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Auctioning off a Non-Rivalrous Good with Interference

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Abstract: Auctions are a prevalent way to exchange goods and are well-studied for the exchange of rivalrous goods, but are less studied for non-rivalrous goods. I examine an auction framework where the good sold can be used simultaneously by multiple bidders if their use does not conflict with others; this simultaneous use directly affects the efficiency of the auction. A timely example includes the auctioning off of a radio spectrum by a licensed primary user to unlicensed secondary users who can use the spectrum simultaneously if they are located far enough apart to not cause interference. I examine a uniform price auction over non-conflicting groups and examine how non-rivalry impacts both efficiency and collusion. Conditions are given under which an auction over groups generates higher social welfare than an individual auction. Additional conditions are given under which collusion in a group auction results in higher prices.

Keywords: auction; collusion; non-rivalrous good

JEL Classification: D44; L14; C7

1. Introduction

Auctions are an important and popular way to sell goods in a variety of areas, such as real estate, art, consumer goods, online advertising, and radio spectra, as auctions can be an efficient way to sell goods; see Vickrey [1] and Dasgupta and Maskin [2]. Although auctions of rivalrous goods are well studied, less attention has been paid to the auctions of goods with non-rivalrous properties; see Klemperer [3], and Milgrom [4] for auction literature overviews and Wang, Umehira, Han, Zhou, Li, and Wu [5] for work on non-rivalrous auctions. In an auction with non-rivalrous properties, groups of agents can simultaneously consume goods, which impacts the efficiency of the auction. I examine a non-rivalrous good with interference, where groups of agents who do not conflict or interfere with each other can simultaneously consume the good. A timely example of such a good is that of secondary radio spectrum, as a spectrum is non-rivalrous for users located far enough apart that they do not interfere with each other. Spectrum use has increased rapidly in recent years due to the commercial demand of smart phone users and IoT devices, as well as government demand for national security and air traffic control; thus, it is vital to find efficient ways to share this finite resource. I analyze a uniform price auction over non-conflicting groups to investigate how non-rivalry impacts both efficiency and collusion.

Specifically, consider an auction where multiple bidders can consume the same unit as long as they do not interfere with each other. Bidders are first placed into non-conflicting groups and then a uniform price auction is run over these groups, where agents bid independently and each group is assigned a bid based on the minimum bid of its members; see Wang, Umehira, Han, Zhou, Li, and Wu [5]. The results show that this group auction can generate higher social welfare than an individual auction as long as its valuations are not spread too far apart. The importance of the conflict network and group construction for efficiency is illustrated when valuations are largely dispersed.

Additionally, I analyze how collusion affects bidding in group auctions. A subgroup of bidders have a common agent or owner who bids on their behalf; see Decarolis, Goldmanis, and Penta [6]. An example could be a cell phone company who owns several cell towers

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and bids for additional spectra on their behalf in a secondary spectrum auction run by a television broadcaster. The results show that if all bidders in a non-conflicting group belong to the same coalition, then the common owner has an incentive to increase this group's minimum bid. This result contrasts with the previous coalition analysis of rivalrous good auctions, where only one agent in the coalition can consume the good, causing most coalition members to depress their bids or refrain from bidding; see Agranov and Yariv [7] and Marshall and Marx [8].

Note that primary spectrum awards are those licenses awarded to commercial and non-commercial spectrum users directly by the FCC, while secondary spectrum awards are spectrum usage rights sold by primary license holders to other providers or users. These secondary awards may be limited to short periods of time and to specific geographical areas.

The most closely related papers are Wang, Umehira, Han, Zhou, Li, and Wu [5] and Decarolis, Goldmanis, and Penta [6]. Wang, Umehira, Han, Zhou, Li, and Wu [5] analyze non-conflicting group auctions for secondary spectrum sharing and focuses on preserving privacy in mechanisms without collusion. Decarolis, Goldmanis, and Penta [6] examine collusion in online ad auctions where bidders share a common agent who bids on their behalf. The current paper differs in that I consider a non-rivalrous good for which the auction takes place over non-conflicting groups.

Related research includes the diverse field of theoretical and empirical auctions; see Khezr and Cumpston [9], Klemperer [3], and Hortaçsu and McAdams [10] for recent surveys. There is also a large body of literature within the realm of auctions that focuses on collusion. Mechanisms for maintaining collusion in rivalrous auctions are examined by Mailath and Zemsky [11], Graham and Marshall [12], McAfee and McMillan [13], and Marshall and Marx [8]. Conley and Decarolis [14] find evidence of bidding rings which both increase and decrease prices in procurement auctions. Asker [15] examines the behavior of bidding rings made by stamp collectors in English auctions. There are many auction papers examining the efficiency of different types of rivalrous auctions; see Pesendorfer and Swinkels [16]; Dasgupta and Maskin [2]; Feldman, Fu, Gravin, and Lucier [17]; and Baisa [18].

There is a large body of literature in electrical engineering examining spectrum-sharing auctions; see Benedetto, Mastroeni, and Quaresima [19] for a review. This literature focuses on various aspects of spectrum sharing such as efficiency, fairness, dynamic access, and privacy; see Wang, Li, Xu, Xu, Gao, and Chen [20]; Wang, Umehira, Han, Zhou, Li, and Wu [5]; Huang, Berry, and Honig [21]; and Khaledi and Abouzeid [22]. Additionally, there is an economics literature on spectrum auctions. Many papers have examined how to design spectrum auctions for the primary spectra awarded by the FCC; see Milgrom and Vogt [23] and Milgrom [24]. While Watts [25] analyzes secondary spectrum auctions with uncertainty, Watts [26] examines secondary spectrum sharing with congestion.

2. Model

There is an owner or primary user (PU) of a reusable resource such as a radio spectrum; the PU has the right to use the resource in a certain area. There are n secondary users (SUs), denoted by $i \in N = \{1, 2, \ldots, n\}$, each of whom would like to access the resource. Each i has a current location and a value for using the resource of v_i, where v_i is private information. Each v_i is a random variable distributed on $[0, \overline{v}]$, where $\overline{v} > 0$. Let $v = (v_1, \ldots, v_n)$. The PU also uses the resource and has both a value and a current location. Assume all locations are common knowledge.

If the SUs are located too close together and access the resource at the same time, then they will cause interference with each other. The SU locations can be used to create a conflict graph, g, which consists of conflict links between the SUs. Formally, if $\ell_{ij} \in g$, then there is a conflict link between i and j, where $i, j \in N$. If i and j are located far enough apart so that no such conflict exists, then there is no link between them and $\ell_{ij} \notin g$. The PU owns κ different channels or radio frequency bands of the resource. If two users who conflict with each other are assigned different channels, then there will be no conflict. Although the

SUs can also conflict with the PU, we can assume that any SU who conflicts with the PU is removed and let N consist only of the remaining SUs.

If the PU allows i access to the spectrum at price p_i, then i receives a payoff of $u_i = v_i - p_i$ while, if access is denied, i receives a payoff of $u_i = 0$. Let the subset of N who receive spectrum access be $\hat{N}$. The PU receives a payoff equal to $u_0 = \sum_{i \in \hat{N}} p_i$ from selling access. Define social welfare to be the sum of all payoffs to the PU and the SUs. The assignment of the spectrum is efficient if social welfare is maximized. Formally, social welfare is defined as $SW = \sum_{j \in N} u_j + u_0$. Note that although the PU also uses the resource, this usage payoff is not included in social welfare; thus, social welfare only includes payoffs from selling secondary access to the resource or spectrum.

I consider two different mechanisms for assigning prices and secondary access to the spectrum.

2.1. Group Auctions

The first mechanism is based on one from Wang, Umehira, Han, Zhou, Li, and Wu [5] and consists of two parts. First, the PU uses graph g to place the SUs into non-conflicting groups. These groups are constructed based on the coloring algorithm of Welsh and Powell [27], which guarantees an upper bound on the number of groups created equal to one plus the largest degree or largest number of conflict links any node has. The algorithm proceeds as follows: For each $i \in N$, define the degree d_i of i as the number of conflicting links i has in g. Order the degrees and rename the SUs such that $d_1 \geq d_2 \geq \cdots \geq d_n$. If two or more agents have equal degrees, then order these specific agents randomly. Next, the non-conflicting groups are formed. Place 1 in group G_1. For agent $i-1$, who has just been placed into a group, consider agent i. Place i into the previous group G_γ with the smallest group number γ as long as i does not have a conflict link with any node in G_γ. If i has a conflict link with at least one member of all existing groups, then assign i to a new group. Let the non-conflicting groups formed be $G \equiv \{G_1, \ldots, G_\mu\}$. Let $|G_\gamma|$ be the number of members of group $G_\gamma \in G$. Assume that G is known to everyone.

Next, the PU holds a uniform price auction for κ units where each non-conflicting group is assigned a bid as follows: Let each bidder i submit bid b_i and let $b = (b_1, \ldots, b_n)$. Define $b_\gamma^{min} = min_{i \in G_\gamma} b_i$. Thus, b_γ^{min} equals the minimum bid of all the SUs in G_γ. For each group $G_\gamma \in G$, let $b_\gamma^G = |G_\gamma| \cdot b_\gamma^{min}$ be the group's bid. Assign spectrum use to the groups with the κ largest group bids and let these groups pay price $b_{\kappa+1}^G$, which is the $\kappa + 1$ highest group bid. Assume the group price is split equally among group members. Thus, each member i of the winning group G_γ pays $\frac{b_{\kappa-1}^G}{|G_\gamma|}$. Note that if $b_\gamma^G = |G_\gamma| \cdot b_\gamma^{min} > b_\eta^G = |G_\eta| \cdot b_\eta^{min}$, then $b_\gamma^{min} > \frac{|G_\eta| \cdot b_\eta^{min}}{|G_\gamma|} = \frac{b_\eta^G}{|G_\gamma|}$. Thus, no member of group G_γ is ever asked to pay more than their bid.

2.2. Individual Auction

A uniform price auction is defined as follows, and no groups are formed: Let each bidder $i \in N$ submit bid b_i. Assign an item or spectrum use to the κ highest bidders. Let the price paid equal the $\kappa + 1$ highest bid, $b_{\kappa+1}$. Thus, there is a single auction where the top κ bidders win an item and each pays the first rejected bid. If $\kappa = 1$, then this auction equals the second price auction.

3. Results

Next I check both mechanisms for truthful bidding and compare social welfare.

Proposition 1. *In both the group auction and individual auction, it is a weakly dominant strategy for each j to choose $b_j = v_j$.*

As truthfulness is an important property and simplifies later analyses, we present a brief proof where the standard argument for uniform price auctions is presented and adjusted to the case of auction groups. Note that the group auction result is similar to the truthfulness result presented in Wang, Umehira, Han, Zhou, Li, and Wu [5].

Proof. First, consider the group auction. We can show that j cannot gain from $b_j > v_j$. If v_j is not the minimum bid in j's group, then increasing the bid beyond v_j will have no affect on winning the auction or on the price. If v_j is the minimum bid in j's group, then bidding $b_j > v_j$ will increase the minimum bid of j's group. Either this will have no affect on j's group or j's group could go from becoming a non-winning group to a winning group. Let $\hat{b}$ be the bid of the current κ highest price group divided by the number of bidders in j's group. Let $b_j > \hat{b} > v_j$. Here j's group does not win a channel in the auction if j bids v_j but does win a channel in the auction if he bids b_j. However, when he bids b_j he will have to pay price $\hat{b}$, which is above v_j. Thus, he is better off bidding v_j. Similarly, j cannot gain from bidding $b_j < v_j$. Again, either this bid will have no effect on j's group winning an item or it could decrease the chance of j's group winning an item. Let v_j be the lowest bid in j's group. Let $\hat{b}$ be the bid of the current $\kappa + 1$ highest price group divided by the number of bidders in j's group. Let $b_j < \hat{b} < v_j$. Here, if j lowers his bid from v_j to b_j, then j's group will go from winning an item at price $\hat{b}$ to not winning an item. Since $\hat{b} < v_j$, j would prefer to win the item. Thus, j is best off bidding v_j and v_j is a weakly dominant strategy for j in the group auction. Next, consider the individual auction. A similar argument shows that j cannot gain from bidding above or below v_j, and so v_j is a weakly dominant strategy in the individual auction. □

Next we compare the social welfare generated by the group auction to that of the individual auction.

Let d_{max} be the largest degree of any j in g. Order agents so that $v_1 \geq v_2 \geq ... \geq v_n$.

Proposition 2. *Let $2v_n \geq v_1$. Then, social welfare is larger with the group auction than with the individual auction.*

If valuations are not too far apart, then a group auction is beneficial as it allows multiple agents with similar valuations to jointly consume the good. Even though these agents may not have maximal valuations, there is not much of a gain from forcing a sale to the highest valuation bidder, as would occur in an individual auction. However, if valuations are spread apart, then it is best to have the highest-value bidder consume the item, which is guaranteed with an individual auction, but not with a group auction.

Proof. First, consider the case of the individual auction. The winning bidders will be agents $\{1, 2, \ldots, \kappa\}$ and they will pay price $v_{\kappa+1}$. The social welfare generated from this auction is $SW = u_0 + \sum_{i=1}^{\kappa} u_i = \sum_{i=1}^{\kappa} v_i - \kappa \cdot v_{\kappa+1} + \kappa \cdot v_{\kappa+1} = \sum_{i=1}^{\kappa} v_i$. Next, consider the case of the group auction. Here, agents are placed into non-conflicting groups and the upper bound on the number of non-conflicting groups is $d_{max} + 1$; see Welsh and Powell [27]. If $d_{max} + 1 = n$, then it is possible that each agent is placed into a non-conflicting group by themselves, which would happen if g equals the complete network. In this case, both auctions would have the same social welfare. Next, assume $d_{max} + 1 < n$. Then at least one non-conflicting group has multiple bidders in it. If a winning group consists of two agents, one of whom is agent n, then this group's bid will equal $2v_n$ and the sum of its valuations will be greater than or equal to $2v_n$. All other groups with multiple agents would either not include n or would have more than two agents and thus would have both a minimum bid and a valuation sum greater than or equal to $2v_n$. Hence, if the winning groups all contain multiple agents, then each has a valuation sum greater than or equal to $2v_n$. Since there are κ winning groups, $SW \geq \kappa \cdot 2v_n$. By this assumption, $2v_n \geq v_1$ and $v_1 \geq v_2 \geq \cdots \geq v_n$; thus, it follows that $\kappa \cdot 2v_n \geq \kappa \cdot v_1 \geq \sum_{i=1}^{\kappa} v_i$. Therefore, $SW \geq \sum_{i=1}^{\kappa} v_i$, and the group auction will create higher social welfare than the individual auction.

Next, consider the case where at least one winning group consists of a single agent. First, let all single agent winners be in the set $\Omega \subset \{1, 2, \ldots, \kappa\}$. Let there be β winning groups with multiple bidders; we already showed that some winners must be in groups with multiple bidders, so $\beta < \kappa$. Social welfare for the group auction will be $SW \geq \beta \cdot 2v_n + \sum_{i \in \Omega} v_i \geq \sum_{i=1}^{\kappa} v_i$, and the group auction has higher social welfare than the individual auction. Second, assume that at least one single agent winner, say $j > \kappa$, is not in the set $\{1, 2, \ldots, \kappa\}$. If $v_j < v_1$, then it must be that all non-conflicting groups with multiple members also win an item, as these group minimum bids are greater than or equal to $2v_n \geq v_1 > v_j$. Thus, if j wins an item, then it must be that all $i < j$ are in groups that have already won an item; if i is in a single-agent group, then his minimum bid would be $v_i \geq v_j$. Therefore, the social welfare generated from the group auction is $SW \geq \sum_{i=1}^{j} v_i \geq \sum_{i=1}^{\kappa} v_i$, and social welfare is higher with the group auction than with the individual auction. □

Note that social welfare is simply the sum of the valuations of the bidders who receive an item and does not depend on price. Thus, differences in social welfare between the group and individual auctions come from two different sources. The first is the fact that the group auction allows multiple non-conflicting agents to consume the same unit of the good by placing agents into non-conflicting groups and then letting these groups bid. The second is each auction's assignment of who receives an item. The individual auction always assigns the highest-value agents an item and thus dominates in the second regard, while the group auction dominates in the first regard. Example 1 illustrates that social welfare is higher with the group auction when valuations are not too spread apart and the conditions of Proposition 2 are met. Example 1 also alters these conditions so that valuations are spread apart in such a way that the individual auction generates higher social welfare even though fewer agents are consuming the good.

Example 1. *Let $n = 5$ and $\kappa = 1$ and let the conflict graph be given by g in Figure 1. Such a conflict graph would occur if agents $\{3, 4, 5\}$ are located in a more densely populated area and 2 is on the outskirts, closer to $\{4, 5\}$, while 1 is on the outskirts closer to 3.*

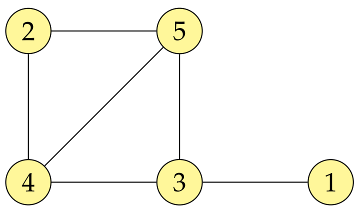

Figure 1. Conflict graph g.

Let $v = (v_1, v_2, v_3, v_4, v_5) = (16, 13, 12, 11, 11)$. First, we construct the non-conflicting groups used in the group auction. In g, SUs 3, 4, and 5 all have three conflicting links or a degree $d_j = 3$, while $d_2 = 2$ and $d_1 = 1$. Thus, the algorithm will first place either 3, 4, or 5 into group G_1, say 3 is randomly chosen. Next, either 4 or 5 will be randomly selected, say 4. As 4 and 3 have a conflict link, 4 cannot be placed into 3's group. It places 4 instead into group G_2. Next we place agent 5 into group G_3, as he conflicts with both 3 and 4. Since 2 has the second highest degree, we place it into group G_1. Lastly, we place 1 into group G_2 as it conflicts with 3 in G_1. Thus, $G_1 = \{2, 3\}$, $G_2 = \{1, 4\}$, and $G_3 = \{5\}$. From Proposition 1, each agent has incentive to bid their own value. Thus, the minimum bids of each group will be 12, 11, and 11, respectively, and group bids will equal $b_1^G = 2 \cdot 12 = 24$, $b_2^G = 2 \cdot 11 = 22$, and $b_3^G = 11$. Group G_1 will be awarded the spectrum and each group member will pay price $p = \frac{b_2^G}{|G_1|} = 22/2 = 11$, and no other spectrum will be awarded. The social welfare equals $SW = v_2 + v_3 = 13 + 12 = 25$.

Note that as $\{3, 4, 5\}$ all have equal degrees, the algorithm randomly selects one of them to be placed into the first group and another into the second. If a different order had been selected, then the final group composition would be dissimilar. One can check that this random selection actually results in only one other group composition, which is $G_1 = \{2, 3\}$, $G_2 = \{1, 5\}$, and $G_3 = \{4\}$. Here, agents 4 and 5 are swapped from the first group composition. As 4 and 5 have the same valuations, the group bids and social welfare will not change.

Now consider the case of an individual auction. Here, each agent has incentive to bid their true values and bidder 1 *wins the auction with a bid of* 16 *and pays the second highest price of* 13*. Social welfare equals* $SW = v_1 = 16 < 25$*, which is less than the social welfare from the group auction as predicted by Proposition 2.*

Next, we alter the example so that the assumptions of Proposition 2 are no longer met. Let v_1 *increase to* $v_1 = 26$*. As* $26 > 2(11) = 22$*, the assumptions are not met, as the highest valuation is now much larger than the lowest valuation. First, consider the group auction. As* g *does not change, the same non-conflicting groups will be formed as before, where* $G_1 = \{2,3\}$*,* $G_2 = \{1,4\}$*, and* $G_3 = \{5\}$*. Here, the lowest bid in each group remains the same and the group bids remain the same. Social welfare does not change and is* $SW = 25$*. However, with an individual auction the social welfare equals* $SW = v_1 = 26 > 25$*. Thus, the individual auction now generates more social welfare. Note here that the composition of the conflict graph matters, as agent* 1*, the highest-value bidder, has no conflict links with the low-value bidders and can be placed into a group with a low-value bidder, ensuring that* 1 *does not win the auction. If the valuations are changed so that* $v = (v_1, v_2, v_3, v_4, v_5) = (13, 26, 12, 11, 11)$*, then agent* 2 *is the new highest-value bidder and has conflict links with both of the lowest-value bidders. Our algorithm forms groups* $G_1 = \{2,3\}$*,* $G_2 = \{1,4\}$*, and* $G_3 = \{5\}$*, and now the highest-value bidder is placed with bidder* 3*, who also has a fairly high value. In fact, group* $G_1 = \{2,3\}$ *will have the highest minimum bid and the highest group bid. Social welfare here equals* $SW = v_2 + v_3 = 26 + 12 = 38$*. This is higher than the social welfare from an individual auction, which equals* $SW = v_2 = 26$*. Thus, when the conditions of Proposition 2 are not met, the conflict graph plays a role in determining the social welfare of the group auction and in whether or not it is higher than the welfare gained from an individual auction.*

Notice that the group auction is not efficient in this example, as constructing groups $\{1,2\}$, $\{3\}$, $\{4\}$, $\{5\}$ would result in $\{1,2\}$ winning the auction with a social welfare of 29, which is larger than the social welfare of 25 found in the example. This inefficiency comes about because the coloring algorithm seeks to minimize the number of groups formed and is based on location or conflicting links and not on bids. Minimizing the number of groups formed should also maximize the number of agents who receive a good, on average, and thus, on average, may lead to more efficient outcomes, although, as was just illustrated, the coloring algorithm often does not result in efficient outcomes for a particular example. In order to increase efficiency, one would need to assign groups based on bids, which is more difficult as one would need to ensure that agents do not have an incentive to misrepresent their bids in order to influence the groups formed. It is possible that an efficient mechanism could be created where groups are formed based on both bids and locations. However, such a mechanism would most likely be quite complex and difficult to implement and is beyond the scope of the current analysis.

Note that with an individual auction only a single bidder consumes the item or spectrum. One could allow other bidders who do not conflict with the winner to also consume spectrum. However, these bidders may have valuations below the $\kappa + 1$ highest bid and thus would not be willing to buy spectrum unless they were charged a lower price. If different prices are paid, then bidders will no longer have an incentive to bid truthfully, which is a desirable property for auctions and for spectrum management.

4. Collusion in Group Auctions

Next, we will investigate how collusion affects bidding in the group auction. Suppose some SUs have a common agent or owner (such as these SUs being cell towers operated by the same mobile company) who bids on their behalf. We will analyze whether or not this group can gain from submitting bids that are different from their valuations. As these SUs have a common owner, the group can be interpreted as a natural cartel who would need no enforcement mechanism; see Decarolis, Goldmanis, and Penta [6], who analyze collusion in online advertising through a common agency. A timely example of a collusive agent acting on behalf of the spectrum seller comes from the FCC's incentive auction where a spectrum from TV broadcasters was resold to mobile broadband providers; see Doraszelski, Seim, Sinkinson, and Wang [28] and Milgrom and Segal [29]. Some TV stations owned multiple licenses and behaved strategically by withholding certain licenses from the sale to drive up the prices of the remaining licenses. Note that our context is different in that we are examining collusive behavior on behalf of the spectrum buyers or secondary users.

Let A be an agent who bids on behalf of a subgroup of bidders $N^A \subseteq N$. As in Decarolis, Goldmanis, and Penta [6], A is able to make proposals of binding agreements to members of N^A subject to the following two stability conditions: The first is that no member of N^A prefers to leave the group and bid on his own. The second is that A's proposal must be consistent with the non-members' equilibrium behavior. All $i \notin N^A$ bid as independents. We refer to N^A as a coalition or cartel.

Agent A chooses both bids and payments for its members. Let b_j^A be the bid proposed by A for $j \in N^A$ and let $p_j^A(x)$ be the payment proposed by A for j if N^A wins an item, and let x be the total payment owed by N^A to the seller. Assume A knows the valuations for all $j \in N^A$ and assume that A's goal is to maximize the sum of its members' payoffs.

Note that the first stability condition assumes that an agent who exits the coalition acts as an independent and bids on his own, while the remaining coalition members stay in the coalition and bid accordingly. This stability condition is in line with Decarolis, Goldmanis, and Penta [6], but is different from coalition stability conditions, which assume that a single coalition member has veto power over the coalition. Our definition makes sense within the context of the coalition being a group or agency where one person's exit does not eliminate the plan or group for others. For example, consider a company or agent that runs multiple wireless networks, where a single network can defect and purchase spectra directly from the secondary spectrum marketplace, and this defection will not affect the networks remaining with the original service provider.

The solution concept which applies to this model is similar to the recursively-stable agency equilibrium of Decarolis, Goldmanis, and Penta [6]. The agent has complete information regarding its members but does not know the valuations of non-members; the agent maximizes the sum of the coalition member's payoffs subject to two stability conditions. Coalition members know that the agent has complete information regarding the members and choose to participate in the coalition or to become an independent bidder. All bidders know the makeup of the non-conflicting groups and know their own valuations. The independent bidders will choose to bid truthfully, as in Proposition 1. These assumptions are captured in the two stability conditions. Proposition 3 shows that if all coalition members belong to the same non-conflicting group, then it is a weakly dominant strategy for the coalition to set the bid of its lowest valued bidders equal to the average value of its members. Thus, Proposition 3 shows that the coalition will deviate from truthful bidding by increasing at least its minimum bid.

Let $b^A_{\gamma\min} = \min_{j \in N^A \cap G_\gamma} b_j^A$. Thus, $b^A_{\gamma\min}$ is the minimum bid A submits for coalition members in the non-conflicting group G_γ.

Proposition 3. *Let $N^A = G_\gamma$ for some non-conflicting group G_γ and let $\{1', 2', \ldots, j'\} = N^A$. Order agents so that $v_{1'} \geq v_{2'} \geq \cdots \geq v_{j'}$ and assume $v_{1'} > v_{j'}$. Then, it is a weakly dominant strategy for $b^A_{\gamma\min} = (\sum_{j=1'}^{j'} v_j)/j'$.*

Notice that $b^A_{\gamma\min} = (\sum_{j=1'}^{j'} v_j)/j' > v_{j'}$ as $v_{1'} > v_{j'}$.

If all coalition members are assigned to the same non-conflicting group, then the coalition has incentive to increase their group bid in order to increase their chance of winning an item. This bid increase is possible because the high-value group bidders can pay more if the group wins an item at a price higher than some members' valuation, which allows low-value bidders to pay below their valuation.

Proof. We show that A can make all members of N^A better off by bidding $b^A_{\gamma\min} = (\sum_{j=1'}^{j'} v_j)/j'$, with carefully chosen payments $p_j^A(x)$, $j \in \{1', 2', \ldots, j'\}$, than they would be by bidding independently. Note that any $j \in G_\gamma$ could choose to become an independent bidder and bid $v_{j'}$. Thus, all coalition members must be made better off than the case of $b^A_{\gamma\text{min}} = v_{j'}$, or when G_γ's minimum bid is $v_{j'}$. First, compare the case of $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$ to that of $b^A_{\gamma\text{min}} = v_{j'}$. If G_γ is one of the κ largest bidders when $b^A_{\gamma\text{min}} = v_{j'}$, then it will remain so when $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$ and the price paid by G_γ, $G_{(\kappa+1)}$'s bid of $b^G_{\kappa+1}$ will remain the same. Suppose instead that G_γ is not one of the κ largest bidders when $b^A_{\gamma\text{min}} = v_{j'}$, but is one of the κ largest bidders when $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$. We show that all agents in G_γ can be made better off. Specifically, we show that there exists $p^A_i(x)$ for all $i \in \{1', 2', \ldots, j'\}$ so that all members of the coalition are better off with $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$ than with $b^A_{\gamma\text{min}} = v_{j'}$. By this assumption, G_γ is one of the κ largest bidders. The largest possible price that G_γ could be assigned is $|G_\gamma| \cdot b^A_{\gamma\text{min}} = j' \cdot b^A_{\gamma\text{min}}$, as we know that $j' \cdot b^A_{\gamma\text{min}}$ is an upper bound on G_γ's group bid and thus the next highest group's bid cannot exceed this. We show that all members of G_γ can be made better off with a group price of $x \leq j' \cdot b^A_{\gamma\text{min}}$. Here, let A assign payments of $p^A_i(x) = v_i - \epsilon_i$ for all $i \in \{1', 2', \ldots, j'\}$ where $\epsilon_i = \frac{v_i}{\sum_{k=1'}^{j'} v_k}(\sum_{k=1'}^{j'} v_k - x)$. Then, i receives a payoff of $v_i - (v_i - \epsilon_i) = \epsilon_i \geq 0$, which is greater than or equal to the payoff of 0 they would receive if $b^A_{\gamma\text{min}} = v_{j'}$ and G_γ was not one of the κ largest bidders. If $x < j' \cdot b^A_{\gamma\text{min}}$, then $\epsilon_i > 0$ and i is strictly better off than if $b^A_{\gamma\text{min}} = v_{j'}$. By construction, the sum of payments equals the amount owed or $\sum_{i=1'}^{j'} p^A_i = x$. Note that $\epsilon_i = \frac{v_i}{\sum_{k=1'}^{j'} v_k}(\sum_{k=1'}^{j'} v_k - x)$ assigns each coalition member i a portion of the coalition surplus $\sum_{k=1'}^{j'} v_k - x$ in proportion to i's valuation.

Thus, by bidding $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j' > v_{j'}$, A can guarantee each coalition member a payoff of $\epsilon_i > 0$ whenever the $(\kappa+1)$ highest bid is $v_{j'} < b_{\kappa+1} < (\sum_{j=1'}^{j'} v_j)/j'$; coalition members would prefer to receive ϵ_i, as bidding independently gives them a payoff of 0. And when $b_{\kappa+1} \leq v_{j'}$, then A can leave payments and surpluses the same as when bidders bid independently. Thus, by bidding $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$, A can guarantee that all $j \in G_\gamma$ prefer to be in the coalition and no one will bid independently.

Next, we show that A will never set $b^A_{\gamma\text{min}} > (\sum_{j=1'}^{j'} v_j)/j'$ or $b^A_{\gamma\text{min}} < (\sum_{j=1'}^{j'} v_j)/j'$. First, assume, to the contrary, that $b^A_{\gamma\text{min}} > (\sum_{j=1'}^{j'} v_j)/j'$. If the $k+1$ highest bid is $b^A_{\gamma\text{min}} \geq b_{k+1} > (\sum_{j=1'}^{j'} v_j)/j'$, then G_γ may win an item and be asked to pay $x = j' \cdot b_{k+1}$. As $b_{k+1} > (\sum_{j=1'}^{j'} v_j)/j'$, we know that $x = j' \cdot b_{k+1} > (\sum_{j=1'}^{j'} v_j)$ and so there is no way to divide the payment x among N^A except by requiring at least one coalition member to pay more than their valuation. If the $(k+1)$ highest bid is $b^A_{\gamma\text{min}} > (\sum_{j=1'}^{j'} v_j)/j' \geq b_{k+1}$, then the winning groups and payments are identical to the $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$ case. Thus, there is no gain to A from setting $b^A_{\gamma\text{min}} > (\sum_{j=1'}^{j'} v_j)/j'$. Next let $b^A_{\gamma\text{min}} < (\sum_{j=1'}^{j'} v_j)/j'$. First, assume $b^A_{\gamma\text{min}} < b_{k+1} < (\sum_{j=1'}^{j'} v_j)/j'$. Then, as $b^A_{\gamma\text{min}} < b_{k+1}$, G_γ does not win an item. But as $b_{k+1} < (\sum_{j=1'}^{j'} v_j)/j'$, the members of N^A would prefer to win the item and divide the payment, as in the $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$ case. Thus, A would be better off setting $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$. If $b_{k+1} < b^A_{\gamma\text{min}} < (\sum_{j=1'}^{j'} v_j)/j'$, then the winning groups and payments are the same as the $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$ case. If $b^A_{\gamma\text{min}} < (\sum_{j=1'}^{j'} v_j)/j' < b_{k+1}$, then G_γ does not win an item and it also does not win an item if $b^A_{\gamma\text{min}} = (\sum_{j=1'}^{j'} v_j)/j'$.

Thus, A cannot gain from setting $b^A_{\gamma^{\min}} < (\sum_{j=1'}^{j'} v_j)/j'$. Therefore, $b^A_{\gamma^{\min}} = (\sum_{j=1'}^{j'} v_j)/j'$ is a weakly dominant strategy. □

Note that we assumed that A knows all the valuations of its members. An interpretation is that the coalition members share a plan owned by A and A has access to information about its members. For instance, consider again an agent that bids on a spectrum for multiple wireless networks; such an agent should have information about the value of the spectrum to its customers. This assumption is similar to that made by Decarolis, Goldmanis, and Penta [6] for the context of an agent bidding for online ads. Additionally, note that we assume that the non-conflicting group assignment is common information. This assumption is made for simplicity, as allowing this information to be private information would complicate the analysis.

Next, we will compare Proposition 3 to the first price auction result, where bidders shade their bids to below their valuations. In the first price auction, bidders control both the probability of winning an item and the price, as the price equals the winning bid. Thus, bidders wish to increase their bids to increase their chance of winning and wish to decrease their bids to increase their surplus or decrease the price paid. In the group auction of Proposition 3, the coalition controls the probability of winning an item with their bid, but does not control the price paid, as it is the highest losing bid or the $\kappa+1$ highest bid. Thus, increasing the coalition's bid can increase the probability of winning the item, but cannot increase the price. The coalition is able to increase the price above the value of its lowest bidder, because it can make internal transfers from high- to low-value coalition members so that all coalition members including the low-value bidder receive a positive surplus.

Next, we illustrate the results of Proposition 3.

Example 2. *Reconsider Example 1, where SUs again have the conflict graph of Figure 1 and* $v = (v_1, v_2, v_3, v_4, v_5) = (16, 13, 12, 11, 11)$. *Let the non-conflicting groups be* $G_1 = \{2,3\}$, $G_2 = \{1,4\}$, *and* $G_3 = \{5\}$. *Let* $N^A = \{1,4\}$. *By Proposition 3, A can increase the payoffs of its members by increasing the bid of its lowest-value member agent* 4. *The maximum value A can choose is* $b^A_4 = \frac{v_1+v_4}{2} = 13.5$ *and A will set* $b^A_1 = v_1 = 16$. *All other agents bid independently and will bid their valuations.* G_1 *will have group bid* $b^G_1 = 2 \cdot 12 = 24$, *while* G_2 *has* $b^G_2 = 2 \cdot 13.5 = 27$, *and* $b^G_3 = 11$. *Thus,* G_2 *wins the auction and must pay the price* 24. *As* $\frac{24}{2} = 12 > v_4 = 11$, *A cannot ask both group members to split the price. Thus, A should assign prices to divide the surplus between the two members. If A chooses to split the surplus as described in the proof of Proposition 3, then the surplus of* $v_1 + v_4 - 24 = 27 - 24 = 3$ *is split between the two agents in proportion to their valuations. Thus, agent* 1 *will pay* $p^A_1 = 14.22$ *and will receive a surplus of* $\frac{v_1}{v_1+v_4} \cdot 3 = 1.78$, *while* 4 *will pay* $p^A_4 = 9.78$ *and will receive surplus of* 1.22.

Next, we compare this result to that of Example 1 when all agents bid independently and there are no coalitions. With no collusion, G_1 *wins the group auction and pays the group price* 22. *Social welfare equals* $SW = v_2 + v_3 = 25$. *While, with collusion,* G_2 *wins the auction and pays group price* 24. *Social welfare here equals* $SW = v_1 + v_4 = 27$. *Thus, both auction revenue and social welfare have increased with the addition of collusion. Note that this comparison is carried out using the weakly dominant strategy outcomes of Propositions 1 and 3.*

Proposition 4. *Let* $\kappa = 1$ *and let the number of non-conflicting groups equal* $\mu > 2$. *Let* $\{1', 2', \ldots, j'\} = G_\gamma$, *let* $\{1'', 2'', \ldots, k''\} = G_\beta$ *for the non-conflicting groups* G_γ *and* G_β, *and let* $N^A = \{1', 2', \ldots, j'\} \cup \{1'', 2'', \ldots, k''\}$. *Order the agents so that* $v_{1'} \geq v_{2'} \geq \cdots \geq v_{j'}$ *and* $v_{1''} \geq v_{2'} \geq \cdots \geq v_{k''}$. *Assume* $v_{1'} > v_{j'}$ *and* $v_{1''} > v_{k''}$. *Let* $\sum_{i=1'}^{j'} v_i > \sum_{i=1''}^{k''} v_i$. *Then, it is a weakly dominant strategy for* $b^A_{\gamma^{\min}} = (\sum_{j=1'}^{j'} v_j)/j'$ *and* $b^A_{i''} = 0$ *for* $i'' \in G_\beta$.

Proposition 4 shows that if an agent controls two groups, then she has incentive to increase the minimum bid of the highest-value group and decrease the bid of the other group.

Proof. As $\kappa = 1$, only one non-conflicting group will be awarded an item and as $\mu > 2$, there is at least one non-conflicting group that A does not control. A has incentive to make sure that either G_γ or G_β wins the item. Since $\sum_{i=1'}^{j'} v_i > \sum_{i=1''}^{k''} v_i$, G_γ can make a higher group bid than G_β can, as $\sum_{i=1'}^{j'} v_i$ is the maximum G_γ can pay for an item. Let A set $b^A_{\gamma\min} = \frac{\sum_{i=1'}^{j'} v_i}{j'}$. And let A set $b^A_{1''} = \cdots = b^A_{k''} = 0$, as this bid will minimize the amount that G_γ must pay. Next, we check that all members of N^A are made better off by belonging to N^A than by bidding independently. If G_γ does not win the item with this bid, then they would also not win the item with a bid of $b^A_{\gamma\min} = v_{j'}$. Consider the case where G_γ does win the item with $b^A_{\gamma\min} = \frac{\sum_{i=1'}^{j'} v_i}{j'}$ but does not win the item with $b^A_{\gamma\min} = v_{j'}$. Let $i' \in \{1', 2', \ldots, j'\}$. If G_γ is asked to pay $\sum_{i=1'}^{j'} v_i$, then A can assign payments $p^A_{i'}$ to i', as in the proof of Proposition 3, so that each i' receives a payoff of $\epsilon_{i'} > 0$, which is larger than the payoff of 0 they would receive from $b^A_{\gamma\min} = v_{j'}$. Next, let $i'' \in \{1'', 2'', \ldots, k''\}$. If i'' leaves the coalition and bids independently so that $b_{i''} > 0$, and all others remain in the coalition, then the minimum bid of G_β will remain at 0 as A has no incentive to change the remaining bids, so i'''s defection will have no effect. Thus, i'''s payoff is always 0 and is not made worse by remaining in the coalition. Thus, no coalition member has an incentive to leave the group. As A's goal is to maximize the sum of the members' payoffs, setting $b^A_{\gamma\min} = (\sum_{j=1'}^{j'} v_j)/j'$ is better than $b^A_{\gamma\min} < (\sum_{j=1'}^{j'} v_j)/j'$, as doing so will decrease the chance that G_γ wins an item; the details are similar to those found in the proof of Proposition 3. Similarly, A will not set $b^A_{\gamma\min} > (\sum_{j=1'}^{j'} v_j)/j'$, as this would require at least one member of G_γ to pay more than their valuation. Thus, A cannot gain from setting $b^A_{\gamma\min} > (\sum_{j=1'}^{j'} v_j)/j'$ or $b^A_{\gamma\min} < (\sum_{j=1'}^{j'} v_j)/j'$ Thus, it is a weakly dominant strategy for $b^A_{\gamma\min} = (\sum_{j=1'}^{j'} v_j)/j'$ and $b^A_{i''} = 0$ for $i'' \in G_\beta$. □

5. Concluding Remarks

The auctions of non-rivalrous goods with interference are examined. The results show that group auctions can achieve higher social welfare than individual auctions if the valuations are not too far apart. Additionally, collusion in group auctions can result in higher bids if the coalition increases their chance of winning the item by allowing high-value bidders to subsidize the payments of low-value bidders.

A possible extension would be to allow for multiple competing collusive agents. If competing agents completely controlled separate non-conflicting groups, then group prices may be bid downward. However, if competing agents had coalition members in the same non-conflicting group, it is possible that bids would be pushed higher. Such an analysis is beyond the scope of the current investigation and I leave this inquiry for future research.

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References

1. Vickrey, W. Counterspeculation, auctions, and competitive sealed tenders. *J. Financ.* **1961**, *16*, 8–37. [CrossRef]
2. Dasgupta, P.; Maskin, E. Efficient auctions. *Q. J. Econ.* **2000**, *115*, 341–388. [CrossRef]
3. Klemperer, P. Auction theory: A guide to the literature. *J. Econ. Surv.* **1999**, *13*, 227–286. [CrossRef]
4. Milgrom, P. Auction research evolving: Theorems and market designs. *Am. Econ. Rev.* **2021**, *111*, 1383–1405. [CrossRef]

5. Wang, X.; Umehira, M.; Han, B.; Zhou, H.; Li, P.; Wu, C. An efficient privacy preserving spectrum sharing framework for internet of things. *IEEE Access* **2020**, *8*, 34675–34685. [CrossRef]
6. Decarolis, F.; Goldmanis, M.; Penta, A. Marketing agencies and collusive bidding in online ad auctions. *Manag. Sci.* **2020**, *66*, 4433–4454. [CrossRef]
7. Agranov, M.; Yariv, L. Collusion through communication in auctions. *Games Econ. Behav.* **2018**, *107*, 93–108. [CrossRef]
8. Marshall, R.C.; Marx, L.M. Bidder collusion. *J. Econ. Theory* **2007**, *133*, 374–402. [CrossRef]
9. Khezr, P.; Cumpston, A. A review of multiunit auctions with homogeneous goods. *J. Econ. Surv.* **2022**, *36*, 1225–1247. [CrossRef]
10. Hortaçsu, A.; McAdams, D. Empirical work on auctions of multiple objects. *J. Econ. Lit.* **2018**, *56*, 157–184. [CrossRef]
11. Mailath, G.J.; Zemsky, P. Collusion in second price auctions with heterogeneous bidders. *Games Econ. Behav.* **1991**, *3*, 467–486. [CrossRef]
12. Graham, D.A.; Marshall, R.C. Collusive bidder behavior at single-object second-price and English auctions. *J. Political Econ.* **1987**, *95*, 1217–1239. [CrossRef]
13. McAfee, R.P.; McMillan, J. Bidding rings. *Am. Econ. Rev.* **1992**, *82*, 579–599.
14. Conley, T.G.; Decarolis, F. Detecting bidders groups in collusive auctions. *Am. Econ. J. Microecon.* **2016**, *8*, 1–38. [CrossRef]
15. Asker, J. A study of the internal organization of a bidding cartel. *Am. Econ. Rev.* **2010**, *100*, 724–762. [CrossRef]
16. Pesendorfer, W.; Swinkels, J.M. Efficiency and information aggregation in auctions. *Am. Econ. Rev.* **2000**, *90*, 499–525. [CrossRef]
17. Feldman, M.; Fu, H.; Gravin, N.; Lucier, B. Simultaneous auctions without complements are (almost) efficient. *Games Econ. Behav.* **2020**, *123*, 327–341. [CrossRef]
18. Baisa, B. Efficient multiunit auctions for normal goods. *Theor. Econ.* **2020**, *15*, 361–413. [CrossRef]
19. Benedetto, F.; Mastroeni, L.; Quaresima, G. Auction-based theory for dynamic spectrum access: A review. In Proceedings of the 2021 44th International Conference on Telecommunications and Signal Processing (TSP), Brno, Czech Republic, 26–28 July 2021; pp. 146–151.
20. Wang, X.; Li, Z.; Xu, P.; Xu, Y.; Gao, X.; Chen, H.H. Spectrum sharing in cognitive radio networks—An auction-based approach. *IEEE Trans. Syst. Man Cybern. Part B (Cybern.)* **2009**, *40*, 587–596. [CrossRef] [PubMed]
21. Huang, J.; Berry, R.A.; Honig, M.L. Auction-based spectrum sharing. *Mob. Netw. Appl.* **2006**, *11*, 405–408. [CrossRef]
22. Khaledi, M.; Abouzeid, A.A. Dynamic spectrum sharing auction with time-evolving channel qualities. *IEEE Trans. Wirel. Commun.* **2015**, *14*, 5900–5912. [CrossRef]
23. Milgrom, P.; Vogt, A. Spectrum Auctions from the Perspective of Matching. Technical Report, Working Paper May. CHOOSiNG AN AUCTiON FORMAT 161. 2021. Available online: https://perma.cc/4LR6-Q9LM (accessed on 5 September 2023).
24. Milgrom, P. Game theory and the spectrum auctions. *Eur. Econ. Rev.* **1998**, *42*, 771–778. [CrossRef]
25. Watts, A. Two ways to auction off an uncertain good. *J. Econ.* **2016**, *119*, 1–15. [CrossRef]
26. Watts, A. Generalized Second Price Auctions over a Network. *Games* **2018**, *9*, 67. [CrossRef]
27. Welsh, D.J.; Powell, M.B. An upper bound for the chromatic number of a graph and its application to timetabling problems. *Comput. J.* **1967**, *10*, 85–86. [CrossRef]
28. Doraszelski, U.; Seim, K.; Sinkinson, M.; Wang, P. *Ownership Concentration and Strategic Supply Reduction*; Technical Report; National Bureau of Economic Research: Cambridge, MA, USA, 2017.
29. Milgrom, P.; Segal, I. Clock auctions and radio spectrum reallocation. *J. Political Econ.* **2020**, *128*, 1–31. [CrossRef]