DOI 10.1007/s11253-024-02270-6
Ukrainian Mathematical Journal, Vol. 75, No. 9, February, 2024 (Ukrainian Original Vol. 75, No. 9, September, 2023)

SCHMIDT RANK AND SINGULARITIES

David Kazhdan,[1] **Amichai Lampert,**[2] **and Alexander Polishchuk**[3,4] UDC 512.5

We revisit Schmidt's theorem connecting the Schmidt rank of a tensor with the codimension of a certain variety and adapt the proof to the case of arbitrary characteristic. We also establish a sharper result for this kind for homogeneous polynomials, assuming that the characteristic does not divide the degree. Further, we use this to relate the Schmidt rank of a homogeneous polynomial (resp., a collection of homogeneous polynomials of the same degree) with the codimension of the singular locus of the corresponding hypersurface (resp., intersection of hypersurfaces). This gives an effective version of Ananyan–Hochster's theorem [*J. Amer. Math. Soc.*, **33**, No. 1, 291–309 (2020), Theorem A].

1. Introduction

Let $\mathbf{k}$ be a field (of any characteristic) and let $P\colon V_1 \times V_2 \times \ldots \times V_d \to \mathbf{k}$ be a polylinear form, where V_i are finite-dimensional vector spaces over $\mathbf{k}$. Equivalently, we consider P as a tensor in $V_1^* \otimes \ldots \otimes V_d^*$.

Definition 1.1.

(i) *We say that $P \neq 0$ has the Schmidt rank 1 if there exist a partition $[1, d] = I \sqcup J$ into two nonempty parts and polylinear forms $P_I(v_{i_1}, \ldots, v_{i_r})$ and $P_J(v_{j_1}, \ldots, v_{j_s})$, where $v_a \in V_a$, $I = \{i_1 < \ldots < i_r\}$, and $J = \{j_1 < \ldots < j_r\}$, such that $P = P_I \cdot P_J$. In general the Schmidt rank of P, denoted by $\mathrm{rk}^S(P)$ is the smallest number r such that $P = \sum_{i=1}^{r} P_i$ with P_i of Schmidt rank 1. For a collection of tensors $\overline{P} = (P_1, \ldots, P_s)$, we define the Schmidt rank $\mathrm{rk}^S(\overline{P})$ as the minimum of Schmidt ranks of the nontrivial linear combinations of (P_i).*

(ii) *Given a collection of nonempty subsets $I_1, \ldots, I_r \subset [1, d]$ and a collection $(P_{I_1}, \ldots, P_{I_r})$, where P_{I_i} is a polylinear form on $\prod_{a \in I_i} V_a$, we denote by $(P_{I_1}, \ldots, P_{I_r}) \subset V_1^* \otimes \ldots \otimes V_d^*$ and call this the tensor ideal generated by $P_{I_1}, \ldots, P_{I_r}$, and the subspace of polylinear forms of the form*

$$P = \sum_{i=1}^{r} P_{I_i} \cdot Q_{J_i},$$

for some polylinear forms Q_{J_i} on $\prod_{b \in J_i} V_b$, where $J_i = [1, d] \setminus I_i$.

The Schmidt rank of a tensor, along with a set of related notions, such as *slice rank*, *G-rank*, *analytic rank*, and a version of Schmidt rank for homogeneous polynomials also known as *strength* (see in what follows) has

[1] Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel; e-mail: kazhdan@math.huji.ac.il.
[2] Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel; Institute for Advanced Study, Princeton, NJ, USA; e-mail: amichai.lam@gmail.com.
[3] Department of Mathematics, University of Oregon, Eugene, USA, National Research University Higher School of Economics, Moscow, Russia; Korea Institute for Advanced Study, Seoul, Republic of Korea; e-mail: apolish@uoregon.edu.
[4] Corresponding author.

Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 75, No. 9, pp. 1248–1266, September, 2023. Ukrainian DOI: 10.37863/umzh.v75i9.7227. Original article submitted May 30, 2022.

 0041-5995/24/7509–1420

been a subject of study in many recent works (see [1, 4, 5, 8, 9] and the references therein). One of the goals of the present paper is to establish a precise relation (in the case of an algebraically closed base field $\mathbf{k}$) between this notion and the codimension of the singular locus of the corresponding hypersurface, thus giving an effective version of Ananyan–Hochster's theorem [2, Theorem A].

We define a subvariety

$$Z_P = Z_P^{V_1} \subset V_2 \times \ldots \times V_d$$

as the set of $(v_2, \ldots, v_d)$ such that $P(v_1, v_2, \ldots, v_d) = 0$ for all $v_1 \in V_1$. Following Schmidt, we set

$$g(P) := \operatorname{codim}_{V_2 \times \ldots \times V_d} Z_P.$$

In [10] (where the authors considered the case $d = 3$), this number is called the *geometric rank* of P. By using [10, Theorem 3.2], we can see that it does not depend on the ordering of the variables $v_1, \ldots, v_d$.

It is easy to see that

$$g(P) \leq \operatorname{rk}^S(P) \tag{1.1}$$

(see Lemma 2.1(i) in what follows or [10, Theorem 1]).

Similarly, for a collection $\overline{P} = (P_1, \ldots, P_s)$, we define $Z_{\overline{P}} \subset V_2 \times \ldots \times V_d$ by the condition on $(v_2, \ldots, v_d)$ that the corresponding map

$$V_1 \to \mathbf{k}^s : v_1 \mapsto (P_i(v_1, v_2, \ldots, v_d))_{1 \leq i \leq s}$$

has $\operatorname{rank} < s$, and we set

$$g(\overline{P}) := \operatorname{codim}_{V_2 \times \ldots \times V_d} Z_{\overline{P}}.$$

The proof of the following theorem closely follows the proof of a similar result presented in [11] for the case where $\mathbf{k} = \mathbb{C}$ and P is symmetric. We modified the proof so that it would work for an arbitrary characteristic and also streamlined some parts of the original arguments. The fact that the original proof can be adapted to an arbitrary characteristic was also pointed out in [11, Section 4].

Theorem 1.1.

(i) Let $g'(P)$ denote the codimension in $V_2 \times \ldots \times V_d$ of the Zariski closure of the set of $\mathbf{k}$-points in Z_P (so that $g(P) \leq g'(P)$ and $g(P) = g'(P)$ if $\mathbf{k}$ is algebraically closed). Then

$$\operatorname{rk}^S(P) \leq C_d g'(P),$$

where

$$C_d = \max(2 + \theta_{d-2}, 2^{d-2} - 1)$$

and θ_n is the number of ordered collections of disjoint nonempty subsets $I_1 \sqcup \ldots \sqcup I_p \subsetneq [1, n]$ (with $p \geq 1$). In particular, $C_3 = 2$, $C_4 = 4$, and $C_5 = 14$.

(ii) Assume that $\mathbf{k}$ is algebraically closed. Then, for a collection $\overline{P} = (P_1, \ldots, P_s)$, the following inequality is true:

$$\operatorname{rk}^S(\overline{P}) \leq C_d(g(\overline{P}) + s - 1).$$

In the appendix, we prove another version of Theorem 1.1 with better bounds for $d \geq 6$. Even though Schmidt applied the above result to the symmetric tensors P corresponding to homogeneous polynomials, we observe that, in the symmetric case, it is natural to modify the relevant variety Z_P and that this leads to much better estimates for the rank.

Let f be a homogeneous polynomial of degree d on a finite-dimensional $\mathbf{k}$-vector space V. The *Schmidt rank* of f, denoted by $\mathrm{rk}^S(f)$, is the minimal number r such that

$$f = \sum_{i=1}^{r} g_i h_i,$$

where g_i and h_i are homogeneous polynomials of positive degrees. Note that if $\mathrm{rk}^S(f) = r$, then, in the terminology of [2], f has *strength* $r-1$. For a collection $\overline{f} = (f_1, \dots, f_s)$, the Schmidt rank $\mathrm{rk}^S(\overline{f})$ is defined as the minimum of Schmidt ranks of the nontrivial linear combinations of f_i.

By $H_f(x)(\cdot,\cdot)$ we denote the Hessian form of f given by the second derivatives of f. This is a symmetric bilinear form on V, which polynomially depends on the point $x \in V$. The symmetric analog of the variety Z_P is a subvariety $Z_f^{\mathrm{sym}} \subset V \times V$ given by

$$Z_f^{\mathrm{sym}} := \big\{(v,x) \in V \times V \mid v \in \ker H_f(x)\big\}.$$

Further, we set

$$g_{\mathrm{sym}}(f) := \mathrm{codim}_{V\times V}(Z_f^{\mathrm{sym}}).$$

The symmetric analog of (1.1) is the following inequality:

$$g_{\mathrm{sym}}(f) \le 4\,\mathrm{rk}^S(f) \tag{1.2}$$

[see Lemma 2.1(ii)].

Similarly, for a collection $\overline{f} = (f_1, \dots, f_s)$ of homogeneous polynomials of degree d, we define a subvariety $Z_{\overline{f}}^{\mathrm{sym}} \subset V \times V$ as the set of (v,x) such that the map

$$V \to \mathbf{k}^s : v' \mapsto (H_{f_i}(x)(v',v))_{1\le i\le s}$$

has the $\mathrm{rank} < s$. By $g_{\mathrm{sym}}(\overline{f})$ we denote the codimension of $Z_{\overline{f}}^{\mathrm{sym}}$ in $V \times V$.

Theorem 1.2.

(i) Assume that $d \ge 3$ and that the characteristic of $\mathbf{k}$ does not divide $(d-1)d$. Let $g'_{\mathrm{sym}}(f)$ denote the codimension in $V \times V$ of the Zariski closure of the set of $\mathbf{k}$-points in Z_f^{sym}. Then

$$\mathrm{rk}^S(f) \le 2^{d-3} g'_{\mathrm{sym}}(f).$$

(ii) Under the same assumptions as in (i), assume, in addition, that $\mathbf{k}$ is algebraically closed. Then

$$\mathrm{rk}^S(\overline{f}) \le 2^{d-3}\big(g_{\mathrm{sym}}(\overline{f}) + s - 1\big).$$

For algebraically closed $\mathbf{k}$, we prove another version of Theorem 1.2 in the appendix with better bounds for $d \ge 6$. The invariant $g_{\mathrm{sym}}(f)$ can be viewed as invariant measuring singularities of the polar map

$$x \mapsto (\partial_i f(x))_{1\le i\le \dim V}$$

of f (see Section 3.3). We also prove that $g_{\mathrm{sym}}(f)$ is related to the codimension of the singular locus of the hypersurface $f=0$. Namely, we set

$$c(f) := \operatorname{codim}_V \operatorname{Sing}(f=0).$$

Under the assumption that $\operatorname{char}(\mathbf{k})$ does not divide $2(d-1)$, we prove that

$$c(f) \le g_{\mathrm{sym}}(f) \le (d+1)c(f) \quad \text{for} \quad \text{even } d,$$

$$c(f) \le g_{\mathrm{sym}}(f) \le dc(f) \quad \text{for} \quad \text{odd } d$$

(see Proposition 3.1).

More generally, for a collection $\overline{f}=(f_1,\dots,f_s)$, we set

$$c(\overline{f}) := \operatorname{codim}_V \operatorname{Sing}\big(V(\overline{f})\big),$$

where $V(\overline{f}) \subset V$ is a subscheme defined by the ideal $(f_1,\dots,f_s)$. We also consider the related invariant

$$c'(\overline{f}) := \operatorname{codim}_V S(\overline{f}),$$

where $S(\overline{f}) \subset V$ is the locus where the Jacobi matrix of $(f_1,\dots,f_s)$ has the $\operatorname{rank} < s$. It is easy to see that

$$c'(\overline{f}) \le c(\overline{f}) \le c'(\overline{f}) + s.$$

Here is our main result concerning the relation between the Schmidt rank and the codimension of the singular locus. It can be regarded as a more precise version of the corresponding result in [9] in the case of an algebraically closed field of sufficiently large (or zero) characteristic, as well as an effective version of a result of Ananyan and Hochster (see [2, Theorem A(a)]) playing a central role in their proof of Stillman's conjecture.

Theorem 1.3. *Assume that* $\operatorname{char}(\mathbf{k})$ *does not divide* d*. Let* $c_{\mathbf{k}}(f)$ *be the codimension in* V *of the Zariski closure of the* $\mathbf{k}$*-points of* $\operatorname{Sing}(f=0)$.

(i) The following inequalities are true:

$$\frac{c(f)}{2} \le \operatorname{rk}^S(f) \le (d-1)c_{\mathbf{k}}(f).$$

(ii) Assume that $\mathbf{k}$ *is algebraically closed. Then, for a collection* $\overline{f}=(f_1,\dots,f_s)$,

$$\operatorname{rk}^S(\overline{f}) \le (d-1)\big(c'(\overline{f}) + s - 1\big).$$

Combining Theorem 1.3(i) with [2, Theorem A(c)], we get the following result:

Corollary 1.1. *Assume that* $\mathbf{k}$ *is algebraically closed and* $\operatorname{char}(\mathbf{k})$ *does not divide* $d!$*. For* $i=2,\dots,d$*, let* $W_i \subset \mathbf{k}[V]_i$ *be a subspace of forms of degree* i*. Also let* $W=\bigoplus_i W_i$*,* $w=\dim W$*. Assume that,* *for some* $m \geq 1$, *the following inequalities are true:*

$$\mathrm{rk}^S(W_i) \geq (i-1)(m+2) + 3(w-1) \quad \textit{for} \quad i = 3, \ldots, d,$$

$$\mathrm{rk}^S(W_2) - 1 \geq \left\lceil \frac{m+1}{2} \right\rceil + 3(w-1).$$

Then every sequence of linearly independent homogeneous forms in W *is regular, and the corresponding complete intersection subscheme in* V *satisfies the Serre condition* R_m.

Note that, without any assumptions on the characteristic on $\mathbf{k}$, we are able to estimate, in terms of $c(f)$, the rank of $H_f(x)(u,v)$ regarded as a polynomial in $(u,v,x) \in V \times V \times V$ (see Remark 3.1).

For a homogeneous polynomial $f(x)$ of degree d on V and a vector $v \in V$, we denote the derivative of f in the direction v by $\partial_v f(x)$. Our next result concerns $\partial_v f$ for generic v.

Theorem 1.4. *Let* f *be a homogeneous polynomial of degree* $d \geq 3$. *Assume that* $\mathbf{k}$ *is algebraically closed of characteristic that does not divide* $(d-1)d$.

(i) For generic $v \in V$, *the following inequality is true:*

$$\mathrm{rk}^S(\partial_v f) \geq 2^{2-d}\,\mathrm{rk}^S(f).$$

(ii) For $s \leq 2^{2-d}\,\mathrm{rk}^S(f) + \frac{1}{2}$ $\left(\textit{resp.}, s \leq 2^{2-d}\,\mathrm{rk}^S(f) - \frac{1}{2}\right)$ *and for generic* $v_1, \ldots, v_s \in V$, *the derivatives* $(\partial_{v_1} f, \ldots, \partial_{v_s} f)$ *define a (resp., normal) complete intersection of the codimension* s *in* V.

In the appendix we prove another version of Theorem 1.4 with better bounds for $d \geq 6$. In Section 3.4, we also discuss the relationship between the invariant $g_{\mathrm{sym}}(f)$, the polar map of f, and the Gauss map of the corresponding projective hypersurface.

2. Schmidt Rank of Polylinear Forms

2.1. Elementary Observations. First, we prove (1.1) and its symmetric version (1.2). We denote by $\mathbf{k}[V]$ the space of polynomial functions on a vector space V. Moreover, by $\mathbf{k}[V]_d \subset \mathbf{k}[V]$ we denote the subspace of homogeneous polynomials of degree d.

Lemma 2.1.

(i) For $P \in V_1^* \otimes \ldots \otimes V_d^*$, *the inequality* $g(P) \leq \mathrm{rk}^S(P)$ *is true.*

(ii) For $f \in \mathbf{k}[V]_d$, *the inequality* $g_{\mathrm{sym}}(f) \leq 4\,\mathrm{rk}^S(f)$ *is true.*

Proof. (i) If $r = \mathrm{rk}^S(P)$, then there exists a decomposition

$$P = \sum_{i=1}^{r} P_{I_i} \cdot Q_{J_i}$$

as in Definition 1.1. Swapping, if necessary, some I_i with J_i, we can assume that $1 \in I_i$ for all i. Then the intersection of r hypersurfaces $Q_{J_i} = 0$ in $V_2 \times \ldots \times V_d$ is contained in Z_P and has the codimension $\leq r$.

(ii) If we have a decomposition

$$f = \sum_{i=1}^{r} g_i h_i,$$

then, over the subvariety $Y = V(g_1, \ldots, g_r, h_1, \ldots, h_r) \subset V$, the symmetric form $H_f(x)$ has the $\mathrm{rank} \leq 2r$: the subspace cut out by $dg_1|_x, \ldots, dg_r|_x, dh_1|_x, \ldots, dh_r|_x$ is contained in its kernel. Since $\mathrm{codim}_V Y \leq 2r$, the preimage of Y in Z_f^{sym} has the codimension $\leq 4r$ in $V \times V$.

Lemma 2.1 is proved.

For a subset of indices $I = \{i_1 < \ldots < i_s\} \subset [1, d]$, we set

$$V_I := V_{i_1} \otimes \ldots \otimes V_{i_s}.$$

Thus, we have the following simple observation:

Lemma 2.2. *Let $V_1' \subset V_1$ be a subspace of codimension c and let $(\ell_1, \ldots, \ell_r)$ be a basis of the subspace orthogonal to V_1' in V_1^*. Suppose that we have tensors*

$$P_{I_s} \in V_1^* \otimes V_{I_s}^* \quad \text{and} \qquad Q_{J_t} \in V_{J_t}^*$$

for some subsets $I_1, \ldots, I_r, J_1, \ldots, J_p \subset [2, \ldots, d]$ such that $P|_{V_1' \times V_2 \times \ldots V_d}$ belongs to the tensor ideal

$$\big(P_{I_s}|_{V_1' \otimes V_{I_s}},\ \ Q_{J_t} \mid s = 1, \ldots, r;\ \ t = 1, \ldots, p\big).$$

Then P belongs to the tensor ideal

$$\big((\ell_i \mid i = 1, \ldots, c),\ \ (P_{I_s}, Q_{J_t} \mid s = 1, \ldots, r;\ \ t = 1, \ldots, p)\big).$$

In particular,

$$\mathrm{rk}^S(P) \leq \mathrm{rk}^S(P|_{V_1' \times V_2 \times \ldots \times V_d}) + c.$$

Proof. This immediately follows from the fact that the tensor ideal $(\ell_i \mid i = 1, \ldots, c)$ is exactly the kernel of the restriction map

$$(V_1 \otimes V_2 \otimes \ldots \otimes V_d)^* \to (V_1' \otimes V_2 \otimes \ldots \otimes V_d)^*.$$

2.2. Determinantal Construction. Let $f\colon V_1 \to V_2$ be a morphism of vector bundles on a scheme X. For every $r \geq 0$, we have a natural morphism

$$\kappa_r\colon \bigwedge^r V_2^\vee \otimes \bigwedge^{r+1} V_1 \to V_1\colon (\phi_1 \wedge \ldots \wedge \phi_r) \otimes \alpha \mapsto \iota_{f^\vee \phi_1} \ldots \iota_{f^\vee \phi_r} \alpha,$$

where, for a section ψ of $V^\vee$, we denote the corresponding contraction operator by $\iota_\psi\colon \bigwedge^i V \to \bigwedge^{i-1} V$.

Lemma 2.3.

(i) Assume that $\bigwedge^{r+1} f = 0$. Then the image of κ_r is contained in $\ker(f)$.

(ii) Assume, in addition, that V_1 and V_2 are trivial vector bundles and, for some point $x \in X$, the rank of $f(x)\colon V_1|_x \to V_2|_x$ is equal to r. Let $n = \operatorname{rk} V_1$. Then there exist $n-r$ global sections $s_1, \ldots, s_{n-r}$ of V_1 such that $f(s_i) = 0$ for all i and $s_1(x), \ldots, s_{n-r}(x)$ is a basis of $\ker f(x)$.

Proof. (i) This is equivalent to the statement that $\iota_{f^\vee \phi_{r+1}} \kappa_r(\alpha) = 0$ for any local section ϕ_{r+1} of $V_2^\vee$. However, $\iota_{f^\vee \phi_1} \ldots \iota_{f^\vee \phi_r} \iota_{f^\vee \phi_{r+1}} = 0$ since $\bigwedge^{r+1} f^\vee = 0$.

(ii) Since V_1 and V_2 are trivial, we can choose splittings $V_1 = \mathcal{K} \oplus \mathcal{W}_1$ and $V_2 = \mathcal{C} \oplus \mathcal{W}_2$ into trivial subbundles such that $\mathcal{K}|_x = \ker f(x)$, $\mathcal{W}_2|_x = \operatorname{im} f(x)$, and $f(x)\colon \mathcal{W}_1|_x \to \mathcal{W}_2|_x$ is an isomorphism. We now consider a composed map

$$s\colon \bigwedge^r \mathcal{W}_2^\vee \otimes \left(\bigwedge^r \mathcal{W}_1 \otimes \mathcal{K}\right) \to \bigwedge^r V_2^\vee \otimes \bigwedge^{r+1} V_1 \xrightarrow{\kappa_r} V_1.$$

Thus, $f \circ s = 0$ and the image of $s(x)$ is exactly $\ker f(x)$. Choosing a trivialization of the target of s, we can represent s as a collection of global sections of V_1, which has the required properties.

Lemma 2.3 is proved.

2.3. Higher Derivatives. Let V be a finite-dimensional vector space and let $\mathbf{k}[V]$ denote the ring of polynomial functions on V.

For each $f \in \mathbf{k}[V]$, each $n \geq 1$, and $v_0 \in V$, we define a homogeneous form $f_{v_0}^{(n)}(v)$ of degree n on V as the nth graded component of $f(v + v_0) \in \mathbf{k}[V]$ (regarded as a function of v for fixed v_0) with respect to the degree grading on $\mathbf{k}[V]$ so that we get a (finite) Taylor decomposition

$$f(v + v_0) = \sum_{n \geq 0} f_{v_0}^{(n)}(v).$$

We refer to $f_{v_0}^{(n)}$ as the nth derivative of f at v_0.

Lemma 2.4. *Let $X \subset V$ be an irreducible closed subvariety of codimension c and let $v_0 \in X$ be a smooth $\mathbf{k}$-point. Also let $g_1, \ldots, g_c$ be the set of elements in the ideal I_X of X with linearly independent differentials at v_0. Then, for any $f \in I_X$ and any $n \geq 1$, the form $f_{v_0}^{(n)} \in \mathbf{k}[V]$ belongs to the ideal in $\mathbf{k}[V]$ generated by $((g_i)_{v_0}^{(j)})_{i=1,\ldots,c; 1 \leq j \leq n}$.*

Proof. Without loss of generality, we can assume that $v_0 = 0$. We set $A = \mathbf{k}[V]$. Also let $\hat{A}$ denote the completion of the origin (the ring of formal power series) with respect to the ideal. Then the keypoint is that $I_X \cdot \hat{A}$ is generated by $g_1, \ldots, g_c$. Indeed, this follows from the fact that the local homomorphism of local regular $\mathbf{k}$-algebras $A_{\mathfrak{m}}/(g_1, \ldots, g_c) \to \mathcal{O}_{X,v_0}$ (where $\mathfrak{m}$ is the maximal ideal of v_0 in A) induces an isomorphism on tangent spaces and, hence, also induces an isomorphism of completions. Note that the higher derivatives are meaningful for elements of $\hat{A}$ (as components in $A_n = \mathbf{k}[V]_n$) and, hence, the assertion follows if we express any element of I_X in the form $\sum_i g_i h_i$ for some $h_i \in \hat{A}$.

Lemma 2.4 is proved.

It is also necessary to consider certain polylinear forms of mixed derivatives. Assume that we have a decomposition $V = V_1 \oplus \ldots \oplus V_n$. Then we get the following induced direct-sum decomposition:

$$\mathbf{k}[V]_m = \bigoplus_{m_1 + \ldots + m_n = m} \mathbf{k}[V_1]_{m_1} \otimes \ldots \otimes \mathbf{k}[V_n]_{m_n}.$$

Further, for $f \in \mathbf{k}[V]_m$ with $m \leq n$ and a subset of indices $1 \leq i_1 < \ldots < i_m \leq n$, we denote the component of f in $\mathbf{k}[V_{i_1}]_1 \otimes \ldots \otimes \mathbf{k}[V_{i_m}]_1$ by $f^{(V_{i_1}, \ldots, V_{i_m})}$. In particular, if we apply this to the mth derivative of f at v_0, then we get a polylinear form

$$f_{v_0}^{(V_{i_1},\ldots,V_{i_m})} := (f_{v_0}^{(m)})^{(V_{i_1},\ldots,V_{i_m})} \in V_{i_1}^* \otimes \ldots \otimes V_{i_m}^*, \tag{2.1}$$

which is called the $(V_{i_1},\ldots,V_{i_m})$*-mixed derivative of* f *at* v_0.

Lemma 2.5. *In the situation of Lemma 2.4, assume, in addition, that* $V = V_1 \oplus \ldots \oplus V_n$. *Then, for any* $f \in \mathbf{k}[V]$ *and any collection of indices* $I = \{i_1 < \ldots < i_m\} \subset [1, n]$, *the polylinear form* $f_{v_0}^{(V_{i_1},\ldots,V_{i_m})}$ *belongs to the tensor ideal generated by* $(g_i)_{v_0}^{(V_{j_1},\ldots,V_{j_s})}$ *for* $i = 1, \ldots, c$ *and* $J = \{j_1 < \ldots < j_s\} \subset I,\ J \neq \varnothing$.

Proof. The proof easily follows from Lemma 2.4.

2.4. Dimension Count. We now change the notation to

$$P\colon U \times V \times W_1 \times \ldots \times W_{d-2} \to \mathbf{k}.$$

Further, we denote $W = W_1 \times \ldots \times W_{d-2}$ and consider the variety $Z_P \subset V \times W$ of all (v, w) such that $P(u, v, w) = 0$ for all $u \in U$.

Let Z be an irreducible component of the Zariski closure of the set of $\mathbf{k}$-points $Z_P(\mathbf{k})$ (with reduced scheme structure) such that $\operatorname{codim}_{V\times W} Z = g'(P)$, and let $Z_W \subset W$ denote the closure of the image of Z under the projection $\pi_W\colon V \times W \to W$ (also with reduced scheme structure). Then the $\mathbf{k}$-points are dense in Z_W.

We can treat P as a linear map from $U \otimes V$ to the space of polynomial functions on W. Hence, it gives a morphism of trivial vector bundles over W,

$$P_W\colon V \otimes \mathcal{O}_W \to U^* \otimes \mathcal{O}_W, \tag{2.2}$$

and $\pi_W^{-1}(w) \cap Z_P$ for $w \in Z_W$ can be identified with $\ker(P_W(w))$.

Let $\mathcal{U} \subset Z_W$ denote a nonempty open subset, where P_W has the maximal rank denoted by r. Then, over $\mathcal{U}$, the cokernel of P_W is locally free over Z_W and, hence, the kernel of P_W is a subbundle $\mathcal{K} \subset V \otimes \mathcal{O}$. We denote by $\operatorname{tot}_{\mathcal{U}}(\mathcal{K})$ the total space of the bundle $\mathcal{K}$ over $\mathcal{U}$ and obtain

$$\operatorname{tot}_{\mathcal{U}}(\mathcal{K}) = \pi_W^{-1}(\mathcal{U}) \cap Z_P \subset V \times W.$$

Note that the $\mathbf{k}$-points are dense in

$$\operatorname{tot}_{\mathcal{U}}(\mathcal{K}) = \pi_W^{-1}(\mathcal{U}) \cap Z_P$$

and, hence, $\pi_W^{-1}(\mathcal{U}) \cap Z$ is an irreducible component of $\pi_W^{-1}(\mathcal{U}) \cap Z_P$. Since $\operatorname{tot}_{\mathcal{U}}(\mathcal{K})$ is irreducible, we get

$$\pi_W^{-1}(\mathcal{U}) \cap Z = \operatorname{tot}_{\mathcal{U}}(\mathcal{K}).$$

Hence, we have

$$\dim Z = \dim Z_W + \dim V - r$$

or, equivalently,

$$\operatorname{codim}_W Z_W + r = \operatorname{codim}_{V\times W} Z = g'(P). \tag{2.3}$$

2.5. Proof of Theorem 1.1. *Step 1. Choosing a General* $\mathbf{k}$*-Point.* Shrinking the open subset $\mathcal{U} \subset Z_W$ considered above, we can assume that $\mathcal{U}$ is smooth. Since $\mathbf{k}$-points are dense in Z_W we can choose a $\mathbf{k}$-point

$$w^0 = (w_1^0, \ldots, w_{d-2}^0) \in \mathcal{U} \subset Z_W.$$

We set

$$S_V := \ker(P_W(w^0) \colon V \to U^*) \qquad \text{and} \qquad S_U := \ker(P_W(w^0)^* \colon U \to V^*).$$

Step 2. The First Set of Key Tensors. Let

$$c := \operatorname{codim}_W Z_W.$$

Since w^0 is a smooth point of Z_W, we can choose c elements $g_1, \ldots, g_c$ in the ideal $I_{Z_W} \subset \mathbf{k}[W]$ with linearly independent derivatives at w^0. Recall that $W = W_1 \times \ldots \times W_{d-2}$. Thus, for each $a = 1, \ldots, c$ and any nonempty subset of indices $I = \{i_1 < \ldots < i_m\} \subset [1, d-2]$, we can consider the polylinear forms, obtained as mixed derivatives at w^0 :

$$g_{a,I} := g_{a,w^0}^{(W_{i_1},\ldots,W_{i_m})} \in W_{i_1}^* \otimes \ldots \otimes W_{i_m}^*.$$

Step 3. Setting Up the Key Identity. We set $k = \dim V - r$. Applying Lemma 2.3(ii) to the morphism of trivial vector bundles (2.2) over Z_W, we find global sections $v_1(w), \ldots, v_k(w) \in V \otimes \mathbf{k}[Z_W]$ such that $v_1(w^0), \ldots, v_k(w^0)$ form a basis of S_V and, moreover,

$$P(u, v_i(w), w) = 0 \quad \text{for any } u \in U \quad \text{and} \quad w \in Z_W, \quad i = 1, \ldots, k.$$

Since $\mathbf{k}[W] \to \mathbf{k}[Z_W]$ is surjective, we can lift $v_i(w)$ to polynomials in $V \otimes \mathbf{k}[W]$, which are denoted in the same way. We now define a collection of U^*-valued polynomials on W as

$$f_i(w) := P(u, v_i(w), w) \in U^* \otimes \mathbf{k}[W]. \tag{2.4}$$

By construction, all $f_i(w)$ belong to $U^* \otimes I_{Z_W} \subset U^* \otimes \mathbf{k}[W]$. Equation (2.4) is the key identity used in the present work.

Step 4. The Second Set of Key Tensors. We consider certain mixed derivatives of $v_i(w)$ viewed as V-valued polynomials on W. Namely, for each

$$I = \{i_1 < \ldots < i_p\} \subset [1, d-2],$$

we set

$$v_{i,I} := v_{i,w^0}^{(W_{i_1},\ldots,W_{i_p})} \in W_I^* \otimes V = \operatorname{Hom}(W_I, V),$$

where

$$W_I := W_{i_1} \otimes \ldots \otimes W_{i_p}.$$

Since $(v_i(w^0))$ form a basis in S_V, there exists a unique operator

$$C_I \colon S_V \to \operatorname{Hom}(W_I, V) \colon v_i(w^0) \mapsto v_{i,I}.$$

We extend C_I in any way to an operator $V \to \mathrm{Hom}(W_I, V)$, which is also denoted by C_I. Note that we can also treat C_I as a linear map

$$C_I\colon V \otimes W_I \to V.$$

For an ordered collection of disjoint subsets $I_1, \dots, I_p \subset [1, d-2]$, we consider a composition

$$C_{I_1} \dots C_{I_p}\colon V \otimes W_{I_1 \sqcup \dots \sqcup I_p} \xrightarrow{C_{I_p}} V \otimes W_{I_1 \sqcup \dots \sqcup I_{p-1}} \to \dots \to V \otimes W_{I_1} \xrightarrow{C_{I_1}} V.$$

The case of an empty collection, i.e., $p = 0$, is allowed. In this case, we just get the identity map $V \to V$.

We choose a basis $\ell_1, \dots, \ell_r \in V^*$ in the subspace orthogonal to S_V. For ordered collections $I_1 \sqcup \dots \sqcup I_p \subset [1, d-2]$ and for $j = 1, \dots, r$, we consider the polylinear forms

$$\ell_j \circ C_{I_1} \dots C_{I_p} \in V^* \otimes W^*_{I_1 \sqcup \dots \sqcup I_p}.$$

Note that, for an empty collection, i.e., for $p = 0$, we just get $\ell_j \in V^*$.

Step 5. Differentiating the Key Identity. For each

$$I = \{i_1 < \dots < i_p\} \subset [1, d-2],$$

we consider the embedding

$$\iota(I)\colon W_I \to W_1 \otimes \dots \otimes W_{d-2},$$

which completes $w_{i_1} \otimes \dots \otimes w_{i_p}$ by the components w_j^0 in the factors W_j with $j \notin I$.

By induction on $p = 0, \dots, d-2$, we prove that, for any $I = \{i_1 < \dots < i_p\} \subset [1, d-2]$, the following inclusion is true:

$$P|_{S_U \otimes V \otimes \iota(I)(W_I)} \in \Big((\ell_j \circ C_{I_1} \dots C_{I_s} \mid I_1 \sqcup \dots \sqcup I_s \subsetneq I,\ \ 1 \le j \le r,\ \ s \ge 0), \\ (g_{a,I'} \mid 1 \le a \le c,\ \ I' \subset I,\ \ I' \ne \varnothing) \Big),$$

where, on the right-hand side, we have the tensor ideal generated by the specified elements. Note that all the subsets I_t are supposed to be nonempty.

The base of induction $p = 0$ is clear because $P(u, v, w_1^0, \dots, w_{d-2}^0) = 0$ for any $u \in S_U$ and $v \in V$. Assume that $p > 0$ and the assertion holds for $p - 1$. We fix a subset $I_0 = \{i_1 < \dots < i_p\} \subset [1, d-2]$.

Further, we equate the $(W_{i_1}, \dots, W_{i_p})$-mixed derivatives at w^0 of both sides of the key identity (2.4). As a result, we get the following equality in $U^* \otimes W^*_{I_0}$:

$$(f_i)_{w^0}^{(W_{i_1}, \dots, W_{i_p})} = P|_{U \otimes v_i(w^0) \otimes \iota(I_0) W_{I_0}} + \sum_{I \sqcup J = I_0, I \ne \varnothing} P|_{U \otimes C_I v_i(w^0) \otimes \iota(J) W_J}. \tag{2.5}$$

Note that, by Lemma 2.5, $(f_i)_{w^0}^{(W_1, \dots, W_p)}$ belong to the tensor ideal generated by $g_{a,I'}$ with $1 \le a \le c$ and $I' \subset I_0$, $I' \ne \varnothing$. We also note that the term in the sum on the right-hand side of (2.5) corresponding to $J = \varnothing$

has zero restriction to S_U. Hence, we get

$$P|_{S_U\otimes S_V\otimes\iota(I_0)W_{I_0}} + \sum_{I\sqcup J=I_0;I,J\neq\varnothing} P|_{(\mathrm{id}_U\otimes C_I)(U\otimes S_V)\otimes\iota(J)W_J} \in (g_{a,I'} \mid 1\le a\le c,\ \ I'\subset I,\ \ I'\neq\varnothing).$$

The induction assumption now implies that $P|_{S_U\otimes S_V\otimes\iota(I_0)W_{I_0}}$ belongs to the tensor ideal generated by $g_{a,I'}$ with $I'\subset I_0$, $I'\neq\varnothing$, and by the restrictions of $\ell_j\circ C_{I_1}\dots C_{I_s}$ with $s\ge 1$ (where $I_1\sqcup\dots\sqcup I_s$ is a proper subset of I_0). By Lemma 2.2, adding (ℓ_j) to the generators of the tensor ideal, we get the required assertion about $P|_{S_U\otimes V\otimes\iota(I_0)W_{I_0}}$.

Step 6. Conclusion of the Proof for a Single Tensor. By using the result of the previous step for $p=d-2$, we now get

$$\mathrm{rk}^S P|_{S_U\otimes V\otimes W_1\otimes\dots\otimes W_{d-2}} \le r(1+\theta_{d-2})+c(2^{d-2}-1),$$

where θ_n is the number of ordered collections of disjoint nonempty subsets $I_1\sqcup\dots\sqcup I_p\subsetneq[1,n]$ (with $p\ge 1$). By Lemma 2.2, this implies that

$$\mathrm{rk}^S P\le r+r(1+\theta_{d-2})+c(2^{d-2}-1).$$

Further, recall that $r+c=g'(P)$ [see (2.3)]. Hence, we get

$$\mathrm{rk}^S P\le (r+c)\max\left(2+\theta_{d-2},2^{d-2}-1\right)=g'(P)C_d,$$

as claimed.

Step 7. The Case of Several Tensors. Now assume that $\mathbf{k}$ is algebraically closed. Suppose that we have a given collection $\overline{P}=(P_1,\dots,P_s)$ of polylinear forms on $V_1\times\dots\times V_d$. For a nonzero collection of coefficients $\overline{c}=(c_1,\dots,c_s)$ in $\mathbf{k}$, we set

$$P_{\overline{c}}=c_1P_1+\dots+c_sP_s.$$

The key observation is that

$$Z_{\overline{P}}=\bigcup_{\overline{c}\neq 0} Z_{P_{\overline{c}}},$$

where we can consider $\overline{c}$ as points in the projective space $\mathbb{P}^{s-1}$. As already proved, for each $\overline{c}$,

$$\mathrm{codim}_{V_2\times\dots\times V_d} Z_{P_{\overline{c}}}\ge C_d^{-1}\,\mathrm{rk}^S(P_{\overline{c}})\ge C_d^{-1}\,\mathrm{rk}^S(\overline{P}).$$

By taking the union over $\overline{c}$ in $\mathbb{P}^{s-1}$, we get

$$\mathrm{codim}_{V_2\times\dots\times V_d} Z_{\overline{P}}\ge C_d^{-1}\,\mathrm{rk}^S(\overline{P})-s+1,$$

as claimed.

3. Symmetric Case

3.1. More on Higher Derivatives. Let $f \in \mathbf{k}[V]_d$. Treating the nth derivative of $f \in \mathbf{k}[V]$ (where $n \leq d$) as a polynomial map of degree $d-n$:

$$V \to \mathbf{k}[V]_n : v_0 \mapsto f_{v_0}^{(n)}$$

we can write it in the form of a tensor

$$f^{(n,d-n)} \in \mathbf{k}[V]_n \otimes \mathbf{k}[V]_{d-n}.$$

By definition,

$$f(v_1+v_2) = \sum_{n=0}^{d} f^{(n,d-n)}(v_1, v_2).$$

Hence, $f^{(n,d-n)}$ is just the component of $f(v_1+v_2)$ of bidegree $(n, d-n)$ in (v_1, v_2).

Similarly, we define an operation for $n_1 + \ldots + n_p = d$,

$$\mathbf{k}[V]_d \to \mathbf{k}[V]_{n_1} \otimes \ldots \otimes \mathbf{k}[V]_{n_p} : f \mapsto f^{(n_1,\ldots,n_p)},$$

by assuming that $f^{(n_1,\ldots,n_p)}$ is the component of multidegree $(n_1, \ldots, n_p)$ in $f(v_1 + \ldots + v_p)$. Thus,

$$f^{(1,1,d-2)} \in V^* \otimes V^* \otimes \mathbf{k}[V]_{d-2}$$

is exactly H_f, i.e., the Hessian symmetric form on V (with polynomial dependence on $x \in V$).

We use two properties of this construction that can be easily checked:

(i) $f^{(n_1,\ldots,n_p)}(x, \ldots, x) = \dfrac{d!}{n_1! \ldots n_p!} f(x)$;

(ii) for $m \leq n_i$, the mth derivative of $f^{(n_1,\ldots,n_p)}(x_1, \ldots, x_p)$ with respect to x_i at $(x_1^0, \ldots, x_p^0)$ is equal to

$$f^{(n_1,\ldots,n_{i-1},m,n_i-m,\ldots,n_p)}\big(x_1^0, \ldots, x_{i-1}^0, v, x_i^0, \ldots, x_p^0\big).$$

3.2. Proof of Theorem 1.2. It is convenient to denote by X one copy of V in the product $V \times V = V \times X$. In addition, we treat $H_f = f^{(1,1,d-2)}$ as a bilinear form on $U \times V$, where $U = V$, so that Z^{sym} consists of pairs $(v, x) \in V \times X$ such that $f^{(1,1,d-2)}(u, v, x) = 0$ for all $u \in U$.

Step 1. Dimension Count and Choosing a General $\mathbf{k}$*-Point*. Let Z be an irreducible component of the Zariski closure of the set of $\mathbf{k}$-points $Z_f^{\mathrm{sym}}(\mathbf{k})$ such that $\operatorname{codim}_{V\times X} Z = g'_{\mathrm{sym}}(f)$ and let $Z_X \subset X$ denote the closure of the image of Z under the projection $p_2 : V \times X \to X$. As earlier, we choose a nonempty smooth open subset $\mathcal{U} \subset Z_X$ over which H_f has the maximal $\operatorname{rank} r$ so that $p_2^{-1}(\mathcal{U}) \cap Z$ is a vector bundle of $\operatorname{rank} \dim V - r$ over $\mathcal{U}$. In particular,

$$\operatorname{codim}_X Z_X + r = g'_{\mathrm{sym}}(f).$$

We choose a $\mathbf{k}$-point x^0 in $\mathcal{U} \subset Z_X$ and set

$$S := \ker(H_f(x^0)) \subset V.$$

Step 2. The First Set of Key Polynomials. We set

$$c := \operatorname{codim}_X Z_X.$$

Since x^0 is a smooth point of Z_X, we can choose c elements $g_1, \ldots, g_c$ in the ideal $I_{Z_X} \subset \mathbf{k}[X]$ with linearly independent derivatives at x^0. Thus, for each $a = 1, \ldots, c$ and for $1 \le i \le d-2$, we consider the derivatives

$$(g_a)^{(i)}_{x^0} \in \mathbf{k}[X]_i.$$

Step 3. Setting up the Key Identity. We set $k = \dim V - r$. Applying Lemma 2.3(ii) to the morphism of trivial vector bundles $V \otimes \mathcal{O} \to V^* \otimes \mathcal{O}$ given by $H_f = f^{(1,1,d-2)}$ over Z_X, we find global sections $v_1(x), \ldots, v_k(x) \in V \otimes \mathbf{k}[Z_X]$, such that $v_1(x^0), \ldots, v_k(x^0)$ form a basis in S and

$$f^{(1,1,d-2)}(u, v_i(x), x) = 0 \quad \text{for any } u \in U \quad \text{and} \quad x \in Z_X, \quad i = 1, \ldots, k.$$

We lift $v_i(x)$ up to polynomials in $V \otimes \mathbf{k}[X]$, which are denoted in the same way. Further, we define the following collection of U^*-valued polynomials on X:

$$f_i(x) := f^{(1,1,d-2)}(u, v_i(x), x) \in U^* \otimes \mathbf{k}[X]. \tag{3.1}$$

By construction, all $f_i(x)$ belong to $U^* \otimes I_{Z_X} \subset U^* \otimes \mathbf{k}[X]$.

Step 4. The Second Set of Key Forms. For each $1 \le m \le d-2$, we consider higher derivatives of v_i at x^0 regarded as V-valued polynomials on X:

$$(v_i)^{(m)}_{x^0} \in V \otimes \mathbf{k}[X]_m.$$

Since $(v_i(x^0))$ form a basis of S, there exists a linear operator

$$C_m \colon S \to V \otimes \mathbf{k}[X]_m \colon v_i(x^0) \mapsto (v_i)^{(m)}.$$

We extend C_m in any way to an operator $V \to V \otimes \mathbf{k}[X]_m$, which is also denoted by C_m. For $m_1 + \ldots + m_p \le d-2$, we consider a composition

$$C_{m_1} \ldots C_{m_p} \colon V \xrightarrow{C_{m_p}} V \otimes \mathbf{k}[X]_{m_p} \to \ldots \to V \otimes \mathbf{k}[X]_{m_2 + \ldots + m_p} \xrightarrow{C_{m_1}} V \otimes \mathbf{k}[X]_{m_1 + \ldots + m_p}.$$

We allow the case of empty collection, i.e., $p = 0$. In this case, we just get the identity map $V \to V$.

Finally, by $\ell_1, \ldots, \ell_r \in V^*$ we denote a basis in the subspace orthogonal to S. For $m_1 + \ldots + m_p \le d-2$ and for $j = 1, \ldots, r$, we consider the elements

$$\ell_j \circ C_{m_1} \ldots C_{m_p} \in V^* \otimes \mathbf{k}[X]_{m_1 + \ldots + m_p}.$$

Note that, for the empty collection, i.e., for $p = 0$, we just get $\ell_j \in V^*$.

Step 5. Differentiating the Key Identity. We proceed by induction on $p = 0, \dots, d-2$ and prove that

$$f^{(1,1,p,d-2-p)}(u,v,x,x^0)|_{S\times V\times X} \in \Big(\big((\ell_j \circ C_{m_1}\dots C_{m_s})(v,x) \mid m_1+\dots+m_s < p,\ \ 1\le j\le r\big), \big((g_a)^{(m)}_{x^0}(x) \mid 1\le a\le c,\ \ 1\le m\le p\big)\Big).$$

Here, on the right-hand side, we have the ideal generated by the specified elements.

The base of induction $p = 0$ is clear, since $f^{(1,1,d-2)}(u,v,x^0) = 0$ for any $u \in S$ and $v \in V$. Assume that $p > 0$ and the assertion holds for $p-1$. We now equate the pth derivatives at $x = x^0$ on both sides of (3.1). As a result, we get the following equality in $U^* \otimes \mathbf{k}[X]_p$:

$$(f_i)^{(p)}_{x^0}(x) = f^{(1,1,p,d-2-p)}(u,v_i(x^0),x,x^0) + \sum_{q=1}^{p} f^{(1,1,p-q,d-2-p+q)}(u, C_q(v_i(x^0),x),x,x^0).$$

The left-hand side belongs to the ideal generated by $(g_a)^{(m)}_{x^0}(x)$ with $1 \le a \le c$ and $1 \le m \le p$. We also note that the term corresponding to $q = p$ on the right-hand side has zero restriction to $u \in S$. Hence, we get

$$f^{(1,1,p,d-2-p)}(u,v,x,x^0)|_{S\times S\times X} + \sum_{q=1}^{p-1} f^{(1,1,p-q,d-2-p+q)}\big(u, C_q(v,x),x,x^0\big)\,\big|_{S\times S\times X} \in \big((g_a)^{(m)}_{x^0}(x) \,\big|\, 1\le a\le c,\ \ 1\le m\le p\big).$$

The induction assumption now implies that $f^{(1,1,p,d-2-p)}(u,v,x,x^0)|_{S\times S\times X}$ belongs to the ideal generated by $(g_a)^{(m)}_{x^0}(x)$ for $1 \le a \le c$ and $1 \le m \le p$ and by the restrictions to $S \times X$ of $(\ell_j \circ C_{m_1}\dots C_{m_s})(v,x)$ with $s \ge 1$, $m_1+\dots+m_s < p$, and $1 \le j \le r$. By Lemma 2.2, adding (ℓ_j) to the generators of the ideal, we get the required assertion for $f^{(1,1,p,d-2-p)}(u,v,x,x^0)|_{S\times V\times X}$.

Step 6. Conclusion of the Proof for a Single Polynomial. By using the result obtained in the previous step for $p = d-2$, we get

$$f^{(1,1,d-2)}(u,v,x)|_{S\times V\times X} \in \Big(\big((\ell_j \circ C_{m_1}\dots C_{m_s})(v,x) \,\big|\, m_1+\dots+m_s < d-2,\ \ 1\le j\le r\big), \big((g_a)^{(m)}_{x^0}(x) \,\big|\, 1\le a\le c,\ \ 1\le m\le d-2\big)\Big).$$

Hence,

$$f^{(1,1,d-2)}(u,v,x) \in \big((\ell_j(u) \,\big|\, 1\le j\le r), \big((\ell_j \circ C_{m_1}\dots C_{m_s})(v,x) \,\big|\, m_1+\dots+m_s < d-2,\ \ 1\le j\le r\big), \big((g_a)^{(m)}_{x^0}(x) \,\big|\, 1\le a\le c,\ \ 1\le m\le d-2\big)\big). \tag{3.2}$$

Thus, setting $u = v = x$, we obtain

$$d(d-1) \cdot f(x) \in \Big(\big(F_{j;m_1,\dots,m_s}(x) \bigm| m_1 + \ldots + m_s < d-2,\ \ 1 \le j \le r \big), \\ \big((g_a)^{(m)}_{x^0}(x) \bigm| 1 \le a \le c,\ \ 1 \le m \le d-2 \big) \Big),$$

where $F_{j;m_1,\dots,m_s}(x) = (\ell_j \circ C_{m_1} \dots C_{m_s})(x,x)$ has the degree $1 + m_1 + \ldots + m_s < d - 1$. This yields

$$\mathrm{rk}^S(f) \le r\big(1 + \theta^{\mathrm{sym}}_{d-2}\big) + c(d-2),$$

where θ^{sym}_n is the number of $(m_1, \dots, m_s)$, with $s \ge 1,\ m_i \ge 1$, and $m_1 + \ldots + m_s < n$. It is easy to see that $\theta^{\mathrm{sym}}_n = 2^{n-1} - 1$. Since $r + c = g'_{\mathrm{sym}}(f)$, we get

$$\mathrm{rk}^S P \le (r+c)\max(2^{d-3}, d-2) = g'_{\mathrm{sym}}(f) \cdot 2^{d-3},$$

as claimed.

Step 7. The Case of Several Polynomials. We now assume that $\mathbf{k}$ is algebraically closed and we have a given collection $\overline{f} = (f_1, \dots, f_s)$ of homogeneous polynomials on V of degree d. For a nonzero collection of coefficients $\overline{c} = (c_1, \dots, c_s)$ in $\mathbf{k}$, we set $f_{\overline{c}} = c_1 f_1 + \ldots + c_s f_s$. As in the nonsymmetric case, the key observation is that

$$Z^{\mathrm{sym}}_{\overline{f}} = \bigcup_{\overline{c} \ne 0} Z^{\mathrm{sym}}_{f_{\overline{c}}}, \tag{3.3}$$

where $\overline{c}$ can be considered as points in the projective space $\mathbb{P}^{s-1}$. By using the case of a single polynomial, we deduce that

$$\mathrm{codim}_{V \times V} Z^{\mathrm{sym}}_{\overline{f}} \ge 2^{-d+3}\, \mathrm{rk}^S(\overline{f}) - s + 1,$$

as claimed.

3.3. Relation to Singularities. We now relate $g_{\mathrm{sym}}(f)$ to the codimension $c(f)$ of the singular locus of the hypersurface $f = 0$ in V.

Proposition 3.1.

(i) The subvariety $Z^{\mathrm{sym}}_f \subset V \times X = V \times V$ contains the singular locus of $f^{(2,d-2)}(v,x) = 0$.

(ii) $g_{\mathrm{sym}}(f) \le (d+1)c(f)$ (resp., $g_{\mathrm{sym}}(f) \le dc(f)$ if d is odd and $\mathrm{char}(\mathbf{k}) \ne 2$).

(iii) If $\mathrm{char}(\mathbf{k})$ does not divide $d-1$, then $c(f) \le g_{\mathrm{sym}}(f)$.

Proof. (i) The first derivative of $f^{(2,d-2)}(v,x)$ along v at (v^0, x^0) is $f^{(1,1,d-2)}(v, v^0, x^0)$. Hence, if (v^0, x^0) is a singular point of $f^{(2,d-2)}(v,x) = 0$, then

$$f^{(1,1,d-2)}(v, v^0, x^0) = 0$$

for all v, i.e., $(v^0, x^0) \in Z^{\mathrm{sym}}_f$.

(ii) Since we are comparing the dimensions of algebraic varieties, without loss of generality, we can assume that $\mathbf{k}$ is algebraically closed.

By part (i), we have $g_{\mathrm{sym}}(f) \le c(F)$, where $F = f^{(2,g-2)}$. It is easy to see that if $F(x) = F_1(x)+\ldots+F_r(x)$, then $c(F) \le c(F_1) + \ldots + c(F_r)$. Moreover, if $A\colon V \to W$ is a linear surjective map and $g \in \mathbf{k}[W]$, then $c(g \circ A) = c(g)$.

Thus, it remains to check that $f^{(2,d-2)}(v,x)$ is a linear combination of $d+1$ (resp., d, if d is odd and $\mathrm{char}(\mathbf{k}) \ne 2$) polynomials of the form $f(A_i(v,x))$ for some linear surjective maps $A_i\colon V \times V \to V$.

We consider $f(v+x)$ as an inhomogeneous function of v, $g(v) = g_0 + g_1 + \ldots + g_d$ of degree $\le d$ (with coefficients in $\mathbf{k}[V]$). Further, picking any $d+1$ distinct elements $\lambda_0, \ldots, \lambda_d \in \mathbf{k}$, we can express $g_0, \ldots, g_d$ as linear combinations of $g(\lambda_0 v), \ldots, g(\lambda_d v)$ (because the corresponding linear change is given by the Vandermonde matrix).

In the case where d is odd and $\mathrm{char}(\mathbf{k}) \ne 2$, we can similarly express the components of even degree, $(g_{2i})_{i \le (d-1)/2}$ as linear combinations of $g_0 = g(0)$ and $(g(\lambda_i v) + g(-\lambda_i v))/2$, for $1 \le i \le (d-1)/2$, where (λ_i) are nonzero constants such that (λ_i^2) are all distinct.

It remains to observe that $g_2 = f^{(2,d-2)}$ and that each $g(\lambda v) = f(\lambda v + x)$ is of the required type.

(iii) This follows from the relation

$$(d-1)f^{(1,d-1)}(v,x) = f^{(1,1,d-2)}(v,x,x).$$

Indeed, this implies that the intersection of Z_f^{sym} with the diagonal $V \subset V \times V$ is exactly the singular locus of $f = 0$, which gives the desired inequality.

Proposition 3.1 is proved.

We now consider the case of a collection $\overline{f} = (f_1, \ldots, f_s)$ of homogeneous polynomials of degree d on V. We consider the corresponding family of hypersurfaces in V, $f_{\overline{c}} = 0$, parametrized by the projective space $\mathbb{P}^{s-1}$. It is clear that, for the locus $S(\overline{f}) \subset V$, where the rank of the Jacobi matrix of $(f_1, \ldots, f_s)$ is $< s$, we have

$$S(\overline{f}) = \bigcup_{\overline{c} \ne 0} \mathrm{Sing}(f_{\overline{c}} = 0).$$

Proposition 3.2.

(i) The following inclusion is true:

$$\bigcup_{\overline{c} \ne 0} \mathrm{Sing}\left(f_{\overline{c}}^{(2,d-2)} = 0\right) \subset Z_{\overline{f}}^{\mathrm{sym}}.$$

(ii) $g_{\mathrm{sym}}(\overline{f}) \le (d+1)c'(\overline{f}) + d(s-1)$ *(resp.,* $g_{\mathrm{sym}}(\overline{f}) \le dc'(\overline{f}) + (d-1)(s-1)$ *if d is odd and* $\mathrm{char}(\mathbf{k}) \ne 2$*).*

(iii) Assume that $(f_1, \ldots, f_s)$ define a complete intersection $V(\overline{f}) \subset V$, i.e., $\mathrm{codim}_V V(\overline{f}) = s$. *Then*

$$c'(\overline{f}) \le c(\overline{f}) \le c'(\overline{f}) + s.$$

Assume, in addition, that $\mathrm{char}(\mathbf{k})$ *does not divide $d-1$. Then*

$$c'(\overline{f}) \le g_{\mathrm{sym}}(\overline{f}).$$

Proof. (i) This follows from Proposition 3.1(i) due to (3.3).

(ii) Since $S(\overline{f})$ has the codimension $c'(\overline{f})$ in V, we conclude that, for some $a \leq s-1$, there exists an a-dimensional subvariety $X \subset \mathbb{P}^{s-1}$ such that

$$c(f_{\overline{c}}) = \operatorname{codim}_V \operatorname{Sing}(f_{\overline{c}}) \leq c'(\overline{f}) + a \quad \text{for} \quad \overline{c} \in X.$$

Applying Proposition 3.1(ii), we see that, for each $\overline{c} \in X$,

$$\operatorname{codim}_{V\times V} Z^{\mathrm{sym}}_{f_{\overline{c}}} \leq (d+1)(c'(\overline{f}) + a)$$

(resp., $\leq d(c'(\overline{f}) + a)$ if d is odd). Hence, by using (3.3), we get

$$\operatorname{codim}_{V\times V} Z^{\mathrm{sym}}_{\overline{f}} \leq (d+1)(c'(\overline{f}) + a) - a$$

(resp., $\leq d(c'(\overline{f}) + a) - a$ if d is odd). Since $a \leq s-1$, this implies the required assertion.

(iii) If $(f_1, \ldots, f_s)$ specify a complete intersection, then, by the Jacobi criterion of smoothness, we obtain

$$\operatorname{Sing} V(\overline{f}) = S(\overline{f}) \cap V(\overline{f}).$$

In particular, we get an inclusion $\operatorname{Sing} V(\overline{f}) \subset S(\overline{f})$. Therefore,

$$c'(\overline{f}) = \operatorname{codim}_V S(\overline{f}) \leq c(\overline{f}).$$

Moreover, we get

$$c(\overline{f}) - s = \operatorname{codim}_{V(\overline{f})} \operatorname{Sing} V(\overline{f}) \leq \operatorname{codim}_V S(\overline{f}) = c'(\overline{f}).$$

If we assume in addition that $\operatorname{char}(\mathbf{k})$ does not divide $d-1$, then the intersection of $Z^{\mathrm{sym}}_{\overline{f}}$ with the diagonal $V \subset V \times V$ is exactly $S(\overline{f})$. Hence, we obtain

$$c'(\overline{f}) = \operatorname{codim}_V S(\overline{f}) \leq g_{\mathrm{sym}}(\overline{f}).$$

Proposition 3.2 is proved.

Proof of Theorem 1.3. (i) If

$$f(x) = \sum_{i=1}^{r} h_i(x) g_i(x),$$

then the locus $h_i(x) = g_i(x) = 0$ for $i = 1, \ldots, r$ is contained in the singular locus of $f(x) = 0$ and, hence, $c(f) \leq 2r$.

Now let, for the other inequality, $c = c_{\mathbf{k}}(f)$ and let X be an irreducible component of codimension c of the Zariski closure of the $\mathbf{k}$-points of $\operatorname{Sing}(f=0)$. Also let $v_0 \in X$ be a smooth $\mathbf{k}$-point and let $g_1, \ldots, g_c \in I(X)$ be defined over $\mathbf{k}$ with linearly independent differentials at v_0. For all $k \in [n]$, we have $\partial_k f \in I(X)$ and, hence, Lemma 2.4 implies that

$$\partial_k f = (\partial_k f)^{(d-1)}_{v_0} \in \left((g_i)^{(j)}_{v_0} \right)_{i\in[c], j\in[d-1]}.$$

By Euler's formula, we get

$$f = \frac{1}{d}\sum_{k=1}^{n} x_k \partial_k f \in \left((g_i)_{v_0}^{(j)}\right)_{i\in[c], j\in[d-1]}.$$

This gives $\mathrm{rk}^S(f) \le (d-1)\cdot c$.

(ii) We deduce the required assertion from the result for a single form as in the proof of Theorem 1.2.

Proof of Corollary 1.1. In the notation of [2, Theorem A] (recalling that the strength of f is $\mathrm{rk}^S(f) - 1$), the inequality of Theorem 1.3(i) implies that we can take

$$^m A(d) = (d-1)(m+2) - 1.$$

It is also well known that, for $d = 2$, we can take

$$^m A(2) = \left\lceil \frac{m+1}{2} \right\rceil$$

(see, e.g., [3, Proposition 4.10]). Thus, the assertion follows from [2, Theorem A(c)].

Remark 3.1. For $\mathbf{k}$ algebraically closed of *arbitrary* characteristic, Eq. (3.2) shows that

$$\mathrm{rk}^S f^{(1,1,d-2)}(u, v, x) \le (2^{d-3}+1)\cdot g_{\mathrm{sym}}(f) \le (2^{d-3}+1)\cdot(d+1)\cdot c(f).$$

The proof of Theorem 1.4 is based on the following geometric observation:

Lemma 3.1. *For generic $v_1, \dots, v_s \in V$, where $s < \dim V$, the following inequality is true:*

$$c'(\partial_{v_1} f, \dots, \partial_{v_s} f) \ge g_{\mathrm{sym}}(f) - s + 1.$$

Proof. By $Z^{(s)} \subset V^s \times V$ we denote a locally closed subvariety formed by $(v_1, \dots, v_s, x)$ such that $v_1, \dots, v_s$ are linearly independent and

$$\dim \mathrm{span}\left(H_f(x)(\cdot, v_1), \dots, H_f(x)(\cdot, v_s)\right) < s$$

(here, we consider $H_f(\cdot, v)$ as a linear form on V). Our aim is to estimate the dimension of $Z^{(s)}$. We get a surjective map (with at least 1-dimensional fibers) $\widetilde{Z}^{(s)} \to Z^{(s)}$, where $\widetilde{Z}^{(s)} \subset V \times V^s \times V$ is given by

$$\widetilde{Z}^{(s)} = \Big\{(v, v_1, \dots, v_s, x) \mid v \in \ker H_f(x),\ \ v \ne 0,\ \ (v_1, \dots, v_s) \text{ are linearly independent},\ \ v \in \mathrm{span}(v_1, \dots, v_s)\Big\}.$$

We have a natural projection

$$\widetilde{Z}^{(s)} \to Z_f^{\mathrm{sym}}\colon (v, v_1, \dots, v_s, x) \mapsto (v, x),$$

which is a locally trivial fibration whose fibers are irreducible of dimension $n(s-1)+s$, where $n = \dim V$.

This yields

$$\dim Z^{(s)} \le \dim \widetilde{Z}^{(s)} - 1 \le \dim Z_f^{\mathrm{sym}} + n(s-1) + s - 1.$$

Hence,

$$\operatorname{codim}_{V^s \times V} Z^{(s)} \ge g_{\mathrm{sym}}(f) - s + 1.$$

Further, we observe that

$$H_f(x)(\cdot, v) = f^{(1,1,d-2)}(\cdot, v, x) = (\partial_v f)^{(1,d-2)}(\cdot, x).$$

Thus, $S(\partial_{v_1} f, \dots, \partial_{v_s} f)$ is exactly the fiber over $(v_1, \dots, v_s)$ of the projection $Z^{(s)} \to V^s$. For generic $v_1, \dots, v_s$, only the components of $Z^{(s)}$ predominant over V^s play a role, and we conclude that

$$c'(\partial_{v_1} f, \dots, \partial_{v_s} f) = \operatorname{codim}_V S(\partial_{v_1} f, \dots, \partial_{v_s} f) \ge \operatorname{codim}_{V^s \times V} Z^{(s)} \ge g_{\mathrm{sym}}(f) - s + 1.$$

Lemma 3.1 is proved.

Proof of Theorem 1.4. (i) By Lemma 3.1 with $s = 1$, we have $c(\partial_v f) \ge g_{\mathrm{sym}}(f)$. Hence, by Theorems 1.3(i) and 1.2,

$$\mathrm{rk}^S(\partial_v f) \ge \frac{1}{2} c(\partial_v f) \ge \frac{1}{2} g_{\mathrm{sym}}(f) \ge 2^{2-d} \,\mathrm{rk}^S(f).$$

(ii) If $c'(\partial_{v_1} f, \dots, \partial_{v_s} f) \ge s$ (resp., $c'(\partial_{v_1} f, \dots, \partial_{v_s} f) \ge s + 2$), then $(\partial_{v_1} f, \dots, \partial_{v_s} f)$ define a (resp., normal) complete intersection of codimension s. Hence, the required assertion follows from Theorem 1.2 and Lemma 3.1.

3.4. Singularities of the Polar Map. Let $f \in \mathbf{k}[V]_d$. Note that $H_f(x)$ can be identified with the map tangent to the *polar map* $\phi_f \colon V \to V^*$ of f sending x to $f_x^{(1)} = df|_x$. Thus, $g_{\mathrm{sym}}(f)$ measures the degeneracy of this map.

More precisely, for any morphism $\phi \colon X \to Y$ between smooth connected varieties, we define the *Thom–Boardman rank*[5] of ϕ denoted by $\mathrm{rk}^{TB}(\phi)$ as follows: Consider a subvariety Z_ϕ in the tangent bundle TX of X consisting of (x, v) such that $d\phi_x(v) = 0$. Then we set

$$\mathrm{rk}^{TB}(\phi) = \operatorname{codim}_{TX} Z_\phi.$$

Note that $\mathrm{rk}^{TB}(\phi) \le r$, where r is the generic rank of the differential of ϕ. However, the inequality can be strict.

By definition,

$$g_{\mathrm{sym}}(f) = \mathrm{rk}^{TB}(\phi_f).$$

It is well known that the generic rank of $d\phi_f = H_f$ is related to the dimension of the projective dual variety X^* of the projective hypersurface associated with f (more precisely, $\dim X^* + 2$ is the generic rank of H_f over the hypersurface $f = 0$). However, it is easy to see that $g^{\mathrm{sym}}(f)$ can be much smaller than the generic rank of ϕ_f.

[5] This name is explained by the relationship with the Thom–Boardman stratification in singularity theory; see [6].

Thus, if $q_1(x)$ and $q_2(y)$ are nondegenerate quadratic forms in two different groups of variables $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, then $\mathrm{rk}^S(q_1(x)q_2(y)) = 1$ and, therefore,

$$g_{\mathrm{sym}}(q_1(x)q_2(y)) \leq 4.$$

On the other hand, the generic rank of $\phi_{q_1(x)q_2(y)}$ is $2n$ (under the assumption that the characteristic of $\mathbf{k}$ is $\neq 2, 3$).

Example 3.1. In the case where $d = 3$, the Schmidt rank of f is equal to its slice $\mathrm{rank}\, s(f)$, i.e., the minimal s such that there exists a linear subspace $L \subset V$ of codimension s contained in $(f = 0)$. Thus, for a cubic form f, we assume that $\mathbf{k}$ is algebraically closed of characteristic $\neq 2, 3$. In this case, it follows from Theorem 1.2 and from (1.2) that

$$s(f) \leq \mathrm{rk}^{TB}(\phi_f) \leq 4s(f).$$

If f is a general homogeneous polynomial of degree d, then we still have $\mathrm{rk}^S(f) = s(f)$ (see [4]). Hence, for this f, under the assumption that $\mathbf{k}$ is algebraically closed of characteristic not dividing $(d-1)d$, we obtain

$$2^{3-d}s(f) \leq \mathrm{rk}^{TB}(\phi_f) \leq 4s(f).$$

It seems that the invariant $\mathrm{rk}^{TB}(\phi)$ deserves to be studied more comprehensively. Indeed, we do not know whether it is always true that $\mathrm{rk}^{TB}(\phi) = \dim X$ for a *finite* morphism ϕ between smooth projective varieties of characteristic zero. We now present the following corollary of Proposition 3.1(iii):

Corollary 3.1. *Assume that* $\mathrm{char}(\mathbf{k})$ *does not divide* $d-1$. *Then*

$$\mathrm{rk}^{TB}(\phi_f) \geq c(f).$$

In particular, if the projective hypersurface associated with f *is smooth, then*

$$\mathrm{rk}^{TB}(\phi_f) = \dim V.$$

Let $V_f \subset \mathbb{P}V$ denote the projective hypersurface associated with f. In [7], the authors considered (for $\mathbf{k} = \mathrm{C}$) the closed locus $S_{\geq r} \subset V_f$, where the co-rank of the Hessian H_f is $\geq r$. They proved that if V_f is smooth, then, for $r(r+1) \leq \dim V$, the subvariety $S_{\geq r}(V)$ is nonempty and

$$\mathrm{codim}_{V_f} S_{\geq r}(V) \leq r(r+1)/2.$$

By using Corollary 3.1, we get the inequality

$$\mathrm{codim}_{V_f} S_{\geq r}(V) \geq r - 1.$$

If V_f is smooth, then the projectivization of the restriction of ϕ_f to $(f = 0)$ can be identified with the *Gauss map*

$$\gamma\colon V_f \to \mathbb{P}V^*.$$

It is easy to check that if $\mathrm{char}(k)$ does not divide $d(d-1)$, then, for any point $x \in (f=0) \subset V$, we have

$$\ker(d(\phi_f)_x) \subset T_x(f=0),$$

and the natural projection

$$\ker(d(\phi_f)_x) \to \ker(d\gamma_x)$$

is an isomorphism. Thus, the above inequalities can be treated as restrictions to possible degeneracies of the Gauss map of V_f (which is finite by a result of Zak from [12]).

Appendix A

This appendix gives alternative versions of Theorems 1.1, 1.2, and 1.4, with better bounds for $d \geq 6$. The second version of Theorem 1.1 is as follows:

Theorem A.1.

(i) Let $g'(P)$ denote the codimension in $V_2 \times \ldots \times V_d$ of the Zariski closure of $Z_P(\mathbf{k})$. Then

$$\mathrm{rk}^S(P) \leq (2^{d-1}-1)g'(P).$$

(ii) Assume that $\mathbf{k}$ is algebraically closed. Then, for a collection $\overline{P} = (P_1, \ldots, P_s)$, the following inequality is true:

$$\mathrm{rk}^S(\overline{P}) \leq (2^{d-1}-1)(g(\overline{P}) + s - 1).$$

For algebraically closed fields, the result presented above matches the result obtained by Cohen and Moshkowitz in [13] but we give a very short proof.

Proof. (i) The proof mimics the proof of Theorem 1.3. We write $g = g'(P)$ and assume that X is an irreducible component of the Zariski closure of $Z_P(\mathbf{k})$ such that

$$\mathrm{codim}_{V_1 \times V_2 \times \ldots \times V_{d-1}} X = g.$$

Let $x_1, \ldots, x_n$ be a basis for V_d^*. We write

$$P = \sum_{k=1}^{n} x_k \cdot Q_k,$$

where $Q_k\colon V_1 \times V_2 \times \ldots \times V_{d-1} \to \mathbf{k}$ are polylinear forms. Let $v_0 \in X$ be a smooth $\mathbf{k}$-point and let $h_1, \ldots, h_g \in I(X)$ be defined over $\mathbf{k}$ with linearly independent differentials at v_0. For all $k \in [n]$, we have

$$Q_k = (Q_k)_{v_0}^{(V_1, V_2, \ldots, V_{d-1})} \in I(X)$$

and, therefore, by Lemma 2.5, this is in the tensor ideal generated by

$$\left((h_i)_{v_0}^{(V_{j_1}, \ldots, V_{j_s})} \right)_{i \in [g], \varnothing \neq \{j_1 < \ldots < j_s\} \subset [d-1]}.$$

By definition, P is in the tensor ideal generated by Q_k and, hence,

$$\mathrm{rk}^S(P) \leq (2^{d-1} - 1) \cdot g.$$

(ii) We deduce this assertion from the result obtained for a single tensor as in the proof of Theorem 1.1.

The second version of Theorem 1.2 is as follows:

Theorem A.2. *Assume that* $\mathbf{k}$ *is algebraically closed of characteristic not dividing* $(d-1)d$.

(i) For a single form f *of degree* d,

$$\mathrm{rk}^S(f) \leq (d-1) g_{\mathrm{sym}}(f).$$

(ii) If $(f_1, \ldots, f_s)$ *specify a complete intersection of codimension* s *in* V, *then*

$$\mathrm{rk}^S(\overline{f}) \leq (d-1)\big(g_{\mathrm{sym}}(\overline{f}) + s - 1\big).$$

Proof. (i) Combine Theorem 1.3(i) with Proposition 3.1(iii).

(ii) Combine Theorem 1.3(ii) with Proposition 3.2(iii).

The second version of Theorem 1.4 is as follows:.

Theorem A.3. *Let* f *be a homogeneous polynomial of degree* d. *Assume that* $\mathbf{k}$ *is algebraically closed of characteristic not dividing* $(d-1)d$.

(i) For generic $v \in V$,

$$\mathrm{rk}^S(\partial_v f) \geq \frac{1}{2d-2} \mathrm{rk}^S(f).$$

(ii) For $s \leq \frac{1}{2d-2} \mathrm{rk}^S(f) + \frac{1}{2}$ $\left(\text{resp., } s \leq \frac{1}{2d-2} \mathrm{rk}^S(f) - \frac{1}{2}\right)$ *and for generic* $v_1, \ldots, v_s \in V$, *the derivatives* $(\partial_{v_1} f, \ldots, \partial_{v_s} f)$ *define a (resp., normal) complete intersection of codimension* s *in* V.

Proof. (i) By Lemma 3.1 with $s = 1$, we get $c(\partial_v f) \geq g_{\mathrm{sym}}(f)$. Hence, by Theorems 1.3(i) and A.2,

$$\mathrm{rk}^S(\partial_v f) \geq \frac{1}{2} c(\partial_v f) \geq \frac{1}{2} g_{\mathrm{sym}}(f) \geq \frac{1}{2d-2} \mathrm{rk}^S(f).$$

(ii) If $c'(\partial_{v_1} f, \ldots, \partial_{v_s} f) \geq s$ (resp., $c'(\partial_{v_1} f, \ldots, \partial_{v_s} f) \geq s + 2$), then $(\partial_{v_1} f, \ldots, \partial_{v_s} f)$ define a (resp., normal) complete intersection of codimension s. Hence, the required assertion follows from Theorem A.2 and Lemma 3.1.

A. Lampert was supported by the National Science Foundation under Grant No. DMS-1926686 and by the Israel Science Foundation under Grant No. 2112/20.

A. Polishchuk was partially supported by the NSF grant DMS-2001224, within the framework of the HSE University Basic Research Program, and by the Russian Academic Excellence Project “5-100”.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

REFERENCES

1. K. Adiprasito, D. Kazhdan, and T. Ziegler, *On the Schmidt and Analytic Ranks for Trilinear Forms*; arXiv:2102.03659.
2. T. Ananyan and M. Hochster, "Small subalgebras of polynomial rings and Stillman's conjecture," *J. Amer. Math. Soc.*, **33**, No. 1, 291–309 (2020).
3. T. Ananyan and M. Hochster, "Strength conditions, small subalgebras, and Stillman bounds in degree ≤ 4," *Trans. Amer. Math. Soc.*, **373**, No. 7, 4757–4806 (2020).
4. E. Ballico, A. Bik, A. Oneto, and E. Ventura, *Strength and Slice Rank of Forms are Generically Equal*; arXiv:2102.11549.
5. E. Ballico, A. Bik, A. Oneto, and E. Ventura, *The Set of Forms with Bounded Strength is Not Closed*; arXiv:2012.01237.
6. J. M. Boardman, "Singularities of differentiable maps," *Publ. Math. Inst. Hautes Édutes Sci.*, **33**, 383–419 (1967).
7. A. D. R. Choudary and A. Dimca, "On the dual and Hessian mappings of projective hypersurfaces," *Math. Proc. Cambridge Philos. Soc.*, **101**, 461–468 (1987).
8. H. Derksen, *The G-Stable Rank for Tensors*; arXiv:2002.08435.
9. D. Kazhdan and T. Ziegler, *On the Codimension of the Singular Locus*; arXiv:1907.11750.
10. S. Kopparty, G. Moshkovitz, and J. Zuiddam, *Geometric Rank of Tensors and Subrank of Matrix Multiplication*; arXiv:2002.09472.
11. W. M. Schmidt, "The density of integer points on homogeneous varieties," *Acta Math.*, **154**, No. 3-4, 243–296 (1985).
12. F. Zak, "Structure of Gauss maps," *Funct. Anal. Appl.*, **21**, 32–41 (1987).
13. A. Cohen and G. Moshkovitz, *Partition and Analytic Rank are Equivalent over Large Fields*; arXiv:2102.10509.