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Outer billiards in the spaces of oriented geodesics of the three-dimensional space forms

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Abstract

Let M_κ be the three-dimensional space form of constant curvature $\kappa = 0, 1, -1$, that is, Euclidean space $\mathbb{R}^3$, the sphere S^3, or hyperbolic space H^3. Let S be a smooth, closed, strictly convex surface in M_κ. We define an outer billiard map B on the four-dimensional space $\mathcal{G}_\kappa$ of oriented complete geodesics of M_κ, for which the billiard table is the subset of $\mathcal{G}_\kappa$ consisting of all oriented geodesics not intersecting S. We show that B is a diffeomorphism when S is quadratically convex. For $\kappa = 1, -1$, $\mathcal{G}_\kappa$ has a Kähler structure associated with the Killing form of $\mathrm{Iso}(M_\kappa)$. We prove that B is a symplectomorphism with respect to its fundamental form and that B can be obtained as an analogue to the construction of Tabachnikov of the outer billiard in $\mathbb{R}^{2n}$ defined in terms of the standard symplectic structure. We show that B does not preserve the fundamental symplectic form on $\mathcal{G}_\kappa$ associated with the cross product on M_κ, for $\kappa = 0, 1, -1$. We initiate the dynamical study of this outer billiard in the hyperbolic case by introducing and discussing a notion of holonomy for periodic points.

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1 | INTRODUCTION

1.1 | Motivation

The *dual* or *outer billiard map* B is defined in the plane as a counterpart to the usual inner billiards. Let $\gamma \subset \mathbb{R}^2$ be a smooth, closed, strictly convex curve, and let p be a point outside of γ. There are two tangent lines to γ through p; choose one of them consistently, say, the right one from the viewpoint of p, and define $B(p)$ as the reflection of p in the point of tangency (see Figure 1).

The study of the dual billiard was originally popularized by Moser [21, 22], who considered the dual billiard map as a crude model for planetary motion and showed that orbits of the map cannot escape to infinity. The outer billiard map has since been studied in a number of settings; see [7, 26, 27, 31] for surveys.

In [28], Tabachnikov generalized planar outer billiards to even-dimensional standard symplectic space $(\mathbb{R}^{2n}, \omega)$ as follows: given a smooth, closed hypersurface M in $\mathbb{R}^{2n}$ that is quadratically convex (that is, the shape operator at any point of M is definite), the restriction of ω to each tangent space T_qM has a one-dimensional kernel, called the *characteristic line*. If ν is the outward-pointing unit normal vector field on M and $\mathbb{R}^{2n}$ is identified with $\mathbb{C}^n$, then $\{q - ti\nu(q) \mid t \in \mathbb{R}\}$ is the characteristic line at $q \in M$. Tabachnikov showed that the collection of (geodesic) rays $\{q - ti\nu(q) \mid t > 0\}$, indexed by $q \in M$, foliate the exterior U of M; in particular, for each $p \in U$ there exists a unique such ray passing through p. This gives a smooth outer billiard map taking p to its reflection in the corresponding tangency point:

$$B : U \to U, \qquad q - ti\nu(q) \mapsto q + ti\nu(q). \tag{1}$$

Tabachnikov proved that the map is a symplectomorphism of the exterior of M, and in [29], he showed that the number of 3 -periodic trajectories of the outer billiard map in $\mathbb{R}^{2n}$ is not less than $2n$.

1.2 | Spaces of geodesics of space forms

The goal of the present article is to define and study outer billiards in another setting: on the space of oriented geodesics of the three-dimensional space form M_κ of constant curvature $\kappa = 0, 1, -1$, that is, Euclidean space $\mathbb{R}^3$, the sphere S^3, or hyperbolic space H^3.

The *space of oriented geodesics* $\mathcal{G}_\kappa$ is a four-dimensional manifold whose elements are the oriented trajectories of complete geodesics in M_κ. Elements of $\mathcal{G}_\kappa$ can also be described as equivalence classes of unit speed geodesics, where $\gamma \sim \sigma$ if $\sigma(t) = \gamma(t + t_o)$ for some $t_o \in \mathbb{R}$. When $\kappa = 0, -1$,

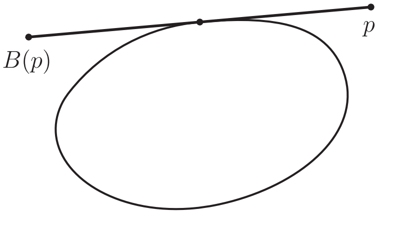

FIGURE 1 The outer billiard map in the plane.

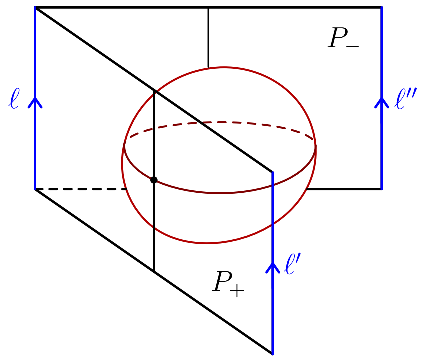

FIGURE 2 The geodesic ℓ is in the outer billiard correspondence with ℓ' and ℓ''.

$\mathcal{G}_\kappa$ is the space of oriented lines in Euclidean or hyperbolic space, which is diffeomorphic to TS^2, and $\mathcal{G}_1$ is the space of oriented great circles of S^3 (or equivalently, the Grassmannian of oriented planes in $\mathbb{R}^4$), which is diffeomorphic to $S^2 \times S^2$ (see [10] or [20]). Historically, the space of oriented geodesics is at the core of symplectic geometry through its relationship with optics. This space possesses a rich geometry (for instance, it admits two natural Kähler structures), whose study began with Hitchin [19] and continued with [1, 2, 8, 14, 23, 24]. It has been useful, for instance, in the characterization of geodesic foliations [12, 13, 15–18, 25].

1.3 | The definition of the outer billiard map on $\mathcal{G}_\kappa$

Let S be a smooth, closed, strictly convex surface in M_κ, that is, for each $p \in S$, the complete totally geodesic surface tangent to p intersects S only at p, near p. By smooth we understand of class C^∞. We denote by $\mathcal{M}$ the three-dimensional space of geodesics that are tangent to S, which can be naturally identified with T^1S, the unit tangent bundle of S (we were inspired by [9]).

We first consider the cases $\kappa = 0, -1$. We define the *billiard table*

$$\mathcal{U} = \mathcal{G}_\kappa - \{\text{oriented geodesics intersecting } S\};$$

this is an open submanifold of $\mathcal{G}_\kappa$ with boundary equal to $\mathcal{M}$.

We say that two distinct geodesics ℓ and ℓ' in $\mathcal{U}$ are in the *outer billiard correspondence* if there exists a complete totally geodesic surface P, tangent to S at a point p, containing ℓ and ℓ', such that ℓ' can be obtained by parallel translating ℓ along the shortest geodesic from ℓ to p, twice the distance from ℓ to p; see Figure 2.

Given $\ell \in \mathcal{U}$, there exist exactly two complete totally geodesic surfaces containing ℓ and tangent to S (the assertion is clear for $\kappa = 0$ and for $\kappa = -1$, for instance, using the Klein ball model of hyperbolic space, see Section 6). So, ℓ is in correspondence with exactly two other elements of $\mathcal{U}$. To define the outer billiard map, we use the orientation of ℓ to choose the surface on the right, say P_+, as follows.

Let $p_\pm$ be the point of tangency of the surface S with $P_\pm$ and let $q_\pm$ be the point on ℓ realizing the distance $d_\pm$ to $p_\pm$ (see Figure 3, left).

Let $\gamma_\pm$ be the geodesic ray joining $q_\pm$ with $p_\pm$, with $\gamma_\pm(0) = q_\pm$ and $\gamma_\pm(d_\pm) = p_\pm$. Let α be a unit speed geodesic such that $\ell = [\alpha]$ and define $t_\pm$ by $\alpha\left(t_\pm\right) = q_\pm$ (see Figure 3, right). Let $W_\pm$ be the

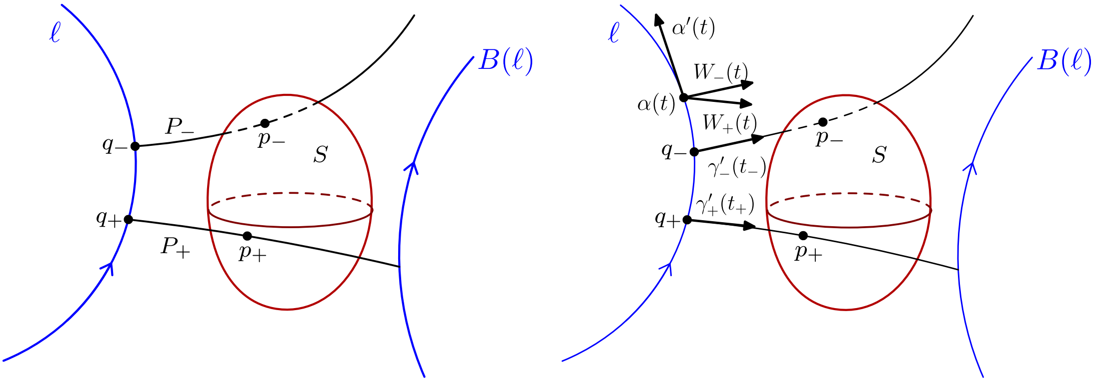

FIGURE 3 The outer billiard map on $\mathcal{L}_\kappa$ associated with S.

parallel vector field along α with $W_\pm(t_\pm) = \gamma'_\pm(0)$. Now choose the sign $+$ so that $\{W_+, W_-, \alpha'\}$ is a positively oriented frame along α. The *outer billiard map* $B : \mathcal{U} \to \mathcal{U}$ can now be defined:

$B(\ell)$ is the oriented line obtained by parallel translating ℓ along γ_+ between 0 and $2d_+$.

Moreover, B is a bijection.

Notice that in the hyperbolic case (in contrast with the Euclidean), parallel translating a line ℓ a distance d along unit speed geodesic rays γ_1 and γ_2 orthogonal to ℓ depends on the initial points $\gamma_1(0)$ and $\gamma_2(0)$ in $\ell = [\alpha]$, even if $\gamma_2'(0)$ is the parallel transport of $\gamma_1'(0)$ along α.

Now we consider the case $\kappa = 1$. We will see that a similar definition of outer billiard map can be given. For an oriented great circle c not intersecting S, there exist exactly two great spheres of S^3 containing c and tangent to S, but there may be more than one (actually, a circle worth of them) shortest geodesics between c and the tangency point in S, so that q_+ or q_- are not well-defined. That is the case when the distance from c to the tangency point in S is $\pi/2$. We call C the set of these oriented great circles. We will describe this set later, in Subsection 2.1, in terms of the Gauss map of S, $q \mapsto T_qS$, using the canonical identification of oriented great circles with oriented planes through the origin in $\mathbb{R}^4$. Although the outer billiard map B will be still well-defined on C as the involution $c \mapsto -c$ (the same circle with opposite orientation), it is easier to exclude C from the domain of definition. Thus, in the spherical case we define

$$\mathcal{U} = \{c \in \mathcal{G}_1 \mid c \text{ does not intersect } S\} - C. \tag{2}$$

Proposition 1.1. *Let S be a strictly convex closed surface in S^3. The analogue of the outer billiard map in the cases $\kappa = 0, -1$ is well-defined on $\mathcal{U}$ for $\kappa = 1$ and is a bijection onto this set.*

For $\kappa = 0, 1, -1$, we call $B : \mathcal{U} \to \mathcal{U}$ as above the *outer billiard map on $\mathcal{G}_\kappa$ associated with S*. We will show that B is a diffeomorphism under the stronger condition that S is quadratically convex (in particular, strictly convex).

Theorem 1.2. *Let S be a smooth, closed, quadratically convex surface in the space form M_κ. The outer billiard map $B : \mathcal{U} \to \mathcal{U}$ associated with S is a diffeomorphism.*

The following proposition shows that strict convexity is not enough for the smoothness of the billiard map. We present the example just for the sake of completeness, as it is essentially the known corresponding fact for plane outer billiards.

Proposition 1.3. *Let S be a closed strictly convex surface in $\mathbb{R}^3$ that is invariant by the reflection with respect to the plane $y = 0$ and contains the graph of the function $\varphi : (-2, 2) \times (-1, 1) \to \mathbb{R}$ defined by $\varphi(x, y) = f(x) + y^2$, where f is a smooth function satisfying $f(0) = f'(0) = f''(0) = 0$. Then the associated outer billiard map B on $\mathcal{G}_0$ is not smooth.*

1.4 | Kähler structures and the analogue of Tabachnikov's construction

The space of oriented geodesics $\mathcal{G}_\kappa$ has one or two canonical Kähler structures (for $\kappa = 0$ or $\kappa = 1, -1$, respectively), so the natural question arises, whether Tabachnikov's construction (1) of the outer billiard map for $\mathbb{R}^{2n}$ can be mimicked.

To deal with this issue, next we briefly introduce Kähler structures on $\mathcal{G}_\kappa$, postponing formal definitions until Section 2.

Given $\ell \in \mathcal{G}_\kappa$, the $\pi/2$-rotation in M_κ that fixes ℓ induces a map on $\mathcal{G}_\kappa$ whose differential at ℓ is a linear operator $\mathcal{J}_\ell$ on $T_\ell\mathcal{G}_\kappa$ that squares to $-$id. It may be visualized by its action on four geodesic variations of ℓ. The geodesic ℓ may be translated in the two directions orthogonal to ℓ or rotated in two planes containing ℓ, and the operator $\mathcal{J}_\ell$ sends translations to translations and rotations to rotations; see Figure 4. The collection of these linear transformations $\mathcal{J}_\ell$ is a complex structure $\mathcal{J}$ on $\mathcal{G}_\kappa$.

For $\kappa = 0, 1, -1$, the manifold $\mathcal{G}_\kappa$ has a pseudo-Riemannian metric $g_\times$ induced by the cross product on M_κ, and $(g_\times, \mathcal{J})$ is a Kähler structure on $\mathcal{G}_\kappa$. For $\kappa = 1, -1$, there is an additional Kähler structure $(g_K, \mathcal{J})$ on $\mathcal{G}_\kappa$, where g_K is induced by the Killing form on Iso(M_κ). In the Euclidean case, the Killing form g_K degenerates. For the formal definitions of $\mathcal{J}$, g_K and $g_\times$ we use the language of Jacobi fields; see Section 2.

Having presented the Kähler structures on $\mathcal{G}_\kappa$, we can give a positive answer to the question in the beginning of the subsection for $\kappa = 1, -1$, using the Kähler structure $(g_K, \mathcal{J})$. In this formulation, we see that B is a direct analogue of the outer billiard map on $(\mathbb{R}^{2n}, \omega)$ described in Subsection 1.1.

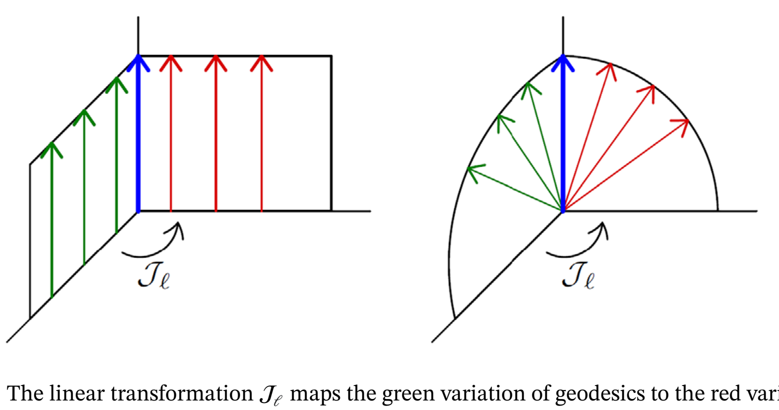

FIGURE 4 The linear transformation $\mathcal{J}_\ell$ maps the green variation of geodesics to the red variation of geodesics.

Suppose that S is a smooth, closed, strictly convex surface in M_κ, and let $\mathcal{U}$ and $\mathcal{M}$ be as above. For $\ell \in \mathcal{M}$ let $\nu(\ell)$ be the outward-pointing unit normal vector to $\mathcal{M}$ at ℓ (outward-pointing means pointing to $\mathcal{U}$). Given $\xi \in T_\ell\mathcal{M}$, let Γ_ξ denote the geodesic in $(\mathcal{G}_\kappa, g_K)$ with initial velocity ξ. Each geodesic $t \mapsto \Gamma_{\mathcal{J}\nu(\ell)}(t)$ traces out a totally geodesic surface in M_κ that is tangent to S. We will show that the collection of geodesic rays $\left\{\Gamma_{\mathcal{J}\nu(\ell)}(t) \mid t < 0\right\}$ indexed by $\ell \in \mathcal{M}$ foliates the exterior $\mathcal{U}$ of $\mathcal{M}$. In particular, for each geodesic $\ell' \in \mathcal{U}$, there exists a unique such ray passing through ℓ'. This induces an outer billiard map $B' : \mathcal{U} \to \mathcal{U}$ taking ℓ' to its “reflection” in the tangent geodesic ℓ:

$$B'(\Gamma_{\mathcal{J}\nu(\ell)}(-t)) = \Gamma_{\mathcal{J}\nu(\ell)}(t)$$

for $t > 0$. In Section 4, after proving that B' is well-defined, we will show that the outer billiard maps B and B' coincide.

Theorem 1.4. *For $\kappa = 1, -1$, let S be a smooth, closed, quadratically convex surface in M_κ, and consider $\mathcal{G}_\kappa$ endowed with the Kähler structure $(g_K, \mathcal{J})$. The map $B' : \mathcal{U} \to \mathcal{U}$ coincides with the outer billiard map B on $\mathcal{U}$ associated with S.*

The following proposition reveals that the Kähler structure $\left(g_\times, \mathcal{J}\right)$ is not appropriate in our setting, as it does not give rise to a billiard map as in Tabachnikov's construction.

Proposition 1.5. *For $\kappa = 0, 1, -1$, consider on $\mathcal{G}_\kappa$ the Kähler structure $\left(g_\times, \mathcal{J}\right)$. Let S be as in the preceding theorem and let $\ell \in \mathcal{M}$. Then the metric $g_\times$ degenerates on $T_\ell\mathcal{M}$, and for each vector N normal to $T_\ell\mathcal{M}$, the image of the geodesic in $\mathcal{G}_\kappa$ with initial velocity $\mathcal{J}(N)$ is disjoint from $\mathcal{U}$.*

1.5 | The symplectic properties of the outer billiard map

The outer billiard map interacts with the symplectic structures on $\mathcal{G}_\kappa$ as follows.

Theorem 1.6. *Let B be the outer billiard map associated with a smooth, closed, quadratically convex surface in M_κ.*

(a) *For $\kappa = 1, -1$, B is a symplectomorphism with respect to the fundamental symplectic form ω_K of $(\mathcal{G}_\kappa, g_K, \mathcal{J})$.*
(b) *For $\kappa = 0, 1, -1$, B does not preserve the fundamental symplectic form $\omega_\times$ of $(\mathcal{G}_\kappa, g_\times, \mathcal{J})$.*

Additional context for Theorem 1.6 is given in Proposition 5.1, which establishes a relationship with plane hyperbolic outer billiards (see [30]) and supports the fact that ω_K (in contrast with $\omega_\times$) is the natural symplectic form in our context.

In the Euclidean case, the outer billiard map B preserves parallelism, yielding an S^2-worth of planar outer billiards as in Figure 1. That is, given a fixed direction v in the two-sphere, the orthogonal projection of S onto any plane P orthogonal to v determines a smooth closed convex curve γ in P (the *shadow* of S with respect to v). These shadows vary smoothly with respect to v, and the outer billiard map B, restricted to lines with direction v, is equivalent to the planar outer billiard in P with respect to γ. In particular, each such restriction is area-preserving.

Recall that for $\kappa = 0$ the Killing form g_K degenerates and so it does not induce a symplectic form on $\mathcal{G}_0$. However, we have a weaker structure, a Poisson bivector field $\mathcal{P}$, also compatible with $\mathcal{J}$, which we define in Subsection 2.3. The proof of the next proposition is immediate from the preceding paragraph.

Proposition 1.7. *The outer billiard map associated with a smooth, closed, quadratically convex surface in $\mathbb{R}^3$ preserves both the canonical Poisson structure $\mathcal{P}$ on $\mathcal{G}_0$ and its symplectic leaves, which are the submanifolds of parallel lines. In particular, the restriction to each such submanifold is a symplectomorphism.*

1.6 | Dynamical properties of the outer billiard B

To begin the study of dynamical properties of the outer billiard map B, we first observe that in the Euclidean case parallelism has consequences for periodic orbits of the outer billiard map B associated with the surface S. Given $v \in S^2$, consider the planar outer billiard system in any plane P orthogonal to v, played outside the shadow of S with respect to v. For this planar system, there exist at least two distinct n-periodic trajectories with rotation number r, for every $n \geqslant 3$ and positive $r \leqslant [(n-1)/2]$ coprime with n (see [31, Theorem 6.2]). Each such periodic orbit lifts to a periodic orbit of the outer billiard map B associated with S. In particular, for each direction $v \in S^2$, there exist n-periodic orbits consisting of geodesics with direction v.

The hyperbolic case is more interesting than the Euclidean one, because unlike in $\mathbb{R}^3$, the lines ℓ, $B(\ell)$, $B^2(\ell), \dots$ in H^3 are (in general) not parallel, in the sense that they are not orthogonal to a fixed totally geodesic surface. Indeed, if a line ℓ in H^3 is parallel transported along a line ℓ' orthogonal to ℓ, then ℓ' is the unique line preserved by the one-parameter group of transvections along ℓ' (that is, isometries of H^3 translating ℓ' whose differentials realize the parallel transport along it). This nonparallel phenomenon is illustrated more explicitly in the following proposition.

Proposition 1.8. *Given $\theta \in (0, \pi/2)$, there exist a quadratically convex closed surface S in H^3 and an oriented line ℓ not intersecting S such that ℓ and $B^3(\ell)$ intersect at a point forming the angle θ.*

Similarly, one can show the existence of a surface S and ℓ such that $B^3(\ell)$ is different from ℓ and asymptotic to it.

We next introduce a notion of holonomy for periodic orbits of the outer billiard map in hyperbolic space. Let $\bar{\ell} = (\ell_0, \dots, \ell_{n-1})$ be an n-periodic orbit of B; in particular, we assume the ℓ_i are distinct. For $0 \leqslant k < n$ let d_k be the signed distance between the points $q_+(\ell_k)$ and $q_-(\ell_k)$, which were defined in Subsection 1.3, that is, if $\ell_k = [\gamma_k]$ with $\gamma_k(0) = q_-$, then $\gamma_k(d_k) = q_+$. Then the *holonomy of the periodic orbit* $\bar{\ell}$ is the number $d = \sum_{k=0}^{n-1} d_k$.

The definition also makes sense in Euclidean space, but every periodic orbit has zero holonomy. We exhibit a periodic orbit in hyperbolic space with nonzero holonomy.

Proposition 1.9. *There exist a quadratically convex closed surface S in H^3 and an oriented line ℓ not intersecting S which is a periodic point of the associated outer billiard map and whose holonomy is not zero.*

We comment on the choice of the word holonomy in this context. For $\kappa \leqslant 0$, let $\varpi : P = T^1M_\kappa \to \mathcal{G}_\kappa$ be the tautological line bundle, that is $\varpi(v) = [\gamma_v]$. It is an $(\mathbb{R}, +)$-principal bundle.

The right action $\rho : P \times \mathbb{R} \to P$ is given by $\rho(u,t) = \gamma_u'(t)$. The line bundle $P_{\mathcal{U}} = \varpi^{-1}(\mathcal{U}) \to \mathcal{U}$ has two distinguished sections: $\sigma_\pm([\gamma]) = \gamma'(0)$ if $\gamma(0) = q_\pm$. The holonomy makes sense only for periodic points and it is not associated with a particular connection, but rather with a combination of the Levi-Civita connection on M_κ and the flat connections induced by $\sigma_\pm$ along the shortest segments joining ℓ_k with ℓ_{k+1}.

Although these results only provide the first steps toward understanding the dynamical properties of the outer billiard map B in hyperbolic space, they also hint at the complexity and richness of the billiard system. Most of the natural dynamical questions for the hyperbolic outer billiard map, for example, regarding the existence of periodic orbits, remain open.

2 | PRELIMINARIES

2.1 | The outer billiard map in the spherical case

We have presented in Subsection 1.3 the outer billiard map associated with S for $\kappa = 0, 1, -1$. The construction is clear for $\kappa = 0, -1$. Now we return to the spherical case. Before proving Proposition 1.1, we comment on the set C that we cut out of the billiard table; see (2). We define the map $\psi : S \to C$ as follows: Given $p \in S$, let $\psi(p)$ be the oriented great circle obtained by intersecting S^3 with the subspace $T_pS \subset \mathbb{R}^4$ (we identify $T_p\mathbb{R}^4$ with $\mathbb{R}^4$ in the usual way), endowed with the orientation induced by that of T_pS. Equivalently, and without using the immersion of the sphere in $\mathbb{R}^4$, for any positively oriented orthonormal basis $\{u, v\}$ of T_pS, $\psi(p) = [C_p]$, where

$$C_p(s) = \mathrm{Exp}_p\left(\tfrac{\pi}{2}(\cos s\, u + \sin s\, v)\right) \cong \cos s\, u + \sin s\, v.$$

With this notation, C is the image of ψ. Notice that if c is an oriented circle in C, then $-c$ is also in C.

Proof of Proposition 1.1. We verify that the same procedure as for the Euclidean and hyperbolic spaces applies here. By [6], S is contained in a hemisphere, say, the northern hemisphere S^3_+.

Let Π denote the central projection from S^3_+ to the tangent plane $\mathbb{R}^3$ at the north pole (the so-called Beltrami map). It preserves strict convexity, as half great spheres in S^3_+ are mapped to affine 2-planes and the order of contact is maintained by diffeomorphisms.

Consider a great circle $c \in \mathcal{G}_1 -$ {oriented great circles intersecting S}. If c is not contained in the equator $S^2 = \partial S^3_+$, then $\Pi(c)$ is an oriented affine line ℓ in $\mathbb{R}^3$. Now, as shown in Subsection 1.3 for $\kappa = 0$, there exist exactly two affine planes $\bar{P}_\pm$ containing ℓ and tangent to $\Pi(S)$, and $\Pi^{-1}(\bar{P}_\pm)$ are the desired great spheres containing c and tangent to S at points $p_\pm$. Notice that $B(c) \in \mathcal{U}$, that is, $B(c)$ is disjoint from S, as this holds for the lines in $\mathbb{R}^3$ corresponding to c and $B(c)$.

Now suppose that c is contained entirely in the equator S^2. As S is a positive distance from the equator S^2, any sufficiently small perturbation of the latter does not intersect S. In particular, we may perturb the equator to a great sphere that does not contain the circle c and argue as in the above paragraph.

Let d be the distance from p_+ to c. If $d < \pi/2$, there exists a unique point $q_+ \in c$ realizing the distance, and the outer billiard map is well-defined. If $d = \pi/2$, then c belongs to C, and so it is not in $\mathcal{U}$. The construction continues as in the Euclidean and hyperbolic cases. □

We observe that if q is any point of $c \in C$, then the parallel transport of c along the geodesic joining q with p_+ between 0 and π is a rotation by π. So the image of c is the same great circle with the opposite orientation (it would hold $B(c) = -c$, had we not excluded C from the billiard table).

2.2 | The Jacobi fields of the three-dimensional space forms

Here we provide a brief review of Jacobi fields, which arise naturally when studying variations of geodesics, and thus play a central role in the proofs of the main theorems. A more thorough treatment can be found in any standard Riemannian geometry text, for example, [5].

Let M be a complete Riemannian manifold and let γ be a unit speed geodesic of M. A Jacobi field J along γ is by definition a vector field along γ arising via a variation of geodesics as follows: Let $\delta > 0$ and $\phi : \mathbb{R} \times (-\delta, \delta) \to M$ be a smooth map such that $r \mapsto \phi(r,s)$ is a geodesic for each $s \in (-\delta, \delta)$ and such that $\phi(r, 0) = \gamma(r)$ for all r. Then

$$J(r) = \left.\tfrac{d}{ds}\right|_0 \phi(r,s).$$

Let $(M_\kappa, \langle , \rangle_\kappa)$ denote the three-dimensional complete simply connected manifold of constant sectional curvature κ. The curvature tensor of M_κ is given by

$$R_\kappa(x,y)z = \kappa(\langle z,x\rangle_\kappa y - \langle z,y\rangle_\kappa x), \tag{3}$$

and the Jacobi fields along a geodesic γ and orthogonal to γ' are exactly the vector fields J along γ satisfying $\left\langle J, \gamma' \right\rangle_\kappa = 0$ and

$$\tfrac{D^2 J}{dr^2} + \kappa J = 0, \tag{4}$$

where $\frac{D}{dr}$ is the covariant derivative associated with the Levi Civita connection of M_κ. Following a common abuse of notation, given a smooth vector field J along a curve γ, we write $J' = \frac{DJ}{dr}$ if there is no danger of confusion.

A Jacobi field J along γ is determined by the values $J(0)$ and $J'(0)$ in the following way. Suppose that a Jacobi field J along γ satisfies $J(0) = u + a\gamma'(0)$ and $J'(0) = v + b\gamma'(0)$ where $a, b \in \mathbb{R}$ and $u, v \in \gamma'^{\perp}$. Let U and V be the parallel vector fields along γ with $U(0) = u$ and $V(0) = v$. Then

$$J(r) = c_\kappa(r)U(r) + s_\kappa(r)V(r) + (a + rb)\gamma'(r), \tag{5}$$

where

$$\begin{aligned} c_1(r) &= \cos r, \quad c_0(r) = 1, \quad c_{-1}(r) = \cosh r, \\ s_1(r) &= \sin r, \quad s_0(r) = r, \quad s_{-1}(r) = \sinh r. \end{aligned}$$

Note that $s_\kappa' = c_\kappa$ and $c_\kappa' = -\kappa s_\kappa$. Equation (5) will allow us to perform most computations without having to resort to coordinates of M_κ or a particular model of it.

Next we see that the tangent vectors to the space $\mathcal{G}_\kappa$ at an oriented geodesic $[\gamma]$ may be identified with Jacobi fields along γ. Let γ be a complete unit speed geodesic of M_κ and let $\mathfrak{J}_\gamma$ be the space of all Jacobi fields along γ that are orthogonal to γ'. There is a canonical isomorphism

$$T_\gamma : \mathfrak{J}_\gamma \to T_{[\gamma]}\mathcal{G}_\kappa, \qquad T_\gamma(J) = \left.\tfrac{d}{ds}\right|_0 [\gamma_s], \tag{6}$$

where γ_s is any variation of γ by unit speed geodesics associated with J. Moreover, if J is the Jacobi field associated with a variation $\phi : \mathbb{R} \times (-\delta, \delta) \to M_\kappa$ of γ by unit speed geodesics (J is not necessarily orthogonal to γ'), then

$$T_\gamma\left(J^N\right) = \left.\tfrac{d}{ds}\right|_0 [\phi_s], \tag{7}$$

where $J^N(r) = J(r) - \langle J(r), \gamma'(r)\rangle_\kappa \gamma'(r)$ (see [19, section 2] or [24]).

We offer one simple but useful application of the isomorphism T_γ. Let S be a smooth, closed, strictly convex surface in M_κ, and let $\mathcal{M} \subset \mathcal{G}_\kappa$ be the collection of oriented geodesics that are tangent to S.

Lemma 2.1. *The isomorphism T_γ identifies $\left\{K \in \mathfrak{J}_\gamma \mid K(0) \in T_{\gamma(0)}S\right\}$ with the tangent space $T_{[\gamma]}\mathcal{M}$.*

Proof. It suffices to show that $T_{[\gamma]}\mathcal{M} \subset T_\gamma(\left\{K \in \mathfrak{J}_\gamma \mid K(0) \in T_{\gamma(0)}S\right\})$, as both spaces have dimension 3. Let $X \in T_{[\gamma]}\mathcal{M}$ and let c be a smooth curve on $\mathcal{M}$ (defined on an interval I containing 0) such that $c(0) = [\gamma]$ and $c'(0) = X$. For each $s \in I$, let $[\gamma_s] \in \mathcal{M}$ such that $\gamma_s(0) \in S$ and $[\gamma_s] = c(s)$. By (7), $X = T_\gamma(J^N)$, where J is given by

$$J(r) = \left.\tfrac{d}{ds}\right|_0 \gamma_s(r).$$

Now $\gamma'(0) \in T_{\gamma(0)}S$, and as $s \mapsto \gamma_s(0)$ is a smooth curve on S, $J(0) = \left.\frac{d}{ds}\right|_0 \gamma_s(0) \in T_{\gamma(0)}S$. Therefore, $J^N(0) = J(0) - \langle J(0), \gamma'(0)\rangle_\kappa \gamma'(0) \in T_{\gamma(0)}S$, as desired. □

2.3 | Kähler structures on the spaces of oriented geodesics

In Subsection 1.2, we introduced the two canonical Kähler structures $(g_K, \mathcal{J})$ and $\left(g_\times, \mathcal{J}\right)$ on the space of oriented geodesics $\mathcal{G}_\kappa$, for $\kappa = 1, -1$, and also the Kähler structure $\left(g_\times, \mathcal{J}\right)$ and the Poisson bivector field $\mathcal{P}$ on $\mathcal{G}_0$. Next we present the precise definitions in terms of the isomorphism (6). We also include the expressions of the associated fundamental forms (see [1, 8, 11, 14, 23, 24]).

Given $\ell = [\gamma] \in \mathcal{G}_\kappa$, the linear complex structure $\mathcal{J}_\ell$ on $\mathfrak{J}_\gamma \cong T_\ell\mathcal{G}_\kappa$, which was described geometrically in Subsection 1.2, is defined by

$$\mathcal{J}_\ell(J) = \gamma' \times J, \qquad \text{for } \; J \in \mathfrak{J}_\gamma \cong T_\ell\mathcal{G}_\kappa, \tag{8}$$

and the square norms of the metrics $g_\times$ and g_K are given by

$$g_\times(J, J) = \langle \gamma' \times J, J'\rangle_\kappa \qquad \text{and} \qquad g_K(J, J) = |J|_\kappa^2 + \kappa |J'|_\kappa^2.$$

Notice that by (4) the right-hand sides are constant functions, so the left-hand sides are well-defined. By polarization, we have

$$2\, g_\times(I,J) = \langle I \times J' + J \times I', \gamma' \rangle_\kappa \qquad \text{for } \kappa = -1, 0, 1; \tag{9}$$

$$g_K(I,J) = \langle I,J\rangle_\kappa + \kappa\langle I',J'\rangle_\kappa \qquad \text{for } \kappa = \pm 1. \tag{10}$$

The second one is the push down onto $\mathcal{G}_\kappa$ of the left-invariant pseudo-Riemannian metric on the Lie group $\mathrm{Iso}_0(M_\kappa)$ given at the identity by a multiple of the Killing form. It is Riemannian for $\kappa = 1$ and split for $\kappa = -1$. [24, Proposition 4] provides a geometric interpretation for the metrics $g_\times$ and g_K in the case $\kappa = -1$, with characterizations of space-like, time-like and null curves in $\mathcal{G}_{-1}$ in both cases.

The associated fundamental forms are given by

$$\omega_\times(I,J) = g_\times(\mathcal{J}(I),J) = \tfrac{1}{2}(\langle I',J\rangle_\kappa - \langle I,J'\rangle_\kappa), \tag{11}$$

$$\omega_K(I,J) = g_K(\mathcal{J}(I),J) = \langle I\times J + \kappa I' \times J', \gamma'\rangle_\kappa. \tag{12}$$

We comment that $p^*\omega_\times$ is a constant multiple of Ω, where $p : T^1M_\kappa \to \mathcal{G}_\kappa$, $v \mapsto [\gamma_v]$ is the canonical submersion and Ω is the restriction to T^1M_κ of the canonical symplectic form on TM_κ (identified with the cotangent bundle T^*M_κ through the Riemannian metric).

The bilinear form g_K degenerates for $\kappa = 0$, but we have the canonical Poisson structure on $\mathcal{G}_0$ well-defined at each ℓ by

$$\mathcal{P}(\ell) = J \wedge \mathcal{J}_\ell(J),$$

where J is any parallel Jacobi field along ℓ, orthogonal to it, with $\|J\| \equiv 1$. Although no such section $\ell \mapsto J_\ell \in T_\ell\mathcal{G}_0$ exists globally (otherwise, it would induce a unit vector field on the 2-sphere), $\mathcal{P}$ is easily seen to be well-defined and smooth; the Schouten bracket $[\mathcal{P},\mathcal{P}]$ vanishes, as the distribution on $\mathcal{G}_0$ induced by $\mathcal{P}$ is integrable. In fact, the symplectic leaves are the submanifolds of parallel lines.

3 | THE SMOOTHNESS OF THE OUTER BILLIARD MAP

Here we establish notation and prove various technical lemmas, working toward the proof of Theorem 1.2. Let S be a closed smooth surface in M_κ. Let n be the inward-pointing unit normal vector field on S. The complex structure i on S is defined by $iz = n(p) \times z$ for $z \in T_pS$.

Given $w \in T^1M_\kappa$, we denote by γ_w the unique geodesic in M_κ with initial velocity w. For $\kappa = 0, -1$, let $T = \infty$, and for $\kappa = 1$, let $T = \pi/2$. Define

$$F : \mathcal{M} \times (-T,T) \cong T^1S \times (-T,T) \to \mathcal{G}_\kappa, \qquad F(u,t) = [\gamma_{u_t}],$$

where u_t is the parallel transport on M_κ of u along γ_{iu} between 0 and t; see Figure 5. Let F_+ and F_- denote the restrictions of F to $T^1S \times (0,T)$ and $T^1S \times (-T,0)$, respectively.

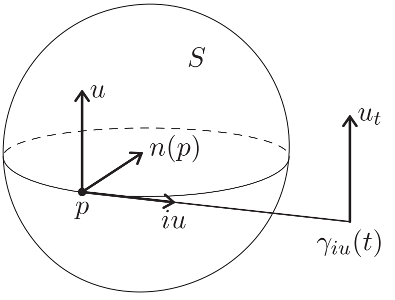

FIGURE 5 The parallel transport of u along γ_{iu} between 0 and t.

By the construction in Subsection 1.3, the outer billiard map $B : \mathcal{U} \to \mathcal{U}$ is equal to the composition

$$B = F_+ \circ g \circ (F_-)^{-1}, \tag{13}$$

where $g : T^1S \times (-T, 0) \to T^1S \times (0, T)$ is defined by $g(u,t) = (u, -t)$. Clearly, F is a smooth function and g is a diffeomorphism, and so the proof of Theorem 1.2 reduces to showing that $F_\pm$ are diffeomorphisms. As they are bijections, we must show that

$$(dF)_{(u,t)} : T_uT^1S \times T_t\mathbb{R} \to T_{F(u,t)}\mathcal{G}_\kappa \cong \mathfrak{J}_{\gamma_{u_t}},$$

is nonsingular for all $u \in T^1S$ and $0 \neq |t| < T$. To verify this, we will compute the differential with respect to certain canonical bases that we introduce next.

Given an oriented geodesic $\ell \in \mathcal{M}$, there are three perturbations of ℓ that stay in $\mathcal{M}$: one that skates along S in the direction of ℓ, one that parallel transports ℓ along S in the direction orthogonal to ℓ, and one that rotates ℓ, maintaining the point of tangency. These three perturbations may be thought of as generating the tangent space $T_\ell\mathcal{M}$. We formalize this intuitive idea below, via the natural identification of $\mathcal{M}$ with T^1S.

In what follows, we fix $u \in T^1_pS$ and $t \neq 0$ and denote $v = iu$.

Given a unit tangent vector $z \in T^1S$, we call σ_z the geodesic of S with initial velocity z. Let $\pi : TS \to S$ be the canonical projection and let $\mathcal{K}_u : T_uTS \to T_pS$ be the connection operator, which is well-defined as follows: Given $\xi \in T_uTS$, let $U : (-\delta, \delta) \to TS$ be a smooth curve with $U(0) = u$ and initial velocity ξ. Then $K_u(\xi) = \frac{DU}{dt}(0)$, where $\frac{D}{dt}$ is the covariant derivative along the foot-point curve $\pi \circ U$.

Lemma 3.1. *For $m = 1, 2, 3$, let $w_m : \mathbb{R} \to T^1S$ be the curve defined by*

$$w_1(s) = \sigma_u'(s), \qquad w_2(s) = \tau_{0,s}^{\sigma_v}(u), \qquad w_3(s) = \cos s\, u + \sin s\, v, \tag{14}$$

where $\tau^\sigma_{0,s}$ denotes the parallel transport on S along σ between 0 and s. Then $\{w_1'(0), w_2'(0), w_3'(0)\}$ is a basis of T_uT^1S.

Proof. We claim that under the linear isomorphism

$$\varphi_u : T_uTS \to T_pS \times T_pS, \qquad \varphi_u(\xi) = (d\pi_u\xi, \mathcal{K}_u\xi)$$

(see, for instance, [3]), $w_1'(0), w_2'(0), w_3'(0)$ are mapped, respectively, to the linearly independent vectors $(u, 0)$, $(v, 0)$ and $(0, v)$. We compute

$$d\pi_u(w_1'(0)) = \frac{d}{ds}\Big|_0 \pi(w_1(s)) = \frac{d}{ds}\Big|_0 \sigma_u(s) = u,$$

and, by definition of $\mathcal{K}_u$,

$$\mathcal{K}_u(w_1'(0)) = \frac{D}{ds}\Big|_0 w_1(s) = \frac{D}{ds}\Big|_0 \sigma_u'(s) = 0.$$

Hence, $\varphi_u(w_1'(0)) = (u, 0)$. The other cases are similar. □

We consider the basis $\mathcal{B}_t = \{W_1, W_2, W_3, W_4\}$ of $T_u T^1 S \times T_t\mathbb{R}$, with

$$W_m = (w_m'(0), 0), \quad \text{for } m = 1, 2, 3 \quad \text{and} \quad W_4 = \left(0, \frac{d}{ds}\Big|_t\right), \tag{15}$$

where w_m are the curves defined in (14). Now, the image of W_m by $(dF)_{(u,t)}$ is a tangent vector to $\mathcal{G}_\kappa$ at $F(u, t)$, and so by (6), it corresponds to a Jacobi field along γ_{u_t} in $\mathfrak{J}_{\gamma_{u_t}}$, which we call J_m. We state this in the following proposition, whose proof is straightforward from the definitions.

Proposition 3.2. *For $m = 1, \ldots, 4$ we have*

$$(dF)_{(u,t)}(W_m) = T_{\gamma_{u_t}}(J_m),$$

where J_m is the normal component of the Jacobi field arising from the geodesic variations of γ_{u_t} given by

$$(s, r) \mapsto \gamma_{(w_m(s))_t}(r)$$

for $m = 1, 2, 3$, and $(s, r) \mapsto \gamma_{u_{t+s}}(r)$ for $m = 4$.

We need J_m explicitly. We consider the parameterized surface

$$f_m : \mathbb{R}^2 \to M_\kappa, \qquad f_m(r, s) = \gamma_{iw_m(s)}(r).$$

In particular, $f_m(r, 0) = \gamma_v(r)$. We write

$$\sigma_m = \pi \circ w_m = f_m(0, \cdot),$$

so $\sigma_1 = \sigma_u, \sigma_2 = \sigma_v$, and $\sigma_3 \equiv p$. Now we can describe the initial conditions of J_m in terms of some vector fields along f_m.

Proposition 3.3. *For $m = 1, 2, 3$, we have*

$$J_m(0) = K_m(t) - \langle K_m(t), u_t\rangle_\kappa u_t \quad \textit{and} \quad J_m'(0) = \frac{D}{ds}\Big|_0 Z_m(t, s),$$

where K_m is the Jacobi vector field along γ_v associated with the geodesic variation f_m, that is,

$$K_m(r) = \frac{d}{ds}\Big|_0 f_m(r, s),$$

and Z_m is the vector field along the surface f_m obtained by parallel transporting $w_m(s)$ on M_κ along the geodesic $\gamma_{iw_m(s)}$ from 0 to r, that is,

$$Z_m(r,s) = P_{0,r}^{\gamma_{iw_m(s)}} w_m(s).$$

Also,

$$J_4(0) = v_t \qquad \text{and} \qquad J_4'(0) = 0,$$

where v_t is the parallel transport on M_κ of v between 0 and t along γ_v.

Proof. Let $\bar{J}_m(r) = \frac{d}{ds}\big|_0 \gamma_{(w_m(s))_t}(r)$. By (7), $J_m(r) = (\bar{J}_m(r))^N$. We compute

$$\bar{J}_m(0) = \frac{d}{ds}\bigg|_0 \gamma_{(w_m(s))_t}(0) = \frac{d}{ds}\bigg|_0 \gamma_{iw_m(s)}(t) = \frac{d}{ds}\bigg|_0 f_m(t,s) = K_m(t).$$

Then $J_m(0)$ is as stated. Also, $J_m'(0) = \bar{J}_m'(0)$, as the geodesics in the variation have unit speed. Now,

$$\bar{J}_m'(0) = \frac{D}{\partial r}\bigg|_0 \frac{\partial}{\partial s}\bigg|_0 \gamma_{(w_m(s))_t}(r) = \frac{D}{\partial s}\bigg|_0 \frac{\partial}{\partial r}\bigg|_0 \gamma_{(w_m(s))_t}(r) = \frac{D}{\partial s}\bigg|_0 (w_m(s))_t = \frac{D}{ds}\Big|_0 Z_m(t,s).$$

The validity of the remaining assertions, involving J_4, follows from similar (simpler) arguments. □

We require explicit formulae for K_m and $\frac{D}{ds}\big|_0 Z_m(t,s)$, which are vector fields along γ_v. In the next three lemmas we compute their coordinates with respect to the basis $\{u_r, v_r, n_r\}$ of $T_{\gamma_v(r)}M_\kappa$, where u_r, v_r, and n_r are obtained by parallel transporting u, v, and $n(p)$ along γ_v, between 0 and r.

Given $p \in S$, the shape operator $A_p : T_pS \to T_pS$ is defined by

$$A_p(x) = -\nabla_x n, \tag{16}$$

where ∇ denotes the Levi–Civita connection of M_κ. In what follows we assume that A_p is positive definite at each $p \in S$ (that is, S is quadratically convex).

We consider the matrix of A_p with respect to the orthonormal basis $\{u, v\}$ and call b_{ij} its entries, that is, $[A_p]_{\{u,v\}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, with $b_{12} = b_{21}$.

Lemma 3.4. *For $m = 1, 2, 3$, the Jacobi vector field K_m along γ_v is given by*

$$\begin{aligned} K_1(r) &= c_\kappa(r)u_r + b_{21}s_\kappa(r)n_r, \\ K_2(r) &= v_r + b_{22}s_\kappa(r)n_r, \\ K_3(r) &= -s_\kappa(r)u_r. \end{aligned} \tag{17}$$

Proof. We compute the initial values of K_m and K_m' and use (5). We write down the details for $m = 1$. The other cases are similar. We compute

$$K_1(0) = \frac{d}{ds}\Big|_0 f_1(0,s) = \frac{d}{ds}\Big|_0 \sigma_1(s) = u$$

and

$$K_1'(0) = \frac{D}{\partial r}\Big|_0 \frac{\partial}{\partial s}\Big|_0 f_1(r,s) = \frac{D}{\partial s}\Big|_0 \frac{\partial}{\partial r}\Big|_0 f_1(r,s) = \frac{D}{\partial s}\Big|_0 iw_1(s).$$

We compute the coordinates of $K_1'(0)$ with respect to the orthonormal basis $\{u, v, n(p)\}$ of T_pM_κ. To obtain $\langle K_1'(0), n(p)\rangle_\kappa$, we observe that $\langle iw_1(s), n(\sigma_1(s))\rangle_\kappa = 0$ for all s. Hence,

$$\left\langle \frac{D}{\partial s}\Big|_0 iw_1(s), n(p)\right\rangle_\kappa = -\left\langle iu, \frac{D}{\partial s}\Big|_0 n(\sigma_1(s))\right\rangle_\kappa = -\langle v, \nabla_u n\rangle_\kappa = \langle v, A_p(u)\rangle_\kappa = b_{21}.$$

In the same way, $\langle K_1'(0), u\rangle_\kappa = 0 = \langle K_1'(0), v\rangle_\kappa$. Therefore, $K_1'(0) = b_{21}n(p)$. Notice that K_m is not necessarily orthogonal to γ_v'. □

For the sake of simplicity of notation, we denote by $Y_m(r) = \frac{D}{ds}\Big|_0 Z_m(r,s)$.

Lemma 3.5. *For $m = 1, 2, 3$, the vector field Y_m along γ_v is given by*

$$Y_1(r) = \kappa s_\kappa(r)v_r + b_{11}n_r, \qquad Y_2(r) = b_{12}n_r, \qquad Y_3(r) = c_\kappa(r)v_r.$$

Before proving the lemma, we introduce the vector field N_m along the surface f_m obtained by parallel transporting $n(\sigma_m(s))$ on M_κ along the geodesic $\gamma_{iw_m(s)}$ from 0 to r, that is,

$$N_m(r,s) = P_{0,r}^{\gamma_{iw_m(s)}} n(\sigma_m(s)).$$

Lemma 3.6. *Let $\zeta_m(r) = \left\langle \frac{D}{ds}\Big|_0 N_m(r,s), u_r\right\rangle_\kappa$. Then $\zeta_1 \equiv -b_{11}$, $\zeta_2 \equiv -b_{12}$, and $\zeta_3 \equiv 0$.*

Proof. We compute

$$\begin{aligned}
\zeta_m'(r) &= \frac{d}{dr}\left\langle \frac{D}{ds}\Big|_0 N_m(r,s), u_r\right\rangle_\kappa = \left\langle \frac{D}{dr}\frac{D}{ds}\Big|_0 N_m(r,s), u_r\right\rangle_\kappa \\
&= \left\langle \frac{D}{ds}\Big|_0 \frac{D}{dr} N_m(r,s) + R_\kappa\left(\frac{\partial f_m}{\partial r}(r,0), \frac{\partial f_m}{\partial s}(r,0)\right)N_m(r,0), u_r\right\rangle_\kappa \\
&= \langle R_\kappa(v_r, K_m(r))n_r, u_r\rangle_\kappa = \kappa\big\langle \langle n_r, v_r\rangle_\kappa K_m(r) - \langle n_r, K_m(r)\rangle_\kappa v_r, u_r\big\rangle_\kappa = 0,
\end{aligned}$$

by (3). Hence, ζ_m is constant, equal to

$$\zeta_m(0) = \left\langle \frac{D}{ds}\Big|_0 N_m(0,s), u_0\right\rangle_\kappa = \left\langle \nabla_{\sigma_m'(0)} n, u\right\rangle_\kappa = -\langle A_p(\sigma_m'(0)), u\rangle_\kappa.$$

Now, the assertions follow from the definition of σ_m and the values of the entries of the matrix $[A_p]_{\{u,v\}}$. □

Proof of Lemma 3.5. Observe that $\langle Z_m, Z_m\rangle_\kappa$, $\langle Z_m, N_m\rangle_\kappa$ and $\langle Z_m, \frac{\partial f_m}{\partial r}\rangle_\kappa$ are constant functions of the second variable s. Hence, we can compute the components $Y_m(r)$ with respect to the basis $\{u_r, v_r, n_r\}$ of $T_{\gamma_v(r)}M_\kappa$ as follows:

$$\langle Y_m(r), u_r\rangle_\kappa = \left\langle \frac{D}{ds}\Big|_0 Z_m(r,s), Z_m(r,0)\right\rangle_\kappa = 0.$$

Also,

$$\langle Y_m(r), v_r\rangle_\kappa = \left\langle \tfrac{D}{ds}\Big|_0 Z_m(r,s), v_r\right\rangle_\kappa = -\left\langle Z_m(r,0), \tfrac{D}{ds}\Big|_0 \gamma'_{iw_m(s)}(r)\right\rangle_\kappa$$
$$= -\left\langle u_r, \tfrac{D}{dr}\tfrac{d}{ds}\Big|_0 \gamma_{iw_m(s)}(r)\right\rangle_\kappa = -\left\langle u_r, \tfrac{D}{dr} K_m(r)\right\rangle_\kappa.$$

By Lemma 3.4, we have

$$-\left\langle u_r, K'_m(r)\right\rangle_\kappa = \begin{cases} \kappa s_\kappa(r), & \text{if } m = 1,\\ 0, & \text{if } m = 2,\\ c_\kappa(r), & \text{if } m = 3.\end{cases}$$

In the same way,

$$\langle Y_m(r), n_r\rangle_\kappa = \left\langle \tfrac{D}{ds}\Big|_0 Z_m(r,s), N_m(r,0)\right\rangle_\kappa = -\left\langle Z_m(r,0), \tfrac{D}{ds}\Big|_0 N_m(r,s)\right\rangle_\kappa = -\zeta_m(r),$$

with ζ_m as in Lemma 3.6. □

With the computational lemmas above, we can present the proof of Theorem 1.2.

Proof of Theorem 1.2. To complete the proof of Theorem 1.2, it remains to show that

$$(dF)_{(u,t)} : T_u T^1 S \times T_t \mathbb{R} \to T_{F(u,t)} \mathcal{G}_\kappa \cong \mathfrak{J}_{\gamma_{u_t}}$$

is nonsingular for all $u \in T^1 S$ and $0 \neq |t| < T$. To verify this, we compute the matrix of $(dF)_{(u,t)}$ with respect to the bases $\mathcal{B}_t$ (given in (15)) of $T_u T^1 S \times T_t \mathbb{R}$ and $\mathcal{E}_t = \{E_1^t, \dots, E_4^t\}$ of $\mathfrak{J}_{\gamma_{u_t}}$, where E_m^t are the Jacobi fields along γ_{u_t} whose initial conditions are

$$\begin{aligned} &E_1^t(0) = 0, && E_2^t(0) = n_t, && E_3^t(0) = 0, && E_4^t(0) = v_t,\\ &(E_1^t)'(0) = n_t, && (E_2^t)'(0) = 0, && (E_3^t)'(0) = v_t, && (E_4^t)'(0) = 0. \end{aligned} \tag{18}$$

By Proposition 3.3 and Lemmas 3.4 and 3.5, we have

$$J_m(0) = \begin{cases} b_{21} s_\kappa(t) n_t, & m = 1,\\ v_t + b_{22} s_\kappa(t) n_t, & m = 2,\\ 0, & m = 3,\\ v_t, & m = 4, \end{cases} \quad \text{and} \quad J'_m(0) = \begin{cases} \kappa s_\kappa(t) v_t + b_{11} n_t, & m = 1,\\ b_{12} n_t, & m = 2,\\ c_\kappa(t) v_t, & m = 3,\\ 0, & m = 4. \end{cases}$$

Hence, calling C_t the matrix of $(dF)_{(u,t)}$ with respect to the bases $\mathcal{B}_t$ and $\mathcal{E}_t$, we obtain that

$$C_t = \begin{pmatrix} b_{11} & b_{12} & 0 & 0\\ b_{21} s_\kappa(t) & b_{22} s_\kappa(t) & 0 & 0\\ \kappa s_\kappa(t) & 0 & c_\kappa(t) & 0\\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{19}$$

Therefore, $\det C_t = c_\kappa(t)s_\kappa(t)b$, where $b = \det\left(A_p\right)$. As, by hypothesis, $0 \neq |t| < T$ and $[A_p]_{\{u,v\}}$ is definite, we have that C_t is nonsingular. $\square$

Proof of Proposition 1.3. By the reflection invariance, B preserves the oriented lines orthogonal to the plane $P = \{(x, y, z) \mid y = 0\}$ and induces in the obvious manner the outer billiard map $\widetilde{B}$ on P determined by the closed strictly convex curve γ with image $S \cap P$, which includes the graph of f. Accordingly, we identify P and the set of oriented lines orthogonal to it with $\mathbb{R}^2$.

Given a small $s \geqslant 0$, we next compute $\widetilde{B}(-1, s)$. Let ℓ_s be the straight line passing through $(-1, s)$ tangent to γ at $(x_s, f(x_s))$, with $-1 < x_s \leqslant 0$. Then ℓ_s can be parameterized by $t \mapsto l_s(t) = (x_s, f(x_s)) + t\left(1, f'(x_s)\right)$ and there exists a unique t_s such that $l_s(t_s) = (-1, s)$. We have

$$x_0 = 0, \qquad x_s + t_s = -1 \quad \text{and} \quad f(x_s) + t_s f'(x_s) = s. \tag{20}$$

Hence,

$$\widetilde{B}(-1, s) = l_s(-t_s) = \left(x_s - t_s, f(x_s) - t_s f'(x_s)\right) = (2x_s + 1, 2f(x_s) - s).$$

Suppose that $\widetilde{B}$ is smooth, then so are $s \mapsto x_s$ and $s \mapsto t_s$. We compute the right derivative at $s = 0$ of both sides of the last equation in (20) and obtain

$$0 = f'(0)x_0' + t_0' f'(0) + t_0 f''(0)x_0' = 1,$$

a contradiction. $\square$

4 | THE KÄHLER FORMULATION OF THE OUTER BILLIARD MAP

To prove Theorem 1.4, we need the presentation of $\mathcal{G}_\kappa$ as a symmetric homogeneous space. The details of the following description can be found, for instance, in [11]. For $\kappa = \pm 1$, we consider the standard presentation of M_κ as a submanifold of $\mathbb{R}^4$: If $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4$, then M_k is the connected component of e_0 of the set

$$\left\{(t, x, y, z) \in \mathbb{R}^4 \mid \kappa t^2 + x^2 + y^2 + z^2 = \kappa\right\}.$$

Let G_κ be the identity component of the isometry group of M_κ, that is, $G_1 = SO_4$ and $G_{-1} = O_o(1, 3)$. The group G_κ acts smoothly and transitively on $\mathcal{G}_\kappa$ as follows: $g \cdot [\gamma] = [g \circ \gamma]$. Let γ_o be the geodesic in M_κ with $\gamma_o(0) = e_0$ and initial velocity $e_1 \in T_{e_0}M_\kappa$ and let H_κ be the stabilizer group of $[\gamma_o]$ in G_κ. Then there exists a diffeomorphism $\phi : G_\kappa/H_\kappa \to \mathcal{G}_\kappa$, given by $\phi(gH_\kappa) = g \cdot [\gamma_o]$.

The Killing form of Lie (G_κ) provides G_κ with a bi-invariant metric and thus there exists a unique pseudo-Riemannian metric $\tilde{g}_K$ on G_κ/H_κ such that the canonical projection $\pi : G_\kappa \to G_\kappa/H_\kappa$ is a pseudo-Riemannian submersion. The diffeomorphism ϕ turns out to be an isometry onto $\mathcal{G}_\kappa$ endowed with a constant multiple of the metric g_K defined in (10).

Besides, it is well-known that $(G_\kappa/H_\kappa, \tilde{g}_K)$ is a pseudo-Riemannian symmetric space. In particular, if Lie $(G_\kappa) =$ Lie $(H_\kappa) \oplus \mathfrak{p}_\kappa$ is the Cartan decomposition determined by $[\gamma_o]$, then for any $Z \in \mathfrak{p}_\kappa$ the curve $t \mapsto \exp(tZ)H_\kappa$ is a geodesic of G_κ/H_κ.

Proof of Theorem 1.4. To see that $B = B'$, as $\mathcal{M} \cong T^1S$, we only need to show that for $u \in T^1_pS$, the curve $\Gamma(t) =: \left[\gamma_{u_t}\right]$ is the geodesic in $(\mathcal{G}_\kappa, g_K)$ with initial velocity $\mathcal{J}\nu(u)$, where $\nu(u)$ is the outward-pointing normal vector of $\mathcal{M}$ at $[\gamma_u]$. First we verify that $\Gamma'(0) = \mathcal{J}\nu(u)$ and afterward that Γ is a geodesic of $\mathcal{G}_\kappa$.

The initial velocity of Γ corresponds, via the isomorphism T_{γ_u} of (6), with the Jacobi field along γ_u given by

$$J(s) = \frac{d}{dt}\Big|_0 \gamma_{u_t}(s).$$

A straightforward computation shows that J is determined by the conditions $J(0) = iu$ and $J'(0) = 0$.

On the other hand, let $I \in \mathfrak{J}_{\gamma_u}$ be the Jacobi field given by the initial conditions $I(0) = -n(p)$ and $I'(0) = 0$. We claim that after the identification with $T_{[\gamma_u]}\mathcal{G}_\kappa$, I corresponds to the unit outward-pointing normal vector field ν on $\mathcal{M}$. Indeed,

$$g_K(I, I) = \langle I, I\rangle_\kappa + \kappa\langle I', I'\rangle_\kappa = (-1)^2\left|n(p)\right|^2_\kappa = 1,$$

and for $K \in T_{[\gamma_u]}\mathcal{M}$, $K(0) \in T_{\gamma_u(0)}S$ by Lemma 2.1, and so

$$g_K(I, K) = -\langle n(p), K(0)\rangle_\kappa = 0.$$

Also, $\nu(u)$ points to $\mathcal{U}$ because n is the inward-pointing unit normal vector field of S.

Now, by the definition of the complex structure $\mathcal{J}$ in (8), the identity $\mathcal{J}_{[\gamma_u]}(\nu_{[\gamma_u]}) = \Gamma'(0)$ translates into $J = \gamma_u' \times I$, which holds because $J(0) = iu = n(p) \times u = \gamma_u'(0) \times I(0)$ and $J'(0) = 0 = I'(0)$.

Next we show that Γ is a geodesic. By homogeneity, we may suppose that $p = e_0$, the inward-pointing unit normal vector of S at e_0 is e_3 and $u = e_1$. Hence, $iu = e_2$ and $u_t = e_1 \in T_{\gamma(t)}M_\kappa$ where $\gamma(t) = c_\kappa(t)e_0 + s_\kappa(t)e_2$.

Let Z be the linear transformation of $\mathbb{R}^4$ defined by $Z(e_0) = e_2$, $Z(e_2) = -\kappa e_0$ and $Z(e_1) = Z(e_3) = 0$. It is easy to verify that $Z \in \mathrm{Lie}(G_\kappa)$ and $\exp(tZ)\,[\gamma_u] = \left[\gamma_{u_t}\right]$ for all t. Now, one can see in the preliminaries of [11] (page 752) that $Z \in \mathfrak{p}_\kappa$, and so Γ is a geodesic by the properties of symmetric spaces presented above. □

Proof of Proposition 1.5. Suppose that $\ell = [\gamma_u]$ with $\gamma_u(0) = p \in S$. The Jacobi field I along γ_u with $I(0) = 0$ and $I'(0) = iu$ spans the normal space to $T_\ell\mathcal{M}$ and is null (see (9)). By Lemma 2.1, I is also tangent to $\mathcal{M}$ at ℓ (this shows, in particular, that $g_\times$ degenerates on $T_\ell\mathcal{M}$). Now,

$$(\mathcal{J}I)(0) = 0 \quad \text{and} \quad (\mathcal{J}I)'(0) = n_p$$

(where n is the inward-pointing unit normal vector field of S, as before). Again by Lemma 2.1, $\mathcal{J}I \in T_\ell\mathcal{M}$.

The geodesic Γ in $\mathcal{G}_\kappa$ with $\Gamma(0) = \ell$ and $\Gamma'(0) = \mathcal{J}I$ consists of oriented geodesics in M_κ rotating around p in the totally geodesic surface orthogonal to S containing the image of γ_u, that is,

$$\Gamma(t) = \left[\gamma_{\cos t\; u + \sin t\; n_p}\right]$$

(this can be verified with computations similar to those we made at the end of the proof of the theorem above). Hence, for each t, the oriented geodesic $\Gamma(t)$ intersects S and so, the image of Γ is disjoint from $\mathcal{U}$. As any normal N is multiple of I, the proof concludes. □

5 | THE SYMPLECTIC PROPERTIES OF THE OUTER BILLIARD MAP

Proof of Theorem 1.6(a). As in (13), we write

$$B = F_+ \circ g \circ F_-^{-1}.$$

Given $\ell \in \mathcal{U}$, suppose that $\ell = F\_(u,-t)$ for some $0 < t < T$. We compute the matrix of $(dB)_\ell$ with respect to the canonical bases $\mathcal{E}_{-t}$ and $\mathcal{E}_t$ of $\mathfrak{J}_{\gamma_{u_{-t}}}$ and $\mathfrak{J}_{\gamma_{u_t}}$ as in (18), respectively, obtaining

$$\begin{aligned}\left[(dB)_\ell\right]_{\mathcal{E}_{-t},\mathcal{E}_t} &= \left[(dF)_{(u,t)}\right]_{\mathcal{B}_t,\mathcal{E}_t}\left[dg_{(u,-t)}\right]_{\mathcal{B}_{-t},\mathcal{B}_t}\left(\left[(dF)_{(u,-t)}\right]_{\mathcal{B}_{-t},\mathcal{E}_{-t}}\right)^{-1} \\ &= C_t\begin{pmatrix} I & 0 \\ 0 & R\end{pmatrix}(C_{-t})^{-1} = \begin{pmatrix} R & 0_2 \\ D & R\end{pmatrix},\end{aligned} \tag{21}$$

where C_t is as in (19), I is the (2×2)-identity matrix, $R = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ and

$$D = \frac{2}{b}\begin{pmatrix} s_\kappa(t)\kappa b_{22} & \kappa b_{12} \\ -b_{21} & -\frac{b_{11}}{s_\kappa(t)}\end{pmatrix},$$

with $b = \det\left(A_p\right)$. Thus,

$$(dB)_\ell\left(E_1^{-t}\right) = E_1^t + \tfrac{2}{b}s_\kappa(t)\kappa b_{22}E_3^t - \tfrac{2}{b}b_{21}E_4^t,$$

$$(dB)_\ell\left(E_2^{-t}\right) = -E_2^t + \tfrac{2}{b}\kappa b_{12}E_3^t - \tfrac{2}{b}\tfrac{b_{11}}{s_\kappa(t)}E_4^t,$$

$$(dB)_\ell\left(E_3^{-t}\right) = E_3^t \qquad \text{and} \qquad (dB)_\ell\left(E_4^{-t}\right) = -E_4^t.$$

Recall from (12) the definition of the symplectic form ω_K. Straightforward computations yield that

$$\left\langle E_i^t(0)\times E_j^t(0), u_t\right\rangle_\kappa = \left\langle \left(E_i^t\right)'(0)\times\left(E_j^t\right)'(0), u_t\right\rangle_\kappa = 0$$

for all $1 \leqslant i < j \leqslant 3$, except for

$$\left\langle E_2^t(0)\times E_4^t(0), u_t\right\rangle_\kappa = \left\langle \left(E_1^t\right)'(0)\times\left(E_3^t\right)'(0), u_t\right\rangle_\kappa = -1.$$

Hence,

$$[\omega_K]_{\mathcal{E}_t} = \begin{pmatrix} 0_2 & \rho \\ -\rho & 0_2\end{pmatrix}$$

with $\rho = \begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix}$. We observe that $[\omega_K]_{\mathcal{E}_{-t}} = [\omega_K]_{\mathcal{E}_t}$. Hence, calling H the matrix in (21), we have to check that

$$H^T [\omega_K]_{\mathcal{E}_t} H = [\omega_K]_{\mathcal{E}_{-t}}$$

The left-hand side equals

$$\begin{pmatrix} -D^T \rho R + R\rho D & \rho \\ -\rho & 0_2 \end{pmatrix}$$

and

$$-D^T \rho R + R\rho D = \frac{2}{a}(1-\kappa^2)\begin{pmatrix} 0 & -b_{12} \\ b_{21} & 0 \end{pmatrix},$$

which is the zero matrix because $\kappa = \pm 1$, as desired. □

Proof of Theorem 1.6(b). Following the computations in the proof of part (a), we have

$$\left[\omega_\times\right]_{\mathcal{E}_t} = \begin{pmatrix} j & 0_2 \\ 0_2 & j \end{pmatrix},$$

where $j = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\left[\omega_\times\right]_{\mathcal{E}_{-t}} = \left[\omega_\times\right]_{\mathcal{E}_t}$. Further computations yield

$$H^T \left[\omega_\times\right]_{\mathcal{E}_t} H = \begin{pmatrix} j & -(RjD)^T \\ RjD & j \end{pmatrix}. \tag{22}$$

Now, $RjD = \frac{1}{b}\begin{pmatrix} -b_{21} & -b_{11}/s_\kappa(t) \\ s_\kappa(t)\,\kappa b_{22} & \kappa b_{12} \end{pmatrix}$ and $b_{11}/s_\kappa(t) \neq 0$ because by the hypothesis S is quadratically convex. Therefore, $RjD \neq 0_2$ and the expression (22) is not equal to $\left[\omega_\times\right]_{\mathcal{E}_{-t}}$. □

We conclude this section with the following proposition, which relates Theorem 1.6 with plane hyperbolic outer billiards [30] and supports the fact that ω_K (in contrast with $\omega_\times$) is the natural symplectic form in our context.

Proposition 5.1. *Let H^2 be a totally geodesic hyperbolic plane in hyperbolic space.*

(a) *Let ν be a unit normal vector field on H^2 and let $f : H^2 \to \mathcal{G}_{-1}$, $f(p) = [\gamma_{\nu(p)}]$. Then $f^*\omega_K$ is the area form on H^2.*
(b) *Let S be a smooth, closed, quadratically convex surface in H^3 that is invariant by the reflection with respect to H^2. Then the outer billiard map on $\mathcal{G}_{-1}$ associated with S preserves the oriented lines orthogonal to H^2 and induces in the obvious manner the outer billiard map on H^2 determined by the closed strictly convex curve with image $S \cap H^2$. This plane outer billiard preserves the area form on H^2.*

Proof. For part (a), let $p \in H^2$ and let $\{z_1, z_2\}$ be a positively oriented (with respect to the orientation on H^2 determined by ν) orthonormal basis of T_pH^2. For $i = 1, 2$, let J_i be the Jacobi field along $\gamma_{\nu(p)}$ satisfying $J_i(0) = z_i$ and $J_i'(0) = 0$. Using (12) and that H^2 is totally geodesic we have that

$$(f^*\omega_K)_p(z_1, z_2) = \omega_K(df_p(z_1), df_p(z_2)) = \omega_K(J_1, J_2) = 1.$$

Part (b) is an immediate consequence of part (a) and Theorem 1.6. □

6 | DYNAMICS OF THE OUTER BILLIARD MAP ON $\mathcal{G}_{-1}$

Before proving Proposition 1.8, we comment on the Klein model of hyperbolic space, that is, the open ball $\mathcal{H}$ centered at the origin with radius 1, where the trajectories of geodesics are the intersections of Euclidean straight lines with the ball. The intersections of $\mathcal{H}$ with Euclidean planes are totally geodesic hyperbolic planes.

We recall the following well-known constructions on $\mathcal{H}$ (see, for instance, [4, chapter 6]). For an oriented line ℓ in $\mathcal{H}$ we call ℓ^+ and ℓ^- its forward and backward ideal end points in the two sphere $\partial\mathcal{H}$.

Let ℓ_1 and ℓ_2 be two coplanar oriented lines in $\mathcal{H}$ such that the corresponding extensions to Euclidean straight lines intersect in the complement of the closure of $\mathcal{H}$. In particular, ℓ_1 and ℓ_2 do not intersect and are not asymptotic and hence there exists the shortest segment joining them with respect to the hyperbolic metric; we call it $s(\ell_1, \ell_2)$.

The hyperbolic midpoint of $s(\ell_1, \ell_2)$ is the intersection of the Euclidean segments joining ℓ_1^+ with ℓ_2^- and ℓ_1^- with ℓ_2^+, or joining ℓ_1^+ with ℓ_2^+ and ℓ_1^- with ℓ_2^-, depending on the orientation of the lines.

Suppose that ℓ_1 and ℓ_2 lie in the plane P, let $D = P \cap \mathcal{H}$, and let C be the boundary of D. We describe the construction of the segment $s(\ell_1, \ell_2)$ in the case when one of the lines, say ℓ_1, is a diameter in D. Let p be the intersection of the tangent lines to C through the ideal end points ℓ_2^+ and ℓ_2^-. Then $s(\ell_1, \ell_2)$ is the segment that joins ℓ_1 and ℓ_2 and is contained in the straight line through p perpendicular to ℓ_1 (the point p is called the pole of ℓ_2 in the plane P).

We recall the formula for the hyperbolic distance between a point in $\mathcal{H}$ and the midpoint of any chord containing it: Let X, Y be two distinct points in $S^2 = \partial\mathcal{H}$ and let C be the midpoint of the segment joining X and Y, that is, $C = \frac{1}{2}(X + Y)$. Then, for any $t \in (0, 1)$,

$$d\left(C, C + t\,\tfrac{Y-X}{2}\right) = \operatorname{arctanh} t, \tag{23}$$

where d is the hyperbolic distance in $\mathcal{H}$ (it is not difficult to deduce the expression from the second displayed formula of [4, Proposition 6.2]).

As quadratic contact is invariant by diffeomorphisms, a quadratically convex surface of $\mathbb{R}^3$ contained in $\mathcal{H}$ is also quadratically convex with the hyperbolic metric. By abuse of notation, we describe an oriented geodesic ℓ in $\mathcal{H}$ by the straight Euclidean line $p + \mathbb{R}u$ containing ℓ, with $p \in \mathcal{H}$ and u a unit vector giving the orientation.

Proof of Proposition 1.8. We use the Klein model of hyperbolic space. Let $\ell = \mathbb{R}e_3$ and $\ell_\theta = \mathbb{R}(\sin\theta\, e_2 + \cos\theta\, e_3)$. We construct a surface S contained in the region $x \geqslant 0, y \geqslant 0$ of $\mathcal{H}$ whose associated outer billiard map B satisfies $B^3(\ell) = \ell_\theta$. We fix a real number r in the interval $(\sin\theta, 1)$.

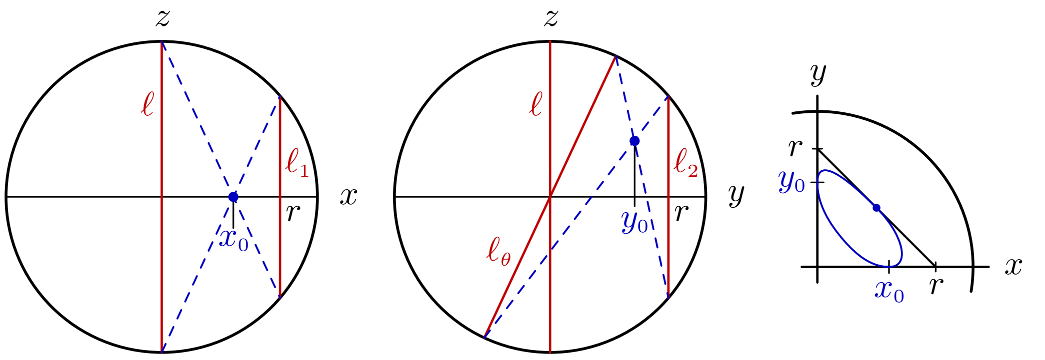

FIGURE 6 Elements for the construction of S.

Let $\ell_1 = re_1 + \mathbb{R}e_3$ and $\ell_2 = re_2 + \mathbb{R}e_3$, and let $p = (x_0, 0, 0)$ and $q = (0, y_0, z_0)$ be the hyperbolic midpoints between ℓ and ℓ_1 and between ℓ_θ and ℓ_2, respectively, (see Figure 6).

There exists a smooth, closed, quadratically convex surface S contained in $\mathcal{H}$ and tangent to the vertical planes $y = 0$, $y + x = r$ and $x = 0$, at the points p, $(r/2, r/2, 0)$ and q, respectively. In fact, consider a quadratically convex compact surface S' tangent to those planes at the points p, $(r/2, r/2, 0)$ and $(0, y_0, 0)$, respectively, such that the absolute value of the height function $z|_{S'}$ is bounded by ε for some $\varepsilon > 0$. Let T be the unique affine transformation of $\mathbb{R}^3$ fixing the vertical plane through p and $(r/2, r/2, 0)$ and sending $(0, y_0, 0)$ to q. Then $S = T\left(S'\right)$ satisfies the desired conditions (in particular, it preserves the vertical planes and the quadratical contact), provided that ε is small enough. By the properties of S, we have that $B(\ell) = \ell_1$, $B^2(\ell) = \ell_2$ and $B^3(\ell) = \ell_\theta$. □

Proof of Proposition 1.9. As in the proof of Proposition 1.8, we use the Klein model $\mathcal{H}$ for hyperbolic space. We write $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. Given $0 < a < \frac{1}{2} < r_o < 1$ and $h_o = \sqrt{1 - r_o^2}$, we consider the straight lines

$$\gamma_0(t) = (t, 0), \qquad \gamma_1(t) = (\tfrac{i}{2} + t(1 - ai), 0),$$
$$\gamma_2(t) = (\tfrac{i}{2} + t(1 - ai), h_o), \quad \gamma_3(t) = (t, h_o).$$

For $k = 0, 1, 2, 3$, let ℓ_k be the corresponding oriented geodesic in $\mathcal{H}$ and set $\ell_4 = \ell_0$. Notice that ℓ_k and ℓ_{k+1} are coplanar for any $k = 0, 1, 2, 3$. We call P_k the hyperbolic plane containing ℓ_k and ℓ_{k+1}. As $0 < 2a < 1$, the lines ℓ_k and ℓ_{k+1} are disjoint and not asymptotic, and so the shortest segment σ_k joining them is well-defined.

We will show the existence of a smooth, closed, quadratically convex surface S in $\mathcal{H}$ not intersecting ℓ_0 such that the associated billiard map B satisfies $B^k(\ell_0) = \ell_k$ for $k = 0, \dots, 4$ and its holonomy at ℓ_0 is not trivial.

We make computations for general values of r and $h = \sqrt{1 - r^2}$, with $0 < a < \frac{1}{2} < r \leqslant 1$, in order to deal simultaneously with ℓ_0 and ℓ_1 on the one hand (case $r = 1$) and ℓ_2 and ℓ_3 on the other (case $r = r_o$), as the former lie in a disc of radius 1 at height 0 and the latter in a disc of radius r_o at height h_o.

The end points of ℓ_1 and ℓ_2 are given by

$$\ell_1^\varepsilon = (z_\varepsilon(1), 0) \quad \text{and} \quad \ell_2^\varepsilon = (z_\varepsilon(r_o), h_o),$$

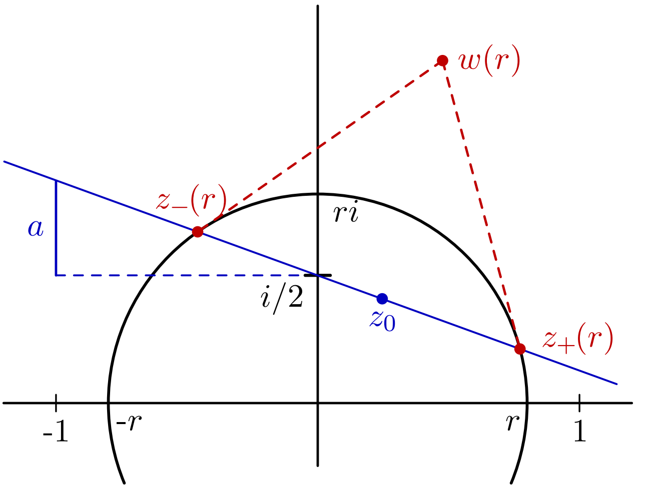

FIGURE 7 Elements for the construction of S.

for $\varepsilon = \pm 1$, where $z_\varepsilon(r) = \frac{i}{2} + t_\varepsilon(r)(1 - ai)$, with $t_-(r) < t_+(r)$ being the solutions of the equation $t^2 + \left(\frac{1}{2} - at\right)^2 = r^2$. As ℓ_1 and ℓ_2 are parallel, the end points of σ_1 are the respective midpoints, whose (common) component in $\mathbb{C}$ is

$$z_o = \tfrac{1}{2}\big(z_+(1) + z_-(1)\big) = \tfrac{1}{2}\big(z_+(r_o) + z_-(r_o)\big) = \tfrac{1}{2(a^2+1)}(a + i).$$

Hence, $q_+(\ell_1) = (z_o, 0)$ and $q_-(\ell_2) = (z_o, h_o)$. Similarly, $q_-(\ell_0) = (0, 0)$ and $q_+(\ell_3) = (0, h_o)$.

Let $p_1 = (w(1), 0)$ and $p_2 = (w(r_o), h_o)$ be the poles of the line ℓ_1 in the plane $\mathbb{C} \times \{0\}$ and of the line ℓ_2 in the plane $\mathbb{C} \times \{h_o\}$, respectively. That is, $w(r)$ is the intersection of the lines tangent to the circle of radius r in $\mathbb{C}$ at the points $z_-(r)$ and $z_+(r)$, in the horizontal plane at height h, for $h = 0, h_o$ (see Figure 7). Using the construction of the shortest segment joining two oriented lines, the segment σ_2 is contained in the line perpendicular to ℓ_3 passing through $(w(r_o), h_o)$, that is, the line $(\operatorname{Re} w(r_o) + \mathbb{R}i, h_o)$. Putting $r = 1$, we get that σ_0 is contained in the line $(\operatorname{Re} w(1) + \mathbb{R}i, 0)$.

We have that $w(r) = z_+(r) + s_o(r) i z_+(r)$, where s_o is the solution of the equation

$$z_+(r) + s i z_+(r) = z_-(r) - s i z_-(r).$$

A straightforward computation yields $\operatorname{Re} w(r) = 2ar^2$.

Now, computing the intersections of the remaining σ_k with the lines ℓ_j, we obtain the rest of the $q_\pm(\ell_j)$:

$$\begin{aligned} q_+(\ell_0) &= (\operatorname{Re} w(1), 0) = 2(a, 0), & q_-(\ell_1) &= 2\left(a + i\left(\tfrac{1}{4} - a^2\right), 0\right) \\ q_+(\ell_2) &= \left(2\left(ar_o + i\left(\tfrac{1}{4} - a^2 r_o^2\right)\right), h_o\right), & q_-(\ell_3) &= (\operatorname{Re} w(r_o), h_o) = \left(2ar_o^2, h_o\right). \end{aligned}$$

As in Proposition 1.8, for each $a > 0$ there exists a quadratically convex surface S_a in $\mathcal{H}$ tangent to the plane P_k at the midpoint of σ_k for any $k = 0, \dots, 3$. The associated billiard map B_a satisfies $(B_a)^4(\ell_0) = \ell_0$ and its holonomy at ℓ_0 turns out to be

$$H(a) = \sum_{k=0}^{3} (-1)^k d(q_+(\ell_k), q_-(\ell_k)).$$

Particularizing $r_o = h_o = 1/\sqrt{2}$, using (23) we obtain that

$$H(a) = \operatorname{arctanh}(2a) - \operatorname{arctanh}\left(a\sqrt{4a^2+3}\right) + \operatorname{arctanh}\left(a\sqrt{2a^2+1}\right) - \operatorname{arctanh}(a).$$

We compute $H(0) = 0$ and $H'(0) \neq 0$. Therefore, for sufficiently small $a > 0$, the holonomy of B_a at ℓ_0 does not vanish. □

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