}{6}\overline{\Psi_3(0)}\left[\mathrm{M}^s_{2nkcc}\left(S_{yz_1}(h^{ccs}_{2nk0110})+S_{yz_2}(h^{ccs}_{2nk1010})\right)+\mathrm{M}^c_{0kcs}S_{yz_3}(h^{ccs}_{0k1100})\right]\\
E^2_{1011} &= \frac{1}{6}\overline{\Psi_3(0)}\left[\mathrm{M}^s_{2nkcc}S_{yz_1}(h^{ccs}_{2nk0011})+\mathrm{M}^c_{0kcs}\left(S_{yz_3}(h^{ccs}_{0k1001})+S_{yz_4}(h^{ccs}_{0k1010})\right)\right].
\end{aligned}
$$

Now, we need to calculate

$$
\begin{aligned}
&h^{ccs}_{0k2000},\ h^{ccs}_{0k1100},\ h^{ccs}_{0k1010},\ h^{ccs}_{0k1001},\ h^{ccs}_{0k0110},\ h^{ccs}_{0k0020},\ h^{ccs}_{0k0011},\\
&h^{ccs}_{2nk2000},\ h^{ccs}_{2nk1100},\ h^{ccs}_{2nk1010},\ h^{ccs}_{2nk1001},\ h^{ccs}_{2nk0110},\ h^{ccs}_{2nk0020},\ h^{ccs}_{2nk0011},\\
&h^{css}_{2nk2000},\ h^{css}_{2nk1100},\ h^{css}_{2nk1010},\ h^{css}_{2nk1001},\ h^{css}_{2nk0110},\ h^{css}_{2nk0020},\ h^{css}_{2nk0011}.
\end{aligned}
$$

From (20) and(23), we get

$$
\begin{aligned}
M_2^2U_2^2(z,0)(\vartheta) =& M_2^2h(z)(\vartheta)\\
=& \begin{cases} D_zh(z)Bz-\tilde{D}_0\Delta h(0)-\tilde{L}_0(h(z)), & \vartheta=0,\\ D_zh(z)Bz-D_\vartheta h(z), & \vartheta\neq 0,\end{cases}\\
=& \begin{cases} \sum_{j\geq 0}\left[D_zh_j(z)\hat{\phi}_{jk}(r,\theta)Bz-\tilde{D}_0\Delta h_j(z)\hat{\phi}_{jk}(r,\theta)-\tilde{L}_0(h_j(z)\hat{\phi}_{jk}(r,\theta))\right], & \vartheta=0,\\ \sum_{j\geq 0}\left[D_zh_j(z)\hat{\phi}_{jk}(r,\theta)Bz-D_\vartheta h_j(z)\hat{\phi}_{jk}(r,\theta)\right], & \vartheta\neq 0,\end{cases}
\end{aligned}
$$

and

$$f_2^2(z,0,0)=\begin{cases}\tilde{F}_2(z,0,0)-\Phi_1(0)f_2^{1(1)}(z,0,0)\hat{\phi}^s_{nm}-\Phi_2(0)f_2^{1(2)}(z,0,0)\hat{\phi}^s_{nm}\\ \qquad -\Phi_3(0)f_2^{1(1)}(z,0,0)\hat{\phi}^c_{nm}-\Phi_4(0)f_2^{1(1)}(z,0,0)\hat{\phi}^c_{nm}, & \vartheta=0,\\ -\Phi_1(\vartheta)f_2^{1(1)}(z,0,0)\hat{\phi}^s_{nm}-\Phi_2(\vartheta)f_2^{1(2)}(z,0,0)\hat{\phi}^s_{nm}\\ \qquad -\Phi_3(\vartheta)f_2^{1(1)}(z,0,0)\hat{\phi}^c_{nm}-\Phi_4(\vartheta)f_2^{1(1)}(z,0,0)\hat{\phi}^c_{nm}, & \vartheta\neq 0.\end{cases}$$

Besides, we have

$$\langle M_2^2\left(U_2^2(z,0)\right),\beta_{jk}\rangle=\langle f_2^2(z,0,0),\beta_{jk}\rangle, \tag{12}$$

with $\beta_{jk}=\frac{\phi_{jk}}{\|\phi_{jk}\|}$.

Thus, the expressions of $h_{jp_1p_2p_3p_4}$ can be obtained. Due to the large number of expressions, we show the specific results in the Section VI.

Noting the fact that $\mathrm{M}^s_{2nkcc}=\mathrm{M}^c_{2nkss}$, therefore, we have

$$E^1_{2100}=E^2_{0021},\ E^1_{2001}=E^2_{0120},\ E^1_{0120}=E^2_{2001},\ E^1_{0021}=E^2_{2100},\ E^1_{1110}=E^2_{1011},\ E^1_{1011}=E^2_{1110},$$

For simplification of notations, we rewrite (11) as

$$\begin{aligned}&\frac{1}{3!}\mathrm{Proj}_{\mathrm{Ker}(\mathrm{M}_3^1)}\left(D_yf_2^1(z,0,0)U_2^2(z,0)\right)\\ =&\begin{pmatrix}E_{2100}z_1^2z_2+E_{2001}z_1^2z_4+E_{0120}z_3^2z_2+E_{0021}z_3^2z_4+E_{1110}z_1z_2z_3+E_{1011}z_1z_3z_4\\ \overline{E_{2100}}z_1z_2^2+\overline{E_{2001}}z_2^2z_3+\overline{E_{0120}}z_4^2z_1+\overline{E_{0021}}z_4^2z_3+\overline{E_{1110}}z_1z_2z_4+\overline{E_{1011}}z_2z_3z_4\\ E_{2100}z_3^2z_4+E_{2001}z_3^2z_2+E_{0120}z_1^2z_4+E_{0021}z_1^2z_2+E_{1110}z_1z_3z_4+E_{1011}z_1z_2z_3\\ \overline{E_{2100}}z_3z_4^2+\overline{E_{2001}}z_4^2z_1+\overline{E_{0120}}z_2^2z_3+\overline{E_{0021}}z_1^2z_2+\overline{E_{1110}}z_2z_3z_4+\overline{E_{1011}}z_1z_2z_4\end{pmatrix}.\end{aligned} \tag{13}$$

Hence, we have

$$\begin{aligned}\frac{1}{3!}g_3^1(z,0,0)=&\frac{1}{3!}\mathrm{Proj}_{\mathrm{Ker}(\mathrm{M}_3^1)}\bar{f}_3^1(z,0,0)\\ =&\begin{pmatrix}B_{2100}z_1^2z_2+B_{2001}z_1^2z_4+B_{0120}z_3^2z_2+B_{0021}z_3^2z_4+B_{1110}z_1z_2z_3+B_{1011}z_1z_3z_4\\ \overline{B_{2100}}z_1z_2^2+\overline{B_{2001}}z_2^2z_3+\overline{B_{0120}}z_4^2z_1+\overline{B_{0021}}z_4^2z_3+\overline{B_{1110}}z_1z_2z_4+\overline{B_{1011}}z_2z_3z_4\\ B_{2100}z_3^2z_4+B_{2001}z_3^2z_2+B_{0120}z_1^2z_4+B_{0021}z_1^2z_2+B_{1110}z_1z_3z_4+B_{1011}z_1z_2z_3\\ \overline{B_{2100}}z_3z_4^2+\overline{B_{2001}}z_4^2z_1+\overline{B_{0120}}z_2^2z_3+\overline{B_{0021}}z_1^2z_2+\overline{B_{1110}}z_2z_3z_4+\overline{B_{1011}}z_1z_2z_4\end{pmatrix},\end{aligned} \tag{14}$$

with $B_{p_1p_2p_3p_4}=C_{p_1p_2p_3p_4}+\frac{3}{2}(D_{p_1p_2p_3p_4}+E_{p_1p_2p_3p_4})$.

II. PROOF OF LEMMA 2

By a smooth transformation

$$\begin{aligned} z_1 &= \zeta_1 + b_1\zeta_1^2\zeta_2 + b_2\zeta_3^2\zeta_2 + b_3\zeta_3^2\zeta_4 + b_4\zeta_1\zeta_3\zeta_4, \\ z_3 &= \zeta_3 + b_1\zeta_3^2\zeta_4 + b_2\zeta_1^2\zeta_4 + b_3\zeta_1^2\zeta_2 + b_4\zeta_1\zeta_2\zeta_3, \end{aligned} \tag{15}$$

with $z_2 = \bar{z}_3,\ z_4 = \bar{z}_1$, we have

$$\begin{aligned} \zeta_1 &= z_1 - b_1 z_1^2 z_2 - b_2 z_3^2 z_2 - b_3 z_3^2 z_4 - b_4 z_1 z_3 z_4 + o(4), \\ \zeta_3 &= z_3 - b_1 z_3^2 z_4 - b_2 z_1^2 z_4 - b_3 z_1^2 z_2 - b_4 z_1 z_2 z_3 + o(4). \end{aligned} \tag{16}$$

Then

$$\begin{aligned} \dot{\zeta}_1 =& \dot{z}_1 - 2b_1 z_1 z_2 \dot{z}_1 - b_1 z_1^2 \dot{z}_2 - 2b_2 z_2 z_3 \dot{z}_3 - b_2 z_3^2 \dot{z}_2 - 2b_3 z_3 z_4 \dot{z}_3 - b_3 z_3^2 \dot{z}_4 \\ &- b_4 z_1 z_3 \dot{z}_4 - b_4 z_1 z_4 \dot{z}_3 - b_4 z_3 z_4 \dot{z}_1 + o(4) \\ =& (\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) z_1 + B_{2100} z_1^2 z_2 + B_{1011} z_1 z_3 z_4 + B_{2001} z_1^2 z_4 + B_{0120} z_3^2 z_2 + B_{0021} z_3^2 z_4 \\ &+ B_{1110} z_1 z_2 z_3 - b_1(2(\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) + (-\mathrm{i}\omega_{\hat{\lambda}} + \overline{B_{11}}\mu)) z_1^2 z_2 \\ &- b_2(2(\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) + (-\mathrm{i}\omega_{\hat{\lambda}} + \overline{B_{11}}\mu)) z_3^2 z_2 - b_3(2(\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) + (-\mathrm{i}\omega_{\hat{\lambda}} + \overline{B_{11}}\mu)) z_3^2 z_4 \\ &- b_4(2(\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) + (-\mathrm{i}\omega_{\hat{\lambda}} + \overline{B_{11}}\mu)) z_1 z_3 z_4 + o(4). \end{aligned}$$

Let

$$\begin{aligned} b_1 &= \frac{B_{2100}}{2(\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) + (-\mathrm{i}\omega_{\hat{\lambda}} + \overline{B_{11}}\mu)},\ b_2 = \frac{B_{0120}}{2(\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) + (-\mathrm{i}\omega_{\hat{\lambda}} + \overline{B_{11}}\mu)}, \\ b_3 &= \frac{B_{0021}}{2(\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) + (-\mathrm{i}\omega_{\hat{\lambda}} + \overline{B_{11}}\mu)},\ b_4 = \frac{B_{1011}}{2(\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) + (-\mathrm{i}\omega_{\hat{\lambda}} + \overline{B_{11}}\mu)}, \end{aligned}$$

then

$$\begin{aligned} \dot{\zeta}_1 =& (\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) z_1 + B_{2001} z_1^2 z_4 + B_{1110} z_1 z_2 z_3 + o(4) \\ =& (\mathrm{i}\omega_{\hat{\lambda}} + B_{11}\mu) \zeta_1 + B_{2001} \zeta_1^2 \zeta_4 + B_{1110} \zeta_1 \zeta_2 \zeta_3 + o(4). \end{aligned}$$

The same is true for $\dot{\zeta}_2, \dot{\zeta}_3$ and $\dot{\zeta}_4$ so that (27) is established.

III. PROOF OF THEOREM 2

We only need to prove the approximate expressions of rotating and standing wave solutions reduced to the center subspace, and the rest of the theorem can be easily obtained from previous analysis.

By (16), (18),(28), we get

$$
\begin{aligned}
U_t(\vartheta)(r,\theta) \approx& \xi e^{i\omega_{\hat\lambda}\vartheta} J_n(\sqrt{\lambda_{nm}}r) e^{in\theta}\rho_1 e^{i\chi_1(t)} + \bar\xi e^{-i\omega_{\hat\lambda}\vartheta} J_n(\sqrt{\lambda_{nm}}r) e^{in\theta}\rho_2 e^{-i\chi_2(t)} \\
&+ \xi e^{i\omega_{\hat\lambda}\vartheta} J_n(\sqrt{\lambda_{nm}}r) e^{-in\theta}\rho_2 e^{i\chi_2(t)} + \bar\xi e^{-i\omega_{\hat\lambda}\vartheta} J_n(\sqrt{\lambda_{nm}}r) e^{-in\theta}\rho_1 e^{-i\chi_1(t)}
\end{aligned}
$$

with $\xi = (p_{11}, p_{12}, \cdots, p_{1n})^{\mathrm{T}}$. For simplicity, we also rewrite p_{1i} in the form of a complex angle as $p_{1i} = |p_{1i}| e^{i\mathrm{Arg}(p_{1i})}$ in the subsequent calculations.

For $(\rho_1, \rho_2) = (0, \sqrt{\frac{-a_1\mu}{a_2}})$,

$$
\begin{aligned}
U_t(\vartheta)(r,\theta) \approx& \xi e^{i\omega_{\hat\lambda}\vartheta} J_n(\sqrt{\lambda_{nm}}r) e^{in\theta}\rho_1 e^{i\chi_1(t)} + \bar\xi e^{-i\omega_{\hat\lambda}\vartheta} J_n(\sqrt{\lambda_{nm}}r) e^{-in\theta}\rho_1 e^{-i\chi_1(t)} \\
\approx& \sum_{i=1}^{n} 2|p_{1i}| \sqrt{\frac{-a_1\mu}{a_2}} J_n(\sqrt{\lambda_{nm}}r) \cos(\mathrm{Arg}(p_{1i}) + \omega_{\hat\lambda}\vartheta + \omega_{\hat\lambda}t + n\theta) e_i.
\end{aligned}
$$

where e_i is the ith unit coordinate vector of $\mathbb{R}^n$. This corresponds to the form of a rotating wave solution in the plane of (z_2, z_3).

For $(\rho_1, \rho_2) = (\sqrt{\frac{-a_1\mu}{a_2}}, 0)$,

$$
\begin{aligned}
U_t(\vartheta)(r,\theta) \approx& \bar\xi e^{-i\omega_{\hat\lambda}\vartheta} J_n(\sqrt{\lambda_{nm}}r) e^{in\theta}\rho_2 e^{-i\chi_2(t)} + \xi e^{i\omega_{\hat\lambda}\vartheta} J_n(\sqrt{\lambda_{nm}}r) e^{-in\theta}\rho_2 e^{i\chi_2(t)} \\
\approx& \sum_{i=1}^{n} 2|p_{1i}| \sqrt{\frac{-a_1\mu}{a_2}} J_n(\sqrt{\lambda_{nm}}r) \cos(\mathrm{Arg}(p_{1i}) + \omega_{\hat\lambda}\vartheta + \omega_{\hat\lambda}t - n\theta) e_i,
\end{aligned}
$$

which corresponds to the form of a rotating wave solution in the opposite direction in the plane of (z_1, z_4).

For $(\rho_1, \rho_2) = (\sqrt{\frac{-a_1\mu}{a_2+a_3}}, \sqrt{\frac{-a_1\mu}{a_2+a_3}})$,

$$
\begin{aligned}
U_t(\vartheta)(r,\theta) \approx& \sum_{i=1}^{n} 2|p_{1i}| \sqrt{\frac{-a_1\mu}{a_2+a_3}} J_n(\sqrt{\lambda_{nm}}r) \cos(\mathrm{Arg}(p_{1i}) + \omega_{\hat\lambda}\vartheta + \omega_{\hat\lambda}t + n\theta) e_i \\
&+ \sum_{i=1}^{n} 2|p_{1i}| \sqrt{\frac{-a_1\mu}{a_2+a_3}} J_n(\sqrt{\lambda_{nm}}r) \cos(\mathrm{Arg}(p_{1i}) + \omega_{\hat\lambda}\vartheta + \omega_{\hat\lambda}t - n\theta) e_i \\
\approx& \sum_{i=1}^{n} 4|p_{1i}| \sqrt{\frac{-a_1\mu}{a_2+a_3}} J_n(\sqrt{\lambda_{nm}}r) \cos(\mathrm{Arg}(p_{1i}) + \omega_{\hat\lambda}\vartheta + \omega_{\hat\lambda}t) \cos(n\theta) e_i,
\end{aligned}
$$

which means when $n\theta = \frac{\pi}{2}$ or $n\theta = \frac{3\pi}{2}$, the form of the solution does not change over time. In other words, in a two-dimensional plane, the image of the solution has a fixed axis, thus, it corresponds to the form of a standing wave solution.

IV. THE CALCULATION FORMULA FOR $A_{p_1p_2p_3p_4}$

A. The calculation formula for $A_{p_1p_2p_3p_4}(p_1+p_2+p_3+p_4=2)$

$$
\begin{aligned}
&A_{2000}=2\hat{\tau}\begin{pmatrix}A^1_{2000}\\ A^2_{2000}\end{pmatrix},\ A_{1100}=2\hat{\tau}\begin{pmatrix}A^1_{1100}\\ A^2_{1100}\end{pmatrix},A_{1010}=2\hat{\tau}\begin{pmatrix}A^1_{1010}\\ A^2_{1010}\end{pmatrix}\\
&A_{1001}=2\hat{\tau}\begin{pmatrix}A^1_{1001}\\ A^2_{1001}\end{pmatrix},\ A_{0020}=2\hat{\tau}\begin{pmatrix}A^1_{0020}\\ A^2_{0020}\end{pmatrix}A_{0011}=2\hat{\tau}\begin{pmatrix}A^1_{0011}\\ A^2_{0011}\end{pmatrix},\\
&A_{0200}=\overline{A_{2000}},\ A_{0101}=\overline{A_{1010}},\ A_{0110}=\overline{A_{1001}},\ A_{0002}=\overline{A_{0020}},
\end{aligned}
$$

with

$$
\begin{aligned}
A^1_{2000}=A^1_{0020}=&F^{(1)}_{20}+F^{(1)}_{11}p_0+F^{(1)}_{02}p_0^2,\\
A^1_{1100}=A^1_{1001}=A^1_{0011}=&2F^{(1)}_{20}+F^{(1)}_{11}(p_0+\bar{p}_0)+2F^{(1)}_{02}p_0\bar{p}_0,\\
A^1_{1010}=&2F^{(1)}_{20}+2F^{(1)}_{11}p_0+2F^{(1)}_{02}p_0^2,\\
A^2_{2000}=A^2_{0020}=&F^{(2)}_{2000}+F^{(2)}_{1100}p_0+F^{(2)}_{0200}p_0^2+F^{(2)}_{0020}\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F^{(2)}_{0011}p_0\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F^{(2)}_{0002}p_0^2\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}\\
&+F^{(2)}_{1010}\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+(F^{(2)}_{1001}+F^{(2)}_{0110})p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F^{(2)}_{0101}p_0^2\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}},\\
A^2_{1100}=A^2_{1001}=A^2_{0011}=&2F^{(2)}_{2000}+F^{(2)}_{1100}(p_0+\bar{p}_0)+2F^{(1)}_{0200}p_0\bar{p}_0+2F^{(2)}_{0020}+F^{(2)}_{0011}(p_0+\bar{p}_0)+2F^{(2)}_{0002}p_0\bar{p}_0\\
&+F^{(2)}_{1010}(\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}})+F^{(2)}_{1001}(p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}})+F^{(2)}_{0110}(p_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+\bar{p}_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}})\\
&+F^{(2)}_{0101}(p_0\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+p_0\bar{p}_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}),\\
A^2_{1010}=&2F^{(2)}_{2000}+2F^{(2)}_{1100}p_0+2F^{(1)}_{0200}p_0^2+2F^{(2)}_{0020}\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+2F^{(2)}_{0011}p_0\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+2F^{(2)}_{0002}p_0^2\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}\\
&+2F^{(2)}_{1010}\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+2(F^{(2)}_{1001}+F^{(2)}_{0110})p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+2F^{(2)}_{0101}p_0^2\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}.
\end{aligned}
$$

B. The calculation formula for $A_{p_1p_2p_3p_4}(p_1+p_2+p_3+p_4=3)$

$$
\begin{aligned}
&A_{2100}=6\hat{\tau}\begin{pmatrix}A^1_{2100}\\ A^2_{2100}\end{pmatrix}A_{2010}=6\hat{\tau}\begin{pmatrix}A^1_{2010}\\ A^2_{2010}\end{pmatrix},A_{2001}=6\hat{\tau}\begin{pmatrix}A^1_{2001}\\ A^2_{2001}\end{pmatrix}A_{1020}=6\hat{\tau}\begin{pmatrix}A^1_{1020}\\ A^2_{1020}\end{pmatrix},\\
&A_{0120}=6\hat{\tau}\begin{pmatrix}A^1_{0120}\\ A^2_{0120}\end{pmatrix}A_{0021}=6\hat{\tau}\begin{pmatrix}A^1_{0021}\\ A^2_{0021}\end{pmatrix},A_{1110}=6\hat{\tau}\begin{pmatrix}A^1_{1110}\\ A^2_{1110}\end{pmatrix}A_{1011}=6\hat{\tau}\begin{pmatrix}A^1_{1011}\\ A^2_{1011}\end{pmatrix},\\
&A_{1200}=\overline{A_{2100}},\ A_{0210}=\overline{A_{2010}},\ A_{0201}=\overline{A_{2001}},\ A_{0002}=\overline{A_{0020}},
\end{aligned}
$$

with

$$
\begin{aligned}
A^1_{2100} = A^1_{2001} = A^1_{0120} = A^1_{0021} =& 3F^{(1)}_{30} + (\bar{p}_0 + 2p_0)F^{(1)}_{21} + (p_0^2 + 2p_0\bar{p}_0)F^{(1)}_{12} + 3p_0^2\bar{p}_0F^{(1)}_{03},\\
A^1_{2010} = A^1_{1020} =& 3F^{(1)}_{30} + 3p_0F^{(1)}_{21} + 3p_0^2F^{(1)}_{12} + 3p_0^3F^{(1)}_{03},\\
A^1_{1110} = A^1_{1011} =& 6F^{(1)}_{30} + (2\bar{p}_0 + 4p_0)F^{(1)}_{21} + (2p_0^2 + 4p_0\bar{p}_0)F^{(1)}_{12} + 6p_0^2\bar{p}_0F^{(1)}_{03},\\
A^2_{2100} = A^2_{2001} = A^2_{0120} = A^2_{0021} =& 3F^{(2)}_{3000} + (\bar{p}_0 + 2p_0)F^{(2)}_{2100} + (p_0^2 + 2p_0\bar{p}_0)F^{(2)}_{1200} + 3p_0^2\bar{p}_0F^{(2)}_{0300}\\
&+ \left(3F^{(2)}_{0030} + (\bar{p}_0 + 2p_0)F^{(2)}_{0021} + (p_0^2 + 2p_0\bar{p}_0)F^{(2)}_{0012} + 3p_0^2\bar{p}_0F^{(2)}_{0003}\right)\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}\\
&+ F^{(2)}_{2010}(\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 2\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}) + F^{(2)}_{2001}(\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 2p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}})\\
&+ F^{(2)}_{0210}(2p_0\bar{p}_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + p_0^2\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}) + F^{(2)}_{0201}(2p_0^2\bar{p}_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + p_0^2\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}})\\
&+ F^{(2)}_{1020}(\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 2) + F^{(2)}_{1002}(p_0^2\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 2p_0\bar{p}_0)\\
&+ F^{(2)}_{0120}(\bar{p}_0\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 2p_0) + F^{(2)}_{0102}(p_0^2\bar{p}_0\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 2p_0^2\bar{p}_0),
\end{aligned}
$$

$$
\begin{aligned}
A^2_{2010} = A^2_{1020} =& 3F^{(2)}_{3000} + 3p_0F^{(2)}_{2100} + 3p_0^2F^{(2)}_{1200} + 3p_0^3F^{(2)}_{0300}\\
&+ \left(3F^{(2)}_{0030} + 3p_0F^{(2)}_{0021} + 3p_0^2F^{(2)}_{0012} + 3p_0^3F^{(2)}_{0003}\right)\mathrm{e}^{-3\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}\\
&+ \left(3F^{(2)}_{2010} + 3p_0F^{(2)}_{2001} + 3p_0^2F^{(2)}_{0210} + 3p_0^3F^{(2)}_{0201}\right)\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}\\
&+ \left(3F^{(2)}_{1020} + 3p_0F^{(2)}_{0120} + 3p_0F^{(2)}_{1002} + +3p_0^3F^{(2)}_{0102}\right)\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}},\\
A^2_{1110} = A^2_{1011} =& 6F^{(2)}_{3000} + (2\bar{p}_0 + 4p_0)F^{(2)}_{2100} + (2p_0^2 + 4p_0\bar{p}_0)F^{(2)}_{1200} + 6p_0^2\bar{p}_0F^{(2)}_{0300}\\
&+ \left(6F^{(2)}_{0030} + (2\bar{p}_0 + 4p_0)F^{(2)}_{0021} + (2p_0^2 + 4p_0\bar{p}_0)F^{(2)}_{0012} + 6p_0^2\bar{p}_0F^{(2)}_{0003}\right)\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}\\
&+ F^{(2)}_{2010}(2\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 4\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}) + F^{(2)}_{2001}(2\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 4p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}})\\
&+ F^{(2)}_{0210}(4p_0\bar{p}_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 2p_0^2\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}) + F^{(2)}_{0201}(4p_0^2\bar{p}_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 2p_0^2\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}})\\
&+ F^{(2)}_{1020}(2\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 4) + F^{(2)}_{1002}(2p_0^2\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 4p_0\bar{p}_0)\\
&+ F^{(2)}_{0120}(2\bar{p}_0\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 4p_0) + F^{(2)}_{0102}(2p_0^2\bar{p}_0\mathrm{e}^{-2\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}} + 4p_0^2\bar{p}_0).
\end{aligned}
$$

V. THE CALCULATION FORMULA FOR $S_{y(0)z_k}$, $S_{y(-1)z_k}$, $k=1,2,3,4$

$$S_{y(0)z_1}=\begin{pmatrix} 2F_{20}^{(1)}+F_{11}^{(1)}p_0 & F_{11}^{(1)}+2F_{02}^{(1)}p_0 \\ S_a & S_b \end{pmatrix},\ S_{y(0)z_2}=\begin{pmatrix} 2F_{20}^{(1)}+F_{11}^{(1)}\bar{p}_0 & F_{11}^{(1)}+2F_{02}^{(1)}\bar{p}_0 \\ S_c & S_d \end{pmatrix},$$
$$S_{y(0)z_3}=\begin{pmatrix} 2F_{20}^{(1)}+F_{11}^{(1)}p_0 & F_{11}^{(1)}+2F_{02}^{(1)}p_0 \\ S_a & S_b \end{pmatrix},\ S_{y(0)z_4}=\begin{pmatrix} 2F_{20}^{(1)}+F_{11}^{(1)}\bar{p}_0 & F_{11}^{(1)}+2F_{02}^{(1)}\bar{p}_0 \\ S_c & S_d \end{pmatrix},$$
$$S_{y(-1)z_1}=\begin{pmatrix} 0 & 0 \\ S_e & S_f \end{pmatrix},\ S_{y(-1)z_2}=\begin{pmatrix} 0 & 0 \\ S_g & S_h \end{pmatrix},\ S_{y(-1)z_3}=\begin{pmatrix} 0 & 0 \\ S_e & S_f \end{pmatrix},\ S_{y(-1)z_4}=\begin{pmatrix} 0 & 0 \\ S_g & S_h \end{pmatrix}.$$

with

$$S_a=2F_{2000}^{(2)}+F_{1100}^{(2)}p_0+F_{1010}^{(2)}\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{1001}^{(2)}p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}},\quad S_b=F_{1100}^{(2)}+2F_{0200}^{(2)}p_0+F_{0110}^{(2)}\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{0101}^{(2)}p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}},$$
$$S_c=2F_{2000}^{(2)}+F_{1100}^{(2)}\bar{p}_0+F_{1010}^{(2)}\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{1001}^{(2)}\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}},\quad S_d=F_{1100}^{(2)}+2F_{0200}^{(2)}\bar{p}_0+F_{0110}^{(2)}\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{0101}^{(2)}\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}},$$
$$S_e=2F_{0020}^{(2)}\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{0011}^{(2)}p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{1010}^{(2)}+F_{0110}^{(2)}p_0,\quad S_f=F_{0011}^{(2)}\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+2F_{0002}^{(2)}p_0\mathrm{e}^{-\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{1001}^{(2)}+F_{0101}^{(2)}p_0,$$
$$S_g=2F_{0020}^{(2)}\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{0011}^{(2)}\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{1010}^{(2)}+F_{0110}^{(2)}\bar{p}_0,\quad S_h=F_{0011}^{(2)}\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+2F_{0002}^{(2)}\bar{p}_0\mathrm{e}^{\mathrm{i}\omega_{\hat{\lambda}}\hat{\tau}}+F_{1001}^{(2)}+F_{0101}^{(2)}\bar{p}_0,$$

VI. THE CALCULATION FORMULA FOR $h_{jp_1p_2p_3p_4}$

$$h_{0k2000}^{ccs}(\vartheta)=-\mathrm{M}_{0kcs}^{c}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{0k}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{2000},$$
$$h_{0k1100}^{ccs}(\vartheta)=-\mathrm{M}_{0kcs}^{c}\left[-\lambda_{0k}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{1100},$$
$$h_{0k1010}^{ccs}(\vartheta)=-\mathrm{M}_{0kcs}^{c}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{0k}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{1010},$$
$$h_{0k1001}^{ccs}(\vartheta)=-\mathrm{M}_{0kcs}^{c}\left[-\lambda_{0k}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{1001},$$
$$h_{0k0110}^{ccs}(\vartheta)=-\mathrm{M}_{0kcs}^{c}\left[-\lambda_{0k}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{0110},$$
$$h_{0k0020}^{ccs}(\vartheta)=-\mathrm{M}_{0kcs}^{c}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{0k}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{0020},$$
$$h_{0k0011}^{ccs}(\vartheta)=-\mathrm{M}_{0kcs}^{c}\left[-\lambda_{0k}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{0011},$$

where $k=0,1,2\cdots$

$$h^{ccs}_{2nk2000}(\vartheta)=-\mathrm{M}^{s}_{2nkcc}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{2000},$$
$$h^{ccs}_{2nk1100}(\vartheta)=-\mathrm{M}^{s}_{2nkcc}\left[-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{1100},$$
$$h^{ccs}_{2nk1010}(\vartheta)=-\mathrm{M}^{s}_{2nkcc}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{1010},$$
$$h^{ccs}_{2nk1001}(\vartheta)=-\mathrm{M}^{s}_{2nkcc}\left[-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{1001},$$
$$h^{ccs}_{2nk0110}(\vartheta)=-\mathrm{M}^{s}_{2nkcc}\left[-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{0110},$$
$$h^{ccs}_{2nk0020}(\vartheta)=-\mathrm{M}^{s}_{2nkcc}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{0020},$$
$$h^{ccs}_{2nk0011}(\vartheta)=-\mathrm{M}^{s}_{2nkcc}\left[-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{0011},$$
$$h^{css}_{2nk2000}(\vartheta)=-\mathrm{M}^{c}_{2nkss}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{2000},$$
$$h^{css}_{2nk1100}(\vartheta)=-\mathrm{M}^{c}_{2nkss}\left[-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{1100},$$
$$h^{css}_{2nk1010}(\vartheta)=-\mathrm{M}^{c}_{2nkss}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{1010},$$
$$h^{css}_{2nk1001}(\vartheta)=-\mathrm{M}^{c}_{2nkss}\left[-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{1001},$$
$$h^{css}_{2nk0110}(\vartheta)=-\mathrm{M}^{c}_{2nkss}\left[-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{0110},$$
$$h^{css}_{2nk0020}(\vartheta)=-\mathrm{M}^{c}_{2nkss}\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}\vartheta}\left[-2\mathrm{i}\omega_{\hat{\lambda}}-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(\mathrm{e}^{2\mathrm{i}\omega_{\hat{\lambda}}}\cdot I_d)\right]^{-1}A_{0020},$$
$$h^{css}_{2nk0011}(\vartheta)=-\mathrm{M}^{c}_{2nkss}\left[-\lambda_{2nk}\tilde{D}_0+\tilde{L}_0(I_d)\right]^{-1}A_{0011}.$$

where $k=1,2,\cdots$.

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