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Data-Driven Safe Control of Discrete-Time Non-Linear Systems

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***Abstract*—This letter proposes a framework to perform verifiably safe control of all discrete-time non-linear systems that are compatible with collected data. Most safety-maintaining control synthesis algorithms (e.g., control barrier functions, density functions) are limited to obtaining theoretical guarantees of safety in continuous-time, even while their implementation on real systems is typically in discrete-time. We first present a sum-of-squares based program to prove the existence of an (acausal) control policy that can safely stabilize all possible data-consistent systems. Causal control policies may be extracted by online optimization, and we provide sufficient conditions for the extraction of this control policy in general scenarios. As a specific case, we introduce a method for tractable online controller recovery when convexity assumptions are imposed on the candidate Lyapunov function and safety region descriptor. Discrete-time safe stabilization is demonstrated on three example systems.**

***Index Terms*—Data-driven control, discrete-time systems, nonlinear systems, safety, stability, sum-of-squares.**

I. Introduction

SAFETY-critical control plays an crucial role in daily lives: we expect systems to avoid unsafe conditions. Over the past decade, intensive studies have been dedicated to safety verification and safe control synthesis.

Barrier Functions [1] represent a type of level-set method certifying the safety of trajectories, which relies on the forward invariance of a super-level set. Control Barrier Functions (CBFs) [2], [3] allow for safe control synthesis by finding a controller that guarantees forward invariance with respect to a candidate barrier function. Safety and stability can be jointly enforced by considering both a CBF and a Control Lyapunov Function (CLF) [4].

These methods depend on perfect knowledge of the system dynamics. In practice, we only possess limited priors of the system structure. When the system is unknown, Data Driven Control (DDC) provides methods that synthesize control laws directly from observations and thus skip a system-identification/control-synthesis pipeline [5]. Amongst the vast literature, the closest DDC approaches related to this letter are those that pursue a set-membership approach [6], [7], [8], [9], [10], [11], [12], [13], which seek a controller that can stabilize all plants compatible with observed data together with a stability certificate (usually a common Lyapunov function).

Recent work on data-driven safe control of continuous-time non-linear systems includes [13], [14], [15], [16]. Reference [14] introduced a learning-based approach that iteratively collects data and updates a controller corresponding to a known CBF to ensure ultimate safety. The work in [13] uses density functions and the Theorems of Alternatives to derive a rational controller together with a density-based safety certificate. The approach in [15] finds an ellipsoidal overapproximation of consistency set and enforces robust invariance to certify safety. The method in [16] performs forward and backward Hamilton-Jacobi reachability analysis on a differential game function and uses data-driven Bayesian inference to construct high probability safety guarantees. Interested readers are referred to [17] and references therein for a comprehensive overview on data-driven safe control using Hamilton-Jacobi reachability, CBFs and predictive control-related techniques.

Unlike the data-driven safe control of continuous-time non-linear systems that is relatively well understood, its discrete-time analogue is considerably less developed and has been approached mostly from a stochastic perspective. The approach in [18] models the system as known dynamics plus an additive residual and synthesizes a control law that guarantees safety up to a given risk level. Reference [19] performs formal synthesis of safety for stochastic systems with a desired confidence level, over a finite time horizon, relying on multiple measurements at each datapoint. While these approaches lead to tractable problems, the safety certificates are probabilistic, which may prevent applicability

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to safety-critical scenarios where it must be guaranteed that the system never enters an unsafe region. Further, none of these approaches guarantees closed-loop stability. On the other hand, the model-based approaches in [20], [21], [22] provide hard certificates and address, in a "soft" fashion, stability, by relaxing the stability constraint to allow a Lyapunov function to increase when it conflicts with the safety requirement. However, these approaches require that both a model of the plant and a known CLF be available. Further, allowing the Lyapunov function to grow may result in limit cycles.

To circumvent these difficulties in this letter we explicitly address both stability and safety in a data-driven fashion without dependence on an existing model or a predetermined CLF. Rather, we construct a CLF and associated control law that are compatible with the safety constraints, that is, the control law guarantees that both, the system never enters the unsafe region and that the Lyapunov function decreases along the flows. To the best of the authors' knowledge, this is the first paper to formulate and solve this problem in a computationally tractable fashion. Specifically the contributions of this letter are:

- Formulate the problem of non-linear, discrete-time data-driven safe control problem with stability constraints as a robust optimization. Notably this approach does not require prior knowledge of a suitable CLF.
- Use a combination of duality and a novel lifting to recast the problem into a sequence of tractable convex optimization problems. This combination results in substantial reduction of computational complexity vis-a-vis direct application of Sum-of-Squares techniques.
- Numerical examples demonstrating the efficacy of safe control on non-linear discrete-time systems, both for convex and non-convex safe sets.

This letter has the following structure: Section II reviews preliminaries such as CLFs and safety certification. Section III presents the proposed data-driven safe control method. Section IV demonstrates the performance of the proposed method on three example systems. Section V concludes this letter and points out to directions for further research.

II. Preliminaries

A. Notation

$\mathbb{R}_{++}, (\mathbb{R}_+)$	Positive (Non-negative) real numbers
$x, \boldsymbol{x}, \boldsymbol{X}$	Scalar, vector, matrix
$\mathbf{1}, \boldsymbol{I}$	Vector of all 1s, identity matrix
$\|\boldsymbol{x}\|_\infty$	L_∞-norm of vector $\boldsymbol{x}$
$\boldsymbol{X} \succeq 0$	$\boldsymbol{X}$ is positive semi-definite
$\otimes$	Kronecker product
$\text{vec}(\boldsymbol{X})$	Vectorized matrix along columns: $\text{vec}(\boldsymbol{X}) = [\boldsymbol{X}(:,1)^T, \ldots, \boldsymbol{X}(:,n)^T]^T$
$R[\boldsymbol{x}]$	Polynomials in the indeterminate $\boldsymbol{x} \in \mathbb{R}^n$
$\mathcal{K}_\infty$	Class $\mathcal{K}$-infinity functions

B. Extended Farkas' Lemma

The following result plays a key role in reducing the safe data-driven control problem to a tractable convex optimization.

Lemma 1 [23]: Consider the polyhedrons $\mathcal{P}_N \doteq \{\boldsymbol{x}\colon \boldsymbol{N}\boldsymbol{x} \le \boldsymbol{\nu}\}$ and $\mathcal{P}_M \doteq \{\boldsymbol{x}\colon \boldsymbol{M}\boldsymbol{x} \le \boldsymbol{\mu}\}$. Then $\mathcal{P}_N \subseteq \mathcal{P}_M$ if and only if there exists a multiplier matrix $\boldsymbol{Y}$ with non-negative entries such that

$$\boldsymbol{Y}\boldsymbol{N} = \boldsymbol{M} \text{ and } \boldsymbol{Y}\boldsymbol{\nu} \le \boldsymbol{\mu}.$$

C. Control Lyapunov Functions

Consider a non-linear discrete-time system of the form

$$\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k) + g(\boldsymbol{x}_k)\boldsymbol{u}_k, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state and $\boldsymbol{u}_k \in \mathbb{R}^m$ is the control.

Definition 1: A continuous function $V(.)\colon \mathbb{R}^n \to \mathbb{R}_+$ is a CLF for the system (1) if for all $\boldsymbol{x} \in \mathbb{R}^n$

a) There exist $\mathcal{K}_\infty$ functions $\alpha_1(.), \alpha_2(.)$ such that $\alpha_1(\|\boldsymbol{x}\|) \le V(\boldsymbol{x}) \le \alpha_2(\|\boldsymbol{x}\|)$,

b) $\inf_{\boldsymbol{u}\in\mathbb{R}^m} V(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}) \le \beta V(\boldsymbol{x}),\ \exists \beta \in [0, 1)$.

As shown in [24], existence of a CLF is equivalent to (weak) uniform global asymptotic stability of the closed-loop system. In the sequel we will work with a slightly stronger condition. We will impose that $\alpha_1 = c_1\|\boldsymbol{x}\|^p$ and $\alpha_2 = c_2\|\boldsymbol{x}\|^q$ for suitable $c_1, c_2, p, q \in \mathbb{R}_{++}$. With these assumptions, the condition above can be relaxed to

$$\inf_{\boldsymbol{u}\in\mathbb{R}^m} V(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}) + c_3\|\boldsymbol{x}\|^q \le V(\boldsymbol{x}),\ c_3 \in \mathbb{R}_{++}. \tag{2}$$

D. Safety Certification

Consider the same system in (1). In the sequel we will denote by $\boldsymbol{x}_k(\boldsymbol{x}_0, \boldsymbol{u})$ the trajectory that starts at the initial condition $\boldsymbol{x}_0$, under the control action $\boldsymbol{u} = \boldsymbol{u}_0, \boldsymbol{u}_1, \ldots$.

Definition 2: Given an initial condition set $\mathcal{X}_0 \subseteq \mathbb{R}^n$ and an unsafe set $\mathcal{X}_u \subseteq \mathbb{R}^n$, system (1) can be rendered safe with respect to $\mathcal{X}_u$, if for all initial conditions $\boldsymbol{x}_0 \in \mathcal{X}_0$, there exists a control sequence $\boldsymbol{u}$ such that $\boldsymbol{x}_k(\boldsymbol{x}_0, \boldsymbol{u}) \notin \mathcal{X}_u$ for all $k \in \mathbb{N}$.

Note that this definition coincides with the forward-invariance-based one used for instance in [18] when the safety set $\mathcal{C} = \mathbb{R}^n \setminus \mathcal{X}_u$ and $\mathcal{X}_0 = \mathcal{C}$. Typically, in the existing literature, safety is certified through the use of barrier functions, defined through the super-level set of a function $h(.)$, $\mathcal{C} = \{\boldsymbol{x}\colon h(\boldsymbol{x}) \ge 0\}$.

Definition 3 [18]: The function $h(.)$ is a control barrier function for the set $\mathcal{C}$ if, for each $\boldsymbol{x} \in \mathcal{C}$ there exist $\boldsymbol{u}$ such that

$$h(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}) \ge \alpha h(\boldsymbol{x}),\ \alpha \in [0, 1]. \tag{3}$$

As shown in [18], existence of a CBF is equivalent to controlled forward invariance of the set $\mathcal{C}$.

The definition above imposes a lower bound on how fast $h(.)$ can decrease along trajectories: $h(\boldsymbol{x}_k) \ge \alpha^k h(\boldsymbol{x}_0)$, and thus may limit performance. To avoid this effect, we will consider the case where $\alpha = 0$.

E. The Data-Driven Discrete-Time Safe Control Problem

The goal of this letter is to design a safe, stabilizing control law based on (noisy) experimental measurements for unknown discrete-time non-linear systems. Specifically, we consider non-linear systems of the form (1). We will assume that the only information available about the system is:

1) A priori information: (i) $f(.)$ can be expressed as linear combinations of functions in a known dictionary $\boldsymbol{\phi}: \mathbb{R}^n \to \mathbb{R}^{d_f}$, that is $f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{\phi}(\boldsymbol{x})$ for some unknown system parameter matrices $\boldsymbol{F} \in \mathbb{R}^{n\times d_f}$; and (ii) $g(.) = \boldsymbol{g}(.)$ is a known matrix.[1]
2) Experimental data $\mathcal{D} = \{(\boldsymbol{x}_k, \boldsymbol{u}_k)\}_{k=0\ldots T-1}$ consisting of T state-input tuples sampled from the trajectories of (1) under some unknown but bounded process disturbance $\boldsymbol{w}$, with $\|\boldsymbol{w}\|_\infty \le \epsilon$, e.g., $\|\boldsymbol{x}_{k+1} - f(\boldsymbol{x}_k) - g(\boldsymbol{x}_k)\boldsymbol{u}_k\|_\infty \le \epsilon$.

In this context, the problem under consideration can be formally stated as:

Problem 1: Given sets $\mathcal{X}_0$, $\mathcal{X}_u$ and T training tuples $\mathcal{D} = \{(\boldsymbol{x}_k, \boldsymbol{u}_k)\}_{k=0\ldots T-1}$, find a state-feedback control law $\boldsymbol{u}(\boldsymbol{x})$ such that all closed-loop systems consistent with the observed data and priors (i) are safe with respect to $\mathcal{X}_0$ and $\mathcal{X}_u$; and (ii) have the origin as a globally asymptotically stable equilibrium point.

III. Problem Solution

We propose to solve Problem 1 by finding a CLF $V(.)$, along with a control action $\boldsymbol{u}(.)$ so that conditions (2) and (3) hold simultaneously.

Define a consistency set $\mathcal{P}_1$ as the set of all matrices $\boldsymbol{F}$ compatible with the observed data in $\mathcal{D}$, that is:

$$\mathcal{P}_1 \doteq \{\boldsymbol{F}: \|\boldsymbol{x}_{k+1} - \boldsymbol{F}\boldsymbol{\phi}(\boldsymbol{x}_k) - \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}_k\|_\infty \le \epsilon, \quad \forall (\boldsymbol{x}_k, \boldsymbol{u}_k) \in \mathcal{D}\}. \tag{4}$$

As shown in [7], the consistency set can be rewritten as

$$\mathcal{P}_1 = \left\{\boldsymbol{f}: \begin{bmatrix}\boldsymbol{A}\\ -\boldsymbol{A}\end{bmatrix}\boldsymbol{f} \le \begin{bmatrix}\epsilon\mathbf{1} + \boldsymbol{\xi}\\ \epsilon\mathbf{1} - \boldsymbol{\xi}\end{bmatrix}\right\}, \tag{5}$$

where $\boldsymbol{f} = \text{vec}(\boldsymbol{F}^T)$ and the matrices $\boldsymbol{A}, \boldsymbol{\xi}$ are functions of the collected data:

$$\boldsymbol{A} \doteq \begin{bmatrix}\boldsymbol{I}\otimes\boldsymbol{\phi}^T(\boldsymbol{x}_0)\\ \vdots\\ \boldsymbol{I}\otimes\boldsymbol{\phi}^T(\boldsymbol{x}_{T-1})\end{bmatrix}, \boldsymbol{\xi} \doteq \begin{bmatrix}\boldsymbol{x}_1 - \boldsymbol{g}(\boldsymbol{x}_0)\boldsymbol{u}_0\\ \vdots\\ \boldsymbol{x}_T - \boldsymbol{g}(\boldsymbol{x}_{T-1})\boldsymbol{u}_{T-1}\end{bmatrix}. \tag{6}$$

In the sequel, we will make the following assumption:

A1: Enough data has been collected so that the polytope $\mathcal{P}_1$ is compact, that is matrix $\boldsymbol{A}$ has full column rank.

This assumption is required to guarantee a finite diameter of the consistency set. Otherwise, the worst case identification error of any interpolatory identification algorithm is unbounded [25] and thus the classical worst-case-identification/control-synthesis pipeline will fail.

In terms of the consistency set $\mathcal{P}_1$ and the safety set $\mathcal{C}$ Problem 1 can be reformulated as:

Problem 2: Find a CLF V and associated control action $\boldsymbol{u}$ such that, for each $\boldsymbol{x} \in \mathcal{C}$ there exists a function $\boldsymbol{u}^*(\boldsymbol{x})$ such that the following two conditions are satisfied for all $\boldsymbol{f} \in \mathcal{P}_1$:

$$V\big(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}^*\big) + c_3\|\boldsymbol{x}\|^q \le V(\boldsymbol{x}), \tag{7}$$

$$h(\boldsymbol{x}) \ge 0 \implies h\big(f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}^*\big) \ge 0. \tag{8}$$

[1]While this assumption seems rather strong, it holds in many practical situations where it is known how the control action affects the dynamics. Alternatively, it can be removed by filtering the control action, for instance to remove high frequency components or to impose integral action.

Problem 2 is a very challenging non-convex feasibility problem. In order to obtain tractable relaxations we will make the following assumptions:

A2: The dictionary $\boldsymbol{\phi}$ is polynomial, with bounded order.
A3: The function h that defines the set $\mathcal{C}$ is polynomial.
A4: The CLF $V(.)$ we are searching over is polynomial.
A5: The sets $\mathcal{X}_0$ and $\mathcal{X}_u$ are each defined by a finite number of bounded degree polynomial inequalities (basic semi-algebraic sets).

Since $\mathcal{P}_1$ is a polytope (and hence semi-algebraic), in principle, under these assumptions Problem 2 can be relaxed to a semi-definite program by imposing that V is a Sum-of-Squares (SoS) function and enforcing the conditions (7)–(8) through Putinar's Positivstellensatz [26]. However, this approach quickly becomes intractable, even for small problems due to the following facts: (i) it requires considering polynomials in the variables $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{f}$; and (ii) due to the polynomial dependence of conditions (7)–(8) on $\boldsymbol{f}$, these polynomials will involve high order monomials in these variables.[2] Further, in order to reduce the problem to an SoS, $\boldsymbol{u}$ must be assumed to be a polynomial function of $\boldsymbol{x}$. In turn, this leads to bilinear expressions involving the coefficients of the polynomials $V(.)$ and $\boldsymbol{u}(.)$.

Next, we indicate how to circumvent these difficulties through a combination of lifting and duality. To this effect, we introduce a new (lifting) variable $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$ that satisfies an associated equality constraint (over the flows of (1)):

$$\tilde{\boldsymbol{x}}_k = \boldsymbol{F}\boldsymbol{\phi}(\boldsymbol{x}_k) + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}_k. \tag{9}$$

Using the properties of the Kronecker product, the constraint above can be rewritten in terms of $\boldsymbol{f}$ as:

$$\tilde{\boldsymbol{x}}_k = \big[\boldsymbol{I}\otimes\boldsymbol{\phi}^T(\boldsymbol{x}_k)\big]\boldsymbol{f} + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}_k. \tag{10}$$

We introduce a vector-valued function $\boldsymbol{\rho}(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ to act as a dual multiplier against the equality constraint in (10). Consider now the following set of polynomial inequalities:

$$\begin{aligned} -\big(\boldsymbol{\rho}^T\otimes\boldsymbol{\phi}^T\big)\boldsymbol{f} &\le V(\boldsymbol{x}) - V(\tilde{\boldsymbol{x}}) - c_3\|\boldsymbol{x}\|^q \\ &\quad - \boldsymbol{\rho}^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + \sum_{i=1}^m \boldsymbol{\psi}_i^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{g}_i(\boldsymbol{x}), \\ -\big(\boldsymbol{\rho}^T\otimes\boldsymbol{\phi}^T\big)\boldsymbol{f} &\le h\big(\tilde{\boldsymbol{x}}\big) - \sigma_1(\boldsymbol{x}, \tilde{\boldsymbol{x}}) - \sigma_2(\boldsymbol{x}, \tilde{\boldsymbol{x}})h(\boldsymbol{x}) \\ &\quad - \boldsymbol{\rho}^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + \sum_{i=1}^m \boldsymbol{\psi}_i^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{g}_i(\boldsymbol{x}). \end{aligned} \tag{11}$$

where σ_i are SoS polynomials, $\boldsymbol{\rho}, \boldsymbol{\psi}_i$ are polynomial vector multipliers, and $\boldsymbol{g}_i$ denotes the i^{th} column of the matrix $\boldsymbol{g}$.

Lemma 2: If there exists a positive definite function $V(.)$ such that the inequalities (11) are satisfied for all $\boldsymbol{f} \in \mathcal{P}_1$, then V is a common CLF and the corresponding control action renders $\mathcal{C}$ safe.

Proof: For a given $\boldsymbol{g}$, define the i^{th} control action:

$$u_i(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \begin{cases} \frac{\boldsymbol{\psi}_i^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{g}_i(\boldsymbol{x})}{\boldsymbol{\rho}^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{g}_i(\boldsymbol{x})} & \text{if } \boldsymbol{\rho}^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{g}_i(\boldsymbol{x}) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

[2]In contrast, in the continuous-time case [7], [13] the corresponding conditions are affine in $\boldsymbol{f}$ which can then be eliminated using the Theorems of Alternatives.

The associated flow satisfies

$$\begin{aligned} V(\boldsymbol{x}_k) - V(\boldsymbol{x}_{k+1}) &\geq c_3\|\boldsymbol{x}_k\|^q + \boldsymbol{\rho}^T(\boldsymbol{x}_k, \boldsymbol{x}_{k+1})\boldsymbol{x}_{k+1} - \\ &\quad \big[\boldsymbol{\rho}^T(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}) \otimes \boldsymbol{\phi}^T(\boldsymbol{x}_k)\big]\boldsymbol{f} - \sum_{i=1}^{m} \boldsymbol{\psi}_i^T(\boldsymbol{x}_k, \boldsymbol{x}_{k+1})\boldsymbol{g}_i(\boldsymbol{x}_k) \\ &= c_3\|\boldsymbol{x}_k\|^q + \boldsymbol{\rho}^T(\boldsymbol{x}_k, \boldsymbol{x}_{k+1})(\boldsymbol{x}_{k+1} - \boldsymbol{F}\boldsymbol{\phi}(\boldsymbol{x}_k) - \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}) \\ &= c_3\|\boldsymbol{x}_k\|^q. \end{aligned} \quad (13)$$

A similar reasoning shows that along the flows:

$$h(\boldsymbol{x}_{k+1}) \geq \sigma_1(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}) + \sigma_2(\boldsymbol{x}_k, \boldsymbol{x}_{k+1})h(\boldsymbol{x}_k). \quad (14)$$

Hence

$$\left\{ \max_{\boldsymbol{u}\in\mathbb{R}^m} V(\boldsymbol{x}_k) - V(\boldsymbol{x}_{k+1}) \text{ subject to } h(\boldsymbol{x}_{k+1}) \geq 0 \right\} \geq c_3\|\boldsymbol{x}_k\|^q.$$

■

Remark 1: The acausal controller construction in (12) is strictly for theoretical results to hold. Section III-A develops methods to extract causal controllers using online optimization.

Next, we exploit Lemma 2 and Farkas' Lemma to construct a common CLF for all $\boldsymbol{f} \in \mathcal{P}_1$.

Theorem 1: V is a common CLF for all $\boldsymbol{f} \in \mathcal{P}_1$ if there exists a matrix function $\boldsymbol{Y}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) \in \mathbb{R}^{2\times 2nT} \geq 0$ such that the following (functional) set of affine constraints is feasible:

$$\boldsymbol{Y}(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{N} = \boldsymbol{r}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) \text{ and } \boldsymbol{Y}(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{e} \leq \boldsymbol{d}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) \quad (15)$$

where for notational simplicity we defined

$$\begin{aligned} \boldsymbol{N} &\doteq \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{bmatrix}, \ \boldsymbol{e} \doteq \begin{bmatrix} \epsilon\boldsymbol{1} + \boldsymbol{\xi} \\ \epsilon\boldsymbol{1} - \boldsymbol{\xi} \end{bmatrix}, \ \boldsymbol{r} \doteq \begin{bmatrix} -\big(\boldsymbol{\rho}^T \otimes \boldsymbol{\phi}^T\big) \\ -\big(\boldsymbol{\rho}^T \otimes \boldsymbol{\phi}^T\big) \end{bmatrix} \\ \boldsymbol{d} &\doteq \big[d_1(\boldsymbol{x}, \tilde{\boldsymbol{x}}), \quad d_2(\boldsymbol{x}, \tilde{\boldsymbol{x}})\big]^T, \\ d_1(\boldsymbol{x}, \tilde{\boldsymbol{x}}) &= V(\boldsymbol{x}) - V(\tilde{\boldsymbol{x}}) - c_3\|\boldsymbol{x}\|^q \\ &\quad -\boldsymbol{\rho}^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + \sum_{i=1}^{m} \boldsymbol{\psi}_i^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{g}_i(\boldsymbol{x}), \\ d_2(\boldsymbol{x}, \tilde{\boldsymbol{x}}) &= h(\tilde{\boldsymbol{x}}) - \sigma_1(\boldsymbol{x}, \tilde{\boldsymbol{x}}) - \sigma_2(\boldsymbol{x}, \tilde{\boldsymbol{x}})h(\boldsymbol{x}) \\ &\quad -\boldsymbol{\rho}^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + \sum_{i=1}^{m} \boldsymbol{\psi}_i^T(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{g}_i(\boldsymbol{x}). \end{aligned} \quad (16)$$

Proof: The proof follows from applying the extended Farkas' Lemma to the polytopes $\mathcal{P}_1$ and $\mathcal{P}_2 \doteq \{\boldsymbol{f}: \boldsymbol{r}\boldsymbol{f} \leq \boldsymbol{d}\}$, i.e., (11) holds. ■

When compared against a straightforward application of Putinar's Positivstellensatz to enforce (7)–(8), Theorem 1 has two main advanatages: a reduction in computational complexity, and freeing from the restriction that $\boldsymbol{u}$ be an explicit function of $\boldsymbol{x}$. The complexity reduction is due to the fact that (15) involves multipliers only in the variables $(\boldsymbol{x}, \tilde{\boldsymbol{x}})$, as opposed to $(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{u})$ with $2n < n + d_f + m$. Specifically, the maximal size Gram-matrix in a degree-$2k$ P-satz in (15) with $(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ will have size $\binom{2n+k}{k}$, while a P-satz $(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{u})$ will have dimension $\binom{n+d_f+m+k}{k}$. Using the P-satz will require postulating that $\boldsymbol{u}$ is a polynomial function of $\boldsymbol{x}$, leading to bilinear products between the coefficients of the unknown $V(.)$ and $\boldsymbol{u}(\boldsymbol{x})$. Theorem 1 avoids this by separating the process of finding the CLF $V(.)$ from that of finding the control action, which is not restricted to be polynomial. On the other hand, these advantages are achieved at the price of having to solve an on-line optimization problem to extract a causal control action $\boldsymbol{u}$.

A. Extracting a Controller via On-Line Optimization

As indicated earlier, the control (12) is mostly of theoretical importance, since, at time k it requires knowledge of both $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$. Next, we discuss how to construct a causal control by solving on-line a robust optimization problem. We will consider first the case where V and $-h$ are convex (for example this assumption holds for the usual positive-definite quadratic Lyapunov functions $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$).

Theorem 2: Assume that $V(\boldsymbol{x})$ is convex and that $h(\boldsymbol{x})$ is concave.[3] Let $\boldsymbol{f}_i$, $i = 1, \dots, n_v$ denote the vertices of the polytope $\mathcal{P}_1$ and consider the following convex optimization program:

$$\begin{aligned} &\boldsymbol{u}_k^* = \arg\min \|\boldsymbol{u}\| \text{ subject to:} \\ &V\big(\big[\boldsymbol{I} \otimes \boldsymbol{\phi}^T(\boldsymbol{x}_k)\big]\boldsymbol{f}_i + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}\big) \leq V(\boldsymbol{x}_k) - c_3\|\boldsymbol{x}_k\|^q, \\ &h\big(\big[\boldsymbol{I} \otimes \boldsymbol{\phi}^T(\boldsymbol{x}_k)\big]\boldsymbol{f}_i + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}\big) \geq 0, \ \forall i = 1, \dots, n_v. \end{aligned} \quad (17)$$

Then the control $\boldsymbol{u}^*$ safely stabilizes $\mathcal{P}_1$.

Proof: Follow from the fact that, from Caratheodory's theorem, any $\boldsymbol{f} \in \mathcal{P}_1$ can be written as

$$\boldsymbol{f} = \sum_{i=1}^{n_v} \lambda_i \boldsymbol{f}_i \text{ with } \sum_{i=1}^{n_v} \lambda_i = 1.$$

Since V and $-h$ are convex in $\boldsymbol{f}$, then, for any $\boldsymbol{f} \in \mathcal{P}_1$

$$\begin{aligned} &V\big(\big[\boldsymbol{I} \otimes \boldsymbol{\phi}^T(\boldsymbol{x}_k)\big]\boldsymbol{f} + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}^*\big) \\ &\quad \leq \sum \lambda_i V\big(\big[\boldsymbol{I} \otimes \boldsymbol{\phi}^T(\boldsymbol{x}_k)\big]\boldsymbol{f}_i + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}^*\big) \\ &\quad \leq V(\boldsymbol{x}_k) - c_3\|\boldsymbol{x}_k\|^q. \end{aligned}$$

A similar reasoning shows that, for all $\boldsymbol{f} \in \mathcal{P}_1$

$$h\big(\big[\boldsymbol{I} \otimes \boldsymbol{\phi}^T(\boldsymbol{x}_k)\big]\boldsymbol{f} + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}^*\big) \geq 0.$$

■

B. The General Case

We now consider the general case where $V, -h$ are not necessarily convex. In order to obtain tractable problems we will make the follow additional assumption:

A6: The control action is constrained to $\boldsymbol{u} \in \mathcal{U}$, compact.

Lemma 3: For each $\boldsymbol{x} \in \mathcal{C}$ define the function $\Delta_{\boldsymbol{x}}$ (parametric in $\boldsymbol{x}$)

$$\Delta_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{f}) \doteq V(\boldsymbol{x}) - V\big(\big[\boldsymbol{I} \otimes \boldsymbol{\phi}^T(\boldsymbol{x})\big]\boldsymbol{f} + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}\big) - c_3\|\boldsymbol{x}\|^q \quad (18)$$

and consider the Linear Program (in $p_{\boldsymbol{x}}(.)$):

$$\begin{aligned} p_{\boldsymbol{x}}^*(\boldsymbol{u}) = &\sup_{p\in R[\boldsymbol{u}]} \int_{\mathcal{U}} p_{\boldsymbol{x}}(\boldsymbol{u}) du_1 \dots du_m \\ &\text{subject to } \forall(\boldsymbol{f}, \boldsymbol{u}) \in \mathcal{P}_1 \times \mathcal{U} \\ &\quad p_{\boldsymbol{x}}(\boldsymbol{u}) \leq \Delta_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{f}) \\ &\quad p_{\boldsymbol{x}}(\boldsymbol{u}) \leq h\big(\big[\boldsymbol{I} \otimes \boldsymbol{\phi}^T(\boldsymbol{x})\big]\boldsymbol{f} + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}\big) \end{aligned} \quad (19)$$

[3]This situation arises for instance when the safe region is convex and h is defined by its level sets.

Then

$$p_{\boldsymbol{x}}(\boldsymbol{u}) = \min_{\boldsymbol{f}\in\mathcal{P}_1} \min\left\{\Delta_{\boldsymbol{x}}(\boldsymbol{u},\boldsymbol{f}), h\left(\left[\boldsymbol{I}\otimes\boldsymbol{\phi}^T(\boldsymbol{x})\right]\boldsymbol{f}+\boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}\right)\right\}$$

almost everywhere in $\mathcal{P}_1 \times \mathcal{U}$.

Proof: The proof follows from applying [27, Lemma 2.5 and Corollary 2.6] to the semi-algebraic function

$$z_{\boldsymbol{x}}(\boldsymbol{f},\boldsymbol{u}) \doteq \min\left\{\Delta_{\boldsymbol{x}}(\boldsymbol{u},\boldsymbol{f}), h\left(\left[\boldsymbol{I}\otimes\boldsymbol{\phi}^T(\boldsymbol{x})\right]\boldsymbol{f}+\boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}\right)\right\}$$

over the compact set $\mathcal{P}_1 \times \mathcal{U}$. ■

Lemma 4: The following controller safely stabilizes $\mathcal{P}_1$

$$\boldsymbol{u}^*(\boldsymbol{x}_k) \doteq \arg\max_{\boldsymbol{u}} \; p^*_{\boldsymbol{x}_k}(\boldsymbol{u}) \text{ subject to } \boldsymbol{u}\in\mathcal{U}. \tag{20}$$

Proof: From Lemma 2, $V(.)$ is such that for all $\boldsymbol{x}\in\mathcal{C}$, there exist $\boldsymbol{u}$ such that (7)–(8) hold for all $\boldsymbol{f}\in\mathcal{P}_1$. Since $\boldsymbol{u}^*$ maximizes $\min_{\boldsymbol{f}\in\mathcal{P}_1} z_{\boldsymbol{x}}(\boldsymbol{f},\boldsymbol{u})$ subject to $\boldsymbol{u}\in\mathcal{U}$, it follows that it renders $\Delta(\boldsymbol{x}_k,\boldsymbol{u}^*)\geq 0$ and $h([\boldsymbol{I}\otimes\boldsymbol{\phi}^T(\boldsymbol{x}_k)]\boldsymbol{f}+\boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}^*)\geq 0$ for all $\boldsymbol{f}\in\mathcal{P}_1$. Thus, the safe set $\mathcal{C}$ is invariant and $V(.)$ decreases along the trajectories. ■

C. Finite Dimensional Approximations

In principle, Lemma 3 requires solving an infinite dimensional linear program. However, under the additional assumption that the sets $\mathcal{U}$ is Archimedean, from [27, Th. 3.3] it follows that the degree n truncation of $p^*_n(\boldsymbol{u})$ converges monotonically to p^*. Hence, for some n large enough, $p^*\geq \min_{\boldsymbol{f}} z_{\boldsymbol{x}}(\boldsymbol{f},\boldsymbol{u})-\delta$. The controller (20) therefore renders the system safe, provided that there exists a control action such that $h(\boldsymbol{x}_{k+1})\geq\delta$ for all $h(\boldsymbol{x}_k)\geq 0$. Further, this controller will drive the system to the ball $V(\boldsymbol{x})\leq\delta$.

IV. Numerical Examples

The proposed method is tested on the following three examples. All experiments are implemented in MATLAB 2022b with Yalmip [28] and are solved by Mosek [29]. Code is publicly available.[4]

Example 1: Consider a discrete-time linear system with

$$f(\boldsymbol{x}_k) = \left[x_2; \quad -2x_1-5x_2\right], g=\left[0; \quad 1\right],$$

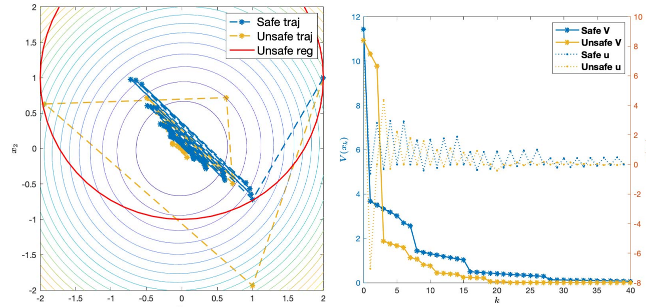
which is open-loop unstable, and a convex safety requirement

$$h(\boldsymbol{x}) = -x_1^2-(x_2-1)^2+4\geq 0.$$

We know as prior knowledge that the system is linear of dimension $n=2$ and that $g=\left[0;1\right]$. 40 noisy datapoints with $\epsilon=0.1$ are collected for the safely stabilizing controller design, yielding a polytope $\mathcal{P}_1$ from (5) with 4 dimensions, 64 (out of 160) nonredundant faces [30], and 36 vertices. Solving Theorem 1 with $c_3=0.1$ for polynomials $\boldsymbol{Y}$, V, $\boldsymbol{\rho}$, $\boldsymbol{\psi}$ of degree ≤ 2 in corresponding dimensions leads to a learned Lyapunov function

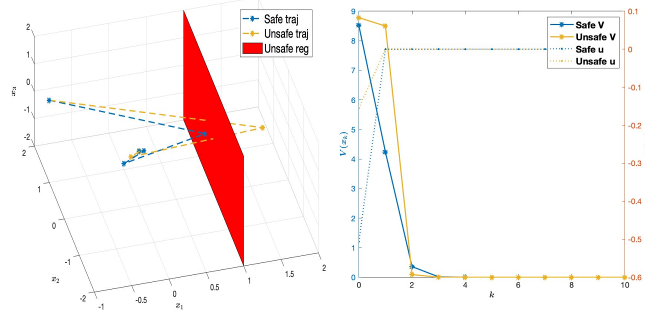
$$V(\boldsymbol{x}) = -0.18x_1x_2+2.39x_1^2+2.24x_2^2.$$

Fig. 1(a) plots 40-step safe/unsafe trajectories in blue/orange starting at $\boldsymbol{x}_0=[2;1]$, and features contour lines of V in colors and unsafe set boundary $h=0$ in red. It clearly illustrates the safety of the closed-loop system under the controller $\boldsymbol{u}$



(a) Safe/unsafe Trajectories (b) V, u along Trajectories

Fig. 1. Results of Example 1.



(a) Safe/unsafe Trajectories (b) V, u along Trajectories

Fig. 2. Results of Example 2.

synthesized using Theorem 2. Note that the unsafe controller (and corresponding V) is designed in the same framework but without considering the safety constraint in Theorem 1 and 2. Fig. 1(b) illustrates the stability from the decreasing trend of Lyapunov traces and control inputs along the trajectories.

Example 2: Consider a 3d discrete-time polynomial system with

$$f(\boldsymbol{x}_k) = \left[x_2^2+x_3; \; x_1x_2^2+x_3; \; 0\right], \quad g=[2; \; -1; \; 1],$$

and a convex safety requirement $h(\boldsymbol{x})=1-x_1\geq 0$.

We know as a prior that f can be represented by the dictionary $\boldsymbol{\phi}=[x_3,x_2^2,x_1x_2^2]$ and that $g=[2;-1;1]$. 8 noisy datapoints with $\epsilon=0.01$ are collected, yielding a polytope $\mathcal{P}_1$ with 9 dimensions, 32 (out of 48) nonredundant faces, and 3584 vertices. Solving Theorem 1 with $c_3=0.01$ for $\boldsymbol{Y}$, V, $\boldsymbol{\rho}$, $\boldsymbol{\psi}$ of degree ≤ 2 leads to a learned Lyapunov function

$$V(\boldsymbol{x}) = -0.00087x_1x_3+2.85x_1^2+2.83x_2^2+2.83x_3^2.$$

Fig. 2(a) plots 10-step safe/unsafe trajectories starting at $\boldsymbol{x}_0=[-1;1;1]$ with designed controller u from Theorem 2. Fig. 2(b) illustrates the decreasing trend of Lyapunov traces along the trajectories.

Example 3: Consider a discrete-time polynomial system

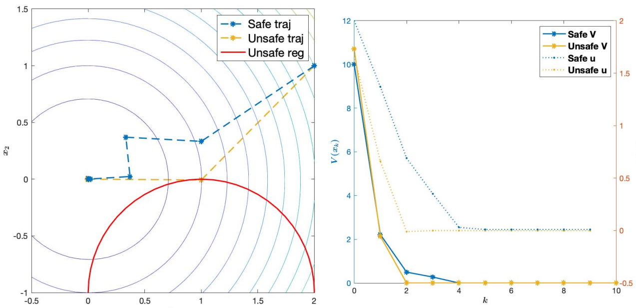
$$f(\boldsymbol{x}_k) = \left[x_2; -x_1+\tfrac{1}{3}x_1^2-x_2\right], \quad g=\left[0; \quad 1\right],$$

with a non-convex safety requirement

$$h(\boldsymbol{x}) = (x_1-1)^2+(x_2+1)^2-1\geq 0.$$

We assume as a prior that f can be represented by the dictionary $\boldsymbol{\phi}=[x_1,x_2,x_1^2]$ and that $g=[0;1]$. 10 noisy

[4] https://github.com/J-mzz/ddc-safety-discrete



(a) Safe/unsafe Trajectories (b) V, u along Trajectories

Fig. 3. Results of Example 3.

datapoints with $\epsilon = 0.01$ are collected for the controller design, yielding a polytope $\mathcal{P}_1$ with 6 dimensions, 22 (out of 40) nonredundant faces, and 60 vertices. Solving Theorem 1 with $c_3 = 0.01$ for $\boldsymbol{Y}, V, \boldsymbol{\rho}, \boldsymbol{\psi}$ of degree ≤ 2 leads to a learned Lyapunov function

$$V(\boldsymbol{x}) = 0.035x_1x_2 + 1.99x_1^2 + 1.99x_2^2.$$

Fig. 3(a) plots 10-step safe/unsafe trajectories starting at $\boldsymbol{x}_0 = [2; 1]$ under the controller $\boldsymbol{u}$ from Lemma 3 and 4 with $\mathcal{U} = [-2, 2]$, and features unsafe set boundary $h = 0$ in red. It illustrates the safety of the closed-loop system with a non-convex safety requirement. Fig. 3(b) illustrates the decreasing trend of Lyapunov traces along the trajectories.

V. Conclusion

Safe stabilization of dynamical systems is generally a challenging problem (when the compatible Lyapunov function must be synthesized), and this difficulty is exacerbated in the discrete-time setting as compared to the continuous-time case. This letter considered safe-stabilization of all possible discrete-time systems consistent with the process-noise corrupted data in $\mathcal{D}$. Control Lyapunov Functions V were generated according to Theorem 1 by solving an SoS program with variables $(\boldsymbol{x}, \tilde{\boldsymbol{x}})$. Feasibility of this program guarantees the existence of a safely stabilizing, albeit acausal, control policy $\boldsymbol{u}(\boldsymbol{x}_k, \boldsymbol{x}_{k+1})$. A causal control $\boldsymbol{u}(\boldsymbol{x}_k)$ can be found by solving an on-line optimization problem.

Future work includes developing methods to relax some of the assumptions imposed in this letter, including synthesis for the case where g is unknown and more general noise models. Other aspects include using duality to reduce the number of constraints in (17).

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