Data-Driven Superstabilizing Control Under Quadratically-Bounded Errors-in-Variables Noise

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Abstract—The Errors-in-Variables model of system identification/control involves nontrivial input and measurement corruption of observed data, resulting in generically nonconvex optimization problems. This letter performs full-state-feedback stabilizing control of all discrete-time linear systems that are consistent with observed data for which the input and measurement noise obey quadratic bounds. Instances of such quadratic bounds include elementwise norm bounds (at each time sample), energy bounds (across the entire signal), and chance constraints arising from (sub)gaussian noise. Superstabilizing controllers are generated through the solution of a sum-of-squares hierarchy of semidefinite programs. A theorem of alternatives is employed to eliminate the input and measurement noise process, thus improving tractability.

Index Terms—Errors-in-variables, data-driven control, robust control, linear systems, sum-of-squares.

I. INTRODUCTION

THIS letter proposes a method to stabilize linear systems corrupted by quadratically bounded Errors-in-Variables (EIV) noise [1]. State and input observations $\mathcal{D} = \{\hat{x}_t, \hat{u}_t\}_{t=1}^T$ are collected along a trajectory of an a-priori unknown n-state m-input linear system $(A_\star, B_\star)$:

$$x_{t+1} = A_\star x_t + B_\star u_t \tag{1a}$$

$$\hat{x}_t = x_t + \Delta x_t, \quad \hat{u}_t = u_t + \Delta u_t, \tag{1b}$$

in which the collected data in $\mathcal{D}$ is corrupted by state noise $\Delta x \in \mathbb{R}^{nT}$ and input noise $\Delta u \in \mathbb{R}^{m(T-1)}$. This letter focuses on the setting where the noise processes $(\Delta x, \Delta u)$ satisfy a collection of L convex quadratic constraints as

$$\forall \ell \in 1..L : \|F_\ell \Delta x + G_\ell \Delta u\|_2 \leq 1, \tag{2}$$

in which the known constraint matrices F_ℓ and G_ℓ have compatible dimensions. These quadratic constraints could arise from deterministic knowledge of $(\Delta x, \Delta u)$, or from high-probability chance-constraints imposed on stochastic processes $(\Delta x, \Delta u)$ if a robust description is overly conservative [2], [3]. Our goal is to find a gain matrix K such that the full-state-feedback control policy $u_t = Kx_t$ can simultaneously stabilize all plants that are consistent with the data in $\mathcal{D}$ under the noise description in (2).

This letter follows the framework of set-membership direct Data Driven Control (DDC). In direct DDC, a control policy is formed from the collected data and modeling assumptions without first performing system identification (and synthesizing a controller for the identified system) [4]. Set-membership DDC has three main ingredients: the set of data-consistent plants (given a noise model), the set of commonly stabilized plants by a designed controller, and the certificate of set-containment that the stabilized-set contains the consistent-set [5]. The Matrix S-Lemma can be used to provide proofs of quadratic and worst-case H_2 or H_∞ robust control when the noise model is defined by a matrix ellipsoid (quadratic matrix inequality) [6], [7]. Farkas-based certificates for polytope-in-polytope membership have been used for robust superstabilization [8] and positive-stabilization [9]. Sum-of-Squares (SOS) certificates of nonnegativity have been employed for stabilization of more general nonlinear systems [10], [11]. We note that other non-set-membership DDC methods include using Virtual Reference Feedback Tuning [12] and Willem's Fundamental Lemma [13].

Most DDC methods focus solely on process-noise corruption, allowing for the synthesis of controllers through the solution of computationally simple convex programs. This letter continues a line of work in addressing the more challenging setting of EIV superstabilization, proposing a method that can handle the more difficult but realistic EIV noise scenario at the cost of more expensive computational requirements. Prior work on superstabilization [14], [15] of EIV systems includes full-state-feedback for polytope-bounded noise [16] (e.g., L_∞-bounds) and dynamic output feedback for SISO plants [17]. In this letter, we will ensure superstabilization

under quadratically bounded noise. This will involve developing matrix-SOS expressions defined for (multiple) quadratic constraints in (2). Computational complexity is reduced by eliminating $(\Delta x, \Delta u)$ using a Theorem of Alternatives [18]. The concurrent and independently developed similar work in [19] performs lossless quadratic stabilization in the presence of a single quadratic-matrix-inequality-representable quadratic constraint, with conservatism added under multiple quadratic constraints.

This letter has the following structure: Section II reviews preliminaries including notation, quadratically constrained noise, superstabilization, and matrix-SOS methods. Section III presents infinite-dimensional Linear Programs (LPs) to perform superstabilization under quadratically-bounded EIV noise, and applies a Theorem of Alternative to eliminate the noise variables $(\Delta x, \Delta u)$ from the linear inequalities. Section IV truncates the infinite-dimensional LPs using the moment-SOS hierarchy of Semidefinite Programs (SDPs), and tabulates computational complexity of different approaches. Section V describes extended superstabilization [20] of data-consistent systems. Section VI demonstrates our method for EIV-tolerant superstabilizing control of example systems. Section VII concludes this letter.

II. PRELIMINARIES

A. Notation

$a..b$	natural numbers between a and b (inclusive)
$\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$)	nonnegative (positive) orthant
π^x	projection operator $(x, y) \rightarrow x$
$A^\top$	Transpose of matrix A
A^+	Pseudoinverse of matrix A
$\mathbb{L}^n$	n-dimensional Second-Order Cone (SOC) $\{(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mid t \geq \|x\|_2\}$
$\mathcal{N}(\mathbf{m}, \Sigma)$	Normal distribution with mean $\mathbf{m}$ and covariance Σ
$\mathbb{R}[x]$	polynomials in indeterminate x
$\deg(h)$	degree of polynomial $h \in \mathbb{R}[x]$
$\mathbb{S}^r[x]$	$r \times r$ symmetric polynomial matrices in x
$\Sigma^r[x]$	SOS symmetric polynomial matrices in x
$\langle A, B \rangle$	Matrix pairing $\text{Tr}(A^\top B) = \sum_{ij} A_{ij} B_{ij}$

B. Quadratic Noise Bounds

This subsection briefly highlights instances of quadratic noise bounds in (2).

1) Elementwise Norm Constraints: A deterministic noise bound could impose that $\forall t \in 1..T : \|\Delta x_t\|_2 \leq \epsilon_x$, and $\forall t \in 1..T-1 : \|\Delta u_t\|_2 \leq \epsilon_u$. Elementwise norm constraints can arise when Δx and Δu are i.i.d. normally distributed random variables ($\Delta x_t \sim \mathcal{N}(\mathbf{0}_n, \Sigma_x)$, $\Delta u_t \sim \mathcal{N}(\mathbf{0}_m, \Sigma_u)$). The statistics $\Delta x_t^T \Sigma_x^{-1} \Delta x_t$ and $\Delta u_t^T \Sigma_u^{-1} \Delta x_t$ are each χ^2-distributed with n and m degrees of freedom respectively.

Let $\varepsilon(\delta; f)$ refer to the quantile statistic of a χ^2 distribution with f degrees of freedom with respect to a probability $\delta \in [0, 1]$ as in $\chi_f^2(p \leq \varepsilon(\delta; f)) = 1 - \delta$ (for a random variable p). For a choice of probabilities $\delta_x, \delta_u \in [0, 1]$, the joint probability of $\Delta x, \Delta u$ lying within the set described by

$$\forall t \in 1..T : \|\Sigma_x^{-1/2} \Delta x_t\|_2 \leq \sqrt{\varepsilon(\delta_x; n)} \tag{3a}$$

$$\forall t \in 1..T-1 : \|\Sigma_u^{-1/2} \Delta u_t\|_2 \leq \sqrt{\varepsilon(\delta_u; m)} \tag{3b}$$

is $P_{\text{joint}} = (1 - \delta_x)^T (1 - \delta_u)^{T-1}$. A controller $u = Kx$ certifiably stabilizes all plants consistent with (1) and (3) will be able to stabilize the true system with probability P_{joint}. Similar elementwise quadratic chance-constraints arise when $\Delta x, \Delta u$ are drawn from i.i.d. subgaussian distributions [21].

2) Energy Constraints: The standard quadratic expression used in a Linear Quadratic Regulator of

$$J = \Delta x_T^\top Q_T \Delta x_T + \left(\sum_{t=1}^{T-1} \Delta x_t^\top Q \Delta x_t + \Delta u_t^\top R \Delta u_t \right) \tag{4}$$

is compatible with the structure of (2) if the cost matrices (Q, R, Q_T) are all Positive Semidefinite (PSD).

C. Superstabilizing Control

Let $W \in \mathbb{R}^{f \times n}$ be a matrix with full column rank such that $\{x \mid \|Wx\|_\infty \leq 1\}$ is a compact set. A discrete-time linear system $x_{t+1} = Ax_t$ is *W-superstable* if $\|Wx\|_\infty$ is a polyhedral Lyapunov function:

$$\|WAW^+\|_\infty < 1 \quad (L_\infty \text{ Operator Norm}). \tag{5}$$

The system is *superstable* if it is W-superstable with $W = I_n$. Any superstable system $x_{t+1} = Ax_t$ obeys the decay bound of $\|x_t\|_\infty \leq \|A\|_\infty^t \|x_0\|_\infty$. This decay bound generalizes to W-superstability as in $\|Wx_t\|_\infty \leq \|WAW^+\|_\infty^t \|Wx_0\|_\infty$. W-superstabilization of the system $x_{t+1} = Ax_t + Bu_t$ proceeds by choosing a gain $K \in \mathbb{R}^{m \times n}$ with $u_t = Kx_t$ such that $\|W(A + BK)W^+\|_\infty < 1$. The W-superstabilization problem of minimizing the decay bound (for fixed W) is a finite-dimensional LP:

$$\lambda^* = \inf_{M,K} \quad \lambda : \quad \sum_{j=1}^f M_{ij} < \lambda \quad \forall i \in 1..f \tag{6a}$$

$$\left| [W(A + BK)W^+]_{ij} \right| \leq M_{ij} \quad \forall i, j \in 1..f. \tag{6b}$$

$$M \in \mathbb{R}^{n \times n}, \quad K \in \mathbb{R}^{m \times n}. \tag{6c}$$

D. Sum-of-Squares Matrices Background

This letter will formulate worst-case-superstabilization problems as infinite-dimensional LPs, which in turn will be truncated using the moment-SOS hierarchy [22]. For any symmetric polynomial-valued-matrix $p \in \mathbb{S}^r[x]$ of size $r \times r$ with indeterminate $x \in \mathbb{R}^n$, the degree of p is the maximum polynomial degree of any one of its entries ($\deg p = \max_{ij} \deg p_{ij}$). A sufficient condition for $p(x) \succeq 0$ over $\mathbb{R}^n$ is if $p(x)$ is an *SOS-matrix*: there exists a polynomial vector $v \in (\mathbb{R}[x])^c$ and a PSD *Gram* matrix $Q \in \mathbb{S}^{rc \times rc}$ such that ([23, Lemma 1])

$$p(x) = (v(x) \otimes I_r)^\top Q(v(x) \otimes I_r). \tag{7}$$

The set of SOS matrices with representation in (7) is $\Sigma^r[x]$, and the subset of SOS matrices with maximal degree $\leq 2k$ is $\Sigma^r[x]_{\leq 2k}$. A constraint region defined by a locus of PSD containments can be constructed from $\{g_j \in \mathbb{S}^{r_j}[x]\}_{j=1}^{N_c}$ as

$$\mathbb{K} = \{x \in \mathbb{R}^n \mid \forall j \in 1..N_c : g_j(x) \succeq 0\}. \tag{8}$$

The matrix $p \in \mathbb{S}^r[x]$ satisfies Polynomial Matrix Inequality (PMI) over the region in (8) if $\forall x \in \mathbb{K}: p(x) \succeq 0$. A sufficient condition for this PMI to hold is that there exist SOS-matrices $\{\sigma_j(x)\}_{j=0}^{N_c}$ and an $\epsilon > 0$ such that (if $r = 1$ or $\forall j: r_j = 1$)

$$p(x) = \sigma_0(x) + \sum_{j=1}^{N_c} g_j(x)\sigma_j(x) + \epsilon I \qquad (9a)$$

$$\sigma_0 \in \Sigma^r[x], \ \forall j \in 1..N_c : \sigma_j \in \Sigma^{r_j}[x], \epsilon > 0. \qquad (9b)$$

The set of matrices in $\mathbb{S}^r[x]$ possessing a representation as in (9a) (existence of $\{\sigma_j\}$) will be written as the Weighted Sum of Squares (WSOS) set $\Sigma^r[\mathbb{K}]$. See [23, Sec. 2] for a generalization of (9) if both $r > 1$ and $\exists j: r_j = 1$. The degree-$2k$-bounded WSOS cone $\Sigma^r[\mathbb{K}]_{\leq 2k}$ is the set of matrices with representation in (9) such that $\deg \sigma_0 \leq 2k$ and $\forall j \in 1..N_c : \deg\langle g_j(x), \sigma_j(x)\rangle \leq 2k$. The representation of $\mathbb{K}$ by polynomial matrices in (8) is *Archimedean* if there exists an $R > 0$ such that the scalar polynomial $p_R(x) = R - \|x\|_2^2$ satisfies $p_R \in \Sigma^r[\mathbb{K}]$. If the representation in (8) is Archimedean, then every $p \in \mathbb{S}^r[x]$ such that $p(x)$ is Positive Definite over K satisfies $(p(x) - \epsilon I_r) \in \Sigma^r[K]$ for some $\epsilon > 0$ (Corollary 1 of [23]). Testing membership of $p \in \Sigma^r[x]_{\leq 2k}$ through (9) can be accomplished by solving Linear Matrix Inequality (LMI) in the Gram matrices for $\{\sigma_j\}$ under coefficient-matching equality constraints. A common choice of polynomial vector $v(x)$ used to represent the SOS matrices in (7) is the vector of all $\binom{n+k}{k}$ monomials of degree $\leq k$. The maximal-size PSD constraint involved in (9) under the monomial choice for $v(x)$ is $r\binom{n+k}{k}$.

III. QUADRATICALLY-BOUNDED LINEAR PROGRAMS

This section presents an infinite-dimensional LP which W-superstabilizes the class of data-consistent plants.

A. Consistency Set

Define $h^0(A, B)$ as the following residual:

$$h_t^0(A, B) = \hat{x}_{t+1} - A\hat{x}_t - B\hat{u}_t \ \forall t \in 1..T-1. \qquad (10)$$

The quantity $h_t^0(A_\star, B_\star)$ will be equal to zero at all $t \in 1..T-1$ if there is no noise present in the system. The joint consistency set of $(A, B, \Delta x, \Delta u)$ compatible with the data in $\mathcal{D}$ can be expressed by defining a matrix-valued function $\Xi(A) : \mathbb{R}^{n \times n} \to \mathbb{R}^{n(T-1) \times nT}$ with

$$\Xi(A) = [(I_{T-1} \otimes A), \mathbf{0}_{n(T-1) \times n}] + [\mathbf{0}_{n(T-1) \times n}, -I_{n(T-1)}],$$

letting n_ℓ be the column dimension of F_ℓ and G_ℓ for each ℓ ($F_\ell \in \mathbb{R}^{n_\ell \times T}$, $G_\ell \in \mathbb{R}^{n_\ell \times m(T-1)}$), and denoting $h^0(A, B)$ as the column-wise vectorization of $\{h_t^0(A, B)\}_{t=1}^{T-1}$. This consistency set can be described by [24]

$$\bar{\mathcal{P}}(A, B, \Delta x, \Delta u) \qquad (11)$$
$$:= \left\{ \begin{array}{l} (A, B, \Delta x, \Delta u) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{nT} \times \mathbb{R}^{m(T-1)} \mid \\ \Xi(A)\Delta x + (I_{T-1} \otimes B)\Delta u + h^0(A, B) = 0 \\ \forall \ell \in 1..L \ (F_\ell \Delta x + G_\ell \Delta u, 1) \in \mathbb{L}^{n_\ell} \end{array} \right\}.$$

The consistency set $\mathcal{P}(A, B) \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ of plants compatible with the data in $\mathcal{D}$ is the following projection,

$$\mathcal{P}(A, B) := \pi^{A,B}\bar{\mathcal{P}}(A, B, \Delta x, \Delta u), \qquad (12)$$

in which there exists bounded noise $(\Delta x, \Delta u)$ following (2) such that $\mathcal{D}$ could have been observed. Our task is as follows:

Problem 1: Find a matrix $K \in \mathbb{R}^{m \times n}$ such that full-state-feedback controller $u_t = Kx_t$ ensures that $A + BK$ is W-superstable for all $(A, B) \in \mathcal{P}$.

B. Full Linear Program

Let $\eta > 0$ be chosen as a margin to ensure stability.

Theorem 1: W-Superstability through a common K (for Problem 1) can be ensured by solving the following infinite-dimensional LP in terms of a matrix $K \in \mathbb{R}^{m \times n}$ and a matrix-valued function $M(A, B) : \mathcal{P} \to \mathbb{R}^{f \times f}$ ([25, eq. (12)]):

$$\underset{M,K}{\text{find}} \text{ s.t. } \forall(A, B, \Delta x, \Delta u) \in \bar{\mathcal{P}}: \qquad (13a)$$

$$\forall i = 1..f: \qquad (13b)$$

$$1 - \eta - \sum_{j=1}^n M_{ij}(A, B) \geq 0$$

$$\forall i = 1..f, \ j = 1..f: \qquad (13c)$$

$$M_{ij}(A, B) - \left(W(A + BK)W^+\right)_{ij} \geq 0$$

$$M_{ij}(A, B) + \left(W(A + BK)W^+\right)_{ij} \geq 0$$

$$M(A, B) : \mathcal{P} \to \mathbb{R}^{f \times f}, \ K \in \mathbb{R}^{m \times n}. \qquad (13d)$$

Proof: The W-superstabilizing linear system in (6) is fulfilled for each $(A, B) \in \mathcal{P}$, as expressed by the $\forall(A, B, \Delta x, \Delta u) \in \bar{\mathcal{P}}$ quantification in (13a). ∎

Remark 1: Under the assumption that $\bar{\mathcal{P}}$ is compact, the function M can be selected to be continuous ([25, Lemma 3.1]) and even polynomial ([25, Lemma 3.2]). The same type of linear system structure for superstabilization in (13d)-(13d) is used (with $W = I_n$) as in [25], but the set $\bar{\mathcal{P}}$ in [25] is defined by polytopic-bounded noise rather than quadratically-bounded noise (11).

C. Robustified Linear Program

Problem (13) involves LP with $2f^2 + f$ infinite-dimensional linear constraints, each posed over the $n(n + T) + m(n + T - 1)$ variables $(A, B, \Delta x, \Delta u)$. Reducing the number of variables involved in any quantification will simplify resultant SOS truncations and increase computational tractability. This subsection will use a Theorem of Alternatives in order to eliminate the noise variables $(\Delta x, \Delta u)$ from Program (13). The maximum number of variables appearing in any infinite-dimensional linear inequality constraint will subsequently drop from $n(n + T) + m(n + T - 1)$ down to $n(n + m)$, which no longer depends on the number of samples T.

Let $q(A, B) : \mathcal{P} \to \mathbb{R}$ be a function that is independent of $(\Delta x, \Delta u)$, such as any one of the left-hand-side elements for constraints (13b)-(13c). The following pair of problems are strong alternatives (exactly one is feasible):

$$\forall(A, B, \Delta x, \Delta u) \in \bar{\mathcal{P}} : q(A, B) \geq 0 \qquad (14a)$$

$$\exists(A, B, \Delta x, \Delta u) \in \bar{\mathcal{P}} : q(A, B) < 0, \qquad (14b)$$

because if $q \geq 0$ for all $(A, B, \Delta x, \Delta u) \in \bar{\mathcal{P}}$ (14a), then there cannot be a point in $\bar{\mathcal{P}}$ where $q(A, B, \Delta x, \Delta u) < 0$ (14b). We can define the following dual variable functions

$$\mu: \mathcal{P} \to \mathbb{R}^{n(T-1)} \qquad (15a)$$

$$(s_\ell, \tau_\ell): \mathcal{P} \to \mathbb{L}^{n_\ell} \forall \ell \in 1..L. \tag{15b}$$

The (A, B) dependence of (μ, s, τ) will be omitted to simplify notation. A weighted sum $\Phi(A, B, \Delta x, \Delta u; \mu, s, \tau)$ may be constructed from q and (μ, s, τ) from (15):

$$\Phi = -q + \mu^\top \left(\Xi(A)\Delta x + (I_{T-1} \otimes B)\, \Delta u + h^0 \right)$$
$$+ \sum_{\ell=1}^{L} \left(\tau_\ell - s_\ell^\top (F_\ell \Delta x + G_\ell \Delta u) \right). \tag{16}$$

The terms in Φ may be rearranged into

$$\Phi = \left(-q + \mu^\top h^0 + \sum_{\ell=1}^{L} \tau_\ell \right)$$
$$+ \left(\Xi(A)^\top \mu - \sum_{\ell=1}^{L} F_\ell^\top s_\ell \right)^\top \Delta x$$
$$+ \left((I_{T-1} \otimes B^\top)\mu - \sum_{\ell=1}^{L} G_\ell^\top s_\ell \right)^\top \Delta u. \tag{17}$$

Expressing the $(\Delta x, \Delta u)$-constant terms of Φ as Q with

$$Q = -q + \mu^\top h^0 + \sum_{\ell=1}^{L} \tau_\ell, \tag{18}$$

the supremal value of Φ w.r.t. $\Delta x, \Delta u$ has a value of

$$\sup_{\Delta x, \Delta u} \Phi = \begin{cases} Q & \text{if } 0 = \Xi(A)^\top \mu - \sum_{\ell=1}^{L} F_\ell^\top s_\ell \\ & \text{if } 0 = (I_{T-1} \otimes B^\top)\mu - \sum_{\ell=1}^{L} G_\ell^\top s_\ell \\ \infty & \text{otherwise.} \end{cases} \tag{19}$$

Theorem 2: Problem (14a) will have the same feasibility (or infeasibility) status as the following program:

$$\text{find} \atop {\mu, s, \tau} \quad \text{s.t.} \quad \forall (A, B) \in \mathcal{P}: \tag{20a}$$

$$q - \sum_{\ell=1}^{L} \tau_\ell - \mu^\top h^0 \geq 0 \tag{20b}$$

$$\Xi(A)^\top \mu - \sum_{\ell=1}^{L} F_\ell^\top s_\ell = 0 \tag{20c}$$

$$\left(I_{T-1} \otimes B^\top \right)\mu - \sum_{\ell=1}^{L} G_\ell^\top s_\ell = 0 \tag{20d}$$

$$(\mu, s, \tau) \text{ from (15).} \tag{20e}$$

Proof: This relationship holds using the convex-duality based Theorem of Alternatives from [18, Sec. V-H], given that all description expressions in q and $\bar{\mathcal{P}}$ are affine in the uncertain terms $(\Delta x, \Delta u)$. ∎

Theorem 3: If $\mathcal{P}$ is compact, then the multipliers (μ, s, τ) can be chosen to be polynomial functions of (A, B).

Proof: The proof is omitted due to page limitations, but follows from [26, Th. 3.3] (generalizing [16, Ths. 4.4 and 4.5] to the conically constrained case). ∎

Remark 2: In the case of probabilistic noise set from (3) with $L = 2T - 1$, the multipliers (s, τ) can be partitioned into $\forall t \in 1..T : (s_t^x, \tau_t^x)$ for (3a) and $\forall t \in 1..T-1 : (s_t^u, \tau_t^u)$ for (3b). The certificate (20) can then be expressed as

$$\text{find} \atop {\mu, s, \tau} \quad \text{s.t.} \quad \forall (A, B) \in \mathcal{P}: \tag{21a}$$

$$q - \sum_{\ell=1}^{T} \sqrt{\varepsilon(\delta_x; n)} \tau_\ell^x + \sum_{\ell=1}^{T-1} \sqrt{\varepsilon(\delta_u; m)} \tau_\ell^u$$
$$- \mu^\top h^0 \geq 0 \tag{21b}$$

$$\Sigma_x^{-1/2} s_1^x = A^\top \mu_1 \tag{21c}$$

$$\forall t = 2..T - 1 : \Sigma_x^{-1/2} s_t^x = A^\top \mu_t - \mu_{t-1} \tag{21d}$$

$$\Sigma_x^{-1/2} s_T^x = -\mu_{T-1} \tag{21e}$$

$$\forall t \in 1..T - 1 : \Sigma_u^{-1/2} s_t^u = B^\top \mu_t \tag{21f}$$

$$(\mu, s, \tau) \text{ from (15).} \tag{21g}$$

IV. TRUNCATED SUM-OF-SQUARES PROGRAMS

This section uses the moment-SOS hierarchy of SDPs to discretize the infinite-dimensional LP in (13) into finite-dimensional convex optimization problems that are more amenable to computation. This discretization will be performed by SOS-matrix truncations.

A. Quadratically-Bounded Truncations

Program (13) has $2f^2 + f$ infinite-dimensional LP constraints posed over $(A, B, \Delta x, \Delta u)$. This subsection applies a degree $2k$ SOS tightening to the constraints in program (13). In each case, the matrix M is restricted to a polynomial $M \in (\mathbb{R}[A, B]_{\leq 2k})^{n \times n}$. The remainder of this section will analyze the computational complexity of an SOS tightening on a single infinite-dimensional constraint from (13) (represented as $q(A, B) \geq 0$ from (14a)). Complexity will be compared according to the PSD Gram matrix of maximal size.

1) Full Program: The full program applies a scalar WSOS constraint $q \in \Sigma^1[\bar{\mathcal{P}}]$ over $(A, B, \Delta x, \Delta u)$. The size of the maximal Gram matrix for each (14a) restriction is

$$p_F = \binom{n(n+T) + m(n+T-1) + k}{k}. \tag{22}$$

2) Alternatives (Dense): The SOC constraint in (15b) can also be expressed by a PSD constraint [27, page 7],

$$(s_\ell, \tau_\ell) \in \mathbb{L}^{n_\ell} \Leftrightarrow \begin{bmatrix} \tau_\ell & s_\ell^\top \\ s_\ell & \tau_\ell I_{n_\ell} \end{bmatrix} \in \mathbb{S}_+^{n_\ell + 1}. \tag{23}$$

In order to prove convergence of the Alternative truncations as the degree $k \to \infty$, we must assume that there exists a known Archimedean set $\Pi(A, B)$ such that $\Pi \supseteq \mathcal{P}$. Such a Π might be known from norm or Lipschitz bounds on (A, B), or other similar knowledge on reasonable plant behavior. If Π is a-priori unknown, then we will use the WSOS symbol $\Sigma^r[\Pi]$ to refer to the SOS set $\Sigma^r[x]$. At the degree-k truncation, the multipliers from (15) can be chosen:

$$\mu \in \left(\mathbb{R}[\Pi]_{\leq 2k-1} \right)^{nT} \tag{24a}$$

$$s_\ell \in \left(\mathbb{R}[\Pi]_{\leq 2k} \right)^{n_\ell} \quad \forall \ell \in 1..L \tag{24b}$$

$$\tau_\ell \in \mathbb{R}[\Pi]_{\leq 2k} \quad \forall \ell \in 1..L. \tag{24c}$$

These multipliers are required to satisfy:

$$\begin{bmatrix} \tau_\ell(A, B) & s_\ell(A, B)^\top \\ s_\ell(A, B) & \tau_\ell(A, B)I_n \end{bmatrix} \in \Sigma^{n_\ell + 1}[\Pi]_{\leq 2k} \quad \forall \ell \in 1..L. \tag{24d}$$

The matrix WSOS constraint in (24d) has PSD Gram matrix of maximal size

$$p_A^\ell = \left(n^\ell + 1 \right) \binom{n(n+m) + k}{k}. \tag{25}$$

The 'dense' nomenclature for this approach will refer to the imposition that (24d) is WSOS over a matrix of size $n_\ell + 1$ for each $\ell = 1..L$.

3) Alternatives (Sparse): The SOC constraint in (23) can be decomposed into 2×2 blocks as in [28, Th. 2.3]:

$$(s_\ell, \tau_\ell) \in \mathbb{L}^{n_\ell} \Leftrightarrow \exists z_{i\ell} : \begin{bmatrix} \tau_\ell & s_{i\ell} \\ s_{i\ell} & z_{i\ell} \end{bmatrix} \in \mathbb{S}_+^2, \quad \tau_\ell = \sum_{i=1}^n z_{i\ell}.$$

The multipliers in (15) can be restricted to

$$\mu \in \left(\mathbb{R}[\Pi]_{\leq 2k-1} \right)^{nT} \tag{26a}$$

$$s_\ell, \, z_\ell \in \left(\mathbb{R}[\Pi]_{\leq 2k} \right)^{n_\ell} \quad \forall \ell \in 1..L \tag{26b}$$

subject to the constraint

$$\begin{bmatrix} \sum_{i=1}^{n_\ell} z_{i\ell}(A, B) & s_{i\ell}(A, B) \\ s_{i\ell}(A, B) & z_{i\ell}(A, B) \end{bmatrix} \in \Sigma^2[\Pi]_{\leq 2k} \, \forall \ell = 1..L. \tag{26c}$$

The Gram matrices from (26c) have a reduced size of

$$p_B = 2 \binom{n(n+m)+k}{k}. \tag{27}$$

Remark 3: The cone constraints (24d) and (26c) are not necessarily equivalent at finite degree $2k$, although they will describe the same set as $k \to \infty$ given that (s, τ) are optimization variables. Refer to [29] for more details on the relationship between matrix SOC cone representations.

B. Computational Complexity

Computational complexity of the Full, Alternatives (Dense), and Alternatives (Decomposed) schemes will be judged comparing the sizes and multiplicities of the largest PSD constraint in any $\bar{\mathcal{P}}$-nonnegativity constraint. As a reminder, the superstabilizing program in (13) has $2n^2 + n$ such $\bar{\mathcal{P}}$-nonnegativity linear inequality constraints.

For each $\bar{\mathcal{P}}$-nonnegativity constraint, the Full program requires 1 block of maximal size p_F from (22). The Alternatives programs require $L+1$ PSD blocks each, in which the Dense program has block sizes of p_A^ℓ from (25) for $\ell = 1..L$ (and a scalar block of size $\binom{n(n+m)+k}{k}$). The sparse program has all L blocks of size p_B from (27) along with a similar scalar block of size $\binom{n(n+m)+k}{k}$. Table I reports the size of the largest PSD matrix constraint for the three approaches under $n = 2, m = 2, k = 2, T = 12$. The considered quadratically-bounding constraints all have $L = 2$, such as in the elementwise-norm $\|\Delta x_t\|_2 \leq \epsilon_x$ and $\|\Delta u_t\|_2 \leq \epsilon_u$ from Section II-B1. We first note that the per-iteration complexity of solving an SDP using an interior point method (with PSD size N having M affine constraints) is $O(N^3 M + N^2 M^2)$ [30]. In the context of Table I, the size of $N = p_F = 1540$ is intractably large for current interior-point methods. The 'Multiplicity' scaling for Dense and Sparse causes computational complexity to grow linearly, while the increasing 'Size' parameter causes polynomial growth in scaling.

V. EXTENDED SUPERSTABILITY

The superstabilization method considered in Sections III and IV rely on the use of a previously given and fixed W matrix. The framework of Extended Superstability [20]

	Size	Multiplicity
Full	$p_F = 1540$	1
Dense	$p_A^\ell = 135$	46
Sparse	$p_B = 90$	46

allows for W to be chosen as a positive-definite n-dimensional diagonal matrix, in which this diagonal-restricted W may be searched over in optimization and is not fixed in advance. The resultant common Lyapunov function $\|Wx\|_\infty$ therefore has hyper-rectangular sublevel sets. Letting $v \in \mathbb{R}^n$, $v > 0$ be a positive vector with matrix $W = \text{diag}(v)^{-1}$, and $x_{t+1} = Ax_t$ be a dynamical system, the W-superstabilization condition $\|WAW^+\|_\infty < 1$ may be expressed as

$$\forall i = 1..n, j = 1..n \, [A \, \text{diag}(v)]_{ij} \leq M_{ij} \tag{28a}$$

$$\forall i = 1..n \quad \sum_{j=1}^n M_{ij} < v_i. \tag{28b}$$

The quadratically-bounded EIV extended superstabilization task involves the following optimization problem with variables $M(A, B) : \mathcal{P} \to \mathbb{R}^{n \times n}, S \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n$:

$$\underset{M, \, S, \, v}{\text{find}} \quad \text{s.t.} \quad \forall (A, B) \in \mathcal{P}: \tag{29a}$$

$$\forall i \in 1..n: \tag{29b}$$

$$v_i > 0, \, \sum_{j=1}^n M_{ij}(A, B) \, < v_i$$

$$\forall i \in 1..n, \, j \in 1..n: \tag{29c}$$

$$|A_{ij}v_j + \sum_{\ell=1}^m B_{i\ell}S_{\ell j}| \leq M_{ij}(A, B)$$

$$M : \mathcal{P} \to \mathbb{R}^{n \times n}, \, S \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n. \tag{29d}$$

If Program (29) is feasible, then the controller K recovered by $K = SW = S(\text{diag}(v))^{-1}$ is guaranteed to W-superstabilize all systems in $\mathcal{P}$. All three methods (Full, Dense, Sparse) for WSOS truncation can be employed to formulate order-k versions of Program (29) (w.r.t. a Psatz in $(A, B, \Delta x, \Delta u)$ for Full or (A, B) for Alternatives) by choosing $M \in (\mathbb{R}[A, B]_{\leq 2k})^{n \times n}$.

VI. NUMERICAL EXAMPLE

Code to generate the following experiment is available at https://github.com/Jarmill/eiv_quad. The SDPs deriving from the SOS programs were synthesized through JuMP [31] and solved using Mosek 10.1 [32].

A. White Noise

This example involves a 2-state 2-input ground-truth plant:

$$A_\star = \begin{bmatrix} 0.6863 & 0.3968 \\ -0.3456 & 1.0388 \end{bmatrix}, \, B_\star = \begin{bmatrix} 0.4170 & 0 \\ 0.7203 & -0.3023 \end{bmatrix}. \tag{30}$$

Data $\mathcal{D}$ is collected from an execution of (30) over a time-horizon of $T = 13$. In the data collection, the EIV noise $\Delta x, \Delta u$ are i.i.d. distributed according to $\Delta x_t \sim \mathcal{N}(0, 0.03^2 I_2), \Delta u_t \sim \mathcal{N}(0, 0.025^2 I_2)$. It is desired to create an extended superstabilizing controller that will succeed in regulating the ground-truth system with joint probability $P_{\text{joint}} = 95\%$. The per-noise probability is chosen as $\delta_x =$

$\delta_u = (0.95)^{1/(2T-1)} = 0.9981$. The (δ_x, δ_u) chance constraint is modeled as $\|\Delta x_t\|_2 \leq 0.03\epsilon(\delta_x; 2) = 0.1056$ and $\|\Delta u_t\|_2 \leq 0.025\epsilon(\delta_u; 2) = 0.08796$ for each t.

Extended superstabilizing control is performed to minimize λ such that $\forall i : \lambda \geq v_i$. The $k = 1$ truncation of (29) using the dense Alternatives method (24) ($p_A = 27$) returns

$$K_{\text{dense}} = \begin{bmatrix} -0.9000 & -0.8807 \\ 0.1564 & 0.3679 \end{bmatrix} \quad v = \begin{bmatrix} 0.5519 \\ 1.4481 \end{bmatrix}. \quad (31)$$

The Full (13), Dense (24), and Sparse (26) superstabilization ($W = I$) programs are all infeasible for $k = 1$. Attempting execution of the $k = 2$ tightening for the Full program ($p_F = \binom{64}{2} = 2016$) results in out-of-memory errors in Mosek. For this specific example, the algorithms from [19, Ths. 1 and 2] both fail to find a common quadratically stabilizing controller.

B. Monte Carlo Test

This second example involves a Monte Carlo test for robust stabilization of 300 randomly generated 2-state 2-input ground-truth plants. Elements of the plant matrices (A, B) are each i.i.d. drawn from a unit normal distribution. The noise Δx_t and Δu_t are drawn i.i.d. uniformly from unit L_2-balls of radius $\epsilon_x = 0.225$ and $\epsilon_u = 0.1$ respectively for a time horizon of $T = 14$. At $k = 1$, dense and sparse superstable SOS restrictions stabilized 48 and 42 systems respectively. Dense and sparse extended superstabilization $k = 1$ SOS restrictions stabilized 71 and 62 systems respectively. [19, Th. 1] was infeasible at each instance. [19, Th. 2] stabilized 61 systems, with an overlap of 31 stabilized systems with the $k = 1$ dense superstability, and 38 systems with the $k = 1$ dense extended superstability.

VII. CONCLUSION

This letter presented a solution approach for data-driven superstabilization in the quadratically-bounded EIV setting. The W-superstabilization problem was formulated as an infinite-dimensional linear program, and was discretized using SOS-matrix methods. A Theorem of Alternative was used to eliminate the $(\Delta x, \Delta u)$ noise terms, resulting in matrix SOS constraints with lower computational complexity.

Future work involves reducing conservatism of EIV-aware control methods and incorporating streaming data for EIV-tolerant model predictive control.

REFERENCES

[1] T. Söderström, *Errors-in-Variables Methods in System Identification.* Cham, Switzerland: Springer, 2018. [Online]. Available: https://link. springer.com/book/10.1007/978-3-319-75001-9

[2] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski, *Robust Optimization*, vol. 28. Princeton, NJ, USA: Princeton Univ. Press, 2009.

[3] T. Martin, T. B. Schön, and F. Allgöwer, "Gaussian inference for data-driven state-feedback design of nonlinear systems," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 4796–4803, 2023.

[4] Z.-S. Hou and Z. Wang, "From model-based control to data-driven control: Survey, classification and perspective," *Inf. Sci.*, vol. 235, pp. 3–35, Jun. 2013.

[5] T. Martin, T. B. Schön, and F. Allgöwer, "Guarantees for data-driven control of nonlinear systems using semidefinite programming: A survey," *Annu. Rev. Control*, vol. 56, Dec. 2023, Art. no. 100911.

[6] H. J. van Waarde, M. K. Camlibel, and M. Mesbahi, "From noisy data to feedback controllers: Nonconservative design via a matrix S-Lemma," *IEEE Trans. Automat. Control*, vol. 67, no. 1, pp. 162–175, Jan. 2022.

[7] A. Bisoffi, C. De Persis, and P. Tesi, "Trade-offs in learning controllers from noisy data," *Syst. Control Lett.*, vol. 154, Aug. 2021, Art. no. 104985.

[8] T. Dai and M. Sznaier, "Data driven robust superstable control of switched systems," *IFAC-PapersOnLine*, vol. 51, no. 25, pp. 402–408, 2018.

[9] J. Miller, T. Dai, M. Sznaier, and B. Shafai, "Data-driven control of positive linear systems using linear programming," in *Proc. 62nd IEEE Conf. Decis. Control (CDC)*, 2023, pp. 1588–1594.

[10] T. Dai and M. Sznaier, "A semi-algebraic optimization approach to data-driven control of continuous-time nonlinear systems," *IEEE Control Syst. Lett.*, vol. 5, no. 2, pp. 487–492, Apr. 2021.

[11] T. Martin and F. Allgöwer, "Data-driven system analysis of nonlinear systems using polynomial approximation," 2021, *arXiv:2108.11298*.

[12] M. C. Campi, A. Lecchini, and S. M. Savaresi, "Virtual reference feedback tuning: A direct method for the design of feedback controllers," *Automatica*, vol. 38, no. 8, pp. 1337–1346, 2002.

[13] J. Coulson, J. Lygeros, and F. Dörfler, "Data-enabled predictive control: In the shallows of the DeePC," in *Proc. 18th Eur. Control Conf. (ECC)*, 2019, pp. 307–312.

[14] M. Sznaier, R. Suárez, S. Miani, and J. Alvarez-Ramírez, "Optimal l_∞ disturbance rejection and global Stabilization of linear systems with saturating control," *IFAC Proc. Vol.*, vol. 29, no. 1, pp. 3550–3555, 1996.

[15] B. T. Polyak and P. S. Shcherbakov, "Superstable linear control systems. I. Analysis," *Autom. Remote Control*, vol. 63, no. 8, pp. 1239–1254, 2002.

[16] J. Miller, T. Dai, and M. Sznaier, "Data-driven superstabilizing control of error-in-variables discrete-time linear systems," in *Proc. IEEE 61st Conf. Decis. Control (CDC)*, 2022, pp. 4924–4929.

[17] J. Miller, T. Dai, and M. Sznaier, "Superstabilizing control of discrete-time ARX models under error in variables," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 2444–2449, 2023.

[18] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge, U.K.: Cambridge Univ. Press, 2004.

[19] A. Bisoffi, L. Li, C. D. Persis, and N. Monshizadeh, "Controller synthesis for input-state data with measurement errors," 2024, *arXiv:2402.04157*.

[20] B. T. Polyak, "Extended superstability in control theory," *Autom. Remote Control*, vol. 65, no. 4, pp. 567–576, 2004.

[21] D. Hsu, S. Kakade, and T. Zhang, "A tail inequality for quadratic forms of subgaussian random vectors," *Electron. Commun. Probab.*, vol. 17, pp. 1–6, Nov. 2012.

[22] J. B. Lasserre, *Moments, Positive Polynomials And Their Applications* (Imperial college press optimization series). Singapore: World Sci. Publ. Co., 2009.

[23] C. W. Scherer and C. W. Hol, "Matrix sum-of-squares relaxations for robust semi-definite programs," *Math. Program.*, vol. 107, no. 1, pp. 189–211, 2006.

[24] V. Cerone, "Feasible parameter set for linear models with bounded errors in all variables," *Automatica*, vol. 29, no. 6, pp. 1551–1555, 1993.

[25] J. Miller, T. Dai, and M. Sznaier, "Data-driven Stabilizing and robust control of discrete-time linear systems with error in variables," 2023, *arXiv:2210.13430*.

[26] J. Miller and M. Sznaier, "Analysis and control of input-affine dynamical systems using infinite-dimensional robust counterparts," 2023, *arXiv:2112.14838*.

[27] F. Alizadeh and D. Goldfarb, "Second-order cone programming," *Math. Program.*, vol. 95, no. 1, pp. 3–51, 2003.

[28] J. Agler, W. Helton, S. McCullough, and L. Rodman, "Positive Semidefinite matrices with a given sparsity pattern," *Linear Algebra its Appl.*, vol. 107, pp. 101–149, Aug. 1988.

[29] D. Papp and F. Alizadeh, "Semidefinite characterization of sum-of-squares cones in algebras," *SIAM J. Optim.*, vol. 23, no. 3, pp. 1398–1423, 2013.

[30] F. Alizadeh, "Interior point methods in semidefinite programming with applications to combinatorial optimization," *SIAM J. Optim.*, vol. 5, no. 1, pp. 13–51, 1995.

[31] M. Lubin, O. Dowson, J. D. Garcia, J. Huchette, B. Legat, and J. P. Vielma, "JuMP 1.0: Recent improvements to a modeling language for mathematical optimization," *Math. Program. Comput.*, vol. 15, pp. 581–589, Sep. 2023.

[32] *The MOSEK Optimization Toolbox for MATLAB Manual, Version 10.1.*, MOSEK ApS, Copenhagen, Denmark, 2023.