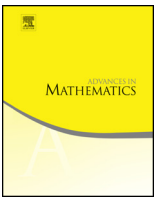
Advances in Mathematics 431 (2023) 109256

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim



$\mathscr{A} = \mathscr{U}$ for cluster algebras from moduli spaces of G-local systems

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ARTICLE INFO

Article history:
Received 28 February 2022
Received in revised form 4 May 2023
Accepted 2 August 2023
Available online 23 August 2023
Communicated by L.K. Williams

Keywords:
Cluster algebra
Moduli space of G-local systems
Skein algebra

ABSTRACT

For a finite-dimensional simple Lie algebra $\mathfrak{g}$ admitting a non-trivial minuscule representation and a connected marked surface Σ with at least two marked points and no punctures, we prove that the cluster algebra $\mathscr{A}_{\mathfrak{g},\Sigma}$ associated with the pair $(\mathfrak{g},\Sigma)$ coincides with the upper cluster algebra $\mathscr{U}_{\mathfrak{g},\Sigma}$. The proof is based on the fact that the function ring $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ of the moduli space of decorated twisted G-local systems on Σ is generated by matrix coefficients of Wilson lines introduced in [23]. As an application, we prove that the Muller-type skein algebras $\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}]$ [32,24,25] for $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$, or $\mathfrak{sp}_4$ are isomorphic to the cluster algebras $\mathscr{A}_{\mathfrak{g},\Sigma}$.

Contents

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https://doi.org/10.1016/j.aim.2023.109256
0001-8708/

1. Introduction

Cluster algebras $\mathscr{A}$, introduced by Fomin and Zelevinsky [15], are a class of commutative algebras with distinguished generators called cluster variables. The cluster variables are grouped into possibly infinitely many collections called *clusters*, which are related by a particular type of transition maps called *cluster transformations*. In [1], Berenstein, Fomin, and Zelevinsky introduced the upper cluster algebras $\mathscr{U}$, defined as the intersections of Laurent polynomial rings associated with clusters. The upper cluster algebras $\mathscr{U}$ are more natural than $\mathscr{A}$ from the perspective of geometry. The Laurent phenomenon of cluster algebras implies that $\mathscr{A} \subset \mathscr{U}$, but in general $\mathscr{A} \neq \mathscr{U}$.

The problem when $\mathscr{A} = \mathscr{U}$ is a mysterious but rather important question in cluster theory. For example, let Q be a quiver with a non-degenerate potential W. Motivated by the representation theory of quivers with potential [9,4], the paper [7] introduces the Caldero-Chapoton algebra $\mathscr{CC}_{Q,W}$. When Q has no loops or 2-cycles, we have

$$\mathscr{A}_Q \subset \mathscr{CC}_{Q,W} \subset \mathscr{U}_Q,$$

where $\mathscr{A}_Q$ and $\mathscr{U}_Q$ are the cluster algebra and the upper cluster algebra associated with Q, respectively. If $\mathscr{A}_Q = \mathscr{U}_Q$, then all the three aforementioned algebras are equal. As an application, after verifying a combinatorial condition on the existence of reddening mutation sequences of Q, the Caldero-Chapoton functions in $\mathscr{CC}_{Q,W}$ provide a natural linear basis on the cluster algebra $\mathscr{A}_Q$, called the generic basis [33]. We refer to [18] for more details on the significance of the $\mathscr{A} = \mathscr{U}$ problem in the study of generic basis of cluster algebras. We refer to [17] for another application of $\mathscr{A} = \mathscr{U}$ on the quantization of cluster algebras.

In this paper, we investigate the $\mathscr{A} = \mathscr{U}$ problem for cluster algebras from moduli spaces of G-local systems. Let G be the simply-connected complex Lie group associated with a simple Lie algebra $\mathfrak{g}$. Let Σ be a surface with punctures and marked points on its boundary. Fock and Goncharov [13] introduced a moduli space $\mathcal{A}_{G,\Sigma}$ of decorated G-local systems over Σ as an algebro-geometric avatar of higher Teichmüller spaces. The moduli space $\mathcal{A}_{G,\Sigma}$ carries a natural cluster structure, constructed by Fock and Goncharov [13] for SL_n, by Le [29] for other type of classical groups, and by Goncharov and the third named author [20] for general groups. As a consequence, the cluster structure on $\mathcal{A}_{G,\Sigma}$

gives rise to a cluster algebra $\mathscr{A}_{\mathfrak{g},\Sigma}$ and an upper cluster algebra $\mathscr{U}_{\mathfrak{g},\Sigma}$ over the ground field $\mathbb{C}$.[1]

Our first main result is as follows.

Theorem 1. *For a finite-dimensional simple Lie algebra $\mathfrak{g}$ admitting a non-trivial minuscule representation (namely, not of type E_8, F_4, G_2) and a connected marked surface Σ with at least two marked points and no punctures, we have*

$$\mathscr{A}_{\mathfrak{g},\Sigma} = \mathscr{U}_{\mathfrak{g},\Sigma}.$$

We prove Theorem 1 in Section 4. Our proof is based on the following geometric considerations on the moduli space $\mathcal{A}_{G,\Sigma}$:

(1) We first show that the function ring $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ is generated by matrix coefficients of *Wilson lines*, which are originally introduced in [23] on the closely related moduli space $\mathcal{P}_{G',\Sigma}$ [20] for adjoint groups G'. In this paper, we introduce a "lifted" version of Wilson lines defined on $\mathcal{A}^{\times}_{G,\Sigma}$, whose values are in the simply-connected group G.
(2) Then we show that the upper cluster algebra $\mathscr{U}_{\mathfrak{g},\Sigma}$ coincides with the function ring $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ over $\mathbb{C}$ by a covering argument up to codimension 2, similarly to the proof of [34, Theorem 1.1]. Therefore we obtain a geometric generating set of the upper cluster algebra provided by the Wilson lines.
(3) Finally, we show that the generalized minors of certain *simple* Wilson lines are single cluster variables, multiplied by several frozen variables (Proposition 4.12). Under the assumption of Theorem 1, these are enough to generate $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) = \mathscr{U}_{\mathfrak{g},\Sigma}$ and thus we get the desired inclusion $\mathscr{U}_{\mathfrak{g},\Sigma} \subset \mathscr{A}_{\mathfrak{g},\Sigma}$.

We remark here that Proposition 4.12 implies that the generalized minors of simple Wilson lines are contained in the *theta basis* [22], and hence they are universally Laurent polynomials with positive integral coefficients. This strengthens the positivity result in [23] for this particular class of Wilson lines and matrix coefficients.

We include a list of results preceding us:

- Muller [31] proved $\mathscr{A} = \mathscr{U}$ for locally acyclic cluster algebras. When $\mathfrak{g} = \mathfrak{sl}_2$ and Σ is unpunctured and contains at least two marked points, the cluster algebra $\mathscr{A}_{\mathfrak{sl}_2,\Sigma}$ is locally acyclic, and hence $\mathscr{A}_{\mathfrak{sl}_2,\Sigma} = \mathscr{U}_{\mathfrak{sl}_2,\Sigma}$.
- Canakci, Lee, and Schiffler [5] prove $\mathscr{A}_{\mathfrak{sl}_2,\Sigma} = \mathscr{U}_{\mathfrak{sl}_2,\Sigma}$, where Σ is an unpunctured surface with one marked point. We expect that the same result can be generalized to arbitrary $\mathfrak{g}$.

[1] In [3], Bucher, Machacek, and Shapiro show that the equality $\mathscr{A} = \mathscr{U}$ depends on the choice of ground ring. In this paper, we always choose the ground field $\mathbb{C}$. By an easy exercise of linear algebra, all the results of the paper can be generalized to $\mathbb{Q}$.

- Goodearl and Yakimov [21] prove a quantum analog of $\mathscr{A} = \mathscr{U}$ for cluster algebras associated with double Bruhat cells. We leave it for a future project to achieve a quantum analog of Theorem 1.
- Shen and Weng [35] prove $\mathscr{A} = \mathscr{U}$ for cluster algebras associated with double Bott-Samelson cells. Examples of double Bott-Samelson cells include all the double Bruhat cells and the augmentation varieties associated with positive-braid Legendrian links. Our Theorem 1 can be viewed as a generalization of the result of [35] from disks to surfaces.
- In the other direction, Berenstein, Fomin and Zelevinsky [1] prove $\mathscr{A}_{\mathfrak{sl}_2,\Sigma} \neq \mathscr{U}_{\mathfrak{sl}_2,\Sigma}$ when Σ is a closed torus with exactly one puncture. Ladkani [28] extended this result for closed surfaces of genus $g \geq 1$ with exactly one puncture. A very recent work of Moon and Wong [30] proves $\mathscr{A}_{\mathfrak{sl}_2,\Sigma} \neq \mathscr{U}_{\mathfrak{sl}_2,\Sigma}$ when Σ is a closed torus with $n \geq 1$ many punctures. They also conjecture $\mathscr{A}_{\mathfrak{sl}_2,\Sigma} \neq \mathscr{U}_{\mathfrak{sl}_2,\Sigma}$ when Σ is a closed surface of genus $g \geq 1$ with $n \geq 1$ many punctures. Therefore our assumption on the absence of punctures is crucial.

Closely related to the moduli space $\mathcal{A}_{G,\Sigma}$ are the *(stated) skein algebras*, which are generated by combinatorial objects called $\mathfrak{g}$-webs modulo a collection of explicit graphical relations. For $\mathfrak{g} = \mathfrak{sl}_2$, the connections between such skein algebras and the cluster algebra $\mathscr{A}_{\mathfrak{sl}_2,\Sigma}$ have been broadly studied in the literature (cf. [2,32,8]). More recently, the study of the relations of skein and cluster algebras and their web bases has been extended to $\mathfrak{sl}_3$ (cf. [10,11,27,24]). The $\mathfrak{sp}_4$-case is also studied in [25].

Following the notation of [24,25], we consider the boundary-localized skein algebra $\mathscr{S}^q_{\mathfrak{g},\Sigma}[\partial^{-1}]$ for $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{sp}_4$. Let $\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}]$ be its classical specialization $q = 1 \in \mathbb{C}$, which is a $\mathbb{C}$-algebra. As an application of Theorem 1, we prove

Theorem 2. *If $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{sl}_3$ or $\mathfrak{sp}_4$ and Σ is an unpunctured surface with at least two marked points, then the skein algebra $\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}]$ is isomorphic to the cluster algebra $\mathscr{A}_{\mathfrak{g},\Sigma}$.*

We prove Theorem 2 in Section 5. The $\mathfrak{sl}_2$-case is exactly the result obtained by [32]. For $\mathfrak{g} = \mathfrak{sl}_3$ or $\mathfrak{sp}_4$, the inclusion $\mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}] \subset \mathscr{A}_{\mathfrak{g},\Sigma}$ is proved in [24,25]. We further verify the inclusion $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) \subset \mathscr{S}_{\mathfrak{g},\Sigma}[\partial^{-1}]$ by verifying that the matrix entries of simple Wilson lines can be written as explicit $\mathfrak{g}$-webs, similarly to the step (3) in the proof of Theorem 1. Then the equality $\mathscr{A}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ over $\mathbb{C}$ allows us to combine these inclusions to get Theorem 2. Theorem 2 confirms a conjecture of the first named author and Yuasa in the classical setting ([24, Conjecture 3]).

Acknowledgments

T. I. is grateful to Wataru Yuasa for a valuable discussion on relations to skein algebras, and giving careful comments on a draft of this paper. The authors are grateful to the anonymous referees for the careful reading and the several comments that significantly improve our paper. T. I. is partially supported by JSPS KAKENHI Grant-in-Aid

for Research Activity Start-up (No. 20K22304). H. O. is supported by JSPS KAKENHI Grant-in-Aid for Early-Career Scientists (No. 19K14515). L. S. is supported by the Collaboration Grants for Mathematicians from the Simons Foundation (# 711926) and the NSF grant DMS-2200738.

2. The function ring $\mathcal{O}(G)$

2.1. Notations from Lie theory

In this section, we briefly recall basic terminologies in Lie theory. We refer the reader to, for example, [26] for missing definitions.

Let G be a simply-connected connected simple algebraic group over $\mathbb{C}$. Let B^+ and H be a Borel subgroup and a maximal torus contained in B^+, respectively. Let U^+ be the unipotent radical of B^+. Let

- $X^*(H) = \mathrm{Hom}(H, \mathbb{G}_m)$ be the weight lattice, $X_*(H) = \mathrm{Hom}(\mathbb{G}_m, H)$ the coweight lattice, and $\langle -, - \rangle$ the natural pairing

$$\langle -, - \rangle \colon \ X_*(H) \times X^*(H) \longrightarrow \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z};$$

- $\Phi \subset X^*(H)$ the root system of (G, H);
- $\Phi_+ \subset \Phi$ the set of positive roots consisting of the H-weights of the Lie algebra of U^+;
- $\{\alpha_s \mid s \in S\} \subset \Phi_+$ the set of simple roots, where S is the index set with $|S| = r$;
- $\{\alpha_s^\vee \mid s \in S\}$ the set of simple coroots.

For $s \in S$, let $\varpi_s \in X^*(H)$ be the s-th fundamental weight such that $\langle \alpha_t^\vee, \varpi_s \rangle = \delta_{st}$. In other words, we have $\alpha_t = \sum_{u \in S} C_{ut} \varpi_u$ for $t \in S$, where $C_{st} := \langle \alpha_s^\vee, \alpha_t \rangle \in \mathbb{Z}$. We have

$$X^*(H) = \sum_{s \in S} \mathbb{Z} \varpi_s.$$

The sub-lattice generated by α_s for $s \in S$ is called the *root lattice*.

For $h \in H$ and $\mu \in X^*(H)$, the evaluation of μ at h is denoted by h^μ. For $s \in S$, we have a pair of root homomorphisms $x_s, y_s \colon \mathbb{G}_a \to G$ such that

$$h x_s(t) h^{-1} = x_s(h^{\alpha_s} t), \qquad h y_s(t) h^{-1} = y_s(h^{-\alpha_s} t).$$

After a suitable normalization, we obtain a homomorphism $\varphi_s \colon SL_2 \to G$ such that

$$\varphi_s \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right) = x_s(a), \qquad \varphi_s \left(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \right) = y_s(a), \qquad \varphi_s \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = \alpha_s^\vee(a).$$

Weyl groups. Let $W(G) := N_G(H)/H$ denote the Weyl group of G, where $N_G(H)$ is the normalizer subgroup of H in G. For $s \in S$, we set

$$\overline{r}_s := \varphi_s \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \in N_G(H).$$

The elements $r_s := \overline{r}_s H \in W(G)$ have order 2, and give rise to a Coxeter generating set for $W(G)$ with the following presentation:

$$W(G) = \langle r_s \ (s \in S) \mid r_s^2 = 1, \quad (r_s r_t)^{m_{st}} = 1 \ (s,t \in S) \rangle,$$

where $m_{st} \in \mathbb{Z}$ is given by the following table

$$\begin{matrix} C_{st}C_{ts} : & 0 & 1 & 2 & 3 \\ m_{st} : & 2 & 3 & 4 & 6 \end{matrix}.$$

There is a left action of $W(G)$ on $X^*(H)$ induced from the (right) conjugation action of $N_G(H)$ on H. The action of each reflection r_s is given by

$$r_s.\mu = \mu - \langle \alpha_s^\vee, \mu \rangle \alpha_s, \qquad \forall s \in S, \quad \forall \mu \in X^*(H).$$

For $w \in W(G)$, a sequence $\boldsymbol{s} = (s_1, \dots, s_\ell)$ of elements of S is called a reduced word of w if $w = r_{s_1} \dots r_{s_\ell}$ and l is the smallest among all the sequences with this property. For a reduced word $\boldsymbol{s} = (s_1, \dots, s_\ell)$ of $w \in W(G)$, the number $l(w) := \ell$ is called the length of w, and set $\overline{w} := \overline{r}_{s_1} \dots \overline{r}_{s_\ell} \in N_G(H)$. Then it turns out that $\overline{w}$ does not depend on the choice of the reduced word.

Let $w_0 \in W(G)$ be the longest element of $W(G)$, and set $s_G := \overline{w_0}^2 \in N_G(H)$. It turns out that $s_G \in Z(G)$, and $s_G^2 = 1$ (cf. [13, §2]). We define an involution $S \to S, s \mapsto s^*$ by

$$\alpha_{s^*} = -w_0.\alpha_s.$$

The Dynkin involution. There exists an anti-involution $\mathsf{T} : G \to G, g \mapsto g^{\mathsf{T}}$ of the algebraic group G given by $\mathsf{T} \circ x_s = y_s$ and $h^{\mathsf{T}} = h$ for $s \in S$, $h \in H$. This is called the *transpose* in G. Let $* : G \to G, g \mapsto g^*$ be a group automorphism defined by

$$g \mapsto \overline{w}_0 (g^{-1})^{\mathsf{T}} \overline{w}_0^{-1}.$$

Then $(g^*)^* = g$ for all $g \in G$. This is called the *Dynkin involution* on G (cf. [19, (2)]).

Irreducible modules and matrix coefficients. Let V be a finite dimensional representation of G (over $\mathbb{C}$). For $f \in V^*$ and $v \in V$, we define the element $c_{f,v}^V \in \mathcal{O}(G)$ by

$$g \mapsto \langle f, g.v \rangle, \qquad \forall g \in G \tag{2.1}$$

An element of this form is called a *matrix coefficient*.

Set $X^*(H)_+ := \sum_{s\in S} \mathbb{Z}_{\geq 0}\varpi_s \subset X^*(H)$. For $\lambda \in X^*(H)_+$, let $V(\lambda)$ be the irreducible representation of G of highest weight λ. A fixed highest weight vector of $V(\lambda)$ is denoted by v_λ. For $\lambda^* = -w_0.\lambda$, there is a natural non-degenerate pairing

$$\langle -, - \rangle : \ V(\lambda^*) \times V(\lambda) \longrightarrow \mathbb{C},$$

which identify $V(\lambda^*)$ with the dual vector space of $V(\lambda)$. We further fix a lowest weight vector of $V(\lambda^*)$, denoted by f_{λ^*}, such that $\langle f_{\lambda^*}, v_\lambda \rangle = 1$.

For $\lambda \in X^*(H)_+$ and $w, w' \in W(G)$, the matrix coefficient

$$\Delta_{w\lambda, w'\lambda}(g) := \langle \overline{w}.f_{\lambda^*}, g\overline{w}'.v_\lambda \rangle \tag{2.2}$$

is called a *generalized minor*.

2.2. The generators of $\mathcal{O}(G)$

Proposition 2.1. *Let G be a semisimple algebraic group over $\mathbb{C}$, and $\rho\colon G \to GL(V)$ a faithful rational representation. Then the ring of regular functions $\mathcal{O}(G)$ is generated by the matrix coefficients of V.*

Proof. Consider the composition map $\det \circ \rho\colon G \to \mathbb{C}^*$. Then it gives a one-dimensional rational representation of G, and it must be trivial since G is semisimple. Therefore, $\rho(G) \subset \operatorname{Ker}\det = SL(V)$. Hence, by [36, Proposition 2.2.5], $\rho(G)$ is a Zariski closed subgroup of $SL(V)$. Here note that G is isomorphic to $\rho(G)$ as an algebraic group since ρ is faithful. By fixing a basis of V, $SL(V)$ is regarded as a Zariski closed set of $\mathbb{C}^{(\dim V)^2}$. Hence, $\rho(G)$ is also considered as a Zariski closed set of $\mathbb{C}^{(\dim V)^2}$, and there exists a surjective algebra homomorphism $R\colon \mathcal{O}(\mathbb{C}^{(\dim V)^2}) = \mathbb{C}[X_{ij} | 1 \leq i, j \leq \dim V] \twoheadrightarrow \mathcal{O}(\rho(G)) \simeq \mathcal{O}(G)$ corresponding to the restriction of the domain. By construction, each $R(X_{ij})$ $(1 \leq i, j \leq \dim V)$ is actually a matrix coefficient of V, hence the claim follows. □

Proposition 2.2. *Let G be a simply-connected simple algebraic group G over $\mathbb{C}$ of type A_n, B_n, C_n, D_n, E_6, or E_7. Then $\mathcal{O}(G)$ is generated by generalized minors.*

Proof. By Proposition 2.1, it suffices to show that there exist minuscule representations $V_1, \ldots, V_k$ such that their direct sum $V_1 \oplus \cdots \oplus V_k$ provides a faithful representation of G. Hence the result follows from the following known facts in representation theory (see, for example, [37, Appendix]):

- When G is of type A_n, the $(n+1)$-dimensional vector representation is a faithful minuscule representation.

- When G is of type B_n, the 2^n-dimensional spin representation is a faithful minuscule representation.
- When G is of type C_n, the $2n$-dimensional vector representation is a faithful minuscule representation.
- When G is of type D_n, there exist two mutually non-isomorphic 2^{n-1}-dimensional minuscule representations, called the half spin representations. Then the direct sum of these two representations is faithful.
- When G is of type E_6, there are two faithful minuscule representations of dimension 27.
- When G is of type E_7, there exists a faithful minuscule representation of dimension 56. □

Remark 2.3. The simple algebraic group of type E_8, F_4, G_2 does not admit non-trivial minuscule representations. This is why we exclude these types from the assumption. However, the statement makes sense also for these types and we do not know whether it holds in these cases.

2.3. Decorated flags and pinnings

The homogeneous spaces $\mathcal{A}_G := G/U^+$ and $\mathcal{B}_G := G/B^+$ are called the *principal affine space* and the *flag variety*, respectively. An element of $\mathcal{A}_G$ (resp. $\mathcal{B}_G$) is called a *decorated flag* (resp. a *flag*). There is a canonical G-equivariant projection $\pi : \mathcal{A}_G \to \mathcal{B}_G$. The basepoint of $\mathcal{A}_G$ is denoted by $[U^+]$. We also adopt the notation $[U^-] := \overline{w_0}.[U^+] \in \mathcal{A}_G$. The flag variety $\mathcal{B}_G$ is identified with the set of connected maximal solvable subgroups of G via $g.B^+ \mapsto gB^+g^{-1}$. The Cartan subgroup H acts on $\mathcal{A}_G$ from the right by $g.[U^+].h := gh.[U^+]$ for $g \in G$ and $h \in H$, making the projection $\pi : \mathcal{A}_G \to \mathcal{B}_G$ a principal H-bundle.

For $k \in \mathbb{Z}_{\geq 2}$, the configuration space of decorated flags is defined to be the stack

$$\mathrm{Conf}_k\mathcal{A}_G := [\overbrace{\mathcal{A}_G \times \cdots \times \mathcal{A}_G}^{k \text{ times}}/G],$$

where we consider the diagonal left action of G.

By the Bruhat decomposition $G = \bigcup_{w \in W(G)} U^+ H \overline{w} U^+$, any G-orbit in the space $\mathrm{Conf}_2\mathcal{A}_G$ has a unique representative of the form $(\mathsf{A}_1, \mathsf{A}_2) = (h.[U^+], \overline{w}.[U^+])$ for some $h \in H$ and $w \in W(G)$. The parameters $h(\mathsf{A}_1, \mathsf{A}_2) := h$ and $w(\mathsf{A}_1, \mathsf{A}_2) := w$ are called the *h-invariant* and the *w-distance* of $(\mathsf{A}_1, \mathsf{A}_2)$, respectively. They only depend on the G-orbit $[\mathsf{A}_1, \mathsf{A}_2]$. See [20, Section 2.2] for details on the properties of these parameters. A pair $(\mathsf{A}_1, \mathsf{A}_2)$ of decorated flags (or its G-orbit) is said to be *generic* if $w(\mathsf{A}_1, \mathsf{A}_2) = w_0$. Genericity only depends on the underlying flags.

Angle invariant. Consider the configuration space $\mathrm{Conf}(\mathcal{A}_G, \mathcal{B}_G, \mathcal{B}_G)$ parametrizing the triples $(\mathsf{A}_1, \mathsf{B}_2, \mathsf{B}_3) \in \mathcal{A}_G \times \mathcal{B}_G \times \mathcal{B}_G$ such that the pairs $(\mathsf{A}_1, \mathsf{B}_2)$ and $(\mathsf{A}_1, \mathsf{B}_3)$ are generic.

The G-orbit of such a triple has a unique representative of the form $(\mathsf{A}_1, \mathsf{B}_2, \mathsf{B}_3) = ([U^+], B^-, u_+B^-)$ for some $u \in U^+$. Then we define the *angle invariant* to be

$$\mathrm{an} : \mathrm{Conf}(\mathcal{A}_G, \mathcal{B}_G, \mathcal{B}_G) \to U^+, \quad [[U^+], B^-, u_+B^-] \mapsto u_+. \tag{2.3}$$

Composing the canonical character $\chi_s := \Delta_{\varpi_s, r_s\varpi_s} : U^+ \to \mathbb{C}$, we get the *potential functions* $W_s := \chi_s \circ \mathrm{an} : \mathrm{Conf}(\mathcal{A}_G, \mathcal{B}_G, \mathcal{B}_G) \to \mathbb{C}$ for $s \in S$.

Pinnings. Following [20], we define a *pinning* to be a generic pair $(\mathsf{A}_1, \mathsf{B}_2) \in \mathcal{A}_G \times \mathcal{B}_G$. Then the space $\mathcal{P}_G$ of pinnings is naturally a principal G-space. We will use the *standard pinning* $p_{\mathrm{std}} := ([U^+], B^-)$ as a basepoint of this space. We define the *opposite pinning* of $p = g.p_{\mathrm{std}}$ as $p^* := g\overline{w_0}^{-1}.p_{\mathrm{std}}$. Alternatively, a pinning is defined to be a generic pair $(\mathsf{A}_1, \mathsf{A}_2) \in \mathcal{A}_G \times \mathcal{A}_G$ such that $h(\mathsf{A}_1, \mathsf{A}_2)$ is trivial. Then we have $p_{\mathrm{std}} = ([U^+], [U^-])$ and $p^*_{\mathrm{std}} = (s_G.[U^-], [U^+])$.

3. Wilson lines on $\mathcal{A}^{\times}_{G,\mathbf{\Sigma}}$

3.1. Notations on marked surfaces

A *marked surface* $(\Sigma, \mathbb{M})$ consists of a compact oriented surface Σ and a fixed non-empty finite set $\mathbb{M} \subset \partial\Sigma$ of *marked points*. In particular, we do not consider marked points in the interior ("punctures"). In this paper, we always assume that

$$\text{each boundary component has at least one marked point.} \tag{3.1}$$

A connected component of $\partial\Sigma \setminus \mathbb{M}$ is called a *boundary interval*. The set of boundary intervals is denoted by $\mathbb{B} = \mathbb{B}(\Sigma)$. When no confusion can occur, we will simply denote a marked surface by Σ. We will always assume that

$$n(\Sigma) := -2\chi(\Sigma) + |\mathbb{M}| > 0. \tag{3.2}$$

This condition ensures that the marked surface Σ admits an ideal triangulation with $n(\Sigma)$ triangles.

The fiber bundle $T'\Sigma := T\Sigma \setminus (0\text{-section})$ with fiber $\mathbb{R}^2 \setminus \{0\}$ is called the *punctured tangent bundle*. The bundle projection $\pi : T'\Sigma \to \Sigma$ induces the exact sequence

$$1 \to \pi_1(\mathbb{R}^2 \setminus \{0\}) \to \pi_1(T'\Sigma) \xrightarrow{\pi_*} \pi_1(\Sigma) \to 1. \tag{3.3}$$

The generator of $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$ is denoted by $\boldsymbol{o}$.

3.2. The boundary-fundamental groupoid

Now we are going to clarify our topological set-up for the definition of Wilson lines on $\mathcal{A}^{\times}_{G,\Sigma}$, for which we need a little care on the fiber direction of $T'\Sigma$. We fix a nowhere-vanishing vector field v_{ori} on $\partial\Sigma$ which induces an orientation on $\partial\Sigma$ compatible with that induced from Σ. Then we get two embeddings $\iota_{\pm} : \partial\Sigma \to T'\Sigma, x \mapsto (x, \pm v_{\mathrm{ori}}(x))$. For each boundary interval $E \in \mathbb{B}$, fix a reference point $x_E \in E$ and define $E^{\pm} := \iota_{\pm}(x_E) \in T'\Sigma$. We regard the points $E^{\pm}$ as representatives of oriented boundary intervals.

Definition 3.1. The *boundary-fundamental groupoid* $\Pi_1(T'\Sigma, \mathbb{B}^{\pm})$ of $T'\Sigma$ is the groupoid where the objects are the points E^{ϵ} for $E \in \mathbb{B}$ and $\epsilon \in \{+, -\}$, and morphisms from $E_1^{\epsilon_1}$ to $E_2^{\epsilon_2}$ are based-homotopy classes of continuous curves in $T'\Sigma$ from $E_1^{\epsilon_1}$ to $E_2^{\epsilon_2}$. We call a morphism $[c] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$ in this groupoid a *framed arc class* in Σ. The composition of framed arc classes $[c_1] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$ and $[c_2] : E_2^{\epsilon_2} \to E_3^{\epsilon_3}$ is given by the concatenation $[c_1] * [c_2] : E_1^{\epsilon_1} \to E_3^{\epsilon_3}$.

Recall from [23] the groupoid $\Pi_1(\Sigma, \mathbb{B})$, whose objects are the points x_E and the morphisms are based-homotopy classes of continuous curves between them. We have a natural projection $\pi_* : \Pi_1(T'\Sigma, \mathbb{B}^{\pm}) \to \Pi_1(\Sigma, \mathbb{B})$.

Definition 3.2 *(transverse immersions and their standard lifts).* A *transverse immersion* is an immersed curve $\overline{c} : [0,1] \to \Sigma$ such that $\overline{c}(0) = x_{E_0}$ and $\overline{c}(1) = x_{E_1}$ for some boundary intervals $E_0, E_1 \in \mathbb{B}$, and transverse to $\partial\Sigma$. Then its *standard lift* is the continuous curve $c : [0,1] \to T'\Sigma$ from E_0^- to E_1^- obtained as the composite of the following two paths:

- the path $t \mapsto (\overline{c}(t), v_{\overline{c}}^{\perp}(t))$ in $T'\Sigma$, where $v_{\overline{c}}^{\perp}(t)$ is a nowhere-vanishing normal vector field along $\overline{c}$ which points toward the left side of $\overline{c}$, and such that $v_{\overline{c}}^{\perp}(0) = E_0^-$, $v_{\overline{c}}^{\perp}(1) = E_1^+$.
- a path from E_1^+ to E_1^- in the tangent space at x_{E_1}, which rotates the tangent vector clockwisely.

See Fig. 1. Note that the homotopy class $[c]$ does not depend on the choice of the vector field $v_{\overline{c}}^{\perp}$ which satisfies the condition.

Remark 3.3. The fundamental groupoid $\Pi_1(T'\Sigma, \mathbb{B}^{\pm})$ has appeared in [8, Section 8] in their study on stated skein algebras. Our standard lifts are their "good lifts" with respect to the negative orientation, which are chosen so that certain matrix coefficients of the Wilson line along them give rise to cluster variables. See Section 4.4.

For later use, we are going to give a good generating set of the fundamental groupoid $\Pi_1(T'\Sigma, \mathbb{B}^{\pm})$. For each boundary interval $E \in \mathbb{B}$, let $\sqrt{\boldsymbol{o}_E}^{\mathrm{in}}, \sqrt{\boldsymbol{o}_E}^{\mathrm{out}} : E^- \to E^+$ denote

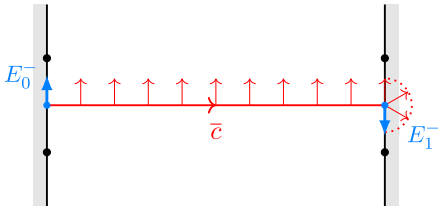


Fig. 1. The standard lift of a transverse immersion.

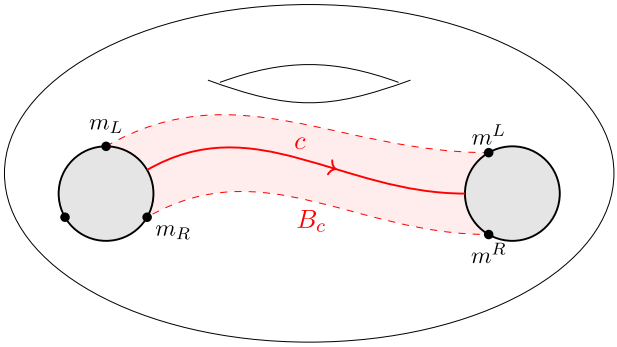


Fig. 2. A simple framed arc class $[c] : E_1^- \to E_2^-$ and a band neighborhood B_c. Here it is allowed that one of $m_L = m^L$ or $m_R = m^R$ holds.

the framed arc classes contained in the tangent space at x_E which go from E^- to E^+ clockwisely and counter-clockwisely, respectively. Then

$$\boldsymbol{o}_{E^-} := \sqrt{\boldsymbol{o}_E}^{\text{in}} * (\sqrt{\boldsymbol{o}_E}^{\text{out}})^{-1} \in \mathrm{Aut}(E^-) \quad \text{and}$$
$$\boldsymbol{o}_{E^+} := (\sqrt{\boldsymbol{o}_E}^{\text{out}})^{-1} * \sqrt{\boldsymbol{o}_E}^{\text{in}} \in \mathrm{Aut}(E^+)$$

represent the clockwise fiber loop. We say that a framed arc class $[c] : E_1^- \to E_2^-$ is *simple* if $E_1 \neq E_2$ and it is represented by the standard lift c of a transverse immersion $\overline{c}$ without self-intersection. Notice that it implies that there exists a band neighborhood B_c of $\overline{c}$ which gives an immersed quadrilateral which is embedded except for its vertices[2], as shown in Fig. 2.

Lemma 3.4. *The fundamental groupoid $\Pi_1(T'\Sigma, \mathbb{B}^{\pm})$ is finitely generated. Namely, there exists a finite set S of framed arc classes such that any framed arc class can be written as a finite concatenation of those in S and their inverses. Moreover if Σ has at least two marked points, then the generating set S can be chosen so that it consists of simple framed arc classes and $\sqrt{\boldsymbol{o}_E}^{\text{in}}, \sqrt{\boldsymbol{o}_E}^{\text{out}}$ for $E \in \mathbb{B}$.*

[2] This is seen as follows. Since the transverse immersion $\overline{c} : [0,1] \to \Sigma$ has no self-intersections, we can surely take an embedding $\iota : [0,1] \times [-\epsilon, \epsilon] \to \Sigma$ so that $\iota(t,0) = \overline{c}(t)$, $I_1 := \iota(\{0\} \times [-\epsilon, \epsilon]) \subset E_1$ and $I_2 := \iota(\{1\} \times [-\epsilon, \epsilon]) \subset E_2$. Since $E_1 \neq E_2$ by assumption, we can modify ('spread out') this embedding in small neighborhoods of E_1 and E_2 so that $I_1 = E_1$ and $I_2 = E_2$. The image of this modified map is B_c.

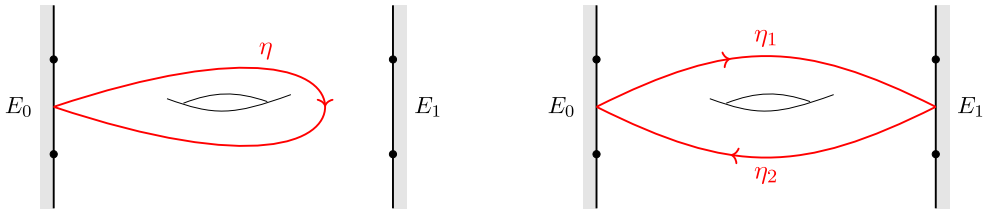


Fig. 3. Decomposition of a loop into two arcs.

Proof. Fix a boundary interval $E_0 \in \mathbb{B}$. We choose a collection of transverse immersions in Σ, as follows.

- Let $S_0 = \{(\overline{\alpha}_i)_{i=1}^g, (\overline{\beta}_i)_{i=1}^g, (\overline{\gamma}_j)_{j=1}^b\}$ be transversely-immersed loops based at $x_0 := x_{E_0}$ without self-intersections, whose homotopy classes generate the fundamental group $\pi_1(\Sigma, x_0)$.
- For each boundary interval $E \neq E_0$, let $\overline{\epsilon}_E$ be a transverse immersion without self-intersection running from x_0 to x_E.

We first claim that the standard lifts of these transverse immersions, together with the framed arc classes $\sqrt{\boldsymbol{o}_E}^{\mathrm{in}}, \sqrt{\boldsymbol{o}_E}^{\mathrm{out}}$ for $E \in \mathbb{B}$, generate the fundamental groupoid. Let $\mathcal{G} \subset \Pi_1(T'\Sigma, \mathbb{B}^{\pm})$ denote the sub-groupoid generated by these framed arc classes.

First note that the fiber loops $\boldsymbol{o}_{E^{\pm}}$ are contained in $\mathcal{G}$. Take an arbitary framed arc class $[c] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$. By concatenating the framed arc classes $\sqrt{\boldsymbol{o}_{E_i}}^{\mathrm{out}}$, $i = 1, 2$ or their inverses if necessary, we may assume $\epsilon_1 = \epsilon_2 = -$. Consider its projection $[\overline{c}] := \pi_*([c])$. Then the concatenation $[\overline{\epsilon}_{E_1}] * [\overline{c}] * [\overline{\epsilon}_{E_2}]^{-1}$ is an element of $\pi_1(\Sigma, x_0)$, which can be written as a concatenation of the elements in S_0. It implies that there exists a framed arc class $[c'] : E_1^- \to E_2^-$ that lies in $\mathcal{G}$ such that $\pi_*([c']) = \pi_*([c])$. Then by the exact sequence (3.3), the arc class $[c]$ can be written as a concatenation of $[c']$ and several powers of fiber loops. Hence $[c]$ belongs to $\mathcal{G}$. Thus the first assertion is proved.

Assume that Σ has at least two marked points. It implies that Σ has at least two boundary intervals, say $E_0 \neq E_1$. Then we can decompose each loop $\eta \in S_0$ into two simple arcs η_1 and η_2, as shown in Fig. 3. Therefore we can replace the standard lift of η with the standard lifts of η_1 and η_2 in the generating set, where the latter two are simple framed arc classes. Thus the second assertion is proved. □

Remark 3.5. For a transverse immersion $\overline{c} : [0,1] \to \Sigma$, let $\overline{c}^{\mathrm{op}} : [0,1] \to \Sigma$ be the transverse immersion in the opposite direction given by $\overline{c}^{\mathrm{op}}(t) := \overline{c}(1-t)$. Let $c : E_0^- \to E_1^-$, $c^{\mathrm{op}} : E_1^- \to E_0^-$ be the standard lifts of $\overline{c}$, $\overline{c}^{\mathrm{op}}$, respectively. Then they satisfy

$$[c] * [c^{\mathrm{op}}] = \boldsymbol{o}_{E_0^-}, \quad [c^{\mathrm{op}}] * [c] = \boldsymbol{o}_{E_1^-}.$$

In particular, note that $[c]^{-1} \neq [c^{\mathrm{op}}]$ as framed arc classes.

3.3. Wilson lines on $\mathcal{A}^{\times}_{G,\Sigma}$

Let G be a semisimple simply-connected algebraic group, Σ a marked surface without punctures. Fix an outward tangent vector field v_{out} on $\partial\Sigma$, which gives rise to an embedding $\iota_{\mathrm{out}} : \partial\Sigma \to T'\Sigma$, $x \mapsto (x, v_{\mathrm{out}}(x))$. Recall from [13, Definition 2.3] that a *twisted G-local system on* Σ is a G-local system on the punctured tangent bundle $T'\Sigma$ with the monodromy s_G along the fiber loop $\boldsymbol{o}$. A *decoration* of a twisted G-local system $\mathcal{L}$ is a flat section α of the associated bundle $\mathcal{L}_{\mathcal{A}} := \mathcal{L} \times_G \mathcal{A}_G$ defined over (a small neighborhood of) $\iota_{\mathrm{out}}(\partial\Sigma \setminus \{x_E\}_{E\in\mathbb{B}})$. In particular, the flat section is defined on a vicinity of the outward lift of each marked point. A decoration is said to be *generic* if for each $E \in \mathbb{B}$, the pair of flat sections defined near the endpoints of E gives rise to a generic pair of decorated flags when they are evaluated at a common point via a parallel-transport.

Definition 3.6 *([13, Definition 2.4]).* Let $\mathcal{A}_{G,\Sigma}$ denote the moduli space of decorated twisted G-local systems on Σ. The open substack consisting of the generically decorated twisted G-local systems is denoted by $\mathcal{A}^{\times}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$.

Let us briefly mention the description of $\mathcal{A}_{G,\Sigma}$ as a quotient stack. Fix a basepoint $\xi = (x, v) \in T'\Sigma$. A *rigidification* of a twisted G-local system $(\mathcal{L}, \alpha)$ at ξ is a choice of a point $s \in \mathcal{L}_\xi$. Let $A_{G,\Sigma}$ denote the set of isomorphism classes of rigidified twisted G-local systems $(\mathcal{L}, \alpha; s)$. The group G acts on $A_{G,\Sigma}$ by $g.[\mathcal{L}, \alpha; s] := [\mathcal{L}, \alpha; s.g]$ for $g \in G$. The following can be verified similarly to [13, Definition 2.2], with a little care on twistings:

Lemma 3.7. *The set $A_{G,\Sigma}$ has a natural structure of quasi-affine G-variety, isomorphic to*

$$\mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \times (\mathcal{A}_G)^{\mathbb{M}}.$$

Here $\mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \subset \mathrm{Hom}(\pi_1(T'\Sigma, \xi), G)$ *denotes the subspace such that* $\rho(\boldsymbol{o}_\xi) = s_G$*;* $\boldsymbol{o}_\xi \in \pi_1(T'\Sigma, \xi)$ *being the fiber loop based at* ξ*.*

Thus we get a stacky definition $\mathcal{A}_{G,\Sigma} := [A_{G,\Sigma}/G]$. In particular, its function ring is $\mathcal{O}(\mathcal{A}_{G,\Sigma}) = \mathcal{O}(A_{G,\Sigma})^G$.

Remark 3.8. Since $\pi_1(\Sigma)$ is a free group, the central extension (3.3) splits. Therefore we have non-canonical isomorphisms $\pi_1(T'\Sigma) \cong \pi_1(\Sigma) \times \mathbb{Z}$ and $\mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \cong \mathrm{Hom}(\pi_1(\Sigma), G) \cong G^{2g+b-1}$, where b denotes the number of boundary components of Σ. In particular, $\mathcal{O}(A_{G,\Sigma})$ is an integral domain, so is its subalgebra $\mathcal{O}(\mathcal{A}_{G,\Sigma})$. The field $\mathcal{K}(\mathcal{A}_{G,\Sigma})$ of rational functions is defined to be its field of fractions.

Example 3.9. Let $\Sigma = \mathbb{D}_k$ be a k-gon, which is a disk with k marked points on the boundary. Choose one distinguished marked point, and let $m_1, \ldots, m_k$ denote the marked points in the counter-clockwise order, where m_1 is the distinguished one. Fix a trivialization

$T'\mathbb{D}_k \cong \mathbb{D}_k \times (\mathbb{R}^2 \setminus \{0\})$, and let $v_j \in T'_x\mathbb{D}_k$ denote a tangent vector points toward m_j for $j = 1, \dots, k$. We may assume that $\xi = (x, v_1)$. Then we choose the following paths in $T'\mathbb{D}_k$:

- The straight line ϵ_1 from ξ to $\iota_{\mathrm{out}}(m_1)$;
- For $j = 2, \dots, k$, the path ϵ_j which first rotates from $\xi = (x, v_1)$ to (x, v_j) in the counter-clockwise direction inside the tangent space at x, and then goes straight from (x, v_j) to $\iota_{\mathrm{out}}(m_j)$.

See the left picture in Fig. 4. Let $(\mathcal{L}, \alpha; s)$ be a rigidified twisted G-local system. The rigidification s determines a trivialization $(\mathcal{L}_{\mathcal{A}})_\xi \cong \mathcal{A}_G$. We have a flat section α_j of $\mathcal{L}_{\mathcal{A}}$ defined near $\iota_{\mathrm{out}}(m_j)$ for $j = 1, \dots, k$. Via the parallel-transport along the path ϵ_j, we can evaluate it at the basepoint ξ, which gives rise to a decorated flag $\alpha_j(\xi) \in (\mathcal{L}_{\mathcal{A}})_\xi \cong \mathcal{A}_G$. Thus we get a G-equivariant isomorphism

$$A_{G,\mathbb{D}_k} \xrightarrow{\sim} \mathcal{A}_G^k, \quad (\mathcal{L}, \alpha; s) \mapsto (\alpha_1(\xi), \alpha_2(\xi), \dots, \alpha_k(\xi)),$$

which descends to an isomorphism $f_{m_1} : \mathcal{A}_{G,\mathbb{D}_k} \xrightarrow{\sim} \mathrm{Conf}_k\mathcal{A}_G := [\mathcal{A}_G^k/G]$ of stacks. Alternatively, one can think that we are choosing a branch cut ℓ in the fiber direction $\mathbb{R}^2 \setminus \{0\}$ as shown in Fig. 4 by waved orange line, and trivializing the twisted local systems on the contractible region $\mathbb{D}_k \times (\mathbb{R}^2 \setminus \ell)$. When we discuss the polygon case, we will only show this branch cut to indicate the isomorphism we use.

When we replace the distinguished marked point m_1 with m_2 and use a similar choice of paths as shown in the right picture in Fig. 4, then we get another isomorphism $f_{m_2} : \mathcal{A}_{G,\mathbb{D}_k} \xrightarrow{\sim} \mathrm{Conf}_k\mathcal{A}_G$. Then the coordinate transformation $f_{m_2} \circ f_{m_1}^{-1}$ is given by the *twisted cyclic shift*

$$\mathcal{S}_k : \mathrm{Conf}_k\mathcal{A}_G \xrightarrow{\sim} \mathrm{Conf}_k\mathcal{A}_G, \quad [\mathsf{A}_1, \dots, \mathsf{A}_{k-1}, \mathsf{A}_k] \mapsto [\mathsf{A}_2, \dots, \mathsf{A}_k, s_G.\mathsf{A}_1].$$

The substack of $\mathrm{Conf}_k\mathcal{A}_G$ corresponding to $\mathcal{A}^\times_{G,\mathbb{D}_k}$ is denoted by $\mathrm{Conf}^\times_k\mathcal{A}_G$.

Pinnings. For each boundary interval $E \in \mathbb{B}$, let $m_E^\pm \in \mathbb{M}$ be its endpoint in the direction $E^\pm$. Given a generic decorated twisted G-local system $(\mathcal{L}, \alpha)$, let $\alpha_E^\pm$ be the flat section defined near $\iota_{\mathrm{out}}(m_E^\pm)$. To the pair (α_E^-, α_E^+), we associate a pinning as follows. Parallel-transport the flat sections $\alpha_E^\pm$ along the line $\iota_{\mathrm{out}}(E)$ to the common point $\iota_{\mathrm{out}}(x_E)$, and continue to transport them along the path in the tangent space at x_E to the point $E^- = \iota_-(x_E)$ in the clockwise direction. Then we get a generic pair $(\mathsf{A}_E^-, \mathsf{A}_E^+)$ in $(\mathcal{L}_{\mathcal{A}})_{E^-}$, and the pinning $p_{E^-} := (\mathsf{A}_E^-, \pi(\mathsf{A}_E^+)) \in (\mathcal{L} \times_G \mathcal{P}_G)_{E^-}$. We may associate another pinning $p_{E^+} := (\hat{\mathsf{A}}_E^+, \pi(\hat{\mathsf{A}}_E^-)) \in (\mathcal{L} \times_G \mathcal{P}_G)_{E^+}$, where the pair $(\hat{\mathsf{A}}_E^-, \hat{\mathsf{A}}_E^+)$ in $(\mathcal{L}_{\mathcal{A}})_{E^+}$ is obtained by the parallel-transport of the pair $(\mathsf{A}_E^-, \mathsf{A}_E^+)$ via the outward path $\sqrt{\boldsymbol{o}_E}^{\mathrm{out}}$.

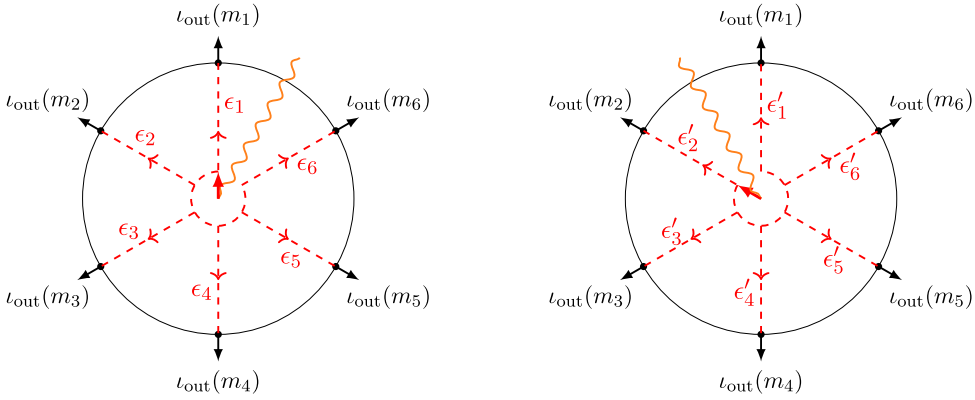


Fig. 4. Construction of isomorphisms $\mathcal{A}_{G,\mathbb{D}_k} \cong \mathrm{Conf}_k \mathcal{A}_G$ with $k = 6$. The flat sections defined near $\iota_{\mathrm{out}}(m_j)$ are parallel-transported along the dashed lines towards the basepoint ξ (shown by the solid arrow in the center of the disk).

Remark 3.10. Thanks to the absence of punctures, the decorated twisted G-local system $(\mathcal{L}, \alpha)$ can be uniquely recovered from the data $(\mathcal{L}, (p_{E^-})_{E \in \mathbb{B}})$ or $(\mathcal{L}, (p_{E^+})_{E \in \mathbb{B}})$. Thus the subspace $\mathcal{A}^{\times}_{G,\Sigma}$ can be identified with a variant $\mathcal{P}_{G,\Sigma}$ of the $\mathcal{P}$-type moduli space in [20] for simply-connected groups G. Then similarly to the arguments in [23, Remark 3.17], one can verify that $\mathcal{A}^{\times}_{G,\Sigma}$ is an algebraic variety.

Now we define the Wilson lines on $\mathcal{A}^{\times}_{G,\Sigma}$ in a similar manner as in [23]. Let $[c] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$ be a framed arc class, and $(\mathcal{L}, \alpha)$ a generically decorated twisted G-local system. Choose a local trivialization s_1 of $\mathcal{L}$ on a vicinity of $E_1^{\epsilon_1}$ so that $p_{E_1^{\epsilon_1}} = p_{\mathrm{std}}$. Extend s_1 via the parallel-transport along the path c, until the terminal point $E_2^{\epsilon_2}$. Then

$$p_{E_2^{\epsilon_2}} = g.p^*_{\mathrm{std}} = g\overline{w_0}^{-1}.p_{\mathrm{std}}$$

for a unique element $g \in G$ under this trivialization. Define $g_{[c]}([\mathcal{L}, \alpha]) := g$. The following can be verified in the same way as [23, Proposition 3.9]:

Lemma-Definition 3.11 *(Wilson lines).* For any framed arc class $[c] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$, the construction above produces a morphism

$$g_{[c]} : \mathcal{A}^{\times}_{G,\Sigma} \to G$$

of stacks, which we call the *Wilson line* along $[c]$. The morphism $g^{\mathrm{tw}}_{[c]} := g_{[c]}\overline{w_0}^{-1}$ is called the *twisted Wilson line*.

These Wilson lines are lifts of those introduced in [23].

Remark 3.12. At first, the twisted Wilson lines $g^{\mathrm{tw}}_{[c]}$ might look more natural than $g_{[c]}$. However, it turns out that the Wilson lines $g_{[c]}$ are compatible with the positivity structure, especially with the cluster structure. The twisted ones have negative coefficients

due to -1 contained in $\overline{w_0}$. The Wilson lines are also defined so as to fit in well with the amalgamation (cf. [23, Proposition 3.11]).

In order to state the relation to the Wilson lines in [23], let $G' = G/Z(G)$ be the adjoint group, and recall the moduli space $\mathcal{P}_{G',\Sigma}$ of framed G'-local systems with pinnings [20].

Proposition 3.13. *For any framed arc class $[c] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}^{\times}_{G,\Sigma} & \xrightarrow{g_{[c]}} & G \\ {\scriptstyle p_\Sigma}\downarrow & & \downarrow \\ \mathcal{P}_{G',\Sigma} & \xrightarrow[g_{[\overline{c}]}]{} & G'. \end{array}$$

Here $p_\Sigma : \mathcal{A}^{\times}_{G,\Sigma} \to \mathcal{P}_{G',\Sigma}$ denotes the projection given in [20, Section 9.2], $G \to G'$ is the canonical projection, and $g_{[\overline{c}]}$ is the Wilson line along the arc class $[\overline{c}] := \pi_*([c])$ [23].

The twisted Wilson lines contain the Wilson loops in the following sense:

Proposition 3.14. *For a free loop $|\gamma|$ on $T'\Sigma$, choose a representative c based at E^ϵ for some $E \in \mathbb{B}$ and $\epsilon \in \{+,-\}$, whose based homotopy class defines a framed arc class $[c] : E^\epsilon \to E^\epsilon$. Then the twisted Wilson line $g^{\mathrm{tw}}_{[c]}$ represents the Wilson line $\rho_{|\gamma|}$, namely, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}^{\times}_{G,\Sigma} & \xrightarrow{g^{\mathrm{tw}}_{[c]}} & G \\ & {\scriptstyle \rho_{|\gamma|}}\searrow & \downarrow \\ & & [G/\mathrm{Ad}G]. \end{array}$$

Here $G \to [G/\mathrm{Ad}G]$ denotes the canonical projection.

Proposition 3.15 *(Internal multiplicativity). For any framed arc classes $[c_1] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$ and $[c_2] : E_2^{\epsilon_2} \to E_3^{\epsilon_3}$, we have*

$$g^{\mathrm{tw}}_{[c_1]*[c_2]} = g^{\mathrm{tw}}_{[c_1]} g^{\mathrm{tw}}_{[c_2]} \quad \textit{or equivalently,} \quad g_{[c_1]*[c_2]} = g_{[c_1]} \overline{w_0}^{-1} g_{[c_2]}.$$

In other words, for any point $[\mathcal{L}, \alpha] \in \mathcal{A}^{\times}_{G,\Sigma}$, the twisted Wilson lines $g^{\mathrm{tw}}_\bullet([\mathcal{L}, \alpha])$ defines a morphism $\Pi_1(T'\Sigma, \mathbb{B}^\pm) \to G$ of groupoids.

Proof. Given $[\mathcal{L}, \alpha] \in \mathcal{A}^{\times}_{G,\Sigma}$, let us prove

$$g^{\mathrm{tw}}_{[c_1]*[c_2]}([\mathcal{L}, \alpha]) = g^{\mathrm{tw}}_{[c_1]}([\mathcal{L}, \alpha]) g^{\mathrm{tw}}_{[c_2]}([\mathcal{L}, \alpha]).$$

Take a local trivialization s_1 of $\mathcal{L}$ so that $(p_{E_1}, p_{E_2}) = (p_{\mathrm{std}}, g_1.p_{\mathrm{std}})$, where $g_1 := g^{\mathrm{tw}}_{[c_1]}([\mathcal{L}, \alpha])$. Under the local trivialization $s_2 := s_1.g_1$, the sections are given by $(p_{E_1}, p_{E_2}, p_{E_3}) = (g_1^{-1}.p_{\mathrm{std}}, p_{\mathrm{std}}, g_2.p_{\mathrm{std}})$, where $g_2 = g^{\mathrm{tw}}_{[c_2]}([\mathcal{L}, \alpha])$ by definition. Going back to the first trivialization, we get $(p_{E_1}, p_{E_3}) = (p_{\mathrm{std}}, g_1 g_2.p_{\mathrm{std}})$, and hence $g_1 g_2 = g^{\mathrm{tw}}_{[c_1]*[c_2]}([\mathcal{L}, \alpha])$. □

Remark 3.16. As a special case of Proposition 3.15, we have

$$g_{[c]^{-1}} = \overline{w_0} g_{[c]}^{-1} \overline{w_0} = (g_{[c]}^{\mathsf{T}})^* \cdot s_G.$$

In view of Remark 3.5 and $g^{\mathrm{tw}}_{\boldsymbol{o}_E} = s_G$, we have $g_{[c^{\mathrm{op}}]} = (g_{[c]}^{\mathsf{T}})^*$, which is a relation preserving the positivity.

Remark 3.17.

(1) Let $(\Sigma_0, \mathbb{M}_0) \subset (\Sigma, \mathbb{M})$ be a sub-marked surface, by which we mean an embedding $\Sigma_0 \subset \Sigma$ which restrict to $\mathbb{M}_0 \subset \mathbb{M}$. Then we have an obvious restriction morphism $\mathrm{res} : \mathcal{A}_{G,\Sigma} \to \mathcal{A}_{G,\Sigma_0}$. Suppose that a framed arc class $[c] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$ on Σ admits a representative contained in $T'\Sigma_0$ and E_1, E_2 are also boundary intervals of Σ_0. Then $[c]$ can be regarded as a morphism in $\Pi_1(T'\Sigma_0; \mathbb{B}_0^{\pm})$, and the Wilson line $g_{[c]}$ factors through $\mathcal{A}^{\times}_{G,\Sigma_0}$:

$$\begin{array}{ccc} \mathcal{A}^{\times}_{G,\Sigma} & \xrightarrow{\mathrm{res}} & \mathcal{A}^{\times}_{G,\Sigma_0} \\ {\scriptstyle g_{[c]}}\downarrow & & \downarrow{\scriptstyle g_{[c]}} \\ G & = & G. \end{array}$$

(2) For a subset $\Xi \subset \mathbb{B}$, let $\mathcal{A}^{\times}_{G,\Sigma;\Xi} \subset \mathcal{A}_{G,\Sigma}$ denote the open substack consisting of decorated twisted G-local systems such that the pair of flat sections associated to each $E \in \Xi$ is generic. We have $\mathcal{A}^{\times}_{G,\Sigma;\emptyset} = \mathcal{A}_{G,\Sigma}$ and $\mathcal{A}^{\times}_{G,\Sigma;\mathbb{B}} = \mathcal{A}^{\times}_{G,\Sigma}$. Then the Wilson line along a framed arc class $[c] : E_1^{\epsilon_1} \to E_2^{\epsilon_2}$ can be defined on $\mathcal{A}^{\times}_{G,\Sigma;\Xi}$ for any subset Ξ containing E_1 and E_2, which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}^{\times}_{G,\Sigma} & \xrightarrow{\mathrm{incl}} & \mathcal{A}^{\times}_{G,\Sigma;\Xi} \\ {\scriptstyle g_{[c]}}\downarrow & & \downarrow{\scriptstyle g_{[c]}} \\ G & = & G. \end{array}$$

Definition 3.18. We give some special names to certain Wilson lines, as follows.

(1) For $E \in \mathbb{B}$, we call the $g_{\sqrt{\boldsymbol{o}_E}^{\mathrm{in}}}$ and $g_{\sqrt{\boldsymbol{o}_E}^{\mathrm{out}}}$ the *boundary Wilson lines* along E.
(2) For $m \in \mathbb{M}$, let $[c_m] : E_1^- \to E_2^+$ denote the framed arc class as shown in Fig. 5 around m. Then we call $g_{[c_m]}$ the *corner Wilson line* around m.

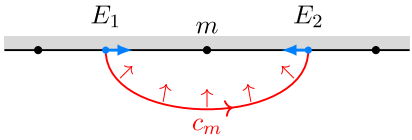


Fig. 5. The corner Wilson line around $m \in \mathbb{M}$.

(3) We call the Wilson lines along simple framed arc classes the *simple Wilson lines*.

Recall the angle invariant (2.3). For a marked point $m \in \mathbb{M}$, let T_m be the unique triangle which contains m as a vertex and the two boundary intervals E_1, E_2 incident to m as its sides. Then by Remark 3.17 above, we have $\mathcal{A}^{\times}_{G,\Sigma} \xrightarrow{\text{res}} \mathcal{A}^{\times}_{G,T_m;\{E_1,E_2\}} \cong \mathrm{Conf}(\mathcal{A}_G, \mathcal{B}_G, \mathcal{B}_G)$. Here the latter isomorphism is obtained by parallel-transport of flags through the interval $\iota_{\mathrm{out}}(\overline{E_1 \cap E_2})$, or equivalently, with respect to the branch-cut on T_m intersecting the opposite side of m. Then we get the angle invariant $\mathrm{an}_m : \mathcal{A}^{\times}_{G,\Sigma} \to U^+$ associated with m as the pull-back of the angle invariant via this morphism.

Proposition 3.19.

(1) *For $E \in \mathbb{B}$, the boundary Wilson line gives the edge invariants $g_{\sqrt{\boldsymbol{o}_E}^{\mathrm{in}}} = h(\mathsf{A}^-_E, \mathsf{A}^+_E)^{-1}$ and $g_{\sqrt{\boldsymbol{o}_E}^{\mathrm{out}}} = (h(\mathsf{A}^+_E, \mathsf{A}^-_E)^*)^{-1}$, where $\mathsf{A}^{\pm}_E$ is the decorated flag assigned to $m^{\pm}_E$.*
(2) *For $m \in \mathbb{M}$, the corner Wilson line gives the angle invariant $g_{[c_m]} = \mathrm{an}_m \cdot \overline{w_0}$ twisted by $\overline{w_0}$.*

Proof. In view of Remark 3.17 (1)(2), the proof reduces to the local computations in Example 3.20 below. □

Example 3.20. Let us consider the case $\Sigma = \mathbb{D}_3$, and choose an isomorphism $\mathcal{A}_{G,\mathbb{D}_3} \cong \mathrm{Conf}_3\mathcal{A}_G$ as described in Example 3.9, corresponding to the branch cut shown in Fig. 6. The boundary intervals are denoted by E_1, E_2, E_3 as shown there. Notice that an arbitrary G-orbit in $\mathrm{Conf}^{\times}_3\mathcal{A}_G$ has a unique representative of the form

$$(\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3) = ([U^+], \overline{w_0}^{-1} h_1.[U^+], u_+ h_2^{-1} \overline{w_0}.[U^+]) \tag{3.4}$$

for $h_1, h_2 \in H$ and $u_+ \in U^+_*$. In this parametrization, we have $h_1 = h(\mathsf{A}_2, \mathsf{A}_1)$, $h_2 = h(\mathsf{A}_1, \mathsf{A}_3)$ and $u_+ = \mathrm{an}_m$. This parametrization can be extended to a parametrization of triples $(\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3)$ such that the pairs $(\mathsf{A}_1, \mathsf{A}_2)$ and $(\mathsf{A}_1, \mathsf{A}_3)$ are generic by $(h_1, h_2, u_+) \in H \times H \times U^+$. Let $\mathsf{B}_i := \pi(\mathsf{A}_i)$ denote the underlying flag for $i = 1, 2, 3$. Then we can compute the associated pinnings as

$$\begin{aligned} p^-_{E_1} &= (\mathsf{A}_1, \mathsf{B}_2) = p_{\mathrm{std}}, \\ p^+_{E_1} &= (\mathsf{A}_2, \mathsf{B}_1) = \overline{w_0}^{-1} h_1.p_{\mathrm{std}}, \\ p^+_{E_2} &= (\mathsf{A}_1, \mathsf{B}_3) = u_+.p_{\mathrm{std}}, \end{aligned}$$

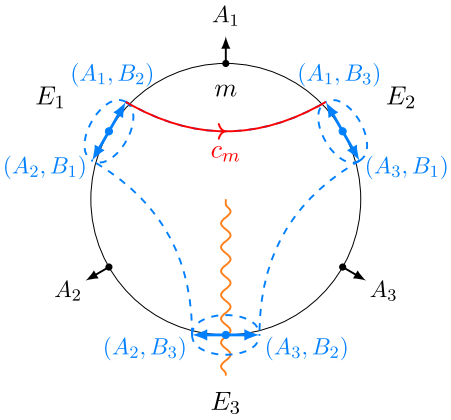


Fig. 6. Computation of Wilson lines on a triangle. Here the framed arc classes giving rise to boundary and corner Wilson lines are shown in dashed blue lines.

$$p^-_{E_2} = (\mathsf{A}_3, \mathsf{B}_1) = u_+ h_2^{-1} \overline{w_0}.p_{\mathrm{std}}$$

Using the relation $p_{\mathrm{std}} = \overline{w_0}.p^*_{\mathrm{std}}$, we get:

- $g_{\sqrt{\boldsymbol{o}_{E_1}}^{\mathrm{out}}} = w_0(h_1) = (h(\mathsf{A}_2, \mathsf{A}_1)^*)^{-1}$, which implies that $g_{\sqrt{\boldsymbol{o}_{E_1}}^{\mathrm{in}}} = w_0(h_1)s_G = h(\mathsf{A}_1, \mathsf{A}_2)^{-1}$.
- $g_{[c_m]} = u_+ \overline{w_0} = \mathrm{an}_m \cdot \overline{w_0}$,

from which one can confirm the statements in Proposition 3.19. Also note that the framed arc class $[c] := [c_m] * [\sqrt{\boldsymbol{o}_{E_2}}^{\mathrm{out}}]^{-1} : E_1^- \to E_2^-$ is simple, and that $g_{[c]} = u_+ h_2^{-1} s_G$.

By Lemma 3.4, $\mathrm{Hom}(\Pi_1(T'\Sigma, \mathbb{B}^\pm), G)$ has a natural structure of affine variety.

Theorem 3.21. *The morphism*

$$\mathcal{A}^\times_{G,\Sigma} \to \mathrm{Hom}\left(\Pi_1(T'\Sigma, \mathbb{B}^\pm), G\right), \quad [\mathcal{L}, \alpha] \mapsto g^{\mathrm{tw}}_\bullet([\mathcal{L}, \alpha])$$

is a closed embedding. The image is characterized by the conditions

$$g^{\mathrm{tw}}_{\boldsymbol{o}_{E^\pm}} = s_G, \quad g^{\mathrm{tw}}_{\sqrt{\boldsymbol{o}_E}^{\mathrm{in}}} \in H\overline{w_0}^{-1}, \quad g^{\mathrm{tw}}_{[c_m]} \in U^+$$

for $E \in \mathbb{B}$ and $m \in \mathbb{M}$.

Proof. Let us prove that a decorated twisted G-local system can be characterized by its (twisted) Wilson lines. Fix a basepoint $E_0^{\epsilon_0}$ for some $E_0 \in \mathbb{B}$ and $\epsilon \in \{+, -\}$. Then by Proposition 3.14, the monodromy homomorphism $\pi_1(T'\Sigma, E_0^{\epsilon_0}) \to G$ of the twisted G-local system can be reconstructed from the twisted Wilson lines. For a marked point $m \in \mathbb{M}$, choose an object E^ϵ, where $E \in \mathbb{B}$ is a boundary interval incident to m and

E^ϵ is the one pointing to m among $E^\pm$. Then the decoration assigned to m can be read off from the Wilson line along a framed arc class of the form $[c] : E_0^{\epsilon_0} \to E^\epsilon$. Since the image is characterized by closed conditions, the assertion is proved. □

Note that the (twisted) Wilson lines define ring homomorphisms

$$g_{[c]}^* : \mathcal{O}(G) \to \mathcal{O}(\mathcal{A}_{G,\Sigma}^\times), \quad (g_{[c]}^{\mathrm{tw}})^* : \mathcal{O}(G) \to \mathcal{O}(\mathcal{A}_{G,\Sigma}^\times).$$

For a matrix coefficient $c_{f,v}^V \in \mathcal{O}(G)$, we write $c_{f,v}^V(g_{[c]}) := g_{[c]}^*(c_{f,v}^V)$ and $c_{f,v}^V(g_{[c]}^{\mathrm{tw}}) := (g_{[c]}^{\mathrm{tw}})^*(c_{f,v}^V) = c_{f,\overline{w_0}.v}^V(g_{[c]})$.

Combining the previous theorem with Lemma 3.4, we get:

Corollary 3.22. *The function ring $\mathcal{O}(\mathcal{A}_{G,\Sigma}^\times)$ is generated by the matrix coefficients of Wilson lines. Moreover if Σ has at least two marked points, then $\mathcal{O}(\mathcal{A}_{G,\Sigma}^\times)$ is generated by matrix coefficients of simple Wilson lines and boundary Wilson lines.*

Remark 3.23. When Σ has punctures, the moduli space $\mathrm{Loc}_{G,\Sigma}^{\mathrm{un}}$ obtained from $\mathcal{A}_{G,\Sigma}^\times$ by forgetting the decorations on punctures is similarly embedded into $\mathrm{Hom}\,(\Pi_1(T'\Sigma, \mathbb{B}^\pm), G)$.

4. Equality of cluster and upper cluster algebras

4.1. Generalities on cluster algebras

We rapidly recall the necessary definitions of cluster algebras $\mathscr{A}$ and upper cluster algebras $\mathscr{U}$ following the notations of [14].

Let $\mathcal{F} = \mathbb{C}(A_1, \ldots, A_n)$ be the field of rational functions in n many independent variables $A_1, \ldots, A_n$ with coefficients in $\mathbb{C}$. Fix a positive integer $m \leq n$. Let $\varepsilon = (\varepsilon_{ij})$ be an $m \times n$ integer matrix such that its $m \times m$ submatrix given by the first m columns is skew-symmetrizable. The set $\mathbf{A} = \{A_1, \ldots, A_n\}$ is called a *cluster chart*. The pair $\mathbf{i} = (\mathbf{A}, \varepsilon)$ is called a *seed* in $\mathcal{F}$. Let $\mathbb{L}_{\mathbf{i}} = \mathbb{C}[A_1^{\pm 1}, \ldots, A_n^{\pm 1}] \subset \mathcal{F}$ be the ring of Laurent polynomials in $A_1, \ldots, A_n$.

For $1 \leq k \leq m$, the *seed mutation* of $\mathbf{i}$ in the direction k produces a new seed $\mathbf{i}_k := (\{A_1', \ldots, A_n'\}, \varepsilon')$ as follows:

$$A_i' = \begin{cases} A_i & \text{if } i \neq k, \\ A_k^{-1}\left(\prod_{j|\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}\right) & \text{otherwise,} \end{cases}$$

$$\varepsilon_{ij}' = \begin{cases} -\varepsilon_{ij}, & \text{if } k \in \{i,j\}, \\ \varepsilon_{ij}, & \text{if } \varepsilon_{ik}\varepsilon_{kj} < 0, \quad k \notin \{i,j\}, \\ \varepsilon_{ij} + |\varepsilon_{ik}|\varepsilon_{kj}, & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0, \quad k \notin \{i,j\}. \end{cases}$$

We say that a seed $\mathbf{i}'$ is mutation equivalent to $\mathbf{i}$, and denote by $\mathbf{i}' \sim \mathbf{i}$, if $\mathbf{i}'$ can be obtained from $\mathbf{i}$ by a sequence of seed mutations. The variables A_i' in each $\mathbf{i}'$ are called cluster

variables. Note that the cluster variables $A_{m+1}, \dots, A_n$ are invariant under mutations and are called frozen variables.

Let us start with an initial seed $\mathbf{i}$. Following [1], there are three versions of commutative algebras associated with $\mathbf{i}$.

Definition 4.1. The *upper bound* associated with a seed $\mathbf{i}$ is a ring

$$\overline{\mathscr{U}}(\mathbf{i}) = \mathbb{L}_{\mathbf{i}} \cap \mathbb{L}_{\mathbf{i}_1} \cap \ldots \cap \mathbb{L}_{\mathbf{i}_m}.$$

The *upper cluster algebra* associated with $\mathbf{i}$ is the intersection of Laurent polynomials for all seeds $\mathbf{i}'$ that are mutation equivalent to $\mathbf{i}$:

$$\mathscr{U}(\mathbf{i}) = \bigcap_{\mathbf{i}' \sim \mathbf{i}} \mathbb{L}_{\mathbf{i}'}.$$

The *cluster algebra* $\mathscr{A}(\mathbf{i})$ is the unital $\mathbb{C}$-subalgebra of $\mathcal{F}$ generated by the cluster variables and the inverses $A_{m+1}^{-1}, \dots, A_n^{-1}$ of the frozen variables.

We frequently write $\mathscr{U}$ and $\mathscr{A}$ instead of $\mathscr{U}(\mathbf{i})$ and $\mathscr{A}(\mathbf{i})$ when there is no confusion.

We have the following inclusion relations

$$\mathscr{A}(\mathbf{i}) \subseteq \mathscr{U}(\mathbf{i}) \subseteq \overline{\mathscr{U}}(\mathbf{i}),$$

where the first inclusion is a consequence of the Laurent phenomenon of the cluster variables [15], and the second inclusion is by definition.

The following result of [1, Corollary 1.9] will be useful in this paper.

Proposition 4.2. *If the exchange matrix ε in $\mathbf{i}$ has full rank m, then the upper cluster algebra $\mathscr{U}(\mathbf{i})$ coincides with the upper bound $\overline{\mathscr{U}}(\mathbf{i})$.*

4.2. The upper cluster algebra coincides with the function ring $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$

Let $\mathcal{T}$ be an ideal triangulation of the marked surface Σ. Let $e(\mathcal{T})$ and $t(\mathcal{T})$ denote the sets of edges and triangles of $\mathcal{T}$. Note that

$$|t(\mathcal{T})| = -2\chi(\Sigma) + |\mathbb{M}|, \qquad |e(\mathcal{T})| = -3\chi(\Sigma) + 2|\mathbb{M}|.$$

For each triangle $T \in t(\mathcal{T})$, we choose a vertex v_T of T together with a reduced decomposition $\mathbf{s}_T$ of w_0. The data

$$\boldsymbol{\mathcal{T}} := (\mathcal{T}, \{v_T\}, \{\mathbf{s}_T\})$$

is called a *decorated ideal triangulation*. Following the construction of [20], every $\boldsymbol{\mathcal{T}}$ gives rise to a seed $\mathbf{i}_{\boldsymbol{\mathcal{T}}}$ in the field $\mathcal{K}(\mathcal{A}_{G,\Sigma})$ of rational functions (recall Remark 3.8). The seeds

obtained from different $\boldsymbol{\mathcal{T}}$ are mutation equivalent, and thus give rise to the canonical cluster algebras

$$\mathscr{A}_{\mathfrak{g},\Sigma} \subseteq \mathscr{U}_{\mathfrak{g},\Sigma} \subset \mathcal{K}(\mathcal{A}_{G,\Sigma}).$$

The exchange matrix ε in each $\mathbf{i}_{\boldsymbol{\mathcal{T}}}$ is an $m \times n$ matrix, where

$$n = |e(\mathcal{T})|r + |t(\mathcal{T})|(l(w_0) - r), \qquad n - m = |\mathbb{M}|r$$

with $r := \operatorname{rank} G = |S|$. Briefly speaking, the construction goes as follows:

- the ideal triangulation $\mathcal{T}$ gives rise to a decomposition of the moduli space $\mathcal{A}_{G,\Sigma}$ into the pieces $\mathcal{A}_{G,T}$ for $T \in t(\mathcal{T})$;
- the vertex v_T determines an isomorphism $f_{v_T} : \mathcal{A}_{G,T} \xrightarrow{\sim} \mathrm{Conf}_3\mathcal{A}_G$ as in Example 3.9;
- the pull-back of the coordinate system on $\mathrm{Conf}_3\mathcal{A}_G$ associated with the reduced word $\boldsymbol{s}_T$ gives a coordinate system on $\mathcal{A}_{G,T}$ for all $T \in t(\mathcal{T})$, which glue together to give a coordinate system on $\mathcal{A}_{G,\Sigma}$, *i.e.*, an open dense embedding

$$i_{\boldsymbol{\mathcal{T}}} : \ (\mathbb{C}^\times)^n \longrightarrow \mathcal{A}^\times_{G,\Sigma}.$$

The transition maps $i^{-1}_{\boldsymbol{\mathcal{T}}'} \circ i_{\boldsymbol{\mathcal{T}}}$ for different ideal triangulations are given by sequences of seed mutations. For details, we refer the reader to [20]. In Section 4.3, we will review the cluster K_2-structure on $\mathcal{A}^\times_{G,Q}$ when $\Sigma = Q$ is a quadrilateral, from which the coordinates on triangles and edges can be also read.

We are going to prove:

Theorem 4.3. *For any marked surface Σ satisfying the assumptions in Section 3.1, the upper cluster algebra $\mathscr{U}_{\mathfrak{g},\Sigma}$ coincides with the function ring $\mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$.*

The proof goes through the same steps as the proof of [34, Theorem 1.1].

Lemma 4.4. *The exchange matrix ε in each $\mathbf{i}_{\boldsymbol{\mathcal{T}}}$ is of full rank.*

Proof. Let G' be the associated adjoint group of G. Fock and Goncharov [13] also considered a moduli space $\mathcal{X}_{G',\Sigma}$, called the moduli space of framed G'-local systems. Similarly to $\mathcal{A}^\times_{G,\Sigma}$, after imposing a generic condition on each boundary interval of Σ, we get an open subspace $\mathcal{X}^\times_{G',\Sigma} \subset \mathcal{X}_{G',\Sigma}$. Every $\boldsymbol{\mathcal{T}}$ equips $\mathcal{X}^\times_{G',\Sigma}$ with a collection of cluster Poisson coordinates $\{X_i \mid 1 \le i \le m\}$. Let $\{A_j \mid 1 \le j \le n\}$ be the cluster coordinates on $\mathcal{A}^\times_{G,\Sigma}$ associated with $\boldsymbol{\mathcal{T}}$. Following [13] and [20], there is a natural map

$$p : \mathcal{A}^\times_{G,\Sigma} \to \mathcal{X}^\times_{G',\Sigma}$$

such that

$$p^* X_i = \prod_{j=1}^{n} A_j^{\varepsilon_{ij}}. \tag{4.1}$$

When Σ has no punctures, the map p is surjective. Therefore the ring homomorphism p^* in (4.1) is injective, and ε is of full rank. □

Lemma 4.5. *Theorem 4.3 holds when $\Sigma = \mathbb{D}_k$ is a disk with k marked points.*

Proof. Let H be the Cartan subgroup of G. If $k = 3$ or 4, then

$$\mathcal{A}^{\times}_{G,\mathbb{D}_3} \cong G^{e,w_0} \times H, \qquad \mathcal{A}^{\times}_{G,\mathbb{D}_4} \cong G^{w_0,w_0} \times H^2,$$

where G^{e,w_0} and G^{w_0,w_0} are double Bruhat cells. After a careful comparison between the cluster structures on double Bruhat cells and on $\mathcal{A}^{\times}_{G,\Sigma}$, the lemma is a direct consequence of [1, Theorem 2.10]. In general, we have

$$\mathcal{A}^{\times}_{G,\mathbb{D}_k} \cong \mathrm{Conf}^{e}_{w_0^{k-2}}(\mathcal{A}_{\mathrm{sc}}) \times H^{k-2},$$

where $\mathrm{Conf}^{e}_{w_0^{k-2}}(\mathcal{A}_{\mathrm{sc}})$ is a double Bott-Samelson variety in [35]. The lemma follows from [35, Theorem 1.1]. □

Fix a decorated ideal triangulation $\boldsymbol{\mathcal{T}}$. Let $E \in e(\mathcal{T})$ be a diagonal (i.e., an internal edge). Let $\mathbf{i}_{\boldsymbol{\mathcal{T}},E}$ be the seed obtained from $\mathbf{i}_{\boldsymbol{\mathcal{T}}}$ by freezing all the mutable vertices that are placed on the diagonals different from E. Let $\mathcal{A}^{\mathcal{T},E}_{G,\Sigma} \subset \mathcal{A}^{\times}_{G,\Sigma}$ be the open subspace such that for every $E' \neq E$ in $e(\mathcal{T})$, its associated pair of decorated flags is generic.

Lemma 4.6 *(Proved in Section 6.1). The coordinate ring $\mathcal{O}(\mathcal{A}^{\mathcal{T},E}_{G,\Sigma})$ coincides with the upper cluster algebra $\mathscr{U}(\mathbf{i}_{\boldsymbol{\mathcal{T}},E})$.*

Now let us go through all the diagonals E of $\mathcal{T}$ and set

$$\widetilde{\mathcal{A}}^{\times}_{G,\Sigma} := \bigcup_{E} \mathcal{A}^{\mathcal{T},E}_{G,\Sigma} \subset \mathcal{A}^{\times}_{G,\Sigma}$$

Lemma 4.7. *We have $\mathcal{O}(\widetilde{\mathcal{A}}^{\times}_{G,\Sigma}) = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$.*

Proof. For a pair E_1, E_2 of diagonals of $\mathcal{T}$, let $\mathcal{A}^{\mathcal{T},E_1,E_2}_{G,\Sigma} \subset \mathcal{A}^{\times}_{G,\Sigma}$ be the subspace such that the pairs of the decorated flags associated with E_1 and E_2 are not generic. The complement of $\widetilde{\mathcal{A}}^{\times}_{G,\Sigma}$ in $\mathcal{A}^{\times}_{G,\Sigma}$ is $\bigcup_{E_1,E_2} \mathcal{A}^{\mathcal{T},E_1,E_2}_{G,\Sigma}$, which is of codimension ≥ 2. See Section 6.2 for a detailed computation of the codimension. Therefore $\mathcal{O}(\widetilde{\mathcal{A}}^{\times}_{G,\Sigma}) = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$. □

Proof of Theorem 4.3. Combining Lemmas 4.4, 4.6 and 4.7, and Proposition 4.2, we have

$$\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) = \mathcal{O}(\widetilde{\mathcal{A}}^{\times}_{G,\Sigma}) = \bigcap_E \mathcal{O}(\mathcal{A}^{\mathcal{T},E}_{G,\Sigma}) = \bigcap_E \mathscr{U}(\mathbf{i}_{\boldsymbol{\tau},E}) = \bigcap_E \overline{\mathscr{U}}(\mathbf{i}_{\boldsymbol{\tau},E}) = \overline{\mathscr{U}}(\mathbf{i}_{\boldsymbol{\tau}}) = \mathscr{U}_{\mathfrak{g},\Sigma}. \quad \square$$

Combining with Corollary 3.22, we get:

Corollary 4.8. *The upper cluster algebra $\mathscr{U}_{\mathfrak{g},\Sigma}$ is generated by the matrix coefficients of Wilson lines. Moreover if Σ has at least two marked points, then $\mathscr{U}_{\mathfrak{g},\Sigma}$ is generated by matrix coefficients of simple Wilson lines and boundary Wilson lines.*

Remark 4.9. As stated in [19, Proposition 3.17 (i)], the upper cluster algebra no longer coincides with $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ when Σ has punctures.

4.3. Cluster K_2-coordinates on $\mathrm{Conf}^{\times}_4\mathcal{A}_G$

For the computations in the next subsection, let us recall the cluster K_2-coordinates on the configuration space $\mathrm{Conf}^{\times}_4\mathcal{A}_G = \mathrm{Conf}^{w_0}_{w_0}\mathcal{A}_G$ from [20, Section 7], which can be regarded as an open subspace of the moduli space $\mathcal{A}_{G,Q}$ for a quadrilateral Q. Here are notations from [20, Section 5]:

- Given $w \in W$, we set

$$S(w) := \{s \in S \mid w\alpha^{\vee}_s < 0\},$$
$$H(w) := \left\{ \prod_{s\in S(w)} \alpha^{\vee}_s(b_s) \mid b_s \in \mathbb{G}_m \right\}.$$

  Then $S(w) = S \iff H(w) = H \iff w = w_0$.
- Given a reduced word $\boldsymbol{s} = (s_1, \dots, s_l)$ of w, we get a sequence of distinct coroots

$$\beta^{\boldsymbol{s}}_k := r_{s_l} \dots r_{s_{k+1}} \alpha^{\vee}_{s_k}, \quad k = 1, \dots, l.$$

  They are precisely the positive coroots $\alpha^{\vee}$ such that $w\alpha^{\vee}$ are negative.
- Recall that we have

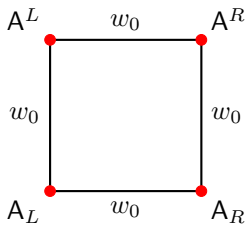
$$\mathcal{O}(\mathrm{Conf}_2\mathcal{A}_G) \cong \bigoplus_{\lambda\in X^*(H)_+} \big(V(\lambda) \otimes V(\lambda^*)\big)^G,$$

  and consider the function $\Delta_s \in \mathcal{O}(\mathrm{Conf}_2\mathcal{A}_G)$ for $s \in S$ such that $\Delta_s \in \big(V(\varpi_s) \otimes V(\varpi^*_s)\big)^G$ and $\Delta_s([U^+], [U^-]) = 1$. We have the relation

$$\Delta_{u\varpi_s, v\varpi_s}(g) = \Delta_s(g\overline{v}.[U^+], \overline{u}.[U^-])$$

  for $g \in G$ and $u, v \in W$.

The configuration space $\mathrm{Conf}_4^{\times}\mathcal{A}_G$ parametrizes the G-orbits of quadruples $(\mathsf{A}^L, \mathsf{A}_L, \mathsf{A}_R, \mathsf{A}^R)$ of decorated flags such that the cyclically consecutive pairs $(\mathsf{A}^L, \mathsf{A}_L)$, $(\mathsf{A}_L, \mathsf{A}_R)$, $(\mathsf{A}_R, \mathsf{A}^R)$, $(\mathsf{A}^R, \mathsf{A}^L)$ are generic. Such a quadruple $(\mathsf{A}^L, \mathsf{A}_L, \mathsf{A}_R, \mathsf{A}^R)$ is illustrated as



Lemma 4.10 *(cf. [20, Lemma-Definition 5.3]). Let $(\mathsf{A}_l, \mathsf{A}_r)$ be a generic pair of decorated flags, and $\boldsymbol{s} = (s_1, \dots, s_N)$ a reduced word of w_0. Then there exists a unique chain $\mathsf{A}_l = \mathsf{A}_0 \overset{s_1}{\leftarrow} \mathsf{A}_1 \overset{s_2}{\leftarrow} \dots \overset{s_N}{\leftarrow} \mathsf{A}_N = \mathsf{A}_r$ of decorated flags such that*

- $w(\mathsf{A}_k, \mathsf{A}_{k-1}) = r_{s_k}$ *for* $k = 1, \dots, N$, *and*
- $h(\mathsf{A}_k, \mathsf{A}_{k-1}) = \alpha^{\vee}_{s_k}(c_k)$.

Here $c_k \in \mathbb{G}_m$ is given by

$$c_k := \begin{cases} h(\mathsf{A}_r, \mathsf{A}_l)^{\varpi_t} & \text{if } \beta^{\boldsymbol{s}}_k = \alpha^{\vee}_t \text{ is simple,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. When $(\mathsf{A}_r, \mathsf{A}_l) = (h.[U^+], \overline{w_0}.[U^+])$ sits in the standard configuration, the intermediate flags are given by

$$\mathsf{A}_k := \overline{r}_{s_N} \dots \overline{r}_{s_{k+1}} h_k.[U^+],$$

where $h_k := u_k \left(\prod_{s \in S(w_0 u_k)} \alpha^{\vee}_s(h(\mathsf{A}_r, \mathsf{A}_l)^{\varpi_s}) \right)$ with $u_k := r_{s_{k+1}} \dots r_{s_N}$. Indeed, we have

$$[\mathsf{A}_k, \mathsf{A}_{k-1}] = [h_k.[U^+], \overline{r}_{s_k} h_{k-1}.[U^+]] = [r_{s_k}(h_{k-1}^{-1}) h_k.[U^+], \overline{r}_{s_k}.[U^+]]$$

and the unique solution of $r_{s_k}(h_{k-1}^{-1}) h_k = \alpha^{\vee}_{s_k}(c_k)$ for $k = 1, \dots, N$ is given by

$$h_k = \prod_{j=1}^{k} r_{s_k} \dots r_{s_{j+1}} \alpha^{\vee}_{s_j}(c_j).$$

Then applying u_k^{-1} to both sides, we get $u_k^{-1}(h_k) = \prod_{j=1}^{k} \beta^{\boldsymbol{s}}_j(c_j) = \prod_{t \in S(w_0 u_k)} \alpha^{\vee}_t(h^{\varpi_t})$, as desired. The intermediate decorated flags for a pair $g.(\mathsf{A}_r, \mathsf{A}_l)$ are given by the translates $g.\mathsf{A}_k$ for $g \in G$. $\square$

Now we choose a double reduced word $\boldsymbol{s}$ of $(w_0, w_0) \in W \times W$. The subword of $\boldsymbol{s}$ consisting of letters in S gives a reduced word $\boldsymbol{s}_\bullet = (s_1, \dots, s_N)$ of w_0; similarly, the $\overline{S}$-part gives another reduced word $\boldsymbol{s}^\bullet = (s^1, \dots, s^N)$. Given a quadruple $(\mathsf{A}^L, \mathsf{A}_L, \mathsf{A}_R, \mathsf{A}^R)$ whose orbit lies in $\mathrm{Conf}_4^\times \mathcal{A}_G$, we apply Lemma 4.10 to the pair $(\mathsf{A}^R, \mathsf{A}^L)$ with the word $\boldsymbol{s}^\bullet$, and to the pair $(\mathsf{A}_L, \mathsf{A}_R)$ with the word $\boldsymbol{s}^*_\bullet$. Then we get the following chains of decorated flags:

$$\begin{aligned} &\mathsf{A}^L = \mathsf{A}^N \xrightarrow{s^N} \mathsf{A}^{N-1} \xrightarrow{s^{N-1}} \dots \xrightarrow{s^1} \mathsf{A}^0 = \mathsf{A}^R, \qquad w(\mathsf{A}^k, \mathsf{A}^{k-1}) = r_{s^k}, \\ &h(\mathsf{A}^k, \mathsf{A}^{k-1}) = \alpha^\vee_{s^k}(c^k), \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\mathsf{A}_L = \mathsf{A}_0 \xleftarrow{s^*_1} \mathsf{A}_1 \xleftarrow{s^*_2} \dots \xleftarrow{s^*_N} \mathsf{A}_N = \mathsf{A}_R, \qquad w(\mathsf{A}_k, \mathsf{A}_{k-1}) = r_{s^*_k}, \\ &h(\mathsf{A}_k, \mathsf{A}_{k-1}) = \alpha^\vee_{s^*_k}(c_k), \end{aligned} \tag{4.3}$$

where c^k and c_k are given by

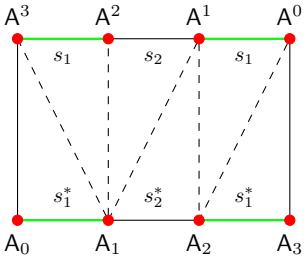
$$c^k := \begin{cases} h(\mathsf{A}^L, \mathsf{A}^R)^{\varpi_t} & \text{if } \beta_k^{\boldsymbol{s}^\bullet} = \alpha_t^\vee \text{ is simple,} \\ 1 & \text{otherwise,} \end{cases}$$

and

$$c_k := \begin{cases} h(\mathsf{A}_R, \mathsf{A}_L)^{\varpi_u} & \text{if } \beta_k^{\boldsymbol{s}^*_\bullet} = \alpha_u^\vee \text{ is simple,} \\ 1 & \text{otherwise,} \end{cases}$$

respectively. Using these chains, the double reduced word $\boldsymbol{s}$ gives rise to a decomposition of the configuration $[\mathsf{A}^L, \mathsf{A}_L, \mathsf{A}_R, \mathsf{A}^R]$ into elementary configurations, as explained by the following example.

Example 4.11 *(Type A_2).* The double reduced word $\boldsymbol{s} = (1, \overline{1}, \overline{2}, 2, \overline{1}, 1)$ gives rise to the decomposition below. The locations where the coroot $\beta_k^{\boldsymbol{s}^\bullet}$ or $\beta_k^{\boldsymbol{s}^*_\bullet}$ becomes simple are shown in green.



Then for each pair $(\mathsf{A}^k, \mathsf{A}_l)$ connected by a dashed line or a vertical solid line, we consider the functions $\Delta_s(\mathsf{A}^k, \mathsf{A}_l)$ for $s \in S$. These functions are not distinct: for example if we have a triple

we have $\Delta_s(\mathsf{A}^k, \mathsf{A}_l) = \Delta_s(\mathsf{A}^k, \mathsf{A}_{l+1})$ for $s \neq s_k^*$. We have a similar relation for a down-pointing triangle. See [20, Lemma 7.13] for details. Collectively, we get $|S| + \#(\text{dashed lines}) + 1$ distinct functions among $\Delta_s(\mathsf{A}^k, \mathsf{A}_l)$'s. They can be assigned to the vertices of the quiver $\mathbf{J}(\boldsymbol{s})$ in [20, Definition 7.5]. We also have additional functions $h(\mathsf{A}^k, \mathsf{A}^{k-1})^{\varpi_t} = h(\mathsf{A}^L, \mathsf{A}^R)^{\varpi_t}$ if $\beta_k^{\boldsymbol{s}^\bullet} = \alpha_t^\vee$ is simple, and $h(\mathsf{A}_l, \mathsf{A}_{l-1})^{\varpi_u^*} = h(\mathsf{A}_R, \mathsf{A}_L)^{\varpi_u^*}$ if $\beta_k^{\boldsymbol{s}^\bullet} = \alpha_u^\vee$ is simple. They are assigned to the green lines, and supply the coordinates on the remaining "extra" vertices in the quiver $\overline{\mathbf{J}}(\boldsymbol{s})$ in [20, Definition 7.5]. The quiver $\overline{\mathbf{J}}(\boldsymbol{s})$ together with the coordinates assigned to its vertices form a cluster K_2-seed in the field $\mathcal{K}(\mathrm{Conf}_4^\times \mathcal{A}_G)$.

We remark that these cluster coordinates are regular functions on a larger space $\mathrm{Conf}_{\mathrm{in}}^{\mathrm{out}} \mathcal{A}_G$ consisting of the G-orbits of quadruples $(\mathsf{A}^L, \mathsf{A}_L, \mathsf{A}_R, \mathsf{A}^R)$ such that the top and bottom pairs $(\mathsf{A}^L, \mathsf{A}^R)$ and $(\mathsf{A}_L, \mathsf{A}_R)$ are generic.

4.4. Generalized minors of simple Wilson lines are cluster variables

Let $[c] : E_1^- \to E_2^-$ be a simple framed arc class and fix its band neighborhood B_c. The band B_c can be regarded as a marked surface (*i.e.*, a disk with four marked points m^L, m_L, m_R, m^R) as shown in Fig. 2. By Remark 3.17 (1)(2), the Wilson line $g_{[c]}$ can be computed on the moduli space $\mathcal{A}^\times_{G,B_c;\{E_1,E_2\}} \cong \mathrm{Conf}_{\mathrm{in}}^{\mathrm{out}} \mathcal{A}_G$, where the identification is determined by the branch cut and the correspondence of flags as shown in Fig. 7. For a weight $\lambda = \sum_{s\in S} a_s \varpi_s$, we write $[\lambda]_+ := \sum_{s\in S} [a_s]_+ \varpi_s$, where $[a_s]_+ := \max\{0, a_s\}$.

Proposition 4.12. *Each generalized minor of a simple Wilson line $g_{[c]}$ is a single cluster variable in $\mathscr{A}_{\mathfrak{g},\Sigma}$ multiplied by inverses of several frozen variables. Specifically, we have*

$$\Delta_{u_{>l}\varpi_s, v_{>k}\varpi_s}(g_{[c]}) = \frac{\Delta_s(\mathsf{A}^k, \mathsf{A}_l)}{h(\mathsf{A}_R, \mathsf{A}_L)^{[u_{>l}\varpi_s]_+^*} h(\mathsf{A}^L, \mathsf{A}^R)^{[v_{>k}\varpi_s]_+}} \tag{4.4}$$

for all $k, l = 0, \dots, N$ and $s \in S$, where $u_{>l} := r_{s_N} \dots r_{s_{l+1}}$ and $v_{>k} := r_{s^N} \dots r_{s^{k+1}}$ for any double reduced word $\boldsymbol{s}$ of $(w_0, w_0) \in W \times W$.

Proof. Any G-orbit $[\mathsf{A}^L, \mathsf{A}_L, \mathsf{A}_R, \mathsf{A}^R]$ contains a unique representative of the form

$$(\mathsf{A}^L, \mathsf{A}_L, \mathsf{A}_R, \mathsf{A}^R) = (gh.[U^+], [U^+], \overline{w_0}^{-1} h'.[U^+], g\overline{w_0}.[U^+])$$

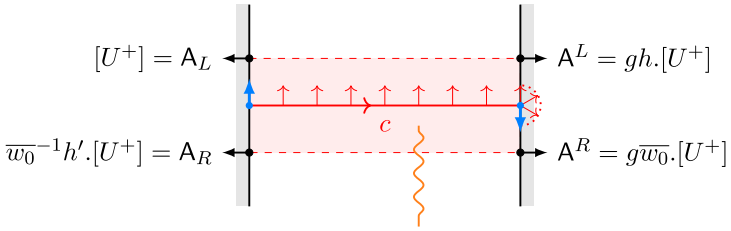


Fig. 7. Computation of the Wilson line $g_{[c]}$ on $\mathcal{A}^{\times}_{G,B_c} \cong \mathrm{Conf}^{\times}_4 \mathcal{A}_G$.

for $h, h' \in H$ and $g \in G^{w_0,w_0}$. In other words, we have an isomorphism $H \times H \times G^{w_0,w_0} \xrightarrow{\sim} \mathrm{Conf}^{\times}_4 \mathcal{A}_G$. In this parametrization, we have $h(\mathsf{A}^L, \mathsf{A}^R) = h$ and $h(\mathsf{A}_R, \mathsf{A}_L) = h'$. Moreover, $(\mathsf{A}_L, \pi(\mathsf{A}_R)) = p_{\mathrm{std}}$ and $(\mathsf{A}^R, \pi(\mathsf{A}^L)) = g\overline{w_0}.p_{\mathrm{std}} = gs_G.p^*_{\mathrm{std}}$; hence $g_{[c]} = gs_G$.

Let $\boldsymbol{s}$ be a double reduced word of $(w_0, w_0) \in W \times W$. Then by the proof of Lemma 4.10, we explicitly get the associated chains $(\mathsf{A}^k)_k$, $(\mathsf{A}_l)_l$ of decorated flags in (4.2) and (4.3) as

$$\mathsf{A}^k = g\overline{r}_{\boldsymbol{s}^N} \dots \overline{r}_{\boldsymbol{s}^{k+1}} h^k.[U^+]$$

with $h^k := w^k \left(\prod_{s \in S(w_0 w^k)} \alpha^\vee_s(h^{\varpi_s})\right)$ and $w^k := r_{\boldsymbol{s}^{k+1}} \cdots r_{\boldsymbol{s}^N}$, and

$$\mathsf{A}_l = \overline{w_0}^{-1} \overline{r}_{\boldsymbol{s}^*_N} \dots \overline{r}_{\boldsymbol{s}^*_{l+1}} h_l.[U^+]$$

with $h_l := w_l \left(\prod_{s \in S(w_0 w_l)} \alpha^\vee_s(h'^{\varpi_s})\right)$ and $w_l := r_{\boldsymbol{s}^*_{l+1}} \cdots r_{\boldsymbol{s}^*_N}$. Then we get

$$\begin{aligned}
\Delta_s(\mathsf{A}^k, \mathsf{A}_l) &= \Delta_s \left(g\overline{r}_{\boldsymbol{s}^N} \dots \overline{r}_{\boldsymbol{s}^{k+1}}.[U^+], \overline{w_0}^{-1} \overline{r}_{\boldsymbol{s}^*_N} \dots \overline{r}_{\boldsymbol{s}^*_{l+1}}.[U^+]\right) \cdot (h^k h_l^*)^{\varpi_s} \\
&= \Delta_s \left(g\overline{r}_{\boldsymbol{s}^N} \cdots \overline{r}_{\boldsymbol{s}^{k+1}}.[U^+], s_G \overline{r}_{\boldsymbol{s}_N} \cdots \overline{r}_{\boldsymbol{s}_{l+1}}.[U^-]\right) \cdot (h^k h_l^*)^{\varpi_s} \\
&= \Delta_{r_{\boldsymbol{s}_N} \cdots r_{\boldsymbol{s}_{l+1}} \cdot \varpi_s, r_{\boldsymbol{s}^N} \cdots r_{\boldsymbol{s}^{k+1}} \varpi_s}(gs_G) \cdot (h^k h_l^*)^{\varpi_s}.
\end{aligned}$$

Thus we get

$$\Delta_{u_{>l}\varpi_s, v_{>k}\varpi_s}(g_{[c]}) = (h^k h_l^*)^{-\varpi_s} \cdot \Delta_s(\mathsf{A}^k, \mathsf{A}_l).$$

The frozen variables are computed as follows:

$$\begin{aligned}
(h^k)^{\varpi_s} &= \prod_{t \in S(w_0 w^k)} \alpha^\vee_t(h^{\varpi_t})^{(w^k)^{-1}\varpi_s} = \prod_{t \in S} \alpha^\vee_t(h^{\varpi_t})^{[(w^k)^{-1}\varpi_s]_+} = h(\mathsf{A}^L, \mathsf{A}^R)^{[v_{>k}\varpi_s]_+}, \\
h_l^{\varpi^*_s} &= \prod_{t \in S(w_0 w_l)} \alpha^\vee_t(h'^{\varpi_t})^{w_l^{-1}\varpi^*_s} = \prod_{t \in S} \alpha^\vee_t(h'^{\varpi_t})^{[w_l^{-1}\varpi^*_s]_+} = h(\mathsf{A}_R, \mathsf{A}_L)^{[u^*_{>l}\varpi^*_s]_+} \\
&= h(\mathsf{A}_R, \mathsf{A}_L)^{[u_{>l}\varpi_s]^*_+}.
\end{aligned}$$

Here for the second equality in the first string: note that writing $(w^k)^{-1}\varpi_s = \sum_{t\in S} c_t\varpi_t$ with $c_t \in \mathbb{Z}$, we have

$$\langle \varpi_s,\ w^k.\alpha_t^\vee\rangle = \langle (w^k)^{-1}.\varpi_s,\ \alpha_t^\vee\rangle = c_t.$$

Hence $t \in S(w_0w^k) \iff w^k.\alpha_t^\vee > 0 \iff c_t > 0$. The second equality in the second string is similarly obtained. Thus the assertion is proved. □

Remark 4.13. Notice that the word $\boldsymbol{s}^*_\bullet$ is read in the reversed order when we "scan" the distances of intermediate flags from the bottom to the top in Fig. 7. Therefore our seed associated with the double reduced word $\boldsymbol{s}$ is the same as the one in [1] associated with $\boldsymbol{s}^\bullet$ and $(\boldsymbol{s}^*_\bullet)_{\mathrm{op}} := (s^*_N, \dots, s^*_1)$. Letting $s'_{N-l+1} := s_l$, the right-hand side of (4.4) is written as $\Delta_{u'_{\le N-l}\varpi_s, v_{>k}\varpi_s}(g_{[c]})$ with $u'_{\le l'} := r_{s'_1}\dots r_{s'_{l'}}$, which has the same form as the cluster variable in [1, (2.11)].

Remark 4.14. When $m^L = m_L =: m$ and $\mathsf{A}^L = \mathsf{A}_L$, the situation reduces to the triangle case (Example 3.20), where the parameters are identified as $h' = h_1$ and $g = u_+h_2^{-1}$. In the description of the cluster variables on $\mathrm{Conf}_3\mathcal{A}_G$ given in [20, Section 6], they are computed as follows:

$$\begin{aligned}\Delta_{\lambda,\mu,\nu}(g\overline{w_0}.[U^+], [U^+], \overline{w_0}^{-1}h'.[U^+]) &= \Delta_{\lambda,\mu,\nu}(g\overline{w_0}^{-1}.[U^+], s_G.[U^+], \overline{w_0}h'.[U^+])\\ &= \Delta_{\lambda,\mu,\nu}(g_{[c]}\overline{w_0}.[U^+], [U^+], \overline{w_0}.[U^+]) \cdot s^\mu_G(h')^\nu\\ &= F_{w\lambda}(g_{[c]}\overline{w_0}.[U^+]) \cdot (h')^\nu\\ &= \Delta_{w\lambda,w_0\lambda}(g_{[c]}) \cdot h(\mathsf{A}_R, \mathsf{A}_L)^\nu.\end{aligned}$$

This formula tells us how to express the cluster variables $\Delta_{\lambda,\mu,\nu}$ associated with an admissible triple (λ,μ,ν) in terms of the matrix coefficients of Wilson lines.

4.5. Proof of Theorem 1

Recall from Corollary 4.8 that the upper cluster algebra $\mathscr{U}_{\mathfrak{g},\Sigma}$ is generated by the matrix coefficients of simple Wilson lines and boundary Wilson lines, under the assumption that Σ has at least two marked points. By Proposition 3.19 (1), the boundary Wilson line $g_{\sqrt{\boldsymbol{o}_E}^{\mathrm{out}}} \in H$ is the inverse of the h-invariant of the pair of decorated flags assigned to a boundary interval $E \in \mathbb{B}$. Hence its non-trivial matrix coefficients $(g_{\sqrt{\boldsymbol{o}_E}^{\mathrm{out}}})^{\varpi_s}$ for $s \in S$ are exactly the inverses of the r many frozen variables on E. By Proposition 2.2 and Proposition 4.12, the matrix coefficients of simple Wilson lines are also cluster variables multiplied by the inverses of frozen variables. Therefore these generators are contained in the cluster algebra $\mathscr{A}_{\mathfrak{g},\Sigma}$, where the frozen variables are invertible. Thus we get $\mathscr{U}_{\mathfrak{g},\Sigma} \subset \mathscr{A}_{\mathfrak{g},\Sigma}$.

5. Examples and skein description

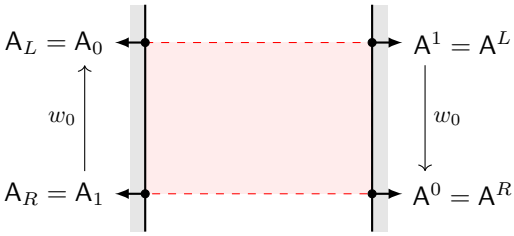
In this section, we give an explicit description of the formula (4.4) in the cases $G = SL_2, SL_3, Sp_4$, and their description in terms of the skein algebras studied in [32,24,25]. Let us consider a simple Wilson line $g_{[c]}$, and use the notations in Section 4.4. We denote the frozen coordinates by

$$A^s_{\mathrm{in}} := h(\mathsf{A}_R, \mathsf{A}_L)^{\varpi_s^*} \quad \text{and} \quad A^s_{\mathrm{out}} := h(\mathsf{A}^L, \mathsf{A}^R)^{\varpi_s}$$

for $s \in S$.

5.1. SL_2-case

The vector representation $V(\varpi_1) = \mathbb{C}^2$ is minuscule. The weights in this representation are given by ϖ_1 and $-\varpi_1 = w_0\varpi_1$. There is a unique choice of reduced word $\boldsymbol{s}^\bullet = \boldsymbol{s}_\bullet = (1)$ of $w_0 \in W$, which gives rise to the chains of decorated flags:



We have $[w_0\varpi_1]_+ = 0$ and $[\varpi_1]_+ = \varpi_1$. Then by the formula (4.4), we get

$$\begin{aligned}
\Delta_{11}(g) &= \Delta_{\varpi_1,\varpi_1}(g) = \frac{\Delta_1(\mathsf{A}^1,\mathsf{A}_1)}{h(\mathsf{A}^L,\mathsf{A}^R)^{\varpi_1} h(\mathsf{A}_R,\mathsf{A}_L)^{\varpi_1}} = \frac{\Delta_1(\mathsf{A}^1,\mathsf{A}_1)}{A^1_{\mathrm{in}} A^1_{\mathrm{out}}},\\
\Delta_{12}(g) &= \Delta_{\varpi_1,w_0\varpi_1}(g) = \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_1)}{h(\mathsf{A}^L,\mathsf{A}^R)^{0} h(\mathsf{A}_R,\mathsf{A}_L)^{\varpi_1}} = \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_1)}{A^1_{\mathrm{in}}},\\
\Delta_{21}(g) &= \Delta_{w_0\varpi_1,\varpi_1}(g) = \frac{\Delta_1(\mathsf{A}^1,\mathsf{A}_0)}{h(\mathsf{A}^L,\mathsf{A}^R)^{\varpi_1} h(\mathsf{A}_R,\mathsf{A}_L)^{0}} = \frac{\Delta_1(\mathsf{A}^1,\mathsf{A}_0)}{A^1_{\mathrm{out}}},\\
\Delta_{22}(g) &= \Delta_{w_0\varpi_1,w_0\varpi_1}(g) = \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_0)}{h(\mathsf{A}^L,\mathsf{A}^R)^{0} h(\mathsf{A}_R,\mathsf{A}_L)^{0}} = \Delta_1(\mathsf{A}^0,\mathsf{A}_0).
\end{aligned}$$

Thus the Wilson line matrix $g_{[c]} \in SL_2(\mathcal{O}(\mathcal{A}^\times_{SL_2,\Sigma}))$ is collectively given by

$$g_{[c]} = \begin{pmatrix} \dfrac{\Delta_1(\mathsf{A}^1,\mathsf{A}_1)}{A^1_{\mathrm{in}} A^1_{\mathrm{out}}} & \dfrac{\Delta_1(\mathsf{A}^0,\mathsf{A}_1)}{A^1_{\mathrm{in}}} \\ \dfrac{\Delta_1(\mathsf{A}^1,\mathsf{A}_0)}{A^1_{\mathrm{out}}} & \Delta_1(\mathsf{A}^0,\mathsf{A}_0) \end{pmatrix}. \tag{5.1}$$

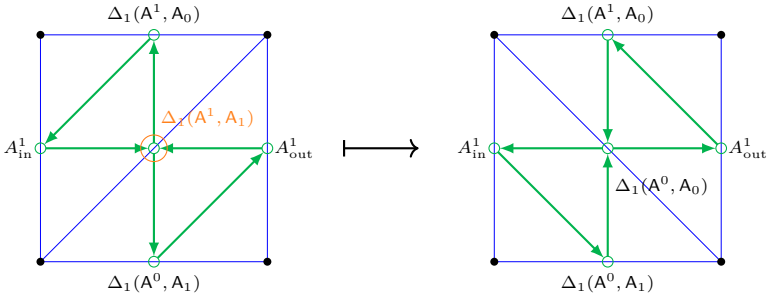
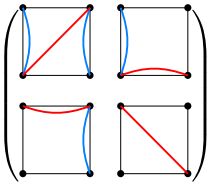


Fig. 8. The two clusters in $\mathcal{O}(\mathrm{Conf}^{\mathrm{out}}_{\mathrm{in}}\mathcal{A}_{SL_2})$ related by a single mutation.

The clusters corresponding to the two ideal triangulations are shown in Fig. 8.

Recall the skein model $\mathscr{A}_{\mathfrak{sl}_2,\Sigma} \cong \mathscr{S}^1_\Sigma[\partial^{-1}]$ given by Muller [32], where the cluster variables are identified with ideal arcs, and the clusters correspond to the ideal triangulations. In this language, the matrix (5.1) is written as

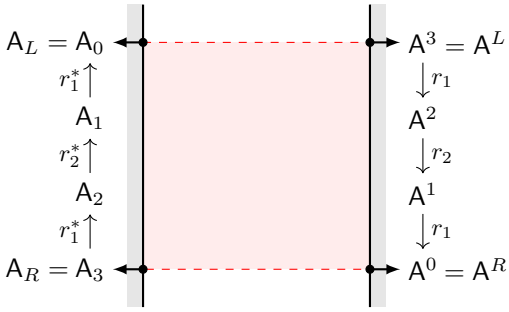
$$g_{[c]} = \left(\quad \right).$$



Here the inverse of an ideal arc is shown in blue.

5.2. SL_3-case

The vector representation $V(\varpi_1) = \mathbb{C}^3$ is minuscule. The weights in this representation are given by ϖ_1, $r_1\varpi_1 = \varpi_2 - \varpi_1$ and $r_2 r_1 \varpi_1 = -\varpi_2 = w_0\varpi_1$. Let us choose the reduced word $\boldsymbol{s}^\bullet = \boldsymbol{s}_\bullet = (1,2,1)$ of $w_0 \in W$, which gives rise to the following chains of decorated flags:



Then the matrix entries of the simple Wilson line $g_{[c]}$ in $V(\varpi_1)$ are computed as follows:

$$\Delta_{11}(g_{[c]}) = \Delta_{\varpi_1,\varpi_1}(g_{[c]}) = \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_3)}{h(\mathsf{A}_R,\mathsf{A}_L)^{\varpi_1^*}h(\mathsf{A}^L,\mathsf{A}^R)^{\varpi_1}} = \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_3)}{A^1_{\mathrm{in}}A^1_{\mathrm{out}}},$$

$$\Delta_{12}(g_{[c]}) = \Delta_{\varpi_1,r_1\varpi_1}(g_{[c]}) = \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_3)}{h(\mathsf{A}_R,\mathsf{A}_L)^{\varpi_1^*}h(\mathsf{A}^L,\mathsf{A}^R)^{\varpi_2}} = \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_3)}{A^1_{\mathrm{in}}A^2_{\mathrm{out}}},$$

$$\Delta_{13}(g_{[c]}) = \Delta_{\varpi_1,w_0\varpi_1}(g_{[c]}) = \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_3)}{h(\mathsf{A}_R,\mathsf{A}_L)^{\varpi_1^*}} = \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_3)}{A^1_{\mathrm{in}}},$$

$$\Delta_{21}(g_{[c]}) = \Delta_{r_1\varpi_1,\varpi_1}(g_{[c]}) = \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_2)}{h(\mathsf{A}_R,\mathsf{A}_L)^{\varpi_2^*}h(\mathsf{A}^L,\mathsf{A}^R)^{\varpi_1}} = \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_2)}{A^2_{\mathrm{in}}A^1_{\mathrm{out}}},$$

$$\Delta_{22}(g_{[c]}) = \Delta_{r_1\varpi_1,r_1\varpi_1}(g_{[c]}) = \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_2)}{h(\mathsf{A}_R,\mathsf{A}_L)^{\varpi_2^*}h(\mathsf{A}^L,\mathsf{A}^R)^{\varpi_2}} = \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_2)}{A^2_{\mathrm{in}}A^2_{\mathrm{out}}},$$

$$\Delta_{23}(g_{[c]}) = \Delta_{r_1\varpi_1,w_0\varpi_1}(g_{[c]}) = \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_2)}{h(\mathsf{A}_R,\mathsf{A}_L)^{\varpi_2^*}} = \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_2)}{A^2_{\mathrm{in}}},$$

$$\Delta_{31}(g_{[c]}) = \Delta_{w_0\varpi_1,\varpi_1}(g_{[c]}) = \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_0)}{h(\mathsf{A}^L,\mathsf{A}^R)^{\varpi_1}} = \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_0)}{A^1_{\mathrm{out}}},$$

$$\Delta_{32}(g_{[c]}) = \Delta_{w_0\varpi_1,r_1\varpi_1}(g_{[c]}) = \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_0)}{h(\mathsf{A}^L,\mathsf{A}^R)^{\varpi_2}} = \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_0)}{A^2_{\mathrm{out}}},$$

$$\Delta_{33}(g_{[c]}) = \Delta_{w_0\varpi_1,w_0\varpi_1}(g_{[c]}) = \Delta_1(\mathsf{A}^0,\mathsf{A}_0) = \Delta_1(\mathsf{A}^0,\mathsf{A}_0).$$

Thus the Wilson line matrix $g_{[c]} \in SL_3(\mathcal{O}(\mathcal{A}^{\times}_{SL_3,\Sigma}))$ is collectively given by

$$g_{[c]} = \begin{pmatrix} \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_3)}{A^1_{\mathrm{in}}A^1_{\mathrm{out}}} & \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_3)}{A^1_{\mathrm{in}}A^2_{\mathrm{out}}} & \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_3)}{A^1_{\mathrm{in}}} \\ \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_2)}{A^2_{\mathrm{in}}A^1_{\mathrm{out}}} & \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_2)}{A^2_{\mathrm{in}}A^2_{\mathrm{out}}} & \frac{\Delta_1(\mathsf{A}^0,\mathsf{A}_2)}{A^2_{\mathrm{in}}} \\ \frac{\Delta_1(\mathsf{A}^3,\mathsf{A}_0)}{A^1_{\mathrm{out}}} & \frac{\Delta_1(\mathsf{A}^2,\mathsf{A}_0)}{A^2_{\mathrm{out}}} & \Delta_1(\mathsf{A}^0,\mathsf{A}_0) \end{pmatrix}. \tag{5.2}$$

The clusters to which the cluster variables appearing above belong are shown in Fig. 9, as well as the mutation sequences relating them. Here we use the fact that the two words $(\cdots s\bar{t}\cdots)$ and $(\cdots \bar{t}s\cdots)$ give rise to the same cluster for $s \neq t$. The corresponding transformations of dashed diagonals connecting the decorated flags are shown in Fig. 10.

Recall the skein model studied in [24]. For any marked surface as in Section 3.1, the first author and W. Yuasa realized the quantum cluster algebra $\mathscr{A}^q_{\mathfrak{sl}_3,\Sigma}$ quantizing $\mathscr{A}_{\mathfrak{sl}_3,\Sigma}$ inside the skew-field of fractions of a certain $\mathfrak{sl}_3$-skein algebra $\mathscr{S}^q_{\mathfrak{sl}_3,\Sigma}$ consisting of $\mathfrak{sl}_3$-webs (*i.e.*, oriented trivalent graphs whose vertices are either sinks  or sources ), and shown the inclusion $\mathscr{S}^q_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subseteq \mathscr{A}^q_{\mathfrak{sl}_3,\Sigma}$. Here $\mathscr{S}^q_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$ denote the localized skein algebra along the boundary $\mathfrak{sl}_3$-webs. It implies $\mathscr{S}^1_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] \subseteq \mathscr{A}_{\mathfrak{sl}_3,\Sigma}$ at the classical specialization $q = 1 \in \mathbb{C}$.

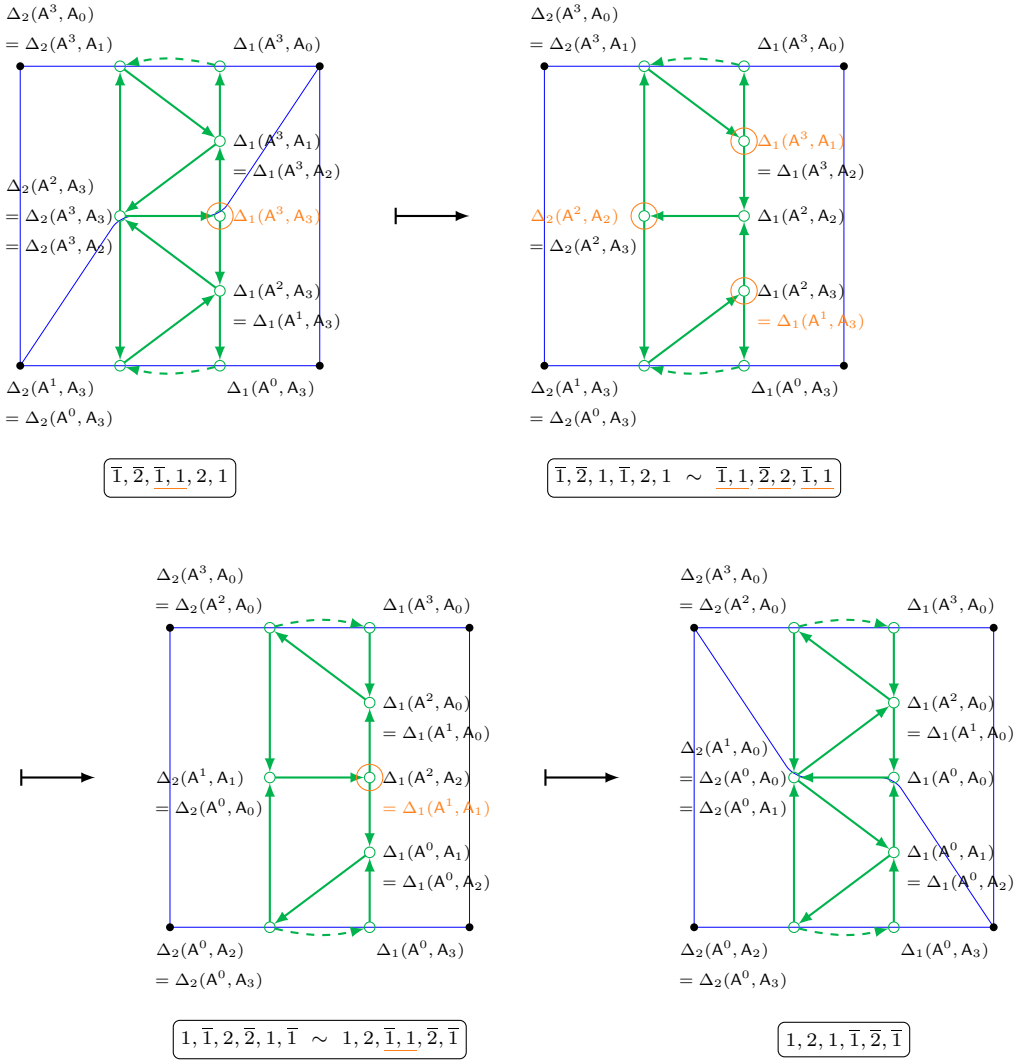


Fig. 9. Some clusters in $\mathcal{O}(\mathrm{Conf}_{\mathrm{in}}^{\mathrm{out}}\mathcal{A}_{SL_3})$ and the mutation sequences relating them. Here the frozen variables/vertices are omitted, and the mutated vertices are shown in orange.

On the other hand, if the marked surface Σ has at least two marked points, then we have $\mathscr{A}_{\mathfrak{sl}_3,\Sigma} = \mathscr{U}_{\mathfrak{sl}_3,\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{SL_3,\Sigma})$ by Theorem 1. Moreover, one can verify that each entry of the matrix (5.2) comes from the localized skein algebra $\mathscr{S}^1_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$ by comparing the clusters in Fig. 9 and the construction in [24, Section 5]. Explicitly, we have

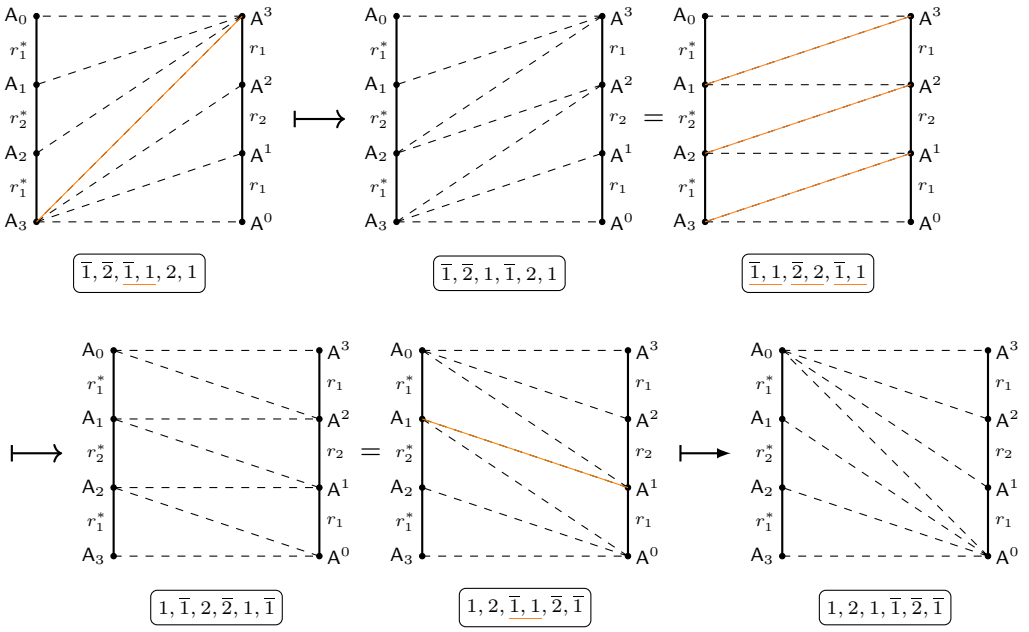
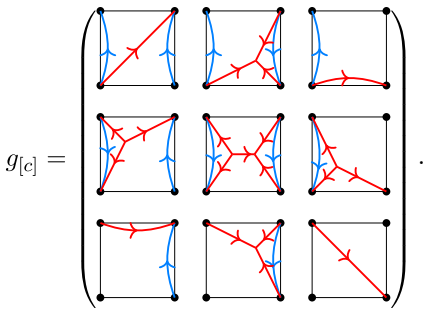


Fig. 10. The transformations of dashed lines connecting the decorated flags corresponding to the mutation sequences in Fig. 9. The flipped diagonals are shown in orange, which are corresponding to mutations.



Here the inverse of a boundary $\mathfrak{sl}_3$-web is shown in blue. Then by the same line of argument as Section 4.5, we see that the inclusion $\mathscr{U}_{\mathfrak{sl}_3,\Sigma} \subset \mathscr{S}^1_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$ holds. Thus we get $\mathscr{S}^1_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}] = \mathscr{A}_{\mathfrak{sl}_3,\Sigma} = \mathscr{U}_{\mathfrak{sl}_3,\Sigma}$, which confirms [24, Conjecture 3] at the classical level.

5.3. Sp_4-case

The vector representation $V(\varpi_1) = \mathbb{C}^4$ of Sp_4 is minuscule. The weights in this representation are given by

$$\varpi_1, \quad r_1\varpi_1 = \varpi_2 - \varpi_1, \quad r_2r_1\varpi_1 = \varpi_1 - \varpi_2, \quad w_0\varpi_1 = -\varpi_1.$$

By convention, we choose the symplectic form to be

$$J := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

which determines the vector representation $Sp_4 \cong Sp(\mathbb{C}^4, J)$. In this case, we need to use both of the reduced words $\boldsymbol{s}^\bullet = (1,2,1,2)$ and $\hat{\boldsymbol{s}}^\bullet = (2,1,2,1)$ of $w_0 \in W$ in order to obtain all the matrix coefficients in $V(\varpi_1)$. They give rise to the chains of decorated flags

$$\mathsf{A}^L = \mathsf{A}^4 \xrightarrow{r_2} \mathsf{A}^3 \xrightarrow{r_1} \mathsf{A}^2 \xrightarrow{r_2} \mathsf{A}^1 \xrightarrow{r_1} \mathsf{A}^0 = \mathsf{A}^R,$$
$$\mathsf{A}^L = \hat{\mathsf{A}}^4 \xrightarrow{r_1} \hat{\mathsf{A}}^3 \xrightarrow{r_2} \hat{\mathsf{A}}^2 \xrightarrow{r_1} \hat{\mathsf{A}}^1 \xrightarrow{r_2} \hat{\mathsf{A}}^0 = \mathsf{A}^R,$$

respectively. Similarly, the words $\boldsymbol{s}_\bullet = (1,2,1,2)$ and $\hat{\boldsymbol{s}}_\bullet = (2,1,2,1)$ give rise to the bottom chains (A_l) and $(\hat{\mathsf{A}}_l)$, respectively. Similarly to the SL_2- and SL_3-cases, we can compute the Wilson line matrix $g_{[c]} \in Sp_4(\mathcal{O}(\mathcal{A}^\times_{Sp_4,\Sigma}))$ and get

$$g_{[c]} = \begin{pmatrix} \dfrac{\Delta_1(\mathsf{A}^4,\mathsf{A}_4)}{A^1_{\mathrm{in}}A^1_{\mathrm{out}}} & \dfrac{\Delta_1(\hat{\mathsf{A}}^3,\mathsf{A}_4)}{A^1_{\mathrm{in}}A^2_{\mathrm{out}}} & \dfrac{\Delta_1(\mathsf{A}^2,\mathsf{A}_4)}{A^1_{\mathrm{in}}A^1_{\mathrm{out}}} & \dfrac{\Delta_1(\mathsf{A}^0,\mathsf{A}_4)}{A^1_{\mathrm{in}}} \\ \dfrac{\Delta_1(\mathsf{A}^4,\hat{\mathsf{A}}_3)}{A^2_{\mathrm{in}}A^1_{\mathrm{out}}} & \dfrac{\Delta_1(\hat{\mathsf{A}}^3,\hat{\mathsf{A}}_3)}{A^2_{\mathrm{in}}A^2_{\mathrm{out}}} & \dfrac{\Delta_1(\mathsf{A}^2,\hat{\mathsf{A}}_3)}{A^2_{\mathrm{in}}A^1_{\mathrm{out}}} & \dfrac{\Delta_1(\mathsf{A}^0,\hat{\mathsf{A}}_3)}{A^2_{\mathrm{in}}} \\ \dfrac{\Delta_1(\mathsf{A}^4,\mathsf{A}_2)}{A^1_{\mathrm{in}}A^1_{\mathrm{out}}} & \dfrac{\Delta_1(\hat{\mathsf{A}}^3,\mathsf{A}_2)}{A^1_{\mathrm{in}}A^2_{\mathrm{out}}} & \dfrac{\Delta_1(\mathsf{A}^2,\mathsf{A}_2)}{A^1_{\mathrm{in}}A^1_{\mathrm{out}}} & \dfrac{\Delta_1(\mathsf{A}^0,\mathsf{A}_2)}{A^1_{\mathrm{in}}} \\ \dfrac{\Delta_1(\mathsf{A}^4,\mathsf{A}_0)}{A^1_{\mathrm{out}}} & \dfrac{\Delta_1(\hat{\mathsf{A}}^3,\mathsf{A}_0)}{A^2_{\mathrm{out}}} & \dfrac{\Delta_1(\mathsf{A}^2,\mathsf{A}_0)}{A^1_{\mathrm{out}}} & \Delta_1(\mathsf{A}^0,\mathsf{A}_0) \end{pmatrix}. \tag{5.3}$$

The investigation of the clusters to which the cluster variables appearing in this expression belong is left to the reader. Apart from the A_n-cases, the choice of distinguished vertices v_T of a triangulation is crucial: we remark here that the cluster for the double reduced word $\boldsymbol{s} = (\overline{1},\overline{2},\overline{1},\overline{2},2,1,2,1)$ corresponds to the decorated triangulation shown in the left of Fig. 11, whose underlying weighted quiver is shown in the right (see [23, Appendix B] for our convention on weighted quivers).

The skein model for the case is studied in [25]. For any marked surface, the quantum cluster algebra $\mathscr{A}^q_{\mathfrak{sp}_4,\Sigma}$ quantizing $\mathscr{A}_{\mathfrak{sp}_4,\Sigma}$ is realized inside the skew-field of fractions of a certain $\mathfrak{sp}_4$-skein algebra consisting of $\mathfrak{sp}_4$-webs. An $\mathfrak{sp}_4$-web is represented by a trivalent graph with two types of edges — and ═ (corresponding to the fundamental representations $V(\varpi_1)$ and $V(\varpi_2)$ respectively), and with trivalent vertices of the form  in the interior.

Similarly to the $\mathfrak{sl}_3$-case, we have an inclusion $\mathscr{S}^q_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subseteq \mathscr{A}^q_{\mathfrak{sp}_4,\Sigma}$ after localizing along the boundary $\mathfrak{sp}_4$-webs. Here $\mathscr{S}^q_{\mathfrak{sp}_4,\Sigma}$ denotes the $\mathbb{Z}_q$-*form* of the $\mathfrak{sp}_4$-skein algebra

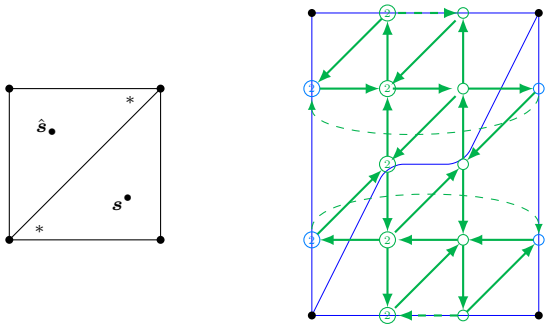
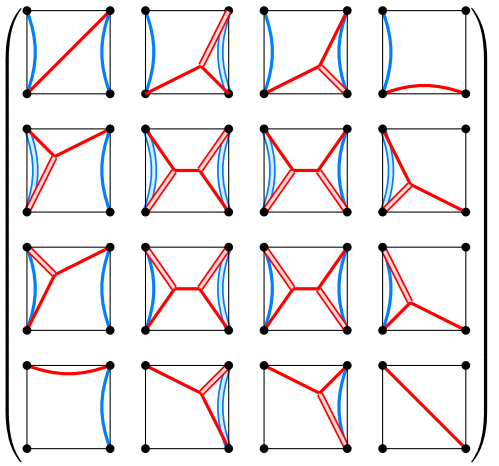


Fig. 11. The decorated triangulation corresponding to the double reduced word $\boldsymbol{s} = (\overline{1}, \overline{2}, \overline{1}, \overline{2}, 2, 1, 2, 1)$ (left) and the underlying weighted quiver (right).

introduced in [25] (namely, $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$ in the notation there). It implies $\mathscr{S}^1_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subseteq \mathscr{A}_{\mathfrak{sp}_4,\Sigma}$ at the classical specialization $q = 1 \in \mathbb{C}$.

Again, each entry of the matrix (5.2) comes from the localized skein algebra $\mathscr{S}^1_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}]$. Explicitly, we have

$g_{[c]} =$  .

Then by exactly the same line of argument as in the $\mathfrak{sl}_3$-case, we get $\mathscr{S}^1_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] = \mathscr{A}_{\mathfrak{sp}_4,\Sigma} = \mathscr{U}_{\mathfrak{sp}_4,\Sigma}$.

6. Supplements to Section 4.2

6.1. Amalgamation of upper cluster algebras

Lemma 6.1. *Let $\mathscr{U}$ be an upper cluster algebra having a cluster $\mathbf{i} = (\{A_i\}_{i\in I}, \varepsilon)$ with* isolated variables, *namely a subset $J \subset I$ such that $\varepsilon_{ij} = 0$ for all $j \in J$ and $i \in I$. Then the quotient of $\mathscr{U}$ by the ideal generated by $A_j - 1$ for $j \in J$ is the upper cluster algebra $\mathscr{U}'$ having the seed obtained from $\mathbf{i}$ by deleting the data for $j \in J$.*

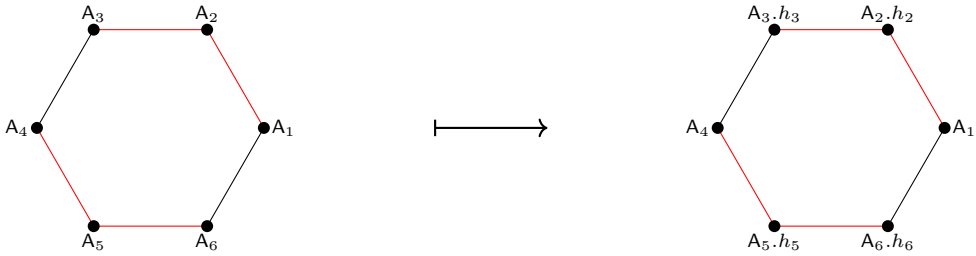


Fig. 12. The change of decorations with $n = 6$ and $J = \{2, 3, 5, 6\}$.

Proof. It is easy because $\mathscr{U} = \mathscr{U}' \otimes \mathbb{C}[A_j^{\pm 1} \mid j \in J]$ in this case. □

Fix a decorated triangulation $\boldsymbol{\mathcal{T}}$ of Σ. Let $\Lambda \subset e(\mathcal{T})$ be a subset of diagonals so that we obtain a collection of disks with marked points when cutting along them. Let $\mathcal{A}^{\times}_{G,\Sigma}[\Lambda] \subset \mathcal{A}^{\times}_{G,\Sigma}$ be the open subspaces such that for every edge in Λ, its associated pair of decorated flags is generic. Let $\mathbf{i}_{\boldsymbol{\mathcal{T}}}[\Lambda]$ be the seed obtained from $\boldsymbol{\mathcal{T}}$ by freezing all the mutable vertices that are placed on the diagonals in Λ. We are going to prove:

Proposition 6.2. $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}[\Lambda]) = \mathscr{U}(\mathbf{i}_{\boldsymbol{\mathcal{T}}}[\Lambda])$.

Lemma 4.6 is covered as the special case $\Lambda = e(\mathcal{T}) \setminus \{E\}$.

Proof. Let N be the number of disks obtained when cutting Σ along the diagonals in Λ. We shall prove Proposition 6.2 via induction on $N \geq 0$. The case $N = 0$ is trivial. For $N \geq 1$, let D be an obtained disk that contains a boundary edge of Σ. Let $\Sigma' \subset \Sigma$ be the marked surface obtained by cutting off D from Σ. The decorated triangulation $\boldsymbol{\mathcal{T}}$ of Σ naturally induces that of Σ', which is denoted by $\boldsymbol{\mathcal{T}}'$. Similarly, we define the spaces $\mathcal{A}^{\times}_{G,\Sigma'}[\Lambda] \subset \mathcal{A}^{\times}_{G,\Sigma'}$ and the seed $\mathbf{i}_{\mathcal{T}'}[\Lambda]$. By induction, we have

$$\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma'}[\Lambda]) = \mathscr{U}(\mathbf{i}_{\boldsymbol{\mathcal{T}}'}[\Lambda]).$$

We label the marked points of D from 1 to n in clockwise order, with the edge $\{1, n\}$ being a boundary interval of Σ. For $i = 2, \ldots, n$, we color the edge $\{i-1, i\}$ of D red if it is glued with either an edge of Σ' or another edge $\{j-1, j\}$ of D with $j < i$. Let J be the set of indices i such that $\{i-1, i\}$ is red. See Fig. 12 for an illustration of the coloring scheme.

We define the cutting map

$$\begin{aligned} \phi : \ & \mathcal{A}^{\times}_{G,\Sigma}[\Lambda] \times H^J \longrightarrow \mathcal{A}^{\times}_{G,\Sigma'}[\Lambda] \times \mathcal{A}^{\times}_{G,D}, \\ & ([\mathcal{L}, \alpha], \{h_j\}_{j \in J}) \longmapsto ([\mathcal{L}', \alpha'], (\mathsf{A}'_1, \ldots, \mathsf{A}'_n)). \end{aligned}$$

Here $[\mathcal{L}', \alpha']$ is the restriction of decorated twisted local system $[\mathcal{L}, \alpha]$ from Σ to Σ'. When restricting $[\mathcal{L}, \alpha]$ to D, we obtain a n-tuple $(\mathsf{A}_1, \ldots, \mathsf{A}_n)$, and set

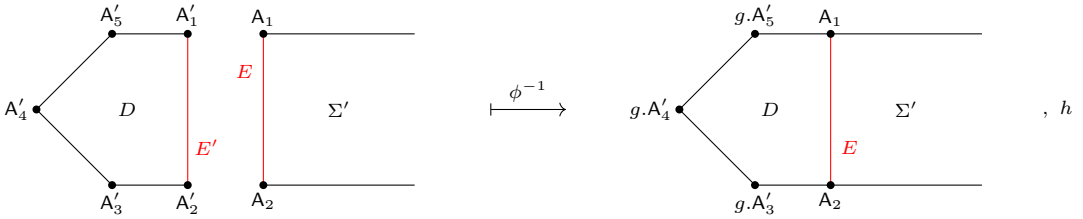


Fig. 13. The gluing map ϕ^{-1}. Here $h \in H$ is uniquely chosen so that $h(\mathsf{A}'_1, \mathsf{A}'_2.h^{-1}) = h(\mathsf{A}_1, \mathsf{A}_2)$. Then $g.(\mathsf{A}'_1, \mathsf{A}'_2.h^{-1}) = (\mathsf{A}_1, \mathsf{A}_2)$ for a unique $g \in G$.

$$\mathsf{A}'_j := \begin{cases} \mathsf{A}_j.h_j & \text{if } j \in J; \\ \mathsf{A}_j & \text{otherwise.} \end{cases}$$

Conversely, given the data $([\mathcal{L}', \alpha'], (\mathsf{A}'_1, \dots, \mathsf{A}'_n))$, one can recursively determine h_j for $j \in J$ such that after rescaling each A'_j by h_j^{-1}, the h-distance for the pair of decorated flags for every red edge coincides with the one that it is glued with. Gluing them back produces the preimage of $([\mathcal{L}', \alpha'], (\mathsf{A}'_1, \dots, \mathsf{A}'_n))$ before ϕ. For example, Fig. 13 is the gluing map along one edge. Hence, the map ϕ is an isomorphism.

The decorated triangulation $\boldsymbol{\mathcal{T}}$ induces a decorated triangulation of D, which further gives rise to a cluster seed $\mathbf{i}_D$ of $\mathcal{A}^{\times}_{G,D}$. By Lemma 4.5, we have $\mathcal{O}(\mathcal{A}^{\times}_{G,D}) = \mathscr{U}(\mathbf{i}_D)$. By [20, Theorem 9.17], every cluster variable A in $\mathbf{i}_D$ is homogeneous with respect to the H^n-actions. Denote by $\lambda(A) := (\lambda_1, \dots, \lambda_n)$ its weight, so that

$$A(\mathsf{A}_1.h_1, \dots, \mathsf{A}_n.h_n) = A(\mathsf{A}_1, \dots, \mathsf{A}_n) \cdot \prod_{i=1}^{n} h_i^{\lambda_i}.$$

The coordinate ring $\mathcal{O}(H^J)$ can be regarded as the upper cluster algebra $\mathscr{U}^{\mathrm{triv}}_m$ with isolated $m := |S| \cdot |J|$ vertices and cluster variables $h_j^{\varpi_s}$ for $j \in J$ and $s \in S$.

Let A be a cluster variable in $\mathbf{i}_{\boldsymbol{\mathcal{T}}'}[\Lambda] \sqcup \mathbf{i}_D$. Under the induced isomorphism

$$\phi^* : \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma'}[\Lambda]) \otimes \mathcal{O}(\mathcal{A}^{\times}_{G,D}) \xrightarrow{\sim} \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}[\Lambda]) \otimes \mathcal{O}(H^J), \tag{6.1}$$

we get

$$\phi^*(A) = \begin{cases} A & \text{if } A \text{ belong to } \mathbf{i}_{\boldsymbol{\mathcal{T}}'}[\Lambda], \\ A \cdot \prod_{j \in J} h_j^{\lambda(A)_j} & \text{if } A \text{ belongs to } \mathbf{i}_D. \end{cases}$$

We claim that ϕ^* is a quasi-isomorphism in the sense of Fraser [16]. Namely, ϕ^* rescales the cluster variables by Laurent monomials of frozen variables in such a way that the unfrozen cluster Poisson variables for algebras on both sides of (6.1) associated with the seeds induced by $\boldsymbol{\mathcal{T}}$ are the same. Indeed, following the geometric description of cluster Poisson charts on the moduli spaces $\mathcal{P}_{G,\Sigma}$ given in [20], the unfrozen cluster Poisson

variables are invariant under the rescaling of decorated flags by $h \in H$. Therefore, the claim follows directly.

As a consequence, the extension of ϕ^* to the fields of rational functions preserves the upper cluster algebras:

$$\phi^* : \mathscr{U}(\mathbf{i}_{\boldsymbol{\mathcal{T}}'}[\Lambda] \sqcup \mathbf{i}_D) \xrightarrow{\sim} \mathscr{U}(\mathbf{i}_{\boldsymbol{\mathcal{T}}}[\Lambda]) \otimes \mathscr{U}_m^{\mathrm{triv}}. \tag{6.2}$$

By the induction assumption, we have $\mathscr{U}(\mathbf{i}_{\boldsymbol{\mathcal{T}}'}[\Lambda] \sqcup \mathbf{i}_D) = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma'}[\Lambda]) \otimes \mathcal{O}(\mathcal{A}^{\times}_{G,D})$. Therefore we get $\mathscr{U}(\mathbf{i}_{\boldsymbol{\mathcal{T}}}[\Lambda]) \otimes \mathscr{U}_m^{\mathrm{triv}} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}[\Lambda]) \otimes \mathcal{O}(H^J)$ by (6.1) and (6.2). Then by applying Lemma 6.1 to the isolated variables $h_j^{\varpi_s}$ in $\mathscr{U}_m^{\mathrm{triv}}$, we get $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}[\Lambda]) = \mathscr{U}(\mathbf{i}_{\boldsymbol{\mathcal{T}}}[\Lambda])$ as desired. □

6.2. Stratifications of $\mathcal{A}_{G,\Sigma}$

Our purpose is to prove that $\mathcal{A}^{\mathcal{T},E_1,E_2}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$ has codimension ≥ 2. Along the way, we also obtain formulae for codimensions of the subspaces in more general stratifications of $\mathcal{A}_{G,\Sigma}$ obtained by prescribing w-distances along arcs and boundary intervals. Recall that the dimension of a quotient stack $\mathcal{X} = [X/G]$ is defined to be $\dim \mathcal{X} := \dim X - \dim G$.

Notation 6.3. For an oriented ideal arc α on Σ, let A^+_α (resp. A^-_α) denote the decorated flag assigned to the initial (resp. terminal) marked point of α. We endow each boundary interval with the orientation induced from the boundary.

Fix two diagonals E_1, E_2 of an ideal triangulation $\mathcal{T}$, and endow them arbitrary orientations. For $u, v \in W$, let $\mathcal{A}^{u,v}_{G,\Sigma} \subset \mathcal{A}^{\times}_{G,\Sigma}$ be the subspace such that $w(\mathsf{A}^+_{E_1}, \mathsf{A}^-_{E_1}) = u$, $w(\mathsf{A}^+_{E_2}, \mathsf{A}^-_{E_2}) = v$. Then we have

$$\mathcal{A}^{\mathcal{T},E_1,E_2}_{G,\Sigma} = \bigsqcup_{u \neq w_0;\ v \neq w_0} \mathcal{A}^{u,v}_{G,\Sigma}.$$

We are going to prove:

Theorem 6.4. *We have* $\dim \mathcal{A}_{G,\Sigma} - \dim \mathcal{A}^{u,v}_{G,\Sigma} = 2l(w_0) - l(u) - l(v)$ *for all* $u, v \in W$. *In particular,* $\mathcal{A}^{\mathcal{T},E_1,E_2}_{G,\Sigma}$ *has codimension* ≥ 2.

It turns out to be useful to include the following more general subspaces into consideration:

Definition 6.5.

- Given a tuple $\boldsymbol{w} = \{w_E\} \in W^{\mathbb{B}(\Sigma)}$, let

$$\mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$$

denote the subspace such that $w(\mathsf{A}_E^+, \mathsf{A}_E^-) = w_E$ for each $E \in \mathbb{B}(\Sigma)$. For $\boldsymbol{w}_0 = \{w_0\} \in W^{\mathbb{B}(\Sigma)}$ that assigns the longest element to each boundary interval, we have $\mathcal{A}^{\times}_{G,\Sigma} = \mathcal{A}^{\boldsymbol{w}_0}_{G,\Sigma}$.

- Given a collection $\mathcal{C}$ of disjoint oriented ideal arcs in Σ and a tuple $\boldsymbol{v}_{\mathcal{C}} = \{v_\alpha\} \in W^{\mathcal{C}}$, let

$$\mathcal{A}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_{\mathcal{C}}] \subset \mathcal{A}_{G,\Sigma}$$

denote the subspace such that $w(\mathsf{A}_\alpha^+, \mathsf{A}_\alpha^-) = v_\alpha$ for each $\alpha \in \mathcal{C}$.

Furthermore, let us write $\mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_{\mathcal{C}}] := \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} \cap \mathcal{A}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_{\mathcal{C}}]$. The original subspace of our interest is $\mathcal{A}^{u,v}_{G,\Sigma} = \mathcal{A}^{\boldsymbol{w}_0}_{G,\Sigma}[E_1, E_2; u, v]$.

Proposition 6.6. *Let $\boldsymbol{w} = \{w_E\} \in W^{\mathbb{B}(\Sigma)}$. If Σ is a polygon, then we assume that there exists $E_0 \in \mathbb{B}(\Sigma)$ such that $w_{E_0} = w_0$. Then $\dim \mathcal{A}_{G,\Sigma} - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} = \sum_{E \in \mathbb{B}(\Sigma)} (l(w_0) - l(w_E))$. In particular, $\mathcal{A}^{\times}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$ is open dense.*

It is proved in Section 6.2.2 below, based on a relation to the *braid varieties*.

Fix an oriented ideal arc α and an element $v_\alpha \in W$. Cutting the surface Σ along α, we obtain a new marked surface Σ', where α is splitted into two boundary intervals α', α'', where α' is the one following the boundary orientation. Then we consider the cutting map

$$\mathrm{cut}_\alpha : \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha] \to \mathcal{A}^{\boldsymbol{w}'}_{G,\Sigma'}, \tag{6.3}$$

where $\boldsymbol{w}' := \boldsymbol{w} \cup \{v_\alpha, v_\alpha^{-1}\}$ under the identification $\mathbb{B}(\Sigma') = \mathbb{B}(\Sigma) \cup \{\alpha', \alpha''\}$, v_α (resp. v_α^{-1}) being assigned to α' (resp. α''). The image of cut_α is characterized by the closed condition

$$h(\mathsf{A}^+_{\alpha'}, \mathsf{A}^-_{\alpha'}) = h(\mathsf{A}^-_{\alpha''}, \mathsf{A}^+_{\alpha''}) \tag{6.4}$$

Proposition 6.7. *Each fiber of cut_α is isomorphic to G_{v_α}, where G_u denotes the stabilizer of the pair $([U^+], \overline{u}.B^+)$ for $u \in W$. In particular if $v_\alpha = w_0$, then cut_α is an isomorphism onto its image.*

It is proved in Section 6.2.1 below, where we explicitly write down the presentation of cut_α on an atlas by fixing a generating system on Σ.

These two propositions might be of independent interest. Assuming them, let us first complete the proof of Theorem 6.4.

Proof of Theorem 6.4. Let us consider the cutting maps

$$\mathcal{A}^{u,v}_{G,\Sigma} = \mathcal{A}_{G,\Sigma}[E_1, E_2; u, v] \xrightarrow{\mathrm{cut}_{E_1}} \mathcal{A}^{\boldsymbol{w}'}_{G,\Sigma'}[E_2; v] \xrightarrow{\mathrm{cut}_{E_2}} \mathcal{A}^{\boldsymbol{w}''}_{G,\Sigma''},$$

where $\Sigma' := \Sigma \backslash E_1$, $\Sigma'' := \Sigma' \backslash E_2$, and E_1 (resp. E_2) is splitted into two boundary intervals E_1', E_1'' in Σ' (resp. E_2', E_2'' in Σ''). The elements $\boldsymbol{w}' = \{u, u^{-1}\}$, $\boldsymbol{w}'' = \{u, u^{-1}, v, v^{-1}\}$ are the ones naturally inherited via the cutting.

Recall that the image of the cutting map has the constraint (6.4). Then from Propositions 6.6 and 6.7, we get

$$\begin{aligned}
\dim \mathcal{A}^{u,v}_{G,\Sigma} &= \mathcal{A}^{\boldsymbol{w}'}_{G,\Sigma'}[E_2; v] - \dim H + \dim G_u \\
&= (\mathcal{A}^{\boldsymbol{w}''}_{G,\Sigma''} - \dim H + \dim G_v) - \dim H + \dim G_u \\
&= \dim \mathcal{A}^{\boldsymbol{w}''}_{G,\Sigma''} - 2 \dim H + \dim G_u + \dim G_v \\
&= (\dim \mathcal{A}_{G,\Sigma''} - 2(2l(w_0) - l(u) - l(v))) - 2 \dim H + \dim G_u + \dim G_v
\end{aligned}$$

Observe that $\dim \mathcal{A}^{\times}_{G,\Sigma''} - 2 \dim H = \dim \mathcal{A}^{\times}_{G,\Sigma}$ by the second statement of Proposition 6.7, and that $\dim \mathcal{A}_{G,X} = \dim \mathcal{A}^{\times}_{G,X}$ for $X = \Sigma, \Sigma''$ by the second statement of Proposition 6.6. We also have

$$\dim G_w = \dim(U^+ \cap \overline{w} U^+ \overline{w}^{-1}) = l(w_0) - l(w) \tag{6.5}$$

for any $w \in W$ [36, Section 8.3]. Thus we get

$$\begin{aligned}
\dim \mathcal{A}^{u,v}_{G,\Sigma} &= \dim \mathcal{A}_{G,\Sigma} - 2(2l(w_0) - l(u) - l(v)) + (l(w_0) - l(u)) + (l(w_0) - l(v)) \\
&= \dim \mathcal{A}_{G,\Sigma} - (2l(w_0) - l(u) - l(v)),
\end{aligned}$$

as desired. □

6.2.1. Proof of Proposition 6.7

Here are preparatory discussions. We may assume that Σ is connected, without loss of generality. Recall from Lemma 3.7 the presentation $\mathcal{A}_{G,\Sigma} = [A_{G,\Sigma}/G]$, where $A_{G,\Sigma}$ is a quasi-affine G-variety that can be identified with $\mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G)$.

More precisely, such an identification is provided if we fix a system $\mathcal{C}_{\mathrm{arc}}$ of arcs in $T'\Sigma$ from the basepoint ξ to the boundary intervals. Let us further specify a collection $\mathcal{C}_{\mathrm{loop}}$ of loops in Σ based at $x := \pi(\xi)$ that generate $\pi_1(\Sigma, x)$. See Fig. 14. Choosing framings of these loops (*i.e.*, lifts of them to $T'\Sigma$), we get a splitting $\pi_1(\Sigma, x) \to \pi_1(T'\Sigma, \xi)$ of the exact sequence (3.3), and hence an isomorphism

$$\mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \cong G^{\mathcal{C}_{\mathrm{loop}}} = G^{-\chi(\Sigma)+1}.$$

Let us call such a collection $\mathcal{C} := \mathcal{C}_{\mathrm{arc}} \cup \mathcal{C}_{\mathrm{loop}}$ a *generating system*. Below we work on the atlases $A_{G,\Sigma} \cong G^{\mathcal{C}_{\mathrm{loop}}} \times \mathcal{A}^{\mathbb{M}}_G$ by choosing an appropriate generating system $\mathcal{C}$ for a given α. The atlases of the relevant moduli spaces are denoted by $A^{\boldsymbol{w}}_{G,\Sigma} \subset A_{G,\Sigma}$, and so on.

Proof of Proposition 6.7. The proof is devided into the three cases.

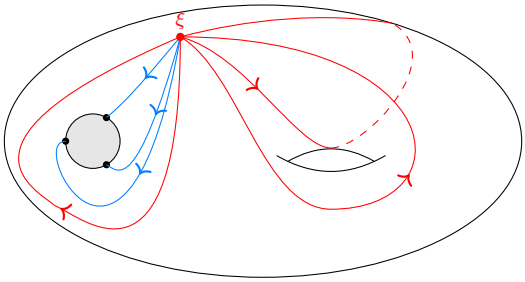


Fig. 14. A generating system $\mathcal{C} = \mathcal{C}_{\mathrm{arc}} \cup \mathcal{C}_{\mathrm{loop}}$ that determines an isomorphism $\mathcal{A}_{G,\Sigma} \cong G^{\mathcal{C}_{\mathrm{loop}}} \times \mathcal{A}_G^{\mathbb{M}}$. Here the framings of curves are omitted.

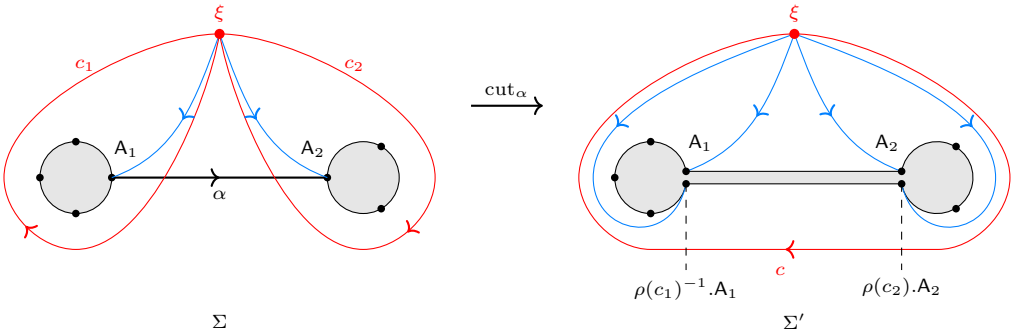


Fig. 15. The choice of curves: Case 1.

Case 1: Σ' is connected, and α connects different boundary components of Σ. Choose the curves shown in the left in Fig. 15, and extend it to a generating system $\mathcal{C}$ on Σ by choosing other curves disjointly from α. The curves disjoint from α naturally descend to Σ'. Together with the curves shown in the right of Fig. 15, they form a generating system $\mathcal{C}'$ on Σ'.

Note that $\mathrm{rank}\pi_1(\Sigma') = \mathrm{rank}\pi_1(\Sigma) - 1$. We have an embedding $\iota : \pi_1(\Sigma', x) \to \pi_1(\Sigma, x)$ given by $\iota(c) := c_2 * c_1$ and $\iota(d) := d$ for $d \in \mathcal{C}'_{\mathrm{loop}} \setminus \{c\}$. It induces a projection

$$\iota^* : \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \to \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma', \xi), G),$$

which is a principal G-bundle. Indeed, we have $\iota^*\rho(c) = \rho(c_2)\rho(c_1)$ and hence the G-action

$$(\rho(c_1), \rho(c_2)) \mapsto (g\rho(c_1), \rho(c_2)g^{-1}), \quad g \in G \tag{6.6}$$

parametrizes the fiber over $\rho \in \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma', \xi), G)$. Then the relevant components of the presentation $\widetilde{\mathrm{cut}}_\alpha : A^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha] \to \mathcal{A}^{\boldsymbol{w}'}_{G,\Sigma'}$ of (6.3) are given by

$$\mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \times \mathcal{A}^2_G \to \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma', \xi), G) \times \mathcal{A}^4_G,$$

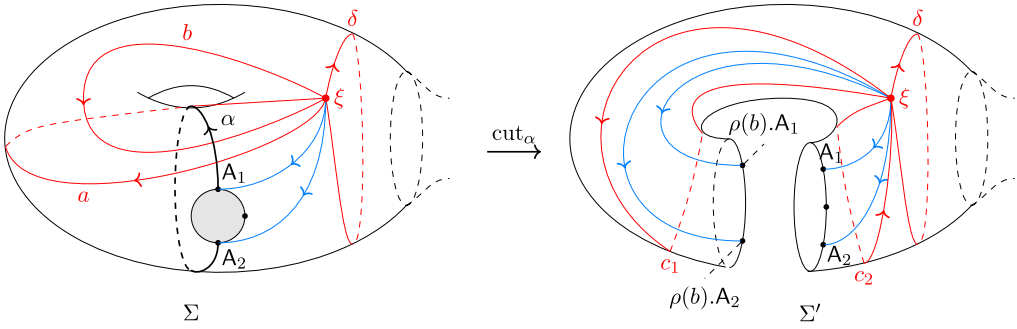


Fig. 16. The choice of curves: Case 2. Here δ can be written as a product of other loops in $\mathcal{C}$.

$$(\rho; \mathsf{A}_1, \mathsf{A}_2) \mapsto (\iota^*\rho; \mathsf{A}_1, \mathsf{A}_2, \rho(c_1)^{-1}.\mathsf{A}_1, \rho(c_2).\mathsf{A}_2).$$

Observe that the G-action (6.6) preserves the $\mathcal{A}^4_G$-component if and only if $g \in \mathrm{Stab}(\mathsf{A}_1, \mathsf{A}_2) \subset G$. Hence the fiber of $\widetilde{\mathrm{cut}}_\alpha$ equals $\mathrm{Stab}(\mathsf{A}_1, \mathsf{A}_2)$, which is isomorphic to G_u.

Case 2: Σ' is connected, and α connects the same boundary component C of Σ. In this case, one can find a handle that contains α as shown in Fig. 16. (Indeed, if we shrink the boundary component C to a puncture p, α becomes a based loop α_p at p. If α_p was non-essential, *i.e.*, belonged to the kernel of the intersection form, then it would cut Σ into a disconnected surface, which contradicts to our assumption. Then it is a standard argument in topology to find such a handle.)

Choose the curves shown in the left in Fig. 16, and extend it to a generating system $\mathcal{C}$ on Σ by choosing other curves disjointly from α. The curves disjoint from α naturally descend to Σ'. Together with the curves shown in the right of Fig. 16, they form a generating system $\mathcal{C}'$ on Σ'.

Note that $\mathrm{rank}\pi_1(\Sigma') = \mathrm{rank}\pi_1(\Sigma) - 1$. We have an embedding $\iota : \pi_1(\Sigma', x) \to \pi_1(\Sigma, x)$ given by $\iota(c_1) := b * a * b^{-1}$, $\iota(c_2) := b * a * b^{-1} * \delta$ and $\iota(d) := d$ for $d \in \mathcal{C}'_{\mathrm{loop}} \setminus \{c_1, c_2\}$. It induces a projection

$$\iota^* : \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \to \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma', \xi), G),$$

which is a principal G-bundle. Indeed, we have $\iota^*\rho(c_1) = \rho(b)\rho(a)\rho(b)^{-1}$ and $\iota^*\rho(c_2) = \rho(b)\rho(a)\rho(b)^{-1}\rho(\delta)$; hence the G-action

$$(\rho(a), \rho(b)) \mapsto (g^{-1}\rho(a)g, \rho(b)g), \quad g \in G \tag{6.7}$$

parametrizes the fiber over $\rho \in \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma', \xi), G)$. Then the relevant components of the presentation $\widetilde{\mathrm{cut}}_\alpha : A^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha] \to \mathcal{A}^{\boldsymbol{w}'}_{G,\Sigma'}$ of (6.3) are given by

$$\mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \times \mathcal{A}^2_G \to \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma', \xi), G) \times \mathcal{A}^4_G,$$
$$(\rho; \mathsf{A}_1, \mathsf{A}_2) \mapsto (\iota^*\rho; \mathsf{A}_1, \mathsf{A}_2, \rho(b).\mathsf{A}_1, \rho(b).\mathsf{A}_2).$$

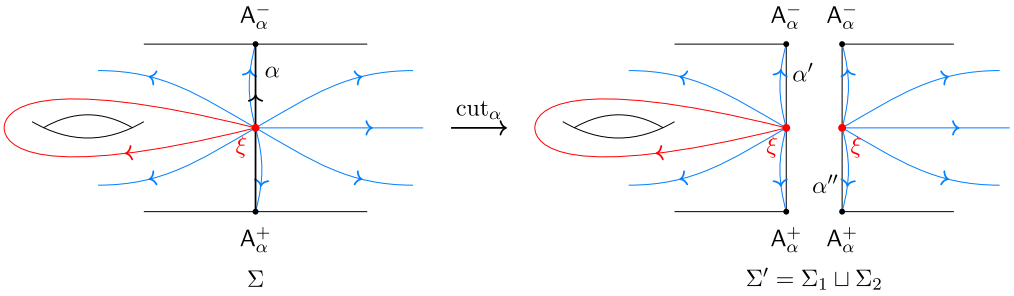


Fig. 17. The choice of curves: Case 3.

Observe that the G-action (6.7) preserves the $\mathcal{A}^4_G$-component if and only if $g \in \mathrm{Stab}(\mathsf{A}_1, \mathsf{A}_2) \subset G$. Hence the fiber of $\widetilde{\mathrm{cut}}_\alpha$ equals $\mathrm{Stab}(\mathsf{A}_1, \mathsf{A}_2)$, which is isomorphic to G_u.

Case 3: Σ' is disconnected. Let $\Sigma' = \Sigma_1 \sqcup \Sigma_2$ be the decomposition into connected components, where Σ_1, Σ_2 are connected marked surfaces. Choose the basepoint x on α. Then $\pi_1(\Sigma, x) = \pi_1(\Sigma_1, x) * \pi_1(\Sigma_2, x)$. We can choose the generating system $\mathcal{C}$ on Σ so that they do not cross α, as shown in Fig. 17. It induces a generating system $\mathcal{C}'$ on Σ', where the two arcs connecting to the endpoints of α are doubled.

Let $\overline{A}^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha] \subset A^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha]$ denote the subspace such that $(\mathsf{A}^+_\alpha, \pi(\mathsf{A}^-_\alpha)) = ([U^+], \overline{u}.B^+)$. Since any point $(\rho, \lambda) \in A^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha]$ can be translated into such a configuration by the G-action, we have

$$\mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha] = [A^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha]/G] = [\overline{A}^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha]/G_u].$$

A similar argument shows

$$\mathcal{A}^{\boldsymbol{w}'}_{G,\Sigma'} = [A^{\boldsymbol{w}'}_{G,\Sigma}/(G \times G)] = [\overline{A}^{\boldsymbol{w}'}_{G,\Sigma}/(G_u \times G_u)].$$

Observe that $\overline{A}^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha]$ is isomorphic via $\widetilde{\mathrm{cut}}_\alpha$ to the closed subspace of $\overline{A}^{\boldsymbol{w}'}_{G,\Sigma}$ characterized by the condition $h(\mathsf{A}^+_{\alpha'}, \mathsf{A}^-_{\alpha'}) = h(\mathsf{A}^-_{\alpha''}, \mathsf{A}^+_{\alpha''})$. Identifying these spaces, we have

$$\mathrm{cut}_\alpha : [\overline{A}^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha]/G_u] \to [\overline{A}^{\boldsymbol{w}}_{G,\Sigma}[\alpha; v_\alpha]/(G_u \times G_u)],$$

which has the fiber $(G_u \times G_u)/G_u \cong G_u$. The assertion is proved. □

6.2.2. Proof of Proposition 6.6

The knowledge on the *braid varieties* is useful below. We refer the reader to [6] for details. Here we introduce a stacky variant of them. Given a positive braid word $\beta = (s_1, \ldots, s_l)$ with $s_1, \ldots, s_l \in S$, the associated *braid stack* is defined to be the quotient stack

$$X(\beta) := \big[\{(\mathsf{A}_0, \mathsf{B}_1, \dots, \mathsf{B}_{l-1}, \mathsf{B}_l) \in \mathcal{A}_G \times \mathcal{B}_G^l \mid \\ w(\mathsf{B}_i, \mathsf{B}_{i-1}) = r_{s_i} \text{ for } i = 1, \dots, l;\ w(\mathsf{B}_0, \mathsf{B}_l) = \delta(\beta)\}/G\big].$$

Here $\delta(\beta) \in W$ denotes the *Demazure product*. A configuration in $X(\beta)$ is illustrated as

$$\begin{array}{ccccc} \mathsf{B}_{l-1} & \xleftarrow{s_{l-1}} & \cdots & \xleftarrow{s_2} & \mathsf{B}_1 \\ {\scriptstyle s_l}\downarrow & & & & \uparrow{\scriptstyle s_1} \\ \mathsf{B}_l & & \xleftarrow[\delta(\beta)]{} & & \mathsf{A}_0. \end{array}$$

We call $\mathsf{B}_l \xleftarrow{\delta(\beta)} \mathsf{A}_0$ the *bottom side* of the configuration. If $\delta(\beta) = w_0$, then the G-action is free and we can uniquely translate the generic pair $(\mathsf{A}_0, \mathsf{B}_{l+1})$ to the standard pinning $([U^+], B^-)$. Hence

$$X(\beta) \cong \{(\mathsf{B}_1, \dots, \mathsf{B}_{l-1}) \in \mathcal{B}_G^{l-1} \mid w(\mathsf{B}_{i-1}, \mathsf{B}_i) = r_{s_i} \text{ for } i = 1, \dots, l,\ \mathsf{B}_0 := B^+, \mathsf{B}_l := B^-\}$$

is an affine algebraic variety. In this case, it is known that $\dim X(\beta) = |\beta| - l(w_0)$ [12, Theorem 20], where $|\beta| = l$ is the length of β.

In general, if $\delta(\beta) = w \in W$, the pair $(\mathsf{A}_0, \mathsf{B}_{l+1})$ can be translated to the position $([U^+], \overline{w}.B^+)$, whose stabilizer subgroup is G_w. Hence we get

$$X(\beta) \cong \big[\{(\mathsf{B}_1, \dots, \mathsf{B}_{l-1}) \in \mathcal{B}_G^{l-1} \mid w(\mathsf{B}_{i-1}, \mathsf{B}_i) = r_{s_i} \\ \text{for } i = 1, \dots, l,\ \mathsf{B}_0 := B^+, \mathsf{B}_l := \overline{w}.\mathsf{B}_0\}/G_w\big]$$

We again know the dimension of the affine algebraic variety before quotient by [12, Theorem 20], and we get

$$\dim X(\beta) = |\beta| - l(w) - \dim G_w = |\beta| - l(w_0). \tag{6.8}$$

Let us drop the condition on the relative position between B_0 and B_l, defining

$$\overline{X}(\beta) := \big[\{(\mathsf{A}_0, \mathsf{B}_1, \dots, \mathsf{B}_{l-1}, \mathsf{B}_l) \in \mathcal{A}_G \times \mathcal{B}_G^l \mid \\ w(\mathsf{B}_i, \mathsf{B}_{i-1}) = r_{s_i} \text{ for } i = 1, \dots, l\}/G\big].$$

Note that before quotient by G, the space is isomorphic to $\mathcal{A}_G \times \mathbb{A}^{|\beta|}$, since a pair of flags of w-distance r_s for $s \in S$ can be translated into the position $(B^+, x_s(t)\overline{r}_s.B^+)$ for some $t \in \mathbb{A}$. Therefore

$$\dim \overline{X}(\beta) = \dim \mathcal{A}_G + |\beta| - \dim G = |\beta| - l(w_0). \tag{6.9}$$

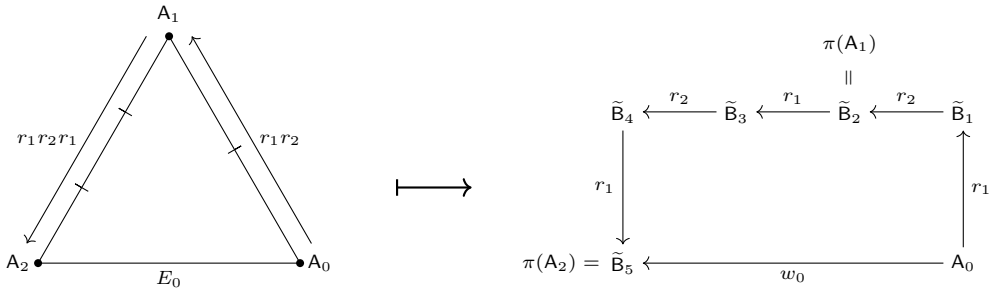


Fig. 18. A map from $\mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}$ to $X(\beta)$. Here Σ is a triangle, and $\boldsymbol{w} = (w_{E_0}, w_{E_1}, w_{E_2}) = (w_0, r_1r_2, r_1r_2r_1)$.

Case 1:* Σ *is a* k*-gon* ($k \geq 3$)*. Enumerate the boundary intervals as $E_0, E_1, \dots, E_{k-1}$ in this counter-clockwise order so that $w_{E_0} = w_0$. Choose a reduced word $\boldsymbol{s}(i)$ of w_{E_i} for $i = 1, \dots, k-1$. Then the concatenation

$$\beta := \boldsymbol{s}(1)\boldsymbol{s}(2)\dots\boldsymbol{s}(k-1)$$

is a positive braid word of length $l = \sum_{i=1}^{k-1} l(w_i)$ such that $\delta(\beta) = w_0$. Let A_0 be the decoration assigned to the terminal endpoint of E_0. Then we have an isomorphism

$$\begin{aligned}&\mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} \xrightarrow{\sim} X(\beta) \times H^{k-1},\\ &[\mathsf{A}_0, \dots, \mathsf{A}_{k-1}] \mapsto ([\mathsf{A}_0, \widetilde{\mathsf{B}}_1, \dots, \widetilde{\mathsf{B}}_{l-1}], h(\mathsf{A}_0, \mathsf{A}_1), \dots, h(\mathsf{A}_{k-2}, \mathsf{A}_{k-1})).\end{aligned}$$

See Fig. 18. Here for each pair $(\mathsf{B}_{i-1}, \mathsf{B}_i)$ with $i = 1, \dots, k-1$, we take the sequence of interpolating flags with their distances given by $\boldsymbol{s}(i)$, and denote by $\widetilde{\mathsf{B}}_1, \dots, \widetilde{\mathsf{B}}_{l-1}$ the resulting sequence from B_0 to B_{k-1}. The bottom side is $\mathsf{B}_{k-1} \xleftarrow{w_0} \mathsf{A}_0$. Observe that the data $[\mathsf{A}_0, \mathsf{A}_1, \dots, \mathsf{A}_{k-1}]$ can be uniquely recovered from $[\mathsf{A}_0, \mathsf{B}_1, \dots, \mathsf{B}_{k-1}]$ and $h(\mathsf{A}_{i-1}, \mathsf{A}_i)$ for $i = 1, \dots, k-1$. It follows that

$$\dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} = \dim X(\beta) + (k-1)\operatorname{rank} G = \sum_{i=1}^{k-1} l(w_i) - l(w_0) + (k-1)\operatorname{rank} G$$

by (6.8). The open dense part $\mathcal{A}^{\times}_{G,\Sigma}$ corresponds to $w_{E_i} = w_0$ for all $i = 1, \dots, k-1$. Therefore

$$\dim \mathcal{A}_{G,\Sigma} - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} = \dim \mathcal{A}^{\times}_{G,\Sigma} - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} = \sum_{i=1}^{k-1} (l(w_0) - l(w_{E_i})),$$

as desired.

Case 2:* Σ *is not a polygon. This case can be reduced to the polygon case, as follows. Let $\partial\Sigma = \bigsqcup_{\nu=0}^{b-1} \partial_\nu$. Choose a special point $m_\nu \in \partial_\nu$ for $\nu = 1, \dots, b-1$, and a *cut system*

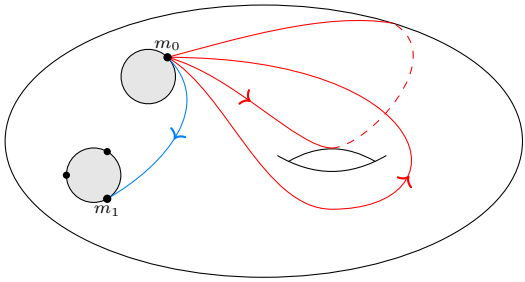


Fig. 19. A cut system $\mathcal{C}$ of Σ, which consists of $2g$ loops based at m_0 and $b-1$ arcs connecting m_0 to m_ν, $\nu = 1, \dots, b-1$.

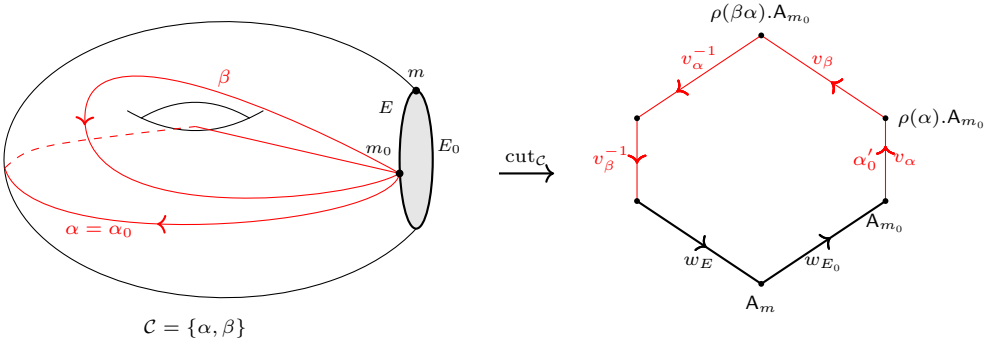


Fig. 20. The cutting map $\mathrm{cut}_{\mathcal{C}}$. In this example, we get two braid words $\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; \alpha_0') = \boldsymbol{s}(v_\beta)\boldsymbol{s}(v_\alpha^{-1})\boldsymbol{s}(v_\beta^{-1})\boldsymbol{s}(w_E)\boldsymbol{s}(w_{E_0})$ and $\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; E_0) = \boldsymbol{s}(v_\alpha)\boldsymbol{s}(v_\beta)\boldsymbol{s}(v_\alpha^{-1})\boldsymbol{s}(v_\beta^{-1})\boldsymbol{s}(w_E)$, where $\boldsymbol{s}(w)$ for $w \in W$ is an arbitrary reduced word.

$\mathcal{C}$ as shown in Fig. 19. The collection $\mathcal{C}$ consists of $2g + b - 1 > 0$ curves. By cutting Σ along the curves in $\mathcal{C}$, we get a polygon Π with $4g + 2(b-1) + |\mathbb{M}|$ sides. Let us consider the subspace $\mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; v_{\mathcal{C}}] \subset \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}$, where $v_{\mathcal{C}} = \{v_\alpha\}_{\alpha\in\mathcal{C}} \in W^{\mathcal{C}}$ and recall Definition 6.5. Let us consider the composite of cutting maps along the arcs in $\mathcal{C}$:

$$\mathrm{cut}_{\mathcal{C}} := \prod_{\alpha\in\mathcal{C}} \mathrm{cut}_\alpha : \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; v_{\mathcal{C}}] \to \mathcal{A}^{\boldsymbol{w}_{\mathcal{C}}}_{G,\Pi}, \tag{6.10}$$

where $\boldsymbol{w}_{\mathcal{C}} := \boldsymbol{w} \cup \{v_\alpha, v_\alpha^{-1} \mid \alpha \in \mathcal{C}\}$ is naturally inherited via the cutting, the pair $\{v_\alpha^{\pm}\}$ being assigned to the two boundary intervals arising from α. See Fig. 20. Observe that the image $\mathrm{Im}(\mathrm{cut}_{\mathcal{C}}) \subset \mathcal{A}^{\boldsymbol{w}_{\mathcal{C}}}_{G,\Pi}$ is characterized by the closed condition (6.4), one for each $\alpha \in \mathcal{C}$. By Proposition 6.7, the fiber of $\mathrm{cut}_{\mathcal{C}}$ is isomorphic to the product $\prod_{\alpha\in\mathcal{C}} G_{v_\alpha}$.

Choose arbitrary reduced words of $\boldsymbol{w}$ and $\boldsymbol{v}_{\mathcal{C}}$. We further choose a side α_0' of the polygon Π which comes from a curve $\alpha_0 \in \mathcal{C}$. Then we get a braid word $\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; \alpha_0')$ by reading off the reduced words assigned to the sides of Π except for α_0' along the boundary orientation of Π.

Then we get an embedding

$$\mathrm{Im}(\mathrm{cut}_{\mathcal{C}}) \hookrightarrow \overline{X}(\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; \alpha_0')) \times H^{2g+b-1+|\mathbb{M}|-1}. \tag{6.11}$$

From this embedding, we can compute

$$\dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; v_{\mathcal{C}}] \leq \sum_{\alpha \in \mathcal{C}} \dim G_{v_\alpha} + (2g + b - 1 + |\mathbb{M}| - 1) \operatorname{rank} G + \dim \overline{X}(\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; \alpha_0')). \tag{6.12}$$

In particular, when $w_{\alpha_0'} = w_0 = \delta(\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; \alpha_0'))$, we get an isomorphism

$$\mathrm{Im}(\mathrm{cut}_{\mathcal{C}}) \xrightarrow{\sim} X(\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; \alpha_0')) \times H^{2g+b-1+|\mathbb{M}|-1}. \tag{6.13}$$

In this case, we get

$$\dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; v_{\mathcal{C}}] = \sum_{\alpha \in \mathcal{C}} \dim G_{v_\alpha} + (2g + b - 1 + |\mathbb{M}| - 1) \operatorname{rank} G + \dim \overline{X}(\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; \alpha_0')). \tag{6.14}$$

Let $\boldsymbol{v}_0 := \{w_0\}_{\alpha \in \mathcal{C}} \in W^{\mathcal{C}}$. We have the following Lemma.

Lemma 6.8.

(1) *Fixing $\boldsymbol{v}_{\mathcal{C}} = \boldsymbol{v}_0$ to be the longest, we get*

$$\begin{aligned} \dim \mathcal{A}^{\times}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_0] - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_0] &= \dim X(\beta(\boldsymbol{w}_0, \boldsymbol{v}_0; \alpha_0')) - \dim X(\beta(\boldsymbol{w}, \boldsymbol{v}_0; \alpha_0')) \\ &= \sum_{E \in \mathbb{B}(\Sigma)} (l(w_0) - l(w_E)). \end{aligned}$$

In particular, the subspace $\mathcal{A}^{\times}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_0] \subset \mathcal{A}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_0]$ is open dense.

(2) *Fixing any distance $\boldsymbol{w} \in W^{\mathbb{B}(\Sigma)}$, we get*

$$\dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_0] - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_{\mathcal{C}}] \geq 0$$

Therefore the subspace $\mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_0] \subset \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}$ has the maximal dimension.

Proof. (1): The first equality follows from (6.14). Since the w-distances on the boundary intervals are set to be the longest, we have $\delta(\beta(\boldsymbol{w}, \boldsymbol{v}_{\mathcal{C}}; \alpha_0')) = w_0$. Then the second equality follows from (6.8).

(2): We apply (6.14) to compute $\dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_0]$ and (6.12) to $\dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C}; \boldsymbol{v}_c]$. By (6.8) and (6.9), we get

$$\begin{aligned}\dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C};\boldsymbol{v}_0] - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C};\boldsymbol{v}_{\mathcal{C}}] &\geq \dim X(\beta(\boldsymbol{w},\boldsymbol{v}_0;\alpha_0')) - \dim \overline{X}(\beta(\boldsymbol{w},\boldsymbol{v}_{\mathcal{C}};\alpha_0')) \\ &\quad - \sum_{\alpha\in\mathcal{C}} \dim G_{v_\alpha} \\ &= |\beta(\boldsymbol{w},\boldsymbol{v}_0;\alpha_0')| - |\beta(\boldsymbol{w},\boldsymbol{v}_{\mathcal{C}};\alpha_0')| - \sum_{\alpha\in\mathcal{C}} (l(w_0) - l(v_\alpha)).\end{aligned}$$

Then the claim follows from

$$|\beta(\boldsymbol{w},\boldsymbol{v}_0;\alpha_0')| - |\beta(\boldsymbol{w},\boldsymbol{v}_{\mathcal{C}};\alpha_0')| = (l(w_0) - l(v_{\alpha_0})) + 2\sum_{\alpha\in\mathcal{C}\setminus\{\alpha_0\}} (l(w_0) - l(v_\alpha)),$$

where notice that the pair $\{v_\alpha, v_\alpha^{-1}\}$ necessarily appears along the braid word $\beta(\boldsymbol{w},\boldsymbol{v}_{\mathcal{C}};E_0)$. □

Now let us complete the proof of Proposition 6.6. By Lemma 6.8 (2), it suffices to compute the codimension

$$\dim \mathcal{A}_{G,\Sigma} - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} = \dim \mathcal{A}_{G,\Sigma}[\mathcal{C};\boldsymbol{v}_0] - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C};\boldsymbol{v}_0].$$

Then by Lemma 6.8 (1), it can be computed as

$$\begin{aligned}\dim \mathcal{A}_{G,\Sigma} - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} &= \dim \mathcal{A}_{G,\Sigma}[\mathcal{C};\boldsymbol{v}_0] - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C};\boldsymbol{v}_0] \\ &= \dim \mathcal{A}^{\times}_{G,\Sigma}[\mathcal{C};\boldsymbol{v}_0] - \dim \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma}[\mathcal{C};\boldsymbol{v}_0] = \sum_{E\in\mathbb{B}(\Sigma)} (l(w_0) - l(w_E)),\end{aligned}$$

as desired.

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