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Data-Driven Control of Positive Linear Systems using Linear Programming

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Abstract— **This paper presents a linear-programming based algorithm to perform data-driven stabilizing control of linear positive systems. A set of state-input-transition observations is collected up to magnitude-bounded noise. A state feedback controller and dual linear copositive Lyapunov function are created such that the set of all data-consistent plants is contained within the set of all stabilized systems. This containment is certified through the use of the Extended Farkas Lemma and solved via Linear Programming. Sign patterns and sparsity structure for the controller may be imposed using linear constraints. The complexity of this algorithm scales in a polynomial manner with the number of states and inputs. Effectiveness is demonstrated on example systems.**

I. Introduction

This paper performs Data-Driven Control (DDC) of Positive Linear Time Invariant (LTI) Continuous-Time Systems (CTSs) and Discrete-Time Systems (DTSs) by finding full-state-feedback stabilizing controllers. These controllers, which stabilize all possible plants that are consistent with observed data, are formulated as the solution to a Linear Program (LP).

Positive systems are a class of dynamical systems whose state and output responses to positive (nonnegative) initial conditions and inputs remain positive (nonnegative) for all time [1], [2], [3], [4]. Instances of positive systems include population models [5], chemical networks [6], radio communications [7], queuing [8], and Markov chains [9]. Full-state-feedback stabilization of known LTI positive systems can be accomplished by solving an LP to find control (dual) linear copositive Lyapunov functions [10]. Alternatively, one can perform stabilization by formulating a Semidefinite Program (SDP) to find a quadratic Lyapunov function [11], [12].

The peak-to-peak ($L_\infty \to L_\infty$ for a CTS or $\ell_\infty \to \ell_\infty$ for a DTS) gain of an extended positive plant can be calculated and regulated using an LP [13], [14], [15], [16], which has also been derived using stability radius formulas [17]. Analysis and stabilization results can be extended to uncertain and switched positive systems [18], as well as time-delay positive systems [19]. The tutorial in [20] is a survey of topics about stabilization and performance regulation for positive linear systems.

DDC is a method that synthesizes controllers for a class of data-consistent plants without first performing a possibly expensive and inaccurate system identification step [21], [22]. Methods that require a reference signal include iterative feedback tuning [23], virtual reference feedback tuning [24], [25], and correlation-based tuning [26], but these algorithms lack stability guarantees for all consistent systems. Data-driven predictive control through input-output data can be accomplished through Willem's Fundamental Lemma, assuming that a rank condition of the Hankel matrices is satisfied (persistency of excitation) [27]. Stabilization, worst-case-optimal control, and Model Predictive Control problems can be solved through the use of this Lemma [28], [29], [30], [31], but the Lemma is vulnerable to noise sensitivity (even with regularization).

Prior knowledge of noise characteristics can be employed to synthesize controllers that will stabilize all plants that are consistent with data. L_∞-bounded noise arises from bounds on the time-derivative of the state (CTS) or discretization of continuous-time finite-difference approximations (DTS). Work addressing DDC of L_∞-bounded noise by solving LPs includes [32] using an Extended Farkas Lemma [33]. Tools from polynomial optimization may be applied to the L_∞ setting, such as for quadratic stabilization [34], switched systems [35], [36], and error-in-variables control [37], [38]. Quadratic Matrix Inequalities may be used to represent consistency sets (including energy-based or L_2-bounded noise) [39], [40], and stabilizing controllers may be synthesized by solving SDPs using a Matrix S-Lemma [41]. The work in [42] employs polynomial optimization for DDC under the assumption that magnitude bounds on Taylor polynomial coefficients and residual terms are known.

The work in [43] utilizes the Fundamental Lemma [27] to perform DDC of positive systems by solving an SDP. System identification of positive systems is performed in [44]. The method in [45] uses data-driven Lyapunov-Metzler inequalities to perform switched linear systems control (including positive linearsystems) at the expense of solving Bilinear Matrix Inequalities.

The contributions of this work are:

- An LP that performs data-driven positive-stabilizing control for all systems consistent with observed data.
- A tabulation of computational complexity required to solve this LP.

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- An extension of this LP towards worst-case peak-to-peak gain minimizing control.

Our work adapts existing model-based methods for positive systems into the DDC setting, and the resultant finite-dimensional LP formulations are comparatively simpler than Semidefinite-Programming-based prior methods in positive DDC.

This paper has the following structure: Section II reviews the preliminaries of notation, positive systems, copositive Lyapunov functions, and the Extended Farkas Lemma. Section III presents an LP to perform data-driven stabilizing control of positive systems. Section IV extends this LP framework to yield controllers that minimize the worst-case peak-to-peak gain between an external input and a controlled output. Section V demonstrates effectiveness of these methods on stabilizing and worst-case-optimal control of example systems. Section VI concludes the paper.

II. Preliminaries

A. Notation

The n-dimensional Euclidean vector space is $\mathbb{R}^n$. Its nonnegative orthant will be written as $\mathbb{R}^n_{\geq 0}$ and its positive orthant will be denoted as $\mathbb{R}^n_{>0}$. The set of $n \times m$ matrices will be $\mathbb{R}^{n \times m}$. The transpose of a matrix $M \in \mathbb{R}^{n \times m}$ is $M^T \in \mathbb{R}^{m \times n}$.

The n-dimensional identity matrix is I_n. The vector of all ones is $\mathbf{1}_n \in \mathbb{R}^n$. The $n \times m$ matrix of all zeros is $\mathbf{0}_{n \times m} \in \mathbb{R}^{n \times m}$. The matrix with $v \in \mathbb{R}^n$ appearing on its main diagonal and zeros elsewhere is $\text{diag}(v) \in \mathbb{R}^{n \times n}$. The Kronecker product of matrices A and B is $A \otimes B$. The column-wise vectorization of a matrix M is $\text{vec}(M)$. The elementwise division between $a, b \in \mathbb{R}^n$ is $a./b$.

The symbol δx will refer to x^+ (next state) in discrete-time or $\dot{x}$ in continuous-time. The symbols $(\circledast, \oplus, \ominus, \odot)$ correspond to an unrestricted (real-valued), a nonnegative, a nonpositive, and a zero-valued element respectively.

B. Positive Systems

A controlled LTI system with states $x \in \mathbb{R}^n$, inputs $u \in \mathbb{R}^m$, and outputs $y \in \mathbb{R}^p$ has the form

$$\delta x = Ax + Bu \qquad\qquad y = Cx + Du. \tag{1}$$

1) Positive System Descriptors:

Definition 2.1: The system (1) is *internally positive* iff for any initial condition $x(0) \in \mathbb{R}^n_{\geq 0}$ and input $u(t) \in \mathbb{R}^m_{\geq 0}$, the state and output responses remain in the positive orthant ($x(t) \in \mathbb{R}^n_{\geq 0},\ y(t) \in \mathbb{R}^p_{\geq 0}\ \forall t \geq 0$) [3].

Internal positivity requires that (B, C, D) are all nonnegative, along with the property that A is Metzler (off-diagonals are nonnegative) for a CTS or that A is nonnegative for a DTS. The system is positive-stable if A is Hurwitz and Metzler (CTS), or Schur and Nonnegative (DTS). For the remainder of this paper, we will assume that $C = I_n$ and $D = \mathbf{0}_{n \times m}$.

The state-feedback control $u = Kx$ with $K \in \mathbb{R}^{m \times n}$ positively-stabilizes (1) if the closed-loop matrix $A + BK$ is Metzler-Hurwitz or Nonnegative-Schur (as appropriate).

2) Copositive Functions:

Definition 2.2: A function $f : \mathbb{R}^n \to \mathbb{R}$ is *copositive* (with respect to the positive orthant) if $\forall x \in \mathbb{R}^n_{>0} : f(x) > 0$.

Copositivity of the linear function $V(x) = v^T x$ and the dual-linear function $V(x) = \max(x./v)$ may be checked by verifying that $v > 0$, but testing copositivity of a matrix function such as $x^T M x$ for some $M \in \mathbb{R}^{n \times n}$ is generically NP-hard [46].

3) Stability of Positive Systems:

Theorem 2.1: Let the system (1) be internally positive. Then it is asymptotically stable (positive-stable) iff one of the following equivalent conditions is satisfied [20]:

- C1) The matrices $-A$ (CTS) or $I_n - A$ (DTS) have positive principal minors.
- C2) There exists a $p \in \mathbb{R}^n_{>0}$ with $P = \text{diag}(p)$ such that $A^T P + PA \prec 0$ (CTS) or $A^T PA - P \prec 0$ (DTS).
- C3) There exists a positive vector $v \in \mathbb{R}^n_{>0}$ with a Linear Copositive Lyapunov Function (LCLF) $v^T x$ such that $A^T v < 0$ (CTS) or $A^T v < v_1$ (DTS).
- C4) There exists a positive vector $v_\infty \in \mathbb{R}^n_{>0}$ with a Dual Linear Copositive Lyapunov Function (DLCLF) $\max(x./v_\infty)$ such that $Av_\infty < 0$ (CTS) or $Av_\infty < v_\infty$ (DTS).

In this paper we will exclusively use Condition C4 of Theorem 2.1 with a DLCLF $\max(x./v_\infty)$. We note that the conditions in Theorem 2.1 strictly treat the case of (dual) LCLFs. Proposition 3.3 of [18] states that every uniformly exponentially stable positive linear system admits a polyhedral Lyapunov function with an undecidable number of facets.

4) Positive System Stabilization: DLCLFs may be employed to find positive-stabilizing controllers $K \in \mathbb{R}^{m \times n}$.

Theorem 2.2 ([10]): The closed-loop system $\delta x = (A + BK)x$ from (1), given a control $u = Kx$, is positive and asymptotically stable if there exists a vector $v \in \mathbb{R}^n_{>0}$ with a diagonal matrix $X = \text{diag}(v)$, and a matrix $Y \in \mathbb{R}^{m \times n}$ such that the gain K satisfies $KX = Y$ and

$$-(AX + BY)\mathbf{1}_n \in \mathbb{R}^n_{>0} \quad AX + BY \text{ is Metzler (CTS)} \tag{2a}$$

$$v - (AX + BY)\mathbf{1}_n \in \mathbb{R}^n_{>0} \quad AX + BY \in \mathbb{R}^{n \times n}_{\geq 0} \text{ (DTS).} \tag{2b}$$

Finding a controller through (2) requires solving an LP with both strict and nonstrict inequality constraints.

5) Structured Control: The stabilization task in (2) may be restricted to a set of controllers that obey sign patterns and sparsity structures. Such sparsity might arise from network information constraints.

Let $\mathcal{S}$ be an $m \times n$ matrix filled with the symbols $(\circledast, \oplus, \ominus, \odot)$. A controller with the structure $K \in \mathcal{S}$ may be constructed by solving (2) under the constraint that $Y \in \mathcal{S}$, given that multiplication by the matrix X with $v \in \mathbb{R}^n_{>0}$ does not change the sign pattern. An unstable internally positive system cannot be positive-stabilized by a nonnegative state feedback controller $K \in \mathbb{R}^{m \times n}_{\geq 0}$.

C. Extended Farkas Lemma

This work will find a state-feedback controller $u = Kx$ such that the set of all K-stabilized systems contains the set of systems consistent with observed data. The method used to enforce this containment is the Extended Farkas Lemma:

Lemma 2.3 (Extended Farkas Lemma [33], [47]): Let $P_1 = \{x \mid G_1 x \leq h_1\}$ and $P_2 = \{x \mid G_2 x \leq h_2\}$ be a pair of polytopes. Then $P_1 \subseteq P_2$ if and only if there exists a nonnegative matrix Z of compatible dimensions such that,

$$ZG_1 = G_2, \qquad Zh_1 \leq h_2. \tag{3}$$

III. Data-Driven Stabilization

This section will detail the data-driven positive-stabilization problem and its solution using robust linear programming.

A. Problem Setting

A set of T observations are recorded of system (1) as corrupted by a noise process $w \in \mathbb{R}^n$,

$$\delta x(t) = Ax(t) + Bu(t) + w(t). \tag{4}$$

These observations are collected into the data $\mathcal{D} = (\mathbf{X}, \mathbf{U}, \mathbf{X}_\delta)$ with the expressions,

$$\begin{array}{rcl} \mathbf{X} & := & [x(0) \quad x(1) \quad \ldots \quad x(T-1)] \\ \mathbf{U} & := & [u(0) \quad u(1) \quad \ldots \quad u(T-1)] \\ \mathbf{X}_\delta & := & [\delta x(0) \quad \delta x(1) \quad \ldots \quad \delta x(T-1)]. \end{array} \tag{5}$$

The discrepancy matrix $\mathbf{W}$ satisfies the relation,

$$\mathbf{W} = \mathbf{X}_\delta - (A\mathbf{X} + B\mathbf{U}). \tag{6}$$

The noise model that we will use is that each $w(t)$ (column of $\mathbf{W}$) is L_∞-norm-bounded by some given $\epsilon \geq 0$ ($\|w(t)\|_\infty \leq \epsilon$).

The set of all system matrices (A, B) that are compatible with the L_∞-corrupted data in $\mathcal{D}$ forms a polytopic consistency set $\Sigma_\mathcal{D}$. If it is known *a priori* that A is Metzler/Nonnegative and/or B is nonnegative, then these constraints in (A, B) may be adjoined to $\Sigma_\mathcal{D}$.

The data-driven positive-stabilization problem is:

Problem 3.1: Find a vector $v \in \mathbb{R}^n_{>0}$ and a controller $K \in \mathcal{S}$ such that $\max(x./v)$ is a common DLCLF ensuring positive-stability of $A + BK$ for all $(A, B) \in \Sigma_\mathcal{D}$.

B. Polytope Description

We will describe K-stabilized and $\mathcal{D}$-consistent polytopes that will be used in solving Problem 3.1 Throughout this section, the column-vectorization of the plant matrices will be defined as $a = \text{vec}(A), b = \text{vec}(B)$. The identity $\text{vec}(UVW) = (W^T \otimes U)\text{vec}(V)$ for matrices (U, V, W) of compatible dimensions will be judiciously used in derivations.

1) Data-Consistent Polytopes: The polytopic set $\Sigma^{\text{data}}_\mathcal{D}$ of plants consistent with the data in $\mathcal{D}$ may be represented as

$$G_1^{\text{data}} = \begin{bmatrix} \mathbf{X}^T \otimes I_n & \mathbf{U}^T \otimes I_n \end{bmatrix} \tag{7a}$$

$$\Sigma^{\text{data}}_\mathcal{D} = \left\{ (A, B) \mid G_1^{\text{data}} \begin{bmatrix} a \\ b \end{bmatrix} \leq \begin{bmatrix} \epsilon \mathbf{1}_{nT} + \text{vec}(\mathbf{X}_\delta) \\ \epsilon \mathbf{1}_{nT} - \text{vec}(\mathbf{X}_\delta) \end{bmatrix} \right\}. \tag{7b}$$

The consistency set of plants $\Sigma_\mathcal{D}$ is the intersection of $\Sigma^{\text{data}}_\mathcal{D}$ and the prior knowledge on system-positivity of (A, B) (linear constraints) described in Σ^{prior}. As an example, where A is a positive system in discrete-time, then $\Sigma^{\text{prior}} = \{A \mid A \in \mathbb{R}^{n \times n}_+\}$, $G_1^{\text{prior}} = -I_{n^2}$, and $h_1^{\text{prior}} = \mathbf{0}_{n^2}$. Let (G_1, h_1) be matrices such that the polytopic data-consistency set $\Sigma_\mathcal{D} = \Sigma^{\text{data}}_\mathcal{D} \cap \Sigma^{\text{prior}}$ can be expressed as

$$\Sigma_\mathcal{D} = P_1 = \left\{ (A, B) \mid G_1 \begin{bmatrix} a \\ b \end{bmatrix} \leq h_1 \right\}. \tag{8a}$$

2) Controller-Stabilizing Polytopes: In order to apply the Extended Farkas Lemma 2.3, we will convert the strict inequalities in (2) and in $v \in \mathbb{R}^n_{>0}$ to non-strict inequalities by utilizing a sufficiently small $\eta > 0$.

$$-(AX + BY)\mathbf{1}_n - \eta \mathbf{1}_n \in \mathbb{R}_{\geq 0} \qquad \text{(CTS)} \tag{9a}$$

$$v - (AX + BY)\mathbf{1}_n - \eta \mathbf{1}_n \in \mathbb{R}_{\geq 0} \qquad \text{(DTS).} \tag{9b}$$

Define the canonical Metzler-indexing matrix $M_n \in \mathbb{R}^{n(n-1) \times n^2}$ as a 0/1-valued matrix that extracts off-diagonal elements, such as

$$M_2 \text{vec}\left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \tag{10}$$

The polytope P_2^C of continuous-time plants (A, B) that can be positive-stabilized via (9a) under a state-feedback controller $K \in \mathcal{S}$ with a DLCLF $\max(x./v)$ such that $Y = KX$ can be described by

$$G_2^C = \begin{bmatrix} v^T \otimes I_n & (Y\mathbf{1}_n)^T \otimes I_n \\ -M_n(X \otimes I_n) & -M_n(Y^T \otimes I_n) \end{bmatrix} \tag{11a}$$

$$P_2^C = \left\{ (A, B) \mid G_2^C \begin{bmatrix} a \\ b \end{bmatrix} \leq \begin{bmatrix} -\eta \mathbf{1}_n \\ \mathbf{0}_{n(n-1)} \end{bmatrix} \right\}. \tag{11b}$$

The top row of G_2^C is the DLCLF stabilization criterion, and the bottom row enforces that $AX + BY$ is Metzler.

The polytope P_2^D of discrete-time plants (A, B) positive-stabilized by (K, Y) under the same conditions is

$$G_2^D = \begin{bmatrix} v^T \otimes I_n & (Y\mathbf{1}_n)^T \otimes I_n \\ -X \otimes I_n & -Y^T \otimes I_n \end{bmatrix} \tag{12a}$$

$$P_2^D = \left\{ (A, B) \mid G_2^D \begin{bmatrix} a \\ b \end{bmatrix} \leq \begin{bmatrix} v - \eta \mathbf{1}_n \\ \mathbf{0}_{n^2} \end{bmatrix} \right\}. \tag{12b}$$

C. Stabilizing Programs using the Extended Farkas Lemma

To unite notation, let P_2 be the appropriate stabilizing polytope for continuous-time (P_2^C) or discrete-time (P_2^D) from Section III-B.2. The number of constraints in the stabilizing polytope P_2 (length of h_2) is $q = n + n(n-1)$ for continuous-time and $q = n + n^2$ for discrete-time. The polytope P_2 has a constraint matrix $G_2 \in \mathbb{R}^{q \times n(n+m)}$ and vector $h_2 \in \mathbb{R}^q$ such that $P_2 = \{(A, B) \mid G_2[a^T \; b^T]^T \leq$

$h_2\}$. The entries in G_2 and h_2 are affinely-dependent on (v, Y).

Problem 3.1 may be expressed in the language of polytope-containment as,

Problem 3.2: Find a vector $v \in \mathbb{R}^n_{>0}$ and a matrix $Y \in \mathcal{S}$ such that $P_1 \subseteq P_2$.

Theorem 3.3: Problem 3.2 (equivalent to (3.1)) has a solution iff the following LP involving variables (v, Y, Z) is feasible:

$$\underset{v,Y,Z}{\text{find}} \quad ZG_1 = G_2(v, Y), \qquad Zh_1 \le h_2(v, Y) \tag{13a}$$

$$v - \eta \mathbf{1}_n \in \mathbb{R}^n_{\ge 0}, \; Y \in \mathcal{S}, \; Z \in \mathbb{R}^{q \times 2nT}_{\ge 0}, \tag{13b}$$

whereby the state-feedback gain $K \in \mathcal{S}$ can be recovered by calculating $K = YX^{-1}$.

Proof: The LP in (13) is a direct application of the Extended Farkas Lemma 2.3 to prove polytope containment $P_1 \subseteq P_2$. If there exists a (v, Y) such that Problem 3.2 is solved, then there exists a Z that renders (13) by the Extended Farkas Lemma. Conversely if there exists a (v, Y, Z) such that (13) is feasible, then the Extended Farkas Lemma proves the containment of polytopes 2.3 with (v, Y). ∎

D. Computational Complexity

Table I computes the number of inequality and equality constraints required to represent Program (13a). The number of equality constraints associated with Y is set to 0 because zero-valued entries of Y will be removed and will not be treated as scalar variables. The LP in (13) has up to $n + mn + (2nT)q$ scalar variables distributed into (v, Y, Z), plus q additional nonnegative slack variables required to represent the inequalities in constraint (13a).

TABLE I: Number of Inequality and Equality constraints in Program (13)

	# Ineq.	# Eq.
v	n	0
Y	$\le mn$	0
Z	$(2nT)q$	0
(13a)	q	$qn(n+m)$

In discrete-time with $q = n^2 + n$ and no value-restrictions on K ($Y \in \mathbb{R}^{m \times n}$), Program (13a) will have $N = (2nT+1)(n^2+n) + (2m+1)n$ nonnegative scalar variables (representing $Y = Y^+ - Y^-$ where both Y^+ and Y^- are nonnegative) and $(n^2+1)n(n+m)$ equality constraints.

The running-time of an Interior Point Method solver for LPs up to γ-optimality is approximately $O(N^{\omega+0.5}|\log(1/\gamma)|)$ [48], where ω is the matrix-multiplication constant. Our DDC algorithm therefore has performance on the order of $(Tn^3)^{\omega+0.5} \sim n^{12.5}$. Significant gains in performance may be realized by noting that the matrices (G_1, G_2) are sparse and are highly structured.

Remark 1: The polytope $\Sigma_{\mathcal{D}}$ may possess a large number of redundant faces. These half-space constraints may be removed to improve computational performance without affecting the description of $\Sigma_{\mathcal{D}}$. Nonredundant faces may be discovered by linear programming over the polytope [49].

Remark 2: An alternative approach is to perform vertex enumeration, in which relations (2) hold at every vertex of $\Sigma_{\mathcal{D}}$. The polytopes $\Sigma_{\mathcal{D}}$ that are gathered as part of the data-acquisition process empirically have a number of vertices that scales exponentially with dimension, for which the face-based approach of the Extended Farkas Lemma is more favorable.

Remark 3: This paper focused on the case of L_∞-bounded noise. This set-containment framework will also be nonconservative when applied to other with other semidefinite-representable noise processes, such as when each column of the discrepency matrix $\mathbf{W}$ in (6) has bounded L_2 norm. The Extended Farkas Lemma 2.3 is a specific instance of a more general Robust Counterpart posed over a system of linear inequalities [50, Theorem 1.3.14]. In the L_2 case, each inequality constraint in the polytope in P_2 over the uncertain (a, b) is replaced via a robust counterpart by $n(n+m)$ second-order-cone variables, $n(n+m)$ linear equality constraints, and one linear inequality constraint. This procedure is performed programmatically in [51] under the 'duality' option.

IV. Peak-to-Peak Gain Regulation

This section performs worst-case peak-to-peak (p2p) gain minimization using the Extended Farkas Lemma.

System (1) may be affected by an external noise process $\xi \in \mathbb{R}^e$ to form dynamics with a controlled output of $z \in \mathbb{R}^p$

$$\delta x(t) = Ax(t) + Bu(t) + E\xi(t) \tag{14a}$$

$$z(t) = Cx(t) + Du(t) + F\xi(t). \tag{14b}$$

For a given set of parameters (A, B, C, D, E, F) with $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, $E \in \mathbb{R}^{n \times e}$, $F \in \mathbb{R}^{p \times e}$, this peak-to-peak gain may be computed by solving an LP,

Lemma 4.1 ([14]): There exists a state-feedback controller $u = Kx$ with $K, Y \in \mathcal{S}$ and $v \in \mathbb{R}^n_{>0}$ such that peak-to-peak gain of (14) is less than or equal to $\gamma \ge 0$ for continuous-time if

$$-(AX + BY)\mathbf{1}_n - E\mathbf{1}_e \in \mathbb{R}^n_{>0} \tag{15a}$$

$$\gamma \mathbf{1}_q - (CX + DY)\mathbf{1}_n - F\mathbf{1}_e \in \mathbb{R}^q_{>0} \tag{15b}$$

$$CX + DY \in \mathbb{R}^{q \times n}_{\ge 0} \tag{15c}$$

$$AX + BY \text{ is Metzler,} \tag{15d}$$

and for discrete-time if

$$v - (AX + BY)\mathbf{1}_n - E\mathbf{1}_e \in \mathbb{R}^n_{>0} \tag{16a}$$

$$\gamma \mathbf{1}_q - (CX + DY)\mathbf{1}_n - F\mathbf{1}_e \in \mathbb{R}^q_{>0} \tag{16b}$$

$$CX + DY \in \mathbb{R}^{q \times n}_{\ge 0} \tag{16c}$$

$$AX + BY \in \mathbb{R}^{n \times n}_{\ge 0}, \tag{16d}$$

whereby the state-feedback gain can be recovered by $K = YX^{-1}$.

We aim to solve the following problem:

Problem 4.2: Find a state-feedback controller $u = Kx$ with $K \in \mathcal{S}$ to minimize the worst-case peak-to-peak gain $\xi \to z$ for any data-consistent plant $(A, B) \in \Sigma_{\mathcal{D}}$.

Remark 4: The ϵ-corrupted data in $\mathcal{D}$ is obtained when $\xi(t) = 0$ at all time samples. It is further assumed that the matrices (C, D, E, F) are all fixed and are known in advance.

Peak-to-peak polytopes for (A, B) in (15) and (16) may be construct in a similar manner to the stabilizing polytopes P_2 in Section III-B.2. The right hand sides of these polytopes for CTSs and DTSs are,

$$h_2^{\text{p2p: }C} = \begin{bmatrix} -\eta \mathbf{1}_n - E\mathbf{1}_e \\ \mathbf{0}_{n(n-1)} \end{bmatrix} \quad h_2^{\text{p2p: }D} = \begin{bmatrix} v - \eta \mathbf{1}_n - E\mathbf{1}_e \\ \mathbf{0}_{n^2} \end{bmatrix}. \tag{17}$$

Theorem 4.3: Problem 4.2 has a solution iff the following LP in variables (v, Y, Z, γ) is feasible,

$$\gamma^* = \min_{\gamma \in \mathbb{R}} \quad \gamma \tag{18a}$$
$$ZG_1 = G_2(v, Y), \qquad Zh_1 \leq h_2^{\text{p2p}}(v, Y) \tag{18b}$$
$$(\gamma - \eta)\mathbf{1}_q - (CX + DY)\mathbf{1}_n - F\mathbf{1}_e \in \mathbb{R}^q_{\geq 0} \tag{18c}$$
$$CX + DY \in \mathbb{R}^{q \times n}_{\geq 0} \tag{18d}$$
$$v - \eta \mathbf{1}_n \in \mathbb{R}^n_{\geq 0}, \ Y \in \mathcal{S}, \ Z \in \mathbb{R}^{q \times 2nT}_{\geq 0}, \tag{18e}$$

whereby the p2p-minimizing state feedback gain $K \in \mathcal{S}$ can be recovered by $K = YX^{-1}$.

Proof: The outer (peak-to-peak) polytope is $P_2^{\text{p2p}} = \{(A, B) \mid G_2[a^T \ b^T]^T \leq h_2^{\text{p2p}}\}$ for the appropriate continuous-time or discrete-time vector in (17), as constructed from conditions (15a) or (16a). The Extended Farkas Lemma 2.3 is then applied in (18b) to ensure that γ is an upper bound for the peak-to-peak gain of all consistent systems (with similar logic as in the proof of Theorem 3.3). The objective in (18a) reduces this gain as much as possible. The minimum is achieved because all constraints in (18) ar nonstrict (due to the given tolerance $\eta > 0$). ∎

V. Numerical Examples

All experiments are written in MATLAB 2021a wit Mosek [52] and YALMIP [53] dependencies. The code i available at https://github.com/jarmill/data_driven_pos. All experiments use a noise level of $\epsilon = 0.$ for the L_∞-norm bound and a tolerance of $\eta = 10^{-3}$ for th strict inequality constraints.

A. Continuous-Time Stabilization

The ground-truth continuous-time system in this example has $n = 3$ inputs and $m = 2$ outputs:

$$A = \begin{bmatrix} -0.55 & 0.3 & 0.65 \\ 0.06 & -1.35 & 0.25 \\ 0.1 & 0.15 & 0.4 \end{bmatrix} \quad B = \begin{bmatrix} 0.18 & 0.08 \\ 0.47 & 0.25 \\ 0.07 & 0.95 \end{bmatrix}. \tag{19}$$

System (19) is internally positive but is open-loop unstable (poles of $0.4907, -0.6055, -1.3851$). The stabilization task in (13) with $T = 5$ and an additional normalization constraint that $\mathbf{1}_n^T v = 1$ results in,

$$v = \begin{bmatrix} 0.5570 & 0.1401 & 0.3029 \end{bmatrix}^T \tag{20a}$$
$$K = \begin{bmatrix} 0.0279 & -0.2660 & 0.5041 \\ 0.0107 & -0.0222 & -0.8650 \end{bmatrix}. \tag{20b}$$

Figure 1 visualizes controlled trajectories of 100 data-consistent plants in which each (A, B) is chosen from the polytope Σ_D by hit-and-run sampling [54] (with implementation in [55]). All closed-loop trajectories of $\dot{x}(t) = (A+BK)x(t)$ begin at the initial point $x(0) = [1; 1; 1]$, share a common K from (20b), and evolve in times $t \in [0, 20]$.

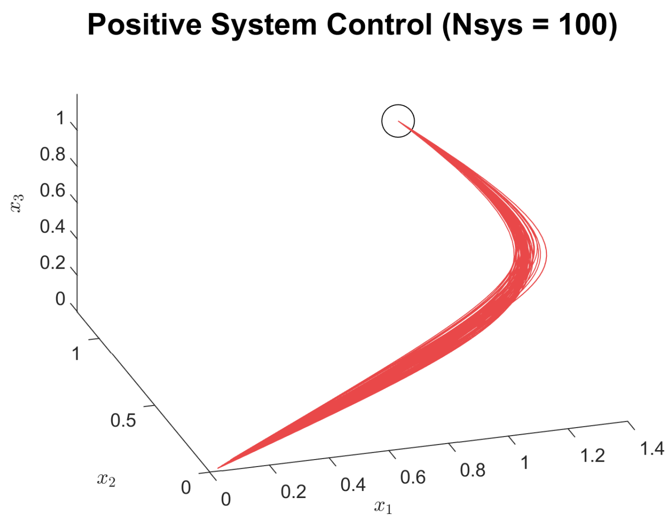

Fig. 1: Application of the controller $u = Kx$ from (20b) to positively-stabilize 100 consistent systems in Σ_D.

Figure 2 plots values of the DLCLF $\max(x./v)$ (for the v in (20a)) along the 100 systems in 1.

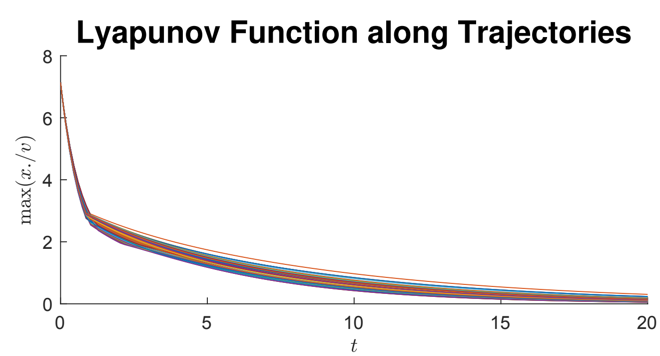

Fig. 2: DLCLF along the 100 trajectories.

B. Discrete-Time Stabilization

This example involves a discrete-time system with $n = 5$ states and $m = 3$ inputs. The ground-truth system is internally positive, and is unstable with poles of $1.3094, \ -0.1218 \pm 0.0992\mathbf{j}, 0.1201 \pm 0.1108\mathbf{j}$. With $T = 60$ observations the following DLCLF and stabilizing controller is recovered:

$$v = \begin{bmatrix}0.2076 & 0.1212 & 0.2651 & 0.2516 & 0\end{bmatrix}^T \tag{21}$$

$$K = \begin{bmatrix} 0.0483 & 0.0088 & -0.1326 & -0.0188 & -0.4273 \\ -0.3243 & 0.0115 & 0.0299 & -0.2980 & 0.0337 \\ 0.1601 & 0.0749 & -0.5962 & -0.3537 & -0.2194 \end{bmatrix}.$$

It is now desired to obtain a stabilizing controller for all consistent plants that obeys the sign pattern

$$\mathcal{S} = \begin{bmatrix} \odot & \odot & \odot & \odot & \ominus \\ \odot & \odot & \circledast & \odot & \oplus \\ \odot & \odot & \odot & \circledast & \circledast \end{bmatrix} \tag{22a}$$

Such a DLCLF certificate and controller is

$$v = \begin{bmatrix}0.2147 & 0.1259 & 0.2448 & 0.2516 & 0.1630\end{bmatrix}^T \tag{22b}$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.6853 \\ 0 & 0 & -0.3206 & 0 & 0.1206 \\ 0 & 0 & 0 & -0.5604 & -0.3317 \end{bmatrix}. \tag{22c}$$

C. Continuous-Time Peak-to-Peak

The following ground-truth positive-stable continuous-time system has $n = 3$ inputs and $m = 2$ outputs

$$A = \begin{bmatrix} -0.2 & 0.2 & 0.2 \\ 0.4 & -0.7 & 0.2 \\ 0 & 0.8 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -0.4 & 0.5 \\ 0.2 & -0.8 \\ -1 & 2 \end{bmatrix}. \tag{23}$$

This system has $e = 2$ external input channels and $p = 5$ controlled outputs with

$$C = \begin{bmatrix} I_3 \\ \mathbf{0}_{2\times 3} \end{bmatrix}, \quad D = \begin{bmatrix} \mathbf{0}_{3\times 2} \\ I_2 \end{bmatrix}, \quad E = \begin{bmatrix} I_2 \\ \mathbf{0}_{1\times 2} \end{bmatrix}, \quad F = \mathbf{0}_{5\times 2}. \tag{24}$$

The peak-to-peak gain of the ground-truth (23) under the parameters in (24) when uncontrolled ($K = \mathbf{0}_{2\times 3}$) is $\gamma^* = 32.178$. Lemma 4.1 synthesizes a controller for the ground-truth system resulting in a gain of $\gamma^* = 3.742$. The constraint $CX + DY \in \mathbb{R}^{q\times n}_{\geq 0}$ with the values in (24) imposes that all elements of Y and K are nonnegative ($\oplus$).

Table II collects the worst-case peak-to-peak gains obtained by (18) as a function of the number of samples T. These gains decrease as T increases and the consistency set $\Sigma_{\mathcal{D}}$ shrinks. The top row of (II) incorporates the prior knowledge that the ground-truth A from (23) is Metzler when constructing the polytope Σ_D. The bottom row does not impose this positivity (Metzler) prior on A, and therefore yields peak to peak bounds that are always greater than or equal to the Metzler-imposed bounds.

TABLE II: Worst-case peak-to-peak gain γ^* computed by (18) decreases as the number of samples T increases

T	20	30	50	80	120
A Metzler	6.4539	5.0182	4.4967	4.0619	4.0028
No Prior	6.4823	5.0719	4.5292	4.0659	4.0029

The system with $T = 50$ and a Metzler-prior on A has a worst-case peak-to-peak gain of $\gamma^* = 4.4967$ and solution outputs of

$$v = \begin{bmatrix}4.4967 & 4.2021 & 0.4303\end{bmatrix}^T \tag{25a}$$

$$K = \begin{bmatrix} 0.5095 & 0.4765 & 0.4727 \\ 0.2587 & 0 & 0 \end{bmatrix}. \tag{25b}$$

The polytope Σ_D under the Metzler-prior has $2nT + (n^2 - n) = 300 + 6 = 306$ faces and 308,672 vertices, of which 62 faces are nonredundant (see Remark 1). The nonnegative Farkas matrix over the nonredundant faces is $Z \in \mathbb{R}^{9\times 62}_{\geq 0}$.

VI. Conclusion

This paper presented an LP-based algorithm (Theorem 3.3) to perform data-driven stabilizing control of positive linear systems. The state-feedback controller K stabilizes all possible systems in the L_∞-norm bounded consistency set $\Sigma_{\mathcal{D}}$, as certified by a common DLCLF function $V(x) = \max(x./v)$ and the Extended Farkas Lemma. There is no conservativeness in such a design: Equation (13) will find a controller iff there exists such a linear copositive Lyapunov function across all consistent systems. This framework can also be used to perform data-driven worst-case peak-to-peak gain minimization using Theorem 4.3.

Future work includes imposing other forms of consistency sets for the noise (e.g. elementwise L2 norm bounds, semidefinite energy-based noise bounds for $\mathbf{W}$). Other avenues include creating controllers in the switched or linear-parameter-varying setting, and extending this method towards Monotone systems.

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