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An Eigenvalue-Based Framework for Constraining Anisotropic Eddy Viscosity

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Key Points:

- Eddy viscosity is usually employed as a scalar coefficient, but its true form is that of a tensor
- Eigenanalysis can reveal new constraints on the coefficients of the eddy viscosity tensor
- Tensor unrolling can help expose the power of the eigenanalysis, but only if done in a particular way

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Abstract Eddy viscosity is employed throughout the majority of numerical fluid dynamical models, and has been the subject of a vigorous body of research spanning a variety of disciplines. It has long been recognized that the proper description of eddy viscosity uses tensor mathematics, but in practice it is almost always employed as a scalar due to uncertainty about how to constrain the extra degrees of freedom and physical properties of its tensorial form. This manuscript borrows techniques from outside the realm of geophysical fluid dynamics to consider the eddy viscosity tensor using its eigenvalues and eigenvectors, establishing a new framework by which tensorial eddy viscosity can be tested. This is made possible by a careful analysis of an operation called tensor unrolling, which casts the eigenvalue problem for a fourth-order tensor into a more familiar matrix-vector form, whereby it becomes far easier to understand and manipulate. New constraints are established for the eddy viscosity coefficients that are guaranteed to result in energy dissipation, backscatter, or a combination of both. Finally, a testing protocol is developed by which tensorial eddy viscosity can be systematically evaluated across a wide range of fluid regimes.

Plain Language Summary Numerical fluid flow solvers need to dissipate energy in order to remain numerically stable, and this is most often achieved by adding a mechanism to the governing equations called eddy viscosity. Generally the implementation of eddy viscosity boils down to specifying a scalar coefficient that governs the rate of energy dissipation. However, the true mathematical form of eddy viscosity is that of a higher-order geometric object called a tensor, and the potential advantages of using this form remain unexplored. This paper uses a generalized version of familiar linear algebra operations (eigenvalues, trace, and determinant) to establish new constraints on the eddy viscosity coefficients that promise to open up this parameterization to renewed scrutiny.

1. Introduction

Ocean turbulence is unsteady (Gerbi et al., 2009), heterogeneous (Klein et al., 2008), and anisotropic (Morrow et al., 1994; Stewart et al., 2015; S. D. Bachman et al., 2020), a far cry from the theories of homogeneous and isotropic turbulence that underpin many eddy parameterizations (e.g., Charney, 1971; Kraichnan, 1967). These theories are most commonly invoked to inform eddy momentum closures, especially scale- and flow-aware forms of eddy viscosity (e.g., Leith, 1996; Smagorinsky, 1963), which are becoming more commonplace as global ocean models progress into eddy-permitting resolutions. Notwithstanding that eddy viscosity may be a dubious choice for a large-scale ocean momentum closure in the first place (Hecht et al., 2008; Mana & Zanna, 2014; Zanna et al., 2017; S. D. Bachman et al., 2018), having a mechanism to dissipate resolved energy is a cornerstone of computational fluid dynamics (Domaradzki, 2010), and the choice of this mechanism can have significant implications for the overall ocean circulation (Jochum et al., 2008; Megann & Storkey, 2021; Pearson et al., 2017).

Most ocean eddy closures are designed with a specific process in mind. Some well-known examples include the stirring of tracers by baroclinic eddies at the mesoscale (Gent et al., 1995; Gent & Mcwilliams, 1990) and submesoscale (Fox-Kemper et al., 2011), mixing along neutral surfaces in the ocean interior (Redi, 1982), or redistribution of momentum by eddy viscosity as is discussed in this paper. Some closures include elements dealing with both tracer and momentum transport, such as those for symmetric instability (S. D. Bachman, Fox-Kemper, Taylor, & Thomas, 2017; Yankovsky et al., 2021). It is anticipated that these parameterizations will be necessary only at grid resolutions coarser than the scale of the process being parameterized, which generally would be $O(10\text{ km})$ for mesoscale closures and $O(1\text{ km})$ for submesoscale closures. When the grid resolution becomes fine enough to begin resolving said process, the parameterization would either be gradually tapered off

(e.g., Hallberg, 2013) or omitted completely. Eddy viscosity closures follow the same logic in that their strength is reduced as the grid resolution becomes finer, except that the actual scale of the process being parameterized (i.e., dissipation by intermolecular forces, at O(1 mm) scales) is so small that it is unlikely to ever be resolved in operational ocean models. That is, eddy viscosity cannot be obviated simply by moving to extremely high resolution, at least within the realm of practical ocean modeling. Furthermore, it is the only type of closure that could be considered strictly *necessary*, due to its role in maintaining the numerical stability of the model.

The above arguments single out eddy viscosity as a high-value target for parameterization development, and motivate progress toward more sophisticated closures than are presently used in practice. The current state-of-the-art closures are scale- and flow-aware, using large eddy simulation techniques that are adapted for large-scale ocean modeling (Griffies & Hallberg, 2000; S. D. Bachman, Fox-Kemper, & Pearson, 2017). These closures prescribe scalar viscosity coefficients that are tuned based on theories of cascading invariants in homogeneous, isotropic turbulence.

The use of a scalar coefficient dates back to the original eddy viscosity hypothesis made by Boussinesq (1877), who proposed a linear relationship between the locally-averaged turbulent stress and strain fields, analogous to the linear constitutive relation for Newtonian fluids (Schmitt, 2007). Richardson (1922) noted through his own experiments and intuition that, due to the small aspect ratio of the atmosphere, the assumption of a single (isotropic) viscosity is questionable and multiple viscosity coefficients might be necessary. He even suggested that "the general theory is found to be ready-made in connection with the elasticity of crystals" (Richardson, 1922, p. 222), hinting at a potential isomorphism with Hooke's Law (Hooke, 1678) in which the stress and strain would be related through a fourth-order viscosity (elastic stiffness) tensor. Several decades passed before this idea began to percolate into studies on turbulent diffusion in the atmosphere (e.g., Kamenkovich, 1967; Kirwan Jr, 1969; Williams, 1972), which was still well before it was considered in the context of ocean modeling (Doney et al., 2003; Griffies, 2004; Large et al., 2001; Smith & McWilliams, 2003; Wajsowicz, 1993).

Despite these advances, the tensorial form of eddy viscosity is used in modern ocean models only in a fairly rudimentary way. Different coefficients are typically used in the vertical and horizontal ("transverse") directions, but both coefficients are typically treated as scalars due to uncertainties about how to set the coefficients of the full viscosity tensor. Throughout the remainder of this paper we will consider the viscosity in the context of its tensor form, and hereafter refer to this tensor as $\mathbf{A}$. In particular, we will be interested in the coefficients of the transverse part of $\mathbf{A}$, and in exploring a new approach to understanding its physics and setting its coefficients.

The goal of this paper is to open a pathway by which the ocean modeling community can move beyond using a scalar coefficient for the horizontal viscosity (i.e., assuming transverse isotropy), whose use is not supported by any observations or theory (Smith & McWilliams, 2003). The main obstacle to achieving this goal thus far is that Hooke's Law is an underdetermined system of equations for solving for the elements of $\mathbf{A}$, and it is not possible to get around this problem by introducing extra tracers as one could do for the thermodynamic state variables (e.g., S. Bachman & Fox-Kemper, 2013; S. D. Bachman et al., 2015). However, since we are primarily interested in $\mathbf{A}$ as a parameterization, we have the freedom to *choose* the structure of $\mathbf{A}$ as long as it satisfies physical constraints on the stress, strain, and energy dissipation. It will be shown in the following sections that one can describe the physics of $\mathbf{A}$ using an eigendecomposition, analogous to how is usually done for matrices and second-order tensors, which is helpful for defining the correct constraints on $\mathbf{A}$. The eigendecomposition also allows us to build a framework for systematically evaluating the different possible structures of $\mathbf{A}$ and their effects on the model solution.

The manuscript is laid out as follows. Section 2 establishes the mathematical background underpinning the formulation of the eddy viscosity tensor. Section 3 introduces the notion of tensor unrolling, which is an important tool for simplifying the mathematics and presentation of the higher-order geometric objects used throughout this paper. Section 4 presents a key result of this paper, which is a generalized set of bounds on the viscosity coefficients that is guaranteed to result in dissipation, backscatter, or both. Section 5 suggests a possible testing protocol for sweeping through and evaluating the possible combinations of the viscosity coefficients. Concluding remarks are made in Section 6.

2. Properties of the Stress, Strain Rate, and Viscosity Tensors

A proper discussion of eddy viscosity taps into the fields of linear algebra and differential geometry and calculus, and since we will be dealing with physical objects it will be important to develop shorthand for identifying tensors

of various orders. Therefore, the notation used throughout the remainder of this manuscript will obey the following rules. Vectors (first-order tensors) will be indicated by boldface and a single underline, for example, $\underline{\boldsymbol{x}}$. Second-order tensors will use boldface and a double underline, for example, $\underline{\underline{\boldsymbol{x}}}$. Fourth-order tensors will use boldface block font with no underline, for example, $\mathbf{x}$. Scalars will have standard font, for example, x, and subscripts will indicate tensor indices.

A deeper look into the nuances of the eddy viscosity parameterization, its tensors, and its underpinnings in differential geometry can be found in Griffies (2004, Ch. 17). Here we will only review the parts of the theory that are necessary to motivate the approach in this paper. Note that only Cartesian tensors will be considered in the following presentation, to avoid needing to distinguish between covariant and contravariant indices. This is done to favor simplicity over mathematical rigor, particularly in cases where additional subscripts and superscripts are helpful to define objects and exponents. A full coordinate-invariant version of these derivations, as found in Griffies (2004, Ch. 17), can be made relatively straightforwardly from the forthcoming results with essentially no alteration to the mathematics.

Eddy viscosity is a parameterization of the frictional stress on fluid parcels, which places certain constraints on the geometry of the viscosity tensor and how it constitutes the relationship between the stress and strain. It is useful to begin by writing the three-dimensional stress tensor $\underline{\underline{\boldsymbol{\tau}}}$ in the general form

$$\tau_{mn} = -p\delta_{mn} + \tilde{\tau}_{mn}, \qquad m,n = \{1,2,3\}, \tag{1}$$

where p is the thermodynamic pressure, δ_{mn} is the Kronecker delta, and $\underline{\underline{\tilde{\boldsymbol{\tau}}}}$ is the frictional stress tensor whose divergence is associated with fluid deformations and energy dissipation. Using the Reynolds averaging axioms,

$$\underline{\underline{\tilde{\boldsymbol{\tau}}}} = -\rho\overline{\underline{\boldsymbol{u}}' \otimes \underline{\boldsymbol{u}}'} \equiv -\rho\overline{u'_m u'_n} \tag{2}$$

is also identified as the Reynolds stress tensor, where the overbar $\overline{(\cdot)}$ indicates a mean, primes indicate deviations away from this mean, $\otimes$ is the outer product operator, ρ is the fluid density, $\underline{\boldsymbol{u}} = (u,v,w)$ is the three-dimensional velocity vector, and the indices m and n range from 1 to 3 as in Equation 1. The identity in Equation 2 holds in general curvilinear coordinates, though one must take care to account for extra terms that appear in this expression when the Reynolds axioms are not satisfied (e.g., Leonard, 1975).

Eddy viscosity aims to parameterize the subgridscale dissipation associated with the frictional stress $\underline{\underline{\tilde{\boldsymbol{\tau}}}}$. Smith and McWilliams (2003) note that it is typical to parameterize only the deviatoric (trace-free) part of $\underline{\underline{\tilde{\boldsymbol{\tau}}}}$ in a nearly incompressible flow, since its divergence is simply the gradient of a scalar field that can be freely absorbed into the pressure without altering the dynamics. Parameterizing only the deviatoric stress is not strictly necessary from either a physical or numerical standpoint, and Smith and McWilliams (2003) characterize the freedom to choose either option as an "uncertainty" in forming the parameterization. However, Griffies (2004) shows that using the deviatoric part reduces the number of unknown elements in $\mathbf{A}$, which will simplify the approach we will develop here. We will thus be considering the deviatoric part of $\underline{\underline{\tilde{\boldsymbol{\tau}}}}$ in the forthcoming derivations.

Following Smagorinsky (1993) we will also proceed using the quasi-hydrostatic constraint, which argues that in a hydrostatic system it is appropriate to separate the horizontal and vertical components of the energy dissipation. For large-scale ocean models this is also justified on the basis that friction is only applied in the horizontal momentum equations, since the vertical momentum equation simply reduces to hydrostatic balance. Griffies (2004) writes the implied frictional stress tensor in the form

$$\tilde{\tau}_{mn} = \begin{bmatrix} \tilde{\tau}_{11} & \tilde{\tau}_{12} & 2\rho\kappa\dfrac{\partial}{\partial z}\overline{u} \\ \tilde{\tau}_{12} & \tilde{\tau}_{22} & 2\rho\kappa\dfrac{\partial}{\partial z}\overline{v} \\ 2\rho\kappa\dfrac{\partial}{\partial z}\overline{u} & 2\rho\kappa\dfrac{\partial}{\partial z}\overline{v} & 0 \end{bmatrix}, \tag{3}$$

where κ is a vertical viscosity coefficient. The deviatoric part of $\underline{\underline{\tilde{\boldsymbol{\tau}}}}$, which we will refer to as $\underline{\underline{\boldsymbol{\sigma}}}$, is

$$\sigma_{mn} = \tilde{\tau}_{mn} - \frac{1}{2}\delta_{mn}\tilde{\tau}_{kk} \tag{4}$$

$$= \begin{bmatrix} \frac{1}{2}(\tilde{\tau}_{11} - \tilde{\tau}_{22}) & \tilde{\tau}_{12} & 2\rho\kappa\frac{\partial}{\partial z}\overline{u} \\ \tilde{\tau}_{12} & \frac{1}{2}(\tilde{\tau}_{22} - \tilde{\tau}_{11}) & 2\rho\kappa\frac{\partial}{\partial z}\overline{v} \\ 2\rho\kappa\frac{\partial}{\partial z}\overline{u} & 2\rho\kappa\frac{\partial}{\partial z}\overline{v} & 0 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} \frac{1}{2}(\overline{v'^2} - \overline{u'^2}) & -\overline{u'v'} & 2\rho\kappa\frac{\partial}{\partial z}\overline{u} \\ -\overline{u'v'} & \frac{1}{2}(\overline{u'^2} - \overline{v'^2}) & 2\rho\kappa\frac{\partial}{\partial z}\overline{v} \\ 2\rho\kappa\frac{\partial}{\partial z}\overline{u} & 2\rho\kappa\frac{\partial}{\partial z}\overline{v} & 0 \end{bmatrix}. \tag{6}$$

The horizontal frictional stresses are described by the upper left 2×2 minor of $\underline{\tilde{\boldsymbol{\tau}}}$, which has been featured in previous studies on eddy geometry and eddy-mean flow interactions (Hoskins et al., 1983; Waterman & Lilly, 2015). For the remainder of this study we will only be concerned with the horizontal part of $\underline{\boldsymbol{\sigma}}$, and will omit the vertical derivative terms in any discussion of $\underline{\boldsymbol{\sigma}}$ or $\mathbf{A}$. As such, we will henceforth treat all tensor indices as having a range from 1 to 2.

2.1. Assumptions for Constructing the Eddy Viscosity Tensor

The standard mathematical form for eddy viscosity is a generalized version of Hooke's Law, where the eddy viscosity tensor $\mathbf{A}$ determines a linear relationship between the eddy stresses, $\underline{\boldsymbol{\sigma}}$, and resolved velocity gradients, $\underline{\underline{\nabla \boldsymbol{u}}}$. Both its index-free and index notation forms, which are

$$\underline{\boldsymbol{\sigma}} = \mathbf{A} : \underline{\underline{\nabla \boldsymbol{u}}} \qquad \text{and} \qquad \sigma_{mn} = A_{mnpq}\partial_p u_q, \tag{7}$$

respectively, will be useful. Here the colon indicates the double contraction operator that reduces the order of the tensor product by two, and is defined so that for two arbitrary second order tensors $\underline{\boldsymbol{A}}$ and $\underline{\boldsymbol{B}}$ and a fourth-order tensor $\mathbf{C}$,

$$\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}} = A_{mn}B_{mn}, \tag{8}$$

$$\mathbf{C} : \underline{\underline{\boldsymbol{B}}} = C_{mnpq}B_{pq}. \tag{9}$$

The double contraction thus represents a Frobenius inner product of two second order tensors, and the action of a linear map for a fourth order tensor on a space of matrices.

The only other tensor operator needed here will be the tensor product, which will be used in the context of multiplying two second order tensors to yield a fourth order tensor,

$$\underline{\underline{\boldsymbol{A}}} \otimes \underline{\underline{\boldsymbol{B}}} = A_{mn}B_{pq} = (AB)_{mnpq}. \tag{10}$$

With this basic notation established, it is useful to review several constraints on the structure of the viscosity tensor that have appeared in previous literature (Griffies, 2004; Smith & McWilliams, 2003; Wajsowicz, 1993). These emerge from a combination of intrinsic symmetries in the eddy stresses, properties that we assume based on physical principles, and properties that we assume for the sake of numerical implementation. To briefly summarize, the structure of $\mathbf{A}$ is determined by the following principles:

- Symmetry of the stress tensor, $\sigma_{mn} = \sigma_{nm}$, which by Equation 7 forces $A_{mnpq} = A_{nmpq}$;

- An assumption that there should be no viscous stress generated as a result of pure rotation, which forces $A_{mnpq} = A_{mnqp}$. This constraint also implies that **A** multiplies only the symmetric part of the velocity gradient tensor in Equation 7, which is the rate of strain tensor

$$\underline{\underline{\dot{\boldsymbol{e}}}} = \frac{1}{2}\left(\underline{\underline{\nabla \boldsymbol{u}}} + \underline{\underline{\nabla \boldsymbol{u}}}^T\right). \tag{11}$$

  Note that, in combination with the previous constraint, **A** possesses the property of *minor symmetry* (Brannon, 2018, Ch.20).
- To ensure that the kinetic energy forcing by the parameterization can be made sign-definite, **A** must possess *major symmetry*, with $A_{mnpq} = A_{pqmn}$. This property has previously been justified by employing an energy dissipation functional (e.g., Green & Zerna, 1992, Ch.5), but it will be shown later that an eigenvalue approach offers a much more flexible and powerful way to understand the kinetic energy forcing.
- The eddy stress tensor used in Equation 7 is trace-free, since for incompressible flow the trace projects onto the pressure field and is thus not frictional in nature. The trace-free condition must apply for arbitrary stresses and strains, which implies that $\delta_{mn}A^{mnpq} = 0$ (Griffies, 2004).

In all, these conditions greatly reduce the number of free parameters in **A** from 81 to 5 in three dimensions, or from 16 to 3 in two dimensions. Note that previous works on this topic (e.g., Smith & McWilliams, 2003; Wajsowicz, 1993) cite these numbers as 81 to 21 and 16 to 6, respectively, since they do not employ the trace-free stress condition. It is thus clear that the trace-free condition is quite powerful, and since the nearly incompressible nature of the ocean allows us to employ it without altering the dynamics (e.g., Smith & McWilliams, 2003), it is sensible to do so.

2.2. Eigendecomposition of Fourth Order Tensors

The approach pursued in this paper differs from previous studies on eddy viscosity, in that we will use an eigendecomposition to determine the structure and properties of **A**. Akin to the well-known eigenvalue and eigenvector methods for second-order tensors, the basis of this approach for fourth-order tensors is the eigenequation

$$\mathbf{A} : \underline{\underline{\boldsymbol{x}}} = \lambda \underline{\underline{\boldsymbol{x}}}, \tag{12}$$

where λ is an eigenvalue and $\underline{\underline{\boldsymbol{x}}}$ is a second-order eigentensor. This equation can also be written

$$(\mathbf{A} - \lambda \mathbf{I}) : \underline{\underline{\boldsymbol{x}}} = 0, \tag{13}$$

where $\mathbf{I}_{mnpq} = \frac{1}{2}(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np})$ is a minor symmetric fourth-order identity tensor. Again in anology with second-order eigenproblems, Equation 13 only has non-trivial solutions if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \tag{14}$$

The characteristic equation (Equation 14) can have several roots, each of which is an eigenvalue of **A** associated with a symmetric eigentensor. The set of eigentensors form a mutually orthogonal basis for **A**, satisfying the condition

$$\underline{\underline{\boldsymbol{x}}}^{(i)} : \underline{\underline{\boldsymbol{x}}}^{(j)} = \underline{\underline{\boldsymbol{\delta}}}^N_{ij}, \tag{15}$$

where the superscripts in parentheses indicate the index of the eigenvalue ($i, j \in \{1, \ldots, N\}$, where $N = 4$ in two dimensions and $N = 9$ in three dimensions). $\underline{\underline{\boldsymbol{\delta}}}^N_{ij}$ indicates the $N \times N$ second-order identity tensor.

Eigendecompositions are very powerful tools for understanding the properties of second-order tensors, and many of these properties have close analogs in fourth-order tensor mathematics. The most important property that will be employed here is *sign-definiteness*. A real-valued second-order tensor $\underline{\underline{\boldsymbol{M}}}$ is *positive definite* if $\underline{\boldsymbol{x}}^T \cdot \underline{\underline{\boldsymbol{M}}} \cdot \underline{\boldsymbol{x}} > 0$ for all nonzero $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, and is *negative definite* if the inequality is reversed. The matrix becomes sign *semi-definite* if the strict inequality is expanded to include equivalence. If $\underline{\underline{\boldsymbol{M}}}$ is symmetric then it has all real eigenvalues, and is

positive (negative) definite only if all of its eigenvalues are positive (negative). Likewise, it is semi-definite if its set of eigenvalues can include zero. Note that these properties hold for any positive integer n, but for our purposes it will suffice to consider $n = \sqrt{N}$.

Fourth-order tensors may also be sign-definite, and similar definitions apply. Namely, a real-valued fourth-order tensor $\mathbf{M}$ is positive definite if $\underline{\underline{x}} : \mathbf{M} : \underline{\underline{x}} > 0$ for all nonzero $\underline{\underline{x}} \in \mathbb{R}^n \times \mathbb{R}^n$, and negative definite if the inequality is reversed. The connection between sign-definiteness and eigenvalues also extends to fourth-order tensors in the same way as is stated above.

The main reason we are interested in these fourth-order eigendecompositions is that they offer a new way to constrain the elements of $\mathbf{A}$. To understand this, consider the requirement from Section 2.1 that $\mathbf{A}$ must be major symmetric, which for dissipative closures is imposed to ensure that the kinetic energy dissipation rate.

$$\mathcal{D} = \frac{1}{2}\dot{\underline{\underline{e}}} : \underline{\underline{\sigma}} \tag{16}$$

$$= \frac{1}{2}\dot{\underline{\underline{e}}} : \mathbf{A} : \dot{\underline{\underline{e}}} \geq 0. \tag{17}$$

Since Equation 17 must be true for arbitrary $\dot{\underline{\underline{e}}}$, this is equivalent to the condition that $\mathbf{A}$ must be positive semi-definite, and hence $\mathbf{A}$ must have non-negative eigenvalues. Likewise, for closures designed to add kinetic energy ("backscatter," e.g. Jansen & Held, 2014; S. D. Bachman, 2019; Juricke et al., 2020; Yankovsky et al., 2024), the inequality in Equation 17 is reversed and $\mathbf{A}$ must be negative semi-definite, and hence $\mathbf{A}$ must have non-positive eigenvalues. We can therefore constrain the elements of $\mathbf{A}$ by employing the usual machinery of eigenproblems. In particular, we will use the characteristic equation (Equation 14) to determine the eigenvalues as a function of the unknown elements of $\mathbf{A}$, after all the constraints in Section 2.1 are applied. It will be shown later that this approach yields both a simple and powerful way to work with both eddy viscosity and backscatter.

For second-order tensors and matrices, the characteristic equation can be solved straightforwardly with pen and paper as long as the matrix dimensions are relatively small. However, the situation is not so simple for fourth-order tensors, owing both to the larger number of eigenvalues (at minimum four, for which an analytic solution to the characteristic equation exists but is extremely cumbersome) and the general difficulty of working with two extra tensor indices. The latter is a primary reason for an operation known as *tensor unrolling*, which takes advantage of the linearity of the relation $\underline{\underline{\sigma}} = \mathbf{A} : \dot{\underline{\underline{e}}}$ to write it as a matrix-vector multiplication instead of a fourth-order tensor multiplying a second-order tensor. Unrolling has been used in previous works on ocean eddy viscosity (e.g., Griffies, 2004; Smith & McWilliams, 2003; Wajsowicz, 1993) because it greatly simplifies the presentation of the problem and makes it very clear which elements of $\mathbf{A}$ multiply which elements of $\dot{\underline{\underline{e}}}$.

Here unrolling will also be employed for these reasons, but we must pay special attention to *how* $\mathbf{A}$ is unrolled to ensure that its eigenvalue properties are preserved through the unrolling. Through this exercise we will strive to combine the best aspects of both unrolling and eigendecompositions, to deconstruct the properties of $\mathbf{A}$ in a way that is easy to understand and implement in models.

3. Tensor Unrolling

Equation 7 simply uses a linear combination of the elements of $\mathbf{A}$ and $\dot{\underline{\underline{e}}}$ to approximate the eddy stresses, so in theory there exist many possible ways to unroll $\mathbf{A}$ into a matrix and $\dot{\underline{\underline{e}}}$ into a vector while still preserving the sums and products in the relation. However, as discussed above, many properties of lower-order tensors have analogous higher-order counterparts, and it is reasonable to desire a method of unrolling that allows these properties to map neatly onto each other when moving from original to unrolled form. Here two common unrolling methods, known respectively as Voigt and Mandel notations, are reviewed to emphasize the importance of this operation in our analysis of $\mathbf{A}$. Both methods originate in the field of plastic mechanics (Dunne & Petrinic, 2005, Ch.4) and have traditionally found the most use for studying the physics of crystals (e.g. Kocks et al., 1998, Ch. 1). However, many concepts from these fields (such as treating a solid as a continuum subject to stress and strain) transfer readily to physical oceanography.

3.1. Voigt Notation

Previous papers on anisotropic eddy viscosity (e.g., Griffies, 2004; Smith & McWilliams, 2003; Wajsowicz, 1993) rewrite Hooke's Law into a form called *Voigt notation*, though in these papers it is not called by name. Here Voigt-unrolled objects will be indicated using a subscript $_V$, and the reader is encouraged to think of applying $_V$ as cutting the order of the tensor in half (from fourth to second, and from second to first). For a minor symmetric fourth-order tensor in two dimensions, unrolling in Voigt notation uses a mapping function $\mu(k)$ to transform single indices in the set $k = \{1, 2, 3\}$ into pairs of indices in the range $\{1, 2\}$, according to the rule.

k	1	2	3
$\mu(k)$	(1, 1)	(2, 2)	(1, 2)

Using the viscosity tensor $\mathbf{A}$ as an example, the elements of a minor-symmetric fourth-order tensor are then laid out in matrix form as follows:

$$\mathbf{A} \to \mathbf{A}_V = \begin{bmatrix} A_{u(1),u(1)} & A_{u(1),u(2)} & A_{u(1),u(3)} \\ A_{u(2),u(1)} & A_{u(2),u(2)} & A_{u(2),u(3)} \\ A_{u(3),u(1)} & A_{u(3),u(2)} & A_{u(3),u(3)} \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} A_{1111} & A_{1122} & A_{1112} \\ A_{2211} & A_{2222} & A_{2212} \\ A_{1211} & A_{1222} & A_{1212} \end{bmatrix}. \tag{19}$$

Note that fourth (second) order tensors can still be Voigt-unrolled even if they are not minor symmetric (symmetric), which simply requires a slight extension of the index table and results in a 4×4 matrix (4×1 vector). Symmetry simply allows us to omit redundant rows, resulting in the smaller matrix shown in Equation 18.

Voigt unrolling may also be used to transform second-order tensors into vector form, with the noteworthy property that stress and strain tensors transform differently:

$$\underline{\underline{\boldsymbol{\sigma}}} \to \underline{\boldsymbol{\sigma}}_V = \begin{bmatrix} \sigma_{\mu(1)} \\ \sigma_{\mu(2)} \\ \sigma_{\mu(3)} \end{bmatrix}, \qquad \underline{\underline{\dot{\boldsymbol{e}}}} \to \underline{\dot{\boldsymbol{e}}}_V = \begin{bmatrix} \dot{e}_{\mu(1)} \\ \dot{e}_{\mu(2)} \\ 2\dot{e}_{\mu(3)} \end{bmatrix}. \tag{20}$$

The difference is accounted for by the coefficient 2 multiplying the third element of $\underline{\dot{\boldsymbol{e}}}_V$, which may seem subtle but is in fact very consequential. It arises because Voigt notation expresses tensors using a non-normalized tensor basis, meaning its covariant and contravariant components are unequal (Helnwein, 2001; Brannon, 2018, Ch.26). In order to avoid the need for a metric that is not the identity tensor, Voigt notation thus traditionally uses different forms to store the stress (contravariant) and strain (covariant) tensors, resulting in the different unrollings that appear in Equation 20. The advantage of this approach is that Voigt notation preserves both Hooke's Law and work conjugacy, which is defined as the inner product between the stress and strain tensors and is equivalent to the twice the dissipation rate (Equation 17):

$$\underline{\underline{\dot{\boldsymbol{e}}}} : \underline{\underline{\boldsymbol{\sigma}}} = \underline{\dot{\boldsymbol{e}}}_V^T \cdot \underline{\boldsymbol{\sigma}}_V \tag{21}$$

$$= \underline{\dot{\boldsymbol{e}}}_V^T \cdot \mathbf{A}_V \cdot \underline{\dot{\boldsymbol{e}}}_V \tag{22}$$

$$= 2D. \tag{23}$$

However, the disadvantage is that Voigt unrolling does not preserve the inner product between two stress or strain tensors,

$$\underline{\underline{\dot{\boldsymbol{e}}}} : \underline{\underline{\dot{\boldsymbol{e}}}} \neq \underline{\underline{\dot{\boldsymbol{e}}}}_V \cdot \underline{\underline{\dot{\boldsymbol{e}}}}_V, \tag{24}$$

$$\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{\sigma}}} \neq \underline{\underline{\boldsymbol{\sigma}}}_V \cdot \underline{\underline{\boldsymbol{\sigma}}}_V, \tag{25}$$

and crucially it does not preserve the eigenvalue relation,

$$\mathbf{A} : \underline{\underline{\boldsymbol{x}}} = \lambda \underline{\underline{\boldsymbol{x}}} \qquad \nrightarrow \qquad \mathbf{A}_V \cdot \underline{\underline{\boldsymbol{x}}}_V = \lambda \underline{\underline{\boldsymbol{x}}}_V. \tag{26}$$

Therefore, despite its prior uses for studying eddy viscosity, Voigt unrolling is not suitable for the objective here, which is to enforce the constraint $\mathcal{D} \geq 0$ through the eigenvalues of $\mathbf{A}_V$.

3.2. Mandel Notation

Many of the difficulties with Voigt notation are solved by choosing an alternative unrolling method known as *Mandel notation* (Mandel, 1965), which uses selectively placed factors of $\sqrt{2}$ to express all tensors using an orthonormal basis (Helnwein, 2001). Like Voigt notation, Mandel notation effectively cuts the order of tensors in half, and uses the same index mapping to transform fourth-order tensors to matrices, and second-order tensors to vectors. Denoting Mandel-unrolled objects with a subscript $_{\mathrm{M}}$, the unrolling results in the following transformations:

$$\mathbf{A} \rightarrow \mathbf{A}_M = \begin{bmatrix} A_{1111} & A_{1122} & \sqrt{2}A_{1112} \\ A_{2211} & A_{2222} & \sqrt{2}A_{2212} \\ \sqrt{2}A_{1211} & \sqrt{2}A_{1222} & 2A_{1212} \end{bmatrix}, \tag{27}$$

$$\underline{\underline{\boldsymbol{\sigma}}} \rightarrow \underline{\underline{\boldsymbol{\sigma}}}_M = \begin{bmatrix} \sigma_{\mu(1)} \\ \sigma_{\mu(2)} \\ \sqrt{2}\,\sigma_{\mu(3)} \end{bmatrix}, \qquad \underline{\underline{\dot{\boldsymbol{e}}}} \rightarrow \underline{\underline{\dot{\boldsymbol{e}}}}_M = \begin{bmatrix} \dot{e}_{\mu(1)} \\ \dot{e}_{\mu(2)} \\ \sqrt{2}\,\dot{e}_{\mu(3)} \end{bmatrix}. \tag{28}$$

Note that, unlike with Voigt unrolling, stress and strain tensors are now unrolled in the same way. Mandel notation still preserves Hooke's Law and work conjugacy, but also now preserves the inner product of stress and strain tensors,

$$\underline{\underline{\dot{\boldsymbol{e}}}} : \underline{\underline{\dot{\boldsymbol{e}}}} = \underline{\underline{\dot{\boldsymbol{e}}}}_M \cdot \underline{\underline{\dot{\boldsymbol{e}}}}_M, \tag{29}$$

$$\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}_M \cdot \underline{\underline{\boldsymbol{\sigma}}}_M, \tag{30}$$

as well as the eigenvalue relation,

$$\mathbf{A} : \underline{\underline{\boldsymbol{x}}} = \lambda \underline{\underline{\boldsymbol{x}}} \qquad \rightarrow \qquad \mathbf{A}_M \cdot \underline{\underline{\boldsymbol{x}}}_M = \lambda \underline{\underline{\boldsymbol{x}}}_M. \tag{31}$$

Mandel notation also possesses the very powerful feature that a tensors's symmetries and spectral properties, such as positive-definiteness, are preserved by the transformation (Brannon, 2018, Ch. 26). It is thus the most natural unrolling method with which to examine the eddy viscosity tensor.

It is also worth noting that it is possible to straightforwardly move between Voigt and Mandel notations via a set of simple matrix operations (Mánik, 2021; Neeman et al., 2008). Defining a transformation matrix

$$\underline{\underline{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \tag{32}$$

objects unrolled in the two notations obey the following equivalences:

$$\mathbf{A}_M = \underline{\underline{T}} \mathbf{A}_V \underline{\underline{T}}, \tag{33}$$

$$\underline{\underline{\sigma}}_M = \underline{\underline{T}}\, \underline{\underline{\sigma}}_V, \tag{34}$$

$$\underline{\underline{\dot{e}}}_M = \underline{\underline{T}}^{-1} \underline{\underline{\dot{e}}}_V. \tag{35}$$

3.3. Unrolling Hooke's Law

Now equipped with these tools to simplify the mathematics and analysis of **A**, it is instructive to explicitly show how tensor unrolling is performed. Both Voigt and Mandel forms will be shown here in parallel, both to connect with previous literature (Voigt) and to lay out the path forward (Mandel).

It will be convenient now to employ the symmetries and assumptions described in Section 2.1. Recall that in the discussion of Voigt and Mandel notation we already applied the assumption that **A** is minor symmetric, which reduced the number of unique elements from 16 to 9. We now add the assumption that **A** is major symmetric, which confers the property that the unrolled, second-order version of **A** is a symmetric matrix. This reduces the number of unique elements from 9 to 6, and, by the properties of symmetric matrices, guarantees that its eigenvalues are real. Lastly, applying the trace-free condition to the eddy stress reduces the number of unique elements from 6 to 3 (e.g., Griffies, 2004). In terms of the unrolled tensor shown in Equation 19, for brevity we will relabel the remaining unique elements as $\alpha = A_{1111}$, $\gamma = A_{1112}$, and $\beta = A_{1212}$.

Using these assumptions we now write Hooke's Law in Voigt and Mandel forms, using Equations 19, 20, 27 and 28. Relabeling the elements as above, unrolling yields.

$$\text{Voigt}: \quad \underline{\underline{\sigma}}_V = \mathbf{A}_V \cdot \underline{\underline{\dot{e}}}_V \rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \alpha & -\alpha & \gamma \\ -\alpha & \alpha & -\gamma \\ \gamma & -\gamma & \beta \end{bmatrix} \begin{bmatrix} \dot{e}_{11} \\ \dot{e}_{22} \\ 2\dot{e}_{12} \end{bmatrix}, \tag{36}$$

$$\text{Mandel}: \quad \underline{\underline{\sigma}}_M = \mathbf{A}_M \cdot \underline{\underline{\dot{e}}}_M \rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\sigma_{12} \end{bmatrix} = \begin{bmatrix} \alpha & -\alpha & \sqrt{2}\gamma \\ -\alpha & \alpha & -\sqrt{2}\gamma \\ \sqrt{2}\gamma & -\sqrt{2}\gamma & 2\beta \end{bmatrix} \begin{bmatrix} \dot{e}_{11} \\ \dot{e}_{22} \\ \sqrt{2}\dot{e}_{12} \end{bmatrix}. \tag{37}$$

It is relatively straightforward to see that the two systems of equations in Equations 36 and 37 are equivalent, but differ only in how factors of $\sqrt{2}$ are distributed between the third row and column. For our purposes, however, a more crucial difference is that the eigenvalues of the unrolled **A** are different between these two forms. Since only Mandel notation preserves eigenvalues through the unrolling, we therefore must consider Equation 37 as the correct system of equations to study.

Before moving on to the eigenanalysis, we will express this system of equations using an alternative basis that was proposed by Smith and McWilliams (2003). This basis confers several attractive invariance properties under coordinate transformations, and underpins the way in which the frictional operator is implemented in most ocean models. The new basis employs the following definitions:

$$\sigma_D = \sigma_{11} + \sigma_{22}, \qquad \sigma_T = \sigma_{11} - \sigma_{22}, \qquad \sigma_S = 2\sigma_{12} \tag{38}$$

$$\dot{e}_D = \dot{e}_{11} + \dot{e}_{22}, \qquad \dot{e}_T = \dot{e}_{11} - \dot{e}_{22}, \qquad \dot{e}_S = 2\dot{e}_{12}. \tag{39}$$

Several of these elements may be familiar upon expanding them out in terms of the velocity derivatives. Notably, $\dot{e}_D$ is the divergence of the horizontal velocity, which is identically zero for quasigeostrophic flow and is often small for the mesoscale flows of most interest here. $\dot{e}_T$ and $\dot{e}_S$ are the horizontal tension and shearing strain, respectively, whose computations are performed quite naturally on Arakawa C- and D-grids (Arakawa & Lamb, 1977). σ_D is the trace of the transverse stress tensor, which we have assumed to be zero based on arguments from Section 2.1.

Transforming Hooke's Law into the new basis requires a simple sequence of row operations from either Voigt or Mandel notation. Denoting the transformed tensors using square brackets, upon transformation both Equations 36 and 37 become

$$\text{Transformed}: \qquad \left[\underline{\underline{\boldsymbol{\sigma}}}\right] = [\mathbf{A}] \cdot \left[\underline{\underline{\dot{\boldsymbol{e}}}}\right] \rightarrow \begin{pmatrix} \sigma_T \\ \sigma_D \\ \sigma_S \end{pmatrix} = \begin{pmatrix} 2\alpha & 0 & 2\gamma \\ 0 & 0 & 0 \\ 2\gamma & 0 & 2\beta \end{pmatrix} \begin{pmatrix} \dot{e}_T \\ \dot{e}_D \\ \dot{e}_S \end{pmatrix}. \tag{40}$$

A few key points must be emphasized about our use of this transformation. First, note that Equation 40 is neither in Voigt or Mandel notation, though we could technically recover either notation by appropriately redefining the identities in Equations 38 and 39 (doing so will be neither advantageous nor necessary moving forward). Second, eigenvalues are generally not preserved by row operations, but this is a unique case where the characteristic equation for Equation 40 is actually the same as for Equation 37. So for the sake of studying the eigenvalues of $\mathbf{A}$, using Equation 40 is warranted. Also, recall that Voigt notation in Equation 36 results in a different set of eigenvalues that are not equal to the eigenvalues of the original $\mathbf{A}$, meaning that transforming into the new basis recovers the "correct" eigenvalues. So by using Voigt unrolling with this new basis we could have naively arrived at the correct characteristic equation without realizing the faults in the Voigt approach. Thus, the reader is encouraged to remember that Mandel notation is the mathematically superior way to unroll Hooke's Law and is the foundation for all of the forthcoming analysis, and that we are only using Equation 40 because of its equivalent characteristic equation and its easier translation into modern ocean model codes.

Lastly, note that standard isotropic viscosity with a coefficient ν is recovered when we set $\alpha = \beta = \nu$ and $\gamma = 0$, yielding the same dissipation derived by Smith and McWilliams (2003),

$$\mathcal{D} = \frac{1}{2}\nu\left(\dot{e}_T^2 + \dot{e}_S^2\right). \tag{41}$$

4. Eigenanalysis, Dissipation, and Backscatter

The tensor $[\mathbf{A}]$ is noninvertible, meaning that it has zero determinant and one of its three eigenvalues is equal to zero (since the determinant is equal to the product of the eigenvalues). This occurs as a consequence of having chosen to use the trace-free part of the eddy stress, but does not impact the ensuing analysis. It will be sufficient to focus our attention on the nonzero eigenvalues of $[\mathbf{A}]$ and its pseudo-determinant (the product of the nonzero eigenvalues) instead.

Using standard techniques for obtaining the eigenvalues of a 3×3 matrix, the two nonzero eigenvalues of $[\mathbf{A}]$ are

$$\lambda = \alpha + \beta \pm \sqrt{(\alpha - \beta)^2 + 4\gamma^2}, \tag{42}$$

whose sum and product are, respectively, the trace and pseudo-determinant,

$$\text{tr}([\mathbf{A}]) = 2\alpha + 2\beta, \tag{43}$$

$$\text{pdet}([\mathbf{A}]) = 4\alpha\beta - 4\gamma^2. \tag{44}$$

Equations 42–44 will now be used to highlight the advantage of our eigenvalue-based analysis of [**A**]. As a point of comparison, consider calculating the dissipation $\mathcal{D}$ using basic matrix multiplication on Equation 40. We have

$$\mathcal{D} = \frac{1}{2}\underline{\underline{\dot{\boldsymbol{e}}}} : \underline{\underline{\boldsymbol{\sigma}}} \tag{45}$$

$$= \frac{1}{2}\underline{\underline{\dot{\boldsymbol{e}}}} : \mathbf{A} : \underline{\underline{\dot{\boldsymbol{e}}}} \tag{46}$$

$$= \frac{1}{4}\left[\underline{\underline{\dot{\boldsymbol{e}}}}\right]^{\mathrm{T}} \cdot [\mathbf{A}] \cdot \left[\underline{\underline{\dot{\boldsymbol{e}}}}\right] \tag{47}$$

$$= \frac{1}{4}\left(2\alpha\dot{e}_T^2 + 4\gamma\dot{e}_T\dot{e}_S + 2\beta\dot{e}_S^2\right). \tag{48}$$

$$= \frac{1}{2}\left[\gamma\left(\dot{e}_T + \dot{e}_S\right)^2 + (\alpha - \gamma)\dot{e}_T^2 + (\beta - \gamma)\dot{e}_S^2\right], \tag{49}$$

from which it is apparent that

$$\mathcal{D} \geq 0 \quad \text{if} \quad \begin{cases} \alpha \geq \gamma \\ \beta \geq \gamma \\ \gamma \geq 0 \end{cases}. \tag{50}$$

Note that this is a more general set of constraints than were derived in Smith and McWilliams (2003) for their study of transverse anisotropic viscosity, which is obscured by their choice to use the full eddy stress instead of only the trace-free part. Also note that the constraints in Equation 50 are necessary and sufficient conditions if we allow $\dot{e}_S$ and $\dot{e}_T$ to be arbitrary.

Now consider establishing constraints on $\mathcal{D}$ using the eigenanalysis. Since $\mathcal{D} \geq 0$ for all strains is equivalent to insisting that **A** is positive-semidefinite, we know that **A** must have two non-negative eigenvalues. The pseudo-determinant is the product of the eigenvalues, and since they are both non-negative pdet([**A**]) ≥ 0, so that by Equation 44 we must have $\alpha\beta \geq \gamma^2$. This can only be true if α and β are both positive or both negative. But since the trace is the sum of the eigenvalues and thus must be non-negative, by Equation 43 we must have that both $\alpha, \beta \geq 0$. The emergent constraints are thus

$$\mathcal{D} \geq 0 \quad \text{if} \quad \begin{cases} \alpha \geq 0 \\ \beta \geq 0 \\ \alpha\beta \geq \gamma^2 \end{cases}, \tag{51}$$

which we can confirm by substituting back into Equation 48 using the third line as an inequality on β:

$$\mathcal{D} \geq \frac{1}{2}\left(\sqrt{\alpha}\,\dot{e}_T + \frac{\gamma}{\sqrt{\alpha}}\dot{e}_S\right)^2 \tag{52}$$

$$\geq 0. \tag{53}$$

Comparison between Equations 50 and 51 leads to several interesting points of discussion. First, the eigenvalue approach has led to a clearly less restrictive set of constraints on the elements of [**A**], assuming that positive dissipation is indeed the goal. A particularly interesting possibility is that, unlike Equation 50 in which all three elements must be non-negative, Equation 51 actually allows us to choose negative γ while still achieving positive $\mathcal{D}$. Furthermore, since we can now understand the properties of [**A**] explicitly through its eigenvalues, we now have a pathway toward other interesting applications. For example, thus far we have required $\mathcal{D} \geq 0$, which is guaranteed if we simply have two non-negative eigenvalues. But what about $\mathcal{D} < 0$, in which the parameterization

injects energy into the flow in a process commonly known as backscatter? Typically this is achieved by using a negative (scalar) viscosity coefficient, but now we can achieve *anisotropic backscatter* using two negative, unequal eigenvalues. It is also possible to achieve a mixture of both dissipation and backscatter if we have one positive and one negative eigenvalue. One could also control the principal directions of dissipation (e.g., Large et al., 2001) using the eigenvectors of $\mathbf{A}$ along with the eigenvalues. Exploring and testing the full range of possibilities is beyond the scope of this paper, but these ideas are mentioned here to seed future explorations into this topic.

5. A Parameter-Based Testing Framework

Now equipped with a robust set of constraints to ensure $[\mathbf{A}]$ is a dissipative tensor, we now may set up a testing framework to explore the effects of permuting its coefficients. While no numerical tests are performed in the present work, the rest of this manuscript and Appendix A are dedicated to making the implementation and testing of anisotropic viscosity as easy as possible for future studies.

It is presently unclear whether any particular combination of $\{\alpha, \beta, \gamma\}$ is superior to the others, in terms of the resultant flow structures, turbulent spectra, or any other numerical outcomes. The aim here will be to develop a fair testing procedure that would allow us to vary α, β, and γ while keeping the overall dissipation by $[\mathbf{A}]$ consistent across all of the experiments. To begin, we again appeal to the eigenvalue-based approach presented in Section 4 and write the unrolled viscosity tensor using a symmetric eigenvalue decomposition,

$$[\mathbf{A}] = \sum_{k=1}^{2} \lambda_k \underline{\mathbf{v}}_k \underline{\mathbf{v}}_k^{\mathrm{T}}, \tag{54}$$

where each $\underline{\mathbf{v}}_k$ represents the unit eigenvector of $[\mathbf{A}]$ corresponding to the eigenvalue λ_k. This decomposition allows us to express the total dissipation as

$$\mathcal{D} = \frac{1}{4}\left[\underline{\underline{\dot{\boldsymbol{e}}}}\right]^{\mathrm{T}} \cdot \sum_{k=1}^{2} \lambda_k \underline{\mathbf{v}}_k \underline{\mathbf{v}}_k^{\mathrm{T}} \cdot \left[\underline{\underline{\dot{\boldsymbol{e}}}}\right] \tag{55}$$

$$= \frac{1}{4} \sum_{k=1}^{2} \lambda_k \left(\underline{\mathbf{v}}_k^{\mathrm{T}} \cdot \left[\underline{\underline{\dot{\boldsymbol{e}}}}\right]\right)^2, \tag{56}$$

which may further manipulated into the form

$$\mathcal{D} = \frac{1}{4} \sum_{k=1}^{2} \lambda_k \left(\dot{e}_T^2 + \dot{e}_S^2\right) \left(\underline{\mathbf{v}}_k^{\mathrm{T}} \cdot \underline{\boldsymbol{x}}\right)^2 \tag{57}$$

by using the condition $\sigma_D = 0$ to rewrite the strain rate vector in terms of a unit vector $\underline{\boldsymbol{x}}$. This form makes explicit that the dissipation is controlled by the eigenvalues λ_k (i.e., the viscosity coefficients), the square of the total deformation, $\dot{e}_T^2 + \dot{e}_S^2$, and the alignment of $\underline{\boldsymbol{x}}$ with the eigenvectors of $[\mathbf{A}]$. In a turbulent fluid the latter is a stochastic quantity whose expectation must be determined statistically. Equation 57 will serve as the basis for the testing framework we will develop, for which it is useful to consider two cases.

5.1. Isotropic Turbulence

Assuming the fluid is isotropic, so that there is no preferred direction in space insofar as the relation between stress and rate of strain is concerned, we may treat $\underline{\boldsymbol{x}}$ as a random vector whose distribution is invariant under orthogonal transformations such as coordinate rotations and reflections (e.g., its distribution is *spherically symmetric*).

Now consider an arbitrary eigenvector $\underline{\mathbf{v}}_k^T$, which can be straightforwardly extended to an orthonormal frame that we will notate as $\underline{\mathbf{v}}_k^T \rightarrow \left\{\underline{\mathbf{v}}_{(1)}, \underline{\mathbf{v}}_{(2)}\right\}$. Since $\underline{\mathbf{v}}_{(1)}$ differs from $\underline{\mathbf{v}}_{(2)}$ only by a simple rotation, and because $\underline{\boldsymbol{x}}$ has a

spherically symmetric distribution, each random variable $\left(\underline{\mathbf{v}}_{(i)} \cdot \underline{\boldsymbol{x}}\right)^2$ has the same distribution. We will define the expected value of this random variable as $\mu = \mathbf{E}\left(\left(\underline{\mathbf{v}}_{(i)} \cdot \underline{\boldsymbol{x}}\right)^2\right)$. Since $\underline{\boldsymbol{x}}$ is a unit vector, we have

$$1 = \underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{x}} = \sum_{i=1}^{2} \left(\underline{\mathbf{v}}_{(i)} \cdot \underline{\boldsymbol{x}}\right)^2, \tag{58}$$

and taking the expected value of both sides gives

$$1 = \mathbf{E}(1) = \mathbf{E}(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{x}}) \tag{59}$$

$$= \mathbf{E}\left[\sum_{i=1}^{2} \left(\underline{\mathbf{v}}_{(i)} \cdot \underline{\boldsymbol{x}}\right)^2\right] \tag{60}$$

$$= \sum_{i=1}^{2} \mathbf{E}\left(\left(\underline{\mathbf{v}}_{(i)} \cdot \underline{\boldsymbol{x}}\right)^2\right) \tag{61}$$

$$= 2\mu. \tag{62}$$

We thus have $\mu = \mathbf{E}\left(\left(\underline{\mathbf{v}}_k^T \cdot \underline{\boldsymbol{x}}\right)^2\right) = 1/2$, so that either eigenvector is equally likely to be aligned with $\underline{\boldsymbol{x}}$. Taking the expected value of Equation 57, we thus have

$$\mathbf{E}(D) = \mathbf{E}\left[\frac{1}{4}\sum_{k=1}^{2} \lambda_k \left(\dot{e}_T^2 + \dot{e}_S^2\right)\left(\underline{\mathbf{v}}_k^{\mathrm{T}} \cdot \underline{\boldsymbol{x}}\right)^2\right] \tag{63}$$

$$= \frac{1}{4}\left(\dot{e}_T^2 + \dot{e}_S^2\right)\sum_{k=1}^{2} \lambda_k \mathbf{E}\left(\left(\underline{\mathbf{v}}_k^{\mathrm{T}} \cdot \underline{\boldsymbol{x}}\right)^2\right) \tag{64}$$

$$= \frac{1}{8}(\lambda_1 + \lambda_2)\left(\dot{e}_T^2 + \dot{e}_S^2\right). \tag{65}$$

Note that this expression is equivalent to Equation 41 in the limit of isotropic viscosity, when $\lambda_1 = \lambda_2 = 2\nu$.

For isotropic turbulence, the key result of this derivation is that *we would expect the same overall dissipation from two different* $\mathbf{A}$ *as long as the sum of their eigenvalues is the same*. Thus one requirement for a fair exploration of the $\{\alpha, \beta, \gamma\}$ parameter space is that $(\lambda_1 + \lambda_2)$ is consistent across all experiments. Note that this argument does not account for any feedbacks onto $\dot{e}_T$ and $\dot{e}_S$ created by our choices of $\{\alpha, \beta, \gamma\}$, which are impossible to anticipate a priori.

It is useful to note here that the eigenvalues (Equation 42) are invariant to coordinate rotations, and furthermore, that the role of α and β inside Equation 42 are interchangeable—it does not matter whether α or β is larger, as they only appear together as a sum or as a squared difference. Also note that the strain terms multiplying α and β, which are $\dot{e}_T$ and $\dot{e}_S$, respectively, are *not* rotationally invariant but the total deformation formed by the sum of their squares, $\left(\dot{e}_T^2 + \dot{e}_S^2\right)^{1/2}$, is. This means that any rotation of the coordinate system will affect the magnitude of $\dot{e}_T$ and $\dot{e}_S$ inversely relative to each other without affecting the expected dissipation (Equation 65) at all. Because in the limit of isotropic turbulence $\dot{e}_T$ and $\dot{e}_S$ are equally likely to occur, we thus can conclude that there is no meaningful difference in choosing α and β to be different from each other, and thus it suffices to set $\alpha = \beta$.

Setting $\alpha = \beta$ allows all of the permissible forms of $\mathbf{A}$ to be described by a very simple test parameter

$$\xi_{iso} = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} = \frac{\gamma}{\alpha}, \tag{66}$$

which has the properties that it is rotationally invariant and ranges between −1 and 1 for dissipative flows obeying Equation 51. A testing framework could thus involve a series of simulations for isotropic turbulence, varying ξ_{iso} across its range to test its effects on key flow metrics (numerical stability, turbulent spectra, dispersion, etc.). With regard to how to choose α, note that the expected dissipation (Equation 65) is set by the sum of the eigenvalues, $\lambda_1 + \lambda_2 = 4\alpha$, which would be equivalent to simply using isotropic viscosity with coefficient $\nu = \alpha$. One might thus consider following any well-tested procedure for setting up a simulation with isotropic viscosity, for example, a doubly-periodic quasigeostrophic simulation with constant lateral buoyancy gradient, in which α could be prescribed according to QG Leith viscosity (S. D. Bachman, Fox-Kemper, & Pearson, 2017).

5.2. Anisotropic Turbulence

Removing the assumption of isotropic turbulence implies that there are meaningful differences between α and β, which would require an expanded testing protocol with extra parameters. However, many of the derivations used above for the isotropic turbulence case no longer apply, as we cannot assume $\dot{e}_T$ and $\dot{e}_S$ have the same probability distribution and the dissipation is no longer proportional to the sum of the eigenvalues as in Equation 65. Rather, we must use the form for the dissipation in Equation 57, and since it involves products of both the eigenvalues and eigenvectors of [**A**] it is not immediately obvious how to set up a testing framework that holds dissipation constant across all experiments.

We gain insight as follows. Because $\mathcal{D}$ is rotationally invariant and the λ_k and $\underline{\mathbf{v}}_k$ are unknown and arbitrary at this point, we are free to rotate the coordinate system however we choose to help us understand how to set up a fair testing framework. So consider a rotation such that the eigenvectors become aligned with the abcissa and ordinate in the Cartesian coordinate system, where we denote the rotated strain rate elements with a tilde. Because the eigenvalues are rotationally invariant, Equation 57 takes the form

$$\mathcal{D} = \frac{1}{4}\left(\lambda_1 \tilde{\dot{e}}_T^2 + \lambda_2 \tilde{\dot{e}}_S^2\right), \tag{67}$$

which clarifies that in order to keep the dissipation constant across all experiments both eigenvalues need to be held constant (note that this requirement applies to each eigenvalue individually, rather than simply holding their sum constant as we did for the isotropic turbulence case). Given the formulas for the eigenvalues in Equation 42, the simplest way to do this is to define two constants,

$$C_1 = \alpha + \beta, \tag{68}$$

$$C_2 = \sqrt{(\alpha - \beta)^2 + 4\gamma^2}, \tag{69}$$

along with a parameter that we will allow to vary in the testing framework,

$$\xi_{aniso} = \frac{\gamma}{\sqrt{\alpha\beta}}. \tag{70}$$

Note the obvious similarity between Equation 70 and ξ_{iso} from the isotropic turbulence case, with the important distinction that Equation 70 is not rotationally invariant. As long as the $\{\alpha, \beta, \gamma\}$ coefficients are chosen such that they satisfy the constraints in Equation 51, and we are not considering the case when $\mathcal{D} = 0$, ξ_{aniso} is bounded such that $-1 < \xi_{aniso} < 1$, which defines the limiting range for the experiments. The basic case of isotropic viscosity is recovered when $\xi_{aniso} = 0$ and $C_2 = 0$. Finally, the above expressions for C_1, C_2, and ξ_{aniso} yield formulas for the viscosity coefficients that are easily applied in a model:

$$\alpha = \frac{1}{2}C_1 \pm \frac{1}{2}\sqrt{C_2^2 - \frac{\xi_{aniso}^2\left(C_2^2 - C_1^2\right)}{\xi_{aniso}^2 - 1}}, \tag{71}$$

$$\beta = \frac{1}{2}C_1 \mp \frac{1}{2}\sqrt{C_2^2 - \frac{\xi_{aniso}^2\left(C_2^2 - C_1^2\right)}{\xi_{aniso}^2 - 1}}, \tag{72}$$

$$\gamma = \pm\frac{1}{2}\sqrt{\frac{\xi_{aniso}^2\left(C_2^2 - C_1^2\right)}{\xi_{aniso}^2 - 1}}. \tag{73}$$

Note that we allow both signs of the square root to be used in all of the above expressions, such that γ can be either positive or negative choice and each value of ξ_{aniso} gives two possible choices for both α and β.

Using these parameters the testing protocol would then proceed as follows. The modeler would first specify the two eigenvalues in an appropriate fashion for the experimental configuration, as in the QG Leith example suggested in Section 5.1. The eigenvalues (Equation 42) would then be used to solve for C_1 and C_2. The user would then conduct a series of simulations varying ξ_{aniso} between -1 and 1, assessing whether any particular choice yields more desirable flow behavior (the specifics of which are left intentionally vague here, since it would depend heavily on both the flow regime and user preference). Because ξ_{aniso} is not rotationally invariant, the modeler should be very deliberate and explicit about whatever direction the anisotropy is imposed in the flow. Any optimal choice for ξ_{aniso} (and hence the structure of $\mathbf{A}$) would then only apply in this direction, so if that $\mathbf{A}$ was applied in a prognostic model it would need to be rotated to accommodate anisotropy in any other direction.

6. Conclusion

Eddy viscosity is a ubiquitous yet mysterious parameterization owing to the difficulty of working with its true form as a fourth-order tensor. Previous studies have identified the potential to study tensorial eddy viscosity by rewriting the underlying algebra in simplified form (i.e., reducing the linear system from fourth-to second-order), but only in an informal way. Though this approach is useful, it can yield misleading results if not done carefully.

Eigenanalysis is an extraordinarily powerful tool used across nearly all scientific disciplines, but has heretofore not been aimed at the problem of understanding eddy viscosity. Traditional eigenanalysis, as applied on matrices and vectors, features concepts such as sign-definiteness that are perfect analogs to some of the properties we desire for the viscosity tensor. However, the disciplines where one might find more comfort with eigenanalysis on higher-order tensors (e.g., elasticity, structural mechanics, etc.) are fairly remote from geophysical fluid dynamics.

This paper has aimed to cross-pollinate these techniques into a new look at eddy viscosity, in the hopes of reigniting consideration about how one could improve modern ocean models. It has been shown that Mandel notation is necessary to transfer all the spectral properties of a fourth-order tensor into its unrolled second-order form. Once the unrolling is complete, it becomes a fairly straightforward exercise to expose the true constraints on the viscosity coefficients. This opens several avenues for further exploration, such as evaluating potential applications and advantages of directionally-oriented viscosity, or marrying directional backscatter and viscosity into a single parameterization (which, for example, could be useful for simulating the process of jet sharpening).

It should be emphasized that, from a physical standpoint, representing the eddy stress-strain relation with Hooke's Law is questionable, since momentum and kinetic energy transfers do not necessarily have to occur locally (e.g., via waves). A fairly significant body of research suggests that inclusion of higher-order effects such as "eddy memory" lead to superior models of turbulent behavior, and various fluid and parameterization models have been developed to explore this possibility (e.g., Rivlin, 1957; Holm et al., 2005; Anstey & Zanna, 2017; S. D. Bachman et al., 2018). Thus, while the eddy stresses and strains themselves are observable in the real ocean, the elements of $\mathbf{A}$ are not, at least in the sense that they may not be physically meaningful. For the purposes of this paper, the tensor $\mathbf{A}$ is an artificial construct that we wish to manipulate to yield numerically stable model solutions with excellent realism. The solutions we may end up with are not necessarily unique, and different combinations of (α, β, γ) may yield identical effects on the flow. Finding any superior combination to isotropic viscosity is the challenge that has prevented anisotropic $\mathbf{A}$ from becoming mainstream, and this work exists to make the search a little easier.

A potential testing protocol has been presented here to lay groundwork for future explorations into this topic, and it is the author's hope that this work will invigorate further research. It should fit seamlessly into the current zeitgeist surrounding both LES-inspired viscous closures and backscatter in eddying climate simulations, which are presently considered state-of-the-art. Though the presentation in this paper is slightly tilted toward ocean

modeling, the mathematics are not specific to any particular type of fluid or coordinate system, and it could find a home in other disciplines as well.

Appendix A: Notes on Model Implementation

A1. Primitive Equation Models

The time tendency of the horizontal velocity, $\underline{\mathbf{u}}_h$, due to the deviatoric stress tensor, $\underline{\underline{\boldsymbol{\sigma}}}$, can be succinctly written as

$$\frac{\partial \underline{\mathbf{u}}_h}{\partial t} = \nabla \cdot \boldsymbol{\sigma}, \tag{A1}$$

where by Equation 4 and the trace-free condition, $\sigma_D = 0$, we have

$$\underline{\underline{\boldsymbol{\sigma}}} = \begin{bmatrix} \frac{1}{2}(\tilde{\tau}_{11} - \tilde{\tau}_{22}) & \tilde{\tau}_{12} \\ \tilde{\tau}_{12} & \frac{1}{2}(\tilde{\tau}_{22} - \tilde{\tau}_{11}) \end{bmatrix} \tag{A2}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \tag{A3}$$

$$= \frac{1}{2}\begin{bmatrix} (\sigma_D + \sigma_T) & \sigma_S \\ \sigma_S & (\sigma_D - \sigma_T) \end{bmatrix} \tag{A4}$$

$$= \frac{1}{2}\begin{bmatrix} \sigma_T & \sigma_S \\ \sigma_S & -\sigma_T \end{bmatrix}. \tag{A5}$$

Then using Equation 40 to substitute for the stresses, the tendencies are

$$\frac{\partial \underline{\mathbf{u}}_h}{\partial t} = \frac{1}{2}\nabla \cdot \begin{bmatrix} \sigma_T & \sigma_S \\ \sigma_S & -\sigma_T \end{bmatrix} \tag{A6}$$

$$= \begin{bmatrix} \partial_x\left(\alpha \dot{e}_T + \gamma \dot{e}_S\right) + \partial_y\left(\gamma \dot{e}_T + \beta \dot{e}_S\right) \\ \partial_x\left(\gamma \dot{e}_T + \beta \dot{e}_S\right) - \partial_y\left(\alpha \dot{e}_T + \gamma \dot{e}_S\right) \end{bmatrix}. \tag{A7}$$

Note that the discretization of Equation A7 does not occur quite as naturally on the Arakawa C-grid as it would for standard isotropic viscosity, when we must have $\gamma = 0$.

A2. Two-Dimensional Spectral Models

Two-dimensional models offer useful testbeds to study the behavior of turbulence and turbulence parameterizations. The defining feature of such models is that the entirety of the dynamic system can be described by a single variable, the vorticity $q = v_x - u_y$, which is related to the horizontal velocities and streamfunction, ψ, by the relations

$$q = \nabla^2 \psi, \qquad u = -\psi_y, \qquad v = \psi_x. \tag{A8}$$

The streamfunction will be the most useful variable for framing the implementation of anisotropic viscosity. Consider a Fourier transform

$$\psi = \hat{\psi} e^{-i(kx+ly-\omega t)}, \tag{A9}$$

for a horizontal wave vector, $\underline{\mathbf{K}} = (k,l)$, and frequency, ω. Then the Fourier transforms of the velocities can be written

$$\hat{u} = il\hat{\psi}, \qquad \hat{v} = -ik\hat{\psi}, \tag{A10}$$

which yields straightforward expressions for the strain rate components,

$$\hat{\hat{e}}_T = \widehat{u_x - v_y} = -2\widehat{\psi_{xy}} = 2kl\hat{\psi}, \tag{A11}$$

$$\hat{\hat{e}}_S = \widehat{v_x + u_y} = \widehat{\psi_{xx}} - \widehat{\psi_{yy}} = \left(-k^2 + l^2\right)\hat{\psi}. \tag{A12}$$

The transformed eddy stresses, σ_T and σ_S, appearing on the right hand side of Equation A7 are then

$$\hat{\sigma}_T = \alpha\hat{\hat{e}}_T + \gamma\hat{\hat{e}}_S = \left(2\alpha kl + \gamma\left(-k^2 + l^2\right)\right)\hat{\psi}, \tag{A13}$$

$$\hat{\sigma}_S = \gamma\hat{\hat{e}}_T + \beta\hat{\hat{e}}_S = \left(2\gamma kl + \beta\left(-k^2 + l^2\right)\right)\hat{\psi}. \tag{A14}$$

The vorticity tendency can be obtained by simple differentiation and algebra on Equation A6, such that

$$\frac{\partial q}{\partial t} = -\partial_{xy}\sigma_T + \frac{1}{2}(\partial_{xx} - \partial_{yy})\sigma_S \tag{A15}$$

$$\frac{\partial \hat{q}}{\partial t} = \left(2\alpha k^2 l^2 + 2\gamma kl\left(-k^2 + l^2\right) + \frac{1}{2}\beta\left(-k^2 + l^2\right)^2\right)\hat{\psi} \tag{A16}$$

$$= -\left(2\alpha k^2 l^2 + 2\gamma kl\left(-k^2 + l^2\right) + \frac{1}{2}\beta\left(-k^2 + l^2\right)^2\right)\left(k^2 + l^2\right)^{-2}\hat{q}. \tag{A17}$$

Note that the implementation of anisotropic viscosity in a layered quasigeostrophic (QG) model would follow the same derivation through (A16), but the relationship between q and ψ in Equation A8 in the QG system would include an extra term representing the vortex stretching and needs to be discretized as a matrix. A QG model implementation may prefer to use the form in Equation A16, which can be done on a layer-by-layer basis and does not incur the extra expense of inverting the matrix to recover q.

Data Availability Statement

No models, data, or analysis software were used to generate the results presented in this paper.

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