Tractable Approximations of LMI Robust Feasibility Sets

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***Abstract*—In this letter, we introduce novel tractable approximations for robust Linear Matrix Inequality (LMI) problems. We present various Quadratic Matrix Inequalities (QMIs) that enable us to characterize the effect of ellipsoidal uncertainty in the robust problem. These formulations are expressed in terms of a set of auxiliary decision variables, which facilitate the derivation of a generalized S-procedure result. This generalization significantly reduces the conservatism of the obtained results, compared with conventional approaches.**

***Index Terms*—Linear matrix inequalities, quadratic matrix inequality, robust semidefinite programming.**

I. Introduction

It IS well known that Linear Matrix Inequalities (LMIs), play a central role in the analysis and design of control systems under the presence of uncertainty [1], [2], [3]. In this context, a robust analysis/synthesis problem consists in solving a feasibility/minimization problem on the decision variables, codified in matrix $X \in \mathcal{X}$ subject to the robust linear matrix inequality in X, i.e., $L(X, \mathbf{w}) \prec 0,\ \forall \mathbf{w} \in \mathcal{W}$, where $\mathbf{w} \in \mathcal{W}$ represents the uncertainty on the model of the system, possible disturbances, noise, etc. Set $\mathcal{X}$ serves to impose the dimension, structure, and additional hard constraints on the decision variable X. This problem, often intractable from a computational point of view [2], [4], [5] is said to be of semi-infinite nature because there is a finite number of decision variables, but an infinite number of constraints. In this letter, we are interested in characterizing the subset of feasible solutions $S_{\mathcal{X}} \doteq \{ X \in \mathcal{X} : L(X, \mathbf{w}) \prec 0,\ \forall \mathbf{w} \in \mathcal{W} \}$, by a reduced number of LMIs that do not depend on $\mathbf{w}$.

In the literature, one often encounters formulations in which $L(X, \mathbf{w})$ depends in an affine way on $\mathbf{w}$, which is constrained to a polytopic set, i.e., $\mathbf{w} \in \mathcal{W} = \operatorname{conv}\{\mathbf{w}_k, k = 1, \ldots, N_w\}$. In this situation, the original robust constraint on X is equivalent [1] to $L(X, \mathbf{w}_k) \prec 0,\ k = 1, \ldots, N_w$.

These assumptions offer a manageable representation of the feasible set $S_{\mathcal{X}}$, only when the number of vertices N_w is not excessively large. For instance, if $\mathcal{W}$ represents matrix interval uncertainty, the number of required vertices explodes exponentially with the dimension of the uncertain matrices, and alternative strategies are required [4], [6], [7]. Another formulation is to consider that $L(X, \mathbf{w})$ exhibits a linear fractional dependence on $\mathbf{w}$, which is constrained to have a block diagonal structure [2]. Each block within this structure is required to have a bounded induced matrix norm, that is

$$\mathcal{W} = \left\{ \operatorname{diag}\left[\Delta_1, \ldots, \Delta_m\right] : \bar{\sigma}(\Delta_i) < 1,\ i = 1, \ldots, m \right\}.$$

In this letter we concentrate on a special case of this framework, in which the dependence with respect to $\mathbf{w}$ is quadratic and $\mathcal{W}$ is defined as an ellipsoidal set, or as the intersection of a finite number of such sets. Our choice is driven by several motivations, which we discuss below. Within the context of system identification, it is possible, under some assumptions on the exciting signals and on the noise/disturbances affecting the system, to employ identification schemes to obtain not only a central estimate for the system's parameters, but also an ellipsoidal bound on them, see, e.g., [8]. Moreover, in order to address complex, interconnected systems, in which each subsystem has a set of parameters, it becomes essential to treat $\mathbf{w}$ as an aggregate of all parameters across systems, with $\mathcal{W}$ the intersection of the ellipsoids corresponding to each subsystem.

The S-lemma [9], [10] provides a powerful tool to convert robustness conditions into a set of quadratic forms. In [11], see also [12], an extension to the classical S-lemma to quadratic matrix inequalities (QMIs) is given. This extension, known as the matrix S-lemma, serves to provide a tractable sufficient condition for the implication of the following two QMIs

$$\begin{bmatrix} \mathbb{I} \\ Z \end{bmatrix}^\top N \begin{bmatrix} \mathbb{I} \\ Z \end{bmatrix} \preceq 0 \Rightarrow \begin{bmatrix} \mathbb{I} \\ Z \end{bmatrix}^\top M \begin{bmatrix} \mathbb{I} \\ Z \end{bmatrix} \prec 0,$$

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where the first QMI represents a bound of the uncertain matrix variable Z and the second one is the robust constraint to check. In this letter, we provide a novel QMI representation of an ellipsoidal set. This innovative representation incorporates auxiliary variables that significantly reduce the conservativeness of the obtained approximation of the feasible set with respect to the standard matrix S-lemma [11].

This letter is organized as follows. In Section II we introduce the problem formulation and we present a motivating example, in Section III we provide a result based on the classical S-procedure, on the same lines of the matrix S-Lemma in [11]. This result is exploited in Section IV, where we introduce two different QMI representations of the uncertainty set $\mathcal{W}$. In particular, the second representation, based on additional variables, allows to derive our main result, presented in Section V. This result is proved to be less conservative than the classical S-procedure, via analytic and numerical examples, discussed in Sections VI and VII, respectively.

Notation: Denote the set of symmetric matrices in $\mathbb{R}^{n\times n}$ by $\mathcal{S}^n$. $\mathcal{S}^n_+ = \{ H \in \mathcal{S}^n : H \succeq 0 \}$ is the set of semi-positive matrices in $\mathcal{S}^n$, and $\mathcal{S}^n_{++} = \{ H \in \mathcal{S}^n : H \succ 0 \}$. The square root of $H \in \mathcal{S}^n_+$ is denoted by $H^{\frac{1}{2}}$, which is the positive semi-definite matrix satisfying $(H^{\frac{1}{2}})^2 = H$. The notation $Tr(H)$ designates the trace of the square matrix H. $\mathbb{I}_n$ is the identity matrix in $\mathcal{S}^n$. $A \otimes B$ is the Kronecker product of matrices A and B. Given $\mathbf{w} \in \begin{bmatrix} w_1 & \dots & w_m \end{bmatrix}^\top \in \mathbb{R}^m$, let us denote

$$\mathbf{w}^n_\otimes \doteq \begin{bmatrix} w_1\mathbb{I}_n & w_2\mathbb{I}_n & \dots & w_m\mathbb{I}_n \end{bmatrix}^\top = \mathbf{w} \otimes \mathbb{I}_n. \tag{1}$$

II. Problem Formulation

In this formulation, we assume that

$$L(X, \mathbf{w}) \in \mathcal{S}^n, \quad \forall X \in \mathcal{X},\ \forall \mathbf{w} \in \mathcal{W} \subset \mathbb{R}^m,$$

and a quadratic dependence on the uncertain vector $\mathbf{w} \in \mathcal{W}$. That is, we assume that there exists a matrix function $M(X)$ such that $L(X, \mathbf{w})$ can be rewritten as

$$L(X, \mathbf{w}) = \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}^n_\otimes \end{bmatrix}^\top M(X) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}^n_\otimes \end{bmatrix},\ \forall X \in \mathcal{X},\ \forall \mathbf{w} \in \mathcal{W},$$

where $\mathbf{w}^n_\otimes$ is defined in (1). Here, we are interested in characterizing the subset of feasible solutions

$$S_\mathcal{X} \doteq \{ X \in \mathcal{X} : L(X, \mathbf{w}) \prec 0,\ \forall \mathbf{w} \in \mathcal{W} \}.$$

We assume that $\mathcal{W}$ is a bounded set that results from the intersection of a finite collection of quadratically constrained sets $\mathcal{W}_i$, $i = 1, \dots, s$. That is, $\mathcal{W} = \bigcap_{j=1,\dots,s} \mathcal{W}_j$, where, given $Q_j \succeq 0$, $j = 1, \dots, s$, we define

$$\mathcal{W}_j = \left\{ \mathbf{w} \in \mathbb{R}^m : \mathbf{w}^\top Q_j \mathbf{w} \leq 1 \right\},\ j = 1, \dots, s.$$

We notice that boundeness of $\mathcal{W}$ is guaranteed if and only if $\sum_{j=1}^s Q_j \succ 0$.

A. Motivating Example: The Discrete Lyapunov Equation

Let us consider the following discrete-time system with affine uncertainty

$$x^+ = \left(A_0 + \sum_{i=1}^m w_i A_i \right) x = A(\mathbf{w})x,$$

where $x \in \mathbb{R}^{n_x}$ and $x^+ \in \mathbb{R}^{n_x}$ are the state and successor state, respectively. The uncertain vector $\mathbf{w} = \begin{bmatrix} w_1 & \dots & w_m \end{bmatrix}^\top \in \mathbb{R}^m$ is constrained to the ellipsoid

$$\mathcal{W} = \left\{ \mathbf{w} : \mathbf{w}^\top Q \mathbf{w} \leq 1 \right\}, \quad Q \succ 0. \tag{2}$$

Given $S \succ 0$, we can define the following robust discrete-time Lyapunov equation for the uncertain system in the variable $P \succ 0$

$$A(\mathbf{w})^\top P A(\mathbf{w}) - P \prec -S,\ \forall \mathbf{w} \in \mathcal{W}.$$

Let us formulate the previous inequality by means of an LMI on the inverse of P. Denote $X = P^{-1} \in \mathcal{S}^{n_x}_{++}$. The previous matrix inequality is equivalent to

$$L(X, \mathbf{w}) = -\begin{bmatrix} X & XA(\mathbf{w})^\top & XS^{\frac{1}{2}} \\ * & X & 0 \\ * & * & \mathbb{I}_{n_x} \end{bmatrix} \prec 0.$$

Suppose that $m = 2$ and set $n = 3n_x$, If we introduce the matrix

$$M(X) = \begin{bmatrix} M_{1,1}(X) & M_{1,2}(X) & M_{1,3}(X) \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \in \mathcal{S}^{(m+1)n}, \tag{3}$$

with

$$M_{1,1}(X) = -\begin{bmatrix} X & XA_0^\top & XS^{\frac{1}{2}} \\ * & X & 0 \\ * & * & \mathbb{I}_{n_x} \end{bmatrix},$$

$$M_{1,2}(X) = -\begin{bmatrix} 0 & XA_1^\top & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix},$$

$$M_{1,3}(X) = -\begin{bmatrix} 0 & XA_2^\top & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix},$$

the inequality $L(X, \mathbf{w}) \prec 0$ can be rewritten as the QMI

$$\begin{bmatrix} \mathbb{I}_n \\ w_1\mathbb{I}_n \\ w_2\mathbb{I}_n \end{bmatrix}^\top M(X) \begin{bmatrix} \mathbb{I}_n \\ w_1\mathbb{I}_n \\ w_2\mathbb{I}_n \end{bmatrix} = \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}^n_\otimes \end{bmatrix}^\top M(X) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}^n_\otimes \end{bmatrix} \prec 0.$$

With this notation, the set of matrices $X \succ 0$ that robustly satisfy the Lyapunov equation is

$$S_\mathcal{X} \doteq \left\{ X \succ 0 : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}^n_\otimes \end{bmatrix}^\top M(X) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}^n_\otimes \end{bmatrix} \prec 0,\ \forall \mathbf{w} \in \mathcal{W} \right\}.$$

III. The S-Procedure and the QMI Representation of the Uncertainty Set $\mathcal{W}$

To provide tractable approximations for set $S_\mathcal{X}$, we will leverage results related to the S-procedure [1], along with a novel scheme that exploits the structure of the uncertainty set $\mathcal{W}$. The following lemma is a direct application of the S-procedure [1, Sec. 2.6.3]. The result allows us to bound the set of feasible solutions $S_\mathcal{X}$, provided that the uncertainty set $\mathcal{W}$ can be described, or bounded, by means of the intersection of a finite number of QMIs with the same structure as the one used to describe $L(X, \mathbf{w})$.

Lemma 1: Suppose that $\mathcal{W} = \bigcap_{j=1,\ldots,s} \mathcal{W}_j$. Given $X \in \mathcal{X}$, suppose that there exists $N_j = N_j^\top$, and $\tau_j \geq 0$, such that, for every $j = 1, \ldots, s$ we have

$$\mathcal{W}_j = \left\{ \mathbf{w} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top N_j \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\}, \tag{4}$$

and $M(X) - \sum_{j=1}^s \tau_j N_j \prec 0$. Then,

$$L(X, \mathbf{w}) = \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top M(X) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \prec 0, \ \forall \mathbf{w} \in \mathcal{W}. \tag{5}$$

Proof: Suppose that there is $\hat{\mathbf{w}} \in \mathcal{W}$ such that

$$\begin{bmatrix} \mathbb{I}_n \\ \hat{\mathbf{w}}_\otimes^n \end{bmatrix}^\top M(X) \begin{bmatrix} \mathbb{I}_n \\ \hat{\mathbf{w}}_\otimes^n \end{bmatrix} \not\prec 0. \tag{6}$$

Since $\hat{\mathbf{w}} \in \mathcal{W}_j$, $j = 1, \ldots, s$, we infer from $\tau_j \geq 0$ and (4) that

$$-\tau_j \begin{bmatrix} \mathbb{I}_n \\ \hat{\mathbf{w}}_\otimes^n \end{bmatrix}^\top N_j \begin{bmatrix} \mathbb{I}_n \\ \hat{\mathbf{w}}_\otimes^n \end{bmatrix} \succeq 0, \ j = 1, \ldots, s.$$

This, along with (6), implies

$$\begin{bmatrix} \mathbb{I}_n \\ \hat{\mathbf{w}}_\otimes^n \end{bmatrix}^\top \left(M(X) - \sum_{j=1}^s \tau_j N_j \right) \begin{bmatrix} \mathbb{I}_n \\ \hat{\mathbf{w}}_\otimes^n \end{bmatrix} \not\prec 0,$$

which contradicts $M(X) - \sum_{j=1}^s \tau_j N_j \prec 0$. ∎

Remark 1: Lemma 1 is similar to the Matrix S-Lemma presented in [11]. The main differences here are that we impose a strict inequality in (5) and we do not require a Slater condition to hold for (4). Note that, since the uncertainty $\mathbf{w}$ is structured, both Lemma 1 and the Matrix S-Lemma yield only sufficient conditions.

IV. Family of QMI Representations of an Ellipsoid

It is clear that, in order to apply Lemma 1, it is necessary to rewrite, or bound, the uncertainty set $\mathcal{W}$, as a QMI on $\mathbf{w}$ with the same structure as in (4). That is, one should find a matrix N, or a family of such matrices, satisfying

$$\mathcal{W} = \left\{ \mathbf{w} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top N \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\}.$$

A. Simple QMI for an Ellipsoidal Set

The following proposition shows that there is a QMI that exactly characterizes an ellipsoidal uncertainty set

$$\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^m : \mathbf{w}^\top Q \mathbf{w} \leq 1\}.$$

Proposition 1: Suppose that $Q \in \mathcal{S}_+^m$ and define

$$N = \begin{bmatrix} -\mathbb{I}_n & 0 \\ 0 & Q \otimes \mathbb{I}_n \end{bmatrix}. \tag{7}$$

Then,

$$\begin{aligned} \mathcal{W} &= \left\{ \mathbf{w} \in \mathbb{R}^m : \mathbf{w}^\top Q \mathbf{w} \leq 1 \right\} \\ &= \left\{ \mathbf{w} \in \mathbb{R}^m : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top N \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\}. \end{aligned}$$

Proof: Denote $Q_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, m$, the elements of matrix $Q \in \mathcal{S}_+^m$. From (1) we have

$$\begin{aligned} \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top N \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} &= \begin{bmatrix} \mathbb{I}_n \\ w_1 \mathbb{I}_n \\ \vdots \\ w_m \mathbb{I}_n \end{bmatrix}^\top \begin{bmatrix} -\mathbb{I}_n & 0 \\ 0 & Q \otimes \mathbb{I}_n \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ w_1 \mathbb{I}_n \\ \vdots \\ w_m \mathbb{I}_n \end{bmatrix} \\ &= -\mathbb{I}_n + \sum_{i=1}^m \sum_{j=1}^m Q_{i,j} w_i w_j \mathbb{I}_n \\ &= \left(\mathbf{w}^\top Q \mathbf{w} - 1 \right) \mathbb{I}_n, \end{aligned}$$

which implies $\begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top N \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \Leftrightarrow \mathbf{w}^\top Q \mathbf{w} - 1 \leq 0$. ∎

We notice that this characterization of an ellipsoidal set, along with Lemma 1, allows us to obtain an approximation of the original robust problem when the uncertainty is given by an ellipsoidal set.

Lemma 2: Suppose that $\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^m : \mathbf{w}^\top Q \mathbf{w} \leq 1\}$, where $Q \succ 0$, N as in (7), and define

$$S_{\mathcal{X}} \doteq \left\{ X \in \mathcal{X} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top M(X) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \prec 0, \ \forall \mathbf{w} \in \mathcal{W} \right\}.$$

Then, $X \in S_{\mathcal{X}}$ if there is $\tau \geq 0$ such that $M(X) - \tau N \prec 0$.

Proof: The proof follows directly from Lemma 1 and direct computations. ∎

In the following we provide a different QMI characterization of the ellipsoidal set $\mathcal{W}$, based on the introduction of a family of matrices $\bar{N}$. This result is instrumental to the derivation of a novel, and less conservative, robust LMI characterization.

Theorem 1: Suppose that $Q \in \mathcal{S}_+^m$, $H \in \mathcal{S}_{++}^n$, and $F_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, m$ are skew-symmetric, i.e., $F_i = -F_i^\top$. Let us define

$$\bar{N}(H, F_1, \ldots, F_m) = \begin{bmatrix} -H & F_1 & \ldots & F_m \\ F_1^\top & & & \\ \vdots & & Q \otimes H & \\ F_m^\top & & & \end{bmatrix}. \tag{8}$$

Then,

$$\mathcal{W} = \left\{ \mathbf{w} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top \bar{N}(H, F_1, \ldots, F_m) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\}.$$

Proof: Consider the matrix $\bar{N}(H, F_1, \ldots, F_m)$ given in (8), and define

$$S_{\text{QMI}} = \left\{ \mathbf{w} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top \bar{N}(H, F_1, \ldots, F_m) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\}.$$

It is clear that

$$\bar{N}(H, F_1, \ldots, F_m) = \bar{N}(H, 0, \ldots, 0) + \bar{N}(0, F_1, \ldots, F_m).$$

We now have that

$$\begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top \bar{N}(0, F_1, \ldots, F_m) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} = \sum_{i=1}^m w_i \left(F_i + F_i^T \right) = 0.$$

Thus,

$$\begin{aligned} S_{\text{QMI}} &= \left\{\mathbf{w} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top N(H, 0, \dots, 0) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\} \\ &= \left\{\mathbf{w} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top \begin{bmatrix} -H & 0 \\ 0 & Q \otimes H \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\}. \end{aligned}$$

We also have

$$\begin{aligned} &\begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top \begin{bmatrix} -H & 0 \\ 0 & Q \otimes H \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top \left(\mathbb{I}_{m+1} \otimes H^{\frac{1}{2}}\right)^\top \begin{bmatrix} -\mathbb{I}_n & 0 \\ 0 & Q \otimes \mathbb{I}_n \end{bmatrix} \left(\mathbb{I}_{m+1} \otimes H^{\frac{1}{2}}\right) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \\ &= H^{\frac{1}{2}} \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top \begin{bmatrix} -\mathbb{I}_n & 0 \\ 0 & Q \otimes \mathbb{I}_n \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} H^{\frac{1}{2}}. \end{aligned}$$

From this, and the fact that $H^{\frac{1}{2}}$ is non-singular, we infer that

$$\begin{aligned} S_{\text{QMI}} &= \left\{\mathbf{w} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top \begin{bmatrix} -\mathbb{I}_n & 0 \\ 0 & Q \otimes \mathbb{I}_n \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\} \\ &= \left\{\mathbf{w} : \left(\mathbf{w}^\top Q \mathbf{w} - 1\right) \mathbb{I}_n \preceq 0 \right\} \\ &= \left\{\mathbf{w} : \mathbf{w}^\top Q \mathbf{w} \le 1 \right\} = \mathcal{W}. \end{aligned}$$

V. Tractable Approximation of the Robust Feasibility Set

The next theorem provides the main contribution of this letter, which is an LMI inequality, not depending on $\mathbf{w}$, that provides a less-conservative characterization of the robust feasibility set of an LMI subject to uncertainty that is bounded by the intersection of s quadratically bounded sets than the one provided in [11].

Theorem 2: Suppose that

$$\mathcal{W} = \bigcap_{j=1,\dots,s} \mathcal{W}_j = \bigcap_{j=1,\dots,s} \{\mathbf{w} \in \mathbb{R}^m : \mathbf{w}^\top Q_j \mathbf{w} \le 1\},$$

where $Q_j \in \mathcal{S}_+^n$, $j = 1, \dots, s$. Given $X \in \mathcal{X}$, suppose that there exists $H_j \in \mathcal{S}_{++}^n$, $j = 1, \dots, s$ and skew-symmetric matrices $F_1, \dots, F_m$ such that

$$M(X) - \begin{bmatrix} -\sum_{j=1}^s H_j & F_1 & \dots & F_m \\ F_1^\top & & & \\ \vdots & & \sum_{j=1}^s (Q_j \otimes H_j) & \\ F_m^\top & & & \end{bmatrix} \prec 0, \tag{9}$$

Then, $L(X, \mathbf{w}) = \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top M(X) \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \prec 0, \ \forall \mathbf{w} \in \mathcal{W}$.

Proof: Denote N_1 as

$$N_1 = \begin{bmatrix} -H_1 & F_1 & \dots & F_m \\ F_1^\top & & & \\ \vdots & & Q_1 \otimes H_1 & \\ F_m^\top & & & \end{bmatrix}.$$

If $s > 1$, for $j = 2, \dots, s$ define N_j as

$$N_j = \begin{bmatrix} -H_j & 0 \\ 0 & Q_j \otimes H_j \end{bmatrix}.$$

The direct application of Theorem 1, provides for $j = 1, \dots, s$, the equalities

$$\mathcal{W}_j = \left\{\mathbf{w} : \mathbf{w}^\top Q_j \mathbf{w} \le 1 \right\} = \left\{\mathbf{w} : \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix}^\top N_j \begin{bmatrix} \mathbb{I}_n \\ \mathbf{w}_\otimes^n \end{bmatrix} \preceq 0 \right\}.$$

These representations for $\mathcal{W}_j$, along with the application of Theorem 2 with $\tau_j = 1$, $j = 1, \dots, s$, provide the following sufficient condition for $L(X, \mathbf{w}) \prec 0$, $\forall \mathbf{w} \in \mathcal{W}$

$$M(X) - \sum_{j=1}^s N_j \prec 0.$$

This concludes the proof because the previous expression is identical to (9) by construction. ■

We note that, compared to [11], the proposed formulation makes use of additional skew-symmetric matrices $F_1, \dots, F_m$. This allows to provide additional degrees of freedom, and thus substantially decrease the conservatism, as discussed in the next section and shown in the numerical examples. Of course, this comes at the cost of increasing the complexity, since the ensuing optimization problem involves additional variables. On the other hand, we remark that the problem is still in a semi-definite programming (SDP) form, and may be solved by interior-point methods that depend mildly (and, in any case, polynomially) on the number of optimization variables.

VI. Improvement With Respect to Standard S-Procedure

We now show that the proposed result provides exact representations of the robust set S_X for some simplified situations. We also show that for those situations, the standard S-procedure, i.e., Lemma 2, fails to provide a sharp representation.

Consider the following robust LMI, where, given $r > 0$, the uncertainty w is a scalar subject to the interval constraint $w \in [-r, r]$, i.e.,

$$M_0 + wM_1 + wM_1^T \prec 0, \ \forall w \in [-r, r]. \tag{10}$$

We remark that both $M_0 \in \mathcal{S}^n$ and $M_1 \in \mathcal{S}^n$ could be affine functions of a given decision variable X. We do not make this dependence explicit to simplify the expressions. Since the dependence on w is affine, the worst-case situations are obtained at the vertices $w = -r$ and $w = r$, see, e.g., [1], [7]. That is, the robust LMI is satisfied if the following deterministic LMIs hold

$$M_0 - rM_1 - rM_1^\top \prec 0, \tag{11}$$

$$M_0 + rM_1 + rM_1^\top \prec 0. \tag{12}$$

Imagine that instead of resorting to this classic vertex result, we formulate both the original robust LMI and the constraint $w \in [w^-, w^+]$ as QMIs on the uncertainty w, and apply Lemma 2. The robust LMI (10) in QMI form is

$$\begin{bmatrix} \mathbb{I}_n \\ w\mathbb{I}_n \end{bmatrix}^\top \begin{bmatrix} M_0 & M_1 \\ M_1^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ w\mathbb{I}_n \end{bmatrix} \prec 0, \ \forall w \in [-r, r]. \tag{13}$$

The constraint $w \in [-r, r]$ can be rewritten as

$$\begin{bmatrix} \mathbb{I}_n \\ w\mathbb{I}_n \end{bmatrix}^\top \begin{bmatrix} -\mathbb{I}_n & 0 \\ 0 & \frac{1}{r^2}\mathbb{I}_n \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ w\mathbb{I}_n \end{bmatrix} \preceq 0.$$

From the application of the S-procedure, we infer that M_0 and M_1 satisfy (10) if there is $\tau \geq 0$ such that

$$\begin{bmatrix} M_0 & M_1 \\ M_1^\top & 0 \end{bmatrix} - \tau \begin{bmatrix} -\mathbb{I}_n & 0 \\ 0 & \frac{1}{r^2}\mathbb{I}_n \end{bmatrix} \prec 0. \tag{14}$$

As we show in the following discussion, this is only a sufficient condition for (10) that can be very conservative. Suppose that $n = 2$, and that

$$M_0 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ M_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In this case, $M_0 + wM_1 + wM_1^\top = M_0 = -\mathbb{I}_2 \prec 0, \ \forall w \in \mathbb{R}$. Thus, in this case, the robust LMI is satisfied for every $r > 0$. We now show that for this particular choice of matrices M_0 and M_1, (14) provides a bounded value for r. We observe that (14) is equivalent to

$$\begin{bmatrix} -M_0 - \tau\mathbb{I}_n & -M_1 \\ -M_1^\top & \frac{\tau}{r^2}\mathbb{I}_n \end{bmatrix} \succ 0.$$

Using the Schur complement [1], we rewrite the previous inequality as $-M_0 - \tau\mathbb{I}_n - \frac{r^2}{\tau}M_1M_1^\top \succ 0$. Taking into consideration the values for M_0 and M_1 we obtain

$$\mathbb{I}_2 - \tau\mathbb{I}_2 - \frac{r^2}{\tau}\mathbb{I}_2 \succ 0.$$

Thus, we conclude that an equivalent condition for (14) is

$$1 - \tau - \frac{r^2}{\tau} > 0. \tag{15}$$

Given r, the optimal value for τ (e.g., maximizing the left term of the previous inequality) is the one for which the derivative is zero. That is, $-1 + \frac{r^2}{\tau^2} = 0$. This means that the optimal value is $\tau = r$. Substituting this optimal value in (15) we obtain the constraint $r < \frac{1}{2}$. This proves that for this example, the standard S-procedure fails to properly characterize the range of values of r for which the robust LMI (10) is satisfied.

In view of Theorem 1, we have that given $F = -F^\top$ and $H \succ 0$, the quadratic constraint $w^2 \leq r^2$ can be rewritten as

$$\begin{bmatrix} \mathbb{I}_n \\ w\mathbb{I}_n \end{bmatrix}^\top \begin{bmatrix} -H & F \\ F^\top & \frac{1}{r^2}H \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ w\mathbb{I}_n \end{bmatrix} \preceq 0.$$

From this and Theorem 2, we deduce that M_0 and M_1 satisfy (10) if there exists $F = -F^\top$ and $H \succ 0$ such that

$$\begin{bmatrix} M_0 + H & M_1 - F \\ M_1^\top - F^\top & -\frac{1}{r^2}H \end{bmatrix} \prec 0.$$

The following lemma shows that this characterization of the robust pairs M_0 and M_1 is exact.

Lemma 3: Matrices M_0 and M_1 satisfy

$$M_0 + wM_1 + wM_1^T \prec 0, \ \forall w \in [-r, r], \tag{16}$$

if and only if there exists $F = -F^\top$ and $H \succ 0$ such that

$$\begin{bmatrix} M_0 + H & M_1 - F \\ M_1^\top - F^\top & -\frac{1}{r^2}H \end{bmatrix} \prec 0. \tag{17}$$

Proof: The implication (17) $\Rightarrow$ (16) has been already discussed (it follows from Theorem 2). We now prove (16) $\Rightarrow$ (17). As commented before (see (11), (12)), (16) is equivalent to

$$M_0 - rM_1 - rM_1^\top \prec 0,$$
$$M_0 + rM_1 + rM_1^\top \prec 0.$$

This can be rewritten as

$$\begin{bmatrix} M_0 - rM_1 - rM_1^\top & 0 \\ 0 & M_0 + rM_1 + rM_1^\top \end{bmatrix} \prec 0.$$

Denote $T = \begin{bmatrix} \mathbb{I}_n & -\mathbb{I}_n \\ \mathbb{I}_n & \mathbb{I}_n \end{bmatrix}$. Pre-multiplying the LMI by $T^\top$ and post-multiplying by T we obtain the equivalent LMI

$$\begin{bmatrix} 2M_0 & 2r(M_1 + M_1^\top) \\ 2r(M_1 + M_1^\top) & 2M_0 \end{bmatrix} \prec 0.$$

Pre and post-multiplying by $\frac{1}{2}\begin{bmatrix} \mathbb{I}_n & 0 \\ 0 & \frac{1}{r}\mathbb{I}_n \end{bmatrix}$, we obtain

$$\begin{bmatrix} \frac{M_0}{2} & \frac{M_1 + M_1^\top}{2} \\ \frac{M_1 + M_1^\top}{2} & \frac{1}{2r^2}M_0 \end{bmatrix} \prec 0.$$

Denote now $H = -\frac{1}{2}M_0$, and $F = \frac{M_1 - M_1^\top}{2}$. With this choice, we can rewrite the LMI as

$$\begin{bmatrix} M_0 + H & M_1 - F \\ M_1^\top - F^\top & -\frac{1}{r^2}H \end{bmatrix} \prec 0.$$

We notice that F is anti-symmetric by construction. Also, $H = -\frac{1}{2}M_0$ is positive definite because of the evaluation of LMI (16) at $w = 0$. ■

VII. Numerical Example

A. Case A

Let us consider the uncertain system

$$x^+ = (A_0 + w_1A_1 + w_2A_2)x = A(\mathbf{w})x,$$
$$A_0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0.2 \\ -0.2 & 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 0.0075 \\ -0.14 & 0.1 \end{bmatrix},$$

with $\mathbf{w} \in \mathcal{W} = \{\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} : \mathbf{w}^\top \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \mathbf{w} \leq 1\}$.

Given $S = \begin{bmatrix} 0.5833 & 0 \\ 0 & 1.2372 \end{bmatrix}$, we aim at minimizing the size of P subject to $A(\mathbf{w})^\top PA(\mathbf{w}) - P \prec -S, \ \forall \mathbf{w} \in \mathcal{W}$, to obtain a sharper bound on

$$J_\infty = \sum_{k=0}^{\infty} x_k^\top Sx_k \leq x_0^\top Px_0, \forall x_0, \forall \mathbf{w} \in \mathcal{W}.$$

It is easy to observe that this can be obtained by maximizing the trace of P^{-1}, i.e., minimizing Tr (P).

Let us denote by N_{MSL} and N_{ISL} the matrices required for the QMI representation of the uncertainty set $\mathcal{W}$ for the standard matrix S-lemma (Lemma 1) and for the improved version proposed in this letter (Theorem 2), respectively, i.e.,

$$N_{MSL} = \begin{bmatrix} -\mathbb{I}_2 & 0 & 0 \\ 0 & Q_{11}\mathbb{I}_{n_x} & Q_{12}\mathbb{I}_{n_x} \\ 0 & Q_{12}\mathbb{I}_{n_x} & Q_{22}\mathbb{I}_{n_x} \end{bmatrix},$$

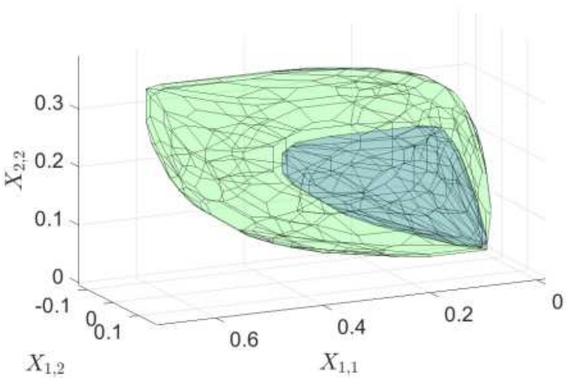

Fig. 1. Feasible sets obtained applying the standard matrix S-lemma (blue) and the improved matrix S-lemma (green).

$$N_{ISL} = \begin{bmatrix} -H & F_1 & F_2 \\ F_1^\top & Q_{11}H & Q_{12}H \\ F_2^\top & Q_{12}H & Q_{22}H \end{bmatrix}.$$

Let us define $X = P^{-1}$. The corresponding optimization problems are

$$X^\star_{MLS} \doteq \max_{X,\tau\geq 0} \ \text{tr(X)} \quad \text{s.t. } M(X) - \tau N_{MSL} \prec 0 \tag{18}$$

$$X^\star_{ILS} \doteq \max_{X,H,F_1,F_2} \ \text{tr(X)} \quad \text{s.t. } M(X) - N_{ISL} \prec 0, \quad H \succ 0, F_1 = -F_1^\top, F_2 = -F_2^\top. \tag{19}$$

where the expression for $M(X)$ can be found in (3). We remark that these problems are in LMI form and, thus, they can be solved by standard SDP tools. Comparing the results obtained with the two methods, we obtain that for the case under analysis, the trace of $X^\star$ achieved applying the proposed approach is equal to $\text{tr}(X^\star_{\text{ILS}}) = 1.0793$ and it is almost 60% larger than the one obtained with the approach proposed in [11], i.e., $\text{tr}(X^\star_{\text{MLS}}) = 0.6346$. Moreover, in Figure 1, we show how the corresponding feasible sets reflect that the relaxed matrix S-lemma is able to reduce some conservativeness of the standard approach thanks to the over-parametrization of the matrix N_{ISL}.

B. Case B

In this second case study, we evaluate how the ratio ρ between $\text{tr}(X^\star_{\text{ILS}})$ and $\text{tr}(X^\star_{\text{MLS}})$ scales with respect to the problem dimension. In particular, we select five case studies with n_x ranging from 2 to 6 and $m = n_x - 1$. For each case study, we run 1000 simulations, where the matrices A_i, $i = 0, \dots, m$, are obtained as sparse random matrices with a sparsity density of 0.4, and with $Q = 10^4 \mathbb{I}_{n_x}$.

Figure 2 shows how increasing the size of the problem implies an improvement in terms of reducing the conservativeness of the standard matrix S-lemma, with the median of the ratio ρ increasing from 1.68 to 2.06.

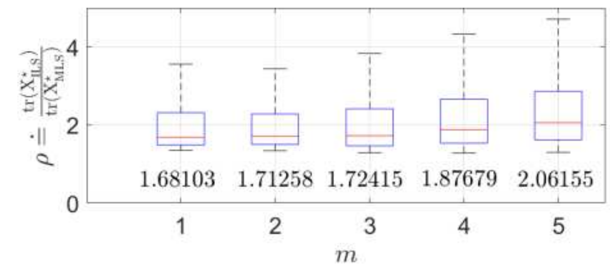

Fig. 2. Evolution of ρ over 1000 runs for $n_x = [2, 6]$ and $m = [1, 5]$.

VIII. Conclusion

We presented a novel approach to design tractable approximations of robust feasible sets of LMIs affected by ellipsoidal uncertainty. These are obtained reformulating the problem in an specifically designed QMI form, and subsequently applying a generalized matrix S-procedure. The result is shown to be significantly less conservative than the classical approach. Future work will be devoted to further reducing conservatism by designing specific versions of our recent approach based on probabilistic scaling [13].

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