

Classification of $\mathbb{Z}/2\mathbb{Z}$ -quadratic unitary fusion categories

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Abstract. A unitary fusion category is called $\mathbb{Z}/2\mathbb{Z}$ -quadratic if it has a $\mathbb{Z}/2\mathbb{Z}$ group of invertible objects and one other orbit of simple objects under the action of this group. We give a complete classification of $\mathbb{Z}/2\mathbb{Z}$ -quadratic unitary fusion categories. The main tools for this classification are skein theory, a generalization of Ostrik’s results on formal codegrees to analyze the induction of the group elements to the center, and a computation similar to Larson’s rank-finiteness bound for $\mathbb{Z}/3\mathbb{Z}$ -near group pseudounitary fusion categories. This last computation is contained in an appendix coauthored with attendees from the 2014 AMS MRC on Mathematics of Quantum Phases of Matter and Quantum Information.

1. Introduction

In the past several decades, unitary fusion categories (UFCs) have seen broad applications to many areas of mathematics and physics, including representation theory, operator algebras, topological quantum field theory (TQFT), theoretical condensed matter, and quantum information. Given the complete list of 6j-symbols for a UFC, one can build unitary TQFTs which compute quantum invariants of links and 3-manifolds [4, 15], together with physical lattice models which realize these TQFTs [31, 32]. These computations are increasingly difficult in the presence of *multiplicity*, i.e., where there is a fusion channel with a dimension greater than 1, a.k.a. a fusion coefficient which is larger than 1.

While many classification techniques work well for multiplicity free fusion categories, more techniques are required to achieve classification in the multiplicity setting. We note that at the time of writing UFCs have only been classified up to rank 3 [48, 49]. For rank 4 fusion categories with a dual pair of simple objects, there is a classification of possible fusion rings in the pseudounitary setting [30]; our Theorem A (and Corollary B) below completes the classification of rank 4 UFCs with a dual pair of simples. The case of rank 4 with 4 self-dual objects still

seems out of reach at this time. Multiplicity free fusion rings up to rank 6 admitting unitary categorifications have been classified [36].

Surprisingly, all currently known fusion categories fit into four families: (1) those related to¹ groups, (2) those related to quantum groups at roots of unity [1, 2, 18, 52, 54–56], (3) the Haagerup-Izumi quadratic categories [3, 12, 13, 20, 24–26, 53], and (4) the Extended Haagerup fusion categories [6, 19]. Given a finite group G , a G -quadratic fusion category is a fusion category $\mathcal{C}$ with a finite group G of simple objects and one other G -orbit $\{g\rho\}_{g \in G}$ of simple objects. (The collection of all G -quadratic fusion categories over all finite groups G is exactly the family (3) above.) The term ‘quadratic’ comes from the existence of a quadratic relation for the self-fusion of an object ρ which generates the other G -orbit. For a family of fusion rings with a fixed rank, we say the family has rank-finiteness if only finitely many of these rings admit a categorification. Surprisingly, for a fixed group G beyond the trivial group, rank-finiteness is not known for G -quadratic fusion categories (for the trivial group, see [49]). The case $G = \mathbb{Z}/2\mathbb{Z}$ is classified in the pivotal setting in [48], and rank-finiteness for $G = \mathbb{Z}/3\mathbb{Z}$ is achieved in the pseudounitary setting in [30].

In this article, we give a complete classification of $\mathbb{Z}/2\mathbb{Z}$ -quadratic unitary fusion categories. While we do not find any new fusion categories in this article, we provide important techniques for finding 6j-symbols for fusion categories with multiplicity. Our main theorem is as follows.

Theorem A. *The complete list of $\mathbb{Z}/2\mathbb{Z}$ quadratic UFCs is as follows.*

3 object categories:

- *the Ising/Tambara-Yamagami categories of the form $\mathcal{TY}(\mathbb{Z}/2\mathbb{Z}, \chi, \pm)$ [53] with A_3 fusion rules, of which there are exactly 2. The case $+$ is Temperley-Lieb-Jones at $q = \exp(2\pi i/8)$, and the case $-$ is $SU(2)_2$.*
- *the three UFCs with $\text{Rep}(S_3)$ fusion rules [11, Remark 6.6] and [25, Theorem 5.1].*
- *the two complex conjugate UFCs with $\text{Ad}(E_6)$ fusion rules [8, 24]. These are exactly the adjoint subcategories of the exceptional quantum subgroups of Temperley-Lieb-Jones at $q = \exp(2\pi i/24)$ and $SU(2)_{10}$ [29, 45].*

4 object categories:

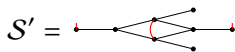
¹Here related to means obtained by iterating known constructions such as equivariantisation, Morita equivalence, ect.

- the pointed categories $\text{Hilb}(\mathbb{Z}/4\mathbb{Z}, \omega)$ where $\omega \in H^3(\mathbb{Z}/4\mathbb{Z}, U(1))$ and $\text{Hilb}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \omega)$ where

$$\omega \in H^3(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, U(1)) / \text{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^3 / \text{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

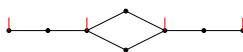
[10, Remark 4.10.4].

- the Deligne products $\text{Fib} \boxtimes \text{Hilb}(\mathbb{Z}/2\mathbb{Z}, \omega)$ for $\omega \in H^3(\mathbb{Z}/2\mathbb{Z}, U(1))$, which have A_4 fusion rules. These two categories are also Temperley-Lieb-Jones at $q = \exp(2\pi i/10)$ and $SU(2)_3$.
- $\text{Ad}(SU(2)_6)$, which is also equivalent to the adjoint subcategory of the A_7 Temperley-Lieb-Jones category with $q = \exp(2\pi i/16)$.
- the even parts of the two complex conjugate subfactor planar algebras with principal graphs



from [26, 34]. These categories are also de-equivariantizations of $2^{\mathbb{Z}/4\mathbb{Z}}1$ near group fusion categories [25, Ex. 9.5] [35, Ex. 2.2].

- the even part of the 2D2 subfactor planar algebra with principal graph



from [39] [26, Cor. 9.3]. This category is also a de-equivariantization of the even part of the $3^{\mathbb{Z}/4\mathbb{Z}}$ subfactor from [26, 51].

All these UFCs are related to quantum groups at roots of unity or near group fusion categories [13, 25].

Remark 1.1. The results of Theorem A make no assumptions on the existence of a braiding on the category. The categories appearing in our classification which do not admit braidings are: the two UFC's with $\text{Rep}(S_3)$ fusion rules which are not equivalent to $\text{Rep}(S_3)$ [46, Section 4.4], the two UFC's with $\text{Ad}(E_6)$ fusion rules [46, Main Theorem], the even parts of the two complex conjugate subfactor planar algebras with principal graphs S' [34], and the even part of the 2D2 subfactor (which can be seen to not admit a braiding from the centre analysis in Subsection 3.2). It is interesting to note that the even parts of the subfactors with S' principal graphs admit σ -braidings as defined in [34, Definition 3.2].

The result [30, Thm. 1.1] gave a finite list of possible fusion rings for rank 4 pseudounitary fusion categories with a dual pair of simple objects, but included one fusion ring not previously known to be categorifiable (the case $c = 2$ from

[30, Thm. 1.1(6)]), and left open the classification of those fusion rings from [30, Thm. 1.1] which were previously known to be categorifiable.

Corollary B. *We have a complete classification of rank 4 unitary fusion categories with a dual pair of simple objects. In particular, there is no UFC with $c = 2$ from [30, Thm. 1.1(6)].*

One tool to prove our classification is an adaptation of Larson’s rank-finiteness bound for $\mathbb{Z}/3\mathbb{Z}$ -near group pseudounitary fusion categories [30, §4]. This adaptation appears in Appendix A below, coauthored with attendees from the 2014 AMS MRC program on the Mathematics of Quantum Phases of Matter and Quantum Information.

Our main new technical tool to achieve Theorem A is a generalization of Ostrik’s results on formal codegrees of a spherical fusion category [47, 48]. We use results of [41, §5], but we use the conventions of [23]. Suppose C is a spherical fusion category, and denote by $\mathcal{F} : Z(C) \rightarrow C$ the forgetful functor and let $\mathcal{I} : C \rightarrow Z(C)$ be its adjoint. Let A be the tube algebra of C , and let $A_{X \leftarrow X}$ be the corner of A corresponding to $X \in \text{Irr}(C)$. We pick a non-degenerate trace Tr_X on $A_{X \leftarrow X}$ given by

$$\text{Tr}_X \left(\begin{array}{c} \text{X} \\ \boxed{f} \\ \text{W} \quad \text{X} \\ \bigcirc \end{array} \right) \overline{\text{W}} := \delta_{\text{W}=\mathbf{1}} \dim(X) \text{tr}_C(f).$$

Given an irreducible representation (V, π_V) of $A_{X \leftarrow X}$, its *formal codegree* [38, 47] with respect to Tr_X is given by

$$f_V := \sum_b \text{Tr}_V(\pi(b)) \pi(b^\vee)$$

where $\{b\}$ is a basis of $A_{X \leftarrow X}$ and $\{b^\vee\}$ is the dual basis with respect to the non-degenerate pairing $(a, b) := \text{Tr}_X(ab)$ on $A_{X \leftarrow X}$. Observe that f_V is independent of the choice of basis $\{b\}$, but depends on the choice of Tr_X . We refer the reader to §2.2 for more details.

Theorem C. *There is a bijective correspondence between irreducible representations (V, π_V) of $A_{X \leftarrow X}$ and simple subobjects $\Gamma_V \subset \mathcal{I}(X) \in Z(C)$. The formal codegree f_V of (V, π) with respect to Tr_X is a scalar, and the categorical dimension of Γ_V is given by $\frac{\dim(C)}{f_V \dim(X)}$. Moreover, if $Y \in \text{Irr}(C)$ and ${}_X \pi_Y$ is the action of $A_{X \leftarrow X}$ on $A_{X \leftarrow Y}$, then*

$$\dim \text{Hom}_C(\mathcal{F}(\Gamma_V) \rightarrow Y) = \dim \text{Hom}(\pi_V \rightarrow {}_X \pi_Y).$$

In the case $X = \mathbf{1} \in \text{Irr}(C)$, this theorem recovers [48, Thm. 2.13], which allowed the computation of the simple decomposition of $\mathcal{I}(\mathbf{1})$ in terms of representations of the fusion algebra of C . Our theorem generalises this result in several ways. The main improvement is that this result allows us to determine the simple decomposition of $\mathcal{I}(X)$ by studying the representations of the corner of the tube algebra $A_{X \leftarrow X}$. When $X = \mathbf{1}$, this algebra is isomorphic to the fusion algebra of C . However, when X is non-trivial, this corner depends on certain 6j-symbols of the category involving X . One immediate application of this theorem comes from the fact that the dimensions of simple objects in $Z(C)$ are highly restricted, which implies the representations of $A_{X \rightarrow X}$ (which depend on the 6j-symbols) are also restricted. Hence we obtain obstructions based on 6j-symbols. We make use of this application in this article to determine several non-trivial 6j-symbols involving the invertible object of a $\mathbb{Z}/2\mathbb{Z}$ -quadratic UFC.

The other improvement Theorem C offers is that for each simple $\Gamma \subset \mathcal{I}(X)$, we can determine $\mathcal{F}(\Gamma) \in C$. This information is encoded in the action of $A_{X \leftarrow X}$ acting on the entire tube algebra A . As these algebras are semi-simple, it is easy to decompose this action into irreducibles, and hence apply Theorem C. A surprising application of this theorem comes from the fact that if we know the action of $A_{X \leftarrow X}$ acting on the entire tube algebra A up to isomorphism, we can often determine the action on the nose. As this action is determined by the 6j-symbols of C , this allows us to find many linear equations involving the 6j-symbols. We use this application later in this article to get our hands on many 6j-symbols. In the general setting, this result allows the combinatorial data of the forgetful functor $Z(C) \rightarrow C$ to be leveraged into the categorical data of the 6j-symbols of C . As the forgetful functor can often be easily determined from the fusion ring of C [40], we expect this application to have many exciting future uses.

2. Preliminaries

We refer the reader to [10] for the basics of fusion categories. We refer the reader to [22, 50] for the basics of unitary fusion categories. In particular, we always assume a unitary fusion category is equipped with its unique unitary spherical structure where the daggers of cups are caps and the quantum dimensions are equal to the Frobenius-Perron dimensions [37].

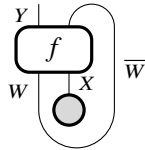
2.1. The tube algebra

One of the key tools in this paper is *Ocneanu's tube algebra* (or equivalently the annular category) of a fusion category. This algebra was first introduced by [44] and [14] in the context of subfactors, and by [23, 24] and [41] in the context of fusion categories.

Definition 2.1. Let C be a spherical fusion category whose spherical trace is denoted tr_C . The tube algebra A of C is the finite dimensional semisimple algebra

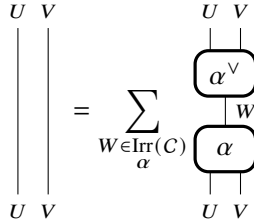
$$\bigoplus_{X, Y \in \text{Irr}(C)} A_{Y \leftarrow X} \quad \text{where} \quad A_{Y \leftarrow X} := \bigoplus_{W \in \text{Irr}(C)} C(W \otimes X \rightarrow Y \otimes W).$$

We graphically represent a fixed basis element of A as



$$f \in C(W \otimes X \rightarrow Y \otimes W).$$

The multiplication on A is defined by composition of the tubes and applying the fusion relation obtained from semisimplicity to the strands around the outside. In more detail, we pick a basis $\{\alpha\} \subset C(U \otimes V \rightarrow W)$ for all $U, V, W \in \text{Irr}(C)$, and let $\{\alpha^\vee\} \subset C(W \rightarrow U \otimes V)$ be the dual basis with respect to the non-degenerate pairing $(\cdot, \cdot) : C(U \otimes V \rightarrow W) \times C(W \rightarrow U \otimes V) \rightarrow \mathbb{C}$ determined by the formula $(h, k) \text{id}_W = h \circ k \in \text{End}_C(W)$. We have the fusion relation



$$\begin{array}{c} U \quad V \\ | \quad | \\ | \quad | \\ U \quad V \end{array} = \sum_{W \in \text{Irr}(C)} \sum_{\alpha} \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{\alpha^\vee} \\ | \quad W \\ \boxed{\alpha} \\ | \quad | \\ U \quad V \end{array}$$

which gives the following formula for composition in the tube algebra, which is independent of the choice of basis $\{\alpha\}$:

$$\begin{array}{c} Z \\ | \\ \boxed{f} \\ / \backslash \\ U \quad Y \\ | \\ \bigcirc \end{array} \bar{U} \cdot \begin{array}{c} Y \\ | \\ \boxed{g} \\ / \backslash \\ V \quad X \\ | \\ \bigcirc \end{array} \bar{V} := \begin{array}{c} Z \\ | \\ \boxed{f} \\ / \backslash \\ U \quad Y \\ | \quad | \\ V \quad X \\ | \quad | \\ \bigcirc \quad \bigcirc \end{array} \bar{U} = \sum_{W \in \text{Irr}(C)} \sum_{\alpha} \begin{array}{c} Z \\ | \\ \boxed{\alpha} \\ | \\ \boxed{f} \\ / \backslash \\ U \quad Y \\ | \quad | \\ V \quad X \\ | \quad | \\ \boxed{\alpha^\vee} \quad \bigcirc \\ | \\ W \end{array} \bar{W}.$$

There is a non-degenerate linear functional ϕ on A given by

$$\begin{array}{c} Y \\ | \\ \boxed{f} \\ / \backslash \\ W \quad X \\ | \\ \bigcirc \end{array} \bar{W} \mapsto \delta_{X=Y} \delta_{W=1} \dim(X) \text{tr}_C(f).$$

Its restriction to $A_{X \leftarrow X}$ is tracial for all $X \in \text{Irr}(C)$, and we denote it by Tr_X .

Note that each of the spaces $A_{X \leftarrow X}$ is the corner $1_X A 1_X$ of A , where we cut down by the idempotent

$$1_X := \begin{array}{c} | \\ \bigcirc \\ X \end{array},$$

and $A_{X \leftarrow X}$ acts on the spaces $A_{X \leftarrow Y}$ by multiplication.

The tube algebra of C is intimately related to the Drinfeld centre $Z(C)$ of C . From the data of $Z(C)$, we obtain a basis of matrix units for the spaces $A_{X \leftarrow Y}$ given by

$$e(\Gamma)_{(X,i),(Y,j)} := \frac{\dim(\Gamma)}{\dim(C) \sqrt{\dim(X) \dim(Y)}} \sum_{W \in \text{Irr } C} \dim(W) \begin{array}{c} Y \\ | \\ \boxed{j'} \\ | \\ \Gamma \\ | \\ \boxed{\beta_{W,\Gamma}} \\ / \backslash \\ W \quad \Gamma \\ | \quad | \\ i \quad X \\ | \\ \bigcirc \end{array} \bar{W}$$

where $(\Gamma, \beta_\Gamma) \in \text{Irr}(Z(C))$, $\{i\}$ is a basis for $C(X \rightarrow \mathcal{F}(\Gamma))$, and $\{j\}$ is a basis for $C(Y \rightarrow \mathcal{F}(\Gamma))$, where $\mathcal{F} : Z(C) \rightarrow C$ is the forgetful functor. Here, $\{j'\} \subset$

$C(\mathcal{F}(\Gamma) \rightarrow Y)$ denotes the dual basis of $\{j\}$ with respect to the pairing $k' \circ j = \delta_{j=k} \text{id}_Y$. With respect to our functional ϕ on A , we have that

$$\phi(e(\Gamma)_{(X,i),(Y,j)}) = \delta_{X,Y} \delta_{i,j} \frac{\dim(X) \dim(\Gamma)}{\dim(C)},$$

and so the dual basis with respect to the ϕ -pairing is given by

$$e(\Gamma)_{(X,i),(Y,j)}^\vee = \frac{\dim(C)}{\dim(X) \dim(\Gamma)} e(\Gamma)_{(Y,j),(X,i)}.$$

The construction above shows us that $Z(C)$ entirely determines the structure of the tube algebra of C . The converse is also true. The tube algebra of C entirely determines the Drinfeld centre of C . The following theorem gives a bijective correspondence between representations of the tube algebra and objects of $Z(C)$.

Theorem 2.2 [23] and [41, §5]. *There is a bijective correspondence between equivalence classes of irreducible representations of the tube algebra of C and isomorphism classes of simple objects in $Z(C)$. This bijection sends*

$$(V, \pi) \mapsto \Gamma_V := \bigoplus_{X \in \text{Irr}(C)} V|_{A_{X \leftarrow X}} \otimes X.$$

Further, we have that the minimal central projection $z_V \in A$ corresponding to (V, π) is given by

$$z_V = \sum_{\substack{X \in \text{Irr}(C), \\ \{i\} \subset C(X \rightarrow \mathcal{F}(\Gamma_V))}} e(\Gamma_V)_{(X,i),(X,i)}.$$

2.2. A new result on formal codegrees

If one knows the full tube algebra of C , then Theorem 2.2 essentially gives you the full data of $Z(C)$. However in many situations, such as in this article, we only know information about some sub-algebra of the tube algebra, and we wish to leverage this information into partial information about $Z(C)$. Towards this goal, we introduce the *formal codegree* of a representation.

Definition 2.3 [38, 47]. Let B be a finite dimensional semisimple algebra equipped with a non-degenerate trace Tr_B , and let (V, π) be a finite dimensional representation of B . We define the *formal codegree* of (V, π) as

$$f_V := \sum_b \text{Tr}_V(\pi(b)) \pi(b^\vee) \in \pi(B) \subset \text{End}(V)$$

where we sum over a basis $\{b\} \subset B$, and $\{b^\vee\}$ denotes the dual basis with respect to the Tr_B -pairing. Observe that f_V is independent of the choice of basis, but depends on the choice of trace Tr_B .

The following theorem allows us to determine the simple summands of $\mathcal{I}(X) \in Z(C)$ by classifying the representations of the subalgebra $A_{X \leftarrow X}$. Here, $\mathcal{I} : C \rightarrow Z(C)$ is the induction functor which is adjoint to the forgetful functor $\mathcal{F} : Z(C) \rightarrow C$. Moreover, we can compute categorical dimensions in terms of formal codegrees of $A_{X \leftarrow X}$ with respect to Tr_X .

Theorem C. *Let C be a spherical fusion category, and let A be the tube algebra of C . Fix $X \in \text{Irr}(C)$. There is a bijective correspondence between equivalence classes of irreducible representations (V, π) of $A_{X \leftarrow X}$ and isomorphism classes of simple subobjects $\Gamma_V \subset \mathcal{I}(X) \in Z(C)$. The formal codegree f_V of (V, π) with respect to Tr_X is a scalar, and the categorical dimension of Γ_V is given by*

$$\dim(\Gamma_V) = \frac{\dim(C)}{f_V \dim(X)}.$$

Moreover, if $Y \in \text{Irr}(C)$ and ${}_X \pi_Y$ is the action of $A_{X \leftarrow X}$ on $A_{X \leftarrow Y}$, then

$$\dim(C(Y \rightarrow \mathcal{F}(\Gamma_V))) = \dim(\text{Hom}(\pi_V \rightarrow {}_X \pi_Y)).$$

Proof. Let (V, π) be an irreducible representation of $A_{X \leftarrow X}$. Since $A_{X \leftarrow X}$ is semisimple, (V, π) corresponds to a simple summand of $A_{X \leftarrow X}$. As $A_{X \leftarrow X}$ is a corner of A , each simple summand of $A_{X \leftarrow X}$ is of the form $A_{X \leftarrow X} z_\Gamma$ for a simple object $(\Gamma, \beta) \in \text{Irr}(Z(C))$. Hence there is a simple $(\Gamma_V, \beta_{\Gamma_V})$ corresponding to (V, π) , and by Theorem 2.2, $z_V 1_X = \sum_i e(\Gamma_V)_{(X,i),(X,i)}$. Moreover, for any other simple object $\Lambda \in \text{Irr}(Z(C))$, we have that $\pi(e(\Lambda)_{(X,i),(X,i)}) = 0$ unless $\Lambda \cong \Gamma_V$. In particular, $\text{Tr}_V(\pi(e(\Lambda)_{(X,i),(X,j)})) = 0$ unless $\Lambda = \Gamma_V$ and $i = j$. We now compute

$$\begin{aligned} f_V &= \sum_{\substack{\Lambda \in \mathcal{I}(X) \\ i,j}} \text{Tr}_V(\pi(e(\Lambda)_{(X,i),(X,j)})) \pi(e(\Lambda)_{(X,i),(X,j)}^\vee) \\ &= \sum_i \frac{\dim(C)}{\dim(X) \dim(\Gamma_V)} \pi(e(\Gamma_V)_{(X,i),(X,i)}) \\ &= \frac{\dim(C)}{\dim(X) \dim(\Gamma_V)} \pi(z_V 1_X). \end{aligned}$$

Thus the formal codegree of (V, π) is given by $f_V = \frac{\dim(C)}{\dim(X) \dim(\Gamma_V)}$, and $\dim(\Gamma_V) = \frac{\dim(C)}{f_V \dim(X)}$. Finally, we observe

$$\begin{aligned}
\dim(\text{Hom}(V \rightarrow {}_X\pi_Y)) &= \dim(\text{Hom}(V \rightarrow 1_X A 1_Y)) \\
&= \dim(\text{Hom}(V \rightarrow z_V 1_X A 1_Y)) \\
&= \dim(\text{Hom}(V \rightarrow z_V 1_X A 1_Y z_V)) \\
&= \sum_j \underbrace{\dim(\text{Hom}(V \rightarrow z_V 1_X A e(\Gamma_V)_{(Y,j),(Y,j)}))}_{=1} \\
&= \dim(C(Y \rightarrow \mathcal{F}(\Gamma_V))). \quad \blacksquare
\end{aligned}$$

Note that if we just consider the subalgebra $A_{\mathbf{1} \leftarrow \mathbf{1}} \cong K_0(C)$, the fusion algebra of C , then the above theorem recovers [48, Theorem 2.13], which shows that irreducible representations of $K_0(C)$ are in bijective correspondence with simple summands of $\mathcal{I}(\mathbf{1})$. Thus our theorem generalises Ostrik's in two ways: (1) it gives us the simple summands of $\mathcal{I}(X)$ where X is any simple object of C , and (2) it tells us the image under the forgetful functor of each of these summands.

2.3. $\mathbb{Z}/2\mathbb{Z}$ -quadratic fusion categories

A $\mathbb{Z}/2\mathbb{Z}$ -quadratic fusion category is a fusion category C whose invertible objects form the group $\mathbb{Z}/2\mathbb{Z}$, i.e., $\text{Inv}(C) = \{1, \alpha\}$ with $\alpha^2 \cong 1$, with one other orbit of simple objects under the $\mathbb{Z}/2\mathbb{Z}$ -action. A simple associativity argument shows we have three cases:

- (Q1) simple objects: $\mathbf{1}, \alpha, \rho$; fusion rules determined by: $\rho^2 \cong 1 \oplus m\rho \oplus \alpha$.
- (Q2) simple objects: $\mathbf{1}, \alpha, \rho, \alpha\rho$, ρ not self-dual; fusion rules determined by: $\rho^2 \cong m\rho \oplus n\alpha\rho \oplus \alpha$.
- (Q3) simple objects: $\mathbf{1}, \alpha, \rho, \alpha\rho$, ρ self-dual; fusion rules determined by: $\rho^2 \cong 1 \oplus m\rho \oplus n\alpha\rho$.

Note that in all three cases we have $\alpha^2 \cong 1$ and $\alpha\rho \cong \rho\alpha$.

2.3.1. Multiplicity bounds and categorifiability. The case (Q1) was classified in the pivotal setting in [48, Thm. 4.1], where it was shown $m \leq 2$. The complete classification of such unitary fusion categories was known prior to this article:

- ($m = 0$) such a fusion category is a Tambara-Yamagami category of the form $\mathcal{T}\mathcal{Y}(\mathbb{Z}/2\mathbb{Z}, \chi, \pm)$ [53], of which there are exactly 2. Both such categories are unitarizable.

- ($m = 1$) such a fusion category has the fusion rules of $\text{Rep}(S_3)$. There are exactly three such unitary fusion categories [25, Theorem 5.1].
- ($m = 2$) such a fusion category has the fusion rules of $\text{Ad}(E_6)$, and there are exactly 4 such fusion categories [21], all within the same Galois orbit, and each admits a spherical structure. Two of these are unitary and complex conjugate to each other [24].

The case (Q2) was studied in the pseudounitary setting ($\dim(C) = \text{FPdim}(C)$) in [30], where it was shown that $m = n \leq 2$. The classification of such fusion categories prior to this article is as follows:

- ($m = 0$) such a fusion category is necessarily pointed with $\mathbb{Z}/4\mathbb{Z}$ fusion rules. It is thus of the form $\text{Vect}(\mathbb{Z}/4\mathbb{Z}, \omega)$ for $\omega \in H^3(\mathbb{Z}/4\mathbb{Z}, U(1)) = \mathbb{Z}/4\mathbb{Z}$, of which there are 4 categories [10, Remark 4.10.4].
- ($m = 1$) this case was open. Two such unitary fusion categories which are complex conjugate were known to exist from [26, 34].
- ($m = 2$) this case was open. No such examples were known to exist.

We finish this classification for unitary fusion categories in Theorem 4.1 below.

In Appendix A, we adapt the results of [30] in the pseudounitary setting to case (Q3), where we prove the following theorem.

Theorem 2.4. *Suppose C is a pseudounitary fusion category with the fusion rules (Q3). Then (m, n) must be equal to one of $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, 2)$.*

Proof. By Theorem A.29 in Appendix A, we must have $m + n \leq 5$. If either m or n is zero, then there is a fusion subcategory with 3 simple objects, so (m, n) must be one of $(0, 0)$, $(0, 1)$, $(1, 0)$ by [49]. If $0 \neq m \neq n \neq 0$, then $m + n \geq 11$ by Remark A.4 in Appendix A. The result follows. ■

The proof that $m + n \leq 5$ that appears in Appendix A below was written by Ryan Johnson, Siu-Hung Ng, David Penneys, Jolie Roat, Matthew Titsworth, and Henry Tucker at the 2014 AMS MRC on The Mathematics of Quantum Phases of Matter and Quantum Information.

By [27, 33] (and applying Galois conjugation), any fusion category with fusion rules (Q3) with $(m, n) \in \{(0, 1), (1, 0)\}$ factorizes as a Deligne product of a fusion category with Fibonacci fusion rules, of which there are two, namely Fib and YL, and a $\mathbb{Z}/2\mathbb{Z}$ -pointed fusion category which must be of the form $\text{Vect}(\mathbb{Z}/2\mathbb{Z}, \omega)$ for $\omega \in H^3(\mathbb{Z}/2\mathbb{Z}, U(1))$, of which there are two. Thus there are exactly 4 such fusion categories, and 2 are unitarizable.

When $m = n \leq 2$, the complete classification of such unitary fusion categories as in Theorem 2.4 is given in Theorem 3.1 below.

3. The self-dual case

In this section we will focus on the unitary categorification of the fusion ring with four simple objects $\mathbf{1}, \alpha, \rho, \alpha\rho$ and fusion rules

$$\alpha \otimes \alpha \cong \mathbf{1} \quad \rho \otimes \rho \cong \mathbf{1} \oplus m\rho \oplus m\alpha\rho. \quad (R(m))$$

Let us write $R(m)$ for such a fusion ring. By Theorem 2.4 above, we know $m \leq 2$. Our main result of this section is as follows.

Theorem 3.1. *The complete classification of unitary fusion categories C_m with $K_0(C_m) \cong R(m)$ for $m \leq 2$ is as follows:*

($m = 0$) C_0 is pointed and thus equivalent to one of the four monoidally distinct categories $\text{Vect}^\omega(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ where

$$\omega \in H^3(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, U(1))/\text{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

[10, Remark 4.10.4].

($m = 1$) C_1 is equivalent to $C(\mathfrak{sl}_2, 7)^{ad}$, which is also equivalent to the even part of the A_7 Temperley-Lieb-Jones category with $q = \exp(2\pi i/16)$ [26, Example 9.1].

($m = 2$) C_2 is equivalent to the even part of the 2D2 subfactor from [26, 39].

Proof. It suffices to restrict our attention to the cases of $m = 1$ and $m = 2$. The first step in our proof is to provide a set of numerical data which fully classifies a categorification of $R(m)$; we do this in §3.1. By describing a sufficient list of local relations in our category, we are able to come up with such a set of numerical data. This data consists of $8m^4$ complex scalars, and a collection of small roots of unity. This data is precisely a subset of the $6j + 4k$ symbols of such a categorification. Using techniques developed in the subfactor classification program, we prove that this subset of the $6j + 4k$ symbols is sufficient to describe the entire category.

In §3.2, in order to get a foothold on the numerical data of a categorification of $R(m)$, we study the Drinfeld centre of such a category. By studying certain small representations of the tube algebra of the category (using basic combinatorial arguments), we are able to deduce a surprising amount of numerical data of the category. This centre analysis tells us nearly all of the small roots of unity in our

numerical data, and even gives us highly non-trivial linear equations involving the $8m^4$ complex scalars.

To reduce the $8m^4$ complex scalars down to a more manageable number, in §3.4, we apply the tetrahedral symmetries of the $6j + 4k$ symbols. These symmetries only apply in the unitary setting, and give S_4 symmetries of these $8m^4$ complex scalars.² This essentially reduces the complexity of the problem by a factor of 24. For example, in the $m = 2$ case, we reduce from 128 complex scalars to roughly 10 (some of the S_4 symmetries are redundant). These symmetries turn an intractable amount of data into a set that can easily be dealt with by hand.

To finish off this section we explicitly solve for the remaining numerical data which describes a categorification of $R(m)$ in §3.5. The results of the previous subsections essentially determine everything except the remaining complex scalars. By evaluating diagrams in our category in multiple ways, we are able to obtain equations of these complex scalars. In the $m = 2$ case we find a single solution, which necessarily has to correspond to the even part of the $2D2$ subfactor. We prove this in Theorem 3.48 below. ■

3.1. Numerical data

Let C_m be a unitary fusion category with $K(C_m) \cong R(m)$, $m \in \{1, 2\}$. The goal of this subsection is to pick nice basis for our morphisms spaces in C_m , and to determine some local relations that these basis elements satisfy. These local relations will depend on the following data:

- two choices of signs $\lambda_\rho, \lambda_\alpha \in \{-1, 1\}$ which are the 2nd Frobenius Schur indicators of α and ρ respectively,
- a choice of sign $\mu \in \{-1, 1\}$,
- $2m$ choices of $\chi_{1,i} \in \{-1, 1\}$ and $\chi_{\alpha,i} \in \{-\sqrt{\lambda_\alpha}, \sqrt{\lambda_\alpha}\}$ for $0 \leq i < m$,
- $2m$ choices of 3rd roots of unity $\omega_{1,i}, \omega_{\alpha,i} \in \{1, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}\}$ for $0 \leq i < m$, and

In the following subsection, we are able to pin down the data μ and χ by analysing the centre of C_m .

To simplify notation, we define $d := \dim(\rho)$, which is the largest solution to $d^2 = 1 + 2md$. If $m = 1$ then $d = 1 + \sqrt{2}$, and if $m = 2$ then $d = 2 + \sqrt{5}$. We choose

² While writing this article, the article [17] was posted to the arXiv, which describes tetrahedral symmetries for general fusion categories. It would be interesting to use their work to extend our results to the non-unitary setting.

orthonormal basexs for our hom spaces

$$\begin{array}{c} \rho \\ \uparrow \\ i \bullet \\ \swarrow \searrow \\ \rho \quad \rho \end{array} \in C_m(\rho \otimes \rho \rightarrow \rho) \quad \begin{array}{c} \alpha \quad \rho \\ \swarrow \searrow \\ \rho \quad \rho \end{array} \in C_m(\rho \otimes \rho \rightarrow \alpha \rho) \quad 0 \leq i < m.$$

We also choose unitary isomorphisms³

$$\begin{array}{c} \bar{\alpha} \\ \uparrow \\ \alpha \end{array} \in C_m(\alpha \rightarrow \bar{\alpha}) \quad \begin{array}{c} \bar{\rho} \\ \uparrow \\ \rho \end{array} \in C_m(\rho \rightarrow \bar{\rho}) \quad \text{and} \quad \begin{array}{c} \bar{\alpha} \quad \rho \\ \swarrow \searrow \\ \rho \quad \alpha \end{array} \in C_m(\rho \otimes \alpha \rightarrow \bar{\alpha} \otimes \rho).$$

We can unambiguously write their inverses as

$$\begin{array}{c} \alpha \\ \uparrow \\ \bar{\alpha} \end{array} \in C_m(\bar{\alpha} \rightarrow \alpha), \quad \begin{array}{c} \rho \\ \uparrow \\ \bar{\rho} \end{array} \in C_m(\bar{\rho} \rightarrow \rho), \quad \text{and} \quad \begin{array}{c} \rho \quad \alpha \\ \swarrow \searrow \\ \bar{\alpha} \quad \rho \end{array} \in C_m(\bar{\alpha} \otimes \rho \rightarrow \rho \otimes \alpha).$$

The duals of these first two isomorphisms are related to their inverses respectively by the Frobenius-Schur indicators of α and ρ , via the following equations:

$$\begin{array}{c} \bar{\alpha} \\ \uparrow \\ \alpha \end{array} = \lambda_\alpha \begin{array}{c} \bar{\alpha} \\ \uparrow \\ \alpha \end{array} \quad \begin{array}{c} \bar{\rho} \\ \uparrow \\ \rho \end{array} = \lambda_\rho \begin{array}{c} \bar{\rho} \\ \uparrow \\ \rho \end{array} \quad \lambda_\alpha, \lambda_\rho \in \{\pm 1\}.$$

We can re-scale the crossing so that

$$\begin{array}{c} \alpha \quad \rho \\ \swarrow \searrow \\ \rho \quad \alpha \end{array} := \begin{array}{c} \alpha \quad \rho \\ \swarrow \searrow \\ \rho \quad \alpha \end{array} = \begin{array}{c} \alpha \quad \rho \\ \swarrow \searrow \\ \rho \quad \alpha \end{array}$$

³Using the convention of switching the orientation of the α -strand through the crossing works better for $\mathbb{Z}/2\mathbb{Z}$ -equivariantization, which is related to the $3^{\mathbb{Z}/4\mathbb{Z}}$ -subfactor [26]. In the non-self-dual case in §4 below, we will use a more natural convention from a diagrammatic point of view which does not change the orientation of the α -strand.

due to the implicit inverses on both sides. Semisimplicity gives us the following local relations.

$$\begin{array}{c}
 \begin{array}{c} \alpha \\ \downarrow \\ \uparrow \\ \bar{\alpha} \end{array} = \begin{array}{c} \alpha \quad \bar{\alpha} \\ \downarrow \quad \downarrow \\ \uparrow \quad \uparrow \\ \alpha \quad \bar{\alpha} \end{array} \quad \begin{array}{c} \rho \quad \rho \\ \downarrow \quad \downarrow \\ \uparrow \quad \uparrow \\ \rho \quad \rho \end{array} = \frac{1}{d} \begin{array}{c} \rho \quad \rho \\ \downarrow \quad \downarrow \\ \uparrow \quad \uparrow \\ \rho \quad \rho \end{array} + \sum_{i=0}^m \begin{array}{c} \rho \quad \rho \\ \downarrow \quad \downarrow \\ \uparrow \quad \uparrow \\ \rho \quad \rho \end{array} + \sum_{i=0}^m \begin{array}{c} \rho \quad \rho \\ \downarrow \quad \downarrow \\ \uparrow \quad \uparrow \\ \rho \quad \rho \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \circlearrowleft \\ \downarrow \\ \uparrow \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ \downarrow \\ \uparrow \\ \circlearrowright \end{array} = 1 \quad \begin{array}{c} \circlearrowleft \\ \downarrow \\ \uparrow \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ \downarrow \\ \uparrow \\ \circlearrowright \end{array} = d
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} i \\ \downarrow \\ \uparrow \\ \rho \end{array} = \begin{array}{c} i \\ \downarrow \\ \uparrow \\ \rho \end{array} = 0 \quad \begin{array}{c} \rho \\ \downarrow \\ \uparrow \\ i \end{array} = 0 \quad \begin{array}{c} \alpha \\ \downarrow \\ \uparrow \\ \rho \end{array} = \begin{array}{c} \alpha \\ \downarrow \\ \uparrow \\ \rho \end{array} = 0 \quad \begin{array}{c} \alpha \\ \downarrow \\ \uparrow \\ \rho \end{array} = \begin{array}{c} \alpha \\ \downarrow \\ \uparrow \\ \rho \end{array} = 0.
 \end{array}$$

(3.2)

Definition 3.3. Let $\mu \in \mathbb{C}^\times$ such that $\begin{array}{c} \bar{\rho} \\ \downarrow \\ \uparrow \\ \alpha \rho \end{array} = \mu \begin{array}{c} \alpha \bar{\rho} \\ \downarrow \\ \uparrow \\ \alpha \rho \end{array}$. Clearly $\mu^2 = 1$.

In order to choose a natural basis for the spaces $C_m(\rho \otimes \rho \rightarrow \rho)$ and $C_m(\rho \otimes \rho \rightarrow \alpha \rho)$, we introduce the following linear operators on these spaces. We often suppress the orientation on the α strands, as it may be inferred from the other orientations in the diagram.

$$K^1 \left(\begin{array}{c} \downarrow \\ \uparrow \\ i \end{array} \right) := \begin{array}{c} \downarrow \\ \uparrow \\ i \end{array} \quad \text{and} \quad K^\alpha \left(\begin{array}{c} \downarrow \\ \uparrow \\ i \end{array} \right) := \begin{array}{c} \downarrow \\ \uparrow \\ i \end{array} .$$

We also define the anti-linear *Frobenius operators*

$$\begin{aligned}
 L^1 \left(\begin{array}{c} \text{diagram} \end{array} \right) &:= \begin{array}{c} \text{diagram} \\ \text{diagram} \end{array} & L^\alpha \left(\begin{array}{c} \text{diagram} \end{array} \right) &:= \begin{array}{c} \text{diagram} \\ \text{diagram} \end{array} \\
 R^1 \left(\begin{array}{c} \text{diagram} \end{array} \right) &:= \begin{array}{c} \text{diagram} \\ \text{diagram} \end{array} & R^\alpha \left(\begin{array}{c} \text{diagram} \end{array} \right) &:= \begin{array}{c} \text{diagram} \\ \text{diagram} \end{array}
 \end{aligned}$$

These operators are unitary with respect to the tracial inner product on hom spaces. By a straightforward but tedious calculation, one proves that these operators satisfy the following relations:

$$\begin{aligned}
 K^1 \circ K^1 &= \lambda_\alpha \text{Id} & K^\alpha \circ K^\alpha &= \text{Id} \\
 L^1 \circ L^1 &= \lambda_\rho \text{Id} = R^1 \circ R^1 & L^\alpha \circ L^\alpha &= \lambda_{\rho\mu} \text{Id} = R^\alpha \circ R^\alpha \\
 K^1 \circ L^1 &= \mu(L^1 \circ K^1) & K^\alpha \circ L^\alpha &= \mu\lambda_\alpha(L^\alpha \circ K^\alpha) \\
 K^1 \circ R^1 &= \mu(R^1 \circ K^1) & K^\alpha \circ R^\alpha &= \mu\lambda_\alpha(R^\alpha \circ K^\alpha) \\
 (R^1 \circ L^1)^3 &= 1 = (L^1 \circ R^1)^3 & (R^\alpha \circ L^\alpha)^3 &= 1 = (L^\alpha \circ R^\alpha)^3.
 \end{aligned}$$

We can diagonalise our basis of $C_m(\rho \otimes \rho \rightarrow \rho)$ and $C_m(\rho \otimes \rho \rightarrow \alpha\rho)$ with respect to these operators to obtain the following.

Lemma 3.4 (α Jellyfish). *We can choose bases for $C_m(\rho \otimes \rho \rightarrow \rho)$ and $C_m(\rho \otimes \rho \rightarrow \alpha\rho)$ such that*

$$\begin{aligned}
 K^1 \left(\begin{array}{c} \text{diagram} \end{array} \right) &= \chi_{1,i} \begin{array}{c} \text{diagram} \end{array} & \text{and} & K^\alpha \left(\begin{array}{c} \text{diagram} \end{array} \right) &= \chi_{\alpha,i} \begin{array}{c} \text{diagram} \end{array} \\
 R^1 \circ L^1 \left(\begin{array}{c} \text{diagram} \end{array} \right) &= \omega_{1,i} \begin{array}{c} \text{diagram} \end{array} & \text{and} & R^\alpha \circ L^\alpha \left(\begin{array}{c} \text{diagram} \end{array} \right) &= \omega_{\alpha,i} \begin{array}{c} \text{diagram} \end{array}
 \end{aligned}$$

where $\chi_{1,i}^2 = \lambda_\alpha$, $\chi_{\alpha,i}^2 = 1$, and $\omega_{1,i}^3 = \omega_{\alpha,i}^3 = 1$.

In particular we have the local relations

$$\begin{aligned}
 \begin{array}{c} \text{diagram} \\ \text{diagram} \end{array} &= \chi_{1,i} \begin{array}{c} \text{diagram} \\ \text{diagram} \end{array} & \text{and} & \begin{array}{c} \text{diagram} \\ \text{diagram} \end{array} &= \chi_{\alpha,i} \begin{array}{c} \text{diagram} \\ \text{diagram} \end{array}
 \end{aligned}$$

Proof. From the above relations, we have that K^1 and $R^1 \circ L^1$ commute. Furthermore, we have $(K^1)^4 = 1$ and $(R^1 \circ L^1)^3 = 1$. Hence we can simultaneously diagonalise these operators to obtain the basis of $C_m(\rho \otimes \rho \rightarrow \rho)$ claimed in the statement of the lemma. As $(K^1)^2 = \lambda_\alpha$, we have that $\chi_{1,i}^2 = \lambda_\alpha$, and as $(R^1 \circ L^1)^3 = 1$, we have that $\omega_{1,i}^3 = 1$. The same argument gives the claimed basis for $C_m(\rho \otimes \rho \rightarrow \alpha\rho)$.

The local relations in the statement of the lemma follow by applying a local α relation to the operators K^1 and K^α . ■

In the case that $m = 1$, we have that the spaces $C_m(\rho \otimes \rho \rightarrow \rho)$ and $C_m(\rho \otimes \rho \rightarrow \alpha\rho)$ are 1-dimensional. Hence the earlier operators are all scalars. In this special case we determine the scalars λ_α , λ_ρ , and μ .

Corollary 3.5. *If $m = 1$, then $\lambda_\alpha = \lambda_\rho = \mu = 1$.*

Proof. As $m = 1$ we have that L^1 acts by a scalar $l^1 \in \mathbb{C}$. As L^1 is anti-linear, the relation $L^1 \circ L^1 = \lambda_\rho$ gives $l^1 \overline{l^1} = \lambda_\rho$. Hence $\lambda_\rho = 1$. The same analysis on the relation $L^\alpha \circ L^\alpha = \lambda_\rho \mu$ gives that $\lambda_\rho \mu = 1$, and so $\mu = 1$. Finally, from Lemma 3.4 the linear operator K^α is a real scalar. The relation $K^\alpha \circ L^\alpha = \mu \lambda_\alpha (L^\alpha \circ K^\alpha)$ then gives us that $\lambda_\alpha = 1$. ■

Note that when $m = 1$ and $\lambda_\alpha = 1$, we have the classification of categories C_1 from [26, Example 9.1]. Hence we have the following Corollary of the above lemma.

Corollary 3.6. *The statement of Theorem 3.1 is true when $m = 1$.*

Hence, for the remainder of this section, we may assume that $m = 2$.

Note that at this point, we can not fully determine the action of the operators L and R on our basis. However, we can make the following observation.

Lemma 3.7. *We have two cases depending on the value of $\lambda_\alpha \mu \in \{1, -1\}$:*

- (1) *If $\lambda_\alpha \mu = 1$ then the operators L^1 and R^1 preserve the eigenspaces of K^1 , and the operators L^α and R^α preserve the eigenspaces of K^α .*
- (2) *If $\lambda_\alpha \mu = -1$ then the operators L^1 and R^1 exchange the eigenspaces of K^1 , and the operators L^α and R^α exchange the eigenspaces of K^α . In particular, $\chi_{1,0} = -\chi_{1,1}$ and $\chi_{\alpha,0} = -\chi_{\alpha,1}$.*

Proof. This follows from the commutation relations above, along with the fact that our L and R operators are anti-linear. Let us illustrate a few examples.

Suppose $\lambda_\alpha \mu = 1$, and let v be an eigenvector for K^1 (with eigenvalue χ). Then the relation $K^1 \circ L^1 = \mu(L^1 \circ K^1)$ gives that

$$K^1 \circ L^1(v) = \lambda_\alpha \bar{\chi} L^1(v).$$

If $\lambda_\alpha = 1$, then χ is real by Lemma 3.4, and we get that $K^1 \circ L^1(v) = \chi L^1(v)$. If $\lambda_\alpha = -1$, then χ is imaginary by Lemma 3.4 and we get that $K^1 \circ L^1(v) = -\bar{\chi} L^1(v) = \chi L^1(v)$. The argument for the eigenspaces of K^α is similar (and easier).

Suppose $\lambda_\alpha \mu = -1$, and let v be an eigenvector for K^1 (again with eigenvalue χ). Now the relation $K^1 \circ L^1 = \mu(L^1 \circ K^1)$ gives that

$$K^1 \circ L^1(v) = -\lambda_\alpha \bar{\chi} L^1(v).$$

The same argument as above shows that $K^1 \circ L^1(v) = -\chi L^1(v)$. Thus L^1 exchanges the eigenspaces of K^1 , which then must be 1-dimensional. In particular we must have that $\chi_{1,0} = -\chi_{1,1}$. The same argument holds for the eigenspaces of K^α .

The above arguments also work verbatim for the operators R^1 and R^α . ■

3.2. Centre analysis

In this subsection we will analyse the Drinfeld centre of the categories C_2 in order to pin down the values of our data χ , μ , and the operators L and R .

Our main tool in this subsection is Theorem 2.2. We remind the reader that this result states that for an object $X \in C$, the irreducible representations V of $A_{X \rightarrow X}$ are in bijective correspondence with simple summands $\Gamma_V \subset I(X) \in Z(C)$, the dimension of Γ_V is given by $\frac{\dim(C)}{\dim(X) f_V}$ where f_V is the formal codegree of the representation V , and the multiplicity of $Y \in \mathcal{F}(\Gamma_V)$ is equal to the multiplicity of V in the left-action of $A_{X \rightarrow X}$ on $A_{X \rightarrow Y}$.

With this tool in mind, we aim to study the tube algebra for C_2 :

$$A = \begin{array}{|c|c|c|c|} \hline A_{1 \leftarrow 1} & A_{1 \leftarrow \alpha} & A_{1 \leftarrow \rho} & A_{1 \leftarrow \alpha\rho} \\ \hline A_{\alpha \leftarrow 1} & A_{\alpha \leftarrow \alpha} & A_{\alpha \leftarrow \rho} & A_{\alpha \leftarrow \alpha\rho} \\ \hline A_{\rho \leftarrow 1} & A_{\rho \leftarrow \alpha} & A_{\rho \leftarrow \rho} & A_{\rho \leftarrow \alpha\rho} \\ \hline A_{\alpha\rho \leftarrow 1} & A_{\alpha\rho \leftarrow \alpha} & A_{\alpha\rho \leftarrow \rho} & A_{\alpha\rho \leftarrow \alpha\rho} \\ \hline \end{array}$$

By determining the irreducible representations of the sub-algebras $A_{X \leftarrow X}$, and their multiplicities in the left action of $A_{X \leftarrow X}$ on $A_{X \leftarrow Y}$, we can determine the simple objects of $Z(C_2)$, and their images under the forgetful functor.

Performing this computation over all of the tube algebra is far too computationally taxing. Instead we restrict our attention to the sub-algebra

$$A_{1 \leftarrow 1} \oplus A_{\alpha \leftarrow \alpha} \oplus A_{1 \leftarrow \rho} \oplus A_{\alpha \leftarrow \rho} \oplus A_{1 \leftarrow \alpha\rho}.$$

We pick the following bases for these spaces:

$$\begin{aligned}
A_{1 \leftarrow 1} &= \text{span} \left\{ \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right\} \\
A_{1 \leftarrow \rho} &= \text{span} \left\{ \begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array} \right\} \\
A_{1 \leftarrow \alpha \rho} &= \text{span} \left\{ \begin{array}{c} \text{red arrows} \\ \text{red arrows} \\ \text{red arrows} \\ \text{red arrows} \end{array} \right\} \\
A_{\alpha \leftarrow \alpha} &= \text{span} \left\{ \begin{array}{c} \text{red arrows} \\ \text{red arrows} \\ \text{red arrows} \\ \text{red arrows} \end{array} \right\} \\
A_{\alpha \leftarrow \rho} &= \text{span} \left\{ \begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array} \right\}.
\end{aligned}$$

By direct computation we obtain that:

- The algebra $A_{1 \leftarrow 1}$ has four 1-dimensional representations, which are:

				
χ_0	1	1	$2 + \sqrt{5}$	$2 + \sqrt{5}$
χ_1	1	1	$2 - \sqrt{5}$	$2 - \sqrt{5}$
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1

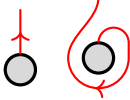
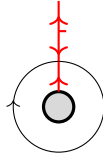
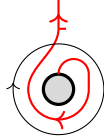
The formal codegrees of these representations are then $20 + 8\sqrt{5}$, $20 - 8\sqrt{5}$, 4, and 4 respectively. Hence by Theorem C the object $\mathcal{I}(1)$ is a direct sum of 4 simple objects X_i with dimensions

$$\dim(X_0) = 1, \quad \dim(X_1) = 9 + 4\sqrt{5}, \quad \text{and} \quad \dim(X_2) = \dim(X_3) = 5 + 2\sqrt{5}.$$

- Direct computation on the basis elements of $A_{\alpha \leftarrow \alpha}$ gives the following multiplications:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_\alpha \\ \mu & 0 & \frac{\chi_1}{\sqrt{\lambda_\alpha}} & \chi_\alpha \\ 0 & \lambda_\alpha \mu & \lambda_\alpha \chi_\alpha & \frac{\chi_1}{\sqrt{\lambda_\alpha}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \lambda_\alpha \mu & \lambda_\alpha \chi_\alpha & \frac{\chi_1}{\sqrt{\lambda_\alpha}} \\ \lambda_\alpha \mu & 0 & \lambda_\alpha \frac{\chi_1}{\sqrt{\lambda_\alpha}} & \lambda_\alpha \chi_\alpha \end{bmatrix}$$

where $\chi_1 := \sqrt{\lambda_\alpha}(\chi_{1,0} + \chi_{1,1}) \in \{-2, 0, 2\}$ and $\chi_\alpha := \chi_{\alpha,0} + \chi_{\alpha,1} \in \{-2, 0, 2\}$. Here we fix our choice of square roots so that $\sqrt{\lambda_\alpha} = 1$ if $\lambda_\alpha = 1$, and $\sqrt{\lambda_\alpha} = i$ if $\lambda_\alpha = -1$. From this, we determine that $A_{\alpha \leftarrow \alpha}$ has the four 1-dimensional representations:

				
τ_0	1	$\sqrt{\lambda_\alpha}$	$\frac{\chi_1 + \chi_\alpha + \sqrt{4\mu\lambda_\alpha + (\chi_1 + \chi_\alpha)^2}}{2\sqrt{\lambda_\alpha}}$	$\frac{\chi_1 + \chi_\alpha + \sqrt{4\mu\lambda_\alpha + (\chi_1 + \chi_\alpha)^2}}{2}$
τ_1	1	$\sqrt{\lambda_\alpha}$	$\frac{\chi_1 + \chi_\alpha - \sqrt{4\mu\lambda_\alpha + (\chi_1 + \chi_\alpha)^2}}{2\sqrt{\lambda_\alpha}}$	$\frac{\chi_1 + \chi_\alpha - \sqrt{4\mu\lambda_\alpha + (\chi_1 + \chi_\alpha)^2}}{2}$
τ_2	1	$-\sqrt{\lambda_\alpha}$	$\frac{\chi_1 - \chi_\alpha + \sqrt{4\mu\lambda_\alpha + (\chi_1 - \chi_\alpha)^2}}{2\sqrt{\lambda_\alpha}}$	$\frac{\chi_1 - \chi_\alpha + \sqrt{4\mu\lambda_\alpha + (\chi_1 - \chi_\alpha)^2}}{-2}$
τ_3	1	$-\sqrt{\lambda_\alpha}$	$\frac{\chi_1 - \chi_\alpha - \sqrt{4\mu\lambda_\alpha + (\chi_1 - \chi_\alpha)^2}}{2\sqrt{\lambda_\alpha}}$	$\frac{\chi_1 - \chi_\alpha - \sqrt{4\mu\lambda_\alpha + (\chi_1 - \chi_\alpha)^2}}{-2}$

Hence by Theorem C the object $\mathcal{I}(\alpha)$ is a direct sum of 4 simple objects Y_i with dimensions

$$\begin{aligned} \dim(Y_0) &= \frac{\dim(C)}{2 + \frac{1}{2} \left| \chi_1 + \chi_\alpha + \sqrt{4\mu\lambda_\alpha + (\chi_1 + \chi_\alpha)^2} \right|^2} \\ \dim(Y_1) &= \frac{\dim(C)}{2 + \frac{1}{2} \left| \chi_1 + \chi_\alpha - \sqrt{4\mu\lambda_\alpha + (\chi_1 + \chi_\alpha)^2} \right|^2} \\ \dim(Y_2) &= \frac{\dim(C)}{2 + \frac{1}{2} \left| \chi_1 - \chi_\alpha + \sqrt{4\mu\lambda_\alpha + (\chi_1 - \chi_\alpha)^2} \right|^2} \\ \dim(Y_3) &= \frac{\dim(C)}{2 + \frac{1}{2} \left| \chi_1 - \chi_\alpha - \sqrt{4\mu\lambda_\alpha + (\chi_1 - \chi_\alpha)^2} \right|^2} \end{aligned}$$

From [23, Lemma 5.4] we have that $t \cdot p_{\tau_i} = \theta_{Y_i} p_{\tau_i}$ where $p_{\tau_i} = \sum_b \tau(b) b^* \in A_{\alpha \leftarrow \alpha}$ is the minimal central idempotent corresponding to τ_i , and in our case,

the operator t is simply right multiplication by the 2nd basis element. Hence we have that

$$\theta_{Y_0} = \theta_{Y_1} = \lambda_\alpha \sqrt{\lambda_\alpha} \quad \text{and} \quad \theta_{Y_2} = \theta_{Y_3} = -\lambda_\alpha \sqrt{\lambda_\alpha}.$$

- Let Z_i be the remaining simple objects of $\mathcal{Z}(C_2)$. Then

$$\mathcal{F}(Z_i) = p_i \rho \oplus q_i \alpha \rho$$

for some $p_i, q_i \in \mathbb{N}$. Further

$$\begin{aligned} \dim \operatorname{Hom}(\mathcal{I}(\rho), \mathcal{I}(\rho)) &= 20 = \dim \operatorname{Hom}(\mathcal{I}(\alpha \rho), \mathcal{I}(\alpha \rho)) \\ \dim \operatorname{Hom}(\mathcal{I}(\alpha \rho), \mathcal{I}(\rho)) &= 16. \end{aligned}$$

- Let $1\pi_\rho$ be the action of $A_{1 \leftarrow 1}$ on $A_{1 \leftarrow \rho}$. Then

$$1\pi_\rho \left(\begin{array}{c} \text{red circle} \\ \text{black circle} \\ \text{trivalent vertex} \end{array} \right) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad 1\pi_\rho \left(\begin{array}{c} \text{black circle} \\ \text{trivalent vertex} \end{array} \right) = \begin{bmatrix} \phi & \phi' \\ \phi' & \phi \end{bmatrix}, \quad \text{and} \quad 1\pi_\rho \left(\begin{array}{c} \text{red circle} \\ \text{black circle} \\ \text{trivalent vertex} \end{array} \right) = \begin{bmatrix} \phi' & \phi \\ \phi & \phi' \end{bmatrix}$$

where ϕ and ϕ' are the operators on $\operatorname{Hom}(\rho \otimes \rho \leftarrow \rho)$ defined by

$$\begin{aligned} \phi \left(\begin{array}{c} \text{trivalent vertex} \\ \text{black circle} \end{array} \right) &= \sum_j \begin{array}{c} \text{trivalent vertex} \\ \text{black circle} \\ \text{red circle} \end{array} \\ \phi' \left(\begin{array}{c} \text{trivalent vertex} \\ \text{black circle} \end{array} \right) &= \sum_j \begin{array}{c} \text{trivalent vertex} \\ \text{black circle} \\ \text{red circle} \end{array} \end{aligned}$$

which we can identify as operators on the two spaces:

$$\left\{ \begin{array}{c} 0 \\ \text{trivalent vertex} \\ \text{black circle} \end{array}, \begin{array}{c} 1 \\ \text{trivalent vertex} \\ \text{black circle} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} 0 \\ \text{trivalent vertex} \\ \text{black circle} \\ \text{red circle} \end{array}, \begin{array}{c} 1 \\ \text{trivalent vertex} \\ \text{black circle} \\ \text{red circle} \end{array} \right\}$$

by local insertion, i.e., the elements of $A_{1 \leftarrow 1}$ which involve ϕ, ϕ' above act on $A_{1 \leftarrow \rho}$ by applying ϕ, ϕ' locally on the trivalent vertices in our standard basis of $A_{1 \leftarrow \rho}$.

- Let $1\pi_{\alpha\rho}$ be the action of $A_{1\leftarrow 1}$ on $A_{1\leftarrow\alpha\rho}$. Then

$$1\pi_{\alpha\rho} \left(\begin{array}{c} \text{red circle} \\ \text{black circle} \end{array} \right) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad 1\pi_{\alpha\rho} \left(\begin{array}{c} \text{red circle} \\ \text{black circle} \end{array} \right) = \begin{bmatrix} \psi' & \psi \\ \psi & \psi' \end{bmatrix} \quad \text{and} \quad 1\pi_{\alpha\rho} \left(\begin{array}{c} \text{red circle} \\ \text{black circle} \end{array} \right) = \begin{bmatrix} \psi & \psi' \\ \psi' & \psi \end{bmatrix}$$

where ψ and ψ' are the operators on $\text{Hom}(\rho \otimes \rho \rightarrow \alpha\rho)$ defined by

$$\psi \left(\begin{array}{c} \text{red arrow} \\ \text{black arrow} \end{array} \right) = \sum_j \begin{array}{c} \text{red wavy line} \\ \text{black wavy line} \end{array}$$

$$\psi' \left(\begin{array}{c} \text{red arrow} \\ \text{black arrow} \end{array} \right) = \sum_j \begin{array}{c} \text{red wavy line} \\ \text{black wavy line} \end{array}.$$

As before, we can naturally identify ψ, ψ' as operators on the following two spaces by local insertion:

$$\left\{ \begin{array}{c} \text{red circle} \\ \text{black circle} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} \text{red circle} \\ \text{black circle} \end{array} \right\}.$$

- Denoting by $\alpha\pi_{\rho}$ the action of $A_{\alpha\rightarrow\alpha}$ on $A_{\alpha\leftarrow\rho}$, we have

$$\alpha\pi_{\rho} \left(\begin{array}{c} \text{red circle} \\ \text{black circle} \end{array} \right) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_{\alpha} & 0 & 0 & 0 \\ 0 & \lambda_{\alpha} & 0 & 0 \end{bmatrix}.$$

We begin by analysing the corner of the tube algebra $A_{1\leftarrow 1}$, and its actions on $A_{1\leftarrow\rho}$ and $A_{1\leftarrow\alpha\rho}$. This gives us the following result.

Lemma 3.8. *There exists $b \in \{0, 1, 2\}$ such that*

$$\mathcal{F}(X_0) = \mathbf{1}$$

$$\mathcal{F}(X_1) = \mathbf{1} \oplus 2\rho \oplus 2\alpha\rho$$

$$\begin{aligned}\mathcal{F}(X_2) &= \mathbf{1} \oplus b\rho \oplus (2-b)\alpha\rho \\ \mathcal{F}(X_3) &= \mathbf{1} \oplus (2-b)\rho \oplus b\alpha\rho.\end{aligned}$$

Furthermore, if $b \in \{0, 2\}$, then the operators ϕ and ψ are both the same scalar

$$\phi = \psi = \frac{1+b-\sqrt{5}}{2},$$

and if $b = 1$, the the operators ϕ and ψ have the two eigenvalues $\frac{1-\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$.

Proof. First note that as X_0 is the tensor unit of $Z(C)$, we have that $\mathbf{1}\pi_\rho$ and $\mathbf{1}\pi_{\alpha\rho}$ contain no copies of χ_0 . From the above computations, we have that

$$\mathrm{Tr}\left(\mathbf{1}\pi_\rho\left(\begin{array}{c} \text{red circle with arrow} \end{array}\right)\right) = 0.$$

As $\mathbf{1}\pi_\rho$ is 4-dimensional, and χ_0 is not a sub-representation, we must have that

$$\mathbf{1}\pi_\rho \cong 2\chi_1 \oplus b\chi_2 \oplus (2-b)\chi_3 \quad \text{where } b \in \{0, 1, 2\}.$$

Thus $\mathcal{F}(X_1)$ contains two copies of ρ , and a dimension count shows that

$$\mathcal{F}(X_1) = \mathbf{1} \oplus 2\rho \oplus 2\alpha\rho.$$

From this we can deduce three possibilities for the restrictions of X_2 and X_3 .

Case 1: $\mathcal{F}(X_2) = \mathcal{F}(X_3) = \mathbf{1} \oplus \rho \oplus \alpha\rho$, in which case $\mathbf{1}\pi_\rho \cong \mathbf{1}\pi_{\alpha\rho} \cong 2\chi_1 \oplus \chi_2 \oplus \chi_3$, and in particular

$$\mathrm{Tr}\left(\mathbf{1}\pi_\rho\left(\begin{array}{c} \text{blue circle with arrow} \end{array}\right)\right) = 4 - 2\sqrt{5},$$

Case 2: $\mathcal{F}(X_2) = \mathbf{1} \oplus 2\rho$ and $\mathcal{F}(X_3) = \mathbf{1} \oplus 2\alpha\rho$, in which case $\mathbf{1}\pi_\rho \cong 2\chi_1 \oplus 2\chi_2$ and $\mathbf{1}\pi_{\alpha\rho} = 2\chi_1 \oplus 2\chi_3$, and in particular

$$\mathrm{Tr}\left(\mathbf{1}\pi_\rho\left(\begin{array}{c} \text{blue circle with arrow} \end{array}\right)\right) = 6 - 2\sqrt{5},$$

Case 3: $\mathcal{F}(X_2) = \mathbf{1} \oplus 2\alpha\rho$ and $\mathcal{F}(X_3) = \mathbf{1} \oplus 2\rho$, in which case $\mathbf{1}\pi_\rho = 2\chi_1 + 2\chi_3$ and $\mathbf{1}\pi_{\alpha\rho} = 2\chi_1 + 2\chi_2$, and in particular

$$\mathrm{Tr}\left(\mathbf{1}\pi_\rho\left(\begin{array}{c} \text{blue circle with arrow} \end{array}\right)\right) = 2 - 2\sqrt{5}.$$

We now aim to deduce more information about the operator ψ . Note that

$$z_1 := \frac{1}{\dim(C)} \left(\text{Diagram 1} + \text{Diagram 2} + (2 + \sqrt{5}) \text{Diagram 3} + (2 + \sqrt{5}) \text{Diagram 4} \right)$$

(Note: Diagram 1 is a grey circle, Diagram 2 is a grey circle with a red border, Diagram 3 is a grey circle with a red arrow pointing to it, and Diagram 4 is a grey circle with a red border and a red arrow pointing to it.)

is the minimal central idempotent corresponding to the representation χ_0 . i.e. $z_1 \cdot x = \chi_0(x) \cdot z_1$. As $1\pi_\rho$ contains no copies of χ_0 , we get $1\pi_\rho(z_1) = 0$, and so

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (2 + \sqrt{5}) \begin{pmatrix} \phi & \phi' \end{pmatrix} = 0 \quad \implies \quad \phi' = - \left(\phi + \begin{bmatrix} \frac{1}{2+\sqrt{5}} & 0 \\ 0 & \frac{1}{2+\sqrt{5}} \end{bmatrix} \right).$$

To solve for ϕ , we use the fusion rule $\rho^2 = \mathbf{1} \oplus 2\rho \oplus 2\alpha\rho$ to get

$$\phi^2 + (\sqrt{5} - 2)\phi = \begin{bmatrix} \frac{1}{2+\sqrt{5}} & 0 \\ 0 & \frac{1}{2+\sqrt{5}} \end{bmatrix}.$$

Together with knowing the trace of $1\pi_\rho(\rho)$ in each of the above cases, we can solve to get the statement of the lemma.

To obtain the statement about ψ we repeat the above analysis with $1\pi_{\alpha\rho}$. ■

This completes our analysis of $I(\mathbf{1})$. We now analyse the object $I(\alpha)$. Our first goal is to show that the object α never lifts to the centre. To begin we prove the following lemma.

Lemma 3.9. *Suppose that α has a lift to the centre $\mathcal{Z}(C_2)$. Then $b = 1$.*

Proof. For a contradiction, suppose that $b = 2$.

By a relabeling we can assume that Y_0 is a lift of α to $\mathcal{Z}(C_2)$, and that $Y_i = Y_0 \otimes X_i$. As $\mathcal{F}$ is a $\otimes$ -functor, this gives us $\mathcal{F}(Y_i)$ in terms of the b from Lemma 3.8. We then have

$$\begin{aligned} I(\rho) &= 2X_1 \oplus 2X_2 \oplus 2Y_1 \oplus 2Y_3 \bigoplus p_i Z_i \\ I(\alpha\rho) &= 2X_1 \oplus 2X_3 \oplus 2Y_1 \oplus 2Y_2 \bigoplus q_i Z_i. \end{aligned}$$

This gives us that

$$\begin{aligned} 20 &= \dim \text{Hom}(I(\rho), I(\rho)) = 16 + \sum p_i^2 \\ 20 &= \dim \text{Hom}(I(\alpha\rho), I(\alpha\rho)) = 16 + \sum q_i^2 \\ 16 &= \dim \text{Hom}(I(\rho), I(\alpha\rho)) = 8 + \sum p_i q_i. \end{aligned}$$

Hence

$$\sum p_i^2 = 4 = \sum q_i^2 \quad \text{and} \quad \sum p_i q_i = 8$$

which is impossible.

The contradiction when $b = 0$ is nearly identical. ■

With this lemma in hand, we can now show that α never lifts to the centre.

Lemma 3.10. *The object α does not have a lift to the centre.*

Proof. For a contradiction, suppose that α lifts to the centre. We have two cases depending on the value of λ_α .

First suppose $\lambda_\alpha = 1$. From Lemma 3.9 we have that $b = 1$, the same style of argument from the proof of this Lemma shows that

$$\begin{aligned} \mathcal{I}(\rho) &= 2X_1 \oplus X_2 \oplus X_3 \oplus 2Y_1 \oplus Y_2 \oplus Y_3 \oplus \bigoplus p_i Z_i \\ \mathcal{I}(\alpha\rho) &= 2X_1 \oplus X_2 \oplus X_3 \oplus 2Y_1 \oplus Y_2 \oplus Y_3 \oplus \bigoplus q_i Z_i \end{aligned}$$

with

$$\sum p_i^2 = 8 = \sum q_i^2 \quad \text{and} \quad \sum p_i q_i = 4.$$

In particular

$$\sum p_i(p_i + q_i) = 12.$$

We now use the Ng-Schauenburg formula for the 2nd Frobenius-Schur indicator of ρ to obtain

$$\begin{aligned} \lambda_\rho \dim(C_2) &= 2(1 + 4d)(\theta_{X_1}^2 + \theta_{Y_1}^2) + (1 + 2d)(\theta_{X_2}^2 + \theta_{Y_2}^2 + \theta_{X_3}^2 + \theta_{Y_3}^2) \\ &\quad + \sum p_i(p_i + q_i)d\theta_{Z_i}^2 \\ &= 8 + 24d + \sum p_i(p_i + q_i)d\theta_{Z_i}^2. \end{aligned}$$

For either case of λ_ρ , we have that $|\sum p_i(p_i + q_i)\theta_{Z_i}^2| \geq 4\sqrt{5} + 8 > 12$. Hence α can not have a lift in this case.

Now suppose that $\lambda_\alpha = -1$. By analysing the dimensions formulas for Y_i over all the different cases of μ , χ_1 , and χ_α , we see that α can only lift if $\mu = -1$, and $|\chi_1| = |\chi_\alpha| = 2$. In this case, we have that the other Y_i have dimensions $9 + 4\sqrt{5}$ and $5 + 2\sqrt{5}$ occuring twice. Furthermore, we see that the dimension 1 object and the dimension $9 + 4\sqrt{5}$ object have same twist. Hence by relabeling, we may assume

$$\begin{aligned} \dim(Y_0) &= 1 & \dim(Y_1) &= 9 + 4\sqrt{5} & \dim(Y_2) &= \dim(Y_3) = 5 + 2\sqrt{5} \\ \theta_{Y_0} &= \theta_{Y_1} = \pm \mathbf{i} & \theta_{Y_2} &= \theta_{Y_3} = \mp \mathbf{i} & Y_i &= Y_0 \otimes X_i. \end{aligned}$$

As $\theta_{Y_0} = \pm \mathbf{i}$ we have that $\langle X_0, Y_0 \rangle$ is a modular subcategory of $\mathcal{Z}(C_2)$. Hence $\mathcal{Z}(C_2)$ factors as $\mathcal{Z}(C_2)_0 \boxtimes \langle X_0, Y_0 \rangle$. Note that we have a simple $W \in \mathcal{Z}(C_2)_0$ iff $\theta_{W \otimes Y_0} = \theta_W \theta_{Y_0}$. This implies that $\{X_0, X_1, Y_2, Y_3\} \in \mathcal{Z}(C_2)_0$, and $\{X_2, X_3, Y_0, Y_1\} \in \mathcal{Z}(C_2)_0 \otimes Y_0$. Let us write $\{Z_i\}_{i \in \Lambda_0}$ for the remaining simple objects of $\mathcal{Z}(C_2)$ which live in $\mathcal{Z}(C_2)_0$. We then have

$$\begin{aligned} I(\rho) &= 2X_1 \oplus X_2 \oplus X_3 \oplus 2Y_1 \oplus Y_2 \oplus Y_3 \oplus \bigoplus_{i \in \Lambda_0} p_i Z_i \oplus \bigoplus_{i \in \Lambda_0} q_i Y_0 \otimes Z_i \\ I(\alpha\rho) &= 2X_1 \oplus X_2 \oplus X_3 \oplus 2Y_1 \oplus Y_2 \oplus Y_3 \oplus \bigoplus_{i \in \Lambda_0} q_i Z_i \oplus \bigoplus_{i \in \Lambda_0} p_i Y_0 \otimes Z_i. \end{aligned}$$

From $20 = \dim \text{Hom}(I(\rho), I(\rho))$ and $16 = \dim \text{Hom}(I(\alpha\rho), I(\rho))$ we obtain

$$\sum_{i \in \Lambda_0} p_i^2 + q_i^2 = 8 \quad \text{and} \quad \sum_{i \in \Lambda_0} 2p_i q_i = 4.$$

Hence

$$\sum_{i \in \Lambda_0} (p_i + q_i)^2 = 12 \quad \text{and} \quad \sum_{i \in \Lambda_0} (p_i - q_i)^2 = 4.$$

By Cauchy-Schwarz applied to the vectors $(p_i + q_i)_i$ and $(|p_i - q_i|)_i$,

$$\sum_{i \in \Lambda_0} |p_i - q_i|^2 = \sum_{i \in \Lambda_0} (p_i + q_i) |p_i - q_i| \leq \sqrt{12 \cdot 4} = 4\sqrt{3}.$$

Again we use the Ng-Schauenburg formula for the 2nd Frobenius-Schur indicator of ρ to obtain

$$\begin{aligned} \lambda_\rho \dim(C_2) &= 2(1 + 4d)(\theta_{X_1}^2 + \theta_{Y_1}^2) + (1 + 2d)(\theta_{X_2}^2 + \theta_{Y_2}^2 + \theta_{X_3}^2 + \theta_{Y_3}^2) \\ &\quad + \sum_{i \in \Lambda_0} p_i(p_i + q_i) d \theta_{Z_i}^2 + \sum_{i \in \Lambda_0} q_i(p_i + q_i) d \theta_{Y_0 \otimes Z_i}^2 \\ &= \sum_{i \in \Lambda_0} (p_i^2 - q_i^2) d \theta_{Z_i}^2. \end{aligned}$$

From this we obtain

$$\sum_{i \in \Lambda_0} |p_i^2 - q_i^2| \geq \frac{\dim(C_2)}{d} = 4\sqrt{5}.$$

Hence α can not have a lift in this case. ■

We can now deduce that both χ_1 and χ_α are 0. That is $\chi_{1,1} = -\chi_{1,2}$ and $\chi_{\alpha,1} = -\chi_{\alpha,2}$.

Lemma 3.11. *We have that $\chi_{1,0} = -\chi_{1,1}$ and $\chi_{\alpha,0} = -\chi_{\alpha,1}$. In particular, we may assume that $\chi_{1,0} = \sqrt{\lambda_\alpha}$, $\chi_{1,1} = -\sqrt{\lambda_\alpha}$, $\chi_{\alpha,0} = 1$, and $\chi_{\alpha,1} = -1$.*

Proof. If $\lambda_\alpha \mu = -1$, then we have the first statement of the lemma from Lemma 3.7. Hence we can assume that $\lambda_\alpha \mu = 1$.

First consider the case that $|\chi_1| = |\chi_\alpha| = 2$. Then the earlier dimension formulas for $\dim(Y_i)$ show that one of these dimensions is 1, which implies that α lifts to the centre. But this contradicts Lemma 3.10.

In the case that $\chi_1 = 0$ and $|\chi_\alpha| = 2$, or $|\chi_1| = 2$ and $\chi_\alpha = 0$, then one of the Y_i 's has dimension $\frac{5+2\sqrt{5}}{2+\sqrt{2}}$, which is impossible.

The only remaining case is that $\chi_1 = 0$ and $\chi_\alpha = 0$ which implies the first statement of the lemma.

As $\chi_{1,i}^2 = \lambda_\alpha$ and $\chi_{\alpha,i}^2 = 1$, we may reorder our basis to give the statement of the lemma. ■

As a result of the above Lemma, we know that the eigenspaces of K^1 and K^α are 1-dimensional. We can pair this information with Lemma 3.7 to obtain the action of the L and R operators on our eigenbasis.

Lemma 3.12. *The basis of the spaces $C_2(\rho \otimes \rho \rightarrow \rho)$ and $C_2(\rho \otimes \rho \rightarrow \alpha\rho)$ from Lemma 3.4 can be chosen so that*

$$\begin{aligned} R^1 \left(\begin{array}{c} \text{diagram with } i \text{ and } \tilde{i} \end{array} \right) &= \lambda_\rho^i \tilde{i} \begin{array}{c} \text{diagram with } \tilde{i} \text{ and } i \end{array} \\ L^1 \left(\begin{array}{c} \text{diagram with } i \text{ and } \tilde{i} \end{array} \right) &= \lambda_\rho^{i+1} \omega_{1,i}^{-1} \tilde{i} \begin{array}{c} \text{diagram with } \tilde{i} \text{ and } i \end{array} \\ R^\alpha \left(\begin{array}{c} \text{diagram with } i \text{ and } \tilde{i} \end{array} \right) &= (\lambda_\rho \mu)^i \tilde{i} \begin{array}{c} \text{diagram with } \tilde{i} \text{ and } i \end{array} \\ L^\alpha \left(\begin{array}{c} \text{diagram with } i \text{ and } \tilde{i} \end{array} \right) &= (\lambda_\rho \mu)^{i+1} \omega_{\alpha,i}^{-1} \tilde{i} \begin{array}{c} \text{diagram with } \tilde{i} \text{ and } i \end{array} \end{aligned}$$

where $i \mapsto \tilde{i}$ is an order two involution on the indexing set $\{0, 1\}$. If $\lambda_\alpha \mu = 1$ then $\tilde{0} = 0$ and $\tilde{1} = 1$. If $\lambda_\alpha \mu = -1$ then $\tilde{0} = 1$ and $\tilde{1} = 0$, and in this case we have that $\omega_{1,0} = \omega_{1,1}$ and $\omega_{\alpha,0} = \omega_{\alpha,1}$. Furthermore, if $\lambda_\rho = -1$, then $\lambda_\alpha \mu = -1$, and if $\mu \lambda_\rho = -1$, then $\lambda_\alpha \mu = -1$.

We are free to exchange our distinguished basis elements, and to rescale them by

$$\begin{aligned} \begin{array}{c} \text{diagram with } i \text{ and } \tilde{i} \end{array} &\mapsto z_{1,i} \begin{array}{c} \text{diagram with } i \text{ and } \tilde{i} \end{array} \\ \begin{array}{c} \text{diagram with } i \text{ and } \tilde{i} \end{array} &\mapsto z_{\alpha,i} \begin{array}{c} \text{diagram with } i \text{ and } \tilde{i} \end{array} \end{aligned} \quad \begin{aligned} z_{1,i} &\in U(1) \\ z_{\alpha,i} &\in U(1). \end{aligned}$$

Proof. Let us begin with the operator R^1 . In the case of $\lambda_\alpha \mu = 1$ we have from Lemma 3.7 that R^1 preserves the eigenspaces of K^1 . As these eigenspaces are 1-dimensional by Lemma 3.11 we have that R^1 is of the form (using linear operator notation, even though R^1 is anti-linear)

$$R^1 = \begin{bmatrix} R_{0,0}^1 & 0 \\ 0 & R_{1,1}^1 \end{bmatrix}.$$

As $(R^1)^4 = 1$, we have that these coefficients are elements of $U(1)$, and as R^1 is anti linear, we can rescale out two basis vectors by $\sqrt{R_{0,0}^1} \in U(1)$ and $\sqrt{R_{1,1}^1} \in U(1)$ to arrange that both these coefficients are 1. Note that this rescaling does not affect the relations of Lemma 3.4 as the operators K^1 and $R^1 \circ L^1$ are linear.

In the case of $\lambda_\alpha \mu = -1$ we have from Lemma 3.7 that R^1 exchanges the eigenspaces of K^1 . We thus have that R^1 is of the form

$$R^1 = \begin{bmatrix} 0 & R_{0,1}^1 \\ R_{1,0}^1 & 0 \end{bmatrix}.$$

By choosing our second basis vector as the image under R^1 of the first, we arrange that $R_{0,1}^1 = 1$. Again, this doesn't affect the relations of Lemma 3.4 as R^1 is unitary. We now use the relation $(R^1)^2 = \lambda_\rho$ to see that $R_{1,0}^1 = \lambda_\rho$.

Together these give the action of R^1 as in the statement of the lemma. The action of L^1 follows from the action of $R^1 \circ L^1$ from Lemma 3.4, along with the relation $(R^1)^2 = \lambda_\rho$. In the case of $\lambda_\alpha \mu = -1$, we can perform that same argument on $L^1 \circ R^1 = (R^1 \circ L^1)^{-1}$ to see that $\omega_{1,0} = \omega_{1,1}$.

Finally, from the relation $(R^1)^2 = \lambda_\rho$, we can see that if $\lambda_\rho = -1$, then only the case $\lambda_\alpha \mu = -1$ is possible.

The same analysis on the operators R^α and L^α gives the remaining statement of the lemma. ■

Now that we have pinned down χ_1 and χ_α , we can describe the objects $Y_i \subset \mathcal{I}(\alpha)$ in more detail.

Lemma 3.13. *We have that*

$$\mathcal{F}(Y_0) = \alpha \oplus c_0 \rho \oplus (2 - c_0) \alpha \rho$$

$$\mathcal{F}(Y_1) = \alpha \oplus (2 - c_0) \rho \oplus c_0 \alpha \rho$$

$$\mathcal{F}(Y_2) = \alpha \oplus c_2 \rho \oplus (2 - c_2) \alpha \rho$$

$$\mathcal{F}(Y_3) = \alpha \oplus (2 - c_2) \rho \oplus c_2 \alpha \rho.$$

where $c_0, c_2 \in \{0, 1, 2\}$.

Proof. As $\chi_{1,1} = -\chi_{1,2}$ and $\chi_{\alpha,1} = -\chi_{\alpha,2}$, we have that $\chi_1 = \chi_\alpha = 0$, and so each of the objects Y_i has dimension $5 + 2\sqrt{5}$. We thus have

$$\mathcal{F}(Y_0) = \alpha \oplus c_0\rho \oplus (2 - c_0)\alpha\rho$$

$$\mathcal{F}(Y_1) = \alpha \oplus c_1\rho \oplus (2 - c_1)\alpha\rho$$

$$\mathcal{F}(Y_2) = \alpha \oplus c_2\rho \oplus (2 - c_2)\alpha\rho$$

$$\mathcal{F}(Y_3) = \alpha \oplus c_3\rho \oplus (2 - c_3)\alpha\rho.$$

For some integers $c_i \in \{0, 1, 2\}$.

From the computations in Subsection 3.2 determining the matrix for the operator ${}_\alpha\pi_\rho$, we see that

$$\text{Tr} \left({}_\alpha\pi_\rho \left(\text{Diagram} \right) \right) = 0.$$

On the other hand from Theorem C we have that

$${}_\alpha\pi_\rho \cong c_0\tau_0 \oplus c_1\tau_1 \oplus c_2\tau_2 \oplus c_3\tau_3.$$

From the earlier tube algebra computations, we know the value of the representations τ_i on the element . In particular, we know the trace of this value.

As traces are preserved under direct sums, we obtain that $c_0 + c_1 - c_2 - c_3 = 0$. From the formula $\dim \text{Hom}(\alpha, \mathcal{RI}(\rho)) = \dim \text{Hom}(\mathcal{I}(\alpha), \mathcal{I}(\rho)) = 4$, we obtain $c_0 + c_1 + c_2 + c_3 = 4$. Together we get the statement of the lemma. ■

With the restrictions of the objects X_i and Y_i now understood, we can give a fairly explicit formula for the even Frobenius-Schur indicators of ρ . This formula will come in handy at several points later in this article.

Lemma 3.14. *We have that*

$$v_{2n}(\rho) \dim(C) = 28 + 12\sqrt{5} + \lambda_\alpha^n (20 + 8\sqrt{5}) + (2 + \sqrt{5}) \sum p_i(p_i + q_i) \theta_{Z_i}^{2n},$$

where p_i and q_i are integers satisfying $\sum p_i(p_i + q_i) = 16$, and the θ_{Z_i} are roots of unity.

Proof. From Lemmas 3.8 and 3.13, we know the image under the forgetful functor of each of the simple objects appearing in $\mathcal{I}(\mathbf{1})$ and $\mathcal{I}(\alpha)$, up to some small integers b, c_0, c_2 . Then we can write

$$\mathcal{I}(\rho) = 2X_1 \oplus bX_2 \oplus (2 - b)X_3 \oplus c_0Y_0 \oplus (2 - c_0)Y_1 \oplus c_2Y_2 \oplus (2 - c_2)Y_3$$

$$\begin{aligned}
& \oplus \bigoplus p_i Z_i \\
I(\alpha\rho) &= 2X_1 \oplus (2-b)X_2 \oplus bX_3 \oplus (2-c_0)Y_0 \oplus c_0Y_1 \oplus (2-c_2)Y_2 \oplus c_2Y_3 \\
& \oplus \bigoplus q_i Z_i.
\end{aligned}$$

Using the fact that $\mathcal{F}(I(\rho)) \cong \bigoplus_{X \in \text{Irr}(C)} X\rho X^*$ we obtain

$$\begin{aligned}
20 &= \dim \text{Hom}(I(\rho), I(\rho)) = 4 + b^2 + (2-b)^2 + c_0^2 + (2-c_0)^2 + c_2^2 + (2-c_2)^2 \\
&+ \sum p_i^2 \\
16 &= \dim \text{Hom}(I(\alpha\rho), I(\rho)) = 4 + 2b(2-b) + 2c_0(2-c_0) + 2c_2(2-c_2) \\
&+ \sum p_i q_i,
\end{aligned}$$

so $36 = \dim \text{Hom}(I(\rho \oplus \alpha\rho), I(\rho)) = 20 + \sum p_i^2 + \sum p_i q_i$, and thus $\sum p_i(p_i + q_i) = 16$.

We have from the earlier computations that $\theta_{X_i}^2 = 1$ and $\theta_{Y_i}^2 = \lambda_\alpha$. We can use the Ng-Shauenburg formula for the $2n$ -th Frobenius-Schur indicator [42, Theorem 4.1] to obtain

$$\begin{aligned}
\nu_{2n}(\rho) \dim(C) &= \sum_{W \in \mathcal{Z}(C)} \dim \text{Hom}(\mathcal{F}(W) \rightarrow \rho) \dim(W) \theta_W^{2n} \\
&= 28 + 12\sqrt{5} + \lambda_\alpha^n (20 + 8\sqrt{5}) + (2 + \sqrt{5}) \sum p_i(p_i + q_i) \theta_{Z_i}^{2n}. \blacksquare
\end{aligned}$$

We finish this subsection by showing that $\mu = 1$ in all cases.

Lemma 3.15. *We have that $\mu = 1$.*

Proof. First suppose that $\lambda_\alpha = \mu$, and for a contradiction suppose that $\mu = -1$, so that $\lambda_\alpha = -1$. We thus have that one of λ_ρ or $\mu\lambda_\rho$ is -1 . We thus get from Lemma 3.12 that $\lambda_\alpha\mu = -1$ which is our contradiction.

Now suppose $\lambda_\alpha = -\mu$, and for a contradiction suppose that $\mu = -1$, so that $\lambda_\alpha = 1$. As $\mu = -1$, we can exchange ρ and $\alpha\rho$ if necessary to arrange $\lambda_\rho = -1$ (as a direct computation shows $\nu_2(\rho) = \mu \cdot \nu_2(\alpha\rho)$).

We can now use Lemma 3.14, along with the fact that the 2nd Frobenius-Schur indicator of ρ is λ_ρ to get the equation

$$-20 - 8\sqrt{5} = 48 + 20\sqrt{5} + (2 + \sqrt{5}) \sum p_i(p_i + q_i) \theta_{Z_i}^2,$$

where $\sum p_i(p_i + q_i) = 16$. Thus

$$\sum p_i(p_i + q_i) \theta_{Z_i}^2 = -4 - 12\sqrt{5}.$$

However Theorem A.26 implies that it takes at least $12\phi(10) = 48$ roots of unity to write $-4 - 12\sqrt{5}$, contradicting $\sum p_i(p_i + q_i) = 16$. Hence $\mu = 1$. $\blacksquare$

3.3. Sufficient relations to evaluate closed diagrams

In this subsection, we will introduce several more 6-j style local relations in our category C_2 , and furthermore show that the full collection of relations described completely determine the category C_2 . We will do this via the standard technique of showing that our relations suffice to evaluate every endomorphism of the tensor unit to a scalar. These additional local relations will be determined by $8m^4$ complex scalars $A_{k,\ell}^{i,j}, B_{k,\ell}^{i,j}, C_{k,\ell}^{i,j}, D_{k,\ell}^{i,j}, \widehat{A}_{k,\ell}^{i,j}, \widehat{B}_{k,\ell}^{i,j}, \widehat{C}_{k,\ell}^{i,j}, \widehat{D}_{k,\ell}^{i,j} \in \mathbb{C}$ for $0 \leq i, j, k, \ell < 2$. These complex scalars are entries of the F -tensors $F_\rho^{\rho,\rho,\rho}, F_{\alpha\rho}^{\rho,\rho,\rho}, F_\rho^{\alpha\rho,\rho,\rho}$, and $F_{\alpha\rho}^{\alpha\rho,\rho,\rho}$.

3.3.1. Jellyfish relations.

Lemma 3.16 (ρ Jellyfish). *There exist scalars*

$$A_{k,\ell}^{i,j}, B_{k,\ell}^{i,j}, C_{k,\ell}^{i,j}, D_{k,\ell}^{i,j}, \widehat{A}_{k,\ell}^{i,j}, \widehat{B}_{k,\ell}^{i,j}, \widehat{C}_{k,\ell}^{i,j}, \widehat{D}_{k,\ell}^{i,j} \in \mathbb{C} \quad 0 \leq i, j, k, \ell < 2$$

such that the following local relations hold in C_2 :

$$\begin{aligned}
 & \begin{array}{c} \bar{\rho} \\ | \\ \rho \quad \rho \end{array} = \frac{\lambda_\rho}{d} \begin{array}{c} \curvearrowright \\ | \\ \rho \quad \rho \end{array} + \sum_i \lambda_\rho^i \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \end{array} + (\lambda_\rho)^i \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \end{array} \\
 & \begin{array}{c} \rho \\ | \\ \rho \quad \rho \quad \rho \end{array} = \frac{\lambda_\rho^{\ell+1} \omega_{1,\ell}}{d} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} + \frac{\lambda_\rho}{\omega_{1,\ell}} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} + \sum_{i,j,k} A_{k,\ell}^{i,j} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} \\
 & \quad + B_{k,\ell}^{i,j} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} + C_{k,\ell}^{i,j} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} + D_{k,\ell}^{i,j} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} \\
 & \begin{array}{c} \alpha \quad \rho \\ | \quad | \\ \rho \quad \rho \quad \rho \end{array} = \frac{\lambda_\rho^{\ell+1} \omega_{\alpha,\ell}}{d} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} + \frac{\lambda_\rho}{d \omega_{\alpha,\ell}} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} + \sum_{i,j,k} \widehat{A}_{k,\ell}^{i,j} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} \\
 & \quad + \widehat{B}_{k,\ell}^{i,j} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} + \widehat{C}_{k,\ell}^{i,j} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array} + \widehat{D}_{k,\ell}^{i,j} \begin{array}{c} \bullet \\ | \\ \rho \quad \rho \quad \rho \end{array}
 \end{aligned}$$

Proof. We provide the proof for the third relation, and the other two are left to the reader. We compute

$$\begin{aligned}
& \text{Diagram with red line } \alpha, \text{ black lines } \rho, \text{ and vertex } \ell = \frac{1}{d} \text{Diagram 1} + \sum_i \text{Diagram 2} + \text{Diagram 3} \\
&= \frac{\mu}{d} \text{Diagram 4} + \sum_i \text{Diagram 5} + \text{Diagram 6} \\
&= \frac{\mu^2}{d} \text{Diagram 7} + L^\alpha(\ell)^* + \underbrace{\sum_i \frac{1}{d} \text{Diagram 8} + \frac{1}{d} \text{Diagram 9}}_{=0 \text{ by (3.2)}} \\
&+ \sum_{i,k} \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12} + \text{Diagram 13} \\
&= \frac{\lambda_\rho^{\ell+1} \omega_{\alpha,\ell}}{d} \text{Diagram 14} + \frac{\lambda_\rho}{d} \sum_i \text{Diagram 15} \quad (3.17) \\
&+ \sum_{i,j,k} \widehat{A}_{k,\ell}^{i,j} \text{Diagram 16} + \widehat{B}_{k,\ell}^{i,j} \text{Diagram 17} + \widehat{C}_{k,\ell}^{i,j} \text{Diagram 18} + \widehat{D}_{k,\ell}^{i,j} \text{Diagram 19}
\end{aligned}$$

For the final four sums of diagrams after the $\sum_{i,k}$ before the final equality above, we express each of the sub-diagrams in the dotted blue boxes in terms of our chosen basis for $C_2(\rho^2 \rightarrow \rho)$ and $C_2(\rho^2 \rightarrow \alpha\rho)$. For instance, for the diagram directly after the $\sum_{i,k}$, the sub-diagram in the blue box lives in $C_2(\alpha\rho^2 \rightarrow \rho) \cong C_2(\rho^2 \rightarrow \alpha\rho)$, so the sub-diagram can be expressed as a linear combination of 4-valent vertices. The coefficients $\widehat{A}_{k,\ell}^{i,j}, \widehat{B}_{k,\ell}^{i,j}, \widehat{C}_{k,\ell}^{i,j}, \widehat{D}_{k,\ell}^{i,j}$ then arise as arbitrary basis coefficients.

For the second diagram in the last line above, we used the relation that for $f \in C_2(\rho^2 \rightarrow \alpha\rho)$,

$$(3.17)$$

which can be verified by a straightforward diagrammatic calculation. We then use the relation in Lemma 3.4 to obtain $(L^\alpha \circ R^\alpha)(f)$. We leave the final simplification of this second diagram using the α Jellyfish Relation of Lemma 3.4 to the reader. $\blacksquare$

Remark 3.18. Recall that the associator F -tensors of a unitary fusion category are determined by the formula

$$F_W^{X,Y,Z} = \sum_{\substack{V \in \text{Irr}(C) \\ 0 \leq i < \dim \text{Hom}(X \otimes Y \rightarrow V) \\ 0 \leq j < \dim \text{Hom}(V \otimes Z \rightarrow W)}} (F_W^{X,Y,Z})^{(V;i,j)}_{(U;k,l)}.$$

We have the following identification between the above 128 complex scalars and certain F -tensors of the category C_2 :

$$\begin{aligned} A_{k,\ell}^{i,j} &= (F_\rho^{\rho,\rho,\rho})_{(\rho;k,\ell)}^{(\rho;i,j)} & \widehat{A}_{k,\ell}^{i,j} &= (F_{\alpha\rho}^{\alpha\rho,\rho,\rho})_{(\alpha\rho;k,\ell)}^{(\rho;i,j)} \\ B_{k,\ell}^{i,j} &= (F_\rho^{\rho,\rho,\rho})_{(\rho;k,\ell)}^{(\alpha\rho;i,j)} & \widehat{B}_{k,\ell}^{i,j} &= (F_{\alpha\rho}^{\alpha\rho,\rho,\rho})_{(\alpha\rho;k,\ell)}^{(\alpha\rho;i,j)} \\ C_{k,\ell}^{i,j} &= (F_{\alpha\rho}^{\rho,\rho,\rho})_{(\rho;k,\ell)}^{(\rho;i,j)} & \widehat{C}_{k,\ell}^{i,j} &= (F_{\alpha\rho}^{\alpha\rho,\rho,\rho})_{(\alpha\rho;k,\ell)}^{(\rho;i,j)} \\ D_{k,\ell}^{i,j} &= (F_{\alpha\rho}^{\rho,\rho,\rho})_{(\rho;k,\ell)}^{(\alpha\rho;i,j)} & \widehat{D}_{k,\ell}^{i,j} &= (F_{\alpha\rho}^{\alpha\rho,\rho,\rho})_{(\alpha\rho;k,\ell)}^{(\alpha\rho;i,j)}. \end{aligned}$$

In the name of readability, we will not use this F -tensor notation in this article.

Remark 3.19. With the above jellyfish relations, we can describe the operators ϕ and ψ from Subsection 3.2 in terms of our free scalars. We have

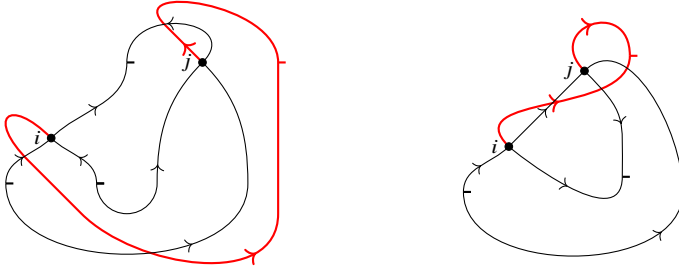
$$\phi \left(\begin{array}{c} \text{tree with root } i \end{array} \right) = \sum_j \begin{array}{c} \text{tree with root } j \end{array} = \sum_{j,k} A_{j,i}^{j,k} \begin{array}{c} \text{tree with root } k \end{array}$$

$$\phi' \left(\begin{array}{c} \text{diagram with vertices } i, j \end{array} \right) = \sum_j \begin{array}{c} \text{diagram with vertices } i, j \end{array} = \chi_{1,i} \sum_{j,k} D_{j,i}^{j,k} \begin{array}{c} \text{diagram with vertex } k \end{array}$$

and

$$\begin{aligned} \psi \left(\begin{array}{c} \text{diagram with vertices } i, j \end{array} \right) &= \sum_j \begin{array}{c} \text{diagram with vertices } i, j \end{array} = \sum_{j,k} \widehat{A}_{j,i}^{j,k} \begin{array}{c} \text{diagram with vertex } k \end{array} \\ \psi' \left(\begin{array}{c} \text{diagram with vertices } i, j \end{array} \right) &= \sum_j \begin{array}{c} \text{diagram with vertices } i, j \end{array} = \lambda_\alpha \chi_{\alpha,i} \sum_{j,k} \widehat{D}_{j,i}^{j,k} \begin{array}{c} \text{diagram with vertex } k \end{array}. \end{aligned}$$

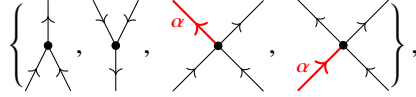
3.3.2. Absorption relations. Using the nomenclature from [6], a closed diagram in our generators is said to be in *jellyfish form* if all trivalent and tetravalent vertices and their labels appear on the external region of the closed diagram. By a slight abuse of nomenclature, we will say that a morphism in a hom space is in jellyfish form (or a *train* in the nomenclature of [7]) if all labels of trivalent and tetravalent vertices in the morphism meet the leftmost region of the morphism. In the examples below, the left diagram is not in jellyfish form, and the right diagram is in jellyfish form.



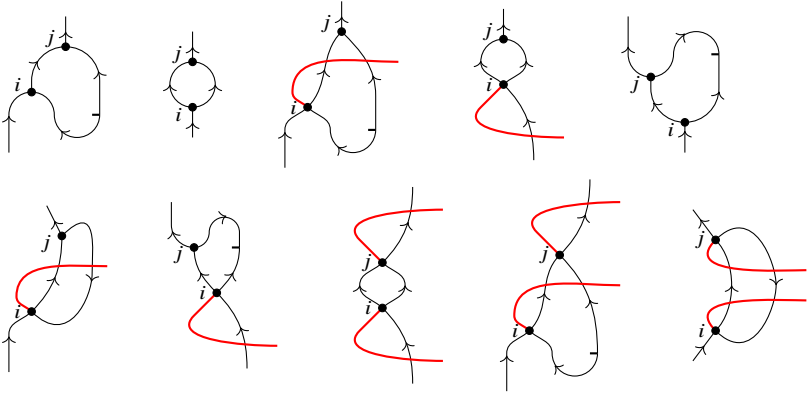
Lemma 3.20 (Absorption). *Using the relations from §3.1 and §3.3.1, any two trivalent/tetravalent vertices in jellyfish form connected by two of their ρ strands*

so that the composite is still in jellyfish form may be simplified into a diagram with no trivalent/tetravalent vertices.

Proof. There are 16 words of length 2 on the symbols



and up to adjoints, 10 are distinct. Given any word of length 2, there is a unique composite in jellyfish form with two ρ strands connected, up to labels and moving tags through crossings. There are thus 10 cases to consider:



We give a full proof for the last case, and the others are similar and omitted:

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = (\lambda_\rho)^{i+j} \text{Diagram 3} = \delta_{i=j} (\lambda_\rho)^{i+j} \text{Diagram 4} . \quad \blacksquare
 \end{aligned}$$

3.3.3. Evaluation algorithm. With these local relations in hand, we can show that the numerical data we have described uniquely determines the category $\mathcal{C}_2$.

Proposition 3.21. *There is at most one unitary fusion category $\mathcal{C}_2$ realising each tuple of data*

$$(\lambda_\alpha, \lambda_\rho, \omega, A, B, C, D, \widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}).$$

Proof. The proof is an adaptation of Bigelow's jellyfish algorithm [5, 6]. Given any closed diagram in our generators, we show it can be evaluated to a scalar using our relations. This immediately implies the stated result by [9, Lem. 2.4] which is

the unshaded pivotal category version of [6, Prop. 3.5] for shaded planar algebras. Indeed, let C' be the quotient of the free category in our generators, modulo the relations corresponding to the data $(\lambda_\alpha, \lambda_\rho, \omega, A, B, C, D, \widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$. If we can show that any closed diagram in our generators can be evaluated to a scalar using the given relations, then we have every ideal of C' is contained in the negligible ideal. We then have an equivalence $C'/\text{Neg}(C') \rightarrow C_2$ which shows that C_2 is uniquely determined by the above tuple of data.

By the jellyfish relations from Lemmas 3.4 and 3.16, it suffices to show we can evaluate any closed diagram in *jellyfish form*, in which all trivalent and tetravalent vertices and their labels appear on the external boundary of the closed diagram. There are 3 cases for such a diagram:

- Case 1: there are no vertices at all in the closed diagram. Then we may use (3.2) to evaluate the closed diagram to a scalar.
- Case 2: there is a trivalent/tetravalent vertex connected to itself. Then we may use (3.2) to show that this closed diagram is equal to zero.
- Case 3: there are two neighboring trivalent/tetravalent vertices that are connected by at least 2 of their ρ strands. Then using the absorption relations from Lemma 3.20, we can express our closed diagram in jellyfish form as a linear combination of diagrams with strictly fewer vertices, which are still in jellyfish form.

We are finished by a simple induction argument on the number of vertices in our closed diagram in jellyfish form. ■

Remark 3.22. We wish to point out that we can also give an existence result for the categories C_m by realising them as actions by endomorphisms on the Cuntz algebras $O_{2m+1} \rtimes \mathbb{Z}_2$. To obtain existence one needs to verify a finite list of polynomial equations that the above tuple needs to satisfy. As we can conclude existence of the examples in this article from the existing literature, we will not include the details of this existence result.

3.4. Symmetries

With the results of the last subsection in hand, the major task in front of us is to determine the 128 complex scalars:

$$A_{k,\ell}^{i,j}, \quad B_{k,\ell}^{i,j}, \quad C_{k,\ell}^{i,j}, \quad D_{k,\ell}^{i,j}, \quad \widehat{A}_{k,\ell}^{i,j}, \quad \widehat{B}_{k,\ell}^{i,j}, \quad \widehat{C}_{k,\ell}^{i,j}, \quad \text{and} \quad \widehat{D}_{k,\ell}^{i,j}.$$

In theory, we could begin evaluating diagrams in our category in multiple ways in order to obtain equations of these variables. However, in practice this task is

too complicated, given that we have 128 unknowns. To make our task of pinning down these scalars easier, we aim to find symmetries between them, and to show that many of them must in fact vanish. The symmetries of these scalars come from the tetrahedral symmetries of the $6j$ symbols, which were rigorously studied in [16], and have been used in previous works of the second author [25, 26]. (See also Footnote 2.)

The main result of this subsection is as follows.

Lemma 3.23. *The scalars $B_{k,\ell}^{i,j}$, $C_{k,\ell}^{i,j}$, $\widehat{B}_{k,\ell}^{i,j}$, $\widehat{C}_{k,\ell}^{i,j}$, $\widehat{D}_{k,\ell}^{i,j}$ can be expressed in terms of the $D_{k,\ell}^{i,j}$ as:*

$$\begin{aligned} B_{k,\ell}^{i,j} &= \lambda_\rho^{1+i+k} \lambda_\alpha \sqrt{\lambda_\alpha} (-1)^\ell \omega_{1,\ell} D_{\bar{i},\ell}^{j,\bar{k}} \\ \widehat{B}_{k,\ell}^{i,j} &= \lambda_\rho^{1+j+\ell} \lambda_\alpha \sqrt{\lambda_\alpha} (-1)^i \omega_{1,j} \omega_{1,i}^2 \omega_{\alpha,\ell} D_{k,\bar{j}}^{\bar{\ell},i}, \\ C_{k,\ell}^{i,j} &= \lambda_\rho^{1+j+k} \lambda_\alpha \sqrt{\lambda_\alpha} (-1)^\ell \omega_{1,\ell}^2 D_{\bar{j},\ell}^{k,i} \\ \widehat{C}_{k,\ell}^{i,j} &= \lambda_\rho^{1+j+k} \sqrt{\lambda_\alpha} (-1)^k \omega_{1,k} \omega_{1,j}^2 \omega_{\alpha,i} D_{\ell,\bar{j}}^{i,\bar{k}}, \\ \widehat{D}_{k,\ell}^{i,j} &= \sqrt{\lambda_\alpha} (-1)^{k+j} \omega_{1,i} \omega_{1,k}^2 \omega_{\alpha,\ell} \omega_{\alpha,j}^2 D_{\ell,k}^{j,i}. \end{aligned}$$

The scalars $A_{k,\ell}^{i,j}$ and $\widehat{A}_{k,\ell}^{i,j}$ satisfy S_4 symmetries generated by the order three rotation:

$$A_{k,\ell}^{i,j} = \lambda_\rho^{1+i+k} \omega_{1,\ell} A_{\bar{i},\ell}^{j,\bar{k}} = \lambda_\rho^{1+j+k} \omega_{1,\ell}^2 A_{\bar{j},\ell}^{k,i} \widehat{A}_{k,\ell}^{i,j} = \lambda_\rho^{1+i+k} \omega_{\alpha,\ell} \widehat{A}_{\bar{i},\ell}^{j,\bar{k}} = \lambda_\rho^{1+j+k} \omega_{\alpha,\ell}^2 \widehat{A}_{\bar{j},\ell}^{k,i}$$

and the order two flips:

$$A_{k,\ell}^{i,j} = \omega_{1,k} \omega_{1,i}^2 A_{\bar{i},\bar{j}}^{k,\bar{\ell}} = \lambda_\rho^{j+\ell} \overline{A_{i,\bar{\ell}}^{k,\bar{j}}} \quad \widehat{A}_{k,\ell}^{i,j} = \omega_{\alpha,k} \omega_{\alpha,i}^2 \widehat{A}_{\bar{i},\bar{j}}^{k,\bar{\ell}} = \lambda_\rho^{j+\ell} \overline{\widehat{A}_{i,\bar{\ell}}^{k,\bar{j}}}.$$

The $D_{k,\ell}^{i,j}$ scalars satisfy the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ symmetries generated by:

$$D_{k,\ell}^{i,j} = \lambda_\rho^{j+\ell} \overline{D_{i,\bar{\ell}}^{k,\bar{j}}} = \lambda_\alpha (-1)^{j+\ell} \omega_{\alpha,k} \omega_{\alpha,i}^2 D_{\bar{i},\bar{j}}^{k,\bar{\ell}} = \lambda_\rho^{i+k} \lambda_\alpha (-1)^{j+\ell} \omega_{\alpha,k} \omega_{\alpha,i}^2 \overline{D_{k,j}^{\bar{i},\bar{\ell}}}.$$

Finally, we have

$$A_{k,\ell}^{i,j} = \widehat{A}_{k,\ell}^{i,j} = D_{k,\ell}^{i,j} = 0 \quad \text{if} \quad i + j + k + \ell \not\equiv 0 \pmod{2}.$$

This result reduces the number of complex scalars to solve for down to 11 in the $\lambda_\alpha = 1$ case, and 7 in the $\lambda_\alpha = -1$ case. This simplification makes it feasible to solve for these scalars in the next subsection.

Lemma 3.24. *We have*

$$A_{k,\ell}^{i,j} = \widehat{A}_{k,\ell}^{i,j} = D_{k,\ell}^{i,j} = 0 \quad \text{if } i + j + k + \ell \not\equiv 0 \pmod{2}.$$

Proof. We will prove the statement of the lemma in the case of the $A_{k,\ell}^{i,j}$ coefficients, as the remaining two cases are nearly identical. We have

$$A_{k,\ell}^{i,j} \begin{array}{c} \text{red strand} \\ \text{black strand} \end{array} = \begin{array}{c} \text{diagram with strands } i, j, k, \ell \end{array} = \frac{\chi_{1,k}\chi_{1,\ell}}{\chi_{1,i}\chi_{1,j}} \begin{array}{c} \text{diagram with strands } i, j, k, \ell \end{array} = \frac{\chi_{1,k}\chi_{1,\ell}}{\chi_{1,i}\chi_{1,j}} A_{k,\ell}^{i,j} \begin{array}{c} \text{red strand} \\ \text{black strand} \end{array} \implies A_{k,\ell}^{i,j} = \frac{\chi_{1,k}\chi_{1,\ell}}{\chi_{1,i}\chi_{1,j}} A_{k,\ell}^{i,j}.$$

Recall from Lemma 3.11 that $\chi_{1,i} = (-1)^i \sqrt{\lambda_\alpha}$. Thus if $i + j + k + \ell \not\equiv 0 \pmod{2}$, then $\frac{\chi_{1,k}\chi_{1,\ell}}{\chi_{1,i}\chi_{1,j}} \neq 1$, which implies $A_{k,\ell}^{i,j} = 0$. $\blacksquare$

Now that we know that half of our coefficients vanish, we move on to describing the symmetries between them. As mentioned before, these symmetries are the standard tetrahedral symmetries of the 6j-symbols. This completes the proof of the main result of this section.

Proof of Lemma 3.23. We include enough examples to illuminate the necessary techniques, all of which involve using the Frobenius maps defined in §3.1. The symmetries of the $A_{k,\ell}^{i,j}$ coefficients are the easiest, as the diagrams only involve ρ strands. We compute the following symmetries:

$$\begin{aligned} A_{k,\ell}^{i,j} &= \frac{1}{d} \begin{array}{c} \text{diagram 1} \end{array} = \frac{\lambda_\rho^{\ell+1}}{d} \begin{array}{c} \text{diagram 2} \end{array} = \frac{\lambda_\rho^\ell}{d} \begin{array}{c} \text{diagram 3} \end{array} = \frac{\lambda_\rho^{\ell+j}}{d} \begin{array}{c} \text{diagram 4} \end{array} \\ &= \lambda_\rho^{\ell+j} \overline{A_{i,\ell}^{k,\bar{j}}}, \end{aligned}$$

$$A_{k,\ell}^{i,j} = \frac{1}{d} \begin{array}{c} \text{diagram 5} \end{array} = \frac{\lambda_\rho^{1+\ell}}{d} \omega_{1,\ell} \begin{array}{c} \text{diagram 6} \end{array} = \frac{\lambda_\rho^{1+\ell}}{d} \omega_{1,\ell} \begin{array}{c} \text{diagram 7} \end{array}$$

$$= \frac{\lambda_\rho^{1+i+\ell}}{d} \omega_{1,\ell} \text{ (diagram) } = \lambda_\rho^{1+i+\ell} \omega_{1,\ell} \overline{A_{j,\tilde{\ell}}^{i,k}},$$

$$A_{k,\ell}^{i,j} = \frac{1}{d} \text{ (diagram) } = \frac{1}{d} \text{ (diagram) } = \frac{\lambda_\rho^k}{d} \omega_{1,k} \text{ (diagram) } = \frac{\lambda_\rho^{i+k}}{d} \frac{\omega_{1,k}}{\omega_{1,i}} \text{ (diagram) }$$

$$= \frac{\lambda_\rho^{i+k}}{d} \frac{\omega_{1,k}}{\omega_{1,i}} \text{ (diagram) } = \lambda_\rho^{i+k} \frac{\omega_{1,k}}{\omega_{1,i}} \overline{A_{\tilde{k},j}^{\tilde{i},\ell}}$$

Together this shows that

$$A_{k,\ell}^{i,j} = \lambda_\rho^{i+k} \omega_{1,\ell} A_{\tilde{i},\tilde{\ell}}^{j,\tilde{k}} = \lambda_\rho^{1+j+k} \omega_{1,\ell}^2 A_{\tilde{j},\tilde{\ell}}^{\tilde{k},i}$$

and

$$A_{k,\ell}^{i,j} = \lambda_\rho^{i+j+k+\ell} \frac{\omega_{1,i}}{\omega_{1,k}} A_{\tilde{i},\tilde{j}}^{\tilde{k},\tilde{\ell}} = \frac{\omega_{1,i}}{\omega_{1,k}} A_{\tilde{i},\tilde{j}}^{\tilde{k},\tilde{\ell}}$$

as claimed. These three tricks work to determine all of the symmetries in the statement of the lemma. In order to show how to deal with α strands, we include one final example.

$$\widehat{A}_{k,\ell}^{i,j} = \frac{1}{d} \text{ (diagram) } = \frac{\lambda_\rho^{\ell+1}}{d} \text{ (diagram) } = \frac{\lambda_\rho^\ell}{d} \text{ (diagram) } = \frac{\lambda_\rho^{\ell+j}}{d} \text{ (diagram) }$$

$$= \frac{\lambda_\rho^{\ell+j}}{d} \text{ (diagram with red loops) } = \lambda_\rho^{\ell+j} \overline{\widehat{A}_{i,\ell}^{k,j}}.$$

We leave the verification of the remaining identities to the reader. ■

To finish off this section, we explicitly compute the 4th Frobenius-Schur indicator of ρ in terms of our free variables. This formula will be useful in the next section.

Lemma 3.25. *We have that*

$$\nu_4(\rho) = \frac{1}{d} + \lambda_\rho \sum_{i,j} \omega_{1,i} \omega_{1,j} A_{i,j}^{i,j} + \lambda_\rho \lambda_\alpha \sum_{i,j} (-1)^{i+j} \omega_{\alpha,i} \omega_{\alpha,j} \widehat{A}_{i,j}^{i,j}.$$

Proof. We pick the following orthonormal basis of $C_2(\rho^{\otimes 4} \rightarrow \mathbf{1})$:

$$\left\{ \frac{1}{d} \text{ (diagram) } \right\} \cup \left\{ \frac{1}{\sqrt{d}} \text{ (diagram) } \right\}_{i,j} \cup \left\{ \frac{1}{\sqrt{d}} \text{ (diagram with red loops) } \right\}_{i,j}.$$

With this basis we compute

$$\begin{aligned} \nu_4(\rho) &= \frac{1}{d^2} \text{ (diagram) } + \frac{1}{d} \sum_{i,j} \text{ (diagram) } + \frac{1}{d} \sum_{i,j} \text{ (diagram with red loops) } \\ &= \frac{1}{d} + \frac{\lambda_\rho}{d} \sum_{i,j} \text{ (diagram) } + \frac{\lambda_\rho}{d} \sum_{i,j} \text{ (diagram with red loops) } \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d} + \frac{\lambda_\rho}{d} \sum_{i,j} \omega_{1,i} \omega_{1,j} \text{ (diagram with two loops)} + \frac{\lambda_\rho}{d} \sum_{i,j} \omega_{\alpha,i} \omega_{\alpha,j} \text{ (diagram with two loops and red circles)} \\
&= \frac{1}{d} + \frac{\lambda_\rho}{d} \sum_{i,j} \omega_{1,i} \omega_{1,j} A_{i,j}^{i,j} + \frac{\lambda_\rho \lambda_\alpha}{d} \sum_{i,j} \chi_{\alpha,i} \chi_{\alpha,j} \omega_{\alpha,i} \omega_{\alpha,j} \text{ (diagram with two loops and red circles)} \\
&= \frac{1}{d} + \lambda_\rho \sum_{i,j} \omega_{1,i} \omega_{1,j} A_{i,j}^{i,j} + \lambda_\rho \lambda_\alpha \sum_{i,j} (-1)^{i+j} \omega_{\alpha,i} \omega_{\alpha,j} \widehat{A}_{i,j}^{i,j}. \quad \blacksquare
\end{aligned}$$

3.5. Classification

In this final subsection, we complete the classification result in the self-dual case (Q3), i.e., we complete the proof of Theorem 3.1, and classify all categorifications of the rings $R(m)$. We have two cases to consider depending on $\lambda_\alpha = \pm 1$.

3.5.1. The case $\lambda_\alpha = 1$. In the case of $\lambda_\alpha = 1$ we have determined that

$$\lambda_\rho = 1, \quad \mu = 1, \quad \tilde{i} = i, \quad \text{and} \quad \chi_{1,i} = \chi_{\alpha,i} = (-1)^i.$$

Thus all that remains is to deduce the 3rd roots of unity $\omega_{1,0}, \omega_{1,1}, \omega_{\alpha,0}, \omega_{\alpha,1}$, along with the free variables $A_{k,\ell}^{i,j}, \widehat{A}_{k,\ell}^{i,j}, D_{k,\ell}^{i,j}$. We express these free variables in the matrix form:

$$X = \begin{bmatrix} X_{0,0}^{0,0} & X_{0,1}^{0,0} & X_{1,0}^{0,0} & X_{1,1}^{0,0} \\ X_{0,0}^{0,1} & X_{0,1}^{0,1} & X_{1,0}^{0,1} & X_{1,1}^{0,1} \\ X_{0,0}^{1,0} & X_{0,1}^{1,0} & X_{1,0}^{1,0} & X_{1,1}^{1,0} \\ X_{0,0}^{1,1} & X_{0,1}^{1,1} & X_{1,0}^{1,1} & X_{1,1}^{1,1} \end{bmatrix} \quad X = A, \widehat{A}, D. \quad (3.26)$$

By applying the symmetries of Lemma 3.23, we have that our free variables are of the form

$$A = \begin{bmatrix} a_0 & 0 & 0 & \omega_{1,0} a_2 \\ 0 & a_2 & \omega_{1,0}^2 a_2 & 0 \\ 0 & \omega_{1,0}^2 a_2 & a_2 & 0 \\ \omega_{1,0} a_2 & 0 & 0 & a_1 \end{bmatrix} \quad \widehat{A} = \begin{bmatrix} \widehat{a}_0 & 0 & 0 & \omega_{\alpha,0} \widehat{a}_2 \\ 0 & \widehat{a}_2 & \omega_{\alpha,0}^2 \widehat{a}_2 & 0 \\ 0 & \omega_{\alpha,0}^2 \widehat{a}_2 & \widehat{a}_2 & 0 \\ \omega_{\alpha,0} \widehat{a}_2 & 0 & 0 & \widehat{a}_1 \end{bmatrix} \quad D = \begin{bmatrix} d_0 & 0 & 0 & -\frac{\omega_{\alpha,1}}{\omega_{\alpha,0}} d_4 \\ 0 & d_2 & d_4 & 0 \\ 0 & -\frac{\omega_{\alpha,0}}{\omega_{\alpha,1}} d_4 & d_3 & 0 \\ \frac{d_4}{\omega_{\alpha,1}} & 0 & 0 & d_1 \end{bmatrix}$$

all of which are real apart from d_4 . If these free coefficients are non-zero, then the tetrahedral symmetries imply conditions on our twists ω . We have

$$\begin{aligned} a_0 \neq 0 &\implies \omega_{1,0} = 1 & a_1 \neq 0 &\implies \omega_{1,1} = 1 \\ \widehat{a}_0 \neq 0 &\implies \omega_{\alpha,0} = 1 & \widehat{a}_1 \neq 0 &\implies \omega_{\alpha,1} = 1 \\ a_2 \neq 0 &\implies \omega_{1,0} = \omega_{1,1} & \widehat{a}_2 \neq 0 &\implies \omega_{\alpha,0} = \omega_{\alpha,1}. \end{aligned}$$

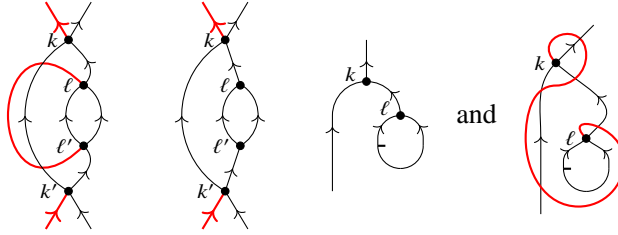
In order to solve for these complex variables, we evaluate certain morphisms in our categories in two ways to obtain equations of these variables. We compute

$$\begin{aligned} \delta_{k,k'} \delta_{\ell,\ell'} \Big| &= \delta_{\ell,\ell'} \Big| = \text{diagram} \\ &= \frac{\omega_{1,\ell}}{2 + \sqrt{5}} \text{diagram} + \sum_{i,j} A_{k,\ell}^{i,j} \text{diagram} + B_{k,\ell}^{i,j} \text{diagram} \\ &= \delta_{k,\ell} \delta_{k',\ell'} \frac{\omega_{1,\ell} \omega_{1,\ell'}^2}{2 + \sqrt{5}} \Big| + \sum_{i,j} A_{k,\ell}^{i,j} \overline{A_{k',\ell'}^{i,j}} \text{diagram} + B_{k,\ell}^{i,j} \overline{B_{k',\ell'}^{i,j}} \text{diagram} \\ &= \left(\delta_{k,\ell} \delta_{k',\ell'} \frac{\omega_{1,\ell} \omega_{1,\ell'}^2}{2 + \sqrt{5}} + \sum_{i',j'} A_{k,\ell}^{i,j} \overline{A_{k',\ell'}^{i',j'}} + B_{k,\ell}^{i,j} \overline{B_{k',\ell'}^{i',j'}} \right) \Big| \\ &= \left(\delta_{k,\ell} \delta_{k',\ell'} \frac{\omega_{1,\ell} \omega_{1,\ell'}^2}{2 + \sqrt{5}} + \sum_{i',j'} A_{k,\ell}^{i,j} \overline{A_{k',\ell'}^{i',j'}} + (-1)^{\ell+\ell'} \omega_{1,\ell} \omega_{1,\ell'}^2 D_{i,\ell}^{j,k} \overline{D_{i,\ell'}^{j,k'}} \right) \Big| \\ &= \left(\delta_{k,\ell} \delta_{k',\ell'} \frac{\omega_{1,\ell} \omega_{1,\ell'}^2}{2 + \sqrt{5}} + \omega_{1,\ell} \omega_{1,\ell'}^2 \sum_{i,j} A_{i,\ell}^{j,k} \overline{A_{i,\ell'}^{j,k'}} + (-1)^{\ell+\ell'} \omega_{1,\ell} \omega_{1,\ell'}^2 D_{i,\ell}^{j,k} \overline{D_{i,\ell'}^{j,k'}} \right) \Big|. \end{aligned}$$

Note that if $\ell \neq \ell'$ then the left hand side vanishes, and we can cancel the $\omega_{1,\ell}\omega_{1,\ell'}^2$ terms. If $\ell = \ell'$ then $\omega_{1,\ell}\omega_{1,\ell'}^2 = 1$. In either case, we can remove the $\omega_{1,\ell}\omega_{1,\ell'}^2$ terms from the above equation. This leaves us with the equation

$$\sum_{i,j} A_{i,\ell}^{j,k} \overline{A_{i,\ell'}^{j,k'}} + (-1)^{\ell+\ell'} \sum_{i,j} D_{k,j}^{\ell,i} \overline{D_{k',j}^{\ell',i}} - (2 - \sqrt{5})\delta_{k,\ell}\delta_{k',\ell'} = \delta_{k,k'}\delta_{\ell,\ell'}.$$

In a similar fashion, we can evaluate the diagrams



in two ways⁴ to obtain

$$\begin{aligned} \sum_{i,j} \widehat{A}_{j,\ell}^{i,k} \overline{\widehat{A}_{j,\ell'}^{i,k'}} + \sum_{i,j} D_{i,\ell}^{j,k} \overline{D_{i,\ell'}^{j,k'}} - (2 - \sqrt{5})\delta_{k,\ell}\delta_{k',\ell'} &= \delta_{k,k'}\delta_{\ell,\ell'} \\ \sum_{i,j} \widehat{D}_{k,\ell}^{i,j} \overline{\widehat{D}_{k',\ell'}^{i,j}} + (-1)^{\ell+\ell'} \omega_{1,\ell}\omega_{1,\ell'}^2 \sum_{i,j} D_{k',\ell'}^{i,j} \overline{D_{k,\ell}^{i,j}} &= \delta_{k,k'}\delta_{\ell,\ell'} \\ \sum_i A_{i,\ell}^{i,k} + (-1)^\ell \sum_i D_{i,\ell}^{i,k} - (2 - \sqrt{5})\delta_{k,\ell} &= 0 \\ \sum_i \widehat{A}_{i,\ell}^{i,k} + \sum_i (-1)^i D_{k,i}^{\ell,i} - (2 - \sqrt{5})\delta_{k,\ell} &= 0. \end{aligned}$$

In terms of our free variables, this gives us the equations:

$$3 - \sqrt{5} = a_0^2 + a_2^2 + d_0^2 + d_3^2 = a_1^2 + a_2^2 + d_1^2 + d_2^2 = \widehat{a}_0^2 + \widehat{a}_2^2 + d_0^2 + d_2^2 = \widehat{a}_1^2 + \widehat{a}_2^2 + d_1^2 + d_3^2 \quad (3.27)$$

$$\frac{1}{2} = a_2^2 + |d_4|^2 = \widehat{a}_2^2 + |d_4|^2 = d_0^2 + |d_4|^2 = d_1^2 + |d_4|^2 = d_2^2 + |d_4|^2 = d_3^2 + |d_4|^2 \quad (3.28)$$

⁴When we say we evaluate a diagram in two ways to obtain a relation, one way is trivial, and the other uses the jellyfish relations from Lemmas 3.4 and 3.16. For the non-self dual case in §4 below, we use the jellyfish relations from (4.3), (4.5), and Lemma 4.7 instead.

$$2 - \sqrt{5} = (a_0 + a_1)a_2 - d_0d_2 - d_1d_3 = (\widehat{a}_0 + \widehat{a}_1)\widehat{a}_2 + d_0d_3 + d_1d_2 \quad (3.29)$$

$$2 - \sqrt{5} = a_0 + a_2 + d_0 + d_3 = a_1 + a_2 - d_1 - d_2 = \widehat{a}_0 + \widehat{a}_2 + d_0 - d_2 = \widehat{a}_1 + \widehat{a}_2 - d_1 + d_3 \quad (3.30)$$

$$\begin{aligned} 0 &= (\omega_{\alpha,0} + \omega_{\alpha,0}^2)a_2^2 + \omega_{\alpha,0}\omega_{\alpha,1}d_4^2 + \omega_{\alpha,0}^2\omega_{\alpha,1}\overline{d_4}^2 \\ &= (\omega_{\alpha,0} + \omega_{\alpha,0}^2)\widehat{a}_2^2 + \omega_{\alpha,0}\omega_{\alpha,1}d_4^2 + \overline{d_4}^2 \end{aligned} \quad (3.31)$$

$$\begin{aligned} 0 &= (1 - \omega_{\alpha,0}\omega_{\alpha,1}^2)(d_2\overline{d_4} - d_3^2) \\ &= d_4(d_1 + d_0) - \overline{d_4}(\omega_{\alpha,0}^2\omega_{\alpha,1}d_0 + \omega_{\alpha,0}\omega_{\alpha,1}^2d_1). \end{aligned} \quad (3.32)$$

While we could begin solving these equations directly, instead we opt for a more measured approach, and use our previous centre analysis to simplify our solution.

Lemma 3.33. *There exists a $\tau \in \{-1, 1\}$ such that*

$$a_0 = a_1 = \widehat{a}_0 = \widehat{a}_1 = \frac{2 + 3\tau - \sqrt{5}}{4}$$

and

$$a_2 = \widehat{a}_2 = d_0 = -d_1 = -d_2 = d_3 = \frac{2 - \tau - \sqrt{5}}{4}.$$

In particular, as $a_0, a_1, \widehat{a}_0$, and $\widehat{a}_1$ are all non-zero, we have $\omega_{1,0} = \omega_{1,1} = \omega_{\alpha,0} = \omega_{\alpha,1} = 1$.

Proof. We first observe from Equation (3.28) that

$$a_2^2 = \widehat{a}_2^2 = d_0^2 = d_1^2 = d_2^2 = d_3^2 = \frac{1}{2} - |d_4|^2,$$

and in particular we have that $a_2, \widehat{a}_2, d_0, d_1, d_2$, and d_3 are real numbers which are equal up to sign. With this information in hand, we can now see from Equation (3.27) that

$$a_0^2 = a_1^2 = \widehat{a}_0^2 = \widehat{a}_1^2 = 3 - \sqrt{5} - 3d_0^2.$$

To make additional progress on solving these equations, we recall the operators ϕ and ψ . In our case, via Equation (3.30), we have that

$$\begin{aligned} \phi &= \begin{bmatrix} a_0 + a_2 & 0 \\ 0 & a_1 + a_2 \end{bmatrix} = \begin{bmatrix} 2 - \sqrt{5} - d_0 - d_3 & 0 \\ 0 & 2 - \sqrt{5} + d_1 + d_2 \end{bmatrix} \\ \psi &= \begin{bmatrix} \widehat{a}_0 + \widehat{a}_2 & 0 \\ 0 & \widehat{a}_1 + \widehat{a}_2 \end{bmatrix} = \begin{bmatrix} 2 - \sqrt{5} - d_0 + d_2 & 0 \\ 0 & 2 - \sqrt{5} + d_1 - d_3 \end{bmatrix}. \end{aligned}$$

From Lemma 3.8, we know that ϕ and ψ have entries in $\{\frac{3-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$, so

$$d_0 + d_3, -d_1 - d_2, d_0 - d_2, d_3 - d_1 \in \left\{ \frac{3 - \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2} \right\}.$$

In particular, as the values d_0, d_1, d_2 , and d_3 are real numbers which are the same up to sign, we have that

$$d_0 = -d_1 = -d_2 = d_3 = \frac{2 - \sqrt{5} - \tau}{4},$$

for some $\tau \in \{-1, 1\}$.

From Equations (3.30) we can deduce that $a_0 = a_1$ and $\widehat{a}_0 = \widehat{a}_1$. We know that a_2 and c_0 are the same up to sign. If we have $a_2 = -c_0$, then Equation (3.30) would imply that

$$a_0 = 2 - \sqrt{5} - d_0.$$

Plugging this value of a_0 into Equation (3.27) gives a contradiction. Thus $a_2 = c_0$, and so Equation (3.30) gives

$$a_0 = 2 - \sqrt{5} - 3d_0 = \frac{2 + 3\tau - \sqrt{5}}{4}.$$

A similar argument shows that $\widehat{a}_2 = c_0$, and thus $\widehat{a}_0 = a_0$. ■

To pin down the value of τ , we return to our analysis of the centre of C_2 . By computing the 4th Frobenius-Schur indicator of ρ in two ways, we can show that $\tau = 1$.

Lemma 3.34. *We have that $\tau = 1$.*

Proof. From Lemma 3.25, we have that $\nu_4(\rho) = 3\tau$. On the other hand, we can use Lemma 3.14 to obtain

$$\nu_4(\rho)(20 + 8\sqrt{5}) = 48 + 20\sqrt{5} + (2 + \sqrt{5}) \sum p_i(p_i + q_i)\theta_i^4$$

where $\sum p_i(p_i + q_i) = 16$ and the θ_i 's are roots of unity. Thus

$$\sum p_i(p_i + q_i)\theta_i^4 = 4(-1 - 2\sqrt{5} + 3\tau\sqrt{5}).$$

If $\tau = -1$, then Theorem A.26 implies that it would take at least 24 roots of unity to write $4(-1 - 2\sqrt{5} + 3\tau\sqrt{5})$, and hence $\sum p_i(p_i + q_i) \geq 24$, giving a contradiction. Thus we must have $\tau = 1$. ■

Now that we know all of our real free variables, we can solve for d_4 , the one complex variable.

Lemma 3.35. *We have that*

$$d_4 = \eta_1 \frac{i}{2} + \eta_2 \frac{1}{2} \sqrt{\frac{-1 + \sqrt{5}}{2}} \quad \text{where } \eta_1, \eta_2 \in \{-1, 1\}.$$

Proof. From Lemmas 3.33 and 3.34, we have $a_2 = \frac{1-\sqrt{5}}{4}$. By Equations (3.28) and (3.31), we have

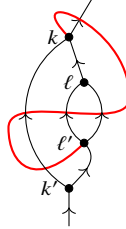
$$|d_4|^2 = \frac{1 + \sqrt{5}}{8} \quad \text{and} \quad d_4^2 + \overline{d_4}^2 = \frac{\sqrt{5} - 3}{4}.$$

The 4 intersection points of this hyperbola and circle yield the statement of the lemma. ■

Now that we have pinned down all of our variables, we can prove part of our main theorem which states that there is no fusion category when $\lambda_\alpha = 1$.

Theorem 3.36. *There is no unitary fusion category that categorifies $R(2)$ with $\lambda_\alpha = 1$.*

Proof. By evaluating the diagram



in two ways (see Footnote 4), we obtain the equation

$$(-1)^{k+l} \sum_{i,j} \overline{D}_{i,j}^{k,\ell} \overline{D}_{\ell',k'}^{i,j} + \sum_{i,j} D_{i,j}^{\ell',k'} D_{k,\ell}^{i,j} = 0.$$

Taking $k = \ell' = 0$ and $\ell = k' = 1$ we see that $\sum_{i,j} D_{i,j}^{0,1} D_{0,1}^{i,j} = d_2^2 - d_4^2 \in \mathbb{R}$. Since $d_2 \in \mathbb{R}$ by Lemma 3.33, this means $d_4^2 \in \mathbb{R}$, which contradicts Lemma 3.35. ■

3.5.2. The case $\lambda_\alpha = -1$. In the case of $\lambda_\alpha = -1$ we have determined that

$$\begin{aligned} \mu &= 1 & \tilde{i} &= 1 - i & \omega_{1,0} &= \omega_{1,1} \\ \omega_{\alpha,0} &= \omega_{\alpha,1} & \chi_{1,i} &= (-1)^i \mathbf{i} & \chi_{\alpha,i} &= (-1)^i. \end{aligned}$$

Thus all that remains is to deduce λ_ρ , the 3rd roots of unity $\omega_{1,0}$ and $\omega_{\alpha,0}$, along with the free variables $A_{k,\ell}^{i,j}$, $\widehat{A}_{k,\ell}^{i,j}$, and $D_{k,\ell}^{i,j}$. By studying the 4th Frobenius-Schur indicator of ρ , we are able to show that $\lambda_\rho = 1$ and $\omega_{\alpha,0} = \omega_{1,0}^2$, along with the values of several of our free variables.

Lemma 3.37. *We have that $\lambda_\rho = 1$, and $\omega_{\alpha,0} = \omega_{1,0}^2$. Further, we have that*

$$A_{0,0}^{0,0} = \frac{3 - \sqrt{5}}{2(1 + \omega_{1,0})} \quad \text{and} \quad \widehat{A}_{0,0}^{0,0} = \frac{3 - \sqrt{5}}{2(1 + \omega_{1,0}^2)}.$$

Proof. Recall the operators ϕ and ψ . By applying the symmetries of Lemma 3.23 we have that

$$\begin{aligned} \phi &= \begin{bmatrix} A_{0,0}^{0,0} + A_{1,0}^{1,0} & 0 \\ 0 & A_{0,1}^{0,1} + A_{1,1}^{1,1} \end{bmatrix} = \begin{bmatrix} A_{0,0}^{0,0}(1 + \lambda_\rho \omega_{1,0}) & 0 \\ 0 & A_{0,0}^{0,0}(1 + \lambda_\rho \omega_{1,0}) \end{bmatrix} \\ \psi &= \begin{bmatrix} \widehat{A}_{0,0}^{0,0} + \widehat{A}_{1,0}^{1,0} & 0 \\ 0 & \widehat{A}_{0,1}^{0,1} + \widehat{A}_{1,1}^{1,1} \end{bmatrix} = \begin{bmatrix} \widehat{A}_{0,0}^{0,0}(1 + \lambda_\rho \omega_{\alpha,0}) & 0 \\ 0 & \widehat{A}_{0,0}^{0,0}(1 + \lambda_\rho \omega_{\alpha,0}) \end{bmatrix}. \end{aligned}$$

Thus the operators ϕ and ψ are scalars, and Lemma 3.8 tells us that

$$A_{0,0}^{0,0} = \frac{2 - \sqrt{5} + \tau}{2(1 + \lambda_\rho \omega_{1,0})} \quad \text{and} \quad \widehat{A}_{0,0}^{0,0} = \frac{2 - \sqrt{5} + \tau}{2(1 + \lambda_\rho \omega_{\alpha,0})}$$

for some $\tau \in \{-1, 1\}$.

From Lemma 3.25, we can write the 4th Frobenius-Schur indicator of ρ as

$$\begin{aligned} \nu_4(\rho) &= \sqrt{5} - 2 + \lambda_\rho \omega_{1,0}^2 (A_{0,0}^{0,0} + A_{0,1}^{0,1} + A_{1,0}^{1,0} + A_{1,1}^{1,1}) \\ &\quad - \lambda_\rho \omega_{\alpha,0}^2 (\widehat{A}_{0,0}^{0,0} - \widehat{A}_{0,1}^{0,1} - \widehat{A}_{1,0}^{1,0} + \widehat{A}_{1,1}^{1,1}) \\ &= \sqrt{5} - 2 + \lambda_\rho \omega_{1,0}^2 A_{0,0}^{0,0} (2 + 2\lambda_\rho \omega_{1,0}) - \lambda_\rho \omega_{\alpha,0}^2 \widehat{A}_{0,0}^{0,0} (2 - 2\lambda_\rho \omega_{\alpha,0}) \\ &= \sqrt{5} - 2 + \lambda_\rho \omega_{1,0}^2 (2 - \sqrt{5} + \tau) - \lambda_\rho \omega_{\alpha,0}^2 (2 - \sqrt{5} + \tau) \frac{1 - \lambda_\rho \omega_{\alpha,0}}{1 + \lambda_\rho \omega_{\alpha,0}} \\ &= \left(-2 + \lambda_\rho \omega_{1,0}^2 (2 + \tau) - \lambda_\rho \omega_{\alpha,0}^2 (2 + \tau) \frac{1 - \lambda_\rho \omega_{\alpha,0}}{1 + \lambda_\rho \omega_{\alpha,0}} \right) \\ &\quad + \sqrt{5} \left(1 - \lambda_\rho \omega_{1,0}^2 + \lambda_\rho \omega_{\alpha,0}^2 \frac{1 - \lambda_\rho \omega_{\alpha,0}}{1 + \lambda_\rho \omega_{\alpha,0}} \right). \end{aligned}$$

As $v_4(\rho) \in \mathbb{Z}[i]$, λ_ρ is a second root of unity, and $\omega_{1,0}$ and $\omega_{\alpha,0}$ are third roots of unity, we have

$$1 - \lambda_\rho \omega_{1,0}^2 + \lambda_\rho \omega_{\alpha,0}^2 \frac{1 - \lambda_\rho \omega_{\alpha,0}}{1 + \lambda_\rho \omega_{\alpha,0}} = 0.$$

This implies that $\lambda_\rho = 1$, and that $\omega_{\alpha,0} = \omega_{1,0}^{-1}$. By simplifying the formula for $v_4(\rho)$ further, we find that $v_4(\rho) = \tau$.

To determine τ we use Lemma 3.14 to write

$$\tau(20 + 8\sqrt{5}) = v_4(\rho) = 48 + 20\sqrt{5} + (2 + \sqrt{5}) \sum p_i(p_i + q_i)\theta_i^4$$

where $\sum p_i(p_i + q_i) = 16$, and the θ_i 's are roots of unity. If $\tau = -1$, then we have

$$\sum p_i(p_i + q_i)\theta_i^4 = -4 - 12\sqrt{5}.$$

However, Theorem A.26 implies that it takes at least 48 roots of unity to write $-4 - 12\sqrt{5}$, giving a contradiction. Thus $\tau = 1$, which gives

$$A_{0,0}^{0,0} = \frac{3 - \sqrt{5}}{2(1 + \omega_{1,0})} \quad \text{and} \quad \widehat{A}_{0,0}^{0,0} = \frac{3 - \sqrt{5}}{2(1 + \omega_{1,0}^2)}.$$

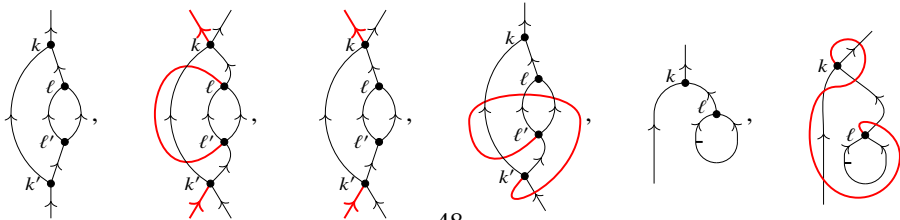
■

Now that we know $\lambda_\rho = 1$, the symmetries of Lemma 3.23 become much simpler. Using the same matrix notation as in the $\lambda_\alpha = 1$ case from (3.26), we can use these symmetries to express our free variables as

$$A = \begin{bmatrix} \omega_{1,0}r & 0 & 0 & \overline{a_1} \\ 0 & \omega_{1,0}^2r & r & 0 \\ 0 & r & \omega_{1,0}^2r & 0 \\ a_1 & 0 & 0 & \omega_{1,0}r \end{bmatrix} \quad \widehat{A} = \begin{bmatrix} \omega_{1,0}^2r & 0 & 0 & \widehat{a_1} \\ 0 & \omega_{1,0}r & r & 0 \\ 0 & r & \omega_{1,0}r & 0 \\ \widehat{a_1} & 0 & 0 & \omega_{1,0}^2r \end{bmatrix} \quad D = \begin{bmatrix} d_0 & 0 & 0 & \overline{d_2} \\ 0 & \overline{d_0} & d_1 & 0 \\ 0 & \overline{d_1} & -\overline{d_0} & 0 \\ d_2 & 0 & 0 & -d_0 \end{bmatrix}$$

where $r = \frac{1}{\omega_{1,0} + \omega_{1,0}^2} \frac{3 - \sqrt{5}}{2} \in \mathbb{R}$, and if either of a_1 or $\widehat{a_1}$ are non-zero, then we have that $\omega_{1,0} = 1$.

Now that we have reduced our free variables down to 5 complex variables, all that remains is to solve for these variables, and to determine the 3rd root of unity $\omega_{1,0}$. As in the $\lambda_\alpha = 1$ case, we get equations of these variables by evaluating the diagrams:



in two ways (see Footnote 4). This gives us the equations:

$$\begin{aligned}
\delta_{k,k'}\delta_{\ell,\ell'} &= \sum_{i,j} A_{j,\ell}^{i,k} \overline{A_{j,\ell'}^{i,k'}} + (-1)^{\ell+\ell'} \sum_{i,j} D_{j,\ell}^{i,k} \overline{D_{j,\ell'}^{i,k'}} - (2 - \sqrt{5})\delta_{k,\ell}\delta_{k',\ell'} \\
\delta_{k,k'}\delta_{\ell,\ell'} &= \sum_{i,j} \widehat{A}_{j,\ell}^{i,k} \overline{\widehat{A}_{j,\ell'}^{i,k'}} + (-1)^{\ell+\ell'} \sum_{i,j} D_{1-k',j}^{1-\ell',i} \overline{D_{1-k,j}^{1-\ell,i}} - (2 - \sqrt{5})\delta_{k,\ell}\delta_{k',\ell'} \\
\delta_{k,k'}\delta_{\ell,\ell'} &= \sum_{i,j} D_{k,\ell}^{i,j} \overline{D_{k',\ell'}^{i,j}} + (-1)^{\ell+\ell'} \sum_{i,j} D_{1-k',1-\ell'}^{i,j} \overline{D_{1-k,1-\ell}^{i,j}} \\
0 &= \sum_{i,j} (-1)^i A_{k,\ell}^{i,j} \overline{D_{k',1-j}^{1-\ell',i}} + (-1)^{\ell+1} \sum_{i,j} D_{1-i,\ell}^{j,1-k} \overline{\widehat{A}_{k,\ell}^{i,j}} - \omega_{1,0}^2 (\sqrt{5} - 2) \mathbf{i} \delta_{k,1-\ell} \delta_{k',1-\ell'} \\
0 &= \sum_i A_{i,\ell}^{i,k} + (-1)^{\ell+1} \mathbf{i} \sum_i D_{i,\ell}^{i,k} - (2 - \sqrt{5})\delta_{k,\ell} \\
0 &= \sum_i \widehat{A}_{i,\ell}^{i,k} + \mathbf{i} \sum_i (-1)^{i+1} D_{1-k,i}^{1-\ell,i} - (2 - \sqrt{5})\delta_{k,\ell}
\end{aligned}$$

In terms of our free variables, this gives us the equations:

$$\operatorname{Im}(d_0) = \frac{1 - \sqrt{5}}{4} = -\frac{\omega_{1,0}^2}{1 + \omega_{1,0}^2} \frac{-1 + \sqrt{5}}{2}, \quad (3.38)$$

$$r^2 + |d_0|^2 = \frac{3 - \sqrt{5}}{2}, \quad (3.39)$$

$$r^2 + |a_1|^2 + |d_1|^2 + |d_2|^2 = r^2 + |\widehat{a}_1|^2 + |d_1|^2 + |d_2|^2 = 1, \quad (3.40)$$

$$|d_1|^2 = |d_2|^2 = \frac{1}{2} - |d_0|^2, \quad (3.41)$$

$$(\omega_{1,0} + \omega_{1,0}^2)r^2 - (d_0^2 + \overline{d_0}^2) = 2 - \sqrt{5}, \quad (3.42)$$

$$ra_1 = d_1d_2 \quad (3.43)$$

$$r\widehat{a}_1 = -\overline{d_1}d_2 \quad (3.44)$$

$$d_2\overline{a_1} + \widehat{a_1}\overline{d_2} = d_1\overline{a_1} + \widehat{a_1}\overline{d_1} = 0 \quad (3.45)$$

Remark 3.46. From Equation (3.38) we see that $\frac{\omega_{1,0}^2}{1+\omega_{1,0}^2} = 1/2$, which implies that $\omega_{1,0} = 1$.

It is now straightforward to solve the above system of equations.

Lemma 3.47. A general solution to Equations (3.38) – (3.45) is given by:

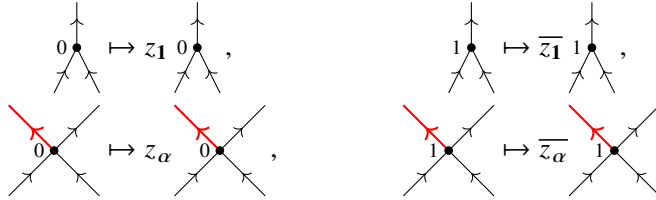
$$a_0 = \widehat{a}_0 = \frac{3 - \sqrt{5}}{4} \quad a_1 = (3 + \sqrt{5})d_1d_2 \quad \widehat{a}_1 = -(3 + \sqrt{5})\overline{d_1}d_2$$

$$|d_1|^2 = |d_2|^2 = \frac{-1 + \sqrt{5}}{8} \quad d_0 = -\frac{1}{2} + \mathbf{i} \frac{1 - \sqrt{5}}{4}.$$

With this lemma in hand, we can show the existence and uniqueness of the unitary fusion category with fusion ring $R(2)$.

Theorem 3.48. *There exists a unique fusion category categorifying the ring $R(2)$ with $\lambda_\alpha = -1$. This unitary fusion category can be realised as the even part of the $2D2$ subfactor.*

Proof. Note that from Lemma 3.12, we are free to re-scale our basis elements of $C_2(\rho \otimes \rho \rightarrow \rho)$ and $C_2(\rho \otimes \rho \rightarrow \alpha\rho)$ by



where $z_1, z_\alpha \in U(1)$. This re-scaling changes the phase of our free variables d_1 and d_2 by $z_1^{-2} z_\alpha^2$ and $z_1^{-2} z_\alpha^{-2}$ respectively. Thus we can arrange so that

$$d_1 = d_2 = \mathbf{i} \frac{1}{2} \sqrt{\frac{1}{2} (-1 + \sqrt{5})}.$$

Hence, up to choice of our basis elements, we have a unique solution of all free parameters determining our category. Thus Proposition 3.21 gives that we have at most one unitary fusion category with fusion ring $R(2)$, and $\lambda_\alpha = -1$.

We know that the even part of the $2D2$ subfactor is a unitary fusion category with fusion ring $R(2)$; hence this must be the unique example. ■

Let us write C_2 for the categorification of $R(2)$ we have classified in this section.

Remark 3.49. We wish to point out the above solutions to our free variables can be used to construct a system of dualizable endomorphisms of the Cuntz algebra $O_5 \rtimes \mathbb{Z}/2\mathbb{Z}$. This gives an independent construction of the category C_2 .

To finish up, we connect C_2 to the even part of the $3^{\mathbb{Z}/4\mathbb{Z}}$ category.

Corollary 3.50. *There is a monoidal $\mathbb{Z}/2\mathbb{Z}$ action on C_2 , such that equivariantisation by this action gives the $3^{\mathbb{Z}/4\mathbb{Z}}$ category of [26, 51].*

Proof. Using the same gauge choice as in the previous Theorem, we can define an order two monoidal equivalence on $\mathcal{C}_2$ by

$$\begin{array}{c} \text{0} \\ \text{1} \end{array} \leftrightarrow \begin{array}{c} \text{1} \\ \text{0} \end{array}, \quad \begin{array}{c} \text{0} \\ \text{1} \end{array} \leftrightarrow \begin{array}{c} \text{1} \\ \text{0} \end{array}, \quad \text{and} \quad \text{red line} \leftrightarrow - \text{red line}.$$

By equivariantising by this order two monoidal auto-equivalence we obtain a unitary fusion category generated by the four morphisms

$$\begin{array}{c} \text{0} \\ \text{1} \end{array} + \begin{array}{c} \text{1} \\ \text{0} \end{array}, \quad \begin{array}{c} \text{0} \\ \text{1} \end{array} + \begin{array}{c} \text{1} \\ \text{0} \end{array}, \quad \begin{array}{c} \text{0} \\ \text{1} \end{array} - \begin{array}{c} \text{1} \\ \text{0} \end{array}, \quad \text{and} \quad \begin{array}{c} \text{0} \\ \text{1} \end{array} - \begin{array}{c} \text{1} \\ \text{0} \end{array},$$

and the isomorphism

$$\text{red loop} : \alpha^{\otimes 4} \sim \mathbf{1}.$$

This is the presentation of the $3^{\mathbb{Z}/4\mathbb{Z}}$ category from [26]. ■

4. The non-self-dual case

In this section we focus on the unitary categorification of the fusion rings with four simple objects $\mathbf{1}, \alpha, \rho, \alpha\rho$ and fusion rules

$$\alpha \otimes \alpha \cong \mathbf{1} \quad \rho \otimes \rho \cong \alpha \oplus m\rho \oplus m\alpha\rho. \quad (S(m))$$

Let us write $S(m)$ for such a fusion ring. By [30] we know that $S(m)$ has a categorification only if $m = 0, 1, 2$.

Our main result of this section is as follows.

Theorem 4.1. *Let $\mathcal{D}_m$ be a unitary fusion category with $K_0(\mathcal{C}) \cong S(m)$. Then either*

- $m = 0$, in which case $\mathcal{D}_0$ is equivalent to one of the four monoidally distinct categories $\text{Hilb}(\mathbb{Z}/4\mathbb{Z}, \omega)$ where $\omega \in H^3(\mathbb{Z}/4\mathbb{Z}, \mathbb{C}^\times)$, or
- $m = 1$, in which case $\mathcal{D}_1$ is equivalent to the monoidally distinct even parts of the two complex conjugate subfactors with principal graphs S' from [26, 34].

In particular the case $m = 2$ from [30, Thm. 1.1(6)] is not categorifiable.

Proof. The $m = 0$ case is easily seen to be pointed, and hence the claim of the above theorem follows from [10, Remark 4.10.4]. Thus it suffices to restrict our attention to the cases of $m = 1$ and $m = 2$.

The general outline of this section follows for the most part as in the self-dual case. In §4.1, We begin by writing down a list of numerical data (essentially the 6j symbols of the category) which fully describe a unitary fusion category with fusion ring $S(m)$. In §4.2, by studying the Drinfeld centre via the tube algebra of the category, we are able to deduce the precise values of some of this numerical data. To reduce the complexity of our numerical data, in §4.3, we use tetrahedral symmetries to essentially cut down the number of free variables in our numerical data by a factor of 24. Finally, in §4.4, we solve for this numerical data by evaluating various morphisms in our categories in multiple ways to obtain equations.

In the case $m = 1$, we reduce our numerical data to two possible solutions, which shows there are at most two distinct unitary fusion categories categorifying $S(1)$. From the subfactor classification literature [26, 34] we know that two such categories exist. We then show there are no solutions to the numerical data in the case $m = 2$, and hence there are no such unitary fusion categories. ■

4.1. Numerical data

We now produce a set of numerical data which completely describes a categorification of the ring $S(m)$. Let us write $\mathcal{D}_m$ for such a unitary fusion category. We will show that the category $\mathcal{D}_m$ can be described by the following data:

- an 8th root of unity $\nu = e^{\pm i\pi\frac{1}{4}}$,
- m choices of signs $\chi_i \in \{-1, 1\}$,
- m choices of 3rd roots of unity $\omega_i \in \{1, e^{2i\pi\frac{1}{3}}, e^{2i\pi\frac{2}{3}}\}$, and
- $8m^4$ complex scalars $A_{k,\ell}^{i,j}, B_{k,\ell}^{i,j}, C_{k,\ell}^{i,j}, D_{k,\ell}^{i,j}, \widehat{A}_{k,\ell}^{i,j}, \widehat{B}_{k,\ell}^{i,j}, \widehat{C}_{k,\ell}^{i,j}, \widehat{D}_{k,\ell}^{i,j} \in \mathbb{C}$ for $0 \leq i, j, k, \ell < m$. These complex scalars are the entries of the F -tensors $F_{\rho}^{\rho,\rho,\rho}$ and $F_{\alpha\rho}^{\rho,\rho,\rho}$

While the 128 complex scalars in the $m = 2$ case seems infeasible to deal with as is, we will use tetrahedral symmetries later on to reduce this 128 to a more workable number.

To simplify notation, we define $d := \dim(\rho)$, which is the largest solution to $d^2 = 1 + 2md$. If $m = 1$ then $d = 1 + \sqrt{2}$, and if $m = 2$ then $d = 2 + \sqrt{5}$. We pick orthonormal bases for the hom spaces

$$\begin{array}{c} \alpha \\ \uparrow \\ \bullet \\ \swarrow \searrow \\ \rho \quad \rho \end{array} \in \mathcal{D}_m(\rho \otimes \rho \rightarrow \alpha), \quad \begin{array}{c} \rho \\ \uparrow \\ \bullet \\ \swarrow \searrow \\ \rho \quad \rho \end{array} \in \mathcal{D}_m(\rho \otimes \rho \rightarrow \rho), \quad \begin{array}{c} \alpha \quad \rho \\ \swarrow \quad \uparrow \\ \bullet \\ \nwarrow \quad \searrow \\ \rho \quad \rho \end{array} \in \mathcal{D}_m(\rho \otimes \rho \rightarrow \alpha\rho)$$

where $0 \leq i \leq m$ so we have the local relations

$$\begin{array}{c}
 \begin{array}{c} \rho \\ \uparrow \\ \rho \end{array} = \begin{array}{c} \rho \quad \rho \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \rho \quad \rho \end{array} + \sum \begin{array}{c} \rho \quad \rho \\ \diagdown \quad \diagup \\ i \\ \diagup \quad \diagdown \\ i \\ \rho \quad \rho \end{array} + \sum \begin{array}{c} \rho \quad \rho \\ \diagdown \quad \diagup \\ i \\ \diagup \quad \diagdown \\ i \\ \rho \quad \rho \end{array} \\
 \\
 \left. \begin{array}{c} \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array} = 1 \\
 \begin{array}{c} \rho \\ \uparrow \\ \rho \end{array} = \begin{array}{c} \rho \\ \uparrow \\ \rho \end{array} = 0 \\
 \begin{array}{c} \rho \\ \uparrow \\ \rho \end{array} = \begin{array}{c} \rho \\ \uparrow \\ \rho \end{array} = 0 \\
 \begin{array}{c} \alpha \\ \uparrow \\ \rho \end{array} = \begin{array}{c} \alpha \\ \uparrow \\ \rho \end{array} = 0 \\
 \begin{array}{c} \alpha \\ \uparrow \\ \rho \end{array} = \begin{array}{c} \alpha \\ \uparrow \\ \rho \end{array} = 0 \end{array} \right\} \quad (4.2)
 \end{array}$$

We also choose unitary isomorphisms

$$\begin{array}{c} \bar{\alpha} \\ \uparrow \\ \alpha \end{array} \in \mathcal{D}_m(\alpha \rightarrow \bar{\alpha}) \quad \text{and} \quad \begin{array}{c} \alpha \quad \rho \\ \diagdown \quad \diagup \\ \rho \quad \alpha \end{array} \in \mathcal{D}_m(\rho \otimes \alpha \rightarrow \alpha \otimes \rho).$$

We normalise this last morphism so that

$$\begin{array}{c} \bar{\alpha} \quad \rho \\ \diagdown \quad \diagup \\ \rho \quad \alpha \end{array} = \begin{array}{c} \bar{\alpha} \quad \rho \\ \diagdown \quad \diagup \\ \rho \quad \alpha \end{array}.$$

We are still free to rescale the crossing up to sign.

Note that as pointed out in the proof of [30, Theorem 5.8], we may assume that α has second Frobenius-Schur indicator -1 , so

$$\begin{array}{c} \bar{\alpha} \\ \uparrow \\ \alpha \end{array} = - \begin{array}{c} \bar{\alpha} \\ \uparrow \\ \alpha \end{array}.$$

Let μ be the scalar defined by

$$\begin{array}{c} \rho \\ \uparrow \\ \rho \end{array} = \mu \begin{array}{c} \rho \\ \uparrow \\ \rho \end{array}. \quad (4.3)$$

Note that from our normalisation we have that $\mu^2 = -1$.

Lemma 4.4. *Without loss of generality, we have the relation*

$$\text{Diagram} = \frac{v}{d} \text{Diagram} \quad \text{where } v = \exp(\pm\pi i/4).$$

Proof. First, by our normalizations for orthonormal bases of hom spaces, we observe that

$$\text{Diagram} = \text{Diagram} = \text{Diagram} = \frac{1}{d} \text{Diagram} = \frac{1}{d} \text{Diagram} \Rightarrow \text{Diagram} = \frac{v}{d} \text{Diagram}$$

for some unimodular scalar v . By computing

$$\frac{v}{d} \text{Diagram} = \text{Diagram} = \mu \text{Diagram} = \frac{\mu \bar{v}}{d} \text{Diagram}$$

we find that $v^2 = \mu$. By re-scaling the crossing by a sign, we may assume that $v = e^{\frac{\pm\pi i}{4}}$. ■

In order to define natural orthonormal bases for the spaces $\mathcal{D}_m(\rho \otimes \rho \rightarrow \rho)$ and $\mathcal{D}_m(\rho \otimes \rho \rightarrow \alpha\rho)$ we define the operators

$$K^1 \left(\begin{array}{c} \text{Diagram} \\ i \end{array} \right) = \text{Diagram} \quad \text{and} \quad K^\alpha \left(\begin{array}{c} \text{Diagram} \\ i \end{array} \right) = \text{Diagram},$$

and the Frobenius operators:

$$R^1 \left(\begin{array}{c} \text{Diagram} \\ i \end{array} \right) = \sqrt{d} \text{Diagram} \quad R^\alpha \left(\begin{array}{c} \text{Diagram} \\ i \end{array} \right) = \sqrt{d} \text{Diagram}$$

$$L^1 \left(\begin{array}{c} \text{diagram} \end{array} \right) = \sqrt{d} \begin{array}{c} \text{diagram} \end{array} \quad L^\alpha \left(\begin{array}{c} \text{diagram} \end{array} \right) = \sqrt{d} \begin{array}{c} \text{diagram} \end{array}$$

on these spaces. Direct computation shows that these operators satisfy the following relations:

$$\begin{aligned} K^1 \circ K^1 &= 1 & K^\alpha \circ K^\alpha &= -1 \\ R^\alpha \circ R^1 &= \nu & R^1 \circ R^\alpha &= \nu^{-1} \\ L^\alpha \circ L^1 &= \nu^{-1} K^1 & L^1 \circ L^\alpha &= \nu^{-1} K^\alpha \\ K^\alpha \circ R^1 &= \mu(R^1 \circ K^1) & K^\alpha \circ L^1 &= \mu(L^1 \circ K^1) \\ (R^\alpha \circ L^1)^3 &= -1 & (R^1 \circ L^\alpha)^3 &= \mu K^\alpha. \end{aligned}$$

As a consequence of these relations, we can diagonalise the action of the operator K^1 , and set

$$\begin{array}{c} \text{diagram} \end{array} := R^1 \left(\begin{array}{c} \text{diagram} \end{array} \right)$$

to obtain that there exist scalars $\chi_i \in \{-1, 1\}$ and $\omega_i \in \{1, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}\}$ such that

$$\begin{aligned} K^1 \left(\begin{array}{c} \text{diagram} \end{array} \right) &= \chi_i \begin{array}{c} \text{diagram} \end{array} & K^\alpha \left(\begin{array}{c} \text{diagram} \end{array} \right) &= \mu \chi_i \begin{array}{c} \text{diagram} \end{array} \\ R^1 \left(\begin{array}{c} \text{diagram} \end{array} \right) &= \begin{array}{c} \text{diagram} \end{array} & R^\alpha \left(\begin{array}{c} \text{diagram} \end{array} \right) &= \nu \begin{array}{c} \text{diagram} \end{array} \\ L^1 \left(\begin{array}{c} \text{diagram} \end{array} \right) &= -\nu \omega_i^2 \begin{array}{c} \text{diagram} \end{array} & L^\alpha \left(\begin{array}{c} \text{diagram} \end{array} \right) &= -\chi_i \omega_i^2 \begin{array}{c} \text{diagram} \end{array}. \end{aligned}$$

In particular, this gives us the local relations

$$\begin{array}{c} \text{diagram} \end{array} = \chi_i \begin{array}{c} \text{diagram} \end{array} \quad \text{and} \quad \begin{array}{c} \text{diagram} \end{array} = \mu \chi_i \begin{array}{c} \text{diagram} \end{array}. \quad (4.5)$$

Remark 4.6. Note we are free to change our basis of $\mathcal{D}_m(\rho \otimes \rho \rightarrow \rho)$ by a unitary which commutes with the operator K^1 . In particular, if $m = 2$ and $\chi_0 = \chi_1$, then we are free to pick any other orthonormal basis of $\mathcal{D}_m(\rho \otimes \rho \rightarrow \rho)$, and if $\chi_0 \neq \chi_1$ then we can only re-scale each basis vector by an element of $U(1)$.

With this special choice of bases, we can determine the following local relations in $\mathcal{D}_m$.

Lemma 4.7. *There are scalars*

$$A_{k,\ell}^{i,j}, B_{k,\ell}^{i,j}, C_{k,\ell}^{i,j}, D_{k,\ell}^{i,j}, \widehat{A}_{k,\ell}^{i,j}, \widehat{B}_{k,\ell}^{i,j}, \widehat{C}_{k,\ell}^{i,j}, \widehat{D}_{k,\ell}^{i,j} \in \mathbb{C} \quad 0 \leq i, j, k, \ell < m$$

such that the following local relations hold in $\mathcal{D}_m$:

The first equation shows a vertex with a blue line labeled α and three legs labeled ρ on the left, equal to $\frac{v}{d}$ times a diagram with a blue line and a vertex, plus $\frac{1}{\sqrt{d}} \sum_{i=0}^1$ times a diagram with a blue line and a vertex labeled i , plus $\frac{v}{\sqrt{d}} \sum_{i=0}^1$ times a diagram with a blue line and a vertex labeled i .

The second equation shows a vertex with a blue line labeled ρ and three legs labeled ρ on the left, equal to $-\frac{\omega_\ell}{v}$ times a diagram with a blue line and a vertex labeled ℓ , minus $v\chi_\ell\omega_\ell^2$ times a diagram with a blue line and a vertex labeled ℓ , plus $\sum_{i,j,k} A_{k,\ell}^{i,j}$ times a diagram with a blue line and a vertex labeled j , plus $B_{k,\ell}^{i,j}$ times a diagram with a blue line and a vertex labeled j .

The third equation shows a vertex with a blue line labeled ρ and three legs labeled ρ on the left, equal to $-\chi_\ell\omega_\ell$ times a diagram with a blue line and a vertex labeled ℓ , minus $v^3\omega_\ell^2$ times a diagram with a blue line and a vertex labeled ℓ , plus $\sum_{i,j,k} \widehat{A}_{k,\ell}^{i,j}$ times a diagram with a blue line and a vertex labeled j , plus $\widehat{B}_{k,\ell}^{i,j}$ times a diagram with a blue line and a vertex labeled j .

The fourth equation shows a vertex with a blue line labeled ρ and three legs labeled ρ on the left, equal to $-\chi_\ell\omega_\ell$ times a diagram with a blue line and a vertex labeled ℓ , minus $v^3\omega_\ell^2$ times a diagram with a blue line and a vertex labeled ℓ , plus $\sum_{i,j,k} \widehat{A}_{k,\ell}^{i,j}$ times a diagram with a blue line and a vertex labeled j , plus $\widehat{B}_{k,\ell}^{i,j}$ times a diagram with a blue line and a vertex labeled j .

Proof. The proof is omitted as it is nearly identical to the proof of Lemma 3.16. ■

Remark 4.8. As in the self-dual case described in Remark 3.18, the above complex scalars are precisely entries of certain F -tensors of $\mathcal{D}_m$.

With these local relations, we can show that our described numerical data fully determines the category $\mathcal{D}_m$.

Proposition 4.9. *There is at most one unitary fusion category $\mathcal{D}_m$ realising each tuple of data*

$$(\nu, \chi, \omega, A, B, C, D, \widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}).$$

Proof. We omit the proof which is nearly identical to the proof of Proposition 3.21 replacing (3.2) with (4.2), the jellyfish relations from Lemmas 3.4 and 3.16 with those from (4.5), (4.3), and Lemma 4.7, and using absorption relations similar to Lemma 3.20. $\blacksquare$

4.2. Centre Analysis

As in the self-dual case, we study the centre of $\mathcal{D}_m$ in order to determine information about our free variables. We restrict our attention to the case of $m = 2$, as this is the most difficult case, and we need as much information about our numerical data as possible in order to make progress on the classification. While we could repeat the analysis for $m = 1$, this is unnecessary as in this case the lack of multiplicity makes it easy to solve for our numerical data.

Our main result of this section is as follows.

Lemma 4.10. *If $m = 2$ and $\chi_0 = \chi_1$, then*

$$\sum_i A_{i,0}^{i,0} = \frac{(2+i) - \sqrt{5}}{2}, \quad \sum_i A_{i,1}^{i,1} = \frac{(2-i) - \sqrt{5}}{2}, \quad \text{and} \quad \sum_i A_{i,0}^{i,1} = \sum_i A_{i,1}^{i,0} = 0.$$

In the case of $\chi_1 = \chi_0$, knowing the above information about the free variables $A_{k,l}^{i,j}$ will be the key starting point in showing non-existence of the category $\mathcal{D}_2$ later on in this paper.

To show this result we study the tube algebra of $\mathcal{D}_2$. As in the self-dual case, we only study a small sub-algebra. We choose the following bases:

$$\begin{aligned} A_{1 \rightarrow 1} &= \text{span} \left\{ \begin{array}{c} \text{Diagram 1: A circle with a dot in the center.} \\ \text{Diagram 2: A circle with a dot in the center and a blue loop around it.} \\ \text{Diagram 3: A circle with a dot in the center and a blue arrow pointing into it.} \\ \text{Diagram 4: A circle with a dot in the center and a blue arrow pointing out of it.} \end{array} \right\} \\ A_{1 \rightarrow \rho} &= \text{span} \left\{ \begin{array}{c} \text{Diagram 5: A circle with a dot in the center and a blue arrow pointing into it, labeled 0.} \\ \text{Diagram 6: A circle with a dot in the center and a blue arrow pointing into it, labeled 1.} \\ \text{Diagram 7: A circle with a dot in the center and a blue arrow pointing into it, labeled 0, with a blue loop around it.} \\ \text{Diagram 8: A circle with a dot in the center and a blue arrow pointing into it, labeled 1, with a blue loop around it.} \end{array} \right\} \\ A_{\alpha \rightarrow \alpha} &= \text{span} \left\{ \begin{array}{c} \text{Diagram 9: A circle with a dot in the center and a blue arrow pointing into it.} \\ \text{Diagram 10: A circle with a dot in the center and a blue arrow pointing into it, with a blue loop around it.} \\ \text{Diagram 11: A circle with a dot in the center and a blue arrow pointing into it, with a blue arrow pointing out of it.} \\ \text{Diagram 12: A circle with a dot in the center and a blue arrow pointing into it, with a blue arrow pointing out of it, and a blue loop around it.} \end{array} \right\}. \end{aligned}$$

By direct computation we obtain that:

(1) The irreducible representations of $A_{1 \rightarrow 1}$ are:

				
χ_0	1	1	$2 + \sqrt{5}$	$2 + \sqrt{5}$
χ_1	1	1	$2 - \sqrt{5}$	$2 - \sqrt{5}$
χ_2	1	-1	$\mathbf{i}$	$-\mathbf{i}$
χ_3	1	-1	$-\mathbf{i}$	$\mathbf{i}$

Hence $\mathcal{I}(\mathbf{1})$ contains 4 simple objects X_i with dimensions

$$\dim(X_0) = 1, \quad \dim(X_1) = 9 + 4\sqrt{5}, \quad \text{and} \quad \dim(X_2) = \dim(X_3) = 5 + 2\sqrt{5}.$$

(2) The irreducible representations of $A_{\alpha \rightarrow \alpha}$ are:

				
τ_0	1	$\mathbf{i}$	$\frac{a(1+\mathbf{i}\mu) + \sqrt{2a^2(1+\mathbf{i}\mu) + 4\mathbf{i}\mu}}{2}$	$\mathbf{i} \frac{a(1+\mathbf{i}\mu) + \sqrt{2a^2(1+\mathbf{i}\mu) + 4\mathbf{i}\mu}}{2}$
τ_1	1	$-\mathbf{i}$	$\frac{a(1-\mathbf{i}\mu) + \sqrt{2a^2(1-\mathbf{i}\mu) - 4\mathbf{i}\mu}}{2}$	$-\mathbf{i} \frac{a(1-\mathbf{i}\mu) + \sqrt{2a^2(1-\mathbf{i}\mu) - 4\mathbf{i}\mu}}{2}$
τ_2	1	$\mathbf{i}$	$\frac{a(1+\mathbf{i}\mu) - \sqrt{2a^2(1+\mathbf{i}\mu) + 4\mathbf{i}\mu}}{2}$	$\mathbf{i} \frac{a(1+\mathbf{i}\mu) - \sqrt{2a^2(1+\mathbf{i}\mu) + 4\mathbf{i}\mu}}{2}$
τ_3	1	$-\mathbf{i}$	$\frac{a(1-\mathbf{i}\mu) - \sqrt{2a^2(1-\mathbf{i}\mu) - 4\mathbf{i}\mu}}{2}$	$-\mathbf{i} \frac{a(1-\mathbf{i}\mu) - \sqrt{2a^2(1-\mathbf{i}\mu) - 4\mathbf{i}\mu}}{2}$

where $a := \chi_0 + \chi_1 \in \{0, \pm 2\}$. Hence $\mathcal{I}(\alpha)$ contains 4 simple objects Y_i with dimensions

$$\frac{20 + 8\sqrt{5}}{2 + \frac{1}{2} \left| a(1 + \mathbf{i}\mu) \pm \sqrt{2a^2(1 + \mathbf{i}\mu) + 4\mathbf{i}\mu} \right|^2}$$

and

$$\frac{20 + 8\sqrt{5}}{2 + \frac{1}{2} \left| a(1 - \mathbf{i}\mu) \pm \sqrt{2a^2(1 - \mathbf{i}\mu) - 4\mathbf{i}\mu} \right|^2}$$

(3) Let ${}_{1}\pi_{\rho}$ be the action of $A_{1 \rightarrow 1}$ on $A_{1 \rightarrow \rho}$. Then

$${}_{1}\pi_{\rho} \left(\begin{array}{c} \text{circle with grey center and blue arrow} \end{array} \right) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad {}_{1}\pi_{\rho} \left(\begin{array}{c} \text{circle with grey center} \end{array} \right) = \begin{bmatrix} \phi & \phi' \\ \phi' & \phi \end{bmatrix} \quad \text{and} \quad {}_{1}\pi_{\rho} \left(\begin{array}{c} \text{circle with grey center and blue arrow} \end{array} \right) = \begin{bmatrix} \phi' & \phi \\ \phi & \phi' \end{bmatrix}$$

where ϕ and ϕ' are the operators on $\text{Hom}(\rho \otimes \rho \rightarrow \rho)$ defined by

$$\begin{aligned}\phi \left(\begin{array}{c} \text{trivalent vertex } i \\ \text{with legs } j, k \end{array} \right) &= \sum_j \begin{array}{c} \text{triangle diagram with vertices } i, j, k \end{array} = \sum_{j,k} A_{j,i}^{j,k} \begin{array}{c} \text{trivalent vertex } k \end{array} \\ \phi' \left(\begin{array}{c} \text{trivalent vertex } i \\ \text{with legs } j, k \end{array} \right) &= \sum_j \begin{array}{c} \text{triangle diagram with vertices } i, j, k \text{ and a blue loop} \end{array} = \chi_{1,i} \sum_{j,k} D_{j,i}^{j,k} \begin{array}{c} \text{trivalent vertex } k \end{array}\end{aligned}$$

which we can naturally identify as operators on the two spaces:

$$\left\{ \begin{array}{c} 0 \\ \text{trivalent vertex } i \end{array} \right\}, \left\{ \begin{array}{c} 1 \\ \text{trivalent vertex } i \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} 0 \\ \text{trivalent vertex } i \text{ with blue loop} \end{array} \right\}, \left\{ \begin{array}{c} 1 \\ \text{trivalent vertex } i \text{ with blue loop} \end{array} \right\}$$

by local insertion. That is, the elements of $A_{1 \leftarrow 1}$ which involve ϕ, ϕ' above act on $A_{1 \leftarrow \rho}$ by applying ϕ, ϕ' locally on the trivalent vertices in our standard basis of $A_{1 \leftarrow \rho}$.

With these computations in hand, we either pin down the scalars χ_0 and χ_1 , or determine the operator ϕ .

Proof of Lemma 4.10. Recall we have three possibilities for $a \in \{-2, 0, 2\}$. If $a = 0$, then we have $\chi_0 = -\chi_1$. Thus we can restrict our attention to the case of $a = \pm 2$.

We begin by determining the decomposition of $1\pi_\rho$ into irreducible representations of $A_{1 \rightarrow 1}$. As X_0 is the tensor unit of $\mathcal{Z}(\mathcal{D}_m)$, its restriction contains no copies of ρ , and thus $1\pi_\rho$ contains no copies of χ_0 . We also know that $\text{Tr}(1\pi_\rho) = 0$, and so from the character table of $A_{1 \rightarrow 1}$ above, we must have that

$$1\pi_\rho \cong 2\chi_1 \oplus k\chi_2 \oplus (2-k)\chi_3$$

with $k \in \{0, 1, 2\}$. In particular, we find that

$$\text{Tr} \left(1\pi_\rho \left(\begin{array}{c} \text{trivalent vertex } i \\ \text{with legs } j, k \end{array} \right) \right) = 4 - 2\sqrt{5} + 2\mathbf{i}(k-1) \quad \implies \quad \text{Tr}(\phi) = 2 - \sqrt{5} + \mathbf{i}(k-1).$$

To determine k we study the restriction of the objects X_i and Y_i . By the above decomposition of $1\pi_\rho$ and from counting dimensions we have

$$\mathcal{F}(X_0) = \mathbf{1} \quad 59$$

$$\begin{aligned}
\mathcal{F}(X_1) &= \mathbf{1} \oplus 2\rho \oplus 2\alpha\rho \\
\mathcal{F}(X_2) &= \mathbf{1} \oplus k\rho \oplus (2-k)\alpha\rho \\
\mathcal{F}(X_3) &= \mathbf{1} \oplus (2-k)\rho \oplus k\alpha\rho.
\end{aligned}$$

By our assumption that $a = \pm 2$, one of the objects Y_i must be invertible. Thus we can label our Y_i so that

$$\begin{aligned}
\mathcal{F}(Y_0) &= \alpha \\
\mathcal{F}(Y_1) &= \alpha \oplus 2\rho \oplus 2\alpha\rho \\
\mathcal{F}(Y_2) &= \alpha \oplus (2-k)\rho \oplus k\alpha\rho \\
\mathcal{F}(Y_3) &= \alpha \oplus k\rho \oplus (2-k)\alpha\rho.
\end{aligned}$$

Hence we now know the restriction of all the objects in both $\mathcal{I}(\mathbf{1})$ and $\mathcal{I}(\alpha)$, up to the integer k . Denote by Z_i the remaining simple objects in $\mathcal{Z}(\mathcal{D})$, i.e. those simple objects such that

$$\mathcal{F}(Z_i) = p_i\rho \oplus q_i\alpha\rho$$

where p_i, q_i are positive integers. This allows us to write

$$\begin{aligned}
\mathcal{I}(\rho) &= 2X_1 + kX_2 + (2-k)X_3 + 2Y_1 + (2-k)Y_2 + kY_3 + \sum p_i Z_i \\
\mathcal{I}(\alpha\rho) &= 2X_1 + (2-k)X_2 + kX_3 + 2Y_1 + kY_2 + (2-k)Y_3 + \sum q_i Z_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
20 &= \dim \operatorname{Hom}(\mathcal{I}(\rho), \mathcal{I}(\rho)) = 4k^2 - 8k + 16 + \sum p_i^2 \\
20 &= \dim \operatorname{Hom}(\mathcal{I}(\alpha\rho), \mathcal{I}(\alpha\rho)) = 4k^2 - 8k + 16 + \sum q_i^2 \\
16 &= \dim \operatorname{Hom}(\mathcal{I}(\rho), \mathcal{I}(\alpha\rho)) = -4k^2 + 8k + 8 + \sum p_i q_i.
\end{aligned}$$

If $k \in \{0, 2\}$ then we get

$$\sum p_i^2 = \sum q_i^2 = 4 \quad \text{and} \quad \sum p_i q_i = 8,$$

which is impossible. Thus we must have $k = 1$, and so $\operatorname{Tr}(\phi) = 2 - \sqrt{5}$. From

$$1\pi_\rho \left(\frac{1}{\dim(\mathcal{D})} \left(\bigcirc + \bigcirc + (2 + \sqrt{5}) \bigcirc + (2 + \sqrt{5}) \bigcirc \right) \right) = 0$$

and

$$1\pi_\rho \left(\bigcirc \right)^2 = 1\pi_\rho \left(\bigcirc \right) + 2 \cdot 1\pi_\rho \left(\bigcirc \right) + 2 \cdot 1\pi_\rho \left(\bigcirc \right),$$

we find that ϕ has the two distinct eigenvalues

$$\frac{(2+i)-\sqrt{5}}{2} \quad \text{and} \quad \frac{(2-i)-\sqrt{5}}{2}.$$

Finally, by Remark 4.6, we are free to unitarily change our basis of $\mathcal{D}_2(\rho \otimes \rho \rightarrow \rho)$ by any element of $U(2)$. In particular we can choose this basis so that ϕ acts diagonally. This gives the statement of the lemma. $\blacksquare$

4.3. Symmetries

We now use the tetrahedral symmetries to determine relations between the 128 complex scalars:

$$A_{k,\ell}^{i,j}, \quad B_{k,\ell}^{i,j}, \quad C_{k,\ell}^{i,j}, \quad D_{k,\ell}^{i,j}, \quad \widehat{A}_{k,\ell}^{i,j}, \quad \widehat{B}_{k,\ell}^{i,j}, \quad \widehat{C}_{k,\ell}^{i,j}, \quad \text{and} \quad \widehat{D}_{k,\ell}^{i,j}.$$

Using the same techniques as in the self-dual case we are able to show the following.

Lemma 4.11. *The scalars $B_{k,\ell}^{i,j}, D_{k,\ell}^{i,j}, \widehat{A}_{k,\ell}^{i,j}, \widehat{C}_{k,\ell}^{i,j}, \widehat{D}_{k,\ell}^{i,j}$ can be expressed in terms of the scalars $A_{k,\ell}^{i,j}$ as:*

$$\begin{aligned} B_{k,\ell}^{i,j} &= -v\chi_\ell\omega_\ell^2 A_{j,\ell}^{k,i} & D_{k,\ell}^{i,j} &= -v^{-1}\omega_\ell A_{i,\ell}^{j,k} \\ \widehat{A}_{k,\ell}^{i,j} &= -v^3\omega_\ell^2 \overline{A_{k,\ell}^{j,i}} & \widehat{C}_{k,\ell}^{i,j} &= -\chi_\ell\omega_\ell \overline{A_{j,\ell}^{i,k}} & \widehat{D}_{k,\ell}^{i,j} &= \overline{A_{i,\ell}^{k,j}}. \end{aligned}$$

The scalars $\widehat{B}_{k,\ell}^{i,j}$ can be expressed in terms of the scalars $C_{k,\ell}^{i,j}$ as:

$$\widehat{B}_{k,\ell}^{i,j} = v^3 \overline{C_{i,\ell}^{k,j}}.$$

The scalars $A_{k,\ell}^{i,j}$ satisfy $\mathbb{Z}/4\mathbb{Z}$ symmetries generated by the relations

$$A_{k,\ell}^{i,j} = -v\chi_i\omega_j\omega_k^2\omega_\ell^2 \overline{A_{j,i}^{k,\ell}} = \chi_i\chi_k\omega_i\omega_\ell\omega_j^2\omega_k^2 A_{\ell,k}^{j,i}.$$

The scalars $C_{k,\ell}^{i,j}$ satisfy S_3 symmetries generated by the order three rotation

$$C_{k,\ell}^{i,j} = \omega_\ell C_{i,\ell}^{j,k} = \omega_\ell^2 C_{j,\ell}^{k,i}$$

and the order two flip

$$C_{k,\ell}^{i,j} = \chi_j\chi_k\omega_i^2\omega_k C_{i,j}^{k,\ell}.$$

Finally we have that if $\chi_0 = -\chi_1$, then

$$A_{k,\ell}^{i,j} = C_{k,\ell}^{i,j} \quad \text{if } i+j+k+\ell \equiv 0 \pmod{2}.$$

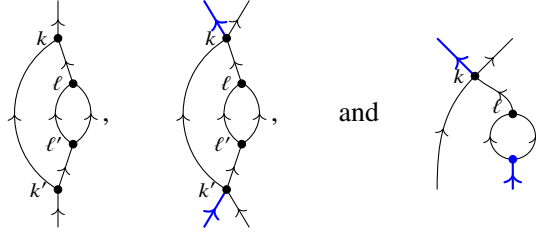
Proof. The proof of this lemma uses the exact same techniques as in the proof of Lemma 3.23. The only real difference is we have different Frobenius operators in this case. ■

4.4. Classification

We now complete the proof of Theorem 4.1 to complete the classification in the non self-dual case. To prove this theorem, we break into three cases: (1) $m = 1$, (2) $m = 2$ and $\chi_0 = -\chi_1$, and (3) $m = 2$ and $\chi_0 = \chi_1$.

The case $m = 1$. If $m = 1$ then from our previous analysis we only have to determine the sign χ_0 , the 3rd root of unity ω_0 , the 8th root of unity ν , and the two complex scalars $a := A_{0,0}^{0,0}$ and $c := C_{0,0}^{0,0}$. Further, we have that if $\omega_0 \neq 1$, then $c = 0$.

By evaluating the diagrams



in two ways (see Footnote 4) we obtain the equations:

$$|a|^2 + |c|^2 = 1 \quad 2|a|^2 = 1 - \frac{1}{1 + \sqrt{2}} \quad \frac{\nu}{1 + \sqrt{2}} = a(\chi_0 \omega_0 - \nu).$$

With the first two of these equations we can solve to find

$$|a|^2 = 1 - \frac{1}{\sqrt{2}} \quad \text{and} \quad |c|^2 = \frac{1}{\sqrt{2}},$$

and thus we have $\omega_0 = 1$. The general solution to these equations is then given by

$$\chi_0 = 1, \quad a = \frac{\nu}{(1 - \nu)}(-1 + \sqrt{2}) \quad \text{and} \quad c = e^{i\theta} 2^{\frac{-1}{4}}$$

where θ is any phase.

Lemma 4.12. *There are exactly two unitary fusion categories, up to monoidal equivalence, which categorify $S(1)$.*

Proof. By unitarily renormalising the basis element

$$\begin{array}{c} \text{diagram} \end{array} \mapsto z \cdot \begin{array}{c} \text{diagram} \end{array} \quad z \in U(1),$$

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we change c to $z^{-2}c$. We can thus renormalise so that $c = 2^{-\frac{1}{4}}$. Hence we have two solutions for our free variables, depending on the choice of $\nu = e^{\pm i\pi\frac{1}{4}}$. By Proposition 4.9, there are at most 2 unitary fusion categories with these fusion rules. These two unitary fusion categories are realised by the even parts of the two subfactors $\mathcal{S}'$ constructed in [34], which are monoidally non-equivalent and complex conjugate to each other. Indeed, they each admit a $\mathbb{Z}/2\mathbb{Z}$ -equivariantization, which produces monoidally non-equivalent $2^{\mathbb{Z}/4\mathbb{Z}}1$ near-group fusion categories which are complex conjugate [35, Ex. 2.2], [25, Ex. 9.5]. ■

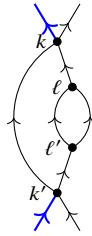
The case $m = 2$ and $\chi_0 = -\chi_1 = 1$. If $m = 2$ and $\chi_0 = -\chi_1 = 1$, then we have to determine the 3rd roots of unity ω_i , the 8th root of unity ν , and the free complex variables $A_{k,l}^{i,j}$ and $C_{k,l}^{i,j}$. We can represent these free complex variables in the same matrix notation as in (3.26) in the self-dual section. After applying the symmetries of Lemma 4.11, we obtain:

$$A = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ 0 & a_1 & -\nu\omega_0^2\overline{a_1} & 0 \\ 0 & \nu\omega_1^2\overline{a_1} & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix} \quad C = \begin{bmatrix} c_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 \end{bmatrix}.$$

Due to the large number of variables which are zero, it is fairly easy to derive a contradiction in this case.

Lemma 4.13. *There is no unitary fusion category that categorifies $S(2)$ with $\chi_0 = -\chi_1 = 1$.*

Proof. By evaluating



in two ways (see Footnote 4), we obtain the equation

$$\chi_\ell \chi_{\ell'} \omega_\ell^2 \omega_{\ell'} \sum_{i,j} A_{j,\ell}^{k,i} \overline{A_{j,\ell'}^{k',i}} + \omega_\ell \omega_{\ell'}^2 \sum_{i,j} A_{i,\ell}^{j,k} \overline{A_{i,\ell'}^{j,k'}} - (2 - \sqrt{5}) \delta_{\ell,k} \delta_{\ell',k'} = \delta_{\ell,\ell'} \delta_{k,k'}.$$

Taking $k = k' = 0$ and $\ell = \ell' = 1$ gives $|a_1|^2 = \frac{1}{2}$, and taking $k = k' = \ell = \ell' = 0$ gives $2|a_0|^2 + 2|a_1|^2 = 3 - \sqrt{5}$. These two equations imply $2|a_0|^2 < 0$, a contradiction. ■

The case $m = 2$ and $\chi_0 = \chi_1$. Finally we deal with the last case where $m = 2$ and $\chi_0 = \chi_1$. Let us again represent our free variables $A_{k,l}^{i,j}$ and $C_{k,l}^{i,j}$ in matrix form as in (3.26). After applying the symmetries of Lemma 4.11, we obtain:

$$A = \begin{bmatrix} a_0 & a_1 & \omega_0 \omega_1^2 a_1 & a_2 \\ -v \chi_0 \omega_1^2 \overline{a_1} & a_3 & -v \chi_0 \omega_0^2 \overline{a_3} & a_4 \\ -v \chi_0 \omega_0^2 \overline{a_1} & -v \chi_0 \omega_1^2 \overline{a_3} & a_3 & \omega_0^2 \omega_1 a_4 \\ -v \chi_0 \omega_0 \omega_1 \overline{a_2} & -v \chi_0 \omega_0^2 \overline{a_4} & -v \chi_0 \omega_1^2 \overline{a_4} & a_5 \end{bmatrix}$$

$$C = \begin{bmatrix} c_0 & c_1 & \omega_0 c_1 & c_2 \\ c_1 & \omega_1^2 c_2 & \omega_0^2 \omega_1^2 c_2 & \omega_1^2 c_3 \\ \omega_0^2 c_1 & \omega_1 c_2 & \omega_1^2 c_2 & c_3 \\ \omega_0 \omega_1^2 c_2 & \omega_1 c_3 & c_3 & c_4 \end{bmatrix}.$$

Recall from Lemma 4.10 that in this case we have

$$\sum_i A_{i,0}^{i,0} = \frac{(2+i) - \sqrt{5}}{2}, \quad \sum_i A_{i,1}^{i,1} = \frac{(2-i) - \sqrt{5}}{2}, \quad \text{and} \quad \sum_i A_{i,0}^{i,1} = \sum_i A_{i,1}^{i,0} = 0,$$

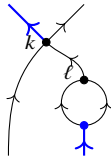
which implies

$$a_0 + a_3 = \frac{(2+i) - \sqrt{5}}{2}, \quad a_3 + a_5 = \frac{(2-i) - \sqrt{5}}{2}, \quad \text{and} \quad a_1 = -a_4 = -\omega_0^2 \omega_1 a_4.$$

With these linear equations in hand, it is straightforward to show non-existence in this case.

Theorem 4.14. *There is no unitary fusion category that categorifies $S(2)$ with $\chi_0 = \chi_1$.*

Proof. Evaluating the diagram



in two ways (see Footnote 4) gives

$$\sum_i A_{i,\ell}^{i,k} + v^3 \chi_\ell \omega_\ell \sum_i A_{i,\ell}^{k,i} - \delta_{\ell,k} (2 - \sqrt{5}) = 0.$$

In terms of our free variables this gives

$$2 - \sqrt{5} = a_0 + a_3 + 2v^3 \chi_0 \omega_0 a_0 + 2\overline{a_3}$$

$$\begin{aligned}
&= \mathbf{i} - 2v^3 \chi_0 \omega_0 \left(2v \chi_0 \omega_0^2 \overline{a_3} + 2a_3 + (-2 - i) + \sqrt{5} \right) \\
2 - \sqrt{5} &= a_5 + a_3 + 2v^3 \chi_0 \omega_1 a_5 + 2\overline{a_3} \\
&= -\mathbf{i} - 2v^3 \chi_0 \omega_1 \left(2v \chi_0 \omega_1^2 \overline{a_3} + 2a_3 + (-2 + i) + \sqrt{5} \right).
\end{aligned}$$

This system of equations of the complex variable a_3 does not hold for any values of our free variables $\chi_0 \in \{-1, 1\}$, $v \in \{e^{\frac{i\pi}{4}}, e^{-\frac{i\pi}{4}}\}$, and $\omega_0, \omega_1 \in \{1, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}\}$. ■

A. A multiplicity bound for $\mathbb{Z}/2\mathbb{Z}$ -quadratic categories

Ryan Johnson, Siu-Hung Ng, David Penneys, Jolie Roat, Matthew Titsworth, and Henry Tucker

In this appendix, we prove Theorem 2.4. That is, given a pseudounitary $\mathbb{Z}/2\mathbb{Z}$ -quadratic fusion category with simple objects $1, \alpha, \rho, \alpha\rho$ with ρ self-dual and fusion rules determined by

$$\alpha^2 \cong 1 \quad \text{and} \quad \rho^2 \cong 1 \oplus m\rho \oplus n\alpha\rho, \quad (\text{Q3})$$

(m, n) must be one of $(0, 0), (0, 1), (1, 0), (1, 1), (2, 2)$.

A.1. Basic number theoretic constraints

Given a $\mathbb{Z}/2\mathbb{Z}$ -quadratic category C with fusion rules (Q3), the fusion matrices are given in the ordering $1, \rho, \alpha\rho, \alpha$ by

$$L_\rho = \left(\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 0 \\ 1 & m & n & 0 & 0 \\ 0 & n & m & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right), \quad L_{\alpha\rho} = \left(\begin{array}{c|cccc} 0 & 0 & 1 & 0 & 0 \\ 0 & n & m & 1 & 0 \\ 1 & m & n & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right), \quad \text{and} \quad L_\alpha = \left(\begin{array}{c|cccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right).$$

Setting $d := \dim(\rho)$, we have $\dim(\alpha) = 1$ and $d^2 = 1 + (m + n)d$, so that

$$d = \frac{1}{2} \left(m + n + \sqrt{4 + (m + n)^2} \right). \quad (\text{A.1})$$

Then

$$\begin{aligned}
\dim(C) &= 2 + 2d^2 = 2 + \frac{(m + n)^2 + 2(m + n)\sqrt{4 + (m + n)^2} + 4 + (m + n)^2}{2} \\
&= 4 + 2(m + n)d.
\end{aligned}$$

Since $K_0(C)$ is abelian of dimension 4, its irreducible representations are all 1-dimensional. Hence by [48, Rem. 2.11], the element

$$R := I + L_\rho^2 + L_{\alpha\rho}^2 + L_\alpha^2 = \begin{pmatrix} 4 & 2m & 2n & 0 \\ 2m & 2n^2 + 2m^2 + 4 & 4mn & 2n \\ 2n & 4mn & 2n^2 + 2m^2 + 4 & 2m \\ 0 & 2n & 2m & 4 \end{pmatrix}$$

is central in $K_0(C)$, and the roots of its characteristic polynomial are called the *formal codegrees* [47] of C :

$$\begin{aligned} f_1 &= 4 + (m+n)^2 + (m+n)\sqrt{4 + (m+n)^2} \\ f_2 &= 4 + (m+n)^2 - (m+n)\sqrt{4 + (m+n)^2} \\ f_3 &= 4 + (m-n)^2 + (m-n)\sqrt{4 + (m-n)^2} \\ f_4 &= 4 + (m-n)^2 - (m-n)\sqrt{4 + (m-n)^2}. \end{aligned}$$

A.2. Computing the induction and forgetful functor

We now assume C is pseudounitary and we analyze the center $Z(C)$, the forgetful functor $\mathcal{F} : Z(C) \rightarrow C$, and the induction functor $\mathcal{I} : C \rightarrow Z(C)$. Recall that

$$\mathcal{F}(\mathcal{I}(c)) \cong \bigoplus_{x \in \text{Irr}(C)} x \otimes c \otimes x^* \quad \forall c \in C \quad (\text{A.2})$$

and that $\mathcal{F}$ is biadjoint to $\mathcal{I}$. We use the notation $(a, b) := \dim(C(a \rightarrow b))$ and $(A, B) := \dim(Z(C)(A \rightarrow B))$.

Lemma A.3 [48, Theorem 2.13]. *There are distinct simple objects $1_{Z(C)}, X_2, X_3, X_4 \in \text{Irr}(Z(C))$ such that*

$$\mathcal{I}(1_C) = 1_{Z(C)} \oplus X_2 \oplus X_3 \oplus X_4 \quad \text{and} \quad \dim(X_k) = \frac{f_1}{f_k}.$$

Setting

$$r := \sqrt{\frac{4 + (m+n)^2}{4 + (m-n)^2}},$$

it is straightforward to calculate that

$$\begin{aligned} \dim(X_2) &= \frac{f_1}{f_2} = \frac{f_1^2}{f_1 f_2} = 1 + (m+n)d = \frac{\dim(C)}{2} - 1 \\ \dim(X_3) &= \frac{f_1}{f_3} = \frac{f_1 f_4}{f_3 f_4} = 1 + \frac{1}{2}(m+n)d - \frac{r}{2}(m-n)d \end{aligned}$$

$$\dim(X_4) = \frac{f_1}{f_4} = \frac{f_1 f_3}{f_3 f_4} = 1 + \frac{1}{2}(m+n)d + \frac{r}{2}(m-n)d.$$

Remark A.4. Since $\dim(X_2), \dim(X_3), \dim(X_4) \in \mathbb{Z}(d)$, it must be the case that either $m = n$ or $r \in \mathbb{Q}(d)$. If $m = 0$ or $n = 0$, then $r = 1$. One may check that if $0 \neq m \neq n \neq 0$ and $r \in \mathbb{Q}(d)$, then $m + n \geq 11$. We will show below in Theorem A.29 that $m + n \leq 5$.

Proposition A.5. *The center $Z(C)$ has 8 distinct simple objects $1_{Z(C)}, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4$ such that*

$$I(1_C) = 1_{Z(C)} \oplus X_2 \oplus X_3 \oplus X_4 \quad \text{and} \quad I(\alpha) = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4.$$

Denote the rest of the simple objects of $Z(C)$ by $\{Z_s\}_{s \in S}$ where S is some finite set. The matrix F of the forgetful functor $\mathcal{F} : Z(C) \rightarrow C$ can then be represented as follows, where zero entries are omitted:

$$F = \begin{array}{c|cccc|cccc|c} & 1_{Z(C)} & X_2 & X_3 & X_4 & Y_1 & Y_2 & Y_3 & Y_4 & Z_s \\ \hline 1_C & 1 & & & & & & & & \\ \alpha & & & & & 1 & 1 & 1 & 1 & \\ \hline \rho & & x_2 & x_3 & x_4 & y_1 & y_2 & y_3 & y_4 & z_s \\ \alpha\rho & & x'_2 & x'_3 & x'_4 & y'_1 & y'_2 & y'_3 & y'_4 & z'_s \end{array}$$

and the induction matrix is given by F^T . Moreover,

$$\sum_{j=2}^4 x_j = 2m \quad \text{and} \quad \sum_{j=2}^4 x'_j = 2n, \quad (\text{A.6})$$

$$x_2 + x'_2 = m + n, \quad (\text{A.7})$$

$$x_3 + x'_3 = \frac{1}{2}(m+n) - \frac{r}{2}(m-n), \quad \text{and} \quad (\text{A.8})$$

$$x_4 + x'_4 = \frac{1}{2}(m+n) + \frac{r}{2}(m-n). \quad (\text{A.9})$$

$$\sum_{j=1}^4 y_j = 2n \quad \text{and} \quad \sum_{j=1}^4 y'_j = 2m. \quad (\text{A.10})$$

Proof. First, $1_{Z(C)}$ decomposes as desired by Lemma A.3. Next, observe that by (A.2),

$$(I(\alpha), I(\alpha)) = 4 \quad \text{and} \quad (\mathcal{F}I(1_C), \alpha) = 0.$$

Since the first Frobenius-Schur indicator ν_1 satisfies $\text{Tr}_{Z(C)}(\theta_{I(\alpha)}) = 0$ [43, Rem. 4.6] (see also [48, Thm. 2.4]), $I(\alpha)$ decomposes as 4 distinct simples which are distinct from $1_{Z(C)}, X_2, X_3, X_4$. Equations (A.6) and (A.10) follow from calculating

$\mathcal{FI}(1_C)$ and $\mathcal{FI}(\alpha)$. Equations (A.7), (A.8), and (A.9) now follow from the formulas for $\dim(X_k)$ for $k = 2, 3, 4$. ■

We now compute the dimensions of all the hom spaces amongst $I(\rho)$ and $I(\alpha\rho)$ in two ways. The first way is by taking adjoints, and using (A.2). The second way is by using the induction matrix F^T computed in Proposition A.5. This gives to see

$$(I(\rho), I(\rho)) = 4 + 2m^2 + 2n^2 = \sum_{j=2}^4 x_j^2 + \sum_{j=1}^4 y_j^2 + \sum_s z_s^2 \quad (\text{A.11})$$

$$(I(\rho), I(\alpha\rho)) = 4mn = \sum_{j=2}^4 x_j x'_j + \sum_{j=1}^4 y_j y'_j + \sum_s z_s z'_s \quad (\text{A.12})$$

$$(I(\alpha\rho), I(\alpha\rho)) = 4 + 2m^2 + 2n^2 = \sum_{j=2}^4 (x'_j)^2 + \sum_{j=1}^4 (y'_j)^2 + \sum_s (z'_s)^2. \quad (\text{A.13})$$

Lemma A.14. *The non-negative integers $x_j, x'_j, y_j, y'_j, z_s, z'_s$ satisfy*

$$8 + \frac{5}{2}(m+n)^2 - \frac{r^2}{2}(m-n)^2 = \sum_{j=1}^4 (y_j + y'_j)^2 + \sum_s (z_s + z'_s)^2 \quad (\text{A.15})$$

$$8 + 4(m-n)^2 = \sum_{j=2}^4 (x_j - x'_j)^2 + \sum_{j=1}^4 (y_j - y'_j)^2 + \sum_s (z_s - z'_s)^2. \quad (\text{A.16})$$

Proof. To get the first equation, sum Equations (A.11) and (A.13) and twice Equation (A.12). Then use Equations (A.7), (A.8), and (A.9) and simplify. The second is similar. ■

Proposition A.17. *We have the following upper bound:*

$$\sum_s (z_s + z'_s)^2 \leq 8 + \frac{3}{2}(m+n)^2. \quad (\text{A.18})$$

Proof. By Equation (A.15), the desired inequality is implied by

$$\sum_{j=1}^4 (y_j + y'_j)^2 \geq \frac{1}{4} \left(\sum_{j=1}^4 y_j + y'_j \right)^2 = (m+n)^2 \geq (m+n)^2 - \frac{r^2}{2}(m-n)^2,$$

which is true. The second equality above holds by Equation (A.10), and the first inequality above follows from the fact that for any real numbers w, x, y, z , we have

$$4(w^2 + x^2 + y^2 + z^2) - (w+x+y+z)^2 =$$

$$(w-x)^2 + (w-y)^2 + (w-z)^2 + (x-y)^2 + (x-z)^2 + (y-z)^2 \geq 0.$$

The proof is complete. $\blacksquare$

Proposition A.19. Denote the twists of $Y_1, \dots, Y_4$ by $\theta_1, \dots, \theta_4$. We have $\theta_1^2 = \theta_2^2 = \theta_3^2 = \theta_4^2 = \pm 1$. Setting $\theta = \theta_1$, without loss of generality, we have $\theta \in \{1, i\}$.

Proof. We calculate the following Frobenius-Schur indicators [43] of α :

$$0 = \text{Tr}_{Z(C)}(\theta_{I(\alpha)}) = \sum_{j=1}^4 \theta_j \dim(Y_j) \quad (\text{A.20})$$

$$\pm \dim(C) = \text{Tr}_{Z(C)}(\theta_{I(Z)}^2) = \sum_{j=1}^4 \theta_j^2 \dim(Y_j). \quad (\text{A.21})$$

Since $\dim(I(\alpha)) = \dim(C) = \sum_{j=1}^4 \dim(Y_k)$, Equation (A.21) implies that $\theta_1^2 = \theta_2^2 = \theta_3^2 = \theta_4^2 = \pm 1$. By Equation (A.20), the θ_j 's split up into two nonempty groups of opposite sign, so without loss of generality, $\theta_1 \in \{1, i\}$. $\blacksquare$

Definition A.22. For $j = 2, \dots, 4$, we let $\epsilon_j \in \{-1, +1\}$ such that $\epsilon_j \theta = \theta_j$ where $\theta = \theta_1$ is the twist of Y_1 . We also set $\gamma := \frac{1}{2}(m+n) + \frac{r}{2}$ and $\bar{\gamma} := \frac{1}{2}(m+n) - \frac{r}{2}$. Notice that $\gamma + \bar{\gamma} = m+n$ and that $\gamma^2 + \bar{\gamma}^2 = \frac{1}{2}(m+n)^2 + \frac{r^2}{2}(m-n)^2$.

Proposition A.23. We have the following equalities:

$$\begin{aligned} - \sum_s \theta_s (z_s + z'_s)^2 d &= \frac{3}{2}(m+n)^2 d + \frac{r^2}{2}(m-n)^2 d + 2(m+n) \\ &\quad + \theta \sum_{j=1}^4 \epsilon_j (y_j + y'_j) (1 + (y_j + y'_j) d) \end{aligned} \quad (\text{A.24})$$

$$\begin{aligned} - \sum_s \theta_s^2 (z_s + z'_s)^2 d &= \Delta + \frac{3}{2}(m+n)^2 d + \frac{r^2}{2}(m-n)^2 d + 2(m+n) \\ &\quad + \theta^2 \sum_{j=1}^4 (y_j + y'_j) (1 + (y_j + y'_j) d) \end{aligned} \quad (\text{A.25})$$

where $\Delta := \pm(\text{Tr}_{Z(C)}(\theta_{I(\rho)}^2) + \text{Tr}_{Z(C)}(\theta_{I(\alpha\rho)}^2)) \in \{0, \pm 2 \dim(C) = \pm(8 + 4(m+n)d)\}$.

Proof. To get Equation (A.24), we add the following two equations for the first Frobenius-Schur indicators [43] of $I(\rho)$ and $I(\alpha\rho)$, and we use Equations (A.7), (A.8), (A.9) in conjunction with Definition A.22.

$$0 = \text{Tr}_{Z(C)}(\theta_{I(\rho)})$$

$$\begin{aligned}
&= x_2(1 + (m+n)d) + x_3(1 + \gamma d) + x_4(1 + \bar{\gamma}d) + \theta \sum_{j=1}^4 \epsilon_j y_j (1 + (y_j + y'_j)d) \\
&+ \sum_s \theta_s z_s (z_s + z'_s) d \\
0 &= \text{Tr}_{Z(C)}(\theta_{I(g\rho)}) \\
&= x'_2(1 + (m+n)d) + x'_3(1 + \gamma d) + x'_4(1 + \bar{\gamma}d) + \theta \sum_{j=1}^4 \epsilon_j y_j (1 + (y_j + y'_j)d) \\
&+ \sum_s \theta_s z'_s (z_s + z'_s) d.
\end{aligned}$$

Obtaining equation (A.25) is similar using the second Frobenius-Schur indicators [43] of ρ and $\alpha\rho$:

$$\begin{aligned}
\pm \dim(C) &= \text{Tr}_{Z(C)}(\theta_{I(\rho)}^2) \\
&= x_2(1+(m+n)d) + x_3(1+\gamma d) + x_4(1+\bar{\gamma}d) + \theta^2 \sum_{j=1}^4 y_j (1+(y_j+y'_j)d) \\
&+ \sum_s \theta_s^2 z_s (z_s + z'_s) d \\
\pm \dim(C) &= \text{Tr}_{Z(C)}(\theta_{I(\alpha\rho)}^2) \\
&= x'_2(1+(m+n)d) + x'_3(1+\gamma d) + x'_4(1+\bar{\gamma}d) + \theta^2 \sum_{j=1}^4 y'_j (1+(y_j+y'_j)d) \\
&+ \sum_s \theta_s^2 z'_s (z_s + z'_s) d.
\end{aligned}$$

Adding the above equations, applying (A.7), (A.8), (A.9) and Definition A.22, and rearranging gives the result. $\blacksquare$

Theorem A.26 [30, Prop. 5.6 and Thm. 5.7]. *Suppose $u, v \in \mathbb{Z}$ and $t \in \mathbb{N}$ is square free.*

- (1) *It requires at least $|u| + 2|v|$ roots of unity to write $u + v\sqrt{2}$ as a sum of roots of unity.*
- (2) *It requires at least $|v|\varphi(2t)$ roots of unity to write $u + v\sqrt{t}$ as a sum of roots of unity.*

Corollary A.27. *Suppose $u \in \mathbb{Q}$, $v \in \mathbb{Z}$, and $t \in \mathbb{N}$ is square free.*

- (1) *It requires at least $|u| + 2|v|$ roots of unity to write $u + v\sqrt{2}$ as a sum of roots of unity.*

(2) It requires at least $|v|\varphi(2t)$ roots of unity to write $u + v\sqrt{t}$ as a sum of roots of unity.

Proof. Suppose $\sum_{i=1}^N \zeta_i = u + v\sqrt{2}$. Write $u = p/q$ in lowest terms with $q > 0$ so that $q \sum_{s=1}^N \zeta_s = p + qv\sqrt{2}$. By Theorem A.26(1), $qN \geq |p| + 2q|v|$, so $N \geq |u| + 2|v|$.

Now suppose $\sum_{i=1}^N \zeta_i = u + v\sqrt{t}$. Again write $u = p/q$ in lowest terms with $q > 0$ so that $q \sum_{s=1}^N \zeta_s = p + qv\sqrt{t}$. By Theorem A.26(2), $qN \geq q|v|\varphi(2t)$, so $N \geq |v|\varphi(2t)$. ■

Lemma A.28. For all $t \in \mathbb{N}$ with $t \neq 1, 2, 3, 6$, $\varphi(2t) \geq \sqrt{\frac{16t}{5}}$.

Proof. By [28], for all $t \in \mathbb{N}$, $\varphi(2t) \geq 2 \left(\frac{t}{3}\right)^{2/3}$. It is straightforward to show that for $t > 42$, $2 \left(\frac{t}{3}\right)^{2/3} \geq \sqrt{16t/5}$. One verifies directly that for $t = 4, 5$ and $7 \leq t \leq 42$, $\varphi(2t) \geq \sqrt{16t/5}$. The result follows. ■

Theorem A.29. If there is a pseudounitary fusion category $\mathcal{C}$ with the fusion rules (Q3), then $(m + n) \leq 5$.

Proof. We consider the two cases for $\theta \in \{1, i\}$ afforded by Proposition A.19.

(1) Suppose $\theta = i$. We add Equation (A.24) to its complex conjugate, divide by d , and simplify to obtain

$$-\sum_s (\theta_s + \overline{\theta_s})(z_s + z'_s)^2 = (m+n)^2 + r^2(m-n)^2 + 2(m+n)\sqrt{4 + (m+n)^2}. \quad (\text{A.30})$$

- **Case 1:** Suppose $(m+n)^2 + 4 = 2v_0^2$ for some integer $v_0 > 0$. Then by Corollary A.27(1) with $v = 2(m+n)v_0$, it requires at least

$$(m+n)^2 + r^2(m-n)^2 + 4(m+n) \underbrace{\sqrt{\frac{4 + (m+n)^2}{2}}}_{=v_0} \geq \underbrace{\left(1 + \frac{4}{\sqrt{2}}\right)}_{=2v} (m+n)^2$$

roots of unity to write the right hand side of Equation (A.30). Together with Inequality (A.18), we see

$$16 + 3(m+n)^2 \geq 2 \sum_s (z_s + z'_s)^2 \geq \left(1 + \frac{4}{\sqrt{2}}\right) (m+n)^2,$$

which implies $m + n \leq 4$.

- **Case 2:** If $4 + (m + n)^2 \neq 2v^2$, then we can write $4 + (m + n)^2 = v^2t$ where v, t are integers with $v > 0$ and $t > 2$ is square free. Then by Corollary A.27(2), it requires at least $2(m + n)v\varphi(2t)$ roots of unity to write the right hand side of Equation (A.30). Since $4 + (m + n)^2 \equiv \pm 1 \pmod{3}$, we know $4 + (m + n)^2 \notin \{1, 2, 3, 6\}$. By Lemma A.28,

$$v^2\varphi(2t)^2 \geq \frac{16v^2t}{5} \iff v\varphi(2t) \geq 4\sqrt{\frac{4 + (m + n)^2}{5}}.$$

Now by inequality (A.18), we see

$$\begin{aligned} 16 + 3(m + n)^2 &\geq 2 \sum_s (z_s + z'_s)^2 \geq 2(m + n)v\varphi(2t) \\ &\geq 8(m + n)\sqrt{\frac{4 + (m + n)^2}{5}}, \end{aligned}$$

which implies $m + n \leq 4$.

- (2) Suppose $\theta = 1$. Then dividing Equation (A.25) by d and simplifying, we get

$$-\sum_s \theta_s^2 (z_s + z'_s)^2 = \frac{\Delta + 4(m + n)}{d} + \frac{3}{2}(m + n)^2 + \frac{r^2}{2}(m - n)^2 + \sum_{j=1}^4 (y_j + y'_j)^2 \quad (\text{A.31})$$

There are now 2 cases depending on the value of Δ .

- **Case 1:** Suppose $\Delta = 0$. Then Equation (A.31) becomes

$$\begin{aligned} -\sum_s \theta_s^2 (z_s + z'_s)^2 &= 2(a + b)\sqrt{4 + (m + n)^2} - \frac{1}{2}(m + n)^2 \\ &\quad + \frac{r^2}{2}(m - n)^2 + \sum_{j=1}^4 (y_j + y'_j)^2. \end{aligned} \quad (\text{A.32})$$

- **Case 2:** Suppose $\Delta = \pm 2 \dim(\mathbb{C}) = \pm(8 + 4(m + n)d)$. Then equation (A.31) becomes

$$\begin{aligned} -\sum_s \theta_s^2 (z_s + z'_s)^2 &= 2(m + n \pm 2)\sqrt{4 + (m + n)^2} - 2(m + n \pm 2)(a + b) \\ &\quad \pm 4(m + n) + \frac{3}{2}(m + n)^2 + \frac{r^2}{2}(m - n)^2 + \sum_{j=1}^4 (y_j + y'_j)^2. \end{aligned} \quad (\text{A.33})$$

In either of the above cases, arguing as in (1) where $\theta = i$, we see that it takes at least

$$\begin{aligned} & \min \left\{ 8(m+n-2)\sqrt{\frac{4+(m+n)^2}{5}}, 4(m+n-2)\sqrt{\frac{4+(m+n)^2}{2}} \right\} \\ & \geq \frac{4}{\sqrt{2}}(m+n)(m+n-2) \end{aligned}$$

roots of unity to write the right hand sides of Equations (A.32) and (A.33). Now by inequality (A.18), we see

$$8 + \frac{3}{2}(m+n)^2 \geq \sum_s (z_s + z'_s)^2 \geq \frac{4}{\sqrt{2}}(m+n)(m+n-2),$$

which implies $m+n \leq 5$. ■

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