Quantum cohomology from mixed Higgs-Coulomb phases

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We generalize Coulomb-branch-based gauged linear sigma model (GLSM)–computations of quantum cohomology rings of Fano spaces. Typically such computations have focused on GLSMs without superpotential, for which the low energy limit of the GLSM is a pure Coulomb branch, and quantum cohomology is determined by the critical locus of a twisted one-loop effective superpotential. We extend these results to cases for which the low energy limit of the GLSM includes both Coulomb and Higgs branches, where the latter is a Landau-Ginzburg orbifold. We describe the state spaces and products of corresponding operators in detail, comparing a geometric phase description, where the operator product ring is quantum cohomology, to the description in terms of Coulomb and Higgs branch states. As a concrete test of our methods, we compare to existing mathematics results for quantum cohomology rings of hypersurfaces in projective spaces.

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1 Introduction

Let X be a smooth compact Kähler manifold. Quantum cohomology is a deformation of an ordinary cohomology of a space X, which is computed mathematically via an analysis of intersection theory on the moduli space of pseudoholomorphic curves on X, and interpreted physically in terms of the operator product of a topological field theory: the A-twisted non-linear sigma model (NLSM) with target X. The physical interpretation has two important consequences. First, it means that there is a set of observables in a non-trivial quantum field theory—the NLSM with target X—that can be calculated using a rigorous mathematical framework. Second, the physical perspective can give new insight into the structure of quantum cohomology.

This is familiar when X can be described as a phase of a gauged linear sigma model (GLSM) [1]. In this situation its quantum cohomology can also be determined in other phases, which may correspond to different geometries or have a non-geometric description. This holds because the GLSM A-model correlation functions are meromorphic functions of the complexified Kähler parameters, and it should be possible to calculate them in any particular phase.[1] When X is a Fano space, the GLSM's complexified Fayet-Iliopoulos (FI) parameters have a non-trivial renormalization group (RG) running. As we review below, it is possible to choose parameters such that for many decades of the RG scale the GLSM RG flow is close to that of the NLSM with target X, and the running FI parameters of the GLSM are mapped to the complexified Kähler parameters of the nonlinear sigma model. The GLSM allows us to follow the running of the FI parameters through a region of strong coupling and provides a description of the low energy physics. Since the topological sector of the theory is RG–invariant, we should be able to extract the quantum cohomology relations from this description. When X can be realized as a phase of a GLSM without superpotential, meaning for example a Fano toric variety, Grassmannian, or flag manifold, then the low energy physics is a set of massive Coulomb vacua, and the quantum cohomology relations can be computed using purely algebraic methods as the Jacobian ideal associated to a one-loop twisted effective superpotential [2]. For Fano spaces X described by GLSMs without a superpotential, these methods are by now standard in the community.

The goal of this paper is to extend these computations of quantum cohomology rings to the case when X is a Fano hypersurface in a projective space, where the low energy dynamics involve both Coulomb and Higgs vacua, with the latter typically giving a non-trivial superconformal theory. Extensive mathematics results on quantum cohomology rings exist for such cases, and we check our computations by comparing to those results. It should be possible to generalize our work to the situation where the projective space is replaced by

[1] This was confirmed for Calabi-Yau hypersurface examples in [2] by summing the gauge instanton sums. In that case a choice of phase determines the topological sectors that contribute to the correlation functions, and while each sum has a different region of convergence in the space of complexified Kähler parameters, they all lead to the same set of meromorphic correlation functions.

a more general toric variety.

We begin by reviewing some aspects of GLSM RG flow and set out our conventions in section 2. In section 3 we outline the basic ideas behind the mixed Higgs-Coulomb branch computations we describe in this paper. We will extensively test our methods in GLSMs describing hypersurfaces in projective spaces, as the mathematics community has extensive results for both quantum cohomology rings and Gromov-Witten invariants in complete intersections in projective spaces (see e.g. [3–6]). To that end, in section 4 we review pertinent mathematics results, against which we will compare later. In section 5 we describe our physical computations. The vector space structure underlying the quantum cohomology rings in the IR phase will arise from a combination of both Coulomb branch states as well as Landau-Ginzburg orbifold states. We extend the dictionary between operators and cohomology classes of [2], and we describe the computation of spectra and give general arguments explaining why those spectrum computations and the operator products should reproduce the known mathematics results reviewed earlier. To further convince the reader, in section 6 we check our methods and computations by applying them to several concrete examples of hypersurfaces in projective spaces. We emphasize that our methods apply in generality.

Finally, in section 7 we outline attempts to apply these same methods to two other analogous quantities which can be computed in GLSMs, namely quantum K theory and quantum sheaf cohomology. In both cases, the methods we have described so far do not completely suffice to fully capture the rings, but we do capture some of the structure. We leave the complete determination of the rings in those cases for the future.

2 GLSM background

2.1 Remarks on phases and RG flow

In this work we consider the case when X is a Fano hypersurface in a projective space, and the non-linear sigma model is asymptotically free. To that end, let us take a moment to review some aspects of RG flow. We define the theory at a fiducial energy scale μ_0 with a Lagrangian based on a metric g_0, and the non-linear sigma model Lagrangian is weakly coupled in the ultraviolet (UV) regime, when the RG scale μ is taken to be large compared to μ_0 and the metric $g(\mu)$ flows to large volume. On the other hand, as we take μ small compared to μ_0, the theory flows to strong coupling, and we cannot use the Lagrangian to determine the full dynamics in the infra-red (IR) regime. The topological twist allows us to isolate a sector of the theory that is insensitive to the RG scale, and together with supersymmetric localization gives a method to calculate correlation functions and operator products of the corresponding set of operators in terms of the quantum cohomology relations.

While this is already a highly non-trivial statement about the low energy limit of the NLSM, it does not determine the nature of the IR fixed point of the RG flow, and we must use other methods to get insight into the low energy dynamics. On the other hand, any other putative description of the IR fixed point should be able to reproduce the topological field theory results. By comparing the topological computations made in the UV NLSM description with those in the IR description we can hope to achieve two goals. First, we can test the claim that the two descriptions really describe the same fixed point. Second, we can find structures in the topological field theory that may not be apparent from the NLSM description. Of course to get the idea off the ground we need a guess or a method to find an alternative description of the low energy dynamics that does not require us to use the strongly-coupled NLSM Lagrangian.

One such method is to embed the RG flow in a gauged linear sigma model. Suppose X is a complete intersection in a toric variety V. Then we can write down a two-dimensional abelian gauge theory with (2,2) worldsheet supersymmetry such that for many decades of the RG scale μ the gauge coupling is small, and the (2,2) vector multiplets are very massive, such that when we integrate out the massive degrees of freedom, the light fields are described by a non-linear sigma model with target space X equipped with a Kähler metric whose Kähler class is determined by the Fayet-Iliopoulos parameters of the gauge theory. It is then reasonable to conjecture that the GLSM and the NLSM will have the same IR fixed point.

When the GLSM is weakly coupled it makes sense to study the pattern of gauge symmetry breaking and its dependence on the Fayet-Iliopoulos (FI) parameters. As is well-known [1,2], the relevant structure is the secondary fan associated to the toric variety, which leads to a division of the linear space of Fayet-Iliopoulos parameters into cones that have come to be called "phases". Classically this is an apt term, since the geometric interpretation of the different cones can be radically different: crossing a phase boundary can lead to topology change, or to a non-geometric regime, or it can even lead to an apparent supersymmetry breaking, and thus the geometric description must break down at the boundary itself. Quantum effects qualitatively modify this classical analysis: the FI parameters are complexified by θ–angles, and while there may be a complex co-dimension 1 locus in this complexified parameter space where the theory is singular, there are no walls separating the various phases—instead the phases are separated by regions of strong coupling.

This discussion is well-known when X is a Calabi-Yau manifold, so that the NLSM flows to a (2,2) superconformal field theory (SCFT), and the FI parameters of the GLSM do not flow under RG. In that situation the different phases describe limits where the GLSM gauge fields can be reliably integrated out to yield an effective description of the low energy physics. Since the phases are smoothly connected, this provides a relationship between disparate corners of the SCFT complexified Kähler moduli space. For example, the Calabi-Yau/Landau-Ginzburg (LG) correspondence can be understood in this fashion.

The results also apply when the FI parameters run, albeit with a different physical interpretation which is perhaps a little bit less familiar.[2] Let us consider the case when $X = \mathbb{P}^n$, where the GLSM is a theory of $n+1$ chiral multiplets, each coupled with charge $+1$ to a single vector multiplet. The theory has a gauge coupling $\mathfrak{e}$ of mass dimension 1 and a complexified Kähler parameter $\tau = ir + \theta/2\pi$, and it is convenient to work with the exponential parameter $q = e^{2\pi i \tau}$. This parameter runs with the renormalization scale μ according to[3]

$$q(\mu) = \left(\frac{\mu_0}{\mu}\right)^{n+1} q(\mu_0) \ . \tag{2.1}$$

Equivalently, the GLSM has a dynamical scale Λ satisfying

$$\Lambda^{n+1} = \mu^{n+1} q(\mu) \ . \tag{2.2}$$

When $\mu \gg \Lambda$, then the gauge degrees of freedom are massive, with mass proportional to $\mathfrak{e}\log(\mu/\Lambda)$, so that we can reliably integrate them out, leading to an effective description of the light degrees of freedom as a (2,2) $\mathbb{P}^n$ NLSM at scale μ with the exponentiated complexified Kähler class given by $e^{2\pi i \tau(\mu)}$. At a fixed scale μ the GLSM and NLSM are of course distinct, and to precisely match the two RG flows we should take the $\mathfrak{e} \to \infty$ limit while holding μ and $\tau(\mu)$ fixed. Although this means the gauge theory is no longer weakly coupled, it is believed that the limit does not lead to any pathology.[4]

Remarkably, we can also obtain an effective description when $\mu \ll \Lambda$, which corresponds to $|q(\mu)| \to \infty$ and $r(\mu) \ll 0$. In this phase there are no "Higgs" vacua, i.e. ones where the chiral multiplets take on non-zero expectation values, and instead there are "Coulomb" vacua, where the complex scalar field σ that resides in the vector multiplet acquires an expectation value determined by an effective twisted superpotential obtained by integrating out the chiral multiplets [2, 7]:

$$\left(\frac{\sigma}{\mu}\right)^{n+1} = q(\mu) \ . \tag{2.3}$$

This result passes an important consistency check: when $|q(\mu)| \gg 1$ these expectation values are such that in every vacuum the chiral multiplets have a mass much larger than μ, and thus could be consistently integrated out. On the other hand, when $|q(\mu)| \ll 1$, these Coulomb vacua are absent. Bringing the μ–dependence to the right-hand-side, we obtain the relation

$$\sigma^{n+1} = \mu^{n+1} q(\mu) = \Lambda^{n+1} \ , \tag{2.4}$$

[2] Although less familiar, the discussion goes back to [1, 2]. A pedagogical presentation is given in [8].

[3] This running is the result of a one-loop computation. Since τ appears as a coupling in a twisted chiral superpotential, we expect there exists a renormalization scheme in which this one-loop running is exact.

[4] An argument in favor of this belief is the successful matching of the A-model topological sector in the two descriptions: this must hold if the $\mathfrak{e} \to \infty$ limit is sensible because in the A-model associated to the GLSM the dependence on $\mathfrak{e}$ is BRST-exact.

and, as pointed out in [9], this way of writing the relation gives it a manifestly RG–invariant form.

It is then natural to conjecture that the low energy limit of the (2,2) $\mathbb{P}^n$ NLSM is a supersymmetric gapped theory with $n+1$ massive vacua characterized by (2.3), and we can test the prediction by comparing the descriptions of the topological sector. Recall that in the A-model the local observables can be written as $\mathcal{O}[\omega]$, where ω is a de Rham cohomology class. For the $\mathbb{P}^n$ the local A-model observables are generated by powers of $\mathcal{O}[\eta]$, where η is the hyper-plane class normalized so that

$$\langle \mathcal{O}[\eta](p_1)\mathcal{O}[\eta](p_2)\cdots\mathcal{O}[\eta](p_n)\rangle = 1 \ . \tag{2.5}$$

Here the p_i denote the insertion points on the worldsheet; since the correlation functions in the topological field theory are independent of the insertion points, we will drop them in what follows. Denoting by t the complexified Kähler class, the remaining non-zero correlation functions take the form

$$\langle \mathcal{O}[\eta]^{n+m(n+1)}\rangle = e^{2\pi i t m} \ . \tag{2.6}$$

This can be reproduced by a localization computation in the IR description based on the Coulomb vacua [9,10]: the local observable is $\widehat{\sigma} = \sigma/\mu$ where μ is the scale at which we match the $\mathbb{P}^n$ NLSM and GLSM descriptions, and the calculation yields the non-zero correlation functions

$$\langle \widehat{\sigma}^{n+m(n+1)}\rangle = q(\mu)^m \ , \tag{2.7}$$

and we can identify the two descriptions as follows:

$$\mathcal{O}[\eta] \leftrightarrow \widehat{\sigma} \ , \qquad\qquad e^{2\pi i t} \leftrightarrow q(\mu) \ . \tag{2.8}$$

In either description, the topological correlation functions are entirely determined by the normalization of the correlator with $m = 0$ and the quantum cohomology relations

$$\mathcal{O}[\eta]^{n+1} = e^{2\pi i t} \ , \qquad\qquad \widehat{\sigma}^{n+1} = q(\mu) \ . \tag{2.9}$$

Note that the $\mathbb{P}^n$ model also has a dynamically generated scale, and in interpreting the first relation in the physical theory we can also express it in terms of that scale. Since we will be making our identifications in the topological theory, we will simply work with the parameter q, and we will also write σ for the observable instead of $\widehat{\sigma}$. It is not difficult to restore the dynamical scale if one wishes to interpret the results in the physical theory.

The analysis of the $\mathbb{P}^n$ theory readily generalizes to Fano toric varieties, Grassmannians, and flag manifolds, and it is possible to extend it to treat the cases when X is a compact complete intersection Calabi-Yau manifold—indeed, that was the primary motivation for the seminal works in the subject [1, 2]. In this paper we generalize the results to the situation when X is a Fano hypersurface in $\mathbb{P}^n$. The main novelty from the physics point of view is that the IR description, based on the GLSM construction, has contributions both from the massive Coulomb vacua and from a superconformal sector—namely (a,c) states of a LG orbifold SCFT. Because the quantum cohomology of such X has been computed in the mathematics literature, we are able to check our computations and realize both of the broad goals mentioned above: first, we provide a non-trivial check on the proposal for the IR dynamics; second, we find a different presentation of the quantum cohomology relations that may provide a useful perspective on quantum cohomology in these and other geometries.

2.2 GLSM Conventions

In this section we lay out our basic conventions for the GLSM. While we will leave details of the supersymmetric action to the references, e.g. [11], we will describe the field content and symmetries of the gauge theory, as well as some terminology that we will use.

The GLSM we consider can be formulated in (2,2) superspace on a Euclidean worldsheet, with superspace coordinates $(z,\theta^-,\overline{\theta}^-;\overline{z},\theta^+,\overline{\theta}^+)$ and corresponding superspace derivatives $D_\pm$ and $\overline{D}_\pm$. Chiral fields are superfields annihilated by $\overline{D}_\pm$.

Consider a (2,2) gauge theory with $n+1$ chiral multiplets X_i and m chiral multiplets P_a coupled to a single vector multiplet V with X_i carrying charge $+1$, and the P_a carrying charge $-d_a$, where d_a are positive integers with sum $d=\sum_a d_a \leq n$, with a standard Kähler potential

$$K=\sum_i |X_i e^V|^2 + \sum_a |P_a e^{-d_a V}|^2 - \frac{1}{4\mathfrak{e}^2}|\Sigma|^2 \ , \tag{2.10}$$

where $\Sigma = \frac{1}{\sqrt{2}}\overline{D}_+ D_- V$ is the gauge-invariant field-strength multiplet—a twisted chiral superfield. The Lagrangian also includes the chiral superpotential of the form

$$W=\sum_{a=1}^{m} P_a G_a(X) \ , \tag{2.11}$$

where the $G_a(X)$ are polynomial in the X fields. Gauge invariance requires G_a to be a homogeneous polynomial of degree d_a. The last term we add is the twisted chiral superpotential

$$\widetilde{W} = \frac{i}{2\sqrt{2}}\tau\Sigma \ . \tag{2.12}$$

The classical Lagrangian density is then written as

$$\mathcal{L} = \frac{1}{2}\int d^4\theta K + \int d\theta^+ d\theta^- W + \int d\theta^+ d\overline{\theta}^- \widetilde{W} + \text{Hermitian conjugate} \ . \tag{2.13}$$

2.3 Key symmetries

Our analysis will extensively use two global symmetries of the GLSM: a continuous $U(1)_V$ symmetry, and a discrete chiral symmetry $\mathbb{Z}_{2(n+1-d)}$. In this section we describe the origin and action of both of these symmetries.

The Lagrangian is invariant under a $U(1)_L \times U(1)_R$ symmetry, which assigns charges $(1,0)$ and $(0,1)$ to the superspace coordinates θ^- and θ^+, respectively, while assigning charge $(0,0)$ to the X_i multiplets and charge $(1,1)$ to the P_a multiplets. More generally, we can assign charges (q_L, q_R) to any gauge-invariant operator. While the vector $U(1)_V$, which assigns charge $q_V = q_R + q_L$, is a symmetry of the quantum theory, the axial $U(1)_A$, with charges $q_A = q_R - q_L$ is anomalous, and only a $\mathbb{Z}_{2(n+1-d)}$ subgroup remains as a symmetry of the effective action. We call the generator of this discrete symmetry ρ and write its action on the gauge-invariant operators as

$$\rho \cdot \mathcal{O}_{q_L,q_R} = \zeta^{q_R - q_L} \mathcal{O}_{q_L,q_R} \ , \tag{2.14}$$

where $\zeta = e^{i\pi/(n+1-d)}$. The action of ρ on the superfields and their components is as follows:

$$\rho \cdot \theta^+ \ = \ \zeta \theta^+, \ \ \rho \cdot \theta^- \ = \ \zeta^{-1}\theta^-, \ \ \rho \cdot P_a \ = \ P_a, \ \ \rho \cdot X_i \ = \ X_i \ . \tag{2.15}$$

We can expand the superfields in components as

$$\begin{aligned} X_i &= x_i + \theta^+ \psi_+^i + \theta^- \psi_-^i + \cdots \ , \\ P_a &= p_a + \theta^+ \psi_+^a + \theta^- \psi_-^a + \cdots \ , \\ \Sigma &= \sigma + \theta^+ \overline{\lambda}_+ + \theta^- \lambda_- + \cdots \ , \end{aligned} \tag{2.16}$$

where the $\cdots$ contain auxiliary fields, the gauge field strength, and other terms determined by supersymmetry from the lower components. We can then see that ρ acts on these fields as in table 1 .[5]

Field	Charge	Field	Charge	Field	Charge	Field	Charge
x, p	0	$\overline{x}$, $\overline{p}$	0	σ	+2	$\overline{\sigma}$	−2
ψ_+	−1	$\overline{\psi}_+$	+1	λ_+	−1	$\overline{\lambda}_+$	+1
ψ_-	+1	$\overline{\psi}_-$	−1	λ_-	+1	$\overline{\lambda}_-$	−1

Table 1: Assignment of ρ charge to the fundamental fields

[5]Since these are not gauge-invariant fields, the action of ρ is ambiguous—we can always combine the symmetry transformation with a gauge transformation. There is no such ambiguity when acting on gauge-invariant fields.

2.4 Phase analysis

The classical phase analysis of this class of theories is simple and involves solving the D-term and F-terms equations that follow from the Lagrangian:

$$\begin{aligned} \sum_i |x_i|^2 - \sum_a d_a |p_a|^2 = r \ , & \qquad \left(\sum_i |x_i|^2 + \sum_a d_a^2 |p_a|^2\right)\sigma = 0 \ , \\ G_a(x) = 0 \ , & \qquad \sum_a p_a \frac{\partial G_a}{\partial x_i} = 0 \ . \end{aligned} \tag{2.17}$$

When $r \gg 0$ and the G_a are chosen to be suitably generic, the low energy degrees of freedom correspond to a non-linear sigma model with target space $\mathbb{P}^n[d_1, d_2, \ldots, d_m]$—a complete intersection defined by the vanishing of m polynomials G_a. On the other hand, when $r \ll 0$, the low energy degrees of freedom are described by a hybrid theory: a Landau-Ginzburg orbifold fibered over a a weighted projective space $\mathbb{P}^{m-1}_{d_1,d_2,\ldots,d_m}$ parameterized by the scalar fields p_a. In general this is a rather complicated quantum field theory involving a LG sector coupled to a non-linear sigma model sector [12–16]. A crucial simplification takes place when $m = 1$. In this case the classical description of the massless degrees of freedom in the $r \ll 0$ phase is a $\mathbb{Z}_d$ LG Orbifold (LGO) which can be studied using the techniques developed in [17–19].

As we reviewed in the previous subsection, this analysis receives two key qualitative corrections. First, the complexified FI parameter runs, so that we cannot meaningfully restrict the analysis to a single phase. Second, the $r \ll 0$ phase has additional Coulomb vacua, and although these are massive, they contribute to the topological sector of the theory, much like the σ vacua in the $r \ll 0$ phase of the $\mathbb{P}^n$ model discussed above. In addition, the RG running of the FI parameter means that, unlike in the Calabi-Yau setting, we cannot simply pick a phase to analyze without reference to an energy scale. However, if we fix a reference scale μ_0 and the corresponding parameter $q(\mu_0)$, we can meaningfully ask about the both the high energy and low energy limits of the flow: in each limit the FI parameter is driven into a particular phase, and we call the former a "UV phase," and the latter an "IR phase." We emphasize that the term "phase" here should be understood as a label for a cone in the secondary fan associated to the toric variety V. We are not discussing any phase transitions, and the only loose analogy with the thermodynamic phases is that there are quantities, like those in the topological sector, which can be calculated using a description based on either the UV or IR degrees of freedom.

In any $U(1)$ GLSM there is a unique UV phase and a unique IR phase.[6] In the example of the $\mathbb{P}^n[d]$ GLSM the UV phase is geometric, in the sense that for sufficiently large μ the light degrees of freedom are described by a weakly coupled NLSM with target space $\mathbb{P}^n[d]$. On the other hand, in the IR phase there are two sectors: a massless sector described by the

[6] In a multi-parameter example the IR and UV phases can depend on the choice of $q(\mu_0)$.

LGO, as well as the Coulomb sector. In terms of the GLSM fields the former arises from the Higgs branch, where the chiral multiplet p acquires an expectation value, while the latter arise when the scalar σ in the vector multiplet acquires an expectation value.

3 A central decoupling observation

For Fano toric varieties, the quantum cohomology ring relations were computed in GLSMs in [2] in two ways: by summing the gauge instantons in the UV geometric phase, and also from the critical locus of the one-loop twisted effective superpotential for the Σ multiplets in the IR phase. For hypersurfaces and complete intersections, the IR phase is no longer a pure Coulomb branch, but is instead a mixture of Coulomb and Higgs branches, and both must be taken into account to reproduce the quantum cohomology structure of the UV phase.

A central idea behind our computations is that deep in the IR, the Coulomb and Higgs branches are disconnected. Correlation functions are computed[7] as a sum of Coulomb branch computations (with vanishing Higgs fields) and Higgs branch computations (with vanishing σ fields). This is ultimately a consequence of certain GLSM bosonic potential terms which in a $U(1)$ GLSM have the form

$$|\sigma|^2 \sum_i Q_i^2 |\phi_i|^2, \tag{3.1}$$

where ϕ_i is the scalar component of the matter chiral superfield of gauge charge Q_i (in our $\mathbb{P}^n[d]$ example the ϕ_i can stand for either the x_i or the p field). As a result,

- If the ϕ_i are nonzero and large, then σ is massive, hence we take $\sigma = 0$, corresponding to the Higgs branch,
- If σ is large, then the Higgs fields are massive, corresponding to the Coulomb branch,

and in any event, both σ and ϕ_i are not simultaneously nonzero at low energies. As a result, in mixed Higgs-Coulomb phases, if t represents a Landau-Ginzburg orbifold field and σ a Coulomb branch field, then

$$\sigma \cdot t \;=\; 0. \tag{3.2}$$

This is an implementation of the 'reliability criterion' described in [7, 10], and we will see that this appears as one of the relations defining OPEs and quantum cohomology rings.

We will combine equation (3.2) and Coulomb branch relations for the σ field, derived as in [2] from a one-loop-exact twisted effective superpotential, with an analysis of the Landau-Ginzburg orbifold states to give an IR phase description of the quantum cohomology ring

[7] On connected worldsheets, which we assume for simplicity.

relations for Fano hypersurfaces in projective space. We will see that although Higgs and Coulomb branches are decoupled, the Dolbeault cohomology classes are represented by linear combinations of contributions from both sectors.

4 Review of pertinent mathematics

The quantum cohomology rings of Fano complete intersections in projective spaces have been well-studied mathematically, see e.g. [3], [4, prop. 1.13], [5, theorem D], [6]. The point of this paper is to understand the corresponding physics computations. In this section we review known mathematical results for these spaces, against which we will compare physics results later.

Thanks to the Lefschetz hyperplane theorem, the cohomology of a complete intersection $\mathbb{P}^n[d_1, \cdots, d_m]$ of all degrees below $n-m$ and above $n+m$ is inherited from that of $\mathbb{P}^n$. In general, there can be additional states in middle degrees, whose index can be computed from the Euler characteristic of the entire space.

For example, for a hypersurface $\mathbb{P}^n[d]$, there are possibly extra states in degree $n-1$, which can be determined from[8]

$$\chi(\mathbb{P}^n[d]) = \sum_{i=0}^{n-1} \left(\begin{array}{c} n+1 \\ i \end{array} \right) (-1)^{n-1-i} d^{n-i} . \tag{4.1}$$

As a result, we see that the primitive cohomology has dimension

$$|\chi(\mathbb{P}^n[d])| - n = \left| \sum_{i=0}^{n-1} \left(\begin{array}{c} n+1 \\ i \end{array} \right) (-1)^{n-1-i} d^{n-i} \right| - n . \tag{4.2}$$

In the quantum field theory quantum cohomology arises as the algebra of local operators of an A-twisted non-linear sigma model associated to the Fano variety X. The space of local operators in such a theory is isomorphic to the de Rham cohomology $H^\bullet(X, \mathbb{R})$, and to each class $[\omega] \in H^k(X, \mathbb{R})$ we associate an operator in the A-model denoted by $\mathcal{O}[\omega]$.

Now, the quantum cohomology and Gromov-Witten theory of Fano complete intersections in projective spaces has been described mathematically in e.g. [3], [4, prop. 1.13], [5, theorem D], [6]. As stated in [3], the quantum cohomology ring of a Fano complete intersection $\mathbb{P}^n[d_1, \cdots, d_m]$ is generated by the hyperplane class η and the primitive cohomology elements

[8] This can be computed by, for example, computing the top Chern class of the tangent bundle from the sequence defining the tangent sheaf of X, and evaluating on the fundamental class of the hyperplane.

$\alpha \in H^{n-m}_{\rm prim}(X,\mathbb{Q})$ with relations[9]

$$\eta^{n+1-m} = q \left(\prod_{a=1}^m (-d_a)^{d_a} \right) \eta^{\sum_a (d_a - 1)}, \quad \eta\alpha = 0, \tag{4.3}$$

$$\alpha \cdot \beta = \frac{(\alpha|\beta)}{d_1 d_2 \cdots d_m} \left(\eta^{n-m} - q \left(\prod_{a=1}^m (-d_a)^{d_a} \right) \eta^{\sum_a (d_a-1)-1} \right), \tag{4.4}$$

for[10]

$$n - m \;\geq\; 2 \sum_a (d_a - 1) - 1, \tag{4.5}$$

where $\alpha, \beta \in H^{n-m}_{\rm prim}(X,\mathbb{Q})$, $(\alpha|\beta)$ is the ordinary classical product in middle cohomology, the "prim" subscript indicates that α, β lie in primitive cohomology, meaning they are annihilated in cohomology by the hyperplane class η:

$$[\alpha \wedge \eta] \;=\; 0 \;=\; [\beta \wedge \eta] \; . \tag{4.6}$$

We shall see that it is no accident that the second relation in (4.3) resembles the decoupling relation (3.2) described earlier.

We should note that there is in general a non-trivial change of coordinates between the GLSM parameter q and the complexified Kähler class of the NLSM due to a semi-classical constant shift (this is equation 4.20 of [2]), as well as to integrating out point-like instantons [1, 2]. In the Calabi-Yau case this is well-known and can be interpreted as the mirror map relating algebraic to special coordinates [2]. Fortunately, in our simple one-parameter example the selection rule based on the anomaly prevents any non-constant corrections to the relation. In this paper we will use the natural GLSM coordinate.

As a consistency check, the reader should note that the ring relations above are linked. For example, from cohomological degrees, it must be the case that

$$\alpha \cdot \beta \;\propto\; \eta^{n-m}, \; q\eta^{\sum_a (d_a-1)-1}. \tag{4.7}$$

Furthermore, from (4.3) (or, ultimately from the physics relation (3.2)),

$$\eta \cdot \alpha \cdot \beta \;=\; 0, \tag{4.8}$$

[9] The reader should note that reference [3] discusses complete intersections in $\mathbb{P}^{n+m}$, whereas we work in $\mathbb{P}^n$, so our "n" differs from that of [3]. We have also normalized q slightly differently, to be consistent with conventions elsewhere in this paper based on the natural GLSM coordinate.

[10] This restriction was included in the statement of the theorem in [3], but our methods based on the IR phase apply to any Fano complete intersection, and indeed we will study examples that lie outside this bound, and the results are consistent with (4.3) and (4.4) without the restriction.

and therefore the linear combination must be in the kernel of η, so from the first part of (4.3), we see

$$\alpha \cdot \beta \, \propto \, \eta^{n-m} - q \left(\prod_{a=1}^{m} (-d_a)^{d_a} \right) \eta^{\sum_a (d_a - 1) - 1}. \tag{4.9}$$

In terms of quantum field theory computations, the quantum cohomology ring is generated by $\mathcal{O}[\eta]$, the operator corresponding to the hyperplane class η, as well as $\mathcal{O}[\alpha]$, where the α belong to the primitive cohomology $H^{n-m}_{\rm prim}(X, \mathbb{Q})$, which obey relations corresponding to the above:

$$(\mathcal{O}[\eta])^{n+1-m} = q \left(\prod_{a=1}^{m} (-d_a)^{d_a} \right) (\mathcal{O}[\eta])^{\sum_a (d_a - 1)} \ , \quad \mathcal{O}[\eta] \cdot \mathcal{O}[\alpha] = 0 \ , \tag{4.10}$$

$$\mathcal{O}[\alpha] \cdot \mathcal{O}[\beta] = \frac{(\alpha|\beta)}{d_1 d_2 \cdots d_m} \left((\mathcal{O}[\eta])^{n-m} - q \left(\prod_{a=1}^{m} (-d_a)^{d_a} \right) (\mathcal{O}[\eta])^{\sum_a (d_a - 1) - 1} \right). \tag{4.11}$$

In the UV phase this is just a restatement of the mathematical results, emphasizing the difference between classical cohomology classes and the corresponding topological field theory operators. Our main goal is to understand how these relations arise in the IR phase. We will see that while the second equation in (4.10) is a consequence of the decoupling between the Higgs and Coulomb branches sketched in (3.2), the remaining equations involve a more intricate interplay between the Coulomb and Landau-Ginzburg orbifold fields.

We will recover these equations from detailed considerations in the next section. Specifically, equation (4.10) will appear later as equations (5.70), (5.73), and equation (4.11) will appear later as equations (5.77). We will also see a nontrivial mixing between Higgs and Coulomb contributions. For example, although (4.10) naively appears to solely result from Coulomb branch computations, later we will see that $\mathcal{O}[\eta]$ is a linear combination of both Coulomb and Higgs branch contributions.

The dimension of the primitive middle cohomology can be obtained from an index computation, as in equation (4.2). For our purposes it will be important to refine the structure further and describe the operators in terms of Dolbeault cohomology $H^{k,\ell}(X)$. In the case of a degree d hypersurface in $\mathbb{P}^{k+\ell+1}$ the dimensions of the primitive Dolbeault cohomology groups can be obtained from an elegant generating function [20, chapter 17, theorem 17.3.4]

$$H(x, y) = \sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \ = \ \frac{(1+y)^{d-1} - (1+x)^{d-1}}{(1+x)^d y - (1+y)^d x}. \tag{4.12}$$

If we let $D_p(n-1, d)$ denote the dimension of the primitive degree-$(n-1)$ cohomology of a degree d hypersurface in $\mathbb{P}^n$, then by setting $y = x$ and extracting the coefficient of x^{n-1} from the generating function above, we obtain

$$D_p(n-1, d) \ = \ \frac{(d-1)^{n+1} + (-)^n}{d} \ + \ (-)^{n-1}. \tag{4.13}$$

5 Physical computations for hypersurfaces

In this section we will describe the mixed Higgs-Coulomb branch computations which give a physical realization of Dolbeault cohomology $H^{k,\ell}$. Our findings can be summarized as follows: for the IR phase corresponding to a hypersurface of degree d in $\mathbb{P}^n$,

- The Coulomb branch will contribute $n+1-d$ states which will correspond to elements of $H^{k,k}$.
- The Higgs branch is a $\mathbb{Z}_d$ Landau-Ginzburg orbifold (LGO), with superpotential W given by the homogeneous polynomial defining the hypersurface. The twisted sectors (indexed by $0 \leq r < d$) contribute states as follows:
  - $r = 0$: one state contributing to $H^{k,k}$ for $2k \equiv 0 \mod 2(n+1-d)$.
  - $r = 1$: For each p such that $p+n+1 \equiv 0 \mod d$, as many states as the dimension of the space of degree p polynomials modulo the ideal (dW), contributing to $H^{k,\ell}$ for

$$\ell - k \;=\; \frac{1}{d}\left(2(p+n+1) - d(n+1)\right), \quad k + \ell \equiv n - 1 \mod 2(n+1-d),$$

  - $1 < r < d$: one state for each r, contributing to $H^{k,k}$ for $2k \equiv 2(n+1-r) \mod 2(n+1-d)$.

The identification of states with elements of Dolbeault cohomology utilizes a nonanomalous vector $U(1)_R$ symmetry and a nonanomalous $\mathbb{Z}_{2(n+1-d)}$ axial R-symmetry. Because one of these is a finite group, the identification of states with elements of Dolbeault cohomology is ambiguous – some elements of Dolbeault cohomology can only be identified with linear combinations of Coulomb and Higgs branch states. We will argue that these states span the quantum cohomology ring.

Applying the methods of [2] to the Coulomb branch, from the one-loop-exact twisted effective superpotential for the GLSM for $\mathbb{P}^n[d_1, \ldots, d_m]$, the σ field obeys

$$\sigma^{n+1-m} \;=\; q\left(\prod_{a=1}^{m} (-d_a)^{d_a}\right)\sigma^{\sum_a (d_a - 1)}. \tag{5.1}$$

Since in the UV (large radius) phase σ can be interpreted as $\mathcal{O}[\eta]$ (the operator associated to the hyperplane class), it may seem that the IR Coulomb branch already appears to match (4.3), and given the proposed OPE (3.2), and we just have to understand (4.4) from a physical perspective. However, as we will see, even for a hypersurface this is far from a complete story because the Coulomb branch only contributes $n+1-d$ states, and when

$d > 1$ these do not span all of $H^{k,k}(X)$. Instead, the Landau-Ginzburg orbifold sector provides the additional states necessary to match both the additional vertical cohomology and the primitive cohomology of X.

The product structure on the quantum cohomology ring is also more involved. To describe it we decompose the IR states (and therefore also the corresponding fields) into three classes:

1. the Coulomb branch vacua;
2. the first twisted sector of the Landau-Ginzburg orbifold;
3. the remaining $d-1$ sectors of the Landau-Ginzburg orbifold.

We will argue in section 5.5 that while the second class corresponds to the primitive cohomology on X, the first and third correspond to the remaining vertical cohomology groups $H^{k,k}(X)$. By studying the spectrum and symmetries of the A-model in the IR phase, we find a field in the IR description that corresponds to the UV field $\mathcal{O}[\eta]$, as well as the IR fields $\Xi[\alpha]$ that map to the $\mathcal{O}[\alpha]$ of the UV description, and these fields have OPE structure compatible with (4.3) and (4.4). Our proposal is summarized in equation (5.81).

5.1 Symmetries of the A-model in the UV description

Specializing to hypersurfaces $\mathbb{P}^n[d]$, we would like to match the A-model local observables or, by the state–operator correspondence in a topological field theory [21,22], the states between the UV and IR descriptions. We will use the vector $U(1)$ symmetry and the discrete $\mathbb{Z}_{2(n+1-d)}$ chiral symmetry described in section 2.3 to facilitate the identification, and in this section we will describe the symmetry action on the local A-model observables.

We start in the UV phase, where the local A-model operators correspond to Dolbeault cohomology classes. Given a class $[\omega] \in H^{k,\ell}(X)$, the operator $\mathcal{O}[\omega]$ carries charge

$$q_V \;=\; q_L + q_R \;=\; \ell - k, \tag{5.2}$$

and $\rho \cdot \mathcal{O}[\omega] = \zeta^{k+\ell}\mathcal{O}[\omega]$. For each of these local operators we also obtain a corresponding state in the A-model, which we denote as $|\omega; \text{uv}\rangle$, so that the A-model Hilbert space has a UV presentation

$$\mathcal{H}_{\text{uv}} \simeq \text{Span}\{\mathcal{O}[\eta]^k|\Omega; \text{uv}\rangle, \quad k = 0, \ldots, n-1, \quad \mathcal{O}[\alpha]|\Omega; \text{uv}\rangle \;, \alpha \in H^{n-1}_{\text{prim}}(M)\} \;, \tag{5.3}$$

where $|\Omega; \text{uv}\rangle$ is the state identified with the identity operator.

5.2 Hyperplane class correlation functions

The operator $\mathcal{O}[\eta]$ associated to the hyperplane class carries exactly the same symmetry charges as the GLSM operator σ, and we expect we should be able to identify the two. In this section we will perform localization computations of correlation functions of operators corresponding to powers of the hyperplane class, in the UV and IR, using the GLSM fields. We will see that they match each other in a non-trivial fashion and reproduce the structure of the "vertical" quantum cohomology generated by $\mathcal{O}[\eta]$, including the quantum cohomology relation (4.3).

UV correlation functions

It is straightforward to apply the methods of [2] to calculate the A-model genus-zero correlation functions of the operator corresponding to the hyperplane class, represented by $\mathcal{O}[\eta]$, in the UV phase. In this large radius phase only Higgs vacua contribute, and we identify the operator $\mathcal{O}[\eta]$ with the GLSM field σ.

The anomalous chiral symmetry of the GLSM yields a selection rule for the non-zero correlation functions. The non-vanishing A-model correlation functions of $\mathcal{O}[\eta]$ are of the form

$$\langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle \;=\; A_m q^m \;, \tag{5.4}$$

where A_m is a q-independent constant. Note that while we restrict to correlation functions that are polynomial in the $\mathcal{O}[\eta]$, or in other words to m in the range $-m \leq (n-1)/(n+1-d)$, the selection rule does in principle allow for correlation functions with $m < 0$. However, each correlation function receives contributions from exactly one instanton number—m, and in fact $m \geq 0$ in the UV phase because for $m < 0$ the instanton moduli space is empty. Denoting the m-th instanton contribution as

$$A_m \;=\; \langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle_m^M \;, \tag{5.5}$$

we now use the restriction result of [2] to relate the correlation function in the A-model for M to that for the A-model for $\mathbb{P}^n$:

$$A_m \;=\; \langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle_m^M \;=\; A_m \;=\; -\langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} (-d\mathcal{O}[\eta])^{1+md} \rangle_m^{\mathbb{P}^n} \;, \tag{5.6}$$

or

$$A_m = \; d(-d)^{dm} \langle \mathcal{O}[\eta]^{n+m(n+1)} \rangle_m^{\mathbb{P}^n} = d(-d)^{dm} \;. \tag{5.7}$$

Putting that together, we obtain the desired correlation functions:

$$\langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle \;=\; \begin{cases} 0 & m < 0 \;, \\ d((-d)^d q)^m & m \geq 0 \;. \end{cases} \tag{5.8}$$

We see that the computation recovers the quantum cohomology relation (4.3) specialized to the hypersurface case:

$$\mathcal{O}[\eta]^n \ = \ (-d)^d q \mathcal{O}[\eta]^{d-1} \ . \tag{5.9}$$

IR correlation functions

By using the GLSM fields and localization, the correlation functions $\langle \mathcal{O}[\eta]^k \rangle$ can also be calculated in the IR by the methods of [2] for the Higgs contribution and those of [10] for the Coulomb contribution. The Coulomb contribution takes the following form:

$$\langle \mathcal{O}[\eta]^k \rangle_{\mathrm{C}} = \sum_{\mathrm{vac}} \frac{1}{H} \sigma^k \ , \tag{5.10}$$

where the sum is over critical points of the effective twisted chiral superpotential for the σ field. In our case these are simply the solutions to

$$\sigma^{n+1-d} = (-d)^d q \ , \tag{5.11}$$

while the measure factor is

$$H \ = \ \left(\frac{n+1}{\sigma} + \frac{d^2}{-d\sigma} \right) \times \sigma^{n+1} \times \frac{1}{d\sigma} \ . \tag{5.12}$$

Let us review how the terms in the measure factor arise.[11]

We perform an A-twist of the GLSM, using a vector R-symmetry that assigns charge 0 to the scalar fields x_i.[12]

1. For a fixed Coulomb vacuum, the first factor comes from the integral over the fluctuations in the zero modes of the gauge sector.

2. The second factor comes from the integral over the zero modes of the A-twisted chiral multiplets X_i: the scalar x_i has a zero mode and a coupling of the form $|\sigma|^2 |x_i|^2$, as do the fermions $\bar{\psi}^i_+$ and ψ^i_-, and the latter have a Yukawa coupling of the form $\bar{\sigma} \psi^i_- \bar{\psi}^i_+$. On the other hand, the fermions ψ^i_+ and $\bar{\psi}^i_-$ do not have zero modes and make no contribution. Performing the Gaussian integrals over these modes produces the second factor.

[11] As in [2, 9, 10] the calculation comes with an overall sign ambiguity that depends on the R-charges of the chiral multiplets. We fix the sign according to the same prescription as followed in those works.

[12] In principle it should be possible to perform the localization computation if we choose a gauge-equivalent twist by shifting the charge of x_i to any integer parameter α, while that of p would become $2 - d\alpha$. The result should be independent of α, but we are not aware of a demonstration of this in the localization literature. We thank the referee for pointing out this out: it would certainly be an interesting point to clarify.

3. The last factor comes from the zero modes of A-twisted chiral multiplet P. Because P carries vector R-charge $+2$ the only fields in the multiplet with zero modes are ψ_+^a and $\bar{\psi}_-^a$, which have the Yukawa coupling $d\sigma\psi_+^a\bar{\psi}_-^a$, and integrating over these leads to the last term.

Performing the sum over the Coulomb vacua we then obtain the Coulomb contribution to the correlation function:

$$\langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle_{\mathrm{C}} = d((-d)^d q)^m \ . \tag{5.13}$$

While superficially agreeing with the UV computation when $m \geq 0$, the result is puzzling, since for $0 < -m \leq (n-1)/(n+1-d)$ there are non-zero contributions to correlation functions with $m < 0$ yet non-negative exponent, i.e. $\langle \mathcal{O}[\eta]^{d-2} \rangle_{\mathrm{C}}$, $\langle \mathcal{O}[\eta]^{2d-3-n} \rangle_{\mathrm{C}}$, ...

Now we move onto the Higgs calculation, following [2]. Here

$$\langle \mathcal{O}[\eta]^p \rangle_{\mathrm{H}} \ = \ \frac{1}{d} q^m \# \left((-\delta_0^2)(\eta^*)^p \chi_m \right)_{\mathcal{M}_m} \ , \tag{5.14}$$

where $\mathcal{M}_m$ is the instanton moduli space, η^* is the pull-back of η to $\mathcal{M}_m$, χ_m is the Euler class of the obstruction bundle, and $\delta_0 = (-d)\eta^*$. In the IR phase we find that $\mathcal{M}_m$ is empty for $m \geq 0$, while for $m < 0$ $\mathcal{M}_m = \mathbb{P}^{-md}$. The Euler class is given entirely by contributions from the X_i and has the form

$$\chi_m \ = \ \left((\eta^*)^{-m-1} \right)^{n+1} \ . \tag{5.15}$$

Note the overall factor of $1/d$: this comes from the unbroken $\mathbb{Z}_d$ gauge group. Finally, the intersection on $\mathcal{M}_m$ is determined by

$$\#(((-d)\eta^*)^{-md}) \ = \ 1 \ , \tag{5.16}$$

and putting all of that together we obtain that the Higgs branch makes non-zero contributions

$$\langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle_{\mathrm{H}} \ = \ -d((-d)^d q)^m \tag{5.17}$$

for $m < 0$. Summing up the contributions we then obtain

$$\begin{aligned} \langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle \ &= \ \langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle_{\mathrm{H}} + \langle \mathcal{O}[\eta]^{n-1+m(n+1-d)} \rangle_{\mathrm{C}} \\ &= \ \begin{cases} 0 & m < 0 \ , \\ d((-d)^d q)^m \ , & m \geq 0 \ . \end{cases} \end{aligned} \tag{5.18}$$

This matches the UV result (5.8), and it does so in a non-trivial fashion: the Higgs sector makes contributions at instanton numbers where the UV computation has no contributions, while the Coulomb sector makes contributions at all instanton numbers. Their sum, however, matches the UV Higgs computation, and this match provides a good check of our understanding of the physics, and in particular the decoupling of the Higgs and Coulomb degrees of freedom in the IR.

The match of the UV and IR correlation functions, and their consistency with the quantum cohomology relation (4.3) supports the proposed decoupling between the Coulomb and Higgs sectors of the IR theory, but it only tests the quantum cohomology relations in the "vertical" column of the Hodge diamond. In principle it should be possible to extend these GLSM computations to include correlation functions with insertions of the $\mathcal{O}[\alpha]$ operators. Unfortunately, while the $\mathcal{O}[\alpha]$ operators have a straightforward realization in the A-twisted non-linear sigma model, it is not obvious how to write these directly in terms of the GLSM fields. This is a manifestation of a general problem in the GLSM description of geometries: a recent discussion may be found in [23]. So, while we can certainly count their multiplicities and organize them by their q_V and ρ charges, we cannot directly reproduce the remaining quantum cohomology relations by working in the UV phase beyond observing that the relations are consistent with the selection rules based on the symmetries. We will see that the IR description gives a complementary perspective on these operators as arising from the twisted sector of the LGO sector.

We now turn to a more detailed study of the IR phase with the goal of describing an isomorphism between $\mathcal{H}_{\text{uv}}$ and the A-model Hilbert space in the IR phase description.

5.3 The Coulomb vacua in the IR phase

As discussed above, we expect that the Coulomb and Higgs sectors decouple, and we can describe the operators and states in the two sectors separately. In this section we will discuss the Coulomb sector, leaving the more involved Higgs phase to the next section.

The Coulomb vacua are labeled by expectation values of the field σ determined by (5.11). The expectation value $\langle\sigma\rangle$ breaks the ρ symmetry to $\mathbb{Z}_2$—the fermion number symmetry of the theory.[13] These $n+1-d$ massive vacua correspond to $n+1-d$ states in the A-model, which we denote by $|t;\text{C}\rangle$, with $t = 0, 1, \ldots, n-d$, and the σ field acts on these states by $\sigma|t;\text{C}\rangle = |t+1;\text{C}\rangle$ for $t = 0, 1, \ldots, n-d-1$. By taking suitable linear combinations we choose the state $|0;\text{C}\rangle$ to be invariant under the chiral ρ–symmetry, and the charges of the remaining states are determined by the action of σ.

These states have the quantum numbers that appear to match those of the UV phase states $|\eta^k;\text{uv}\rangle$ for $k = 0, \ldots, n-d$, but we see that to match the remaining states in $\mathcal{H}_{\text{uv}}$ additional states are needed in the IR description—states that have the quantum numbers

[13] When working in a quantum field theory in infinite spatial volume we would have super-selection sectors labeled by the σ expectation values. In the finite-dimensional Hilbert space of the A-model we do not have such super-selection sectors, and instead the field σ relates one vacuum to another.

of $|\eta^k; \mathrm{uv}\rangle$ with

$$n + 1 - d \leq k \leq n \ , \tag{5.19}$$

as well as states corresponding to the $|\alpha; \mathrm{uv}\rangle$. These additional states are provided by the Higgs vacua in the IR phase, and their physics is described by a Landau-Ginzburg orbifold.

5.4 The LGO sector

In the Higgs sector of the IR phase the chiral superfield field P acquires an expectation value which breaks the gauge symmetry to a subgroup $G = \mathbb{Z}_d$. In the low energy limit the massive P and vector multiplet degrees decouple, leaving the low energy physics of a (2,2) supersymmetric Landau-Ginzburg orbifold (LGO), where the superpotential W is a degree d function of $n+1$ chiral superfields $X_0, \cdots, X_n$, and the orbifold group is the unbroken gauge symmetry G.

Starting with the seminal work [24, 25], there is by now ample evidence that the theory has non-trivial IR dynamics of a (2,2) superconformal theory, with left and right central charges given by

$$c \ = \ c_L \ = \ c_R \ = \ 3 \sum_i \left(1 - \frac{2}{d}\right) \ = \ 3(n+1) \left(\frac{d-2}{d}\right) , \tag{5.20}$$

For generic superpotential this is a non-trivial superconformal field theory (although it is a solvable Gepner-like theory when W is taken to be Fermat). We do not expect to be able to relate the dynamics of the full theory to the quantum geometry of the Fano hypersurface in the UV phase. However, as we will see, the A-model topological sector of the LGO will provide exactly the states we need to match those of the UV A-model.

We will compute the (a,c) ring of the (2,2) SCFT appearing as the IR limit of the $\mathbb{Z}_d$ orbifold of the Landau-Ginzburg theory above, as those states are the ones that contribute to the A model. Our analysis will follow [18], to which we refer the reader for additional details.

We should note one important difference between this LGO and those that are typically discussed in the context of the Calabi-Yau/LGO correspondence, regarding the non-local spectral flow fields $\mathcal{U}_{\alpha_L,\alpha_R}$ that shift the charges of operators according to

$$(q_L, q_R) \ \mapsto \ (q_L + \alpha_L \, c/3, q_R + \alpha_R \, c/3). \tag{5.21}$$

In the more familiar setting of the Calabi-Yau/LGO correspondence the SCFT has integral q_L and q_R charges after the orbifold projection, and thus chiral spectral flow operators $\mathcal{U}_{1/2,0}$, $\mathcal{U}_{0,1/2}$ that can be used to relate chiral R and NS sectors, and construct the R-NS and NS-R sectors and a consistent string vacuum, as discussed in [17]. In our context the projection will not lead to integral q_L and q_R charges essentially because $c/3 \notin \mathbb{Z}$ in general, and hence we do not have chiral spectral flow operators. Nevertheless, the twisting procedure for the $\mathbb{Z}_d$ orbifold can still be formally implemented using the spectral flow operator $\mathcal{U}_{-1/2,1/2}$ and its inverse. For the particular orbifold we consider this leads to an important isomorphism between the unprojected RR states and the unprojected (a,c) ring states: if we let $\mathcal{H}_{ac}^{\text{unproj}}(r)$ denote the (a,c) ring in the r twisted sector before projecting to G-invariant states, and $\mathcal{H}_{RR}^{\text{unproj}}(r)$ denote the RR sector states in the r twisted sector before projecting to G-invariant states, then

$$\mathcal{U}_{1/2,-1/2} : \mathcal{H}_{ac}^{\text{unproj}}(r) \xrightarrow{\sim} \mathcal{H}_{RR}^{\text{unproj}}(r-1). \tag{5.22}$$

As discussed in [18], the isomorphism of the unprojected (a,c) states in a sector twisted by an element h and that of the RR states in a sector twisted by an element hj^{-1} holds in general LGOs. The special simplifying feature in our case, where the orbifold is by the $\mathbb{Z}_d$ symmetry generated by j is that $\mathcal{U}_{1/2,-1/2}$ is an operator in the projected theory and yields an isomorphism for the projected states as well.

With this in mind, the algorithm we will follow to compute the twisted-sector (a,c) states is to first classify the unprojected RR twisted sector states, then apply spectral flow $\mathcal{U}_{-1/2,1/2}$ to get the unprojected (a,c) ring states, and finally take G-invariants to get the desired A model contributions.

In our examples $G = \mathbb{Z}_d$ with generator j, which acts on the chiral superfields X_i by

$$j \cdot X_i = \exp\left(2\pi i \theta_i\right) X_i \ , \qquad \theta_i = \frac{1}{d} \ . \tag{5.23}$$

Let $|r;\text{RR}\rangle$ denote the RR sector vacuum in the r twisted sector with $0 \leq r < d$. The (unprojected) states in the r twisted RR sector are of the form

$$f_p(x)|r;\text{RR}\rangle \tag{5.24}$$

for $f_p(x)$ a homogeneous polynomial of degree p in the x_i—the zero modes of the scalar fields in the multiplets X_i. If $r > 0$, then because of the form of the G action, all of the x_i have nonzero moding, and so the only possible ground states have $p = 0$. On the other hand, for $r = 0$ every x_i has a zero mode, and the ideal

$$(dW) = \left(\frac{\partial W}{\partial x_0}, \frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n} \right) \in \mathbb{C}[x_0, x_1, \ldots, x_n] \tag{5.25}$$

annihilates all of the states, so that we can take $f_p(x) \in \mathcal{R}$, where $\mathcal{R}$ is the quotient ring $\mathcal{R} = \mathbb{C}[x_0, x_1, \ldots, x_n]/\left(dW\right)$. When we write $f_p(x)$ it will be with this understanding.

Unprojected (a,c) states in the r twisted sector are of the form

$$|f_p;r;\text{ac}\rangle = \mathcal{U}_{-1/2,1/2}\, f_p(x)|r-1;\text{RR}\rangle. \tag{5.26}$$

To obtain the projected states we need the action of j in the twisted sectors. This is worked out in [18], and for our group action the result simplifies to

$$j \cdot |f_p; r; \mathrm{ac}\rangle \;=\; \exp\left[2\pi i \left(\theta_T^j(r-1) + \frac{p}{d}\right)\right]\, |f_p; r; \mathrm{ac}\rangle \;, \tag{5.27}$$

where $\theta_T^j(r-1)$ denotes the contribution from the action of j on the Landau-Ginzburg fields X_i which are invariant under j^{r-1}.

We can summarize results as follows:

- $r=0$: There is one untwisted (a,c) state that survives the orbifold.
- $r=1$: Here,
$$\theta_T^j(r-1) \;=\; \sum_i \theta_i \;=\; \frac{n+1}{d} \tag{5.28}$$
(each of $n+1$ Landau-Ginzburg fields x_i contributes $\theta_i = 1/d$). Therefore, we project onto states for which
$$p+n+1 \equiv 0 \mod d \;. \tag{5.29}$$
- $1<r<d$: Here, only states with $p=0$ contribute, and since no Landau-Ginzburg fields are invariant under j^{r-1}, $\theta_T^j(r-1)=0$, and each state $|r;\mathrm{ac}\rangle$ is G–invariant.

The Landau-Ginzburg orbifold has a $U(1)_L \times U(1)_R$ symmetry, and we classify states according to their charges q_L, q_R. These are determined as follows. First, the R-charges of the RR vacuum $|r,\mathrm{RR}\rangle$ are given by [26]

$$q_L \;=\; +\sum_{i\notin\mathcal{T}_r}\left(\theta_i^r - [\theta_i^r] - \frac{1}{2}\right) \;+\; \sum_{i\in\mathcal{T}_r}\left(\theta_i - \frac{1}{2}\right), \tag{5.30}$$

$$q_R \;=\; -\sum_{i\notin\mathcal{T}_r}\left(\theta_i^r - [\theta_i^r] - \frac{1}{2}\right) \;+\; \sum_{i\in\mathcal{T}_r}\left(\theta_i - \frac{1}{2}\right), \tag{5.31}$$

where here $\theta_i^r = r/d$ for all i, $[x]$ denotes the smallest integer part, and

$$\mathcal{T}_r \;=\; \{i \in \{1,\cdots,n+1\} \,|\, \theta_i^r \in \mathbb{Z}\}. \tag{5.32}$$

We can simplify this expression for charges of RR vacua by distinguishing two cases:

- $r=0$: Here, the RR vacuum $|0,\mathrm{RR}\rangle$ has charges
$$q_L \;=\; +q_R \;=\; (n+1)\left(\frac{1}{d} - \frac{1}{2}\right) \;=\; -\frac{c}{6}. \tag{5.33}$$

- $0 < r < d$: Here,

$$q_L \;=\; -q_R \;=\; (n+1)\left(\frac{r}{d}-\frac{1}{2}\right). \tag{5.34}$$

The RR state $f_p(x)|r,\mathrm{RR}\rangle$ has $q_{L,R}$ differing from those of $|r,\mathrm{RR}\rangle$ by the addition of p/d, the R-charge of $f_p(x)$. The spectral flow operator $\mathcal{U}_{-1/2,+1/2}$ has

$$(q_L, q_R) \;=\; \left(-\frac{c}{6}, +\frac{c}{6}\right) \tag{5.35}$$

and from this we deduce that the r-twisted (a,c) state $|f_p; r; \mathrm{ac}\rangle$ has R-charges given as follows:

- $r = 0$: Here, the corresponding RR state is in sector $d-1$, and we find

$$q_L \;=\; +q_R \;=\; \frac{p}{d}. \tag{5.36}$$

- $r = 1$:

$$q_L \;=\; \frac{p}{d} - \frac{c}{3} \;=\; \frac{p}{d} - (n+1)\left(\frac{d-2}{d}\right), \quad q_R \;=\; \frac{p}{d}. \tag{5.37}$$

- $1 < r < d$:

$$q_L \;=\; \frac{p}{d} + (n+1)\left(\frac{r}{d}-1\right), \quad q_R \;=\; \frac{p}{d} - (n+1)\left(\frac{r}{d}-1\right). \tag{5.38}$$

Applying the algorithm we then obtain the Hilbert spaces of the projected (a,c) states $\mathcal{H}_{\mathrm{ac}}(r)$ in the r-th twisted sector:

$$\begin{aligned} &\mathcal{H}_{\mathrm{ac}}(0) = \mathrm{Span}\{|0;\mathrm{ac}\rangle\}\ , \\ &\mathcal{H}_{\mathrm{ac}}(1) = \mathrm{Span}\{|f_p;1;\mathrm{ac}\rangle \;\;|\;\; f_p \in \mathcal{R}\ , p+n+1 \equiv 0 \mod d\}\ , \end{aligned} \tag{5.39}$$

and for $1 < r < d$

$$\mathcal{H}_{\mathrm{ac}}(r) = \mathrm{Span}\{|r;\mathrm{ac}\rangle\}\ . \tag{5.40}$$

Matching the UV and IR symmetries

We now have both UV and IR descriptions of the A-model states space. We will use symmetries preserved along the RG flow of the physical theory to match the two descriptions. Recall that we identified a continuous non-chiral vector symmetry, as well as a discrete chiral R-symmetry in section 2.3.

The vector symmetry with charges q_V is present throughout the RG flow, and we know precisely how it acts on the GLSM fields in either phase. We therefore expect that the $q_V = q_R + q_L$ in the LGO theory as well. With this identification made we see that the only states in the LGO that can match the middle cohomology states $|\alpha; \text{uv}\rangle$ are those from the $r = 1$ twisted sector.

In the UV phase we also defined the discrete symmetry with generator ρ. In the IR its fate is complicated: it is spontaneously broken to a $\mathbb{Z}_2$ in the Coulomb sector, and while it remains unbroken in the LGO vacuum, it is not a priori clear how to identify it with symmetries in that vacuum. The relevant symmetries in the LGO vacuum are the chiral R-symmetry with charges $q_R - q_L$ and the quantum symmetry of the orbifold, which assigns a charge $\exp\{2\pi i r/d\}$ to the r-th twisted sector. To make the identification, we note that the ρ-charge of $\mathcal{O}[\alpha]$ for $\alpha \in H^{n-1}(X)$ is ζ^{n-1}, and as we just observed, we expect these states to map to the first twisted sector of the LGO. Moreover, for all LGO states $q_R - q_L + rc/3 \in \mathbb{Z}$. Putting these two observations together, we arrive at the following proposal for the action of ρ on the LGO states:

$$
\begin{aligned}
\rho|f_p; r; \text{ac}\rangle &= \exp\left[\frac{\pi i}{n+1-d}\left(q_R - q_L + \frac{2(n+1-d)r}{d}\right)\right]|f_p; r; \text{ac}\rangle \\
&= \zeta^{q_R - q_L} e^{2\pi i r/d}|f_p; r; \text{ac}\rangle \ .
\end{aligned} \tag{5.41}
$$

It is easy to check that this generates a $\mathbb{Z}_{2(n+1-d)}$ action on all of the LGO states, and we will see that it leads to a nice matching between the UV and IR descriptions of the A-model's Hilbert space.

Multiplicities in the $r = 1$ twisted sector

As we just saw, the description of the LGO states is simple, and the only non-universal feature concerns the multiplicities of the states in the $r = 1$ twisted sector. While calculating these multiplicities in particular examples is easily accomplished, the explicit enumeration is awkward. Fortunately, there is an elegant generating function that encodes all of the multiplicities.

Let us start by defining a generating function that counts the projected (a,c) states in the first twisted sector, i.e.

$$
F_{n+1}(t, \bar{t}) = \text{Tr}_{\mathcal{H}_{\text{ac}}(1)} t^{dq_L} \bar{t}^{dq_R} = \sum_{q_L, q_R} \mu_{dq_L, dq_R} t^{dq_L} \bar{t}^{dq_R} \ . \tag{5.42}
$$

Equivalently we can write this as a sum over the unprojected states with an insertion of a projector onto states where[14]

$$dq_L - (n+1) = 0 \mod d\ . \tag{5.43}$$

This leads to

$$F_{n+1}(t,\bar{t}) = \frac{1}{d}\sum_{k=0}^{d-1} \mathrm{Tr}_{\mathcal{H}_{\rm ac}^{\rm unproj}(1)} \left[t^{dq_L}\bar{t}^{dq_R}\xi^{k(dq_L-(n+1))}\right]\ , \tag{5.44}$$

where the trace is now over the unprojected states, and $\xi = e^{2\pi i/d}$. We know the explicit generating function for the unprojected states [18,27]:

$$M(t,\bar{t}) = \mathrm{Tr}_{\mathcal{H}_{\rm ac}^{\rm unproj}(1)} t^{dq_L}\bar{t}^{dq_R} = t^{-dc/3}\mathrm{Tr}_{\mathcal{H}_{\rm cc}^{\rm unproj}(1)} t^{dq_L}\bar{t}^{dq_R} = t^{-dc/3}\left(\frac{(1-(t\bar{t})^{d-1})}{1-t\bar{t}}\right)^{n+1}\ , \tag{5.45}$$

so we have

$$F_{n+1}(t,\bar{t}) = \frac{1}{d}\sum_{k=0}^{d-1} \xi^{-k(n+1)} M(t\xi^k,\bar{t})\ . \tag{5.46}$$

Simplifying further,

$$F_{n+1}(t,\bar{t}) = \frac{1}{d}\sum_{k=0}^{d-1}\left(\frac{\xi^k t^{2-d} - t\bar{t}^{d-1}}{1-\xi^k t\bar{t}}\right)^{n+1}\ . \tag{5.47}$$

For any fixed n this gives a nice way of listing the projected states, but it is awkward to implement the projection. Much the same awkwardness arises when trying to calculate the middle cohomology Hodge numbers $h^{k,l}(X)$ reviewed above, and this inspires us to instead consider a generating function obtained by summing over all possible n:

$$G(s,t) = \sum_{n=0}^{\infty} s^n F_{n+2}(t,t)\ . \tag{5.48}$$

Notice that we also set $t=\bar{t}$, so that the coefficient of $(t^d)^a$ in the expansion counts the multiplicity of states with a fixed value of the q_V charge $q_V = q_L + q_R = a$.

After performing the geometric sum on n we can simplify $G(s,t)$ and carry out the sum that implements the projection through repeated use of the identity

$$\sum_{l=0}^{d-1} a^{d-1-l}z^l = \frac{a^d - z^d}{a-z}\ , \tag{5.49}$$

[14]This form of the projection is equivalent to the one given above for $f_p|1;\mathrm{ac}\rangle$ as $p+n+1=0 \mod d$, but it is the more convenient form for the generating function manipulations that will follow.

which allows us to shift all of the ξ–dependence into the numerator of the rational function. The result is

$$G(s,t) = \frac{(1+st^{-d})^{d-1} - (1+st^{d})^{d-1}}{(1+st^{d})^{d} st^{-d} - (1+st^{-d})^{d} st^{d}} \ . \tag{5.50}$$

We now compare this to the generating function $H(x,y)$ for the primitive Hodge numbers from (4.12) above and observe that

$$H(s/t^d, st^d) = G(s,t) \ . \tag{5.51}$$

But now since

$$H(s/t^d, st^d) = \sum_{k,l} (h^{k,l} - \delta^{k,l}) s^{k+l} t^{d(l-k)} \ , \tag{5.52}$$

we see that the multiplicities of the $r = 1$ twisted sector states in the LGO precisely match the primitive Hodge numbers once we make the identification $n = k+l+1$ and $q_L + q_R = l - k$.

5.5 Matching the UV and IR descriptions

Having analyzed the A-model's Hilbert space of states from both the UV and IR points of view, we will now use the identification of symmetries to match, as far as possible, the two presentations. To do so, we first review our findings, organizing the states by their charges.

In the UV description we have the presentation of the A-model Hilbert space as

$$\mathcal{H}_{\text{uv}} \simeq \mathcal{H}_{\text{uv}}^{\text{vert}} \oplus \textstyle\bigoplus_{k+l=n-1} \mathcal{H}_{\text{uv}}^{k,l} \ , \tag{5.53}$$

with

$$\begin{aligned} \mathcal{H}_{\text{uv}}^{\text{vert}} &= \text{Span}\{\mathcal{O}[\eta]^k |\Omega; \text{uv}\rangle, \quad k = 0, \ldots, n-1\} \ , \\ \mathcal{H}_{\text{uv}}^{k,l} &= \text{Span}\{\mathcal{O}[\alpha_{k,l}] |\Omega; \text{uv}\rangle \ , \alpha \in H_{\text{prim}}^{k,l}(X)\} \ , \end{aligned} \tag{5.54}$$

and we characterize the states according to their symmetry charges as

$$\begin{array}{ccc} \text{state} & \mathcal{O}[\eta]^k|\Omega;\text{uv}\rangle & \mathcal{O}[\alpha_{k,l}]|\Omega;\text{uv}\rangle^{\oplus \mu_{k,l}} \\ q_V & 0 & l-k \\ \rho & \zeta^{2k} & \zeta^{l+k} \end{array} \ . \tag{5.55}$$

The observables have a more complicated description in the IR variables. The Hilbert space is a direct sum of the Coulomb and LGO factors:

$$\mathcal{H}_{\text{ir}} = \mathcal{H}_{\text{C}} \oplus \mathcal{H}_{\text{LGO}} \ , \tag{5.56}$$

and the latter has a further decomposition into the orbifold sectors:

$$\mathcal{H}_{\text{LGO}} = \mathcal{H}_{\text{ac}}(1) \oplus \bigoplus_{r=2}^{d} \mathcal{H}_{\text{ac}}(r) \ . \tag{5.57}$$

Note that we included the untwisted sector as $r = d$. While the $r > 1$ spaces are all one-dimensional and carry $q_V = 0$, $\mathcal{H}_{\text{ac}}(1)$ can be graded further by the q_V charges:

$$\mathcal{H}_{\text{ac}}(1) = \bigoplus_{q_V} \mathcal{H}_{\text{ac}}^{q_V}(1) \ . \tag{5.58}$$

We determined the action of ρ in the LGO sector by requiring that the states in $\mathcal{H}_{\text{ac}}(1)$ carry ρ–charge ζ^{n-1}, and we showed above that

$$\dim \mathcal{H}_{\text{ac}}^{l-k}(1) = \dim H^{k,l}_{\text{prim}}(X) \ , \tag{5.59}$$

and we therefore propose that there is an isomorphism $\mathcal{H}_{\text{uv}}^{k,l} \simeq \mathcal{H}_{\text{ac}}^{l-k}(1)$.[15]

The remaining states in the IR description consist of the $n+1-d$ Coulomb states $|t;\text{C}\rangle$ and the $d-1$ LGO states from the untwisted sector $|0;\text{ac}\rangle$ and the higher twisted sectors $|r;\text{ac}\rangle$ with $1 < r < d$. All of these states have $q_V = 0$, while their ρ–charges are as follows:

$$\begin{array}{cccccc} \text{state} & |0;\text{C}\rangle & \cdots & |t;\text{C}\rangle & \cdots & |n-d;\text{C}\rangle \\ \rho & \zeta^0 & \cdots & \zeta^{2t} & \cdots & \zeta^{2(n-d)} \end{array} \ , \tag{5.60}$$

and

$$\begin{array}{cccccc} \text{state} & |j^d;\text{ac}\rangle & \cdots & |j^{d-r};\text{ac}\rangle & \cdots & |j^2;\text{ac}\rangle \\ \rho & \zeta^0 & \cdots & \zeta^{2r} & \cdots & \zeta^{2(d-2)} \end{array} \ . \tag{5.61}$$

We would like to match these n states to those in $\mathcal{H}_{\text{uv}}^{\text{vert}}$. The dimensions match, but symmetry alone does not determine the correspondence between states and cohomology. Instead, we have at least a two-fold ambiguity in sectors with ρ charge ζ^{2r} for $r = 0, \ldots, d-2$, in both the IR description, as well as in the UV description, since both $\mathcal{O}[\eta^r]$ and $\mathcal{O}[\eta^{n+1-d+r}]$ have ρ–charge ζ^{2r}. There will be yet further ambiguity in the spectrum if[16] $n < 2d-2$.

To proceed with the identification we observe $\mathcal{H}_{\text{uv}}^{\text{vert}}$ is generated by powers of a single operator $\mathcal{O}[\eta]$ acting on a vacuum state $|\Omega;\text{uv}\rangle$ that we associate to the identity operator. Similarly, the Coulomb Hilbert space is generated by the action of σ:

$$\mathcal{H}_{\text{C}} \simeq \text{Span}\{\sigma^k|0;\text{C}\rangle, \quad k = 0, \ldots, n-d\} \ . \tag{5.62}$$

[15] Note that we defined $\mathcal{H}_{\text{uv}}^{k,l}$ as the primitive Dolbeault cohomology of X. It is possible to grade these spaces further by the large permutation symmetry enjoyed by the A-model and thereby refine the isomorphism.

[16] We observe that this is the same constraint (4.5) that appears in [3], albeit the origin here is completely different.

In the LGO sector we have the state $|0;\mathrm{ac}\rangle$, and by the state-operator correspondence we can find an operator Ψ of ρ–charge ζ^2 and quantum symmetry charge $e^{-2\pi i/d}$ such that

$$|d-1;\mathrm{ac}\rangle = \Psi|0;\mathrm{ac}\rangle\ . \tag{5.63}$$

Taking further powers we obtain states $\Psi^k|0;\mathrm{ac}\rangle$ with the same quantum numbers as $|d-k;\mathrm{ac}\rangle$ for $0\le k\le d-2$, while for $\Psi^{d-1}|0;\mathrm{ac}\rangle$ we obtain a state with the quantum symmetry charge of the first twisted sector but ρ–charge $\zeta^{2(d-1)}$. This is is inconsistent with the ρ–charge in the first twisted sector, which is given by ζ^{n-1} unless

$$2(d-1) = n-1 \mod (n+1-d)\ . \tag{5.64}$$

But this is impossible, since it is equivalent to $d = m(n+1-d)$ for some positive integer m, and that is inconsistent with our basic assumption $d < n+1$. Thus it must be $\Psi^{d-1} = 0$. We will assume that $\Psi^k \neq 0$ for $k < d-1$, so that Ψ generates all of the higher twisted states:

$$\oplus_{r=2}^{d}\mathcal{H}_{\mathrm{ac}}(r) = \mathrm{Span}\{\Psi^k|0;\mathrm{ac}\rangle \ \mid\ 0\le k\le d-2\}\ . \tag{5.65}$$

Having made these identifications we now proceed to describe our proposal for the isomorphism

$$\mathcal{H}_{\mathrm{uv}}^{\mathrm{vert}} \simeq \mathcal{H}_{\mathrm{C}} \oplus \oplus_{r=2}^{d}\mathcal{H}_{\mathrm{ac}}(r)\ . \tag{5.66}$$

We suppose that the ground state can be written as

$$|\Omega;\mathrm{uv}\rangle = |0;\mathrm{C}\rangle + |0;\mathrm{ac}\rangle\ , \tag{5.67}$$

and identify

$$\mathcal{O}[\eta] = \sigma + x\Psi\ , \tag{5.68}$$

where x is a parameter that we will constrain further momentarily. Our fields have the following properties:

$$\sigma^{n+1-d} = (-d)^d q\ ,\qquad \Psi^{d-1}=0\ ,\qquad \sigma\Psi = 0\ ,\qquad \sigma|0;\mathrm{ac}\rangle = 0\ ,\qquad \Psi|0;\mathrm{C}\rangle = 0\ , \tag{5.69}$$

where the last three relations follow from the decoupling between the Higgs and Coulomb sectors.[17]

We now calculate, for $d>1$,

$$\mathcal{O}[\eta]^n|\Omega;\mathrm{uv}\rangle = \sigma^n|0;\mathrm{C}\rangle = \sigma^{n+1-d}\sigma^{d-1}|\Omega;\mathrm{uv}\rangle = (-d)^d q\mathcal{O}[\eta]^{d-1}|\Omega;\mathrm{uv}\rangle\ , \tag{5.70}$$

[17]These relations show that there is no significance to the relative phase or normalization between the two states in $|\Omega;\mathrm{uv}\rangle$, at least as far as the topological sector is concerned, because there is no operator that can take $|0;\mathrm{C}\rangle$ to $|0;\mathrm{ac}\rangle$, i.e. the two states belong to two different superselection sectors.

and this is one of the UV quantum cohomology relations (4.10).

Next, we return to the states in $\mathcal{H}_{\rm ac}(1)$, which we already identified with $H^{n-1}_{\rm prim}(X)$. With this identification the state-operator correspondence implies that for each $\alpha \in H^{n-1}_{\rm prim}(X)$ there is a LGO field $\Xi[\alpha]$ of ρ-charge ζ^{n-1} such that

$$\mathcal{O}[\alpha] = \Xi[\alpha]|0;{\rm ac}\rangle \ . \tag{5.71}$$

Applying the selection rules as before we conclude that

$$\Xi[\alpha]\Psi = 0 \ . \tag{5.72}$$

Furthermore, the Coulomb/Higgs decoupling implies

$$\Xi[\alpha]\sigma = 0 \qquad \text{and} \qquad \Xi[\alpha]|0;{\rm C}\rangle = 0 \ . \tag{5.73}$$

This is another one of the UV quantum cohomology relations (4.10).

On the other hand, $\Xi[\alpha]\Xi[\beta]$ carries quantum symmetry charge $e^{4\pi i/d}$ and ρ–charge $\zeta^{2(n-1)} = \zeta^{2(d-2)}$, so for $d > 2$ it must be that

$$\Xi[\alpha]\Xi[\beta]|\Omega;{\rm uv}\rangle = C(\alpha,\beta)\Psi^{d-2}|\Omega;{\rm uv}\rangle \ . \tag{5.74}$$

But now we observe, with $d > 2$,

$$\begin{aligned}\mathcal{O}[\eta]^{n-1}|\Omega;{\rm uv}\rangle &= (\sigma + x\Psi)^{n-1}|\Omega;{\rm uv}\rangle = \sigma^{n-1}|\Omega;{\rm uv}\rangle = (-d)^d q\sigma^{d-2}|\Omega;{\rm uv}\rangle \ ,\\ \mathcal{O}[\eta]^{d-2}|\Omega;{\rm uv}\rangle &= \left(\sigma^{d-2} + x^{d-2}\Psi^{d-2}\right)|\Omega;{\rm uv}\rangle \ ,\end{aligned} \tag{5.75}$$

and putting these two statements together,

$$\left(\mathcal{O}[\eta]^{n-1} - (-d)^d q\mathcal{O}[\eta]^{d-2}\right)|\Omega;{\rm uv}\rangle = -(-d)^d q x^{d-2}\Psi^{d-2}|\Omega;{\rm uv}\rangle \ . \tag{5.76}$$

Eliminating Ψ^{d-2}, we therefore obtain

$$\Xi[\alpha]\Xi[\beta]|\Omega;{\rm uv}\rangle = \frac{C(\alpha,\beta)}{-(-d)^d q x^{d-2}}\left(\mathcal{O}[\eta]^{n-1} - (-d)^d q\mathcal{O}[\eta]^{d-2}\right)|\Omega;{\rm uv}\rangle \ . \tag{5.77}$$

This equation nearly corresponds to one of the desired quantum cohomology relations (4.11).

To complete the match, we need to determine $C(\alpha,\beta)$, as well as the constant x introduced in (5.68). If our matching of the $\Xi[\alpha]$ and $\mathcal{O}[\alpha]$ is correct, then we know that in the $q \to 0$ limit $C(\alpha,\beta)$ should be proportional to the $(\alpha|\beta)$ pairing on the primitive cohomology classes. On the other hand, $C(\alpha,\beta)$ is determined entirely in the LGO sector, and therefore should not depend on q. We can fix the proportionality constant to match the conventions of (4.11) by requiring

$$\frac{C(\alpha,\beta)}{-(-d)^d q x^{d-2}} = \frac{(\alpha|\beta)}{d}. \tag{5.78}$$

For example, if $C(\alpha, \beta) = (\alpha|\beta)$, then then this will reproduce the quantum cohomology relation (4.11) obtained in [3] provided that

$$-(-d)^d q x^{d-2} = d, \tag{5.79}$$

or equivalently

$$x = -\frac{1}{d}(-dq)^{\frac{1}{d-2}} . \tag{5.80}$$

It would be illuminating to obtain the pairing $C(\alpha, \beta)$ from a direct LGO computation.

To summarize, we proposed a detailed isomorphism $\mathcal{H}_{\rm uv} \simeq \mathcal{H}_{\rm ir}$ largely determined by symmetries, and we explained how the quantum cohomology structure on $\mathcal{H}_{\rm uv}$ can be understood in the IR phase as arising from selection rules, the decoupling of the Coulomb and Higgs sectors, and the structure of the Coulomb vacua. Our proposal passes a number of consistency checks and relies on the identifications

$$\begin{aligned} |\Omega; \mathrm{uv}\rangle &= |0; \mathrm{C}\rangle + |0; \mathrm{ac}\rangle \ , \\ \mathcal{O}[\eta] &= \sigma - \frac{1}{d}(-dq)^{\frac{1}{d-2}}\Psi \ , \\ \mathcal{O}[\alpha] &= \Xi[\alpha] \ . \end{aligned} \tag{5.81}$$

We have recovered the predicted OPE relations in the mixed Higgs/Coulomb states (4.10), (4.11), namely

$$(\mathcal{O}[\eta])^{n+1-m} = q\left(\prod_{a=1}^{m}(-d_a)^{d_a}\right)(\mathcal{O}[\eta])^{\sum_a(d_a-1)} \ , \quad \mathcal{O}[\eta]\cdot\mathcal{O}[\alpha] = 0 \ , \tag{5.82}$$

$$\mathcal{O}[\alpha]\cdot\mathcal{O}[\beta] = (\alpha|\beta)\frac{1}{d}\left((\mathcal{O}[\eta])^{n-m} - q\left(\prod_{a=1}^{m}(-d_a)^{d_a}\right)(\mathcal{O}[\eta])^{\sum_a(d_a-1)-1}\right), \tag{5.83}$$

corresponding to the product structure in the cohomology ring, above as equations (5.70), (5.73), and (5.77). Furthermore, we note that the physical origin as Coulomb/Higgs branch states defines a distinction in the cohomology that is different from the role played by primitive cohomology.

Before leaving this discussion, we comment on two issues. First, the calculation of the correlation functions of $\mathcal{O}[\eta]$ and σ fixes their normalization, but our proposal does not fix the normalizations of Ψ or $\Xi[\alpha]$. These should also be fixed by calculating correlation functions, and it would be useful to develop the technology to do so.

6 Examples

Next, we will compute the predicted cohomology ring explicitly in a number of examples. We will verify in each case that the predictions match known mathematics results. For degrees $d \geq 3$, to compute the Landau-Ginzburg states, we will specialize to Fermat (diagonal) hypersurfaces. Since the A-model is independent of the precise form of the chiral superpotential this choice is just a matter of convenience.

6.1 Example: hyperplanes

In this section we will consider the GLSM for a hyperplane, namely $\mathbb{P}^n[1]$. The case of hyperplanes is particularly simple, as $\mathbb{P}^n[1] = \mathbb{P}^{n-1}$. We shall quickly walk through the details to check that this special case is correctly reproduced. (This analysis has also appeared in e.g. [28, section 5.2].)

From equation (4.13),

$$D_p(n, 1) \; = \; 0. \tag{6.1}$$

Expanding the generating function (4.12), we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \;\; = \;\; 0. \tag{6.2}$$

This means that the Dolbeault cohomology of $\mathbb{P}^n[1]$ is completely diagonal ($h^{p,q} = \delta_{p,q}$), as expected for $\mathbb{P}^{n-1}$.

In physics, the Coulomb branch relation (5.1) becomes

$$\sigma^{n+1-1} \; = \; -q\sigma^0, \tag{6.3}$$

or more simply, $\sigma^n = -q$. Hence, the σ fields describe Dolbeault cohomology $H^{k,k}$ for $k < n$, which matches the quantum cohomology ring of $\mathbb{P}^{n-1}$, with no need for any additional contributions.

In this special case, the Landau-Ginzburg model has a linear superpotential, which does not have any supersymmetric vacua (as $dW \neq 0$), hence there is no contribution from the Landau-Ginzburg model, consistent with the Coulomb branch computation.

6.2 Example: quadric hypersurfaces

Next, we turn to degree two hypersurfaces in $\mathbb{P}^n$. (This case was also discussed in e.g. [28, section 5.3]; we include this case for completeness.) We assume the hypersurfaces are Fano.

First, we discuss the mathematics. From equation (4.13),

$$D_p(n, 2) \; = \; \frac{1 + (-)^n}{2} + (-)^{n-1} \; = \; \begin{cases} 0 & n \text{ even}, \\ 1 & n \text{ odd}. \end{cases} \tag{6.4}$$

Expanding the generating function (4.12), we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell}\right) x^k y^\ell \;=\; \frac{(1+y)-(1+x)}{(1+x)^2 y - (1+y)^2 x}, \tag{6.5}$$

$$= \frac{y-x}{(y-x) - xy(y-x)}, \tag{6.6}$$

$$= \frac{1}{1-xy} \;=\; \sum_{k=0}^{\infty} (xy)^k. \tag{6.7}$$

This is interpreted to mean

$$h^{n-1}\left(\mathbb{P}^n[2]\right) \;=\; \begin{cases} 1 & n-1 \text{ odd}, \\ 2 & n-1 \text{ even}, \end{cases} \tag{6.8}$$

where the extra contribution for $n-1$ even corresponds to monomials $x^{n-1}y^{n-1}$.

Mathematically, in the case that $n+1$ is odd, the entire cohomology ring is a restriction of the cohomology ring of $\mathbb{P}^n$ to the hypersurface, specifically,

$$\mathbb{C}[x]/(x^n - q), \tag{6.9}$$

and in this case, we will see that it can be completely understood from the Coulomb branch. The quantum cohomology ring of $\mathbb{P}^n[2]$ for $n+1$ even can be presented as

$$\mathbb{C}[y,t,q]/\langle y^{2k+1} - 4qy, \quad yt = 0, \quad t^2 = (-1)^k (y^{2k} - 4q)\rangle \tag{6.10}$$

where $n = 2k+1$, $y = 1 - x$.

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{n+1-1} \;=\; q(-2)^2 \sigma, \tag{6.11}$$

or equivalently for our purposes

$$\sigma^{n-1} \;=\; (-2)^2 q. \tag{6.12}$$

The Coulomb branch contributes $n-1$ states to the Dolbeault cohomology groups $H^{k,k}$, of which there are a total of n (corresponding to $H^{k,k}$ for $k \neq n-1$). We will see that the remaining state arises from the Landau-Ginzburg orbifold (LGO).)

Next, we consider the Landau-Ginzburg $\mathbb{Z}_2$ orbifold. Here, for later use, note $\rho \in \mathbb{Z}_{2(n-1)}$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 0 \mod 2(n-1)$, hence a linear combination of $H^{p,p}$ for $2p \equiv 0 \mod 2(n-1)$.

- $r = 1$: states of the form

$$\mathcal{U}_{-1/2,1/2}\, f_p(x)|0, \mathrm{RR}\rangle \tag{6.13}$$

with $f_p \sim f_p + dW$ for p satisfying (5.29). Since the ideal (dW) is generated by linear monomials x_i, this can only receive contributions from states with $p = 0$, which will only happen if $(n+1)/2 \in \mathbb{Z}$, meaning $n+1$ is even. In that case, if $n+1$ is even, we get one state of this form, with charges $(q_L, q_R) = (0, 0)$, and ρ eigenvalue ζ^{n-1}, where ζ generates $\mathbb{Z}_{2(n-1)}$. This state contributes to $H^{k,\ell}$ for $\ell - k = 0$, $k + \ell = n - 1$ mod $2(n-1)$, hence $H^{(n-1)/2,(n-1)/2}$.

Thus, if $n+1$ is even, the Landau-Ginzburg orbifold contributes two states (one from each of $r = 0$ and $r = 1$), and if $n+1$ is odd, it only contributes one (from $r = 0$). This matches the results in [28, section 5.3], [29–31], and also completes the description of the physical origin of the cohomology.

6.3 Example: cubic hypersurfaces

Next, we describe the contributions from the Landau-Ginzburg orbifold for a cubic hypersurface in $\mathbb{P}^n$.

From equation (4.13),

$$D_p(n, 3) \;=\; \frac{2}{3}\left(2^n \;+\; (-)^{n-1}\right). \tag{6.14}$$

Expanding the generating function (4.12), we have

$$\begin{aligned}
\sum_{k,\ell}\left(h^{k,\ell} - \delta_{k,\ell}\right) x^k y^\ell \;&=\; \frac{(1+y)^2 - (1+x)^2}{(1+x)^3 y - (1+y)^3 x}, && (6.15)\\
&=\; \frac{2 + x + y}{1 - x^2 y - 3xy - xy^2}, && (6.16)\\
&=\; (2 + x + y)\sum_{k=0}^{\infty}\left(x^2 y + 3xy + xy^2\right)^k, && (6.17)\\
&=\; 2 \;+\; (x+y) \;+\; 2(x^2 y + 3xy + xy^2)\\
&\qquad + (x+y)(x^2 y + 3xy + xy^2) \;+\; \cdots && (6.18)\\
&=\; 2 \;+\; (x+y) \;+\; (6xy) \;+\; (5x^2 y + 5xy^2) \;+\; \cdots. && (6.19)
\end{aligned}$$

We list some special cases below, comparing results from both physics and mathematics. In each case, we will take the Landau-Ginzburg superpotential to be of Fermat type, meaning

$$W \;=\; \sum_{i=0}^{n} x_i^d. \tag{6.20}$$

Note that in the $r = 1$ sector, since for a Fermat cubic the ideal (dW) includes all degree-two terms of the form $(x_i)^2$, the number of surviving f_p, modulo the ideal (dW), for any p is

$$\left(\begin{array}{c} n+1 \\ p \end{array} \right) \; = \; \frac{(n+1)!}{(n+1-p)!\,p!}. \tag{6.21}$$

In each case, we will begin by listing mathematics results for the Dolbeault cohomology groups, and then give the corresponding physics. We will count states arising from both the Coulomb and the Higgs (Landau-Ginzburg orbifold) branches. For the latter, we will use symmetries to determine which Dolbeault groups $H^{k,\ell}$ the states should contribute to, using the relation (5.2)

$$\ell - k \; = \; q_L + q_R, \tag{6.22}$$

and the fact that the ρ eigenvalue determines $k + \ell \mod 2(n+1-d)$, using the ρ action in equation (5.41).

6.3.1 $n = 3$

In this subsection we consider[18] $\mathbb{P}^3[3]$. Mathematically, $D_p(n,3) = 6$, and expanding the generating function, for fixed $n = 3$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \; = \; 6xy, \tag{6.23}$$

from which we deduce

$$h^{1,1} = 7, \tag{6.24}$$

plus

$$h^{0,0} \; = \; 1 \; = \; h^{2,2}. \tag{6.25}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{4-1} \; = \; q(-3)^3 \sigma^2, \tag{6.26}$$

or equivalently for our purposes

$$\sigma \; = \; (-3)^3 q. \tag{6.27}$$

The Coulomb branch only contributes one state to the Dolbeault cohomology groups $H^{p,p}$, which will be a linear combination of that one state and certain Landau-Ginzburg orbifold states.

Applying our earlier analysis, the Landau-Ginzburg orbifold contributes

[18] In passing, we note that this example lies outside the bound (4.5).

- $r=0$: one state of charge $(q_L, q_R) = (0,0)$ and ρ eigenvalue 1,
- $r=1$: six states arising in the RR sector in the form
$$\mathcal{U}_{-1/2,1/2}\, f_p(x) |0, \mathrm{RR}\rangle \tag{6.28}$$
for $p=2$ (satisfying (5.29)) and $f_p \sim f_p + dW$, meaning that the f_2 are linear combinations of
$$x_0 x_1, \quad x_0 x_2, \quad x_0 x_3, \quad x_1 x_2, \quad x_1 x_3, \quad x_2 x_3 \tag{6.29}$$
of charge $(q_L, q_R) = (-2/3, +2/3)$ and ρ eigenvalue 1,
- $r=2$: one state of charge $(q_L, q_R) = (-4/3, +4/3)$ and ρ eigenvalue 1.

Now, let us interpret these Landau-Ginzburg states mathematically. First, recall states contribute to $H^{p,q}$ for $q-p = q_L + q_R$, but all the Landau-Ginzburg states have $q_L + q_R = 0$, hence they all contribute to $H^{p,p}$ for various p. Also, they all have ρ eigenvalue $+1$, and as ρ generates a $\mathbb{Z}_2$, this means they all are in even-degree cohomology – as is true of all of the cohomology of $\mathbb{P}^3[3]$. As a result, in this case we cannot distinguish which Landau-Ginzburg states contribute to which precise Dolbeault cohomology groups; however, it is clear that the total collection of Landau-Ginzburg states plus the Coulomb branch state spans the dimension of the cohomology groups, as expected.

We summarize our results below on a Hodge diamond, using σ to indicate Coulomb-branch contributions from σ fields, and LG to indicate Higgs-branch contributions from Landau-Ginzburg orbifold states:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} = \begin{array}{ccccc} & & \mathrm{LG}_{r=0,1,2},\, \sigma & & \\ & 0 & & 0 & \\ 0 & & \mathrm{LG}_{r=0,1,2},\, \sigma & & 0 \\ & 0 & & 0 & \\ & & \mathrm{LG}_{r=0,1,2},\, \sigma & & \end{array}$$

6.3.2 $n=4$

In this subsection we consider $\mathbb{P}^4[3]$. Mathematically, $D_p(n,3) = 10$, and expanding the generating function, for fixed $n=4$, we have
$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \;\; = \;\; 5x^2 y + 5xy^2, \tag{6.30}$$
from which we deduce
$$h^{2,1} = 5, \quad h^{1,2} = 5, \tag{6.31}$$

plus

$$h^{0,0} \;=\; 1 \;=\; h^{1,1} \;=\; h^{2,2} \;=\; h^{3,3}. \tag{6.32}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{5-1} \;=\; q(-3)^3 \sigma^2, \tag{6.33}$$

or equivalently for our purposes

$$\sigma^2 \;=\; (-3)^3 q. \tag{6.34}$$

The Coulomb branch contributes two states to the Dolbeault cohomology groups $H^{p,p}$. (We will see that the remainder arise from Landau-Ginzburg orbifold states.)

Next, we consider the Landau-Ginzburg orbifold. Here, for later use, note $\rho \in \mathbb{Z}_4$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 0 \mod 4$, hence a linear combination of $H^{0,0}$, $H^{2,2}$.
- $r = 1$: states of the form
$$\mathcal{U}_{-1/2,1/2}\, f_p(x) |0, \mathrm{RR}\rangle \tag{6.35}$$
with $f_p \sim f_p + dW$ for p satisfying (5.29):
  - five states for $p = 1$ of the form $f_p = x_i$ of charge $(q_L, q_R) = (-4/3, +1/3)$, ρ eigenvalue ζ^3, which contribute to $H^{k,\ell}$ with $\ell - k = -1$ and $k + \ell \equiv 3 \mod 4$, hence $H^{2,1}$,
  - five states for $p = 4$ of the form $f_p = x_i x_j x_k x_\ell$ of charge $(q_L, q_R) = (-1/3, +4/3)$, ρ eigenvalue ζ^3, which contribute to $H^{k,\ell}$ with $\ell - k = +1$ and $k + \ell \equiv 3 \mod 4$, hence $H^{1,2}$.
- $r = 2$: one state of charge $(q_L, q_R) = (-5/3, +5/3)$, ρ eigenvalue ζ^6, which contribute to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 2 \mod 4$, hence a linear combination of $H^{1,1}$, $H^{3,3}$.

We summarize our results below on a Hodge diamond, using σ to indicate Coulomb-branch contributions from σ fields, and LG to indicate Higgs-branch contributions from Landau-Ginzburg orbifold states:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \;=\; \begin{array}{ccccccc} & & & \mathrm{LG}_{r=0},\, \sigma & & & \\ & & 0 & & 0 & & \\ & 0 & & \mathrm{LG}_{r=2},\, \sigma & & 0 & \\ 0 & & \mathrm{LG} & & \mathrm{LG} & & 0 \\ & 0 & & \mathrm{LG}_{r=0},\, \sigma & & 0 & \\ & & 0 & & 0 & & \\ & & & \mathrm{LG}_{r=2},\, \sigma & & & \end{array}$$

6.3.3 $n = 5$

In this subsection we consider $\mathbb{P}^5[3]$. Mathematically, $D_p(n,3) = 22$, and expanding the generating function, for fixed $n = 5$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \;=\; x^3 y + 20 x^2 y^2 + x y^3, \tag{6.36}$$

from which we deduce

$$h^{3,1} = 1, \quad h^{2,2} = 21, \quad h^{1,3} = 1, \tag{6.37}$$

plus

$$h^{0,0} \;=\; 1 \;=\; h^{1,1} \;=\; h^{2,2} \;=\; h^{3,3} \;=\; h^{4,4}. \tag{6.38}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{6-1} \;=\; q(-3)^3 \sigma^2, \tag{6.39}$$

or equivalently for our purposes

$$\sigma^3 \;=\; (-3)^3 q. \tag{6.40}$$

The Coulomb branch contributes three states to $H^{p,p}$, and we will see the remainder arise from Landau-Ginzburg orbifold states.

Next we turn to the Landau-Ginzburg orbifold. In passing, this orbifold (a $\mathbb{Z}_3$ orbifold of a superpotential that is degree 3 in six variables, is related to the SCFT for a K3 surface, implemented as an orbifold of a product of two elliptic curves $\mathbb{P}^2[3]$, consistent with the fact that this theory flows to a SCFT with $c/3 = 2$, from (5.20). (Cubic fourfolds have also been of recent interest in algebraic geometry, see e.g. [32].)

Next, we consider the Landau-Ginzburg orbifold states. Here, for later use, note that $\rho \in \mathbb{Z}_6$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 0 \mod 6$, hence a linear combination of $H^{0,0}$, $H^{3,3}$.
- $r = 1$: states of the form

$$\mathcal{U}_{-1/2,1/2}\, f_p(x) | 0, \mathrm{RR} \rangle \tag{6.41}$$

  with $f_p \sim f_p + dW$ for p satisfying (5.29):

  - one state for $p = 0$ of the form $f_p = 1$ of charge $(q_L, q_R) = (-2, 0)$, ρ eigenvalue ζ^4, which contribute to $H^{k,\ell}$ with $\ell - k = -2$ and $k + \ell \equiv 4 \mod 6$, hence $H^{3,1}$.

  – 20 states for $p = 3$ of the form $f_p = x_i x_j x_k$ of charge $(q_L, q_R) = (-1, +1)$, ρ eigenvalue ζ^4, which contribute to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 4 \mod 6$, hence $H^{2,2}$.
  – one state for $p = 6$ of the form $f_p = x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6}$ of charge $(q_L, q_R) = (0, +2)$, ρ eigenvalue ζ^4, which contributes to $H^{k,\ell}$ with $\ell - k = +2$ and $k + \ell \equiv 4 \mod 6$, hence $H^{1,3}$.

- $r = 2$: one state of charge $(q_L, q_R) = (-2, +2)$, ρ eigenvalue ζ^8, which contributes to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 8 \mod 6$, hence a linear combination of $H^{1,1}$, $H^{4,4}$.

We summarize our results below on a Hodge diamond, using σ to indicate Coulomb-branch contributions from σ fields, and LG to indicate Higgs-branch contributions from Landau-Ginzburg orbifold states:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
\;=\;
\begin{array}{ccccccccc}
 & & & & \mathrm{LG}_{r=0},\,\sigma & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & \mathrm{LG}_{r=2},\,\sigma & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & \mathrm{LG} & & \mathrm{LG}_{r=1},\,\sigma & & \mathrm{LG} & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & \mathrm{LG}_{r=0},\,\sigma & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & \mathrm{LG}_{r=2},\,\sigma & & & &
\end{array}
$$

6.3.4 $n = 6$

In this subsection we consider $\mathbb{P}^6[3]$. Mathematically, $D_p(n, 3) = 42$, and expanding the generating function, for fixed $n = 6$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell}\right) x^k y^\ell \;=\; 21 x^3 y^2 + 21 x^2 y^3, \tag{6.42}$$

from which we deduce

$$h^{3,2} = 21, \quad h^{2,3} = 21, \tag{6.43}$$

plus

$$h^{0,0} \;=\; 1 \;=\; h^{1,1} \;=\; h^{2,2} \;=\; h^{3,3} \;=\; h^{4,4} \;=\; h^{5.5}. \tag{6.44}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{7-1} \;=\; q(-3)^3 \sigma^2, \tag{6.45}$$

or equivalently for our purposes

$$\sigma^4 \;=\; (-3)^3 q. \tag{6.46}$$

The Coulomb branch contributes four states to $H^{k,k}$, and we will see the remainder arise from Landau-Ginzburg orbifold states.

Next we turn to the Landau-Ginzburg orbifold. Here, for later use, note that $\rho \in \mathbb{Z}_8$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0, 0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ for $\ell - k = 0$ and $k + \ell \equiv 0 \mod 8$, hence a linear combination of $H^{0,0}$, $H^{4,4}$.
- $r = 1$: states of the form
$$\mathcal{U}_{-1/2,1/2}\, f_p(x) |0, \mathrm{RR}\rangle \tag{6.47}$$
with $f_p \sim f_p + dW$ for p satisfying (5.29):
  - 21 states for $p = 2$ of the form $f_p = x_i x_j$ of charge $(q_L, q_R) = (-5/3, +2/3)$, ρ eigenvalue ζ^5, which contribute to $H^{k,\ell}$ for $\ell - k = -1$ and $k + \ell \equiv 5 \mod 8$, hence $H^{3,2}$.
  - 21 states for $p = 5$ of the form $f_p = x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5}$ of charge $(q_L, q_R) = (-2/3, 5/3)$, ρ eigenvalue ζ^5, which contribute to $H^{k,\ell}$ for $\ell - k = +1$ and $k + \ell \equiv 5 \mod 8$, hence $H^{2,3}$.
- $r = 2$: one state of charge $(q_L, q_R) = (-7/3, +7/3)$, ρ eigenvalue ζ^{10}, which contributes to $H^{k,\ell}$ for $\ell - k = 0$ and $k + \ell \equiv 10 \mod 8$, hence a linear combination of $H^{1,1}$, $H^{5,5}$.

In terms of the Hodge diamond for $\mathbb{P}^6[3]$,

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 0 & & 21 & & 21 & & 0 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

the physical origin of the cohomology is described as

$$
\begin{array}{ccccccccccc}
 & & & & & \mathrm{LG}_{r=0},\ \sigma & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & \mathrm{LG}_{r=2},\ \sigma & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & \sigma & & 0 & & 0 & \\
0 & & 0 & & \mathrm{LG} & & \mathrm{LG} & & 0 & & 0 \\
 & 0 & & 0 & & \sigma & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & \mathrm{LG}_{r=0},\ \sigma & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & \mathrm{LG}_{r=2},\ \sigma & & & & &
\end{array}
$$

6.3.5 $n = 7$

In this subsection we consider $\mathbb{P}^7[3]$. Mathematically, $D_p(n,3) = 86$, and expanding the generating function, for fixed $n = 7$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \; = \; 8x^4y^2 + 70x^3y^3 + 8x^2y^4, \tag{6.48}$$

from which we deduce

$$h^{4,2} = 8, \quad h^{3,3} = 71, \quad h^{2,4} = 8, \tag{6.49}$$

plus

$$h^{0,0} \; = \; 1 \; = \; h^{1,1} \; = \; h^{2,2} \; = \; h^{4,4} \; = \; h^{5.5} \; = \; h^{6,6}. \tag{6.50}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{8-1} \; = \; q(-3)^3 \sigma^2, \tag{6.51}$$

or equivalently for our purposes

$$\sigma^5 \; = \; (-3)^3 q. \tag{6.52}$$

The Coulomb branch contributes five states to $H^{k,k}$, and we will see that the remainder arise from Landau-Ginzburg orbifold states.

Next we turn to the Landau-Ginzburg orbifold. Here, for later use, note that $\rho \in \mathbb{Z}_{10}$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ for $\ell - k = 0$ and $k + \ell \equiv 0 \mod 10$, hence a linear combination of $H^{0,0}$, $H^{5,5}$.

- $r = 1$: states of the form
$$\mathcal{U}_{-1/2,1/2}\, f_p(x)|0,\mathrm{RR}\rangle \tag{6.53}$$
with $f_p \sim f_p + dW$ for p satisfying (5.29):
  - 8 states for $p = 1$ of the form $f_p = x_i$ of charge $(q_L, q_R) = (-7/3, +1/3)$, ρ eigenvalue ζ^6, which contribute to $H^{k,\ell}$ for $\ell - k = -2$ and $k + \ell \equiv 6 \mod 10$, hence $H^{4,2}$.
  - 70 states for $p = 4$ of the form $f_p = x_{i_1}x_{i_2}x_{i_3}x_{i_4}$ of charge $(q_L, q_R) = (-4/3, +4/3)$, ρ eigenvalue ζ^6, which contribute to $H^{k,\ell}$ for $\ell - k = 0$ and $k + \ell \equiv 6 \mod 10$, hence $H^{3,3}$.
  - 8 states for $p = 7$ of the form $f_p = x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6}x_{i_7}$ of charge $(q_L, q_R) = (-1/3, +7/3)$, ρ eigenvalue ζ^6 which contribute to $H^{k,\ell}$ for $\ell - k = +2$ and $k + \ell \equiv 6 \mod 10$, hence $H^{2,4}$.
- $r = 2$: one state of charge $(q_L, q_R) = (-8/3, +8/3)$, ρ eigenvalue ζ^{12}, which contributes to $H^{k,\ell}$ for $\ell - k = 0$ and $k + \ell \equiv 12 \mod 10$, hence a linear combination of $H^{1,1}$, $H^{6,6}$.

In terms of the Hodge diamond for $\mathbb{P}^7[3]$,

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 1 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 1 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 8 & & 71 & & 8 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 1 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 1 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$

the physical origin of the cohomology is described as

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathrm{LG}_{r=0},\,\sigma & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & \mathrm{LG}_{r=2},\,\sigma & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & \sigma & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & \mathrm{LG} & & \mathrm{LG}_{r=1},\,\sigma & & \mathrm{LG} & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \sigma & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & \mathrm{LG}_{r=0},\,\sigma & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & \mathrm{LG}_{r=2},\,\sigma & & & & & &
\end{array}
$$

6.3.6 $n = 8$

In this subsection we consider $\mathbb{P}^8[3]$. Mathematically, $D_p(n,3) = 170$, and expanding the generating function, for fixed $n = 8$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \;=\; x^5 y^2 + 84 x^4 y^3 + 84 x^3 y^4 + x^2 y^5, \tag{6.54}$$

from which we deduce

$$h^{5,2} = 1, \quad h^{4,3} = 84, \quad h^{3,4} = 84, \quad h^{2,5} = 1, \tag{6.55}$$

plus

$$h^{0,0} \;=\; 1 \;=\; h^{1,1} \;=\; h^{2,2} \;=\; h^{3,3} \;=\; h^{4,4} \;=\; h^{5.5} \;=\; h^{6,6} \;=\; h^{7,7}. \tag{6.56}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{9-1} \;=\; q(-3)^3 \sigma^2, \tag{6.57}$$

or equivalently for our purposes

$$\sigma^6 \;=\; (-3)^3 q. \tag{6.58}$$

The Coulomb branch contributes six states to $H^{k,k}$, and we will see that the remainder are contributed by Landau-Ginzburg orbifold states.

Next we turn to the Landau-Ginzburg orbifold. Here, for later use, note that $\rho \in \mathbb{Z}_{12}$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ for $\ell - k = 0$ and $k + \ell \equiv 0 \mod 12$, hence a linear combination of $H^{0,0}$, $H^{6,6}$.

- $r=1$: states of the form

$$\mathcal{U}_{-1/2,1/2}\, f_p(x)|0,\mathrm{RR}\rangle \tag{6.59}$$

  with $f_p \sim f_p + dW$ for p satisfying (5.29):

  - one state for $p=0$ of the form $f_p = 1$ of charge $(q_L,q_R)=(-3,0)$, ρ eigenvalue ζ^7, which contributes to $H^{k,\ell}$ for $\ell-k=-3$ and $k+\ell \equiv 7 \mod 12$, hence $H^{5,2}$.
  - 84 states for $p=3$ of the form $f_p = x_{i_1}x_{i_2}x_{i_3}$ of charge $(q_L,q_R)=(-2,+1)$, ρ eigenvalue ζ^7, which contribute to $H^{k,\ell}$ for $\ell-k=-1$ and $k+\ell \equiv 7 \mod 12$, hence $H^{4,3}$.
  - 84 states for $p=6$ of the form $f_p = x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6}$ of charge $(q_L,q_R)=(-1,+2)$, ρ eigenvalue ζ^7, which contribute to $H^{k,\ell}$ for $\ell-k=+1$ and $k+\ell\equiv 7 \mod 12$, hence $H^{3,4}$.
  - one state for $p=9$ of the form $f_p = x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6}x_{i_7}x_{i_8}x_{i_9}$ of charge $(q_L,q_R)=(0,+3)$, ρ eigenvalue ζ^7, which contributes to $H^{k,\ell}$ for $\ell-k=+3$ and $k+\ell\equiv 7 \mod 12$, hence $H^{2,5}$.

- $r=2$: one state of charge $(q_L,q_R)=(-3,+3)$, ρ eigenvalue ζ^{14}, which contributes to $H^{k,\ell}$ for $\ell-k=0$ and $k+\ell\equiv 14 \mod 12$, hence a linear combination of $H^{1,1}$, $H^{7,7}$.

In terms of the Hodge diamond for $\mathbb{P}^8[3]$,

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&0&&0&&&&&& \\
&&&&&0&&1&&0&&&&& \\
&&&&0&&0&&0&&0&&&& \\
&&&0&&0&&1&&0&&0&&& \\
&&0&&0&&0&&0&&0&&0&& \\
&0&&0&&0&&1&&0&&0&&0& \\
0&&0&&1&&84&&84&&1&&0&&0 \\
&0&&0&&0&&1&&0&&0&&0& \\
&&0&&0&&0&&0&&0&&0&& \\
&&&0&&0&&1&&0&&0&&& \\
&&&&0&&0&&0&&0&&&& \\
&&&&&0&&1&&0&&&&& \\
&&&&&&0&&0&&&&&& \\
&&&&&&&1&&&&&&&
\end{array}$$

the physical origin of the cohomology is described as

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \mathrm{LG}_{r=0},\,\sigma & & & & & & & \\
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & 0 & & \mathrm{LG}_{r=2},\,\sigma & & 0 & & & & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & 0 & & 0 & & \sigma & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & \sigma & & 0 & & 0 & & 0 & \\
0 & & 0 & & \mathrm{LG} & & \mathrm{LG} & & \mathrm{LG} & & \mathrm{LG} & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & \sigma & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & \sigma & & 0 & & 0 & & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & & & 0 & & \mathrm{LG}_{r=0},\,\sigma & & 0 & & & & & \\
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & & & \mathrm{LG}_{r=2},\,\sigma & & & & & & &
\end{array}
$$

6.4 Example: degree four hypersurfaces

From equation (4.13),

$$D_p(n,3) \;=\; \frac{3}{4}\left(3^n + (-)^{n-1}\right). \tag{6.60}$$

Expanding the generating function (4.12), we have

$$\sum_{k,\ell}\left(h^{k,\ell} - \delta_{k,\ell}\right)x^k y^\ell \;=\; \frac{(1+y)^3 - (1+x)^3}{(1+x)^4 y - (1+y)^4 x}, \tag{6.61}$$

$$=\; \frac{3 + 3(y+x) + (x^2+xy+y^2)}{1 - 6xy - 4xy(y+x) - xy(x^2+xy+y^2)}. \tag{6.62}$$

We list some special cases below, comparing results from both physics and mathematics. As in the degree three examples, we will take the Landau-Ginzburg superpotential to be of Fermat type, meaning

$$W \;=\; \sum_{i=0}^{n} x_i^d. \tag{6.63}$$

In each case, we will begin by listing mathematics results for the Dolbeault cohomology groups, and then give the corresponding physics. As before, we will count states arising from both the Coulomb and the Higgs (Landau-Ginzburg orbifold) branches, and in the latter, determine which Dolbeault groups $H^{k,\ell}$ the states should contribute to using the relation (5.2)

$$\ell - k \;=\; q_L + q_R, \tag{6.64}$$

and the fact that the ρ eigenvalue determines $k+\ell \mod 2(n+1-d)$, using the ρ action in equation (5.41).

6.4.1 $n = 4$

In this subsection we consider[19] $\mathbb{P}^4[4]$. Mathematically, $D_p(n,4) = 60$, and expanding the generating function, for fixed $n = 4$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \;=\; 30x^2y + 30xy^2, \tag{6.65}$$

from which we deduce

$$h^{2,1} = 30, \quad h^{1,2} = 30. \tag{6.66}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{5-1} \;=\; q(-4)^4 \sigma^3, \tag{6.67}$$

or equivalently for our purposes

$$\sigma \;=\; (-4)^4 q. \tag{6.68}$$

The Coulomb branch only contributes one state to $H^{k,k}$. We will see that the remainder are contributed by the Landau-Ginzburg orbifold.

Next, we consider the Landau-Ginzburg orbifold. Here, for later use, note $\rho \in \mathbb{Z}_2$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 0 \mod 2$, hence $H^{k,k}$.
- $r = 1$: states of the form
$$\mathcal{U}_{-1/2,1/2}\, f_p(x) |0, \mathrm{RR}\rangle \tag{6.69}$$
with $f_p \sim f_p + dW$, for $p = 3, 7$, satisfying (5.29).
  - For $p = 3$, possible f_p (modulo the ideal) are of the form
  $$x_i x_j x_k, \quad x_i x_j^2, \tag{6.70}$$
  for $i \neq j \neq k$, of which there are 30 possible terms. The corresponding states have charge $(q_L, q_R) = (-7/4, +3/4)$ and ρ eigenvalue ζ^3, so they contribute to $H^{k,\ell}$ for $\ell - k = -1$ and $k + \ell \equiv 1 \mod 2$, hence $H^{2,1}$.

[19] In passing, we observe that this example lies outside the bound (4.5).

– For $p = 7$, possible f_p (modulo the ideal) are of the form

$$x_i^2 x_j^2 x_k^2 x_\ell, \quad x_i x_j x_k x_\ell^2 x_m^2, \tag{6.71}$$

for $i \neq j \neq k \neq \ell \neq m$, of which there are 30 possible terms. The corresponding states have charge $(q_L, q_R) = (-3/4, +7/4)$ and ρ eigenvalue ζ^3, so they contribute to $H^{k,\ell}$ for $\ell - k = +1$ and $k + \ell \equiv 1 \mod 2$, hence $H^{1,2}$.

- $r = 2$: one state of charge $(q_L, q_R) = (-5/2, +5/2)$, ρ eigenvalue 1, so they contribute to $H^{k,\ell}$ for $\ell - k = 0$ and $k + \ell \equiv 0 \mod 2$, hence $H^{k,k}$.
- $r = 3$: one state of charge $(q_L, q_R) = (-5/4, +5/4)$, ρ eigenvalue 1, so they contribute to $H^{k,\ell}$ for $\ell - k = 0$ and $k + \ell \equiv 0 \mod 2$, hence $H^{k,k}$.

The $r = 0$, $r = 2$, and $r = 3$ states could each contribute to any linear combination of $H^{1,1}$, $H^{2,2}$, $H^{3,3}$; symmetries alone do not suffice to further distinguish.

We summarize our results below on a Hodge diamond,

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array}$$

using σ to indicate Coulomb-branch contributions from σ fields, and LG to indicate Higgs-branch contributions from Landau-Ginzburg-orbifold states:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 30 & & 30 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} = \begin{array}{ccccccc} & & & \mathrm{LG}_{r=0,2,3},\ \sigma & & & \\ & & 0 & & 9 & & \\ & 0 & & \mathrm{LG}_{r=0,2,3},\ \sigma & & 0 & \\ 0 & & \mathrm{LG} & & \mathrm{LG} & & 0 \\ & 0 & & \mathrm{LG}_{r=0,2,3},\ \sigma & & 0 & \\ & & 0 & & 0 & & \\ & & & \mathrm{LG}_{r=0,2,3},\ \sigma & & & \end{array}$$

6.4.2 $n = 5$

In this subsection we consider[20] $\mathbb{P}^5[4]$. Mathematically, $D_p(n,4) = 183$, and expanding the generating function, for fixed $n = 5$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \;=\; 21 x^3 y + 141 x^2 y^2 + 21 x y^3, \tag{6.72}$$

from which we deduce

$$h^{3,1} = 21, \quad h^{2,2} = 142, \quad h^{1,3} = 21, \tag{6.73}$$

plus

$$h^{0,0} \;=\; 1 \;=\; h^{1,1} \;=\; h^{3,3} \;=\; h^{4,4}. \tag{6.74}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{6-1} \;=\; q(-4)^4 \sigma^3, \tag{6.75}$$

or equivalently for our purposes

$$\sigma^2 \;=\; (-4)^4 q. \tag{6.76}$$

The Coulomb branch only contributes two states to $H^{k,k}$. We shall see the remainder arise as Landau-Ginzburg orbifold states.

Next, we consider the Landau-Ginzburg orbifold. Here, for later use, note $\rho \in \mathbb{Z}_4$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 0 \mod 4$, hence to a linear combination of $H^{0,0}$, $H^{2,2}$, $H^{4,4}$.
- $r = 1$: states of the form

$$\mathcal{U}_{-1/2,1/2}\, f_p(x) |0, \mathrm{RR}\rangle \tag{6.77}$$

  with $f_p \sim f_p + dW$, for $p = 2, 6, 10$, satisfying (5.29).

  - For $p = 2$, possible f_p (modulo the ideal) are of the form

$$x_i x_j, \quad x_i^2, \tag{6.78}$$

    for $i \neq j$, of which there are 21 possible terms. The corresponding states have charge $(q_L, q_R) = (-5/2, +1/2)$, ρ eigenvalue 1, and contribute to $H^{k,\ell}$ for $\ell - k = -2$ and $k + \ell \equiv 0 \mod 4$, hence $H^{3,1}$.

[20] In passing, we observe that this example lies outside the bound (4.5).

– For $p = 6$, possible f_p (modulo the ideal) are of the form

$$x_i^2 x_j^2 x_k^2, \quad x_i x_j x_k^2 x_\ell^2, \quad x_i x_j x_k x_\ell x_m^2, \quad x_i x_j x_k x_\ell x_m x_n, \tag{6.79}$$

for distinct factors, of which there are 141 possible terms. The corresponding states have charge $(q_L, q_R) = (-3/2, +3/2)$, ρ eigenvalue 1, and contribute to $H^{k,\ell}$ with $\ell - k = 0$, $k + \ell \equiv 0 \mod 4$, hence a linear combination of $H^{0,0}$, $H^{2,2}$, and $H^{4,4}$.

– For $p = 10$, possible f_p (modulo the ideal) are of the form

$$x_i^2 x_j^2 x_k^2 x_\ell^2 x_m^2, \quad x_i x_j x_k^2 x_\ell^2 x_m^2 x_n^2, \tag{6.80}$$

of which there are 21 possible terms. The corresponding states have charge $(q_L, q_R) = (-1/2, +5/2)$, ρ eigenvalue 1, and contribute to $H^{k,\ell}$ for $\ell - k = +2$, $k + \ell \equiv 0 \mod 4$, hence $H^{1,3}$.

- $r = 2$: one state of charge $(q_L, q_R) = (-3, +3)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ with $\ell - k = 0$, $k + \ell \equiv 0 \mod 4$, hence a linear combination of $H^{0,0}$, $H^{2,2}$, and $H^{4,4}$.

- $r = 3$: one state of charge $(q_L, q_R) = (-3/2, +3/2)$, ρ eigenvalue ζ^2, which contributes to $H^{k,\ell}$ with $\ell - k = 0$, $k + \ell \equiv 2 \mod 4$, hence a linear combination of $H^{1,1}$ and $H^{3,3}$.

We summarize our results below on a Hodge diamond, using σ to indicate Coulomb-branch contributions from σ fields, and LG to indicate Higgs-branch contributions from Landau-Ginzburg-orbifold states:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 21 & & 142 & & 21 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array} \; = \; \begin{array}{ccccccccc} & & & & \mathrm{LG}_{r=0,1,2},\,\sigma & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & \mathrm{LG}_{r=3},\,\sigma & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & \mathrm{LG} & & \mathrm{LG}_{r=0,1,2},\,\sigma & & \mathrm{LG} & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & \mathrm{LG}_{r=3},\,\sigma & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & \mathrm{LG}_{r=0,1,2},\,\sigma & & & & \end{array}$$

6.4.3 $n = 6$

In this subsection we consider $\mathbb{P}^6[4]$. Mathematically, $D_p(n, 4) = 546$, and expanding the generating function, for fixed $n = 6$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \;=\; 7x^4 y + 266x^3 y^2 + 266x^2 y^3 + 7xy^4, \tag{6.81}$$

from which we deduce

$$h^{4,1} = 7, \quad h^{3,2} = 266, \quad h^{2,3} = 266, \quad h^{1,4} = 7, \tag{6.82}$$

plus

$$h^{0,0} \; = \; 1 \; = \; h^{1,1} \; = \; h^{2,2} \; = \; h^{3,3} \; = \; h^{4,4} \; = \; h^{5,5}. \tag{6.83}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{7-1} \; = \; q(-4)^4 \sigma^3, \tag{6.84}$$

or equivalently for our purposes

$$\sigma^3 \; = \; (-4)^4 q. \tag{6.85}$$

The Coulomb branch only contributes three states to $H^{k,k}$. We shall see the remainder arise as Landau-Ginzburg orbifold states.

Next, we consider the Landau-Ginzburg orbifold. Here, for later use, note $\rho \in \mathbb{Z}_6$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0, 0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ with $\ell - k = 0$ and $k + \ell \equiv 0 \mod 6$, hence a linear combination of $H^{0,0}$, $H^{3,3}$.
- $r = 1$: states of the form

$$\mathcal{U}_{-1/2,1/2} \, f_p(x) | 0, \mathrm{RR} \rangle \tag{6.86}$$

  with $f_p \sim f_p + dW$, for $p = 1, 5, 9, 13$, satisfying (5.29).

  - For $p = 1$, possible f_p (modulo the ideal) are of the form

$$x_i \tag{6.87}$$

    of which there are 7 possible terms. The corresponding states have charge $(q_L, q_R) = (-13/4, +1/4)$, ρ eigenvalue ζ^5, and contribute to $H^{k,\ell}$ for $\ell - k = -3$, $k + \ell \equiv 5 \mod 6$, hence $H^{4,1}$.
  - For $p = 5$, possible f_p (modulo the ideal) are of the form

$$x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5}, \quad x_{i_1} x_{i_2} x_{i_3} x_{i_4}^2, \quad x_{i_1} x_{i_2}^2 x_{i_3}^2, \tag{6.88}$$

    of which there are $21 + 140 + 105 = 266$ possible terms. The corresponding states have charge $(q_L, q_R) = (-9/4, +5/4)$, ρ eigenvalue ζ^5, and contribute to $H^{k,\ell}$ for $\ell - k = -1$, $k + \ell \equiv 5 \mod 6$, hence $H^{3,2}$.

– For $p = 9$, possible f_p (modulo the ideal) are of the form

$$x_{i_1}^2 x_{i_2}^2 x_{i_3} \cdots x_{i_7}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4} x_{i_5} x_{i_6}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}, \tag{6.89}$$

of which there are $21 + 140 + 105 = 266$ possible terms. The corresponding terms have charge $(q_L, q_R) = (-5/4, +9/4)$, ρ eigenvalue ζ^5, and contribute to $H^{k,\ell}$ for $\ell - k = +1$, $k + \ell \equiv 5 \mod 6$, hence $H^{2,3}$.

– For $p = 13$, possible f_p (modulo the ideal) are of the form

$$x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}^2 x_{i_6}^2 x_{i_7}, \tag{6.90}$$

of which there are 7 possible terms. The corresponding states have charge $(q_L, q_R) = (-1/4, +13/4)$, ρ eigenvalue ζ^5, and contribute to $H^{k,\ell}$ for $\ell - k = +3$, $k + \ell \equiv 5 \mod 6$, hence $H^{1,4}$.

- $r = 2$: one state of charge $(q_L, q_R) = (-7/2, +7/2)$, ρ eigenvalue ζ^{10}, which contributes to $H^{k,\ell}$ for $\ell - k = 0$, $k + \ell \equiv 10 \mod 6$, hence a linear combination of $H^{2,2}$, $H^{5,5}$.

- $r = 3$: one state of charge $(q_L, q_R) = (-7/4, +7/4)$, ρ eigenvalue ζ^8, which contributes to $H^{k,\ell}$ for $\ell - k = 0$, $k + \ell \equiv 2 \mod 6$ hence a linear combination of $H^{1,1}$, $H^{4,4}$.

The Hodge diamond for $\mathbb{P}^6[4]$ is

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 7 & & 266 & & 266 & & 7 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

We summarize the physical origin of these states below, using σ to indicate Coulomb-branch contributions from σ fields, and LG to indicate Higgs-branch contributions from Landau-Ginzburg-orbifold states:

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathrm{LG}_{r=0},\,\sigma & & & & & & \\
 & & & & 0 & & & & 0 & & & & \\
 & & & 0 & & & \mathrm{LG}_{r=3},\,\sigma & & & 0 & & & \\
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & 0 & & 0 & & & \mathrm{LG}_{r=2},\,\sigma & & & 0 & & 0 & \\
0 & & \mathrm{LG} & & \mathrm{LG} & & & & \mathrm{LG} & & \mathrm{LG} & & 0 \\
 & 0 & & 0 & & & \mathrm{LG}_{r=0},\,\sigma & & & 0 & & 0 & \\
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & & 0 & & & \mathrm{LG}_{r=3},\,\sigma & & & 0 & & & \\
 & & & & 0 & & & & 0 & & & & \\
 & & & & & & \mathrm{LG}_{r=2},\,\sigma & & & & & &
\end{array}
$$

6.4.4 $n = 7$

In this subsection we consider $\mathbb{P}^7[4]$. Mathematically, $D_p(n,4) = 1641$, and expanding the generating function, for fixed $n = 7$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \; = \; x^5 y + 266 x^4 y^2 + 1107 x^3 y^3 + 266 x^2 y^4 + x y^5, \tag{6.91}$$

from which we deduce

$$h^{5,1} = 1, \quad h^{4,2} = 266, \quad h^{3,3} = 1108, \quad h^{2,4} = 266, \quad h^{1,5} = 1, \tag{6.92}$$

plus

$$h^{0,0} \; = \; 1 \; = \; h^{1,1} \; = \; h^{2,2} \; = \; h^{4,4} \; = \; h^{5,5} \; = \; h^{6,6}. \tag{6.93}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{8-1} \; = \; q(-4)^4 \sigma^3, \tag{6.94}$$

or equivalently for our purposes

$$\sigma^4 \; = \; (-4)^4 q. \tag{6.95}$$

The Coulomb branch only contributes four states to $H^{k,k}$. We will see the remainder arise as Landau-Ginzburg orbifold states.

Next, we consider the Landau-Ginzburg orbifold. Here, for later use, note $\rho \in \mathbb{Z}_8$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ for $\ell - k = 0$, $k + \ell \equiv 0 \mod 8$, hence a linear combination of $H^{0,0}$, $H^{4,4}$.

- $r=1$: states of the form
$$\mathcal{U}_{-1/2,1/2}\, f_p(x)|0,\mathrm{RR}\rangle \tag{6.96}$$
with $f_p \sim f_p + dW$, for $p = 0, 4, 8, 12, 16$, satisfying (5.29).
  - For $p=0$, $f_p \propto 1$, hence one possible term. The corresponding state has charge $(q_L, q_R) = (-4, 0)$, ρ eigenvalue ζ^6, and contributes to $H^{k,\ell}$ for $\ell - k = -4$, $k+\ell \equiv 6 \mod 8$, hence $H^{5,1}$.
  - For $p=4$, possible f_p (modulo the ideal) are of the form
$$x_{i_1}x_{i_2}x_{i_3}x_{i_4}, \quad x_{i_1}^2 x_{i_2}x_{i_3}, \quad x_{i_1}^2 x_{i_2}^2 \tag{6.97}$$
of which there are $70+168+28 = 266$ possible terms. The corresponding states have charge $(q_L, q_R) = (-3, +1)$, ρ eigenvalue ζ^6, and contribute to $H^{k,\ell}$ for $\ell - k = -2$, $k+\ell \equiv 6 \mod 8$, hence $H^{4,2}$.
  - For $p=8$, possible f_p (modulo the ideal) are of the form
$$\begin{array}{c} x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6}x_{i_7}x_{i_8}, \quad x_{i_1}^2 x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6}x_{i_7}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3}x_{i_4}x_{i_5}x_{i_6}, \\ x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}x_{i_5}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2, \end{array} \tag{6.98}$$
of which there are $1+56+420+560+70 = 1107$ possible terms. The corresponding states have charge $(q_L, q_R) = (-2, +2)$, ρ eigenvalue ζ^6, and contribute to $H^{k,\ell}$ for $\ell - k = 0$, $k+\ell \equiv 6 \mod 8$, hence $H^{3,3}$.
  - For $p=12$, possible f_p (modulo the ideal) are of the form
$$x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}^2 x_{i_6}^2, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}^2 x_{i_6}x_{i_7}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}x_{i_6}x_{i_7}x_{i_8}, \tag{6.99}$$
of which there are $28+168+70 = 266$ possible terms. The corresponding states have charge $(q_L, q_R) = (-1, +3)$, ρ eigenvalue ζ^6, and contribute to $H^{k,\ell}$ for $\ell - k = +2$, $k+\ell \equiv 6 \mod 8$, hence $H^{2,4}$.
  - For $p=16$, possible f_p (modulo the ideal) are of the form
$$x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}^2 x_{i_6}^2 x_{i_7}^2 x_{i_8}^2, \tag{6.100}$$
of which there is only one term. The corresponding state has charge $(q_L, q_R) = (0, +4)$, ρ eigenvalue ζ^6, and contributes to $H^{k,\ell}$ for $\ell - k = +4$, $k+\ell \equiv 6 \mod 8$, hence $H^{1,5}$.
- $r=2$: one state of charge $(q_L, q_R) = (-4, +4)$, ρ eigenvalue ζ^{12}, which contributes to $H^{k,\ell}$ for $\ell - k = 0$, $k+\ell \equiv 12 \mod 8$ hence a linear combination of $H^{2,2}$ and $H^{6,6}$.
- $r=3$: one state of charge $(q_L, q_R) = (-2, +2)$, ρ eigenvalue ζ^{10}, which contributes to $H^{k,\ell}$ for $ell - k = 0$, $k+\ell \equiv 10 \mod 8$, hence a linear combination of $H^{1,1}$ and $H^{5,5}$.

The Hodge diamond for $\mathbb{P}^7[4]$ is

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&& \\
&&&&&0&&0&&&&& \\
&&&&0&&1&&0&&&& \\
&&&0&&0&&0&&0&&& \\
&&0&&0&&1&&0&&0&& \\
&0&&0&&0&&0&&0&&0& \\
0&&1&&266&&1108&&266&&1&&0 \\
&0&&0&&0&&0&&0&&0& \\
&&0&&0&&1&&0&&0&& \\
&&&0&&0&&0&&0&&& \\
&&&&0&&1&&0&&&& \\
&&&&&0&&0&&&&& \\
&&&&&&1&&&&&&
\end{array}$$

We summarize the physical origin of these states below, using σ to indicate Coulomb-branch contributions from σ fields, and LG to indicate Higgs-branch contributions from Landau-Ginzburg-orbifold states:

$$\begin{array}{ccccccccccccc}
&&&&&&\text{LG}_{r=0},\ \sigma&&&&&& \\
&&&&&0&&0&&&&& \\
&&&&0&&\text{LG}_{r=3},\ \sigma&&0&&&& \\
&&&0&&0&&0&&0&&& \\
&&0&&0&&\text{LG}_{r=2},\ \sigma&&0&&0&& \\
&0&&0&&0&&0&&0&&0& \\
0&&\text{LG}&&\text{LG}&&\text{LG}_{r=1},\ \sigma&&\text{LG}&&\text{LG}&&0 \\
&0&&0&&0&&0&&0&&0& \\
&&0&&0&&\text{LG}_{r=0},\ \sigma&&0&&0&& \\
&&&0&&0&&0&&0&&& \\
&&&&0&&\text{LG}_{r=3},\ \sigma&&0&&&& \\
&&&&&0&&0&&&&& \\
&&&&&&\text{LG}_{r=2},\ \sigma&&&&&&
\end{array}$$

6.4.5 $n = 8$

In this subsection we consider $\mathbb{P}^8[4]$. Mathematically, $D_p(n, 4) = 4920$, and expanding the generating function, for fixed $n = 8$, we have

$$\sum_{k,\ell} \left(h^{k,\ell} - \delta_{k,\ell} \right) x^k y^\ell \; = \; 156 x^5 y^2 + 2304 x^4 y^3 + 2304 x^3 y^4 + 156 x^2 y^5, \tag{6.101}$$

from which we deduce

$$h^{5,2} = 156, \quad h^{4,3} = 2304, \quad h^{3,4} = 2304, \quad h^{2,5} = 156, \tag{6.102}$$

plus

$$h^{0,0} \;=\; 1 \;=\; h^{1,1} \;=\; h^{2,2} \;=\; h^{3,3} \;=\; h^{4,4} \;=\; h^{5,5} \;=\; h^{6,6} \;=\; h^{7,7}. \tag{6.103}$$

Now, let us turn to the corresponding physics. From the Coulomb branch, there are σ fields, obeying the relation (5.1)

$$\sigma^{9-1} \;=\; q(-4)^4 \sigma^3, \tag{6.104}$$

or equivalently for our purposes

$$\sigma^5 \;=\; (-4)^4 q. \tag{6.105}$$

The Coulomb branch only contributes five states to $H^{k,k}$. We shall see that the remainder arise as Landau-Ginzburg orbifold states.

Next, we consider the Landau-Ginzburg orbifold. Here, for later use, note $\rho \in \mathbb{Z}_{10}$, whose generator we label ζ. The states are as follows:

- $r = 0$: one state of charge $(q_L, q_R) = (0,0)$, ρ eigenvalue 1, which contributes to $H^{k,\ell}$ for $\ell - k = 0$, $k + \ell \equiv 0 \mod 10$, hence a linear combination of $H^{0,0}$, $H^{5,5}$.
- $r = 1$: states of the form

$$\mathcal{U}_{-1/2,1/2}\, f_p(x) |0, \mathrm{RR}\rangle \tag{6.106}$$

  with $f_p \sim f_p + dW$, for $p = 3, 7, 11, 15$, satisfying (5.29).

  - For $p = 3$, possible f_p (modulo the ideal) are of the form

$$x_{i_1}^2 x_{i_2}, \quad x_{i_1} x_{i_2} x_{i_3} \tag{6.107}$$

    of which there are $72 + 84 = 156$ possible terms. The corresponding states have charge $(q_L, q_R) = (-15/4, +3/4)$, ρ eigenvalue ζ^7, and contribute to $H^{k,\ell}$ for $\ell - k = -3$, $k + \ell \equiv 7 \mod 10$, hence $H^{5,2}$.
  - For $p = 7$, possible f_p (modulo the ideal) are of the form

$$x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3} x_{i_4} x_{i_5}, \quad x_{i_1}^2 x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6}, \quad x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} x_{i_7}, \tag{6.108}$$

    of which there are $504+1260+504+36 = 2304$ possible terms. The corresponding states have charge $(q_L, q_R) = (-11/4, +7/4)$, ρ eigenvalue ζ^7, and contribute to $H^{k,\ell}$ for $\ell - k = -1$, $k + \ell \equiv 7 \mod 10$, hence $H^{4,3}$.
  - For $p = 11$, possible f_p (modulo the ideal) are of the form

$$\begin{array}{c} x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}^2 x_{i_6}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5} x_{i_6} x_{i_7}, \\ x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4} x_{i_5} x_{i_6} x_{i_7} x_{i_8}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3} x_{i_4} x_{i_5} x_{i_6} x_{i_7} x_{i_8} x_{i_9}, \end{array} \tag{6.109}$$

    of which there are $504+1260+504+36 = 2304$ possible terms. The corresponding states have charge $(q_L, q_R) = (-7/4, +11/4)$, ρ eigenvalue ζ^7, and contribute to $H^{k,\ell}$ for $\ell - k = +1$, $k + \ell \equiv 7 \mod 10$, hence $H^{3,4}$.

– For $p = 15$, possible f_p (modulo the ideal) are of the form

$$x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}^2 x_{i_6}^2 x_{i_7}^2 x_{i_8}, \quad x_{i_1}^2 x_{i_2}^2 x_{i_3}^2 x_{i_4}^2 x_{i_5}^2 x_{i_6}^2 x_{i_7} x_{i_8} x_{i_9}, \tag{6.110}$$

of which there are $72 + 84 = 156$ possible terms. The corresponding states have charge $(q_L, q_R) = (-3/4, +15/4)$, ρ eigenvalue ζ^7, and contribute to $H^{k,\ell}$ for $\ell - k = +3$, $k + \ell \equiv 7 \mod 10$, hence $H^{2,5}$.

- $r = 2$: one state of charge $(q_L, q_R) = (-9/2, +9/2)$, ρ eigenvalue ζ^{14}, which contributes to $H^{k,\ell}$ for $\ell - k = 0$, $k + \ell \equiv 14 \mod 10$, hence a linear combination of $H^{2,2}$, $H^{7,7}$.
- $r = 3$: one state of charge $(q_L, q_R) = (-9/4, +9/4)$, ρ eigenvalue ζ^{12}, which contributes to $H^{k,\ell}$ for $\ell - k = 0$, $k + \ell \equiv 12 \mod 10$, hence a linear combination of $H^{1,1}$, $H^{6,6}$.

The Hodge diamond for $\mathbb{P}^8[4]$ is

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & 0 & & 1 & & 0 & & & & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & 0 & & 0 & & 1 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 156 & & 2304 & & 2304 & & 156 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 1 & & 0 & & 0 & & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & & & \\
 & & & & & 0 & & 1 & & 0 & & & & & \\
 & & & & & & 0 & & 0 & & & & & & \\
 & & & & & & & 1 & & & & & & &
\end{array}$$

We summarize the physical origin of these states below, using σ to indicate Coulomb-branch contributions from σ fields, and LG to indicate Higgs-branch contributions from Landau-Ginzburg-orbifold states:

							$\mathrm{LG}_{r=0}$, σ							
						0		0						
					0		$\mathrm{LG}_{r=3}$, σ		0					
				0		0		0		0				
			0		0		$\mathrm{LG}_{r=2}$, σ		0		0			
		0		0		0		0		0		0		
	0		0		0		σ		0		0		0	
0		0		LG		LG		LG		LG		0		0
	0		0		0		σ		0		0		0	
		0		0		0		0		0		0		
			0		0		$\mathrm{LG}_{r=0}$, σ		0		0			
				0		0		0		0				
					0		$\mathrm{LG}_{r=3}$, σ		0					
						0		0						
							$\mathrm{LG}_{r=2}$, σ							

7 Other theories

In this section we will apply the same ideas to computations in quantum K theory (realized via three-dimensional GLSMs) and quantum sheaf cohomology (realized via two-dimensional GLSMs with (0,2) supersymmetry). We will see that the structure of the mathematical results is consistent with the form of the computations described earlier; unfortunately, in both cases there is insufficient symmetry to make a prediction just from the physics we currently understand.

7.1 Quantum K theory

Quantum K theory can be computed from three-dimensional GLSMs, see e.g. [28, 33–39]. Consider a three-manifold of the form $S^1 \times \Sigma$, for a Riemann surface Σ. The quantum K theory relations emerge as the OPEs of Wilson lines wrapped on S^1 as they are brought closer along Σ. By performing a Kaluza-Klein reduction to Σ (and resumming, not truncating, the higher modes), one recovers an effective two-dimensional theory in which OPEs of local operators encode the Wilson line OPEs of the three-dimensional theory.

For GLSMs without superpotential describing Fano target spaces, these ideas have been used to compute quantum K theory ring relations using Coulomb branch relations, in the same pattern as [2]. It is natural to try to extend these computations to Fano spaces described by GLSMs with superpotential, using the methods described earlier.

Unfortunately, for quantum K theory, in general we are only able to use the physical methods above to make weaker statements, essentially because the observables in the quantum K theory ring do not have well-defined $U(1)_R$ eigenvalues, so internal consistency is a weaker constraint. We describe some computations below.

For $\mathbb{P}^n[d]$, the quantum K theory ring determined solely by the Coulomb branch (σ fields) has the form [28, equ'n (5.7)], [34, equ'n (2.24)], namely

$$(1-x)^{n+1} \;=\; (-)^d q (1-x^d)^d, \tag{7.1}$$

and we also assume (3.2) that $\sigma \cdot t = 0$, or more explicitly $(1-x)t = 0$, for any Landau-Ginzburg orbifold state. (we use here $\sigma \sim 1-x$.) This will determine some, but not all, of the quantum K theory ring relations.

Let us briefly walk through two examples.

First, consider the quantum K theory of a hyperplane, $\mathbb{P}^n[1]$. For the same reasons discussed earlier in section 6.1 and in [28, section 5.2], the superpotential acts as a mass term, removing both p and x_{n+1}, reducing this theory to the $\mathbb{P}^{n-1}$ model, with quantum K theory given by that of $\mathbb{P}^{n-1}$.

Next, we consider degree two hypersurfaces. Our analysis in this case closely follows that of quantum cohomology for degree two hypersurfaces. In the Landau-Ginzburg phase, for a nondegenerate quadric Q, the x_i are massive, and there are $n+1$ of them.

Just as in our discussion of quantum cohomology of quadrics, if $n+1$ is odd, taking the $\mathbb{Z}_2$ orbifold results in a single vacuum, whereas if $n+1$ is even, taking the $\mathbb{Z}_2$ orbifold results in a pair of vacua. Specializing (7.1), the σ field contribution obeys

$$(1-x)^{n+1} \;=\; q\left(1-x^2\right)^2, \tag{7.2}$$

For $n+1$ odd, as there are no nontrivial Landau-Ginzburg orbifold vacua, one expects that this is the complete quantum K theory ring of $\mathbb{P}^n[2]$. Indeed, for $n+1$ odd, this matches known results for this case (see e.g. [28] and references therein), under the dictionary $1-x = \mathcal{O}_{\square}$.

For $n+1$ even, for the same reason as in quantum cohomology computations, physics predicts that the quantum K theory ring of $\mathbb{P}^n[2]$ has an extra generator t in (middle) degree $k = n-1$ and relations

$$(1-x)^{n+1} \;=\; q\left(1-x^2\right)^2, \tag{7.3}$$

$$(1-x)\cdot t \;=\; 0, \tag{7.4}$$

using (3.2), plus a relation involving t^2, to which we turn next.

To get the t^2 relation, we can try to follow the same procedure as for quantum cohomology. Write $y = 1-x$. Since the pertinent operators do not have well-defined $U(1)_R$ eigenvalues, we can only say that t^2 is some polynomial in y. Applying $yt^2 = 0$, requires that

$$yt^2 \propto f(y) \left(y^n - qy(y-2)^2 \right), \tag{7.5}$$

for some unknown function $f(y)$ We can then read off that

$$t^2 = f(y) \left(y^{n-1} - q(y-2)^2 \right) \tag{7.6}$$

(up to a proportionality factor which can be absorbed into the definition of t).

To completely determine the relation, we would need to determine the function $f(y)$. In quantum cohomology, $f(y)$ was determined on symmetry grounds, but that is not an option here. We leave the physical determination of $f(y)$ for future work.

We can compare to existing mathematics results[21] for the case $n+1$ even, which determine $f(y)$ to be a constant. Specifically a presentation of the quantum K theory ring of $\mathbb{P}^n[2]$ for $n+1$ even is given by

$$\mathbb{C}[y,t,q]/\langle y^{2k+1} - qy(y-2)^2, \quad yt = 0, \quad t^2 = (-1)^k (y^{2k} - q(y-2)^2) \rangle \tag{7.7}$$

under the dictionary

$$y = 1 - x = \mathcal{O}_{\square}, \quad t = \mathcal{O}_k - \mathcal{O}_{k-1,1}. \tag{7.8}$$

Here the notation $\mathcal{O}$ indicates a Schubert class

$$\mathcal{O}_0, \quad \mathcal{O}_1 = \mathcal{O}_{\square}, \quad \cdots, \quad \mathcal{O}_k, \quad \mathcal{O}_{k-1,1}, \quad \mathcal{O}_{k,1}, \quad \mathcal{O}_{k,k}, \tag{7.9}$$

where $n = 2k+1$. Linearizing this result reproduces the quantum cohomology ring (6.10) of the quadric $\mathbb{P}^n[2]$ for n odd, namely

$$\mathbb{C}[y,t,q]/\langle y^{2k+1} - 4qy, \quad yt = 0, \quad t^2 = (-1)^k (y^{2k} - 4q) \rangle \tag{7.10}$$

where $n = 2k+1$, $y = 1 - x$.

These matters have also been considered in [40], where further subtleties are discussed.

7.2 Quantum sheaf cohomology

Quantum sheaf cohomology [41] is a quantum-corrected sheaf cohomology ring, realized physically via twisted two-dimensional (0,2) supersymmetric theories. It can also be realized in gauged linear sigma models. For Fano toric varieties, it was computed using GLSM techniques in [42, 43], and mathematically in [44, 45]. See also [46, 47] for computations on Grassmannians, and [48] for computations on flag manifolds. See also [49–51] and references therein for reviews and additional references.

[21] We would like to thank L. Mihalcea and W. Xu for discussions of the quantum K theory ring of $\mathbb{P}^n[2]$.

For quantum sheaf cohomology, we expect an analogue of (3.2), the statement that $\sigma t = 0$ in OPE rings, essentially because of the (0,2) GLSM bosonic potential term $|E(\phi)|^2$, where on the (2,2) locus,

$$E(\phi_i) \; = \; \sigma Q_i \phi_i. \tag{7.11}$$

Proceeding as before, one expects that the OPE ring should include the product $\sigma t = 0$.

That said, the methods we have described so far do not suffice to describe the quantum cohomology ring of a (0,2) deformation of a hypersurface. The reason[22] is that ultimately (0,2) deformations of hypersurfaces are encoded in J deformations, whereas typical quantum sheaf cohomology computations have only been worked out for E deformations, as we shall discuss. Now, there is a duality that suggests one may be able to make progress: the Grassmannian $G(2,4) = \mathbb{P}^5[2]$, and (0,2) deformations of $G(2,4)$ are E-deformations and hence computable. Unfortunately, we shall see that the dictionary to $\mathbb{P}^5[2]$ is sufficiently obscure to make any possible conclusions about quantum sheaf cohomology of J deformations of $\mathbb{P}^5[2]$, much less hypersurfaces of other degrees, unreachable at this time. For completeness, we will quickly walk through both of these points.

7.2.1 Basic setup

In (0,2) language, in a $U(1)$ theory with chiral fields Φ of charge Q,

$$E \; = \; Q \sigma \Phi, \quad J \; = \; \frac{\partial W}{\partial \Phi}. \tag{7.12}$$

on the (2,2) locus. A quadric in $\mathbb{P}^n$ is described in (0,2) language on the (2,2) locus as

$$\begin{array}{rclrcl} E_p & = & -2\sigma p, & J_p & = & Q(x_i), \\ E_i & = & \sigma x_i, & J_i & = & p \, \partial_i Q(x_i). \end{array} \tag{7.13}$$

We can deform off the (2,2) locus by adding terms to the J's, such that

$$\sum_{\text{field } \alpha} E_\alpha J_\alpha \; = \; 0. \tag{7.14}$$

In the present case, for (0,2) deformations of $\mathbb{P}^n[2]$, this can be accomplished by taking

$$\begin{array}{rclrcl} E_p & = & -2\sigma p, & J_p & = & Q(x_i), \\ E_i & = & \sigma x_i, & J_i & = & p \, \partial_i Q(x_i) \; + \; \sum_j p A_{ij} x_j, \end{array} \tag{7.15}$$

[22] Well, one reason. As is well-known, for (2,2) cases, the dimension of the chiral ring is invariant under deformations. For (0,2) theories, matters are slightly more subtle. For example, in geometric realizations, the states correspond to sheaf cohomology, which can jump due to the semicontinuity theorem, see e.g. [52, section III.12]. One particularly vivid recent example of this subtlety in (0,2) theories along RG flows specifically is described in [53]. In any event, our central concerns here are different in nature.

where A_{ij} is an antisymmetric $(n+1)\times(n+1)$ matrix. It is straightforward to check that this satisfies (7.14).

We can simplify the expression above by observing that

$$\partial_i Q \;=\; \sum_j S_{ij} x_j \tag{7.16}$$

for a symmetric matrix S_{ij}, hence

$$J_i \;=\; p \sum_j M_{ij} x_j, \tag{7.17}$$

where M_{ij} is a general $(n+1)\times(n+1)$ matrix whose symmetric part is determined by derivatives of Q.

In any event, we now see the basic problem: current techniques for computing quantum sheaf cohomology are restricted to E deformations, but the (0,2) deformations of a hypersurface in $\mathbb{P}^n$ are instead deformations of J, for which no computational methods currently exist.

In the next subsection, we will discuss a possible workaround in a special case, for which a duality exists to a different description, in which the (0,2) deformations are realized by E deformations, not J deformations, and so quantum sheaf cohomology is computable.

7.2.2 Special case: $\mathbb{P}^5[2] = G(2,4)$

One family of cases for which we have results are (0,2) deformations of $\mathbb{P}^5[2]$, using the fact that $\mathbb{P}^5[2]$ is the same as the Grassmannian $G(2,4)$, and results on quantum sheaf cohomology for Grassmannians are known [46, 47]. We shall rewrite known results for $G(2,4)$ in the form of results for the quadric $\mathbb{P}^5[2]$. We shall see that when expressed in the language of the presentation $\mathbb{P}^5[2]$, the quantum sheaf cohomology ring can be described in terms of generators σ, t (on the Coulomb and Higgs branches, respectively), such that $\sigma\cdot t = 0$, as expected for a mixed branch OPE. Unfortunately, the form of t is obscure, and so we are not able to use this as a key for predicting results for more general hypersurfaces.

Before describing (0,2) deformations, let us first quickly review the (2,2) supersymmetric $G(2,4)$ and its (ordinary) quantum cohomology ring. The cohomology ring of any Grassmannian $G(k,n)$ is described additively by Young tableau fitting inside a $k\times n$ box, which for $G(2,4)$ means the cohomology ring has the additive generators

$$1, \quad \sigma_{(1)} = \sigma_{\square}, \quad \sigma_{(2)} = \sigma_{\square\square}, \quad \sigma_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}, \quad \sigma_{\begin{smallmatrix}\square\square\\ \square\;\;\end{smallmatrix}}, \quad \sigma_{\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}}, \tag{7.18}$$

with cohomological degree given by twice the number of boxes. Multiplicatively, this ring has relations

$$\begin{aligned} \sigma_{\square}\sigma_{\square\square} &= \sigma_{\square}\sigma_{\substack{\square\\\square}}, & (7.19)\\ \sigma_{\substack{\square\\\square}} &= \sigma_{(1)}^2 - \sigma_{(2)}, & (7.20)\\ \sigma_{\substack{\square\square\\\square}} &= \sigma_{(1)}\sigma_{\substack{\square\\\square}}, & (7.21)\\ \sigma_{\substack{\square\square\\\square\square}} &= \sigma_{(2)}^2 - \sigma_{(1)}^2\sigma_{(2)} + \sigma_{(1)}^2\sigma_{\substack{\square\\\square}}, & (7.22)\\ &= \sigma_{\substack{\square\\\square}}^2, & (7.23) \end{aligned}$$

$$\sigma_{\square}^4 + \sigma_{\square}^2\sigma_{\square\square} + \sigma_{\square\square}^2 - \sigma_{\square}^2\sigma_{\substack{\square\\\square}} = -q. \tag{7.24}$$

Note that there are two degree four elements, $\sigma_{\square\square}$ and $\sigma_{\substack{\square\\\square}}$, and as we now argue, the nontrivial Landau-Ginzburg field t arising in the presentation $\mathbb{P}^5[2]$ is a linear combination of these elements. It is natural to identify the $\mathbb{Z}_2$ quantum symmetry of the Landau-Ginzburg orbifold with a transpose operation on the Young tableau, which leaves elements of all other degrees invariant and exchanges $\sigma_{\square\square}$ and $\sigma_{\substack{\square\\\square}}$. Up to scalar multiples, we then identify

$$t = \sigma_{\square\square} - \sigma_{\substack{\square\\\square}}\ . \tag{7.25}$$

As there is only one generator of cohomological degree 2, namely $\sigma_{\square}$, we identify, up to scalar multiplication,

$$\sigma = \sigma_{\square}. \tag{7.26}$$

The reader should note that equation (7.19) then implies equation (3.2), namely,

$$\sigma \cdot t = 0. \tag{7.27}$$

Now, (0,2) deformations of a Grassmannian $G(k,n)$ are defined by a traceless $n \times n$ matrix B, as described in [46, 47]. The resulting quantum sheaf cohomology ring is defined in terms of functions I_j defined [46, section 2.2] to be the coefficients of the characteristic polynomial of B:

$$\det(tI + B) = \sum_{i=0}^{n} I_{n-i} t^i. \tag{7.28}$$

For example,

$$I_0 = 1, \quad I_1 = \operatorname{tr} B, \quad I_n = \det\, B. \tag{7.29}$$

For the special case of (0,2) deformations of $G(2,4)$, the quantum sheaf cohomology ring computed in [46] can be described in terms of the additive generators

$$1, \quad \sigma_{(1)}, \quad \sigma_{(2)}, \quad \sigma_{\substack{\square\\\square}}, \quad \sigma_{\substack{\square\square\\\square}}, \quad \sigma_{\substack{\square\square\\\square\square}}, \tag{7.30}$$

by the relations

$$\sigma_{(1)}\sigma_{(2)}\left[1+I_1+I_2+I_3\right] \;=\; \sigma_{(1)}\sigma_{\yng(1,1)}\left[1-I_2-I_3\right], \tag{7.31}$$

$$\sigma_{\yng(1,1)} \;=\; \sigma_{(1)}^2 - \sigma_{(2)}, \tag{7.32}$$

$$\sigma_{\yng(2,1)} \;=\; \sigma_{(1)}\sigma_{\yng(1,1)}, \tag{7.33}$$

$$\sigma_{\yng(2,2)} \;=\; \sigma_{(2)}^2 - \sigma_{(1)}^2\sigma_{(2)} + \sigma_{(1)}^2\sigma_{\yng(1,1)}, \tag{7.34}$$

$$\;=\; \sigma_{\yng(1,1)}^2, \tag{7.35}$$

$$\sigma_{(1)}^4\left[1+I_3+I_4\right] + \sigma_{(1)}^2\sigma_{(2)}\left[-1+I_1+I_2\right] + \sigma_{(2)}^2 - \sigma_{(1)}^2\sigma_{\yng(1,1)}\left[2+I_1\right] \;=\; -q. \tag{7.36}$$

As a consistency check, on the (2,2) locus, the relations above reduce to

$$\begin{aligned} \sigma_{(1)}\sigma_{(2)} \;=\; \sigma_{(1)}\sigma_{\yng(1,1)} \;&=\; \sigma_{\yng(2,1)}\;, \\ \sigma_{\yng(2,2)} \;=\; \sigma_{(2)}^2 \;&=\; \sigma_{\yng(1,1)}^2\;, \\ \sigma_{(2)}\sigma_{\yng(1,1)} \;&=\; q\;, \\ \sigma_{(1)}\sigma_{\yng(2,1)} \;&=\; \sigma_{\yng(2,2)} + q\;, \end{aligned} \tag{7.37}$$

which is a presentation of the ordinary quantum cohomology ring of $G(2,4)$.

If we naively identify

$$t \;=\; \sigma_{(2)} \;-\; \sigma_{\yng(1,1)}\;, \tag{7.38}$$

as we did on the (2,2) locus, then we see from (7.31) that $\sigma t \neq 0$. However, there is no fundamental reason why t should be identified in the same fashion as on the (2,2) locus. Instead, it is also clear from (7.31) that if we instead pick

$$t \;=\; \sigma_{(2)}\left[1+I_1+I_2+I_3\right] \;-\; \sigma_{\yng(1,1)}\left[1-I_2-I_3\right], \tag{7.39}$$

then $\sigma \cdot t = 0$, as expected.

We believe that the correct identification with the (0,2) deformation of $\mathbb{P}^5[2]$ is the relation (7.39) above, which is consistent with expectations. Clearly, renormalization group flow is being shifted by the (0,2) deformations in such a way as to make the identification of t somewhat obscure. Unfortunately, we do not know of a way to identify t from first principles, and so the arguments of this paper are not sufficient to make predictions for quantum sheaf cohomology in mixed Higgs-Coulomb phases.

8 Conclusions

In this paper, we have generalized the Coulomb-branch based computations of quantum cohomology described in [2] to cases with mixed Higgs-Coulomb branches, discussing in detail the case of GLSMs for hypersurfaces in projective spaces, $\mathbb{P}^n[d]$. We have described in detail how the vector space and product structures are reproduced by physical computations in the IR phase with both Higgs and Coulomb branches.

One of the conclusions of this work is that the cohomology of a Fano hypersurface has a decomposition corresponding to the Coulomb and Higgs sectors that is distinct from the decomposition of Dolbeault cohomology into primitive and non-primitive subspaces. It may be interesting to explore the mathematical implications of this Higgs/Coulomb decomposition further.

In terms of generalizing our results, perhaps the most immediate next step is to consider GLSMs for complete intersection Fano varieties in $\mathbb{P}^n$. In this case the Higgs sector of the IR phase is a hybrid theory, and a detailed comparison of the well-understood UV phase may clarify a number of aspects of hybrid CFTs. As these are perhaps the closest to a "generic" Higgs branch of a GLSM, it would be quite useful to have more detailed studies of their structure in a controlled setting.

It would also be interesting and reasonably straightforward to repeat the correlation function computations of section 5 using the technology of supersymmetric localization, as in e.g. [9], as well as to generalize our results to hypersurfaces in (resolved) weighted projective spaces. It should also be instructive to consider higher rank GLSMs, where the Higgs/Coulomb decomposition will become more intricate and there will be genuine mixed sectors, where some σ fields have large expectation values, while others are set to zero.

Finally, it remains an open problem to extend these ideas to quantum K theory and quantum sheaf cohomology computations for Fano spaces described by GLSMs with superpotential. In both cases, as we have seen, the ideas we have described in this paper are not sufficient to completely predict the OPE ring. We leave for future work the question of extending these methods to cover those cases.

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