



Role of Subgradients in Variational Analysis of Polyhedral Functions

Nguyen T. V. Hang¹ · Woosuk Jung² · Ebrahim Sarabi²

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Abstract

Understanding the role that subgradients play in various second-order variational analysis constructions can help us uncover new properties of important classes of functions in variational analysis. Focusing mainly on the behavior of the second subderivative and subgradient proto-derivative of polyhedral functions, i.e., functions with polyhedral convex epigraphs, we demonstrate that choosing the underlying subgradient, utilized in the definitions of these concepts, from the relative interior of the subdifferential of polyhedral functions ensures stronger second-order variational properties such as strict twice epi-differentiability and strict subgradient proto-differentiability. This allows us to characterize continuous differentiability of the proximal mapping and twice continuous differentiability of the Moreau envelope of polyhedral functions. We close the paper with proving the equivalence of metric regularity and strong metric regularity of a class of generalized equations at their nondegenerate solutions.

Keywords Polyhedral functions · Reduction lemma · Nondegenerate solutions · Strict proto-differentiability · Strict twice Epi-differentiability · Proximal mappings · Strong metric regularity

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✉ Ebrahim Sarabi
sarabim@miamioh.edu

Nguyen T. V. Hang
ntvhang@math.ac.vn

Woosuk Jung
jungw2@miamioh.edu

¹ Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam

² Department of Mathematics, Miami University, Oxford, OH 45065, USA

1 Introduction

Second-order variational constructions such as the second subderivative and subgradient proto-derivative play an important role in parametric optimization and convergence analysis of important numerical algorithms such as the augmented Lagrangian method [6, 30]. Given an extended-real-valued function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$, these second-order variational constructions are defined at a point (x, v) in the graph of the subgradient mapping of f . One may wonder what impacts the selection of the subgradient v can have in these constructions. Our main goal in this paper is to study the underlying role that the selection of a subgradient can play for such constructions. To achieve this goal, we focus mainly on a particular class of convex functions called polyhedral functions. Recall that a proper function $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is called polyhedral if its epigraph is a polyhedral convex set. Given $\bar{z} \in \mathbb{R}^m$ with $g(\bar{z})$ finite, consider a subgradient $\bar{\lambda} \in \partial g(\bar{z})$. It is well known (cf. [31, Proposition 13.9]) that the polyhedral function g is twice epi-differentiable at \bar{z} for $\bar{\lambda}$, that the subgradient mapping ∂g is proto-differentiable at \bar{z} for $\bar{\lambda}$, and that its proximal mapping is directionally differentiable at $\bar{z} + \bar{\lambda}$; see Sects. 3 and 4 for the definitions of these concepts.

Should we expect further properties if, in addition, we assume that $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$? This is the main question that we are going to investigate for this class of convex functions. Note that such a relative interior condition has been utilized in several studies related to different numerical methods, including the partial smoothness [14] and the \mathcal{U} -Lagrangian of convex functions [12]. Therefore understanding the role that is played by this condition in second-order variational analysis can lead to stronger stability properties for important classes of functions in variational analysis as demonstrated in Sects. 4 and 5.

Following this question, we uncover new second-order variational properties of polyhedral functions, including strict twice epi-differentiability and strict subgradient proto-differentiability, under this extra assumption. These findings allow us to achieve a useful characterization of continuous differentiability of the proximal mapping and twice continuous differentiability of the Moreau envelope of polyhedral functions. As an important application, we turn to study stability properties of the solution mapping to the generalized equation

$$0 \in \psi(x) + \partial g(x), \quad (1)$$

where $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a \mathcal{C}^1 mapping and $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function. Our interest is mainly in examining the relationship between metric regularity and strong metric regularity for (1). The seminal work of Donchev and Rockafellar in [3] demonstrated that these properties are equivalent for (1) when the polyhedral function g is the indicator function of a polyhedral convex set. Employing our new developments under the relative interior condition, we are going to show that if \bar{x} is a *nondegenerate* solution to the generalized equation (1), meaning that it satisfies the condition

$$-\psi(\bar{x}) \in \text{ri } \partial g(\bar{x}), \quad (2)$$

then under some verifiable assumptions the solution mapping to the canonical perturbation of (1) has a Lipschitz continuous single-valued localization, which is continuously differentiable. The latter smoothness of a localization of solution mappings of gen-

eralized equations resembles a similar conclusion from the classical inverse mapping theorem. Note that the nondegeneracy condition in (2) was taken from [5, Definition 3.4.1] and is different from the nondegeneracy constraint qualification in the sense of [2, Definition 4.70].

The rest of the paper is organized as follows. Section 2 contains definitions of important concepts, used in this paper. We also establish some properties of polyhedral functions. Section 3 begins with a new proof of the reduction lemma for polyhedral functions and then we present its important consequences in various second-order variational constructions. In particular, we show that under the relative interior condition, the subgradient mappings of polyhedral functions are strictly proto-differentiable. Section 4 is devoted to study strict twice epi-differentiability of polyhedral functions. As an important consequence, we characterize continuous differentiability of the proximal mapping and twice continuous differentiability of the Moreau envelope of polyhedral functions. The final section, Sect. 5, concerns the equivalence of metric regularity and strong metric regularity for the generalized equation (1). Using this equivalence and (2), we present sufficient conditions for a smooth single-valued localization of the solution mapping to the canonical perturbation of (1).

2 Notation and Preliminary Results

In what follows, we denote by \mathbb{B} the closed unit ball in the space in question and by $\mathbb{B}_r(x) := x + r\mathbb{B}$ the closed ball centered at x with radius $r > 0$. In the product space $\mathbb{R}^n \times \mathbb{R}^m$, we use the norm $\|(w, u)\| = \sqrt{\|w\|^2 + \|u\|^2}$ for any $(w, u) \in \mathbb{R}^n \times \mathbb{R}^m$. Given a nonempty set $C \subset \mathbb{R}^n$, the symbols $\text{int } C$, $\text{ri } C$, $\text{cone } C$, and $\text{conv } C$ signify its interior, relative interior, convex conic hull, and convex hull, respectively. For any set C in \mathbb{R}^n , its indicator function is defined by $\delta_C(x) = 0$ for $x \in C$ and $\delta_C(x) = \infty$ otherwise. We denote by P_C the projection mapping onto C and by $\text{dist}(x, C)$ the distance between $x \in \mathbb{R}^n$ and a set C . For a vector $w \in \mathbb{R}^n$, the subspace $\{tw \mid t \in \mathbb{R}\}$ is denoted by $[w]$. The domain and range of a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are defined, respectively, by $\text{dom } F = \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ and $\text{rge } F = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ with } y \in F(x)\}$.

In this paper, the convergence of a family of sets is always understood in the sense of Painlevé-Kuratowski (cf. [31, Definition 4.1]). This means that the inner limit set of a parameterized family of sets $\{C^t\}_{t>0}$ in \mathbb{R}^d , denoted $\liminf_{t \searrow 0} C^t$, is the set of points x such that for every sequence $t_k \searrow 0$, x is the limit of a sequence of points $x^{t_k} \in C^{t_k}$. The outer limit set of this family of sets, denoted $\limsup_{t \searrow 0} C^t$, is the set of points x such that there exist sequences $t_k \searrow 0$ and $x^{t_k} \in C^{t_k}$ such that $x^{t_k} \rightarrow x$ as $k \rightarrow \infty$. A sequence $\{f^k\}_{k \in \mathbb{N}}$ of functions $f^k : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is said to *epi-converge* to a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ if we have $\text{epi } f^k \rightarrow \text{epi } f$ as $k \rightarrow \infty$, where $\text{epi } f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq \alpha\}$ is the epigraph of f ; see [31, Definition 7.1] for more details on the epi-convergence of a sequence of extended-real-valued functions.

Given a nonempty set $\Omega \subset \mathbb{R}^n$ with $\bar{x} \in \Omega$, the tangent cone to Ω at \bar{x} , denoted $T_\Omega(\bar{x})$, is defined by

$$T_\Omega(\bar{x}) = \limsup_{t \searrow 0} \frac{\Omega - \bar{x}}{t}. \quad (3)$$

The regular/Fréchet normal cone $\widehat{N}_\Omega(\bar{x})$ to Ω at \bar{x} is defined by $\widehat{N}_\Omega(\bar{x}) = T_\Omega(\bar{x})^*$, the polar of the tangent cone (3). The (limiting/Mordukhovich) normal cone $N_\Omega(\bar{x})$ to Ω at \bar{x} is the set of all vectors $\bar{v} \in \mathbb{R}^n$ for which there exist sequences $\{x^k\}_{k \in \mathbb{N}}$ and $\{v^k\}_{k \in \mathbb{N}}$ with $v^k \in \widehat{N}_\Omega(x^k)$ such that $(x^k, v^k) \rightarrow (\bar{x}, \bar{v})$. When Ω is convex, both normal cones boil down to that of convex analysis. Given a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ and a point $\bar{x} \in \mathbb{R}^n$ with $f(\bar{x})$ finite, the subderivative function $df(\bar{x}) : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is defined by

$$df(\bar{x})(w) = \liminf_{\substack{t \searrow 0 \\ w' \rightarrow w}} \frac{f(\bar{x} + tw') - f(\bar{x})}{t}.$$

A vector $v \in \mathbb{R}^n$ is called a subgradient of f at \bar{x} if $(v, -1) \in N_{\text{epi } f}(\bar{x}, f(\bar{x}))$. The set of all subgradients of f at \bar{x} is denoted by $\partial f(\bar{x})$. When f is convex, the latter set reduces to the well-known subdifferential of convex analysis (cf. [26, page 214]), namely $v \in \partial f(\bar{x})$ if $f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle$ for any $x \in \mathbb{R}^n$.

As pointed out earlier, a proper function $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is called *polyhedral* if $\text{epi } g$ is a polyhedral convex set. According to [31, Theorem 2.49], this class of convex functions enjoys the representation

$$g(z) = \begin{cases} \max_{j \in J} \{ \langle a^j, z \rangle - \alpha_j \} & \text{if } z \in \text{dom } g, \\ \infty & \text{otherwise,} \end{cases}$$

where $J = \{1, \dots, l\}$ for some $l \in \mathbb{N}$, $a^j \in \mathbb{R}^m$ and $\alpha_j \in \mathbb{R}$ for all $j \in J$, and where $\text{dom } g = \{z \in \mathbb{R}^m \mid g(z) < \infty\}$ is a polyhedral convex set with the representation

$$\text{dom } g = \{z \in \mathbb{R}^m \mid \langle b^i, z \rangle \leq \beta_i, i \in I = \{1, \dots, s\}\}, \tag{4}$$

where $s \in \mathbb{N}$ and $b^i \in \mathbb{R}^m$ and $\beta_i \in \mathbb{R}$ for all $i \in I$. Thus, we can equivalently express a polyhedral function g as

$$g(z) = \max_{j \in J} \{ \langle a^j, z \rangle - \alpha_j \} + \delta_{\text{dom } g}(z), \quad z \in \mathbb{R}^m. \tag{5}$$

It was observed in [18, Proposition 3.2] that $\text{dom } g$ can be expressed as the finite union of the polyhedral convex sets $C_j, j \in J$, defined by

$$\begin{aligned} C_j &= \{z \in \text{dom } g \mid g(z) = \langle a^j, z \rangle - \alpha_j\} \\ &= \{z \in \text{dom } g \mid \langle a^i - a^j, z \rangle \leq \alpha_i - \alpha_j, i \in J\}. \end{aligned}$$

Pick $\bar{z} \in \text{dom } g$ and define the sets of active indices at \bar{z} corresponding to the representation (4) and to the expression of $\text{dom } g$ via the finite union of C_j 's by

$$I(\bar{z}) = \{i \in I \mid \langle b^i, \bar{z} \rangle = \beta_i\} \quad \text{and} \quad J(\bar{z}) = \{j \in J \mid \bar{z} \in C_j\}. \tag{6}$$

These sets allow us to conclude from [18, Proposition 3.3] that the subdifferential of g at \bar{z} can be calculated as

$$\partial g(\bar{z}) = \text{conv} \{a^j \mid j \in J(\bar{z})\} + \text{cone} \{b^i \mid i \in I(\bar{z})\}, \quad (7)$$

which tells us that any $\bar{\lambda} \in \partial g(\bar{z})$ can be written as

$$\bar{\lambda} = \sum_{j \in J(\bar{z})} \bar{\sigma}_j a^j + \sum_{i \in I(\bar{z})} \bar{\tau}_i b^i \quad \text{with} \quad \bar{\sigma}_j, \bar{\tau}_i \geq 0 \quad \text{and} \quad \sum_{j \in J(\bar{z})} \bar{\sigma}_j = 1. \quad (8)$$

For simplicity, we denote by $\bar{\sigma} \in \mathbb{R}^l$ and $\bar{\tau} \in \mathbb{R}^s$ the vectors with components $\bar{\sigma}_j$ and $\bar{\tau}_i$, respectively, where $\bar{\sigma}_j$ is taken from (8) if $j \in J(\bar{z})$ and $\bar{\sigma}_j = 0$ if $j \notin J(\bar{z})$ and likewise, $\bar{\tau}_i$ is taken from (8) if $i \in I(\bar{z})$ and $\bar{\tau}_i = 0$ if $i \notin I(\bar{z})$. Pick the given representation of $\bar{\lambda}$ in (8) and define the sets of positive coefficients at \bar{z} for $\bar{\sigma}$ and $\bar{\tau}$, respectively, by

$$J_+(\bar{z}, \bar{\sigma}) = \{j \in J(\bar{z}) \mid \bar{\sigma}_j > 0\} \quad \text{and} \quad I_+(\bar{z}, \bar{\tau}) = \{i \in I(\bar{z}) \mid \bar{\tau}_i > 0\}. \quad (9)$$

Note that the representation of $\bar{\lambda}$ in (8) is not unique. We demonstrate in the next result that all these representations of $\bar{\lambda}$ enjoy an interesting property, important for our proof of the reduction lemma in the next section. A similar result was established in [18, Theorem 3.4] but the neighborhood obtained therein depends on a chosen decomposition of the subgradient $\bar{\lambda}$ in (8). Next, we show that such a neighborhood can be chosen to be independent of a given representation of $\bar{\lambda}$. We also simplify the proof presented in [18] significantly.

Lemma 2.1 *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and that $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then there exists $r > 0$ such that for any decomposition of $\bar{\lambda}$ as (8) and any $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$, we have*

$$J_+(\bar{z}, \bar{\sigma}) \subset J(z) \quad \text{and} \quad I_+(\bar{z}, \bar{\tau}) \subset I(z). \quad (10)$$

Proof Suppose by contradiction that for each $k \in \mathbb{N}$, there are a decomposition of $\bar{\lambda}$ as

$$\bar{\lambda} = \sum_{j \in J(\bar{z})} \bar{\sigma}_j^k a^j + \sum_{i \in I(\bar{z})} \bar{\tau}_i^k b^i \quad \text{with} \quad \bar{\sigma}_j^k, \bar{\tau}_i^k \geq 0 \quad \text{and} \quad \sum_{j \in J(\bar{z})} \bar{\sigma}_j^k = 1, \quad (11)$$

and $(z^k, \lambda^k) \in \text{gph } \partial g$ such that $(z^k, \lambda^k) \rightarrow (\bar{z}, \bar{\lambda})$ as $k \rightarrow \infty$ with $J_+^k \not\subset J(z^k)$ or $I_+^k \not\subset I(z^k)$, where $J_+^k := J_+(\bar{z}, \bar{\sigma}^k)$ and $I_+^k := I_+(\bar{z}, \bar{\tau}^k)$ are defined via (9). Since $z^k \rightarrow \bar{z}$ as $k \rightarrow \infty$, the inclusions $J(z^k) \subset J(\bar{z})$ and $I(z^k) \subset I(\bar{z})$ hold for all k sufficiently large. Passing to a subsequence if necessary, we can assume that there exist subsets $\bar{J} \subset J(\bar{z})$ and $\bar{I} \subset I(\bar{z})$ such that

$$J(z^k) = \bar{J} \quad \text{and} \quad I(z^k) = \bar{I} \quad \text{for all } k \in \mathbb{N},$$

which, together with (7), lead us to

$$\partial g(z^k) = \text{conv} \{a^j \mid j \in \bar{J}\} + \text{cone} \{b^i \mid i \in \bar{I}\} =: \Omega \quad \text{for all } k \in \mathbb{N}.$$

Since $\lambda^k \in \partial g(z^k) = \Omega$ and $\lambda^k \rightarrow \bar{\lambda}$ as $k \rightarrow \infty$, we arrive at $\bar{\lambda} \in \Omega = \partial g(z^k)$ for all k sufficiently large. Fix such a $k \in \mathbb{N}$ and deduce from $\bar{\lambda} \in \partial g(z^k)$ that

$$\langle \bar{\lambda}, \bar{z} - z^k \rangle \leq g(\bar{z}) - g(z^k). \tag{12}$$

By the decomposition in (11), we obtain

$$\begin{aligned} \langle \bar{\lambda}, z^k - \bar{z} \rangle &= \sum_{j \in J(\bar{z})} \bar{\sigma}_j^k \langle a^j, z^k - \bar{z} \rangle + \sum_{i \in I(\bar{z})} \bar{\tau}_i^k \langle b^i, z^k - \bar{z} \rangle \\ &= \sum_{j \in J(\bar{z})} \bar{\sigma}_j^k (\langle a^j, z^k \rangle - \alpha_j - g(\bar{z})) + \sum_{i \in I(\bar{z})} \bar{\tau}_i^k (\langle b^i, z^k \rangle - \beta_i), \\ &\leq \sum_{j \in J(\bar{z})} \bar{\sigma}_j^k (g(z^k) - g(\bar{z})), \end{aligned} \tag{13}$$

where the second equality results from (6) and the last inequality comes from the fact that $z^k \in \text{dom } g$, combined with (4)–(5). If $J_+^k \not\subset J(z^k)$, there exists $j_0 \in J_+^k$ such that $z^k \notin C_{j_0}$, meaning that

$$\bar{\sigma}_{j_0}^k (\langle a^{j_0}, z^k \rangle - \alpha_{j_0} - g(\bar{z})) < \bar{\sigma}_{j_0}^k (g(z^k) - g(\bar{z})).$$

If $I_+^k \not\subset I(z^k)$, we find $i_0 \in I_+^k$ such that $\langle b^{i_0}, z^k \rangle < \beta_{i_0}$, which implies that

$$\bar{\tau}_{i_0}^k (\langle b^{i_0}, z^k \rangle - \beta_{i_0}) < 0,$$

which tells us that in both cases the last inequality in (13) is strict. Thus, our assumption that either $J_+^k \not\subset J(z^k)$ or $I_+^k \not\subset I(z^k)$, together with the last condition in (11), yields

$$\langle \bar{\lambda}, z^k - \bar{z} \rangle < \sum_{j \in J(\bar{z})} \bar{\sigma}_j^k (g(z^k) - g(\bar{z})) = g(z^k) - g(\bar{z}),$$

which contradicts (12) and therefore completes the proof. □

We next record some first-order variational properties of polyhedral functions, important for our developments in this paper. Recall from [31, Exercise 6.47] that for any polyhedral convex set $C \subset \mathbb{R}^n$ and $\bar{x} \in C$, we can find a neighborhood \mathcal{O} of $0 \in \mathbb{R}^n$ for which we have

$$T_C(\bar{x}) \cap \mathcal{O} = (C - \bar{x}) \cap \mathcal{O}. \tag{14}$$

Proposition 2.1 *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and $\bar{z} \in \text{dom } g$. Then the following properties hold.*

(a) The domain of the subderivative function $dg(\bar{z})$ can be calculated by

$$\text{dom } dg(\bar{z}) = T_{\text{dom } g}(\bar{z}) = \bigcup_{j \in J(\bar{z})} T_{C_j}(\bar{z}).$$

(b) If $w \in T_{C_j}(\bar{z})$ for some $j \in J(\bar{z})$, then we have $dg(\bar{z})(w) = \langle a^j, w \rangle$. Moreover, there exists $r > 0$ such that for any $w \in T_{\text{dom } g}(\bar{z}) \cap \mathbb{B}_r(0)$, the representation

$$g(\bar{z} + w) = g(\bar{z}) + dg(\bar{z})(w) \quad (15)$$

holds.

Proof The first equality in (a) was established in [31, Proposition 10.21]. The alternative representation of $\text{dom } dg(\bar{z})$ as union of $T_{C_j}(\bar{z})$, $j \in J(\bar{z})$, and the first claim in (b) can be found again in the proof of [31, Proposition 10.21]. To prove (15), recall that for any $j \in J(\bar{z})$, C_j is a polyhedral convex set. Employing (14) for these sets, we can find $r > 0$ such that

$$T_{C_j}(\bar{z}) \cap \mathbb{B}_r(0) = (C_j - \bar{z}) \cap \mathbb{B}_r(0) \quad \text{for all } j \in J(\bar{z}).$$

Pick any $w \in T_{\text{dom } g}(\bar{z}) \cap \mathbb{B}_r(0)$, and conclude from (a) that $w \in T_{C_{j_0}}(\bar{z}) \cap \mathbb{B}_r(0)$ for some $j_0 \in J(\bar{z})$ and from the first claim in (b) that $dg(\bar{z})(w) = \langle a^{j_0}, w \rangle$. Thus, we get $\bar{z} + w \in C_{j_0}$, which, together with the definition of C_{j_0} and (6), leads us to

$$g(\bar{z} + w) = \langle a^{j_0}, \bar{z} + w \rangle - \alpha_{j_0} = g(\bar{z}) + \langle a^{j_0}, w \rangle = g(\bar{z}) + dg(\bar{z})(w),$$

and hence completes the proof. \square

Given a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ and a point $\bar{x} \in \mathbb{R}^n$ with $f(\bar{x})$ finite, the critical cone of f at \bar{x} for \bar{v} with $\bar{v} \in \partial f(\bar{x})$ is defined by

$$K_f(\bar{x}, \bar{v}) = \{w \in \mathbb{R}^n \mid \langle \bar{v}, w \rangle = df(\bar{x})(w)\}.$$

When $f = \delta_\Omega$, where Ω is a nonempty subset of \mathbb{R}^n , the critical cone of δ_Ω at \bar{x} for \bar{v} is denoted by $K_\Omega(\bar{x}, \bar{v})$. In this case, the above definition of critical cone of a function boils down to the well-known concept of critical cone to a set (see [4, page 109]), namely $K_\Omega(\bar{x}, \bar{v}) = T_\Omega(\bar{x}) \cap [\bar{v}]^\perp$ due to $d\delta_\Omega(\bar{x}) = \delta_{T_\Omega(\bar{x})}$. If f is convex and $\partial f(\bar{x}) \neq \emptyset$, it follows from [31, Theorem 8.30] that its subderivative function is the support function of $\partial f(\bar{x})$, that is

$$df(\bar{x})(w) = \sup \{\langle v, w \rangle \mid v \in \partial f(\bar{x})\},$$

which in turn allows us to equivalently describe the critical cone $K_f(\bar{x}, \bar{v})$ as

$$K_f(\bar{x}, \bar{v}) = N_{\partial f(\bar{x})}(\bar{v}). \quad (16)$$

Proposition 2.2 *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then the following properties are equivalent:*

- (a) $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$;
- (b) *the critical cone $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace.*

Proof By (16), we have $K_g(\bar{z}, \bar{\lambda}) = N_{\partial g(\bar{z})}(\bar{\lambda})$. The equivalence of (a) and $N_{\partial g(\bar{z})}(\bar{\lambda})$ being a subspace is a well-known fact from convex analysis. \square

The following equivalent description of the critical cone of a polyhedral function was established in [19, Proposition 3.2].

Proposition 2.3 (*critical cone of polyhedral functions*) *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. If $\bar{\lambda}$ has the decomposition in (8), then $w \in K_g(\bar{z}, \bar{\lambda})$ if and only if w satisfies the conditions*

$$\begin{aligned} \langle a^i - a^j, w \rangle &= 0, & \text{for } i, j \in J_+(\bar{z}, \bar{\sigma}), \\ \langle a^i - a^j, w \rangle &\leq 0, & \text{for } i \in J(\bar{z}) \setminus J_+(\bar{z}, \bar{\sigma}) \text{ and } j \in J_+(\bar{z}, \bar{\sigma}), \\ \langle b^i, w \rangle &= 0, & \text{for } i \in I_+(\bar{z}, \bar{\tau}), \\ \langle b^i, w \rangle &\leq 0, & \text{for } i \in I(\bar{z}) \setminus I_+(\bar{z}, \bar{\tau}). \end{aligned}$$

3 Reduction Lemma for Polyhedral Functions and Its Applications

We begin this section by providing an extension of the reduction lemma for polyhedral functions. The reduction lemma, established first by Robinson in [25, Proposition 4.4] for polyhedral convex sets, shows that the graph of the normal cone to a polyhedral convex set coincides locally with that of the normal cone to its critical cone. Robinson’s proof of this result relies upon his sticky face lemma, which was established in [24, Lemma 3.5] with a rather involved proof. A simpler proof of this result was presented by Dontchev and Rockafellar in [4, Lemma 2E.4]. Recently, the reduction lemma was extended for an important class of convex functions, called piecewise linear-quadratic (cf. [31, Definition 10.20]), by the third author in [33, Theorem 2.3]. This class of convex functions clearly encompasses polyhedral functions and thus the result below can be derived from the recent result in [33]. However, the presented proof in [33] relies upon two major results: 1) Robinson’s reduction lemma for polyhedral convex sets and 2) the fact that the graph of subgradient mappings of convex piecewise linear-quadratic functions can be expressed as a finite union of polyhedral convex sets (see the proof of [31, Theorem 11.14(b)]). Below, we present a direct proof of the reduction lemma for polyhedral functions, which is solely based on Lemma 2.1.

Theorem 3.1 (Reduction lemma) *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then there exists $r > 0$ such that*

$$((\text{gph } \partial g) - (\bar{z}, \bar{\lambda})) \cap \mathbb{B}_r(0, 0) = (\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}) \cap \mathbb{B}_r(0, 0). \tag{17}$$

Proof Suppose that the polyhedral function g has the representation (5). We first show that there is $r > 0$ for which the inclusion ‘ \supset ’ in (17) holds. To do so, using (14) for the polyhedral convex set $\partial g(\bar{z})$, we find $r_0 > 0$ such that

$$T_{\partial g(\bar{z})}(\bar{\lambda}) \cap \mathbb{B}_{r_0}(0) = (\partial g(\bar{z}) - \bar{\lambda}) \cap \mathbb{B}_{r_0}(0). \quad (18)$$

Pick now a pair (w, u) from the right-hand side of (17) with $r = r_0$ therein, which implies that the conditions $w \in K_g(\bar{z}, \bar{\lambda}) \cap \mathbb{B}_{r_0}(0)$, $u \in K_g(\bar{z}, \bar{\lambda})^* \cap \mathbb{B}_{r_0}(0)$, and $\langle u, w \rangle = 0$ are satisfied. Moreover, using (16) for the polyhedral function g , we get $K_g(\bar{z}, \bar{\lambda})^* = T_{\partial g(\bar{z})}(\bar{\lambda})$. Therefore u belongs to the left-hand side of (18), and thus $\bar{\lambda} + u \in \partial g(\bar{z})$. We now show that $(\bar{z} + w, \bar{\lambda} + u) \in \text{gph } \partial g$. To this end, take $z \in \mathbb{R}^m$ and observe that

$$\langle \bar{\lambda} + u, z - (\bar{z} + w) \rangle = \langle \bar{\lambda} + u, z - \bar{z} \rangle - \langle \bar{\lambda}, w \rangle - \langle u, w \rangle \leq g(z) - g(\bar{z}) - \text{d}g(\bar{z})(w), \quad (19)$$

where the last inequality is deduced from the facts that $\bar{\lambda} + u \in \partial g(\bar{z})$, $w \in K_g(\bar{z}, \bar{\lambda})$, and $\langle u, w \rangle = 0$. Choosing a smaller radius r_0 if necessary, we can assume that (15) is valid on $T_{\text{dom } g}(\bar{z}) \cap \mathbb{B}_{r_0}(0)$. It follows from Proposition 2.1(a) and $w \in K_g(\bar{z}, \bar{\lambda}) \cap \mathbb{B}_{r_0}(0)$ that $w \in T_{\text{dom } g}(\bar{z}) \cap \mathbb{B}_{r_0}(0)$, which in turn allows us to conclude via (15) that $g(\bar{z}) + \text{d}g(\bar{z})(w) = g(\bar{z} + w)$. Combining this and (19) leads us to

$$\langle \bar{\lambda} + u, z - (\bar{z} + w) \rangle \leq g(z) - g(\bar{z} + w)$$

for any arbitrary $z \in \mathbb{R}^m$, and thus to $\bar{\lambda} + u \in \partial g(\bar{z} + w)$. This proves that (w, u) belongs to the left-hand side of (17) with $r = r_0$.

We now proceed with proving the inclusion ‘ \subset ’ in (17) for some $r > 0$. Represent $\bar{\lambda} \in \partial g(\bar{z})$ in the form of (8), pick r_0 from (18), and choose $r \in (0, r_0]$ such that the inclusions in (10) hold for all $(z, v) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$. Pick now a pair (w, u) from the left-hand side of (17), and conclude that $(w, u) \in \mathbb{B}_r(0, 0)$ with $\bar{\lambda} + u \in \partial g(\bar{z} + w)$. We are going to show that $u \in N_{K_g(\bar{z}, \bar{\lambda})}(w)$, which amounts via [4, Proposition 2A.3] to the conditions

$$w \in K_g(\bar{z}, \bar{\lambda}), \quad u \in K_g(\bar{z}, \bar{\lambda})^*, \quad \text{and} \quad \langle u, w \rangle = 0. \quad (20)$$

Taking a smaller radius r if necessary, we can assume that $\partial g(\bar{z} + w) \subset \partial g(\bar{z})$ (cf. [18, Proposition 3.3(a)]). Thus, $u \in (\partial g(\bar{z}) - \bar{\lambda}) \cap \mathbb{B}_r(0)$. Since $r \leq r_0$, we deduce from (18) that $u \in T_{\partial g(\bar{z})}(\bar{\lambda}) = K_g(\bar{z}, \bar{\lambda})^*$, which justifies the second inclusion in (20). By shrinking r again if necessary, we can derive from $(\bar{z} + w, \bar{\lambda} + u) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$ and (10) that

$$J_+(\bar{z}, \bar{\sigma}) \subset J(\bar{z} + w) \subset J(\bar{z}) \quad \text{and} \quad I_+(\bar{z}, \bar{\tau}) \subset I(\bar{z} + w) \subset I(\bar{z}).$$

To prove that $w \in K_g(\bar{z}, \bar{\lambda})$, we use Proposition 2.3 in which an equivalent description of $K_g(\bar{z}, \bar{\lambda})$ was given. We break this task into four cases as follows:

(i) $i, j \in J_+(\bar{z}, \bar{\sigma})$. In this case, we obtain from the inclusions above that $i, j \in J(\bar{z} + w) \subset J(\bar{z})$. These inclusions bring us via (6) to

$$\langle a^i, \bar{z} \rangle - \alpha_i = \langle a^j, \bar{z} \rangle - \alpha_j \quad \text{and} \quad \langle a^i, \bar{z} + w \rangle - \alpha_i = \langle a^j, \bar{z} + w \rangle - \alpha_j.$$

Combining these confirms that $\langle a^i - a^j, w \rangle = 0$ for all $i, j \in J_+(\bar{z}, \bar{\sigma})$.

(ii) $i \in J(\bar{z}) \setminus J_+(\bar{z}, \bar{\sigma})$ and $j \in J_+(\bar{z}, \bar{\sigma})$. In this case, we arrive at $i, j \in J(\bar{z})$ and $j \in J(\bar{z} + w)$, which imply via (6) that

$$\langle a^i, \bar{z} \rangle - \alpha_i = \langle a^j, \bar{z} \rangle - \alpha_j \quad \text{and} \quad \langle a^i, \bar{z} + w \rangle - \alpha_i \leq \langle a^j, \bar{z} + w \rangle - \alpha_j.$$

Combining these confirms that $\langle a^i - a^j, w \rangle \leq 0$ for all $i \in J(\bar{z}) \setminus J_+(\bar{z}, \bar{\sigma})$ and $j \in J_+(\bar{z}, \bar{\sigma})$.

(iii) $i \in I_+(\bar{z}, \bar{\tau})$. In this case, we have $i \in I(\bar{z})$ and $i \in I(\bar{z} + w)$, which result in $\langle b^i, \bar{z} \rangle = \langle b^i, \bar{z} + w \rangle = \beta_i$. Combining these confirms that $\langle b^i, w \rangle = 0$ for all $i \in I_+(\bar{z}, \bar{\tau})$.

(iv) $i \in I(\bar{z}) \setminus I_+(\bar{z}, \bar{\tau})$. In this case, we deduce from $\bar{z} + w \in \text{dom } g$ and $i \in I(\bar{z})$ that $\langle b^i, \bar{z} + w \rangle \leq \beta_i = \langle b^i, \bar{z} \rangle$, which in turn yields $\langle b^i, w \rangle \leq 0$ for all $i \in I(\bar{z}) \setminus I_+(\bar{z}, \bar{\tau})$.

In summary, we showed that w satisfies the equivalent description of $K_g(\bar{z}, \bar{\lambda})$ from Proposition 2.3, and so $w \in K_g(\bar{z}, \bar{\lambda})$. It remains to demonstrate that $\langle w, u \rangle = 0$. Recall that $\bar{\lambda} + u \in \partial g(\bar{z} + w)$. Using (7), we can express $\bar{\lambda} + u$ as

$$\bar{\lambda} + u = \sum_{j \in J(\bar{z} + w)} \sigma_j a^j + \sum_{i \in I(\bar{z} + w)} \tau_i b^i \quad \text{with} \quad \sigma_j, \tau_i \geq 0 \quad \text{and} \quad \sum_{j \in J(\bar{z} + w)} \sigma_j = 1.$$

This and the decomposition of $\bar{\lambda}$ as in (8) allow us to conclude that

$$\begin{aligned} \langle u, w \rangle &= \langle \bar{\lambda} + u, w \rangle - \langle \bar{\lambda}, w \rangle \\ &= \sum_{j \in J(\bar{z} + w)} (\sigma_j - \bar{\sigma}_j) \langle a^j, w \rangle + \sum_{i \in I(\bar{z} + w)} (\tau_i - \bar{\tau}_i) \langle b^i, w \rangle \\ &= \sum_{j \in J(\bar{z} + w)} (\sigma_j - \bar{\sigma}_j) [\langle a^j, \bar{z} + w \rangle - \alpha_j - (\langle a^j, \bar{z} \rangle - \alpha_j)] \\ &\quad + \sum_{i \in I(\bar{z} + w)} (\tau_i - \bar{\tau}_i) [\langle b^i, \bar{z} + w \rangle - \langle b^i, \bar{z} \rangle] \\ &= (g(\bar{z} + w) - g(\bar{z})) \sum_{j \in J(\bar{z} + w)} (\sigma_j - \bar{\sigma}_j) + \sum_{i \in I(\bar{z} + w)} (\tau_i - \bar{\tau}_i) (\beta_i - \beta_i) \\ &= 0, \end{aligned}$$

where the second equality results from the inclusions $J_+(\bar{z}, \bar{\sigma}) \subset J(\bar{z} + w)$ and $I_+(\bar{z}, \bar{\tau}) \subset I(\bar{z} + w)$, where the fourth equality comes from the definitions of the active index sets $J(\bar{z} + w)$ and $I(\bar{z} + w)$ as well as the inclusions $J(\bar{z} + w) \subset J(\bar{z})$ and $I(\bar{z} + w) \subset I(\bar{z})$, and where the last equality follows from the assumption on σ_j

and $\bar{\sigma}_j$. This proves the inclusion ‘ \subset ’ in (17) for some $r > 0$ and thus completes the proof. \square

The rest of this section will be devoted to presenting several important consequences of Theorem 3.1 for different second-order variational constructions of polyhedral functions. We begin with a duality relation between critical cones of a polyhedral function and its (Fenchel) conjugate in the sense of convex analysis. Both results in the following corollary were observed by Rockafellar in [30, equations (3.23) & (3.29)]. Since an explicit proof was not presented for (21) in [30] and since this result plays an important role in strict twice epi-differentiability of polyhedral functions in the next section, we supply a short proof for readers’ convenience. Recall that, for any function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, its conjugate $f^* : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is defined by $f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle y, x \rangle - f(x)\}$. To proceed, let us recall also an important result, used in the proof of next corollary, saying that the conjugate function g^* of a polyhedral function g is again a polyhedral function, cf. [31, Theorem 11.14(a)].

Corollary 3.1 (polar relation of critical cones) *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then $K_g(\bar{z}, \bar{\lambda})$ enjoys the duality relationship*

$$K_g(\bar{z}, \bar{\lambda}) = K_{g^*}(\bar{\lambda}, \bar{z})^*. \quad (21)$$

Consequently, we have

$$\bar{\lambda} \in \text{ri } \partial g(\bar{z}) \text{ if and only if } \bar{z} \in \text{ri } \partial g^*(\bar{\lambda}). \quad (22)$$

Proof Let $r > 0$ be such that (17) holds. Then, we conclude from (16) that

$$\begin{aligned} w \in K_g(\bar{z}, \bar{\lambda}) &\iff (w, 0) \in \text{gph } N_{K_g(\bar{z}, \bar{\lambda})} \\ &\iff \exists r' > 0 : t(w, 0) \in (\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}) \cap \mathbb{B}_r(0, 0) \quad \forall t \in [0, r') \\ &\iff \exists r' > 0 : (\bar{z} + tw, \bar{\lambda}) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda}) \quad \forall t \in [0, r') \\ &\iff \exists r' > 0 : \bar{\lambda} \in \partial g(\bar{z} + tw) \quad \forall t \in [0, r') \\ &\iff \exists r' > 0 : \bar{z} + tw \in \partial g^*(\bar{\lambda}) \quad \forall t \in [0, r') \\ &\iff w \in T_{\partial g^*(\bar{\lambda})}(\bar{z}) = (N_{\partial g^*(\bar{\lambda})}(\bar{z}))^* = K_{g^*}(\bar{\lambda}, \bar{z})^*, \end{aligned}$$

where the third equivalence follows from (17) and the last equivalence relies on the polyhedrality of $\partial g^*(\bar{\lambda})$. To justify (22), it follows from Proposition 2.2 that $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace if and only if $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$. A similar observation can be made for $K_{g^*}(\bar{\lambda}, \bar{z})$, which together with (21) completes the proof of (22). \square

We proceed to characterize critical cones of polyhedral functions for points nearby a given point in the graph of their subgradient mappings. Such a result for polyhedral convex sets can be found in [4, Lemma 4H.2]. Recall that a closed face F of a polyhedral convex cone $C \subset \mathbb{R}^d$ is defined by

$$F = C \cap [v]^\perp \quad \text{for some } v \in C^*.$$

Proposition 3.1 *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then there exists $r > 0$ such that for all $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$ the corresponding critical cone $K_g(z, \lambda)$ enjoys the following representation*

$$K_g(z, \lambda) = F_1 - F_2,$$

for some faces F_1, F_2 of $K_g(\bar{z}, \bar{\lambda})$ with $F_2 \subset F_1$. Conversely, for any pair of faces F_1, F_2 of $K_g(\bar{z}, \bar{\lambda})$ with $F_2 \subset F_1$ and any sufficiently small real number $r > 0$, there exists $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$ with $K_g(z, \lambda) = F_1 - F_2$.

Proof Let $r > 0$ be such that (17) holds. Choosing a smaller radius r if necessary, we can assume without loss of generality that the inclusions $\partial g(z) \subset \partial g(\bar{z})$ and $N_{\partial g(z)}(\lambda) \subset N_{\partial g(\bar{z})}(\bar{\lambda})$ hold for any $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$, which can be guaranteed by g being a polyhedral function. We first show that for any $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$, we have

$$K_g(z, \lambda) = K_g(\bar{z}, \bar{\lambda}) \cap [\lambda - \bar{\lambda}]^\perp + [z - \bar{z}]. \tag{23}$$

To this end, pick $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$ and observe that the inclusion

$$N_{\partial g(z)}(\lambda) \subset N_{\partial g(z)}(\bar{\lambda}) \tag{24}$$

holds. To justify it, it is not hard to see $\partial g(z) = \partial g(\bar{z}) \cap D$ with $D := \{v \in \mathbb{R}^m \mid \langle v, z - \bar{z} \rangle = g(z) - g(\bar{z})\}$. It follows from $\lambda \in \partial g(z) \subset D$ and $\lambda - \bar{\lambda} \in N_{K_g(\bar{z}, \bar{\lambda})}(z - \bar{z})$ that $\bar{\lambda} \in D$. This, together with the definition of D , leads us to $N_D(\lambda) = N_D(\bar{\lambda}) = [z - \bar{z}]$. Since both $\partial g(\bar{z})$ and D are polyhedral and convex, we conclude from the intersection rule for normal cones, holding without any constraint qualification for polyhedral convex sets (cf. [26, Corollary 23.8.1]), that

$$\begin{aligned} N_{\partial g(z)}(\lambda) &= N_{\partial g(\bar{z}) \cap D}(\lambda) = N_{\partial g(\bar{z})}(\lambda) + N_D(\lambda) \\ &\subset N_{\partial g(\bar{z})}(\bar{\lambda}) + N_D(\bar{\lambda}) = N_{\partial g(\bar{z}) \cap D}(\bar{\lambda}) = N_{\partial g(z)}(\bar{\lambda}), \end{aligned}$$

proving our claimed inclusion. Since g^* is a polyhedral function as well, using Lemma 2.1 for g^* at $(\bar{\lambda}, \bar{z}) \in \text{gph } \partial g^*$ and shrinking r , if necessary, ensure that $\bar{z} \in \partial g^*(\lambda)$. Moreover, it results from (24) that $N_{\partial g^*(\lambda)}(z) \subset N_{\partial g^*(\lambda)}(\bar{z})$, which in combination with (16) leads us to

$$K_{g^*}(\lambda, z) = N_{\partial g^*(\lambda)}(z) = N_{\partial g^*(\lambda)}(\bar{z}) \cap [z - \bar{z}]^\perp = K_{g^*}(\lambda, \bar{z}) \cap [z - \bar{z}]^\perp.$$

Using this and (21) brings us to

$$K_g(z, \lambda) = K_{g^*}(\lambda, z)^* = (K_{g^*}(\lambda, \bar{z}) \cap [z - \bar{z}]^\perp)^* = K_g(\bar{z}, \lambda) + [z - \bar{z}], \tag{25}$$

where in the last equality, we employ [31, Corollary 11.25(b)] and the fact that sum of two polyhedral convex sets $K_g(\bar{z}, \lambda)$ and $[z - \bar{z}]$ is polyhedral, cf. [26, Corollary 19.3.2],

and therefore closed. Similarly, by (16) and the definition of the normal cone to the polyhedral convex set $\partial g(\bar{z})$, we obtain

$$K_g(\bar{z}, \lambda) = N_{\partial g(\bar{z})}(\lambda) = N_{\partial g(\bar{z})}(\bar{\lambda}) \cap [\lambda - \bar{\lambda}]^\perp = K_g(\bar{z}, \bar{\lambda}) \cap [\lambda - \bar{\lambda}]^\perp,$$

which, together with (25), proves (23).

After these preparations, we are in a position to justify the claimed descriptions of critical cones of g . Pick $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$ and set $F_1 := K_g(\bar{z}, \bar{\lambda}) \cap [\lambda - \bar{\lambda}]^\perp$, which is clearly a face of $K_g(\bar{z}, \bar{\lambda})$. Because $\lambda - \bar{\lambda} \in N_{K_g(\bar{z}, \bar{\lambda})}(z - \bar{z})$, resulting from (17), we conclude that $z - \bar{z} \in F_1$. Since the relative interiors of nonempty faces of F_1 form a partition of this set (cf. [26, Theorem 18.2]), we find a face of F_1 , denoted by F_2 , that $z - \bar{z} \in \text{ri } F_2$. This tells us that $F_2 \subset F_1$. Moreover, F_2 is a face of $K_g(\bar{z}, \bar{\lambda})$ as well. By (23), the inclusion $K_g(z, \lambda) \subset F_1 - F_2$ clearly holds. To get the opposite inclusion, pick $x_i \in F_i$ for $i = 1, 2$. It follows from $z - \bar{z} \in \text{ri } F_2$ and [26, Theorem 6.4] that there is $t > 0$ such that $(1 + t)(z - \bar{z}) - tx_2 \in F_2 \subset F_1$, which yields

$$t(x_1 - x_2) = tx_1 + (1 + t)(z - \bar{z}) - tx_2 - (1 + t)(z - \bar{z}) \in F_1 + [z - \bar{z}], \quad (26)$$

which confirms via (23) that the inclusion $F_1 - F_2 \subset K_g(z, \lambda)$ holds. This shows that $K_g(z, \lambda) = F_1 - F_2$.

Assume now that F_1 and F_2 are faces of $K_g(\bar{z}, \bar{\lambda})$ with $F_2 \subset F_1$. Pick $r > 0$ from Theorem 3.1 and choose $u \in K_g(\bar{z}, \bar{\lambda})^*$ with $\|u\| < r/2$ such that $F_1 = K_g(\bar{z}, \bar{\lambda}) \cap [u]^\perp$. Since $u \in K_g(\bar{z}, \bar{\lambda})^* = N_{K_g(\bar{z}, \bar{\lambda})}(0)$, it follows from Theorem 3.1 that $\bar{\lambda} + u \in \partial g(\bar{z})$. Pick now $w \in \text{ri } F_2$ with $\|w\| < r/2$ and observe from $w \in F_1$ that $(w, u) \in \text{gph } N_{K_g(\bar{z}, \bar{\lambda})}$. By Theorem 3.1, we arrive at $(\bar{z} + w, \bar{\lambda} + u) \in \text{gph } \partial g$. Since $w \in \text{ri } F_2$, as in (26), we can show that $F_1 - F_2 = F_1 + [w]$. Employing this and (23) leads us to

$$F_1 - F_2 = F_1 + [w] = K_g(\bar{z}, \bar{\lambda}) \cap [u]^\perp + [w] = K_g(\bar{z} + w, \bar{\lambda} + u),$$

which completes the proof. \square

Corollary 3.2 *For a polyhedral function $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$, the following properties hold.*

- (a) *There exists $r > 0$ such that for all $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$ the following inclusions hold:*

$$K_g(z, \lambda) \subset K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda}), \quad (27a)$$

$$K_g(\bar{z}, \bar{\lambda}) \cap -K_g(\bar{z}, \bar{\lambda}) \subset K_g(z, \lambda). \quad (27b)$$

- (b) *If, in addition, $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$, then there exists $r > 0$ such that for all $(z, \lambda) \in (\text{gph } \partial g) \cap \mathbb{B}_r(\bar{z}, \bar{\lambda})$, we have $\lambda \in \text{ri } \partial g(z)$ and*

$$K_g(z, \lambda) = K_g(\bar{z}, \bar{\lambda}). \quad (28)$$

Proof The inclusion in (27a) for any $(z, \lambda) \in \text{gph } \partial g$ sufficiently close to $(\bar{z}, \bar{\lambda})$ falls directly from Proposition 3.1. Also, according to Proposition 3.1, for any such $(z, \lambda) \in \text{gph } \partial g$, one can find some faces F_1, F_2 of $K_g(\bar{z}, \bar{\lambda})$ with $F_2 \subset F_1$ such that $K_g(z, \lambda) = F_1 - F_2$. Since $K_g(\bar{z}, \bar{\lambda}) \cap -K_g(\bar{z}, \bar{\lambda})$ is the smallest face of $K_g(\bar{z}, \bar{\lambda})$, we get $K_g(\bar{z}, \bar{\lambda}) \cap -K_g(\bar{z}, \bar{\lambda}) \subset F_i$ for $i = 1, 2$, which in turn justify the inclusion in (27b) for all $(z, \lambda) \in \text{gph } \partial g$ sufficiently close to $(\bar{z}, \bar{\lambda})$.

To justify (b), it follows from $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$ and Proposition 2.2 that $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace. This and the inclusions (27a) and (27b) tell us that

$$K_g(\bar{z}, \bar{\lambda}) = K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda}) \supset K_g(z, \lambda) \supset K_g(\bar{z}, \bar{\lambda}) \cap -K_g(\bar{z}, \bar{\lambda}) = K_g(\bar{z}, \bar{\lambda})$$

for all $(z, \lambda) \in \text{gph } \partial g$ sufficiently close to $(\bar{z}, \bar{\lambda})$, which in turn confirms that $K_g(z, \lambda)$ is a linear subspace. Using Proposition 2.2 again, we have $\lambda \in \text{ri } \partial g(z)$ and hence complete the proof of (b). □

Both inclusions in Corollary 3.2(a) were perviously established in [4, Proposition 2E.10] for polyhedral convex sets without appealing to the reduction lemma, an important tool used in our proof. Note that while the observation in Corollary 3.2(b) is a simple and direct consequence of the inclusions (27a)–(27b), it plays an indispensable role in the next two sections in which we are going to study strict twice epi-differentiability of polyhedral functions. In fact, this result reveals that when we are converging in the graph of subgradient mappings of polyhedral functions to a given point therein under the extra relative interior condition, all those points, used in this convergence, enjoy this relative interior condition. This has a major implication for second-order variational constructions such as limiting coderivatives and strict subgradient graphical derivatives; see Theorem 3.2(c) and Theorem 3.3(c).

We proceed with two other important applications of the established reduction lemma for polyhedral functions in calculating the proto-derivative and coderivative of subgradient mappings of this class of functions. To this end, consider a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. According to [31, Definition 8.33], the *graphical derivative* of F at \bar{x} for \bar{y} with $(\bar{x}, \bar{y}) \in \text{gph } F$ is the set-valued mapping $DF(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined via the tangent cone to $\text{gph } F$ at (\bar{x}, \bar{y}) by

$$u \in DF(\bar{x}, \bar{y})(w) \iff (w, u) \in T_{\text{gph } F}(\bar{x}, \bar{y}),$$

or, equivalently, $\text{gph } DF(\bar{x}, \bar{y}) = T_{\text{gph } F}(\bar{x}, \bar{y})$. Using the definition of the tangent cone, we can present an alternative definition of $DF(\bar{x}, \bar{y})$ in terms of graphical limits as

$$\text{gph } DF(\bar{x}, \bar{y}) = \limsup_{t \searrow 0} \frac{\text{gph } F - (\bar{x}, \bar{y})}{t}. \tag{29}$$

The set-valued mapping F is said to be *proto-differentiable* at \bar{x} for \bar{y} if the outer graphical limit in (29) is actually a full limit. If F is proto-differentiable at \bar{x} for \bar{y} , its graphical derivative $DF(\bar{x}, \bar{y})$ is called the *proto-derivative* of F at \bar{x} for \bar{y} . When $F(\bar{x})$ is a singleton consisting of \bar{y} only, the notation $DF(\bar{x}, \bar{y})$ is simplified to $DF(\bar{x})$. It is easy to see that for a single-valued function F , which is differentiable at \bar{x} , the graphical derivative $DF(\bar{x})$ boils down to the Jacobian matrix of F at \bar{x} , denoted

by $\nabla F(\bar{x})$. Recall from [31, Definition 9.53] that the strict graphical derivative of a set-valued mapping F at \bar{x} for \bar{y} with $(\bar{x}, \bar{y}) \in \text{gph } F$, is the set-valued mapping $\widetilde{D}F(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, defined by

$$\text{gph } \widetilde{D}F(\bar{x}, \bar{y}) = \limsup_{\substack{t \searrow 0 \\ (x,y) \xrightarrow{\text{gph } F} (\bar{x}, \bar{y})}} \frac{\text{gph } F - (x, y)}{t}. \quad (30)$$

The set-valued mapping F is said to be *strictly* proto-differentiable at \bar{x} for \bar{y} if the outer graphical limit in (30) is a full limit. When F is strictly proto-differentiable at \bar{x} for \bar{y} , its strict graphical derivative $\widetilde{D}F(\bar{x}, \bar{y})$ is called the *strict* proto-derivative of F at \bar{x} for \bar{y} . To elaborate more, take a set $\Omega \subset \mathbb{R}^d$ and $\bar{x} \in \Omega$. The regular (Clarke) tangent cone and the paratingent cone to Ω at \bar{x} are defined, respectively, by

$$\widehat{T}_\Omega(\bar{x}) = \liminf_{x \xrightarrow{\Omega} \bar{x}, t \searrow 0} \frac{\Omega - x}{t} \quad \text{and} \quad \widetilde{T}_\Omega(\bar{x}) = \limsup_{x \xrightarrow{\Omega} \bar{x}, t \searrow 0} \frac{\Omega - x}{t}. \quad (31)$$

The strict proto-differentiability of F at \bar{x} for \bar{y} amounts to the coincidence $\widehat{T}_{\text{gph } F}(\bar{x}, \bar{y}) = \widetilde{T}_{\text{gph } F}(\bar{x}, \bar{y})$. Observe also that since the inclusions $\widehat{T}_{\text{gph } F}(\bar{x}, \bar{y}) \subset T_{\text{gph } F}(\bar{x}, \bar{y}) \subset \widetilde{T}_{\text{gph } F}(\bar{x}, \bar{y})$ always hold, strict proto-differentiability of F at \bar{x} for \bar{y} implies that its proto-derivative and strict proto-derivative coincide, namely we have

$$DF(\bar{x}, \bar{y}) = \widetilde{D}F(\bar{x}, \bar{y}). \quad (32)$$

Strict proto-differentiability of a set-valued mapping has far-reaching consequences. In fact, it allows us to evaluate the strict proto-derivative of a set-valued mapping, which can be utilized to characterize its strong metric regularity; see [4, Theorem 4D.1] and Sect. 5 for more details on this application. The intriguing question is whether strict proto-differentiability holds for any class of set-valued mappings among those commonly seen in constrained and composite optimization. In the next theorem, we are going to show that subgradient mappings of polyhedral functions enjoy this property when a relative interior condition is satisfied for subgradients under consideration. Furthermore, we will prove that such a relative interior condition, indeed, characterizes strict proto-differentiability of these set-valued mappings. While it is not easy to study strict proto-differentiability of set-valued mappings, its weaker version, namely proto-differentiability, has been well understood for many important classes of functions that are important for various applications; see [16, 17, 29, 31] for more details and examples. In particular, it is well known (cf. [31, Corollary 13.41]) that subgradient mappings of polyhedral functions are always proto-differentiable.

Theorem 3.2 *For a polyhedral function $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$, the following properties hold.*

(a) *The graphical derivative of ∂g at \bar{z} for $\bar{\lambda}$ can be obtained by*

$$\text{gph } D(\partial g)(\bar{z}, \bar{\lambda}) = T_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \text{gph } N_{K_g}(\bar{z}, \bar{\lambda}). \quad (33)$$

(b) The strict graphical derivative of ∂g at \bar{z} for $\bar{\lambda}$ can be obtained by

$$\text{gph } \tilde{D}(\partial g)(\bar{z}, \bar{\lambda}) = \text{gph } N_{K_g(\bar{z}, \bar{\lambda})} - \text{gph } N_{K_g(\bar{z}, \bar{\lambda})}. \tag{34}$$

(c) The condition $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$ amounts to the existence of a neighborhood O of $(\bar{z}, \bar{\lambda})$ such that for any $(z, \lambda) \in O \cap \text{gph } \partial g$, ∂g is strictly proto-differentiable at z for λ and its proto-derivative and strict proto-derivative coincide, namely

$$\tilde{D}(\partial g)(z, \lambda) = D(\partial g)(z, \lambda). \tag{35}$$

Proof Observe that the first equality in (33) is the definition of graphical derivative. Using the equality (17) and the definition of the tangent cone to $\text{gph } \partial g$ at $(\bar{z}, \bar{\lambda})$, we immediately arrive at the second equality in (33).

To justify (b), we conclude from Theorem 3.1 and [4, Proposition 4 H.10], respectively, that

$$\begin{aligned} \tilde{T}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) &= \text{gph } \tilde{D}(\partial g)(\bar{z}, \bar{\lambda}) = \text{gph } \tilde{D}N_{K_g(\bar{z}, \bar{\lambda})}(0, 0) \\ &= \text{gph } N_{K_g(\bar{z}, \bar{\lambda})} - \text{gph } N_{K_g(\bar{z}, \bar{\lambda})}. \end{aligned} \tag{36}$$

Finally, we proceed with the proof of (c). Suppose that $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$. By Corollary 3.2(b), we find $r > 0$ such that for any $(z, \lambda) \in \mathbb{B}_r(\bar{z}, \bar{\lambda}) \cap \text{gph } \partial g$, the property $K_g(z, \lambda) = K_g(\bar{z}, \bar{\lambda})$ holds. Take any such a pair (z, λ) and conclude from part (a) that

$$T_{\text{gph } \partial g}(z, \lambda) = \text{gph } N_{K_g(z, \lambda)} = \text{gph } N_{K_g(\bar{z}, \bar{\lambda})}.$$

It follows then from [31, Theorem 6.26] that

$$\widehat{T}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \liminf_{(z, \lambda) \xrightarrow{\text{gph } \partial g} (\bar{z}, \bar{\lambda})} T_{\text{gph } \partial g}(z, \lambda) = \text{gph } N_{K_g(\bar{z}, \bar{\lambda})}.$$

Since $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$, we conclude that $\text{gph } N_{K_g(\bar{z}, \bar{\lambda})} = K_g(\bar{z}, \bar{\lambda}) \times K_g(\bar{z}, \bar{\lambda})^\perp$ is a linear subspace. These, coupled with (36), lead us to $\widehat{T}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \tilde{T}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda})$, meaning that ∂g is strictly proto-differentiable at \bar{z} for $\bar{\lambda}$. To achieve a similar conclusion for any pair $(z, \lambda) \in \text{gph } \partial g$ sufficiently close to $(\bar{z}, \bar{\lambda})$, observe from Corollary 3.2(b) that for any such a pair, we have $\lambda \in \text{ri } \partial g(z)$. A similar argument as the one presented above for $(\bar{z}, \bar{\lambda})$, shows that ∂g is strictly proto-differentiable at z for λ whenever $(z, \lambda) \in \text{gph } \partial g$ is sufficiently close to $(\bar{z}, \bar{\lambda})$. Finally, (35) results from (32). Conversely, if ∂g is strictly proto-differentiable at \bar{z} for $\bar{\lambda}$, our discussion prior to (32) implies that

$$\text{gph } N_{K_g(\bar{z}, \bar{\lambda})} = T_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \tilde{T}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \text{gph } N_{K_g(\bar{z}, \bar{\lambda})} - \text{gph } N_{K_g(\bar{z}, \bar{\lambda})},$$

where the last equality results from (36). This clearly tells us that $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace and hence it follows from Proposition 2.2 that $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$, which completes the proof. \square

We continue with one more direct application of the reduction lemma for polyhedral functions in finding the regular and limiting normal cones to $\text{gph } \partial g$.

Theorem 3.3 (Regularity of subgradient mappings of polyhedral functions) *For a polyhedral function $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$, the following properties hold.*

(a) *The regular normal cone to $\text{gph } \partial g$ at $(\bar{z}, \bar{\lambda})$ can be calculated by*

$$\widehat{N}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = K_g(\bar{z}, \bar{\lambda})^* \times K_g(\bar{z}, \bar{\lambda}). \quad (37)$$

(b) *The (limiting) normal cone to $\text{gph } \partial g$ at $(\bar{z}, \bar{\lambda})$ can be calculated by*

$$N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \bigcup_{\substack{F_1, F_2 \subset \mathcal{F}_{K_g(\bar{z}, \bar{\lambda})} \\ F_2 \subset F_1}} (F_1 - F_2)^* \times (F_1 - F_2), \quad (38)$$

where $\mathcal{F}_{K_g(\bar{z}, \bar{\lambda})}$ is the collection of all faces of $K_g(\bar{z}, \bar{\lambda})$.

(c) *The condition $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$ is equivalent to*

$$N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \widehat{N}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}). \quad (39)$$

Proof We begin with the proof of (a). It follows from Theorem 3.1 that $\text{gph } \partial g - (\bar{z}, \bar{\lambda})$ and $\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}$ coincide locally around $(0, 0)$, which leads us via the definition of the regular normal cone to

$$\widehat{N}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \widehat{N}_{\text{gph } \partial g - (\bar{z}, \bar{\lambda})}(0, 0) = \widehat{N}_{\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}}(0, 0).$$

Since $K_g(\bar{z}, \bar{\lambda})$ is a polyhedral convex cone, it results from [4, page 264] that

$$\widehat{N}_{\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}}(0, 0) = K_{K_g(\bar{z}, \bar{\lambda})}(0, 0)^* \times K_{K_g(\bar{z}, \bar{\lambda})}(0, 0) = K_g(\bar{z}, \bar{\lambda})^* \times K_g(\bar{z}, \bar{\lambda}).$$

Combining these proves (a).

To justify (b), observe that $(u, w) \in N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda})$ if and only if there are sequences $(z^k, \lambda^k) \rightarrow (\bar{z}, \bar{\lambda})$ with $(z^k, \lambda^k) \in \text{gph } \partial g$ and $(u^k, w^k) \rightarrow (u, w)$ with $(u^k, w^k) \in \widehat{N}_{\text{gph } \partial g}(z^k, \lambda^k)$. Using the definition of the regular normal cone and (17), we conclude for any k sufficiently large that

$$\begin{aligned} (u^k, w^k) \in \widehat{N}_{\text{gph } \partial g}(z^k, \lambda^k) &\iff (u^k, w^k) \in \widehat{N}_{\text{gph } \partial g - (\bar{z}, \bar{\lambda})}(z^k - \bar{z}, \lambda^k - \bar{\lambda}) \\ &\iff (u^k, w^k) \in \widehat{N}_{\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}}(z^k - \bar{z}, \lambda^k - \bar{\lambda}). \end{aligned}$$

By the definition of the limiting normal cone, we conclude from the last inclusion that $(u, w) \in N_{\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}}(0, 0)$ and thus get the inclusion $N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) \subset N_{\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}}(0, 0)$. A similar argument via (17) can be used to justify the opposite inclusion and obtain

$$N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = N_{\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}}(0, 0).$$

Recall that $K_g(\bar{z}, \bar{\lambda})$ is a polyhedral convex cone. Employing the representation of the limiting normal cone to the normal cone to the graph of a polyhedral convex set, obtained in the proof of [3, Theorem 2], we arrive at the representation on the right-hand side of (38) for $N_{\text{gph } N_{K_g(\bar{z}, \bar{\lambda})}}(0, 0)$, which completes the proof of (b).

To prove (c), observe that it follows from $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$ and Corollary 3.2(b) that there exists a neighborhood O of $(\bar{z}, \bar{\lambda})$ such that for any $(z, \lambda) \in O \cap \text{gph } \partial g$, we have $K_g(z, \lambda) = K_g(\bar{z}, \bar{\lambda})$. This and (37) bring us to

$$\widehat{N}_{\text{gph } \partial g}(z, \lambda) = \widehat{N}_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) \quad \text{for all } (z, \lambda) \in O \cap \text{gph } \partial g, \tag{40}$$

which, together with the definition of the limiting normal cone, justifies the claimed equality in (39). Conversely, suppose that (39) holds. We are going to show that $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace. To this end, it suffices to show that if $w \in K_g(\bar{z}, \bar{\lambda})$, then $-w \in K_g(\bar{z}, \bar{\lambda})$. Pick $w \in K_g(\bar{z}, \bar{\lambda})$ and conclude from (33) that there is $u \in \mathbb{R}^m$ such that $(w, u) \in T_{\text{gph } \partial g}(\bar{z}, \bar{\lambda})$. Appealing now to [31, Theorem 13.57] implies that $(u, -w) \in N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda})$. This, together with (39) and (37), tells us that $-w \in K_g(\bar{z}, \bar{\lambda})$. Remember that $K_g(\bar{z}, \bar{\lambda}) = N_{\partial g(\bar{z})}(\bar{\lambda})$. Since $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace, we arrive at $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$, which completes the proof of (c). \square

The description (37) of the regular normal cone to $\text{gph } \partial g$ in terms of the critical cone of g was established in [18, Theorem 4.3(i)] using a different approach. Our current proof relies upon the reduction lemma, which allows us to simplify the proof of this result. We should mention that a similar result was established for polyhedral convex sets using Robinson’s reduction lemma for polyhedral convex sets in the proof of [3, Theorem 2]. A similar expression of the limiting normal cone to $\text{gph } \partial g$ was achieved in [18, Theorem 5.1] via a lengthy direct argument. Our proof, which heavily uses Theorem 3.1, reduces the calculation to the case of a polyhedral convex cone and then utilizes the available result for this setting. Thus, Theorem 3.3 can be considered as a generalization of Dontchev and Rockafellar’s result, obtained in [3, Theorem 2], for polyhedral convex sets. Note that part (c) of Theorem 3.3 offers a new piece of information about $\text{gph } \partial g$, which has not been observed before to the best of our knowledge. Indeed, it tells us that $\text{gph } \partial g$ is *regular* at $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$ in the sense of [31, Definition 6.4] if and only if the subgradient $\bar{\lambda}$ is taken from the relative interior of $\partial g(\bar{z})$.

Recall that for a polyhedral function g with $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$, the coderivative mapping of ∂g at \bar{z} for $\bar{\lambda}$, denoted $D^*(\partial g)(\bar{z}, \bar{\lambda})$, is defined by

$$u \in D^*(\partial g)(\bar{z}, \bar{\lambda})(w) \iff (u, -w) \in N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}).$$

It is well known (cf. [31, Theorem 13.57]) that the inclusion $D(\partial g)(\bar{z}, \bar{\lambda})(w) \subset D^*(\partial g)(\bar{z}, \bar{\lambda})(w)$, for any $w \in \mathbb{R}^m$, always holds for a polyhedral function g . Below, we show that this inclusion becomes equality provided that the subgradient $\bar{\lambda}$ is taken from the relative interior of $\partial g(\bar{z})$.

Corollary 3.3 *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and that $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$ if and only if*

$$D(\partial g)(\bar{z}, \bar{\lambda})(w) = D^*(\partial g)(\bar{z}, \bar{\lambda})(w) \quad \text{for all } w \in \mathbb{R}^m. \tag{41}$$

Proof Suppose that $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$. Thus the critical cone $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace. By Theorem 3.2(a), we have $T_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = \text{gph } N_{K_g(\bar{z}, \bar{\lambda})} = K_g(\bar{z}, \bar{\lambda}) \times K_g(\bar{z}, \bar{\lambda})^\perp$. Moreover, we conclude from Theorem 3.3(c) that $N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda}) = K_g(\bar{z}, \bar{\lambda})^\perp \times K_g(\bar{z}, \bar{\lambda})$. Combining these with the definitions of coderivative and proto-derivative justifies (41).

Assume now that (41) holds. We are going to show that $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace. To this end, it suffices to show that if $w \in K_g(\bar{z}, \bar{\lambda})$, then $-w \in K_g(\bar{z}, \bar{\lambda})$. Pick $w \in K_g(\bar{z}, \bar{\lambda})$ and conclude from (33) that there is $u \in \mathbb{R}^m$ such that $u \in D(\partial g)(\bar{z}, \bar{\lambda})(w) = N_{K_g(\bar{z}, \bar{\lambda})}(w)$, which yields that $(w, u) \in K_g(\bar{z}, \bar{\lambda}) \times K_g(\bar{z}, \bar{\lambda})^*$. This, combined with (38), tells us that $(u, w) \in K_g(\bar{z}, \bar{\lambda})^* \times K_g(\bar{z}, \bar{\lambda}) \subset N_{\text{gph } \partial g}(\bar{z}, \bar{\lambda})$. Employing the definition of coderivative and (41), we obtain $u \in D^*(\partial g)(\bar{z}, \bar{\lambda})(-w) = D(\partial g)(\bar{z}, \bar{\lambda})(-w) = N_{K_g(\bar{z}, \bar{\lambda})}(-w)$, which clearly confirms that $-w \in K_g(\bar{z}, \bar{\lambda})$. Since $K_g(\bar{z}, \bar{\lambda}) = N_{\partial g(\bar{z})}(\bar{\lambda})$ and $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace, we obtain $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$, which completes the proof. \square

4 Strict Twice Epi-Differentiability of Polyhedral Functions

In this section, we study another important second-order variational property, called strict twice epi-differentiability, for polyhedral functions and continuous differentiability of proximal mappings for this class of functions. Once again, our results rely heavily on our extension of the reduction lemma for polyhedral functions. To achieve our goal, consider a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ with $\bar{x} \in \mathbb{R}^n$ with $f(\bar{x})$ finite and define the parametric family of second-order difference quotients of f at \bar{x} for $\bar{v} \in \partial f(\bar{x})$ by

$$\Delta_t^2 f(\bar{x}, \bar{v})(w) = \frac{f(\bar{x} + tw) - f(\bar{x}) - t\langle \bar{v}, w \rangle}{\frac{1}{2}t^2}$$

for any $w \in \mathbb{R}^n$ and $t > 0$. The second subderivative of f at \bar{x} for \bar{v} , denoted $d^2 f(\bar{x}, \bar{v})$, is an extended-real-valued function defined by

$$d^2 f(\bar{x}, \bar{v})(w) = \liminf_{\substack{t \searrow 0 \\ w' \rightarrow w}} \Delta_t^2 f(\bar{x}, \bar{v})(w'), \quad w \in \mathbb{R}^n.$$

Following [31, Definition 13.6], f is said to be twice epi-differentiable at \bar{x} for \bar{v} if the functions $\Delta_t^2 f(\bar{x}, \bar{v})$ epi-converge to $d^2 f(\bar{x}, \bar{v})$ as $t \searrow 0$. Further, we say that f is *strictly* twice epi-differentiable at \bar{x} for \bar{v} if the functions $\Delta_t^2 f(x, v)$ epi-converge to a function as $t \searrow 0$, $(x, v) \rightarrow (\bar{x}, \bar{v})$ with $f(x) \rightarrow f(\bar{x})$ and $(x, v) \in \text{gph } \partial f$. If this condition holds, the limit function is then the second subderivative $d^2 f(\bar{x}, \bar{v})$. Twice epi-differentiability of extended-real-valued functions, introduced by Rockafellar in [27], has been investigated for important classes of functions appearing in constrained and composite optimization problems in [1, 16, 17, 28]. Its strict version, introduced in [21], was only studied in [20] for nonlinear programming and minimax problems. In the context of numerical algorithms, strict twice epi-differentiability was recently

exploited to design a linesearch algorithm for minimizing the sum of nonconvex functions in [34, 35]. It is then tempting to ask when this property holds and to explore its applications in parametric optimization. To this end, we define the *strict second subderivative* of f at \bar{x} for \bar{v} with $\bar{v} \in \partial f(\bar{x})$ at $w \in \mathbb{R}^n$ by

$$d_s^2 f(\bar{x}, \bar{v})(w) = \liminf_{\substack{t \searrow 0, w' \rightarrow w \\ (x, v) \xrightarrow[\text{gph } \partial f]{f(x) \rightarrow f(\bar{x})} (\bar{x}, \bar{v})}} \Delta_t^2 f(x, v)(w').$$

When f is subdifferentially continuous at \bar{x} for \bar{v} in the sense of [31, Definition 13.28], we can drop the requirement $f(x) \rightarrow f(\bar{x})$ in the definition of the strict second subderivative. According to [31, Example 13.30], convex functions are always subdifferentially continuous. Clearly, we always have $d_s^2 f(\bar{x}, \bar{v})(w) \leq d^2 f(\bar{x}, \bar{v})(w)$ for any $w \in \mathbb{R}^n$. If f is strictly twice epi-differentiable at \bar{x} for \bar{v} , the latter inequality becomes equality. Being able to calculate the strict second subderivative of a function and comparing with its second subderivative can tell us when strict twice epi-differentiability should be expected for such a function. Note also that the second subderivative was exploited to characterize the quadratic growth condition for extended-real-valued functions in [31, Theorem 13.24(c)]. Similarly, we can use the strict second subderivative to achieve a characterization of the uniform quadratic growth condition (cf. [2, Definition 5.16]), which plays an important role in parametric optimization. This is beyond the scope of this paper and so we postpone it to our future research. Below, we use the characterization of the faces of the critical cone of a polyhedral function in Proposition 3.1 to find its strict second subderivative.

Proposition 4.1 *For a polyhedral function $g : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$ and $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$, its strict second subderivative can be calculated by*

$$d_s^2 g(\bar{z}, \bar{\lambda})(w) = \delta_{K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})}(w) \quad \text{for all } w \in \mathbb{R}^m. \tag{42}$$

Proof We first prove that $d_s^2 g(\bar{z}, \bar{\lambda})(w) = 0$ for all $w \in K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})$. First, observe that the convexity of g yields $d_s^2 g(\bar{z}, \bar{\lambda})(w) \geq 0$ for all $w \in \mathbb{R}^m$. To obtain the opposite inequality, pick $w \in K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})$. By Proposition 3.1, we find a sequence $(z^k, \lambda^k) \rightarrow (\bar{z}, \bar{\lambda})$ such that $(z^k, \lambda^k) \in \text{gph } \partial g$ and $K_g(z^k, \lambda^k) = K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})$. This implies that $w \in K_g(z^k, \lambda^k)$, which is equivalent to saying that $dg(z^k)(w) = \langle \lambda^k, w \rangle$. Employing Proposition 2.1(b), we find a sequence $t_k \searrow 0$ such that $g(z^k + t_k w) - g(z^k) = dg(z^k)(t_k w) = t_k dg(z^k)(w)$. By definition, we get $\Delta_{t_k}^2 g(z^k, \lambda^k)(w) = 0$, which results in

$$d_s^2 g(\bar{z}, \bar{\lambda})(w) \leq \lim_{k \rightarrow \infty} \Delta_{t_k}^2 g(z^k, \lambda^k)(w) = 0.$$

This confirms that $d_s^2 g(\bar{z}, \bar{\lambda})(w) = 0$ for any $w \in K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})$. To obtain (42), it suffices to justify that $\text{dom } d_s^2 g(\bar{z}, \bar{\lambda}) \subset K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})$. Taking any $w \in \mathbb{R}^m$ with $d_s^2 g(\bar{z}, \bar{\lambda})(w) < \infty$, we can find sequences $t_k \searrow 0, w^k \rightarrow w$, and $(z^k, \lambda^k) \rightarrow (\bar{z}, \bar{\lambda})$ with $(z^k, \lambda^k) \in \text{gph } \partial g$ such that

$$\lim_{k \rightarrow \infty} \frac{g(z^k + t_k w^k) - g(z^k) - t_k \langle \lambda^k, w^k \rangle}{\frac{1}{2} t_k^2} < \infty.$$

This, together with the convexity of g , allows us to find a constant $M > 0$ such that

$$0 \leq \frac{g(z^k + t_k w^k) - g(z^k) - t_k \langle \lambda^k, w^k \rangle}{t_k} \leq M t_k$$

for all k sufficiently large and therefore to obtain

$$\lim_{k \rightarrow \infty} \frac{g(z^k + t_k w^k) - g(z^k) - t_k \langle \lambda^k, w^k \rangle}{t_k} = 0. \quad (43)$$

Take an arbitrary

$$u \in (K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda}))^\perp = K_g(\bar{z}, \bar{\lambda})^* \cap -K_g(\bar{z}, \bar{\lambda})^* = T_{\partial g(\bar{z})}(\bar{\lambda}) \cap -T_{\partial g(\bar{z})}(\bar{\lambda}),$$

where the last equality results from (16). We now show that $\langle u, w \rangle = 0$, which in turn yields $w \in K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})$. To this end, by (14), we find $\alpha > 0$ such that $\bar{\lambda} \pm \alpha u \in \partial g(\bar{z})$. This clearly implies that

$$\langle \bar{\lambda} \pm \alpha u, z^k + t_k w^k - \bar{z} \rangle \leq g(z^k + t_k w^k) - g(\bar{z}). \quad (44)$$

Employing Theorem 3.1, we can conclude that $(z^k - \bar{z}, \lambda^k - \bar{\lambda}) \in \text{gph } N_{K_g(\bar{z}, \bar{\lambda})}$ for all k sufficiently large. This tells us that $z^k - \bar{z} \in K_g(\bar{z}, \bar{\lambda}) = N_{\partial g(\bar{z})}(\bar{\lambda})$ for all k sufficiently large. Moreover, Lemma 2.1 confirms that $\bar{\lambda} \in \partial g(z^k)$ for all k sufficiently large. Combining these, we can conclude that

$$\langle u, z^k - \bar{z} \rangle = 0 \quad \text{and} \quad \langle \bar{\lambda}, z^k - \bar{z} \rangle = g(z^k) - g(\bar{z}).$$

These, together with (44), yield

$$\langle \bar{\lambda} \pm \alpha u, t_k w^k \rangle \leq g(z^k + t_k w^k) - g(z^k),$$

and therefore we get

$$\langle \bar{\lambda} - \lambda^k \pm \alpha u, w^k \rangle \leq \frac{g(z^k + t_k w^k) - g(z^k) - t_k \langle \lambda^k, w^k \rangle}{t_k}.$$

Passing to the limit as $k \rightarrow \infty$ and employing (43), we arrive at $\pm \alpha \langle u, w \rangle \leq 0$, meaning $\langle u, w \rangle = 0$. This confirms that $w \in K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})$ and hence completes the proof. \square

The established formula for the strict second subderivative of a polyhedral function in (42) suggests a path forward in the study of strict twice epi-differentiability of this class of functions. As pointed out earlier, strict twice epi-differentiability requires that

the second subderivative and strict second subderivative coincide. For polyhedral functions, Proposition 4.1 immediately suggests that the given subgradient $\bar{\lambda}$ must belong to $\text{ri } \partial g(\bar{z})$; see the proof of the implication (a) \implies (b) in Theorem 4.1 for a detailed proof. One may wonder whether the opposite holds as well, namely the relative interior condition $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$ implies strict twice epi-differentiability of polyhedral functions. Our next goal is to indeed demonstrate that this is true. In doing so, we rely heavily upon the reduction lemma, obtained in Theorem 3.1, as well as a characterization of strict twice epi-differentiability of prox-regular functions from [22]. The next result is a simplified version of [22, Corollary 4.3], and presents a useful characterization of strict twice epi-differentiability of convex functions, which comprise an important subclass of prox-regular functions according to [31, Example 3.30].

Proposition 4.2 (*Characterization of strict twice epi-differentiability*) *Assume that $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, $\bar{x} \in \mathbb{R}^n$ with $f(\bar{x})$ finite, and $\bar{v} \in \partial f(\bar{x})$ and that f is a convex function. Then there is a neighborhood O of (\bar{x}, \bar{v}) such that for any $(x, v) \in O \cap \text{gph } f$, the following properties are equivalent:*

- (a) f is strictly twice epi-differentiable at x for v ;
- (b) ∂f is strictly proto-differentiable at x for v .

The next result is an immediate consequence of Theorem 3.2(c) and reveals that polyhedral functions are always strictly twice epi-differentiable under a relative interior condition.

Theorem 4.1 (*Strict twice epi-differentiability of polyhedral functions*) *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and that $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then the following properties are equivalent:*

- (a) *there is a neighborhood O of $(\bar{z}, \bar{\lambda})$ such that for any $(z, \lambda) \in O \cap \text{gph } \partial g$, g is strictly twice epi-differentiable at z for λ ;*
- (b) $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$.

Proof We begin with the implication (b) \implies (a). By Proposition 4.2, (a) is equivalent to strict proto-differentiability of ∂g at z for λ for any $(z, \lambda) \in O \cap \text{gph } \partial g$. It follows from Theorem 3.2(c) that the latter condition holds since $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$. This shows that (b) yields (a). To prove the opposite implication, suppose that (a) holds. As pointed out earlier, when strict twice epi-differentiability holds for a function, its second subderivative and strict second subderivative coincide. Strict twice epi-differentiability of g at \bar{z} for $\bar{\lambda}$, together with (42), tells us that

$$\delta_{K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda})} = d_s^2 g(\bar{z}, \bar{\lambda}) = d^2 g(\bar{z}, \bar{\lambda}) = \delta_{K_g(\bar{z}, \bar{\lambda})},$$

where the last equality comes from [31, Propsoition 13.9]. This implies that $K_g(\bar{z}, \bar{\lambda}) - K_g(\bar{z}, \bar{\lambda}) = K_g(\bar{z}, \bar{\lambda})$, meaning that $K_g(\bar{z}, \bar{\lambda})$ is a linear subspace. By Proposition 2.2, we get $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$, which completes the proof. \square

Combining the obtained characterization of strict twice epi-differentiability of polyhedral functions with (22) allows us to conclude that this property is preserved under the Fenchel conjugate for polyhedral functions, as shown below.

Corollary 4.1 *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and that $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then the following properties are equivalent:*

- (a) *there is a neighborhood O of $(\bar{z}, \bar{\lambda})$ such that for any $(z, \lambda) \in O \cap \text{gph } \partial g$, g is strictly twice epi-differentiable at z for λ ;*
- (b) *there is a neighborhood U of $(\bar{\lambda}, \bar{z})$ such that for any $(\lambda, z) \in U \cap \text{gph } \partial g^*$, g^* is strictly twice epi-differentiable at λ for z ;*
- (c) $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$;
- (d) $\bar{z} \in \text{ri } \partial g^*(\bar{\lambda})$.

Proof We know from [31, Theorem 11.14(a)] that g^* is a polyhedral function. We obtain the equivalence of (a) and (c) and of (b) and (d) from Theorem 4.1. Corollary 3.1 also tells us that (c) and (d) are equivalent, which completes the proof. \square

We close this section with another consequence of the reduction lemma for polyhedral function about smoothness of the Moreau envelope and proximal mapping for polyhedral functions. To this end, recall that for a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ and parameter value $r > 0$, the Moreau envelope function $e_r f$ and the proximal mapping $\text{prox}_{r,f}$ are defined, respectively, by

$$e_r f(x) = \inf_{w \in \mathbb{R}^n} \left\{ f(w) + \frac{1}{2r} \|w - x\|^2 \right\},$$

and

$$\text{prox}_{r,f}(x) = \underset{w \in \mathbb{R}^n}{\text{argmin}} \left\{ f(w) + \frac{1}{2r} \|w - x\|^2 \right\}.$$

When f is convex, the subdifferential sum rule from convex analysis implies that

$$\text{prox}_{r,f}(x) = (I + r \partial f)^{-1}(x), \quad x \in \mathbb{R}^n, \quad (45)$$

where I stands for the $n \times n$ identity matrix. Furthermore, it is known that the envelope function $e_r f$ is continuously differentiable (cf. [31, Theorem 2.26]) for any convex function f . If, in addition, f is a polyhedral function, we deduce from [31, Proposition 13.9] and [31, Exercise 13.45] that the proximal mapping $\text{prox}_{r,f}$ is semidifferentiable; cf. the discussion prior to [31, Exercise 9.25(a)] or the exercise itself for definition and characterization of semidifferentiability of a vector-valued mapping. According to [4, Proposition 2D.1], the latter is equivalent to directional differentiability of $\text{prox}_{r,f}$. Below, we present a simple but useful characterization of continuous differentiability of the proximal mapping of polyhedral functions.

Theorem 4.2 *Assume that $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is a polyhedral function and that $(\bar{z}, \bar{\lambda}) \in \text{gph } \partial g$. Then the following properties are equivalent:*

- (a) $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$;
- (b) *for any $r > 0$, the envelope function $e_r g$ is \mathcal{C}^2 in a neighborhood of $\bar{z} + r\bar{\lambda}$;*
- (c) *for any $r > 0$, the proximal mapping $\text{prox}_{r,g}$ is \mathcal{C}^1 in a neighborhood of $\bar{z} + r\bar{\lambda}$.*

Furthermore, if $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$ and $r > 0$, then for any x sufficiently close to $\bar{z} + r\bar{\lambda}$, the Jacobian matrix $\nabla(\text{prox}_{r_g})(x)$ and the Hessian matrix $\nabla^2(e_r g)(x)$ can be calculated, respectively, by

$$\nabla(\text{prox}_{r_g})(x) = P_{K_g(\bar{z}, \bar{\lambda})} \quad \text{and} \quad \nabla^2(e_r g)(x) = \frac{1}{r} (I - P_{K_g(\bar{z}, \bar{\lambda})}) = \frac{1}{r} P_{K_{g^*}(\bar{\lambda}, \bar{z})},$$

where P_S , for a subspace S of \mathbb{R}^m , is understood to be the matrix associated with the linear mapping P_S , the projection onto S .

Proof Set $\varphi(z) = g(z) - \langle \bar{\lambda}, z \rangle$ for any $z \in \mathbb{R}^m$. Since $\bar{\lambda} \in \partial g(\bar{z})$, we get $0 \in \partial \varphi(\bar{z})$. This, together with the convexity of φ , tells us that \bar{z} is a global minimizer of φ over \mathbb{R}^m . Observe also that the condition $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$ is equivalent to $0 \in \text{ri } \partial \varphi(\bar{z})$ and that

$$\text{prox}_{r_\varphi}(z) = \text{prox}_{r_g}(z + r\bar{\lambda}) \quad \text{for all } z \in \mathbb{R}^m. \tag{46}$$

Picking $r > 0$ and employing [22, Theorem 4.4] imply that prox_{r_φ} is C^1 in a neighborhood of \bar{z} if and only if φ is strictly twice epi-differentiable at z for v for all $(z, v) \in \text{gph } \partial \varphi$ sufficiently close to $(\bar{z}, 0)$. Note that the parameter r in [22, Theorem 4.4] should be chosen sufficiently small since the function under consideration in [22] is prox-regular. It is, however, well known that such a restriction on r for convex functions is not necessary. By Theorem 4.1, the latter property of φ amounts to the condition $0 \in \text{ri } \partial \varphi(\bar{z})$. It is not hard to see that φ is strictly twice epi-differentiable at z for v for any $(z, v) \in \text{gph } \partial \varphi$ sufficiently close to $(\bar{z}, 0)$ if and only if g enjoys the same property at z for λ for any $(z, \lambda) \in \partial g$ sufficiently close to $(\bar{z}, \bar{\lambda})$. Combining these with (46) and Theorem 4.1, we conclude the equivalence of (a) and (c). To obtain the equivalence of (a) and (b), one can see that

$$e_r \varphi(z) = e_r g(z + r\bar{\lambda}) - \langle \bar{\lambda}, z \rangle - \frac{r}{2} \|\bar{\lambda}\|^2 \quad \text{for all } z \in \mathbb{R}^m.$$

This, combined with a similar argument via [22, Theorem 4.4], confirms that (a) and (b) are equivalent.

Finally, pick $r > 0$ and suppose that $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$. By (b) and (c), there exists a neighborhood U of $\bar{z} + r\bar{\lambda}$ such that prox_{r_g} is C^1 and $e_r g$ is C^2 on U . Take $x \in U$ and set $y = \text{prox}_{r_g}(x)$. By (45), we get $v := r^{-1}(x - y) \in \partial g(y)$. It also follows from (45) and the definition of graphical derivative that

$$D(\text{prox}_{r_g})(x)(w) = (I + rD(\partial g)(y, v))^{-1}(w) \quad \text{for all } w \in \mathbb{R}^m,$$

which, together with (33), brings us to

$$D(\text{prox}_{r_g})(x)(w) = (I + rN_{K_g(y, v)})^{-1}(w) = (I + N_{K_g(y, v)})^{-1}(w) = P_{K_g(y, v)}(w)$$

for any $w \in \mathbb{R}^m$. By (c), the proximal mapping prox_{r_g} is differentiable at x and thus we get $D(\text{prox}_{r_g})(x) = \nabla(\text{prox}_{r_g})(x)$. Combining these confirms, the claimed

formula for the Jacobian matrix $\nabla(\text{prox}_{r_g})(x)$. Recall also from [31, Theorem 2.26] that $\nabla(e_r g)(x) = r^{-1}(x - \text{prox}_{r_g}(x))$ for any $x \in \mathbb{R}^m$. By (b), the envelope function $e_r g$ is twice differentiable at x and thus we have

$$\nabla^2(e_r g)(x) = \frac{1}{r}(I - \nabla(\text{prox}_{r_g})(x)) = \frac{1}{r}(I - P_{K_g(y,v)}) = \frac{1}{r}P_{K_g^*(v,y)},$$

where the last equality comes from the identity $P_{K_g(y,v)} + P_{K_g(y,v)^*} = I$ together with (21). Now, we claim that $K_g(y, v) = K_g(\bar{z}, \bar{\lambda})$ whenever $x \in U$. This can be accomplished via Theorem 3.2(b) provided that we show $(y, v) \in \text{gph } \partial g$ is sufficiently close to $(\bar{z}, \bar{\lambda})$. Since the proximal mapping is nonexpansive and since $\text{prox}_{r_g}(\bar{z} + r\bar{\lambda}) = \bar{z}$, we get

$$\|y - \bar{z}\| = \|\text{prox}_{r_g}(x) - \text{prox}_{r_g}(\bar{z} + r\bar{\lambda})\| \leq \|x - \bar{z} - r\bar{\lambda}\|.$$

Moreover, we have

$$\|v - \bar{\lambda}\| = \|r^{-1}(x - y) - \bar{\lambda}\| \leq r^{-1}\|x - \bar{z} - r\bar{\lambda}\| + r^{-1}\|y - \bar{z}\| \leq 2r^{-1}\|x - \bar{z} - r\bar{\lambda}\|.$$

Using these estimates and shrinking U , if necessary, confirm our claim and hence prove the claimed formulas for the Jacobian matrix $\nabla(\text{prox}_{r_g})(x)$ and the Hessian matrix $\nabla^2(e_r g)(x)$. \square

We should mention that smoothness of projection mapping onto a closed convex set was first studied by Holmes in [8] in Hilbert spaces. His main result, [8, Theorem 2], states that if $C \subset \mathbb{R}^d$ is a closed convex set, $x \in \mathbb{R}^d$, the boundary of C is a \mathcal{C}^2 -smooth manifold (cf. [31, Example 6.8]) around $y = P_C(x)$, then the projection mapping P_C is \mathcal{C}^1 in a neighborhood of the open normal ray $\{y + t(x - y) \mid t > 0\}$. As pointed out by Hiriart-Urruty in [7], when the projection point y is a *corner point*, Holmes's result can not be utilized to study smoothness of the projection mapping because the boundary of C fails to be a \mathcal{C}^2 smooth manifold around y . In contrast, Theorem 4.2 goes beyond the projection mapping and provides a characterization of smoothness of the proximal mapping of a polyhedral function via a verifiable condition. While our result is limited to polyhedral functions, our approach via second-order variational analysis opens a new door to study smoothness of projection mappings of convex sets. It is important to emphasize that our approach to characterize smoothness of proximal mappings demonstrates that instead of expecting smoothness of the boundary of the convex set under consideration, we should look for a second-order regularity condition, which seems to be the driving force for such a result.

Corollary 4.2 *Assume that $C \subset \mathbb{R}^m$ is a polyhedral convex set and $x \in \mathbb{R}^m$. Then P_C is \mathcal{C}^1 in a neighborhood of x if and only if $x - z \in \text{ri } N_C(z)$, where $z = P_C(x)$.*

Proof Applying Theorem 4.2 to the polyhedral function $g = \delta_C$ proves the claimed equivalence. \square

Note that a characterization of differentiability *at* a point, but not continuous differentiability *around* a point, of the projection mapping P_C , C being a polyhedral convex set, via the same relative interior condition as in Corollary 4.2 can be found in [5, Corollary 4.1.2]. Not only is our proof different from the one in [5], but Corollary 4.2 also improves the latter result by showing that differentiability of the projection mapping at a given point can be strengthened to the \mathcal{C}^1 property of this mapping in a neighborhood of that point.

Remark 4.1 Assume that the set $C \subset \mathbb{R}^m$ considered in Corollary 4.2 is an affine subspace. The normal cone $N_C(z)$ is then a subspace for any $z \in C$, and, the relative interior condition in the latter corollary therefore holds automatically. Thus, we can conclude that the projection onto an affine subspace is \mathcal{C}^1 in a neighborhood of any point in \mathbb{R}^m , and hence \mathcal{C}^1 everywhere. Pick $z \in C$, $\lambda \in N_C(z) = C^\perp$, and $r > 0$ arbitrarily, and observe by Theorem 4.2 that $\nabla P_C(x) = P_{K_C(z,\lambda)} = P_{\text{par}\{C\}}$ for all x sufficiently close to $z + r\lambda$, where $\text{par}\{C\}$ denotes the linear space parallel to C . We then infer that the Jacobian mapping ∇P_C is constant everywhere and arrive at a well-known fact that the projection onto an affine subspace is an affine mapping.

5 Regularity Properties of Variational Systems

In this section, we aim to explore the relationship between important regularity properties of the solution mapping to the generalized equation (1). When $g = \delta_C$ with C being a polyhedral convex set, the generalized equation (1) will be an example of variational inequalities. In this case, the seminal paper [3] revealed for the first time that strong metric regularity and metric regularity of the solution mapping to the canonical perturbation of (1) are equivalent; see below for the definitions of both concepts. We aim to present a similar result for the generalized equation (1), which is based upon the reduction lemma for polyhedral functions. Furthermore, we show that for nondegenerate solutions to (1), meaning solutions that satisfy the condition (2), one can find conditions under which the solution mapping to the generalized equation (1) is continuously differentiable.

To explore regularity properties of (1), define the set-valued mapping $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ by

$$G(x) := \psi(x) + \partial g(x), \quad x \in \mathbb{R}^m, \quad (47)$$

with $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ being a \mathcal{C}^1 mapping and $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ being a polyhedral function, and then consider the solution mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ to the canonical perturbation of the generalized equation (1) by

$$S(y) := G^{-1}(y) = \{x \in \mathbb{R}^m \mid y \in \psi(x) + \partial g(x)\}, \quad y \in \mathbb{R}^m. \quad (48)$$

Recall that a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *metrically regular* at \bar{x} for $\bar{y} \in F(\bar{x})$ if there exist $\kappa \geq 0$ and neighborhoods U of \bar{x} and V of \bar{y} such that the distance estimate

$$\text{dist}(x, F^{-1}(y)) \leq \kappa \text{dist}(y, F(x)) \quad (49)$$

holds for all $(x, y) \in U \times V$. The mapping is called *strongly metrically regular* at \bar{x} for \bar{y} if F^{-1} admits a Lipschitz continuous single-valued localization around \bar{y} for \bar{x} , which means that there exist neighborhoods U of \bar{x} and V of \bar{y} such that the mapping $y \mapsto F^{-1}(y) \cap U$ is single-valued and Lipschitz continuous on V . According to [4, Proposition 3G.1], strong metric regularity of F at \bar{x} for \bar{y} amounts to F being metrically regular at \bar{x} for \bar{y} and its inverse F^{-1} admitting a single-valued localization around \bar{y} for \bar{x} .

Theorem 5.1 (Equivalence between metric regularity and strong metric regularity) *Assume that \bar{x} is a solution to the generalized equation (1). Then the mapping G , taken from (47), is metrically regular at \bar{x} for 0 if and only if it is strongly metrically regular at \bar{x} for 0.*

Proof Note that one can argue via [4, Corollary 3F.5] that (strong) metric regularity of the mapping G from (47) at \bar{x} for 0 is equivalent to that of the mapping $x \mapsto \psi(\bar{x}) + \nabla\psi(\bar{x})(x - \bar{x}) + \partial g(x)$ at \bar{x} for 0. Employing then Theorem 3.1 tells us that (strong) metric regularity of the latter amounts to the same property of the mapping $\Phi(w) := \nabla\psi(\bar{x})w + N_{\bar{K}}(w)$ with $\bar{K} = K_g(\bar{x}, -\psi(\bar{x}))$ and $w \in \mathbb{R}^m$ at 0 for 0. Since Φ fits into the framework of [3], one can conclude via [3, Theorem 3] (or [9, Corollary 9.7]) that metric regularity and strong metric regularity are equivalent for Φ at 0 for 0, which implies that these properties are equivalent for G at \bar{x} for 0 and hence completes the proof. \square

We proceed with an application of Theorem 5.1 in studying regularity properties of the solution mapping to the KKT system of the composite minimization problem

$$\text{minimize } \varphi(x) + (g \circ \Phi)(x), \quad \text{subject to } x \in \mathbb{R}^n, \quad (50)$$

where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are C^2 functions and $g : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$ is a polyhedral function. The KKT system associated with the composite problem (50) is given by

$$0 = \nabla_x L(x, \lambda), \quad \lambda \in \partial g(\Phi(x)), \quad (51)$$

where $L(x, \lambda) := \varphi(x) + \langle \lambda, \Phi(x) \rangle$ with $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ is the Lagrangian of (50). A pair $(\bar{x}, \bar{\lambda})$ is called a *KKT point* of (50) provided that it satisfies the KKT system (51). Define the mapping $\Psi : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ by

$$\Psi(x, \lambda) := \begin{bmatrix} \nabla_x L(x, \lambda) \\ -\Phi(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \partial g^*(\lambda) \end{bmatrix} \quad (52)$$

and observe that $(\bar{x}, \bar{\lambda})$ is a KKT point if and only if $(0, 0) \in \Psi(\bar{x}, \bar{\lambda})$. We aim at finding conditions under which the solution mapping to the canonical perturbed of the KKT system (51), defined by

$$S_{\text{KKT}}(p, q) := \Psi^{-1}(p, q) = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \mid (p, q) \in \Psi(x, \lambda)\},$$

has a Lipschitz continuous single-valued localization. As shown below, this can be distilled from Theorem 5.1.

Theorem 5.2 *Let $(\bar{x}, \bar{\lambda})$ be a KKT point of (50). Then the following properties are equivalent:*

- (a) *the mapping Ψ is metrically regular at $(\bar{x}, \bar{\lambda})$ for $(0, 0)$;*
- (b) *the mapping Ψ is strongly metrically regular at $(\bar{x}, \bar{\lambda})$ for $(0, 0)$;*
- (c) *the solution mapping S_{KKT} has a Lipschitz continuous single-valued localization around $(0, 0)$ for $(\bar{x}, \bar{\lambda})$;*
- (d) *the implication*

$$\begin{aligned} \nabla_{xx}^2 L(\bar{x}, \bar{\lambda})w + \nabla\Phi(\bar{x})^*w' = 0, \\ w' \in D^*(\partial g)(\Phi(\bar{x}), \bar{\lambda})(\nabla\Phi(\bar{x})w) \implies (w, w') = (0, 0) \end{aligned}$$

holds.

Proof Set

$$\psi(x, \lambda) := \begin{bmatrix} \nabla_x L(x, \lambda) \\ -\Phi(x) \end{bmatrix} \quad \text{and} \quad \hat{g}(x, \lambda) := g^*(\lambda), \quad (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m,$$

and observe that the KKT system (51) can be written as the generalized equation

$$(0, 0) \in \Psi(x, \lambda) = \psi(x, \lambda) + \partial\hat{g}(x, \lambda). \tag{53}$$

It follows from [31, Theorem 11.14(a)] that g^* is a polyhedral function and so is \hat{g} . Also, we deduce from [31, Proposition 10.5] that $\partial\hat{g}(x, \lambda) = \{0\} \times \partial g^*(\lambda)$. The equivalence of (a)–(c) comes directly from Theorem 5.1. Part (d) is a direct consequence of the coderivative criterion from [31, Theorem 9.40] for metric regularity of the mapping Ψ at $(\bar{x}, \bar{\lambda})$ for $(0, 0)$:

$$(0, 0) \in D^*\Psi((\bar{x}, \bar{\lambda}), (0, 0))(w, w') \implies (w, w') = (0, 0),$$

which translates as the implication

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} \nabla_{xx}^2 L(\bar{x}, \bar{\lambda})w - \nabla\Phi(\bar{x})^*w' \\ \nabla\Phi(\bar{x})w + D^*(\partial g^*)(\bar{\lambda}, \Phi(\bar{x}))(w') \end{bmatrix} \implies (w, w') = (0, 0).$$

The second inclusion in this implication amounts to

$$-w' \in D^*(\partial g)(\Phi(\bar{x}), \bar{\lambda})(\nabla\Phi(\bar{x})w).$$

Combining this with the above implication confirms that (d) and (a) are equivalent and hence completes the proof. □

For classical nonlinear programming problems (NLPs), it is well known that metric regularity and strong metric regularity of KKT systems are equivalent; see [4, Theorem 4I.2] and [11, Section 7.5]. Theorem 5.2 extends this result for the composite problem (50).

According to Theorem 5.1, if \bar{x} is a solution to (1) and if the mapping G in (47) is metrically regular at \bar{x} for 0, then the solution mapping S in (48) has a Lipschitz continuous single-valued localization around 0 for \bar{x} . Our final goal in this section is to show that if, in addition, \bar{x} is a nondegenerate solution to (1) which means it satisfies (2), then the aforementioned localization of S is continuously differentiable around 0. To this end, we begin with showing that strict proto-differentiability is preserved for the sum of two functions.

Proposition 5.1 (Sum rule for strict proto-derivative) *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be C^1 around \bar{x} and let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $\bar{y} \in F(\bar{x})$. Then we have*

$$\tilde{D}(f + F)(\bar{x}, f(\bar{x}) + \bar{y})(w) = \nabla f(\bar{x})w + \tilde{D}F(\bar{x}, \bar{y})(w) \quad \text{for all } w \in \mathbb{R}^n.$$

Moreover, F is strictly proto-differentiable at \bar{x} for \bar{y} if and only if $f + F$ is strictly proto-differentiable at \bar{x} for $f(\bar{x}) + \bar{y}$.

Proof The given formula for $\tilde{D}(f + F)(\bar{x}, f(\bar{x}) + \bar{y})$ was already established in [31, Exercise 10.43(b)]. To justify the second claim, suppose that F is strictly proto-differentiable at \bar{x} for \bar{y} . Let $(w, u) \in \text{gph } \tilde{D}(f + F)(\bar{x}, \bar{y} + f(\bar{x}))$ and take arbitrary sequences $t_k \searrow 0$, and $(x^k, z^k) \rightarrow (\bar{x}, \bar{y} + f(\bar{x}))$ with $\{(x^k, z^k)\}_{k \in \mathbb{N}} \subset \text{gph}(f + F)$. The latter tells us that $(x^k, y^k) \rightarrow (\bar{x}, \bar{y})$ with $y^k := z^k - f(x^k)$. By the sum rule for the strict graphical derivative, we get $(w, u - \nabla f(\bar{x})w) \in \text{gph } \tilde{D}F(\bar{x}, \bar{y})$. Since F is strictly proto-differentiable at \bar{x} for \bar{y} , we find a sequence $(w^k, v^k) \rightarrow (w, u - \nabla f(\bar{x})w)$ such that $y^k + t_k v^k \in F(x^k + t_k w^k)$ for all $k \in \mathbb{N}$, which in turn implies for any $k \in \mathbb{N}$ that

$$z^k + t_k u^k \in (f + F)(x^k + t_k w^k) \quad \text{with } u^k := \frac{f(x^k + t_k w^k) - f(x^k)}{t_k} + v^k.$$

Since $\{(w^k, u^k)\}_{k \in \mathbb{N}}$ converges to (w, u) , we conclude that $f + F$ is strictly proto-differentiable at \bar{x} for $f(\bar{x}) + \bar{y}$.

Assume now that $f + F$ is strictly proto-differentiable at \bar{x} for $f(\bar{x}) + \bar{y}$. By the argument above, we can conclude that $F = -f + (f + F)$ is strictly proto-differentiable at \bar{x} for \bar{y} , which completes the proof. \square

Recall from [27, page 173] that a set $C \subset \mathbb{R}^d$ is called *smooth* at $\bar{x} \in C$ if the tangent cone $T_C(\bar{x})$ is a linear subspace of \mathbb{R}^d and the “lim sup” in (3) is the “lim.” It is called *strictly smooth* at \bar{x} if $\widehat{T}_C(\bar{x}) = \tilde{T}_C(\bar{x})$, where both tangent cones were defined in (31). It follows from [27, Proposition 3.1] that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz continuous around $\bar{x} \in \mathbb{R}^n$, then $\text{gph } f$ is (strictly) smooth at $(\bar{x}, f(\bar{x}))$ if and only if f is (strictly) differentiable at \bar{x} .

Theorem 5.3 *Assume that \bar{x} is a nondegenerate solution to the generalized equation (1). Then the solution mapping S to (1) has a Lipschitz continuous single-valued localization σ around $0 \in \mathbb{R}^m$ for \bar{x} if and only if one of the equivalent conditions*

$$(a) \quad (\nabla \psi(\bar{x})\bar{K}) + \bar{K}^\perp = \mathbb{R}^m;$$

$$(b) \{w \in \mathbb{R}^m \mid \nabla\psi(\bar{x})^*w \in \bar{K}^\perp\} \cap \bar{K} = \{0\},$$

holds, where $\bar{K} = K_g(\bar{x}, -\psi(\bar{x}))$. In this case, the function σ is \mathcal{C}^1 in a neighborhood of 0 and

$$\nabla\sigma(y) = B(B^*\nabla\psi(\sigma(y))B)^{-1}B^*$$

for all y sufficiently close to 0, where $B \in \mathbb{R}^{m \times s}$ is a matrix whose columns form a basis for the linear subspace \bar{K} with $s = \dim \bar{K}$.

Proof The equivalence of the conditions (a) and (b) falls directly from [31, Corollary 11.25(c)]. Part (b) is also a translation of the coderivative criterion from [31, Theorem 9.40] for metric regularity of the mapping G in (47) at \bar{x} for 0. Indeed, we have for any $w \in \mathbb{R}^m$ that

$$D^*G(\bar{x}, 0)(w) = \nabla\psi(\bar{x})^*w + D^*(\partial g)(\bar{x}, -\psi(\bar{x}))(w) = \nabla\psi(\bar{x})^*w + N_{\bar{K}}(w),$$

where the last equality results from Theorems 3.3 and 3.2(a). This reads as $u \in D^*G(\bar{x}, 0)(w)$ if and only if $u \in \nabla\psi(\bar{x})^*w + \bar{K}^\perp$ and $w \in \bar{K}$. Using this, one can see that (b) amounts to saying that G is metrically regular at \bar{x} for 0. By Theorem 5.1, the latter is equivalent to strong metric regularity of G at \bar{x} for 0, a property that amounts to saying that the solution mapping S in (48) has a Lipschitz continuous single-valued localization around 0 for \bar{x} . So we find neighborhoods U of \bar{x} and V of 0 such that the mapping $y \mapsto S(y) \cap U$ is single-valued and Lipschitz continuous on V . Define the function $\sigma : V \rightarrow U$ by $\sigma(y) = S(y) \cap U$ for any $y \in V$. We are going to demonstrate that σ is \mathcal{C}^1 in a neighborhood of 0. To this end, it follows from the nondegeneracy condition (2) and Theorem 3.2(c) that ∂g is strictly proto-differentiable at x for z whenever $(x, z) \in \text{gph } \partial g$ is sufficiently close to $(\bar{x}, -\psi(\bar{x}))$, which, together with Proposition 5.1, tells us that G is strictly proto-differentiable at x for y whenever $(x, y) \in \text{gph } G$ is sufficiently close to $(\bar{x}, 0)$. Suppose without loss of generality that G is strictly proto-differentiable at x for y whenever $(x, y) \in (U \times V) \cap \text{gph } G$. Choose a pair $(x, y) \in (U \times V) \cap \text{gph } G$ and observe that σ is strictly proto-differentiable at y for x . This implies that $\text{gph } \sigma$ is strictly smooth at (y, x) . Since σ is Lipschitz continuous on V , it follows from [27, Proposition 3.1] that σ is strictly differentiable at y . This means that σ is strictly differentiable on V , a property equivalent to saying that σ is \mathcal{C}^1 on V (cf. [4, Exercise 1D.8]).

Finally, to justify the claimed formula for the Jacobian matrix of σ , take $y \in V$. Thus, for any $u \in \mathbb{R}^n$ we have $w = \nabla\sigma(y)u = D\sigma(y)(u)$, which is equivalent to $u \in DG(x, y)(w)$, where $x = \sigma(y)$. By the definition of G from (47), the sum rule for the graphical derivative from [31, Exercise 10.43(b)], and Theorem 3.2(a), we obtain for any $(x, y) \in \text{gph } G$ and $w \in \mathbb{R}^m$ that

$$\begin{aligned} DG(x, y)(w) &= \nabla\psi(x)w + D(\partial g)(x, y - \psi(x))(w) \\ &= \nabla\psi(x)w + N_{K_g(x, y - \psi(x))}(w). \end{aligned}$$

Recall that \bar{x} is a nondegenerate solution to (1). Shrinking the neighborhoods U and V , if necessary, we deduce from Corollary 3.2(b) that for any $(x, y) \in (U \times V) \cap \text{gph } G$,

we have $K_g(x, y - \psi(x)) = K_g(\bar{x}, -\psi(\bar{x})) = \bar{K}$. Combining these with \bar{K} being a linear subspace brings us to

$$u \in DG(x, y)(w) \iff u \in \nabla\psi(x)w + \bar{K}^\perp, \quad w \in \bar{K}. \quad (54)$$

By the definition of B , we have $\bar{K} = \text{rge } B$, or equivalently, $\bar{K}^\perp = \ker B^*$. Thus, we find $q \in \mathbb{R}^s$ such that $w = Bq$ and that $u - \nabla\psi(x)Bq \in \ker B^*$, or equivalently, $B^*u = B^*\nabla\psi(x)Bq$. We claim that the matrix $B^*\nabla\psi(x)B$ is nonsingular, which results in

$$\nabla\sigma(y)u = w = Bq = B(B^*\nabla\psi(x)B)^{-1}B^*u,$$

and hence confirms the claimed formula for $\nabla\sigma(y)$. To justify our claim, we conclude from metric regularity of G at \bar{x} for 0 and its equivalent description by (a) that such a condition holds for all $x \in \text{dom } \partial g$ sufficiently closed to \bar{x} , due to the robustness of metric regularity. Thus, for any $u \in \mathbb{R}^m$, there exist $w \in \bar{K}$ and $w' \in \bar{K}^\perp$ such that $u = \nabla\psi(x)w + w'$. Since $\bar{K} = \text{rge } B$, we find $q \in \mathbb{R}^s$ such that $w = Bq$ and hence $B^*u = B^*(\nabla\psi(x)Bq + w') = B^*\nabla\psi(x)Bq$. Since $u \in \mathbb{R}^m$ was taken arbitrary, the latter equality leads us to

$$\text{rge } (B^*\nabla\psi(x)B) = B^*\mathbb{R}^m = \text{rge } B^* = \mathbb{R}^s,$$

where the last equality comes from B having full column rank. This confirms that $B^*\nabla\psi(x)B$ is an $s \times s$ nonsingular matrix and completes the proof. \square

Theorem 5.3 can be viewed as an extension of the classical inverse mapping theorem for generalized equations. This well-known result ensures under the nonsingularity of the Jacobian matrix that the inverse of a \mathcal{C}^1 mapping has a Lipschitz continuous single-valued localization, which is continuously differentiable. Robinson, in his landmark paper [23], showed that for generalized equations, one can expect under appropriate conditions that their solution mappings have a Lipschitz continuous single-valued localization. Theorem 5.3 demonstrates that for nondegenerate solutions to some particular class of generalized equations such a localization can be continuously differentiable as well.

6 Summary and Discussion

Employing reduction lemma for polyhedral functions, we comprehensively answered the question arising at the beginning of this paper: “Should we expect further properties, if, in addition, we assume that $\bar{\lambda} \in \text{ri } \partial g(\bar{z})$?” Not only did we show that such a relative interior condition leads to strict twice epi-differentiability of polyhedral functions, but we also proved that it actually characterizes this property in a neighborhood of the reference point as well as \mathcal{C}^1 -smoothness of the proximal mapping and \mathcal{C}^2 -smoothness of the Moreau envelope. We were then able to establish an appealing extension of the classical inverse mapping theorem for variational systems associated

with subgradient mapping of polyhedral functions, namely, under the relative interior assumption, the solution mapping to such generalized equations has a \mathcal{C}^1 single-valued localization. Since polyhedral functions are encountered in a wide variety of bundle-type methods (see, e.g., [10, 13, 15]), which rely on polyhedral approximations of the original model, the obtained results are potentially beneficial to numerical aspects of constrained and composite optimization problems. For instance, the \mathcal{C}^2 -smoothness of the Moreau envelope can be exploited to accelerate the local convergence of a proximal bundle method.

In our subsequent works, we demonstrate that the obtained characterization of strict twice epi-differentiability holds for certain nonconvex and nonpolyhedral functions such as \mathcal{C}^2 -decomposable function defined by Shapiro in [32]. We don't know whether similar results can be obtained for nonconvex polyhedral functions such as those appearing in mathematical programs with complementarity constraints (MPCCs). This is an important direction for our future research.

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