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Analysis and application of a time-domain finite element method for the Drude metamaterial perfectly matched layer model☆

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ABSTRACT

In this paper, we develop a finite element method (FEM) to solve the Drude metamaterial perfectly matched layer (PML) model by edge elements. Both the stability analysis and error estimate for the scheme are established. To our best knowledge, this is the first paper on developing and analyzing a FEM for this Drude PML model. Numerical results are presented to justify the theoretical error estimate and demonstrate the effectiveness of this PML in absorbing outgoing waves in the Drude metamaterial.

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1. Introduction

The perfectly matched layer (PML) technique was first proposed in 1994 by Berenger [1] to solve the time-dependent Maxwell's equations in unbounded domains. Over the last three decades, many PML models have been developed for solving Maxwell's equations in both time domain [2–6] and frequency domain [7,8]. Due to their demonstrated effectiveness in absorbing the outgoing waves over a wide frequency range, the PML technique has been quickly extended to solve other types of wave propagation problems [9–12], such as acoustics, elastodynamics [13], elasticity, anisotropic dispersive media and metamaterials. The readers can find more details and references on PMLs for Maxwell's equations in books ([14, Ch.7], [15, Ch.8]), the review papers [16].

The newly discovered metamaterials (cf. [15,17,18] and references cited therein) have many potential exciting applications such as invisibility cloaks and subwavelength imaging, and have inspired a growing interest in the study of wave propagation in metamaterials (e.g., [19,20]). Some early applications of PMLs for metamaterials [21–23] found that special PML models are needed to avoid unstability. In 2015, Bécache et al. [11] presented a rigorous way in developing stable PMLs for the Drude metamaterials. In 2022, we [24] developed and analyzed a FDTD method for solving a Drude

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metamaterial PML model proposed in [11]. For potential broader application of the Drude PML model in complex geometry problems, in this paper we develop and analyze a finite element method to solve the Drude PML model. We like to emphasize that though there are many published papers on the analysis and applications of Drude model (e.g., [19,20]) and PML models for wave interaction with vacuum (e.g., [2–6]), this is the first paper on finite element analysis and application for the Drude PML model.

The rest of the paper is organized as follows. In Section 2, we present the Drude PML model, and construct a fully-discrete finite element scheme for solving the model. In Section 3, we establish the stability analysis and error estimate for our proposed scheme. In Section 4, some numerical results are presented to confirm our error estimate and further apply our scheme to simulate two practical wave propagation problems to show the effectiveness of wave absorbing property of PML model. We conclude the paper in Section 5.

2. The Drude metamaterial PML model and a fully-discrete finite element scheme

Here we consider a 2-D transverse electric (TE) metamaterial PML model developed by Bécache et al. [11] and solved by the FDTD scheme [24]. Denoting the electric field $\mathbf{E} = (E_x, E_y)'$, the induced electric current $\mathbf{J} = (J_x, J_y)'$, the magnetic field $H = H^x + H^y$, the induced magnetic current (K^x, K^y), and $\boldsymbol{\sigma} = diag(\sigma_y, \sigma_x)$, we can write the governing equations for the TEz PML model as follows [24, (2.1)]: For any $(x, y, t) \in \Omega \times (0, T]$,

$$\partial_t \mathbf{E} + \omega_e^2 \mathbf{J} + \varepsilon_0^{-1} \boldsymbol{\sigma} \mathbf{E} = \varepsilon_0^{-1} \nabla \times H, \tag{1a}$$

$$\partial_t \mathbf{J} - \mathbf{E} = 0, \tag{1b}$$

$$\partial_t H^x + \omega_m^2 K^x + \mu_0^{-1} \sigma_y H^x = \mu_0^{-1} \partial_y E_x, \tag{1c}$$

$$\partial_t K^x - H^x = 0, \tag{1d}$$

$$\partial_t H^y + \omega_m^2 K^y + \mu_0^{-1} \sigma_x H^y = -\mu_0^{-1} \partial_x E_y, \tag{1e}$$

$$\partial_t K^y - H^y = 0, \tag{1f}$$

where $\sigma_x(x) \geq 0$ and $\sigma_y(y) \geq 0$ are the damping functions in the x and y directions, ϵ_0 and μ_0 are the permittivity and permeability in free space, and $\omega_e > 0$ and $\omega_m > 0$ are the electric and the magnetic plasma frequencies in the Drude model described as follows:

$$\epsilon(\omega) = \epsilon_0 (1 - \frac{\omega_e^2}{\omega^2}), \quad \mu(\omega) = \mu_0 (1 - \frac{\omega_m^2}{\omega^2}).$$

Considering that the PML is often used as a rectangle surrounding the interested physical domain, we assume that Ω is a rectangular domain.

To complete the setup of the model, we assume that the model problem (1a)–(1f) is subject to the initial conditions

$$\boldsymbol{E}(x, y, 0) := \boldsymbol{E}_0(x, y) = (E_{x0}(x, y), E_{y0}(x, y))', \quad \boldsymbol{J}(x, y, 0) := \boldsymbol{J}_0(x, y) = (J_{x0}(x, y), J_{y0}(x, y))', \tag{2}$$

$$H^x(x, y, 0) = H_{x0}(x, y), H^y(x, y, 0) = H_{y0}(x, y), K^x(x, y, 0) = K_{x0}(x, y), K^y(x, y, 0) = K_{y0}(x, y), \tag{3}$$

and the perfect conduct (PEC) boundary condition

$$\boldsymbol{E} \cdot \hat{\tau} = 0, \quad \text{on } \partial\Omega, \tag{4}$$

where $E_{x0}, E_{y0}, J_{x0}, J_{y0}, H_{x0}, H_{y0}, K_{x0}$, and K_{y0} are some properly given functions. Moreover, $\hat{\tau}$ denotes the unit tangential vector along $\partial\Omega$.

In order to develop a finite element scheme for solving (1a)–(1f), we partition the domain Ω by a family of regular triangular mesh T_h with maximum mesh size h, and adopt the rth $(r \geq 1)$ order Raviart–Thomas–Nédélec (RTN) mixed finite element spaces on triangular elements:

$$\boldsymbol{U}_h = \{\boldsymbol{\phi}_h \in H(curl; \Omega) : \ \boldsymbol{\phi}_h|_K \in (p_{r-1})^2 \oplus S_r, \ \forall K \in T_h\}, \ S_r = \{\vec{p} \in (\tilde{p}_r)^2, \boldsymbol{x} \cdot \vec{p} = 0\},$$
$$V_h = \{\psi_h \in L^2(\Omega) : \ \psi_h|_K \in p_{r-1}, \ \forall K \in T_h\},$$

where $\tilde{p}_r$ denotes the space of homogeneous polynomials of degree r, and p_r denotes the space of polynomials of degree less than or equal to r in variables x, y, respectively. To impose the PMC boundary condition, we denote the subspace

$$\boldsymbol{U}_h^0 = \{\boldsymbol{\phi}_h \in \boldsymbol{U}_h : \ \boldsymbol{\phi}_h \cdot \hat{n} = 0 \ \ on \ \ \partial\Omega\}.$$

To define a fully discrete scheme, we divide the time interval $[0, T]$ into N_t uniform subintervals by points $t_k = k\tau, k = 0, 1, \ldots, N_t$, where the time step size $\tau = \frac{T}{N_t}$. Moreover, for the time solution $u^k = u(\cdot, k\tau)$, we introduce the following central difference and average operators:

$$\delta_\tau u^n = (u^{n+\frac{1}{2}} - u^{n-\frac{1}{2}})/\tau, \quad \bar{u}^n = \frac{1}{2}(u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}).$$

Using the finite element spaces U_h^0 and V_h,we can construct the following leapfrog scheme for solving (1a)–(1f): For any $n \geq 0$, find $\boldsymbol{E}_h^{n+1} = (E_{xh}, E_{yh})', \boldsymbol{J}_h^{n+\frac{1}{2}} \in U_h^0$, and $H_h^{x,n+\frac{3}{2}}, H_h^{y,n+\frac{3}{2}}, K_h^{x,n+1}, K_h^{y,n+1} \in V_h$ such that:

$$\varepsilon_0(\delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}, \boldsymbol{\phi}_h) + \varepsilon_0\omega_e^2(\boldsymbol{J}_h^{n+\frac{1}{2}}, \boldsymbol{\phi}_h) + (\boldsymbol{\sigma}\overline{\boldsymbol{E}}_h^{n+\frac{1}{2}}, \boldsymbol{\phi}_h) = (H_h^{n+\frac{1}{2}}, \nabla \times \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{U}_h^0, \tag{5a}$$

$$(\delta_\tau \boldsymbol{J}_h^n, \tilde{\boldsymbol{\phi}}_h) - (\boldsymbol{E}_h^n, \tilde{\boldsymbol{\phi}}_h) = 0, \quad \forall \tilde{\boldsymbol{\phi}}_h \in \boldsymbol{U}_h^0, \tag{5b}$$

$$\mu_0(\delta_\tau H_h^{x,n+1}, \psi_h) + \mu_0\omega_m^2(K_h^{x,n+1}, \psi_h) + (\sigma_y \overline{H}_h^{x,n+1}, \psi_h) = (\partial_y E_{xh}^{n+1}, \psi_h), \quad \forall \psi_h \in V_h, \tag{5c}$$

$$(\delta_\tau K_h^{x,n+\frac{1}{2}}, \psi_{1h}) - (H_h^{x,n+\frac{1}{2}}, \psi_{1h}) = 0, \quad \forall \psi_{1h} \in V_h, \tag{5d}$$

$$\mu_0(\delta_\tau H_h^{y,n+1}, \tilde{\psi}_h) + \mu_0\omega_m^2(K_h^{y,n+1}, \tilde{\psi}_h) + (\sigma_x \overline{H}_h^{y,n+1}, \tilde{\psi}_h) = -(\partial_x E_{yh}^{n+1}, \tilde{\psi}_h), \quad \forall \tilde{\psi}_h \in V_h, \tag{5e}$$

$$(\delta_\tau K_h^{y,n+\frac{1}{2}}, \tilde{\psi}_{1h}) - (H_h^{y,n+\frac{1}{2}}, \tilde{\psi}_{1h}) = 0, \quad \forall \tilde{\psi}_{1h} \in V_h, \tag{5f}$$

where the needed initial approximations $\boldsymbol{E}_h^0, H_h^{x,\frac{1}{2}}, H_h^{y,\frac{1}{2}}, \boldsymbol{J}_h^{-\frac{1}{2}}, K_h^{x,0}$ and $K_h^{y,0}$ can be obtained as follows:

$$\boldsymbol{E}_h^0 = \Pi_h^c \boldsymbol{E}_0(\boldsymbol{x}), \quad K_h^{x,0} = \Pi_h^2 K_{x0}(\boldsymbol{x}), \quad K_h^{y,0} = \Pi_h^2 K_{y0}(\boldsymbol{x}), \quad \boldsymbol{J}_h^{-\frac{1}{2}} = \Pi_h^c[\boldsymbol{J}_0 - \frac{\tau}{2}\boldsymbol{E}_0](\boldsymbol{x}), \tag{6a}$$

$$H_h^{x,\frac{1}{2}} = \Pi_h^2\left[(H^x + \frac{\tau}{2}\partial_t H^x)(\boldsymbol{x}, 0)\right] = \Pi_h^2\left[H_{x0} + \frac{\tau}{2}(\mu_0^{-1}\partial_y E_{x0} - \omega_m^2 K_{x0} - \mu_0^{-1}\sigma_y H_{x0})\right], \tag{6b}$$

$$H_h^{y,\frac{1}{2}} = \Pi_h^2\left[(H^y + \frac{\tau}{2}\partial_t H^y)(\boldsymbol{x}, 0)\right] = \Pi_h^2\left[H_{y0} + \frac{\tau}{2}(-\mu_0^{-1}\partial_x E_{y0} - \omega_m^2 K_{y0} - \mu_0^{-1}\sigma_x H_{y0})\right], \tag{6c}$$

where Π_h^c and Π_h^2 denote the standard Nédélec interpolation into the space $\boldsymbol{U}_h$ and the standard L^2 projection into the space V_h, respectively.

We like to remark that the scheme (5a)–(5f) is very efficient in decoupling the original coupled system (1a)–(1f). The scheme can be implemented as follows: for any $n \geq 0$, at each time step,

Step 1. Solve (5b) for $\boldsymbol{J}_h^{n+\frac{1}{2}}$;

Step 2. Solve (5a) for $\boldsymbol{E}_h^{n+1}$; Solve (5d) for $K_h^{x,n+1}$; Solve (5f) for $K_h^{y,n+1}$. Note that these three equations can be done in parallel.

Step 3. Solve (5c) for $H_h^{x,n+\frac{3}{2}}$; Solve (5e) for $H_h^{y,n+\frac{3}{2}}$. These two equations can be solved in parallel too.

3. Stability analysis and error estimate for the scheme

In this section, we will present the stability analysis and the error estimate for our proposed scheme (5a)–(5f).

3.1. The stability analysis

Denote $c_v = \frac{1}{\sqrt{\epsilon_0\mu_0}}$ for the speed of light in vacuum, and the positive constant c_{inv} appearing in the standard inverse estimate [15, p. 98–101]:

$$\|\nabla \times \boldsymbol{u}_h\| \leq c_{inv} h^{-1}\|\boldsymbol{u}_h\|, \quad \forall \boldsymbol{u}_h \in \boldsymbol{U}_h, \tag{7}$$

here and below $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ denotes the L^2 norm over Ω.

Theorem 1. *For the solution $(\boldsymbol{E}_h^{n+1}, \boldsymbol{J}_h^{n+\frac{1}{2}}, H_h^{x,n+\frac{3}{2}}, H_h^{y,n+\frac{3}{2}}, K_h^{x,n+1}, K_h^{y,n+1})$ of scheme (5a)–(5f), we define the following discrete energy:*

$$\mathcal{E}_{dis}(n) := \frac{\varepsilon_0}{2}\|\boldsymbol{E}_h^{n+1}\|^2 + \frac{\epsilon_0\omega_e^2}{2}\|\boldsymbol{J}_h^{n+\frac{1}{2}}\|^2 + \frac{\mu_0}{2}\|(H_h^x + H_h^y)^{n+\frac{3}{2}}\|^2 + \frac{\mu_0\omega_m^2}{2}\|(K_h^x + K_h^y)^{n+1}\|^2. \tag{8}$$

Then for any $m \in [1, N_t - 1]$ and any nonnegative functions $\sigma_x(x)$ and $\sigma_y(y)$, we have the discrete energy identity:

$$\begin{aligned}
&\frac{1}{\tau}[\mathcal{E}_{dis}(m) - \mathcal{E}_{dis}(-1)] \\
&+ \sum_{0\leq n\leq m}\|\sigma^{\frac{1}{2}}\overline{\mathbf{E}}_h^{n+\frac{1}{2}}\|^2 + \sum_{0\leq n\leq m}\left[\|\sigma_y^{\frac{1}{2}}\overline{H}_h^{x,n+1}\|^2 + \|\sigma_x^{\frac{1}{2}}\overline{H}_h^{y,n+1}\|^2 + \left((\sigma_x + \sigma_y)\overline{H}_h^{x,n+1}, \overline{H}_h^{y,n+1}\right)\right] \\
&+ \frac{\mu_0\omega_m^2}{2}\left[\left((K_h^x + K_h^y)^{m+1}, (H_h^x + H_h^y)^{m+\frac{3}{2}}\right) - \left((K_h^x + K_h^y)^0, (H_h^x + H_h^y)^{\frac{1}{2}}\right)\right] \\
&+ \frac{\epsilon_0\omega_e^2}{2}\left[(\mathbf{J}_h^{m+\frac{1}{2}}, \mathbf{E}_h^{m+1}) - (\mathbf{J}_h^{-\frac{1}{2}}, \mathbf{E}_h^0)\right] = \frac{1}{2}\left[(H_h^{\frac{1}{2}}, \nabla \times \mathbf{E}_h^0) - (H_h^{m+\frac{3}{2}}, \nabla \times \mathbf{E}_h^{m+1})\right].
\end{aligned} \tag{9}$$

Furthermore, when $\sigma_x = \sigma_y = \sigma$ is a positive constant, under the time step constraint:

$$\tau \leq \min\{\frac{1}{\sqrt{2}\omega_e}, \frac{1}{\sqrt{2}\omega_m}, \frac{h}{2c_v c_{inv}}\}, \tag{10}$$

we have the following discrete stability for scheme (5a)–(5f):

$$\mathcal{E}_{dis}(m) \leq 3\mathcal{E}_{dis}(-1), \quad \forall m \in [1, N_t - 1]. \tag{11}$$

Proof. To make our proof easy to follow, we divide the proof into several major steps.

(I) Choosing $\phi_h = \overline{\boldsymbol{E}}_h^{n+\frac{1}{2}}$ in (5a), we obtain

$$\frac{\varepsilon_0}{2\tau}\left(\|\boldsymbol{E}_h^{n+1}\|^2 - \|\boldsymbol{E}_h^{n}\|^2\right) + \varepsilon_0\omega_e^2(\boldsymbol{J}_h^{n+\frac{1}{2}}, \overline{\boldsymbol{E}}_h^{n+\frac{1}{2}}) + \|\boldsymbol{\sigma}^{\frac{1}{2}}\overline{\boldsymbol{E}}_h^{n+\frac{1}{2}}\|^2 = (H_h^{n+\frac{1}{2}}, \nabla \times \overline{\boldsymbol{E}}_h^{n+\frac{1}{2}}). \tag{12}$$

Choosing $\tilde{\phi}_h = \varepsilon_0\omega_e^2\bar{\boldsymbol{J}}_h^n$ in (5b), we have:

$$\frac{\varepsilon_0\omega_e^2}{2\tau}\left(\|\boldsymbol{J}_h^{n+\frac{1}{2}}\|^2 - \|\boldsymbol{J}_h^{n-\frac{1}{2}}\|^2\right) - \varepsilon_0\omega_e^2(\boldsymbol{E}_h^n, \bar{\boldsymbol{J}}_h^n) = 0. \tag{13}$$

Adding (12) and (13) together, then summing up the result from $n = 0$ to any $m < N_t - 1$, and using the following identity:

$$(\boldsymbol{J}_h^{n+\frac{1}{2}}, \overline{\boldsymbol{E}}_h^{n+\frac{1}{2}}) - (\boldsymbol{E}_h^n, \bar{\boldsymbol{J}}_h^n) = \frac{1}{2}\left((\boldsymbol{J}_h^{n+\frac{1}{2}}, \boldsymbol{E}_h^{n+1}) - (\boldsymbol{J}_h^{n-\frac{1}{2}}, \boldsymbol{E}_h^n)\right), \tag{14}$$

we have:

$$\begin{aligned}&\frac{\varepsilon_0}{2\tau}\left(\|\boldsymbol{E}_h^{m+1}\|^2 - \|\boldsymbol{E}_h^{0}\|^2\right) + \frac{\varepsilon_0\omega_e^2}{2\tau}\left(\|\boldsymbol{J}_h^{m+\frac{1}{2}}\|^2 - \|\boldsymbol{J}_h^{-\frac{1}{2}}\|^2\right) + \sum_{0\leq n\leq m}\|\boldsymbol{\sigma}^{\frac{1}{2}}\overline{\boldsymbol{E}}_h^{n+\frac{1}{2}}\|^2 \\ &+ \frac{\varepsilon_0\omega_e^2}{2}\left((\boldsymbol{J}_h^{m+\frac{1}{2}}, \boldsymbol{E}_h^{m+1}) - (\boldsymbol{J}_h^{-\frac{1}{2}}, \overline{\boldsymbol{E}}_h^{0})\right) = \sum_{0\leq n\leq m}(H_h^{n+\frac{1}{2}}, \nabla \times \overline{\boldsymbol{E}}_h^{n+\frac{1}{2}}).\end{aligned} \tag{15}$$

(II) Choosing $\psi_h = (\overline{H}_h^x + \overline{H}_h^y)^{n+1}$ in (5c), we obtain

$$\begin{aligned}&\frac{\mu_0}{2\tau}\left(\|H_h^{x,n+\frac{3}{2}}\|^2 - \|H_h^{x,n+\frac{1}{2}}\|^2\right) + \mu_0(\delta_\tau H_h^{x,n+1}, \overline{H}_h^{y,n+1}) \\ &+ \mu_0\omega_m^2(K_h^{x,n+1}, \overline{H}_h^{x,n+1} + \bar{H}_h^{y,n+1}) + \|\sigma_y^{\frac{1}{2}}\bar{H}_h^{x,n+1}\|^2 + (\sigma_y\overline{H}_h^{x,n+1}, \overline{H}_h^{y,n+1}) \\ =&(\partial_y E_{xh}^{n+1}, \overline{H}_h^{x,n+1} + \overline{H}_h^{y,n+1}).\end{aligned} \tag{16}$$

Choosing $\tilde{\psi}_{1h} = \mu_0\omega_m^2(\overline{K}_h^x + \overline{K}_h^y)^{n+\frac{1}{2}}$ in (5d), we have

$$\begin{aligned}&\frac{\mu_0\omega_m^2}{2\tau}\left(\|K_h^{x,n+1}\|^2 - \left\|K_h^{x,n}\right\|^2\right) + \mu_0\omega_m^2(\delta_\tau K_h^{x,n+\frac{1}{2}}, \overline{K}_h^{y,n+\frac{1}{2}}) \\ &- \mu_0\omega_m^2(H_h^{x,n+\frac{1}{2}}, \overline{K}_h^{x,n+\frac{1}{2}} + \overline{K}_h^{y,n+\frac{1}{2}}) = 0.\end{aligned} \tag{17}$$

Now adding (16) and (17) together, then summing up the result from $n = 0$ to any $m < N_t - 1$, and using the identity:

$$(K_h^{x,n+1}, \overline{H}_h^{x,n+1}) - (H_h^{x,n+\frac{1}{2}}, \overline{K}_h^{x,n+\frac{1}{2}}) = \frac{1}{2}\left[(K_h^{x,n+1}, H_h^{x,n+\frac{3}{2}}) - (K_h^{x,n}, H_h^{x,n+\frac{1}{2}})\right] \tag{18}$$

we have

$$\begin{aligned}
&\frac{\mu_0}{2\tau}\left(\|H_h^{x,m+\frac{3}{2}}\|^2-\|H_h^{x,\frac{1}{2}}\|^2\right)+\sum_{0\le n\le m}\mu_0(\delta_\tau H_h^{x,n+1},\overline{H}_h^{y,n+1})+\sum_{0\le n\le m}\|\sigma_y^{\frac{1}{2}}\overline{H}_h^{x,n+1}\|^2\\
&\quad+\frac{\mu_0\omega_m^2}{2}\left((K_h^{x,m+1},H_h^{x,m+\frac{3}{2}})-(K_h^{x,0},H_h^{x,\frac{1}{2}})\right)+\sum_{0\le n\le m}\mu_0(\sigma_y\overline{H}_h^{x,n+1},\overline{H}_h^{y,n+1})\\
&\quad+\frac{\mu_0\omega_m^2}{2\tau}\left(\left\|K_h^{x,m+1}\right\|^2-\left\|K_h^{x,0}\right\|^2\right)+\sum_{0\le n\le m}\mu_0\omega_m^2(\delta_\tau K_h^{x,n+\frac{1}{2}},\overline{K}_h^{y,n+\frac{1}{2}})\\
&\quad+\sum_{0\le n\le m}\mu_0\omega_m^2\left((K_h^{x,n+1},\overline{H}_h^{y,n+1})-(H_h^{x,n+\frac{1}{2}},\overline{K}_h^{y,n+\frac{1}{2}})\right)\\
&=\sum_{0\le n\le m}(\partial_y E_{xh}^{n+1},\overline{H}_h^{x,n+1}+\overline{H}_h^{y,n+1}).
\end{aligned}\tag{19}$$

Similarly, from (5e) and (5f) we can obtain

$$\begin{aligned}
&\frac{\mu_0}{2\tau}\left(\|H_h^{y,m+\frac{3}{2}}\|^2-\|H_h^{y,\frac{1}{2}}\|^2\right)+\sum_{0\le n\le m}\mu_0(\delta_\tau H_h^{y,n+1},\overline{H}_h^{x,n+1})+\sum_{0\le n\le m}\|\sigma_x^{\frac{1}{2}}\overline{H}_h^{y,n+1}\|^2\\
&+\frac{\mu_0\omega_m^2}{2}\left((K_h^{y,m+1},H_h^{y,m+\frac{3}{2}})-(K_h^{y,0},H_h^{y,\frac{1}{2}})\right)+\sum_{0\le n\le m}\mu_0(\sigma_x\overline{H}_h^{y,n+1},\overline{H}_h^{x,n+1})\\
&+\frac{\mu_0\omega_m^2}{2\tau}\left(\left\|K_h^{y,m+1}\right\|^2-\left\|K_h^{y,0}\right\|^2\right)+\sum_{0\le n\le m}\mu_0\omega_m^2(\delta_\tau K_h^{y,n+\frac{1}{2}},\overline{K}_h^{x,n+\frac{1}{2}})\\
&+\sum_{0\le n\le m}\mu_0\omega_m^2\left((K_h^{y,n+1},\overline{H}_h^{x,n+1})-(H_h^{y,n+\frac{1}{2}},\overline{K}_h^{x,n+\frac{1}{2}})\right)\\
&=-\sum_{0\le n\le m}(\partial_x E_{yh}^{n+1},\bar{H}_h^{x,n+1}+\bar{H}_h^{y,n+1}).
\end{aligned}\tag{20}$$

(III) Adding up the first and second terms of (19) and (20) gives us:

$$\begin{aligned}
&\frac{\mu_0}{2\tau}\left(\|H_h^{x,m+\frac{3}{2}}\|^2-\|H_h^{x,\frac{1}{2}}\|^2\right)+\frac{\mu_0}{2\tau}\left(\|H_h^{y,m+\frac{3}{2}}\|^2-\|H_h^{y,\frac{1}{2}}\|^2\right)\\
&\quad+\mu_0\sum_{0\le n\le m}\left((\delta_\tau H_h^{x,n+1},\overline{H}_h^{y,n+1})+(\delta_\tau H_h^{y,n+1},\overline{H}_h^{x,n+1})\right)\\
&=\frac{\mu_0}{2\tau}\left(\|H_h^{x,m+\frac{3}{2}}\|^2-\|H_h^{x,\frac{1}{2}}\|^2\right)+\frac{\mu_0}{2\tau}\left(\|H_h^{y,m+\frac{3}{2}}\|^2-\|H_h^{y,\frac{1}{2}}\|^2\right)+\frac{\mu_0}{\tau}\left((H_h^{y,m+\frac{3}{2}},H_h^{x,m+\frac{3}{2}})-(H_h^{y,\frac{1}{2}},H_h^{x,\frac{1}{2}})\right)\\
&=\frac{\mu_0}{2\tau}\left(\|H_h^{x,m+\frac{3}{2}}+H_h^{y,m+\frac{3}{2}}\|^2-\|H_h^{x,\frac{1}{2}}+H_h^{y,\frac{1}{2}}\|^2\right).
\end{aligned}\tag{21}$$

Similarly, adding up the sixth and seventh terms of (19) and (20) leads to :

$$\begin{aligned}
&\frac{\mu_0\omega_m^2}{2\tau}\left[\left(\left\|K_h^{x,m+1}\right\|^2-\left\|K_h^{x,0}\right\|^2\right)+\left(\left\|K_h^{y,m+1}\right\|^2-\left\|K_h^{y,0}\right\|^2\right)\right.\\
&\quad\left.+2\tau\sum_{0\le n\le m}\left((\delta_\tau K_h^{x,n+\frac{1}{2}},\overline{K}_h^{y,n+\frac{1}{2}})+(\delta_\tau H_h^{y,n+\frac{1}{2}},\overline{H}_h^{x,n+\frac{1}{2}})\right)\right]\\
&=\frac{\mu_0\omega_m^2}{2\tau}\left[\left(\left\|K_h^{x,m+1}\right\|^2-\left\|K_h^{x,0}\right\|^2\right)+\left(\left\|K_h^{y,m+1}\right\|_0^2-\left\|K_h^{y,0}\right\|^2\right)+2\left((K_h^{x,m+1},K_h^{y,m+1})-(K_h^{x,0},K_h^{y,0})\right)\right]\\
&=\frac{\mu_0\omega_m^2}{2\tau}\left(\left\|K_h^{x,m+1}+K_h^{y,m+1}\right\|^2-\left\|K_h^{x,0}+K_h^{y,0}\right\|^2\right).
\end{aligned}\tag{22}$$

Also note that adding up the fourth and eighth terms of (19) and (20) yields:

$$\begin{aligned}
&\frac{\mu_0\omega_m^2}{2}\left[(K_h^{x,m+1},H_h^{x,m+\frac{3}{2}})-(K_h^{x,0},H_h^{x,\frac{1}{2}})+(K_h^{y,m+1},H_h^{y,m+\frac{3}{2}})-(K_h^{y,0},H_h^{y,\frac{1}{2}})\right]\\
&\quad+\sum_{0\le n\le m}\mu_0\omega_m^2\left[(K_h^{x,n+1},\overline{H}_h^{y,n+1})-(H_h^{x,n+\frac{1}{2}},\overline{K}_h^{y,n+\frac{1}{2}})+(K_h^{y,n+1},\overline{H}_h^{x,n+1})-(H_h^{y,n+\frac{1}{2}},\overline{K}_h^{x,n+\frac{1}{2}})\right]\\
&=\frac{\mu_0\omega_m^2}{2}\left[(K_h^{x,m+1},H_h^{x,m+\frac{3}{2}})-(K_h^{x,0},H_h^{x,\frac{1}{2}})+(K_h^{y,m+1},H_h^{y,m+\frac{3}{2}})-(K_h^{y,0},H_h^{y,\frac{1}{2}})\right]\\
&+\frac{\mu_0\omega_m^2}{2}\left[\left((K_h^{x,m+1},H_h^{y,m+\frac{3}{2}})-(K_h^{x,0},H_h^{y,\frac{1}{2}})\right)+\left((K_h^{y,m+1},H_h^{x,m+\frac{3}{2}})-(K_h^{y,0},H_h^{x,\frac{1}{2}})\right)\right]\\
&=\frac{\mu_0\omega_m^2}{2}\left[(K_h^{x,m+1},H_h^{x,m+\frac{3}{2}}+H_h^{y,m+\frac{3}{2}})-(K_h^{x,0},H_h^{x,\frac{1}{2}}+H_h^{y,\frac{1}{2}})\right.\\
&\quad\left.+(K_h^{y,m+1},H_h^{x,m+\frac{3}{2}}+H_h^{y,m+\frac{3}{2}})-(K_h^{y,0},H_h^{x,\frac{1}{2}}+H_h^{y,\frac{1}{2}})\right]\\
&=\frac{\mu_0\omega_m^2}{2}\left[(K_h^{x,m+1}+K_h^{y,m+1},H_h^{x,m+\frac{3}{2}}+H_h^{y,m+\frac{3}{2}})-(K_h^{x,0}+K_h^{y,0},H_h^{x,\frac{1}{2}}+H_h^{y,\frac{1}{2}})\right].
\end{aligned}\tag{23}$$

Hence, adding (19) and (20) together, and using the identities (21)–(23), we have

$$\begin{aligned}
&\frac{\mu_0}{2\tau}\left(\|H_h^{x,m+\frac{3}{2}}+H_h^{y,m+\frac{3}{2}}\|^2-\|H_h^{x,\frac{1}{2}}+H_h^{y,\frac{1}{2}}\|^2\right)\\
&+\frac{\mu_0\omega_m^2}{2\tau}\left(\left\|K_h^{x,m+1}+K_h^{y,m+1}\right\|^2-\left\|K_h^{x,0}+K_h^{y,0}\right\|^2\right)\\
&+\frac{\mu_0\omega_m^2}{2}\left[(K_h^{x,m+1}+K_h^{y,m+1},H_h^{x,m+\frac{3}{2}}+H_h^{y,m+\frac{3}{2}})-(K_h^{x,0}+K_h^{y,0},H_h^{x,\frac{1}{2}}+H_h^{y,\frac{1}{2}})\right]\\
&+\sum_{0\le n\le m}\left[\|\sigma_y^{\frac{1}{2}}\overline{H}_h^{x,n+1}\|^2+\|\sigma_x^{\frac{1}{2}}\overline{H}_h^{y,n+1}\|^2+((\sigma_x+\sigma_y)\overline{H}_h^{x,n+1},\overline{H}_h^{y,n+1})\right]\\
&=\sum_{0\le n\le m}\left[(\partial_y E_{xh}^{n+1},\overline{H}_h^{x,n+1}+\overline{H}_h^{y,n+1})-(\partial_x E_{yh}^{n+1},\overline{H}_h^{x,n+1}+\overline{H}_h^{y,n+1})\right].
\end{aligned}\tag{24}$$

(IV) Note that adding up the right hand side (RHS) terms of (15) and (24) give us:

$$\begin{aligned}
&\sum_{0\le n\le m}(H_h^{n+\frac{1}{2}},\nabla\times\overline{\mathbf{E}}_h^{n+\frac{1}{2}})+\sum_{0\le n\le m}\left[(\partial_y E_{xh}^{n+1},\overline{H}_h^{x,n+1}+\overline{H}_h^{y,n+1})-(\partial_x E_{yh}^{n+1},\overline{H}_h^{x,n+1}+\overline{H}_h^{y,n+1})\right]\\
&=\sum_{0\le n\le m}(H_h^{n+\frac{1}{2}},\nabla\times\overline{\mathbf{E}}_h^{n+\frac{1}{2}})+\sum_{0\le n\le m}(\partial_y E_{xh}^{n+1}-\partial_x E_{yh}^{n+1},\overline{H}_h^{x,n+1}+\overline{H}_h^{y,n+1})\\
&=\sum_{0\le n\le m}\left[(H_h^{n+\frac{1}{2}},\nabla\times\overline{\mathbf{E}}_h^{n+\frac{1}{2}})-(\overline{H}_h^{n+1},\nabla\times\mathbf{E}_h^{n+1})\right]\\
&=\sum_{0\le n\le m}\frac{1}{2}\left[\left(H_h^{n+\frac{1}{2}},(\nabla\times\mathbf{E}_h^{n+1}+\nabla\times\mathbf{E}_h^{n})\right)-\left((H_h^{n+\frac{3}{2}}+H_h^{n+\frac{1}{2}}),\nabla\times\mathbf{E}_h^{n+1}\right)\right]\\
&=\sum_{0\le n\le m}\frac{1}{2}\left[(H_h^{n+\frac{1}{2}},\nabla\times\mathbf{E}_h^{n})-(H_h^{n+\frac{3}{2}},\nabla\times\mathbf{E}_h^{n+1})\right]\\
&=\frac{1}{2}\left[(H_h^{\frac{1}{2}},\nabla\times\mathbf{E}_h^{0})-(H_h^{m+\frac{3}{2}},\nabla\times\mathbf{E}_h^{m+1})\right].
\end{aligned}\tag{25}$$

Now adding up (15) and (24), and using (25), we obtain

$$\begin{aligned}
&\frac{\varepsilon_0}{2\tau}\left(\|\mathbf{E}_h^{m+1}\|^2-\|\mathbf{E}_h^{0}\|^2\right)+\frac{\varepsilon_0\omega_e^2}{2\tau}\left(\|\mathbf{J}_h^{m+\frac{1}{2}}\|^2-\|\mathbf{J}_h^{-\frac{1}{2}}\|^2\right)+\frac{\mu_0}{2\tau}\left(\|H_h^{x,m+\frac{3}{2}}+H_h^{y,m+\frac{3}{2}}\|^2-\|H_h^{x,\frac{1}{2}}+H_h^{y,\frac{1}{2}}\|^2\right)\\
&+\frac{\mu_0\omega_m^2}{2\tau}\left(\left\|K_h^{x,m+1}+K_h^{y,m+1}\right\|^2-\left\|K_h^{x,0}+K_h^{y,0}\right\|^2\right)+\frac{\varepsilon_0\omega_e^2}{2}\left[(\mathbf{J}_h^{m+\frac{1}{2}},\mathbf{E}_h^{m+1})-(\mathbf{J}_h^{-\frac{1}{2}},\bar{\mathbf{E}}_h^{0})\right]\\
&+\sum_{0\le n\le m}\left[\|\sigma_y^{\frac{1}{2}}\overline{H}_h^{x,n+1}\|^2+\|\sigma_x^{\frac{1}{2}}\overline{H}_h^{y,n+1}\|^2+\left((\sigma_x+\sigma_y)\overline{H}_h^{x,n+1},\overline{H}_h^{y,n+1}\right)\right]+\sum_{0\le n\le m}\|\sigma^{\frac{1}{2}}\overline{\mathbf{E}}_h^{n+\frac{1}{2}}\|^2
\end{aligned}$$

$$+\frac{\mu_0\omega_m^2}{2}\left[(K_h^{x,m+1}+K_h^{y,m+1},H_h^{x,m+\frac{3}{2}}+H_h^{y,m+\frac{3}{2}})-(K_h^{x,0}+K_h^{y,0},H_h^{x,\frac{1}{2}}+H_h^{y,\frac{1}{2}})\right]$$
$$=\frac{1}{2}\left[(H_h^{\frac{1}{2}},\nabla\times\mathbf{E}_h^0)-(H_h^{m+\frac{3}{2}},\nabla\times\mathbf{E}_h^{m+1})\right].$$

Using the discrete energy definition (8), we conclude the proof of the discrete energy identity (9).

(V) When $\sigma_x=\sigma_y=\sigma$ is a positive constant, using the identity:

$$\|\sigma_y^{\frac{1}{2}}\overline{H}_h^{x,n+1}\|^2+\|\sigma_x^{\frac{1}{2}}\overline{H}_h^{y,n+1}\|^2+\left((\sigma_x+\sigma_y)\overline{H}_h^{x,n+1},\overline{H}_h^{y,n+1}\right)=\|\sigma^{\frac{1}{2}}(\overline{H}_h^{x,n+1}+\overline{H}_h^{y,n+1})\|^2,$$

and dropping the second and third terms on the left hand side of (9), we have

$$\begin{aligned}&\mathcal{E}_{\text{dis}}(m)-\mathcal{E}_{\text{dis}}(-1)\\&\le-\frac{\tau\mu_0\omega_m^2}{2}\left[\left((K_h^x+K_h^y)^{m+1},(H_h^x+H_h^y)^{m+\frac{3}{2}}\right)-\left((K_h^x+K_h^y)^0,(H_h^x+H_h^y)^{\frac{1}{2}}\right)\right]\\&-\frac{\tau\epsilon_0\omega_e^2}{2}\left[(\mathbf{J}_h^{m+\frac{1}{2}},\mathbf{E}_h^{m+1})-(\mathbf{J}_h^{-\frac{1}{2}},\mathbf{E}_h^0)\right]+\frac{\tau}{2}\left[(H_h^{\frac{1}{2}},\nabla\times\mathbf{E}_h^0)-(H_h^{m+\frac{3}{2}},\nabla\times\mathbf{E}_h^{m+1})\right].\end{aligned}\tag{26}$$

Now we just need to estimate those right hand side terms of (26). By Young's inequality, we easily have

$$\frac{\tau\mu_0\omega_m^2}{2}\left((K_h^x+K_h^y)^{m+1},(H_h^x+H_h^y)^{m+\frac{3}{2}}\right)$$
$$\le\frac{\mu_0\omega_m^2}{4}\|\left(K_h^x+K_h^y\right)^{m+1}\|^2+\frac{(\tau\omega_m)^2\mu_0}{4}\|\left(H_h^x+H_h^y\right)^{m+\frac{3}{2}}\|^2,\tag{27a}$$

$$\frac{\tau\epsilon_0\omega_e^2}{2}(\boldsymbol{J}_h^{m+\frac{1}{2}},\boldsymbol{E}_h^{m+1})\le\frac{\varepsilon_0\omega_e^2}{4}\|\boldsymbol{J}_h^{m+\frac{1}{2}}\|^2+\frac{(\tau\omega_e)^2\varepsilon_0}{4}\left\|\boldsymbol{E}_h^{m+1}\right\|^2,\tag{27b}$$

$$\begin{aligned}&\frac{\tau}{2}(H_h^{m+\frac{3}{2}},\nabla\times\boldsymbol{E}_h^{m+1})=(\sqrt{\mu_0}H_h^{m+\frac{2}{3}},\frac{\tau c_v\sqrt{\varepsilon_0}}{2}\nabla\times\boldsymbol{E}_h^{m+1})\\&\le\frac{\mu_0}{8}\|H_h^{m+\frac{3}{2}}\|^2+\frac{(\tau c_v)^2\varepsilon_0}{2}\|\nabla\times\boldsymbol{E}_h^{m+1}\|^2\\&\le\frac{\mu_0}{8}\|H_h^{m+\frac{3}{2}}\|^2+\frac{(\tau c_v c_{inv})^2}{2h^2}\varepsilon_0\|\boldsymbol{E}_h^{m+1}\|^2,\end{aligned}\tag{27c}$$

where we used the inverse estimate (7) and the notation $c_v=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$ to derive the last inequality.

Substituting (27a)–(27c) and similar estimates for the rest two terms into (26), we obtain

$$\begin{aligned}&\mathcal{E}_{\text{dis}}(m)-\mathcal{E}_{\text{dis}}(-1)\\&\le\frac{\mu_0\omega_m^2}{4}\|\left(K_h^x+K_h^y\right)^{m+1}\|^2+\frac{(\tau\omega_m)^2\mu_0}{4}\|(H_h^x+H_h^y)^{m+\frac{3}{2}}\|^2+\frac{\mu_0\omega_m^2}{4}\|\left(K_h^x+K_h^y\right)^0\|^2\\&+\frac{(\tau\omega_m)^2}{4}\mu_0\|(H_h^x+H_h^y)^{\frac{1}{2}}\|^2+\frac{\varepsilon_0\omega_e^2}{4}\|\mathbf{J}_h^{m+\frac{1}{2}}\|^2+\frac{(\tau\omega_e)^2\varepsilon_0}{4}\left\|\mathbf{E}_h^{m+1}\right\|^2+\frac{\varepsilon_0\omega_e^2}{4}\|\mathbf{J}_h^{-\frac{1}{2}}\|^2+\frac{(\tau\omega_e)^2\varepsilon_0}{4}\left\|\mathbf{E}_h^0\right\|^2\\&+\frac{\mu_0}{8}\|(H_h^x+H_h^y)^{m+\frac{3}{2}}\|^2+\frac{(\tau c_v c_{inv})^2\varepsilon_0}{2h^2}\|\mathbf{E}_h^{m+1}\|^2+\frac{\mu_0}{8}\|(H_h^x+H_h^y)^{\frac{1}{2}}\|^2+\frac{(\tau c_v c_{inv})^2\varepsilon_0}{h^2}\|\mathbf{E}_h^0\|^2\\&=\left[\frac{(\tau\omega_e)^2}{2}+\frac{(\tau c_v c_{inv})^2}{h^2}\right]\cdot\frac{\varepsilon_0}{2}\left\|\mathbf{E}_h^{m+1}\right\|^2+\frac{1}{2}\cdot\frac{\varepsilon_0\omega_e^2}{2}\|\mathbf{J}_h^{m+\frac{1}{2}}\|^2+\left[\frac{1}{4}+\frac{(\tau\omega_m)^2}{2}\right]\cdot\frac{\mu_0}{2}\|H_h^{m+\frac{3}{2}}\|^2\\&+\frac{1}{2}\cdot\frac{\mu_0\omega_m^2}{2}\|(K_h^x+K_h^y)^{m+1}\|^2+\left[\frac{(\tau\omega_e)^2}{2}+\frac{(\tau c_v c_{inv})^2}{h^2}\right]\cdot\frac{\varepsilon_0}{2}\left\|\mathbf{E}_h^0\right\|^2+\frac{1}{2}\cdot\frac{\varepsilon_0\omega_e^2}{2}\|\mathbf{J}_h^{-\frac{1}{2}}\|^2\\&+\left[\frac{1}{4}+\frac{(\tau\omega_m)^2}{2}\right]\cdot\frac{\mu_0}{2}\|(H_h^x+H_h^y)^{\frac{1}{2}}\|^2+\frac{1}{2}\cdot\frac{\mu_0\omega_m^2}{2}\|(K_h^x+K_h^y)^0\|^2.\end{aligned}\tag{28}$$

If we choose τ satisfying the following constraints (which is equivalent to (10)):

$$\frac{(\tau\omega_e)^2}{2}\le\frac{1}{4},\quad\frac{(\tau c_v c_{inv})^2}{h^2}\le\frac{1}{4},\quad\frac{(\tau\omega_m)^2}{2}\le\frac{1}{4},$$

from (28) we obtain

$$\mathcal{E}_{\text{dis}}(m)-\mathcal{E}_{\text{dis}}(-1)\le\frac{1}{2}\mathcal{E}_{\text{dis}}(m)+\frac{1}{2}\mathcal{E}_{\text{dis}}(-1),$$

which concludes our proof of (11). □

3.2. The error estimate

To simplify the error analysis, we introduce the following solution error notations

$$\begin{aligned}
&\mathcal{E}_h^n := \mathbf{E}^n - \mathbf{E}_h^n = \left(\Pi_h^c \mathbf{E}^n - \mathbf{E}_h^n\right) - \left(\Pi_h^c \mathbf{E}^n - \mathbf{E}^n\right) = \mathcal{E}_\eta^n - \mathcal{E}_\xi^n,\\
&\mathcal{J}_h^{n+\frac{1}{2}} := \mathbf{J}^{n+\frac{1}{2}} - \mathbf{J}_h^{n+\frac{1}{2}} = \left(\Pi_h^c \mathbf{J}^{n+\frac{1}{2}} - \mathbf{J}_h^{n+\frac{1}{2}}\right) - \left(\Pi_h^c \mathbf{J}^{n+\frac{1}{2}} - \mathbf{J}^{n+\frac{1}{2}}\right) = \mathcal{J}_\eta^{n+\frac{1}{2}} - \mathcal{J}_\xi^{n+\frac{1}{2}},\\
&\mathcal{H}_h^{x,n+\frac{1}{2}} := H^{x,n+\frac{1}{2}} - H_h^{x,n+\frac{1}{2}} = \left(\Pi_h^2 H^{x,n+\frac{1}{2}} - H_h^{x,n+\frac{1}{2}}\right) - \left(\Pi_h^2 H^{x,n+\frac{1}{2}} - H^{x,n+\frac{1}{2}}\right) = \mathcal{H}_\eta^{x,n+\frac{1}{2}} - \mathcal{H}_\xi^{x,n+\frac{1}{2}},\\
&\mathcal{H}_h^{y,n+\frac{1}{2}} := H^{y,n+\frac{1}{2}} - H_h^{y,n+\frac{1}{2}} = \left(\Pi_h^2 H^{y,n+\frac{1}{2}} - H_h^{y,n+\frac{1}{2}}\right) - \left(\Pi_h^2 H^{y,n+\frac{1}{2}} - H^{y,n+\frac{1}{2}}\right) = \mathcal{H}_\eta^{y,n+\frac{1}{2}} - \mathcal{H}_\xi^{y,n+\frac{1}{2}},\\
&\mathcal{K}_h^{x,n} := K^{x,n} - K_h^{x,n} = \left(\Pi_h^2 K^{x,n} - K_h^{x,n}\right) - \left(\Pi_h^2 K^{x,n} - K^{x,n}\right) = \mathcal{K}_\eta^{x,n} - \mathcal{K}_\xi^{x,n},\\
&\mathcal{K}_h^{y,n} := K^{y,n} - K_h^{y,n} = \left(\Pi_h^2 K^{y,n} - K_h^{y,n}\right) - \left(\Pi_h^2 K^{y,n} - K^{y,n}\right) = \mathcal{K}_\eta^{y,n} - \mathcal{K}_\xi^{y,n},
\end{aligned} \tag{29}$$

where we denote $\mathbf{E}^n := E(\boldsymbol{x}, t_n)$ and similar notations for other variables. Recall that Π_h^c and Π_h^2 denote the standard Nédélec interpolation into the space $\boldsymbol{U}_h$ and the standard L^2 projection into the space V_h, respectively, and they satisfy the following error estimates [25]: For any $r \geq 1$,

$$\left\|\Pi_h^c \boldsymbol{u} - \boldsymbol{u}\right\|_{H(\,\mathrm{curl};\Omega)} \leq \mathrm{ch}^r \left(\|\boldsymbol{u}\|_{(H^r(\Omega))^2} + \|\,\mathrm{curl}\,\boldsymbol{u}\|_{H^r(\Omega)}\right), \quad \forall \boldsymbol{u} \in \boldsymbol{U}_h, \tag{30}$$

$$\left\|\Pi_h^2 v - v\right\|_{L^2(\Omega)} \leq \mathrm{ch}^r \|v\|_{H^r(\Omega)} \quad \forall v \in V_h. \tag{31}$$

To carry out the error estimate for our scheme (5a)–(5f), we need the following lemma.

Lemma 1 ([15, Ch.3]). *Denote $u^k := u(\cdot, t_k)$. Then we have:*

$$||\delta_\tau u^{k+\frac{1}{2}}||^2 = ||\frac{u^k - u^{k-1}}{\tau}||_0^2 \leq \frac{1}{\tau}\int_{t_{k-1}}^{t_k} ||\partial_t u(s)||^2 ds, \quad \forall\, u \in H^1\left(0, T; L^2(\Omega)\right), \tag{32a}$$

$$||u^k - \frac{1}{\tau}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} u(s)ds||^2 \leq \frac{\tau^3}{4}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} ||\partial_{tt} u(s)||^2 ds, \quad \forall\, u \in H^2\left(0, T; L^2(\Omega)\right), \tag{32b}$$

$$||\frac{u^{k-\frac{1}{2}} + u^{k+\frac{1}{2}}}{2} - \frac{1}{\tau}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} u(s)ds||^2 \leq \frac{\tau^3}{4}\int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} ||\partial_{tt} u(s)||^2 ds, \quad \forall\, u \in H^2\left(0, T; L^2(\Omega)\right). \tag{32c}$$

Theorem 2. *Suppose that the solution of (1a)–(1f) satisfy the following regularity requirements,*

$$\begin{aligned}
&\partial_t \boldsymbol{E}, \partial_t \boldsymbol{J} \in L^2(0, T; H^r(curl; \Omega)), \quad \boldsymbol{E}, \boldsymbol{J} \in L^\infty(0, T; H^r(curl; \Omega)), \quad H, K \in L^\infty(0, T; H^r(\Omega)),\\
&\partial_{tt}\boldsymbol{E}, \partial_{tt}\boldsymbol{J}, \partial_{tt}\nabla \times H \in L^2(0, T; (L^2(\Omega))^2), \quad \partial_{tt} H, \partial_{tt} K, \partial_{tt}\nabla \times \boldsymbol{E} \in L^2(0, T; L^2(\Omega)),
\end{aligned} \tag{33}$$

then under the same time step constraint as given in Theorem 1, the solution of the scheme (5a)–(5f) subject to the initial approximations (6a)–(6c) satisfies the following optimal error estimate: For any $m \geq 0$,

$$\epsilon_0 \left\|\mathcal{E}_h^{m+1}\right\|^2 + \mu_0 \left\|(\mathcal{H}_h^x + \mathcal{H}_h^y)^{m+\frac{3}{2}}\right\|^2 + \epsilon_0 \omega_e^2 \left\|\mathcal{J}_h^{m+\frac{1}{2}}\right\|^2 + \mu_0 \omega_m^2 \left\|(\mathcal{K}_h^x + \mathcal{K}_h^y)^{m+1}\right\|^2 \leq C(\tau^2 + h^r)^2,$$

where the positive constant C is independent of h and τ, and $r \geq 1$ denotes the order of the basis functions in the finite element spaces $\boldsymbol{U}_h$ and V_h.

Proof. First, let us derive the error equations.

(I) Integrating (1a) with respect to t from t_n to t_{n+1}, then multiplying the result by any $\frac{\boldsymbol{\phi}_h}{\tau} \in \boldsymbol{U}_h^0$ and integrating over Ω, we obtain

$$\varepsilon_0 \left(\delta_\tau \mathbf{E}^{n+\frac{1}{2}}, \boldsymbol{\phi}_h\right) = \left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}} H ds, \nabla \times \boldsymbol{\phi}_h\right) - \left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}} (\varepsilon_0 \omega_e^2 \mathbf{J} + \boldsymbol{\sigma}\mathbf{E}) ds, \boldsymbol{\phi}_h\right). \tag{34}$$

Subtracting (5a) from (34), and using the error notations introduced in (29), we obtain the E-error equation:

$$\begin{aligned}
\varepsilon_0 \left(\delta_\tau \mathcal{E}_h^{n+\frac{1}{2}}, \boldsymbol{\phi}_h\right) = &\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}} (H - H_h^{n+\frac{1}{2}}) ds, \nabla \times \boldsymbol{\phi}_h\right)\\
&- \left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}} \varepsilon_0 \omega_e^2 (\boldsymbol{J} - \boldsymbol{J}_h^{n+\frac{1}{2}}) ds + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \boldsymbol{\sigma}(\boldsymbol{E} - \bar{\boldsymbol{E}}_h^{n+\frac{1}{2}}) ds, \boldsymbol{\phi}_h\right),
\end{aligned} \tag{35}$$

which can be written as

$$\begin{aligned}
&\varepsilon_0\left(\delta_\tau\mathcal{E}_\eta^{n+\frac{1}{2}},\boldsymbol{\phi}_h\right)-\left(\mathcal{H}_\eta^{n+\frac{1}{2}},\nabla\times\boldsymbol{\phi}_h\right)+\left(\varepsilon_0\omega_e^2\mathcal{J}_\eta^{n+\frac{1}{2}}+\boldsymbol{\sigma}\bar{\mathcal{E}}_\eta^{n+\frac{1}{2}},\boldsymbol{\phi}_h\right)\\
=&\varepsilon_0\left(\delta_\tau\mathcal{E}_\xi^{n+\frac{1}{2}},\boldsymbol{\phi}_h\right)-\left(\mathcal{H}_\xi^{n+\frac{1}{2}},\nabla\times\boldsymbol{\phi}_h\right)+\left(\varepsilon_0\omega_e^2\mathcal{J}_\xi^{n+\frac{1}{2}}+\boldsymbol{\sigma}\bar{\mathcal{E}}_\xi^{n+\frac{1}{2}},\boldsymbol{\phi}_h\right)+\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(H-H^{n+\frac{1}{2}})ds,\nabla\times\boldsymbol{\phi}_h\right)\\
&-\varepsilon_0\omega_e^2\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(\mathbf{J}-\mathbf{J}^{n+\frac{1}{2}})ds,\boldsymbol{\phi}_h\right)-\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\boldsymbol{\sigma}(\mathbf{E}-\bar{\mathbf{E}}^{n+\frac{1}{2}})ds,\boldsymbol{\phi}_h\right).
\end{aligned}\tag{36}$$

(II) Similarly, integrating (1b) from $t_{n-\frac{1}{2}}$ to $t_{n+\frac{1}{2}}$, then multiplying the result by any $\frac{\tilde{\boldsymbol{\phi}}_h}{\tau}\in\boldsymbol{U}_h^0$ and integrating over Ω, we have

$$\left(\delta_\tau\mathbf{J}^n,\tilde{\boldsymbol{\phi}}_h\right)=\left(\frac{1}{\tau}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\mathbf{E}ds,\tilde{\boldsymbol{\phi}}_h\right).\tag{37}$$

Subtracting (5b) from (37), and using the error notations introduced in (29), we have

$$\left(\delta_\tau\mathcal{J}_h^n,\tilde{\boldsymbol{\phi}}_h\right)=\left(\frac{1}{\tau}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}(\mathbf{E}-\mathbf{E}_h^n)ds,\tilde{\boldsymbol{\phi}}_h\right),\tag{38}$$

which can be written as

$$\left(\delta_\tau\mathcal{J}_\eta^n,\tilde{\boldsymbol{\phi}}_h\right)-\left(\mathcal{E}_\eta^n,\tilde{\boldsymbol{\phi}}_h\right)=\left(\delta_\tau\mathcal{J}_\xi^n,\tilde{\boldsymbol{\phi}}_h\right)-\left(\mathcal{E}_\xi^n,\tilde{\boldsymbol{\phi}}_h\right)+\left(\frac{1}{\tau}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}(\mathbf{E}-\mathbf{E}^n)ds,\tilde{\boldsymbol{\phi}}_h\right).\tag{39}$$

(III) Similarly, integrating (1c) from $t_{n+\frac{1}{2}}$ to $t_{n+\frac{3}{2}}$, then multiplying the result by any $\frac{\psi_h}{\tau}\in V_h$ and integrating over Ω, we have

$$\mu_0(\delta_\tau H^{x,n+1},\psi_h)+\mu_0\omega_m^2(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}K^x ds,\psi_h)+(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\sigma_y H^x ds,\psi_h)=(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\partial_y E_x ds,\psi_h).\tag{40}$$

Subtracting (5c) from (40), and using the error notations introduced in (29), we obtain the H^x-error equation:

$$\begin{aligned}
&\mu_0\left(\delta_\tau\mathcal{H}_\eta^{x,n+1},\psi_h\right)+\mu_0\omega_m^2\left(\mathcal{K}_\eta^{x,n+1},\psi_h\right)+\left(\sigma_y\bar{\mathcal{H}}_\eta^{x,n+1},\psi_h\right)-\left(\partial_y\mathcal{E}_{x,\eta}^{n+1},\psi_h\right)\\
=&\mu_0\left(\delta_\tau\mathcal{H}_\xi^{x,n+1},\psi_h\right)+\mu_0\omega_m^2\left(\mathcal{K}_\xi^{x,n+1},\psi_h\right)+\left(\sigma_y\bar{\mathcal{H}}_\xi^{x,n+1},\psi_h\right)-\left(\partial_y\mathcal{E}_{x,\xi}^{n+1},\psi_h\right)\\
&-\mu_0\omega_m^2\left(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}(K^x-K^{x,n+1})ds,\psi_h\right)-\left(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\sigma_y(H^x-\bar{H}^{x,n+1})ds,\psi_h\right)\\
&+\left(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\partial_y(E_x-E_x^{n+1})ds,\psi_h\right).
\end{aligned}\tag{41}$$

(IV) Similarly, integrating (1d) from t_n to t_{n+1}, then multiplying the result by any $\frac{\psi_{1h}}{\tau}\in V_h$ and integrating over Ω, we have

$$\left(\delta_\tau K^{x,n+\frac{1}{2}},\psi_{1h}\right)=\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}H^x ds,\psi_{1h}\right).\tag{42}$$

Subtracting (5d) from (42), and using the error notations introduced in (29), we have the K^x-error equation:

$$\begin{aligned}
&\left(\delta_\tau\mathcal{K}_\eta^{x,n+\frac{1}{2}},\psi_{1h}\right)-\left(\mathcal{H}_\eta^{x,n+\frac{1}{2}},\psi_{1h}\right)\\
=&\left(\delta_\tau\mathcal{K}_\xi^{x,n+\frac{1}{2}},\psi_{1h}\right)-\left(\mathcal{H}_\xi^{x,n+\frac{1}{2}},\psi_{1h}\right)+\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(H^x-H^{x,n+\frac{1}{2}})ds,\psi_{1h}\right).
\end{aligned}\tag{43}$$

(V) By symmetry, we immediately obtain the H^y error equation

$$\begin{aligned}
&\mu_0\left(\delta_\tau\mathcal{H}_\eta^{y,n+1},\tilde{\psi}_h\right)+\mu_0\omega_m^2\left(\mathcal{K}_\eta^{y,n+1},\tilde{\psi}_h\right)+\left(\sigma_x\bar{\mathcal{H}}_\eta^{y,n+1},\psi_h\right)+\left(\partial_x\mathcal{E}_{y,\eta}^{n+1},\tilde{\psi}_h\right)\\
=&\mu_0\left(\delta_\tau\mathcal{H}_\xi^{y,n+1},\tilde{\psi}_h\right)+\mu_0\omega_m^2\left(\mathcal{K}_\xi^{y,n+1},\tilde{\psi}_h\right)+\left(\sigma_x\bar{\mathcal{H}}_\xi^{y,n+1},\psi_h\right)+\left(\partial_x\mathcal{E}_{y,\xi}^{n+1},\tilde{\psi}_h\right)\\
&-\mu_0\omega_m^2\left(\frac{1}{\tau}\int_{t_{n+\frac12}}^{t_{n+\frac32}}(K^y-K^{y,n+1})ds,\tilde{\psi}_h\right)-\left(\frac{1}{\tau}\int_{t_{n+\frac12}}^{t_{n+\frac32}}\sigma_x(H^y-\bar{H}^{y,n+1})ds,\tilde{\psi}_h\right)\\
&-\left(\frac{1}{\tau}\int_{t_{n+\frac12}}^{t_{n+\frac32}}\partial_x(E_y-E_y^{n+1})ds,\tilde{\psi}_h\right).\quad\forall\ \tilde{\psi}_h\in V_h,
\end{aligned}\tag{44}$$

and the K^y-error equation:

$$\begin{aligned}
&\left(\delta_\tau\mathcal{K}_\eta^{y,n+\frac12},\tilde{\psi}_{1h}\right)-\left(\mathcal{H}_\eta^{y,n+\frac12},\tilde{\psi}_{1h}\right)\\
&=\left(\delta_\tau\mathcal{K}_\xi^{y,n+\frac12},\tilde{\psi}_{1h}\right)-\left(\mathcal{H}_\xi^{y,n+\frac12},\tilde{\psi}_{1h}\right)+\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(H^y-H^{y,n+\frac12})ds,\tilde{\psi}_{1h}\right),\quad\forall\ \tilde{\psi}_{1h}\in V_h.
\end{aligned}\tag{45}$$

(VI) Note that all the left hand side terms in the above error equations have the same forms as the original finite element scheme (5a)–(5f), and all the right hand side terms are extra error terms caused by spatial and time approximations. Moreover, all the extra error terms can be bounded by either $O(h^r)$ or $O(\tau^2)$.

The rest of the proof follows exactly the same as the stability analysis given in Theorem 1 by choosing $\phi_h=2\tau\bar{\mathcal{E}}_\eta^{n+\frac12}$ in (36), $\tilde{\phi}_h=2\tau\varepsilon_0\omega_e^2\bar{\mathcal{J}}_\eta^n$ in (39), $\psi_h=2\tau(\bar{\mathcal{H}}_\eta^{x,n+1}+\bar{\mathcal{H}}_\eta^{y,n+1}):=2\tau\bar{\mathcal{H}}_\eta^{n+1}$ in (41), $\psi_{1h}=2\tau(\bar{\mathcal{K}}_\eta^{x,n+\frac12}+\bar{\mathcal{K}}_\eta^{y,n+\frac12}):=2\tau\bar{\mathcal{K}}_\eta^{n+\frac12}$ in (43), $\tilde{\psi}_h=2\tau\bar{\mathcal{H}}_\eta^{n+1}$ in (44), and $\tilde{\psi}_{1h}=2\tau\bar{\mathcal{K}}_\eta^{n+\frac12}$ in (45). adding the results together, then summing them up from $n=0$ to any $m<N_t-1$, we obtain the left hand side (LHS) as

$$\begin{aligned}
LHS=&\varepsilon_0\left(\left\|\mathcal{E}_\eta^{m+1}\right\|^2-\left\|\mathcal{E}_\eta^0\right\|^2\right)+\varepsilon_0\omega_e^2\left(\left\|\mathcal{J}_\eta^{m+\frac12}\right\|^2-\left\|\mathcal{J}_\eta^{-\frac12}\right\|^2\right)+2\tau\sum_{0\le n\le m}\left\|\boldsymbol{\sigma}^{\frac12}\bar{\mathcal{E}}_\eta^{n+\frac12}\right\|^2\\
&+\mu_0\left(\left\|\mathcal{H}_\eta^{x,m+\frac32}+\mathcal{H}_\eta^{y,m+\frac32}\right\|^2-\left\|\mathcal{H}_\eta^{x,\frac12}+\mathcal{H}_\eta^{y,\frac12}\right\|^2\right)+\tau\varepsilon_0\omega_e^2\left[\left(\mathcal{J}_\eta^{m+\frac12},\mathcal{E}_\eta^{m+1}\right)-\left(\mathcal{J}_\eta^{-\frac12},\mathcal{E}_\eta^0\right)\right]\\
&+\mu_0\omega_m^2\left[\left\|\mathcal{K}_\eta^{x,m+1}+\mathcal{K}_\eta^{y,m+1}\right\|^2-\left\|\mathcal{K}_\eta^{x,0}+\mathcal{K}_\eta^{y,0}\right\|^2\right]\\
&+2\tau\sum_{0\le n\le m}\left[\left\|\sigma_y^{\frac12}\bar{\mathcal{H}}_\eta^{x,n+1}\right\|^2+\left\|\sigma_x^{\frac12}\bar{\mathcal{H}}_\eta^{y,n+1}\right\|^2+\left(\left(\sigma_x+\sigma_y\right)\bar{\mathcal{H}}_\eta^{x,n+1},\bar{\mathcal{H}}_\eta^{y,n+1}\right)\right]\\
&+\tau\mu_0\omega_m^2\left[\left(\mathcal{K}_\eta^{x,m+1}+\mathcal{K}_\eta^{y,m+1},\mathcal{H}_\eta^{x,m+\frac32}+\mathcal{H}_\eta^{y,m+\frac32}\right)-\left(\mathcal{K}_\eta^{x,0}+\mathcal{K}_\eta^{y,0},\mathcal{H}_\eta^{x,\frac12}+\mathcal{H}_\eta^{y,\frac12}\right)\right]\\
&-\tau\left[\left(\mathcal{H}_\eta^{\frac12},\nabla\times\mathcal{E}_\eta^0\right)-\left(\mathcal{H}_\eta^{m+\frac32},\nabla\times\mathcal{E}_\eta^{m+1}\right)\right].
\end{aligned}$$

Then using the assumption that $\sigma_x=\sigma_y=\sigma_0$ is a positive constant and the property of operator Π_h^2, we have

$$\begin{aligned}
&\varepsilon_0\left(\|\mathcal{E}_\eta^{m+1}\|^2-\|\mathcal{E}_\eta^0\|^2\right)+\varepsilon_0\omega_e^2\left(||\mathcal{J}_\eta^{m+\frac12}||^2-||\mathcal{J}_\eta^{-\frac12}||^2\right)\\
&+\mu_0\left(||\mathcal{H}_\eta^{m+\frac32}||^2-||\mathcal{H}_\eta^{\frac12}||^2\right)+\mu_0\omega_m^2\left(\left\|\mathcal{K}_\eta^{m+1}\right\|^2-\left\|\mathcal{K}_\eta^0\right\|^2\right)\\
\le&2\tau\varepsilon_0\sum_{n=0}^m\left(\delta_\tau\mathcal{E}_\xi^{n+\frac12},\bar{\mathcal{E}}_\eta^{n+\frac12}\right)+2\tau\sum_{n=0}^m\left(\varepsilon_0\omega_e^2\mathcal{J}_\xi^{n+\frac12}+\sigma_0\bar{\mathcal{E}}_\xi^{n+\frac12},\bar{\mathcal{E}}_\eta^{n+\frac12}\right)\\
&+2\tau\sum_{n=0}^m\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(H-H^{n+\frac12})ds,\nabla\times\bar{\mathcal{E}}_\eta^{n+\frac12}\right)-2\tau\varepsilon_0\omega_e^2\sum_{n=0}^m\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(\mathbf{J}-\mathbf{J}^{n+\frac12})ds,\bar{\mathcal{E}}_\eta^{n+\frac12}\right)\\
&-2\tau\sigma_0\sum_{n=0}^m\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(\mathbf{E}-\bar{\mathbf{E}}^{n+\frac12})ds,\bar{\mathcal{E}}_\eta^{n+\frac12}\right)+2\tau\varepsilon_0\omega_e^2\sum_{n=0}^m\left(\delta_\tau\mathcal{J}_\xi^n,\bar{\mathcal{J}}_\eta^n\right)-2\tau\varepsilon_0\omega_e^2\sum_{n=0}^m\left(\mathcal{E}_\xi^n,\bar{\mathcal{J}}_\eta^n\right)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\tau\varepsilon_0\omega_e^2\sum_{n=0}^{m}\left(\frac{1}{\tau}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}(\mathbf{E}-\mathbf{E}^n)ds, \overline{\mathcal{J}}_\eta^n\right)\\
&+ 2\tau\mu_0\omega_m^2\sum_{n=0}^{m}\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(H-H^{n+\frac{1}{2}})ds, \overline{\mathcal{K}}_\eta^{n+\frac{1}{2}}\right)+2\tau\left(\nabla\times\mathcal{E}_\xi^{n+1}, \overline{\mathcal{H}}_\eta^{n+1}\right)\\
&- 2\tau\mu_0\omega_m^2\sum_{n=0}^{m}\left(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}(K-K^{n+1})ds, \overline{\mathcal{H}}_\eta^{n+1}\right)-2\tau\sigma_0\sum_{n=0}^{m}\left(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}(H-H^{n+1})ds, \overline{\mathcal{H}}_\eta^{n+1}\right)\\
&- 2\tau\sum_{n=0}^{m}\left(\frac{1}{\tau}\int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}}\nabla\times(\mathbf{E}-\mathbf{E}^{n+1})ds, \overline{\mathcal{H}}_\eta^{n+1}\right)-\tau\varepsilon_0\omega_e^2\left[(\mathcal{J}_\eta^{m+\frac{1}{2}}, \mathcal{E}_\eta^{m+1})-(\mathcal{J}_\eta^{-\frac{1}{2}}, \mathcal{E}_\eta^0)\right]\\
&- \tau\mu_0\omega_m^2\left[(\mathcal{K}_\eta^{m+1}, \mathcal{H}_\eta^{m+\frac{3}{2}})-(\mathcal{K}_\eta^0, \mathcal{H}_\eta^{\frac{1}{2}})\right]-\tau\left[(\mathcal{H}_\eta^{m+\frac{3}{2}}, \nabla\times\mathcal{E}_\eta^{m+1})-(\mathcal{H}_\eta^{\frac{1}{2}}, \nabla\times\mathcal{E}_\eta^0)\right]:=\sum_{i=1}^{16}Err_i. \qquad (46)
\end{aligned}
$$

(VII) Now we just need to estimate each Err_i. Using the arithmetic–geometric mean inequality

$$(a,b)\le\delta\|a\|^2+\frac{1}{4\delta}\|b\|^2,\quad\forall\,\delta>0, \qquad (47)$$

Lemma 1 and the interpolation error estimate (30), we have

$$
\begin{aligned}
\mathrm{Err}_1&\le\tau\varepsilon_0\sum_{n=0}^{m}\left(2\delta_1\left\|\overline{\mathcal{E}}_\eta^{n+\frac{1}{2}}\right\|^2+\frac{1}{2\delta_1}\left\|\delta_\tau\mathcal{E}_\xi^{n+\frac{1}{2}}\right\|^2\right)\\
&\le\tau\varepsilon_0\left[\sum_{n=0}^{m}\delta_1(\|\mathcal{E}_\eta^{n+1}\|^2+\|\mathcal{E}_\eta^n\|^2)+\sum_{n=0}^{m}\frac{1}{2\delta_1\tau}\int_{t_n}^{t_{n+1}}\|\partial_t\mathcal{E}_\xi(t)\|^2dt\right]\\
&\le\tau\delta_1\varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2+2\tau\delta_1\varepsilon_0\sum_{n=0}^{m}\|\mathcal{E}_\eta^n\|^2+\frac{1}{\delta_1}\cdot Ch^{2r}\int_0^T\|\partial_t\mathbf{E}\|^2_{H^r(curl;\Omega)}dt. \qquad (48)
\end{aligned}
$$

Similarly, by the arithmetic–geometric mean inequality (47) and the interpolation error estimate (30), we have

$$
\begin{aligned}
\mathrm{Err}_2&\le\tau\sum_{n=0}^{m}\left(2\delta_2\varepsilon_0\|\overline{\mathcal{E}}_\eta^{n+\frac{1}{2}}\|^2+\frac{1}{2\delta_2\varepsilon_0}\|\varepsilon_0\omega_e^2\mathcal{J}_\xi^{n+\frac{1}{2}}+\sigma_0\overline{\mathcal{E}}_\xi^{n+\frac{1}{2}}\|^2\right)\\
&\le\tau\delta_2\varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2+2\tau\delta_2\varepsilon_0\sum_{n=0}^{m}\|\mathcal{E}_\eta^n\|^2+\frac{T}{\delta_2\varepsilon_0}\cdot Ch^{2r}\left(\|\mathbf{J}\|^2_{L^\infty(0,T;H^r(curl;\Omega))}+\|\mathbf{E}\|^2_{L^\infty(0,T;H^r(curl;\Omega))}\right). \qquad (49)
\end{aligned}
$$

Using integration by parts, Lemma 1, and the arithmetic–geometric mean inequality (47), we obtain

$$
\begin{aligned}
\mathrm{Err}_3&=2\tau\sum_{n=0}^{m}\left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\nabla\times(H-H^{n+\frac{1}{2}})ds, \overline{\mathcal{E}}_\eta^{n+\frac{1}{2}}\right)\\
&\le\tau\sum_{n=0}^{m}\left(2\delta_3\varepsilon_0\|\overline{\mathcal{E}}_\eta^{n+\frac{1}{2}}\|^2+\frac{1}{2\delta_3\varepsilon_0}\|\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\nabla\times(H-H^{n+\frac{1}{2}})ds\|^2\right)\\
&\le\tau\delta_3\varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2+2\tau\delta_3\varepsilon_0\sum_{n=0}^{m}\|\mathcal{E}_\eta^n\|^2+\frac{\tau^4}{8\delta_3\varepsilon_0}\int_0^T\|\partial_{tt}\nabla\times H\|^2dt. \qquad (50)
\end{aligned}
$$

Similarly, by the arithmetic–geometric mean inequality (47) and Lemma 1, we have

$$\mathrm{Err}_4\le\tau\delta_4\varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2+2\tau\delta_4\varepsilon_0\sum_{n=0}^{m}\|\mathcal{E}_\eta^n\|^2+\frac{\tau^4\varepsilon_0\omega_e^4}{8\delta_4}\int_0^T\|\partial_{tt}\boldsymbol{J}\|^2dt, \qquad (51)$$

$$\mathrm{Err}_5\le\tau\delta_5\varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2+2\tau\delta_5\varepsilon_0\sum_{n=0}^{m}\|\mathcal{E}_\eta^n\|^2+\frac{\tau^4\sigma_0}{8\delta_5\varepsilon_0}\int_0^T\|\partial_{tt}\boldsymbol{E}\|^2dt. \qquad (52)$$

Using the arithmetic–geometric mean inequality (47), Lemma 1, and the interpolation error estimate (30), we have

$$\mathrm{Err}_6\le\tau\delta_6\varepsilon_0\omega_e^2\|\mathcal{J}_\eta^{m+\frac{1}{2}}\|^2+2\tau\delta_6\varepsilon_0\omega_e^2\sum_{n=0}^{m}\|\mathcal{J}_\eta^{n-\frac{1}{2}}\|^2+\frac{1}{\delta_6}\cdot Ch^{2r}\int_0^T\|\partial_t\boldsymbol{J}\|^2_{H^r(curl;\Omega)}dt. \qquad (53)$$

Similarly, we have

$$\mathrm{Err}_7 \le \tau\delta_7\varepsilon_0\omega_e^2\|\mathcal{J}_\eta^{m+\frac{1}{2}}\|^2 + 2\tau\delta_7\varepsilon_0\omega_e^2\sum_{n=0}^{m}\|\mathcal{J}_\eta^{n-\frac{1}{2}}\|^2 + \frac{T}{\delta_7}\cdot Ch^{2r}\|\boldsymbol{E}\|^2_{L^\infty(0.T;H^r(curl;\Omega))}, \tag{54}$$

$$\mathrm{Err}_8 \le \tau\delta_8\varepsilon_0\omega_e^2\|\mathcal{J}_\eta^{m+\frac{1}{2}}\|^2 + 2\tau\delta_8\varepsilon_0\omega_e^2\sum_{n=0}^{m}\|\mathcal{J}_\eta^{n-\frac{1}{2}}\|^2 + \frac{\tau^4\varepsilon_0\omega_e^2}{8\delta_8}\int_0^T\|\partial_{tt}\boldsymbol{E}\|^2dt, \tag{55}$$

$$\mathrm{Err}_9 \le \tau\delta_9\mu_0\omega_m^2\|\mathcal{K}_\eta^{m+1}\|^2 + 2\tau\delta_9\mu_0\omega_m^2\sum_{n=0}^{m}\|\mathcal{K}_\eta^{n}\|^2 + \frac{\tau^4\mu_0\omega_m^2}{8\delta_9}\int_0^T\|\partial_{tt}H\|^2dt. \tag{56}$$

Using the arithmetic–geometric mean inequality (47), and the interpolation error estimate (30), we have

$$\begin{aligned}\mathrm{Err}_{10} &\le \tau\sum_{n=0}^{m}\left(2\delta_{10}\mu_0\|\overline{\mathcal{H}}_\eta^{n+1}\|^2 + \frac{1}{2\delta_{10}\mu_0}\|\nabla\times\mathcal{E}_\xi^{n+1}\|^2\right)\\ &\le \tau\left(\delta_{10}\mu_0\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|^2 + 2\delta_{10}\mu_0\sum_{n=0}^{m}\|\mathcal{H}_\eta^{n+\frac{1}{2}}\|^2\right) + \frac{T}{2\delta_{10}\mu_0}\cdot Ch^{2r}\|\boldsymbol{E}\|^2_{L^\infty(0.T;H^r(curl;\Omega))}.\end{aligned} \tag{57}$$

By the arithmetic–geometric mean inequality (47) and Lemma 1, we have

$$\mathrm{Err}_{11} \le \tau\delta_{11}\mu_0\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|^2 + 2\tau\delta_{11}\mu_0\sum_{n=0}^{m}\|\mathcal{H}_\eta^{n+\frac{1}{2}}\|^2 + \frac{\tau^4\mu_0\omega_m^4}{8\delta_{11}}\int_0^T\|\partial_{tt}K\|^2dt, \tag{58}$$

$$\mathrm{Err}_{12} \le \tau\delta_{12}\mu_0\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|^2 + 2\tau\delta_{12}\mu_0\sum_{n=0}^{m}\|\mathcal{H}_\eta^{n+\frac{1}{2}}\|^2 + \frac{\tau^4\sigma_0^2}{8\delta_{12}\mu_0}\int_0^T\|\partial_{tt}H\|^2dt, \tag{59}$$

$$\mathrm{Err}_{13} \le \tau\delta_{13}\mu_0\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|^2 + 2\tau\delta_{13}\mu_0\sum_{n=0}^{m}\|\mathcal{H}_\eta^{n+\frac{1}{2}}\|^2 + \frac{\tau^4}{8\delta_{13}\mu_0}\int_0^T\|\partial_{tt}\nabla\times\boldsymbol{E}\|^2dt. \tag{60}$$

Using the Cauchy–Schwarz inequality and the fact that $\mathcal{E}_\eta^0 = \Pi_h^c\boldsymbol{E}^0 - \boldsymbol{E}_h^0 = 0$, we have

$$\mathrm{Err}_{14} = -\tau\varepsilon_0\omega_e^2(\mathcal{J}_\eta^{m+\frac{1}{2}}, \mathcal{E}_\eta^{m+1}) \le \frac{\varepsilon_0\omega_e^2}{2}\|\mathcal{J}_\eta^{m+\frac{1}{2}}\|^2 + \frac{(\tau\omega_e)^2}{2}\varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2. \tag{61}$$

Similarly, using the fact that $\mathcal{K}_\eta^0 = \Pi_h^2K^0 - K_h^0 = 0$, we have

$$\mathrm{Err}_{15} = -\tau\mu_0\omega_m^2(\mathcal{K}_\eta^{m+1}, \mathcal{H}_\eta^{m+\frac{3}{2}}) \le \frac{\mu_0\omega_m^2}{2}\|\mathcal{K}_\eta^{m+1}\|^2 + \frac{(\tau\omega_m)^2}{2}\mu_0\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|^2. \tag{62}$$

Using the notation $c_v = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$, and the inverse estimate (7), we have

$$\begin{aligned}\mathrm{Err}_{16} &= -\tau(\mathcal{H}_\eta^{m+\frac{3}{2}}, \nabla\times\mathcal{E}_\eta^{m+1}) \le \tau c_v\cdot\sqrt{\mu_0}\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|\cdot c_{inv}h^{-1}\sqrt{\varepsilon_0}\|\mathcal{E}_\eta^{m+1}\|\\ &\le \frac{\tau c_v c_{inv}h^{-1}}{2}\left(\mu_0\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|^2 + \varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2\right).\end{aligned} \tag{63}$$

(VIII) Substituting the above estimates of Err_i into (46), we have

$$\begin{aligned}&\varepsilon_0\left(\|\mathcal{E}_\eta^{m+1}\|^2 - \|\mathcal{E}_\eta^0\|^2\right) + \varepsilon_0\omega_e^2\left(||\mathcal{J}_\eta^{m+\frac{1}{2}}||^2 - ||\mathcal{J}_\eta^{-\frac{1}{2}}||^2\right)\\ &+ \mu_0\left(||\mathcal{H}_\eta^{m+\frac{3}{2}}||^2 - ||\mathcal{H}_\eta^{\frac{1}{2}}||^2\right) + \mu_0\omega_m^2\left(\left\|\mathcal{K}_\eta^{m+1}\right\|^2 - \left\|\mathcal{K}_\eta^{0}\right\|^2\right)\\ &\le \varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2\left[\tau(\delta_1+\delta_2+\delta_3+\delta_4+\delta_5) + \frac{(\tau\omega_e)^2}{2}\right]\\ &+ \varepsilon_0\omega_e^2\|\mathcal{J}_\eta^{m+\frac{1}{2}}\|^2\left[\tau(\delta_6+\delta_7+\delta_8) + \frac{1}{2}\right]\\ &+ \mu_0\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|^2\left[\tau(\delta_{12}+\delta_{13}+\delta_{10}+\delta_{11}) + \frac{\tau c_v c_{inv}h^{-1}}{2} + \frac{(\tau\omega_m)^2}{2}\right]\end{aligned}$$

$$
\begin{aligned}
&+ \mu_0\omega_m^2\|\mathcal{K}_\eta^{m+1}\|^2\left(\tau\delta_9+\frac{1}{2}\right)\\
&+\tau\Big(\sum_{n=0}^{m}\varepsilon_0\|\mathcal{E}_\eta^n\|^2\Big)(2\delta_1+2\delta_2+2\delta_3+2\delta_4+2\delta_5)\\
&+\tau\Big(\sum_{n=0}^{m}\varepsilon_0\omega_e^2\|\mathcal{J}_\eta^{n-\frac{1}{2}}\|^2\Big)(2\delta_6+2\delta_7+2\delta_8)\\
&+\tau\Big(\sum_{n=0}^{m}\mu_0\|\mathcal{H}_\eta^{n+\frac{1}{2}}\|^2\Big)(2\delta_{12}+2\delta_{13}+2\delta_{10}+2\delta_{11})\\
&+\tau\Big(\sum_{n=0}^{m}\mu_0\omega_m^2\|\mathcal{K}_\eta^{n}\|^2\Big)(2\delta_9)+C(h^{2r}+\tau^4).
\end{aligned}
\tag{64}
$$

Now choosing the time step size τ satisfying the same constraint (10) given in Theorem 1, and δ_i properly such as

$$
\begin{aligned}
&\delta_1=\delta_2=\delta_3=\delta_4=\delta_5=\frac{1}{10},\quad \delta_6=\delta_7=\delta_8=\frac{1}{12},\\
&\delta_9=\frac{1}{4},\quad \delta_{10}=\delta_{11}=\delta_{12}=\delta_{13}=\frac{1}{16},
\end{aligned}
\tag{65}
$$

then using the facts that $\mathcal{E}_\eta^0=\mathbf{0}$, $\mathcal{K}_\eta^0=0$, $\mathcal{J}_\eta^{-\frac{1}{2}}=O(\tau^2+h^r)$, $\mathcal{H}_\eta^{\frac{1}{2}}=O(\tau^2+h^r)$, and the discrete Gronwall inequality for (64), we obtain

$$
\varepsilon_0\|\mathcal{E}_\eta^{m+1}\|^2+\varepsilon_0\omega_e^2\|\mathcal{J}_\eta^{m+\frac{1}{2}}\|^2+\mu_0\|\mathcal{H}_\eta^{m+\frac{3}{2}}\|^2+\mu_0\omega_m^2\|\mathcal{K}_\eta^{m+1}\|^2\le C(\tau^2+h^r)^2. \tag{66}
$$

Finally, using the triangle inequality, the interpolation error estimate (30), and the projection error estimate (31) for (66), we complete the proof. □

4. Numerical results

Here we present several numerical results to justify the error estimate of our proposed scheme, and show the effectiveness of this PML model for absorbing outgoing wave in the Drude metamaterial. For simplicity, our numerical results are carried out in Matlab by using the lowest-order triangular edge element.

4.1. Example 1. Testing convergence rate

To rigorously check the convergence rate obtained by our scheme (5a)–(5f), we construct an analytical solution of (1a) and (1f), with extra source functions $f(x,y,t)$, $g_1(x,y,t)$ and $g_2(x,y,t)$ added to the right hand sides of (5a) and (5f). More specifically, the original scheme (5a)–(5f) is modified as

$$
\begin{aligned}
&\varepsilon_0(\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}},\boldsymbol{\phi}_h)+\varepsilon_0\omega_e^2(\boldsymbol{J}_h^{n+\frac{1}{2}},\boldsymbol{\phi}_h)+(\boldsymbol{\sigma}\overline{\boldsymbol{E}}_h^{n+\frac{1}{2}},\boldsymbol{\phi}_h)=(H_h^{n+\frac{1}{2}},\nabla\times\boldsymbol{\phi}_h)+(\boldsymbol{f}(\cdot,t_{n+\frac{1}{2}}),\boldsymbol{\phi}_h),\quad \forall\boldsymbol{\phi}_h\in\boldsymbol{U}_h^0,\\
&(\delta_\tau\boldsymbol{J}_h^n,\tilde{\boldsymbol{\phi}}_h)-(\boldsymbol{E}_h^n,\tilde{\boldsymbol{\phi}}_h)=0,\quad \forall\tilde{\boldsymbol{\phi}}_h\in\boldsymbol{U}_h^0,\\
&\mu_0(\delta_\tau H_h^{x,n+1},\psi_h)+\mu_0\omega_m^2(K_h^{x,n+1},\psi_h)+(\sigma_y\overline{H}_h^{x,n+1},\psi_h)=(\partial_y E_{xh}^{n+1},\psi_h)+(g_1(\cdot,t_{n+1}),\psi_h),\quad \forall\psi_h\in V_h,\\
&(\delta_\tau K_h^{x,n+\frac{1}{2}},\psi_{1h})-(H_h^{x,n+\frac{1}{2}},\psi_{1h})=0,\quad \forall\psi_{1h}\in V_h,\\
&\mu_0(\delta_\tau H_h^{y,n+1},\tilde{\psi}_h)+\mu_0\omega_m^2(K_h^{y,n+1},\tilde{\psi}_h)+(\sigma_x\overline{H}_h^{y,n+1},\tilde{\psi}_h)=-(\partial_x E_{yh}^{n+1},\tilde{\psi}_h)+(g_2(\cdot,t_{n+1}),\tilde{\psi}_h),\quad \forall\tilde{\psi}_h\in V_h,\\
&(\delta_\tau K_h^{y,n+\frac{1}{2}},\tilde{\psi}_{1h})-(H_h^{y,n+\frac{1}{2}},\tilde{\psi}_{1h})=0,\quad \forall\tilde{\psi}_{1h}\in V_h.
\end{aligned}
$$

In our simulation, we choose the physical domain $\Omega=[0,1]^2$ partitioned by uniformly refined triangular meshes, the model parameters $\omega_e=\omega_m=\pi$, $\epsilon_0=\mu_0=1$, $\sigma_x=\sin^2(\pi x)$, $\sigma_y=\sin^2(\pi y)$, the analytical solution given as:

$$
E=\begin{pmatrix}E_x\\E_y\end{pmatrix}=\begin{pmatrix}e^{-\pi t}\cos(\pi x)\sin(\pi y)\\e^{-\pi t}\sin(\pi x)\cos(\pi y)\end{pmatrix},J=\begin{pmatrix}J_x\\J_y\end{pmatrix}=\begin{pmatrix}-\frac{1}{\pi}e^{-\pi t}\cos(\pi x)sin(\pi y)\\-\frac{1}{\pi}e^{-\pi t}\sin(\pi x)cos(\pi y)\end{pmatrix},
$$

$$
H^x=H^y=e^{-\pi t}\cos(\pi x)\cos(\pi y),K^x=K^y=-\frac{1}{\pi}e^{-\pi t}\cos(\pi x)\cos(\pi y)
$$

Table 1
The L_2 errors obtained for Example 1 with $N_t = 10\,000$, $\tau = 10^{-5}$.

h	$\|\boldsymbol{E} - \boldsymbol{E}_h^{n+1}\|$	Rate	$\|H - H_h^{n+\frac{3}{2}}\|$	Rate	$\|\boldsymbol{J} - \boldsymbol{J}_h^{n+\frac{1}{2}}\|$	Rate	$\|K - K_h^{n+1}\|$	Rate
1/4	0.2049	–	0.1940	–	0.2087	–	0.0604	–
1/8	0.1011	1.0199	0.0959	1.0169	0.1031	1.0175	0.0304	0.9922
1/16	0.0507	0.9946	0.0479	1.0016	0.0517	0.9944	0.0152	0.9981
1/32	0.0254	0.9996	0.0240	0.9961	0.0259	0.9995	0.0076	0.9995
1/64	0.0127	0.9980	0.0122	0.9821	0.0130	0.9981	0.0038	0.9996
1/128	0.0064	0.9921	0.0064	0.9321	0.0065	0.9924	0.0019	0.9990

Table 2
The L_2 errors obtained for the second exact solution case with $N_t = 10\,000$, $\tau = 10^{-5}$.

h	$\|\mathbf{E} - \mathbf{E}_\mathbf{h}^{\mathbf{n+1}}\|$	Rate	$\|H - H_h^{n+\frac{3}{2}}\|$	Rate	$\|\mathbf{J} - \mathbf{J}_\mathbf{h}^{\mathbf{n+\frac{1}{2}}}\|$	Rate	$\|K - K_h^{n+1}\|$	Rate
1/4	0.2054	–	0.1107	–	0.2057	–	0.0530	–
1/8	0.1025	1.0035	0.0560	0.9827	0.1026	1.0035	0.0266	0.9967
1/16	0.0512	1.0011	0.0281	0.9957	0.0513	1.0011	0.0133	0.9992
1/32	0.0256	1.0003	0.0141	0.9989	0.0256	1.0003	0.0067	0.9998
1/64	0.0128	1.0001	0.0070	0.9997	0.0128	1.0001	0.0033	0.9999
1/128	0.0064	1.0000	0.0035	0.9999	0.0064	1.0000	0.0017	1.0000

and the source functions given as

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} e^{-\pi t}\sin^2(\pi y)\cos(\pi\sin(\pi y)) \\ e^{-\pi t}\sin(\pi x)\cos(\pi y)[\sin^2(\pi x) - 4\pi] \end{pmatrix}$$

$$g_1 = e^{-\pi t}\cos(\pi x)\cos(\pi y)[\sin^2(\pi y) - 3\pi],\ g_2 = e^{-\pi t}\cos(\pi x)\cos(\pi y)[\sin^2(\pi x) - \pi].$$

The obtained convergence rates with a fixed small time step $\tau = 10^{-5}$ and varying mesh sizes are given in Table 1, which demonstrate $O(h)$ convergence in the L^2 norm. This is consistent with our theoretical analysis.

Since the convergence rates of the last example decrease slightly as mesh gets finer, to see the convergence rates clearly we create another analytical solution of (1a)–(1f) given as follows:

$$E = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \frac{2n\pi}{w}cos(m\pi x)sin(n\pi y)cos(wt) \\ -\frac{2m\pi}{w}sin(m\pi x)cos(n\pi y)cos(wt) \end{pmatrix}, J = \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \frac{2n\pi}{w^2}cos(m\pi x)sin(n\pi y)sin(wt) \\ -\frac{2m\pi}{w^2}sin(m\pi x)cos(n\pi y)sin(wt) \end{pmatrix},$$

$$H^x = H^y = cos(m\pi x)cos(n\pi y)sin(wt),\ K^x = K^y = -\frac{1}{w}cos(m\pi x)cos(n\pi y)cos(wt)$$

and the source functions given as

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} [\frac{2n\pi}{w^2}sin(wt) + \frac{2n\pi}{w}\sigma_y cos(wt)]cos(m\pi x)sin(n\pi y) \\ [-\frac{2m\pi}{w^2}sin(wt) - \frac{2m\pi}{w}\sigma_x cos(wt)]sin(m\pi x)cos(n\pi y) \end{pmatrix}$$

$$g_1 = [(w - \frac{1}{w} - \frac{2n^2\pi^2}{w})cos(wt) + \sigma_y sin(wt)]cos(m\pi x)cos(n\pi y)$$

$$g_2 = [(w - \frac{1}{w} - \frac{2m^2\pi^2}{w})cos(wt) + \sigma_x sin(wt)]cos(m\pi x)cos(n\pi y)$$

where $w = \pi\sqrt{m^2 + n^2}$. In our computation, we use $m = n = 1$ and the same model parameters as the last example. The obtained convergence rates with fixed time step $\tau = 10^{-5}$ and varying mesh sizes are given in Table 2, which demonstrate $O(h)$ convergence clearly in the L^2 norm.

4.2. Example 2. Wave absorbing capability

This example is developed to test the effectiveness of the PML model (1a)–(1f) in absorbing outgoing waves. Here we simulate a source wave propagating in a Drude metamaterial domain of dimension $[-17, 17]^2$. We surround the Drude metamaterial region (governed by (1a)–(1f) with $\sigma_x = \sigma_y = \sigma = 0$) by a PML governed by (1a)–(1f). The PML thickness is $d = 15h$, where h denotes the mesh size. Along the outside boundaries of PMLs, we use the perfect conductor boundary condition.

The incident source wave is imposed in the H^x Eq. (5c) as a source function

$$g_1(x, y, t) = g(x, y)h(t),$$

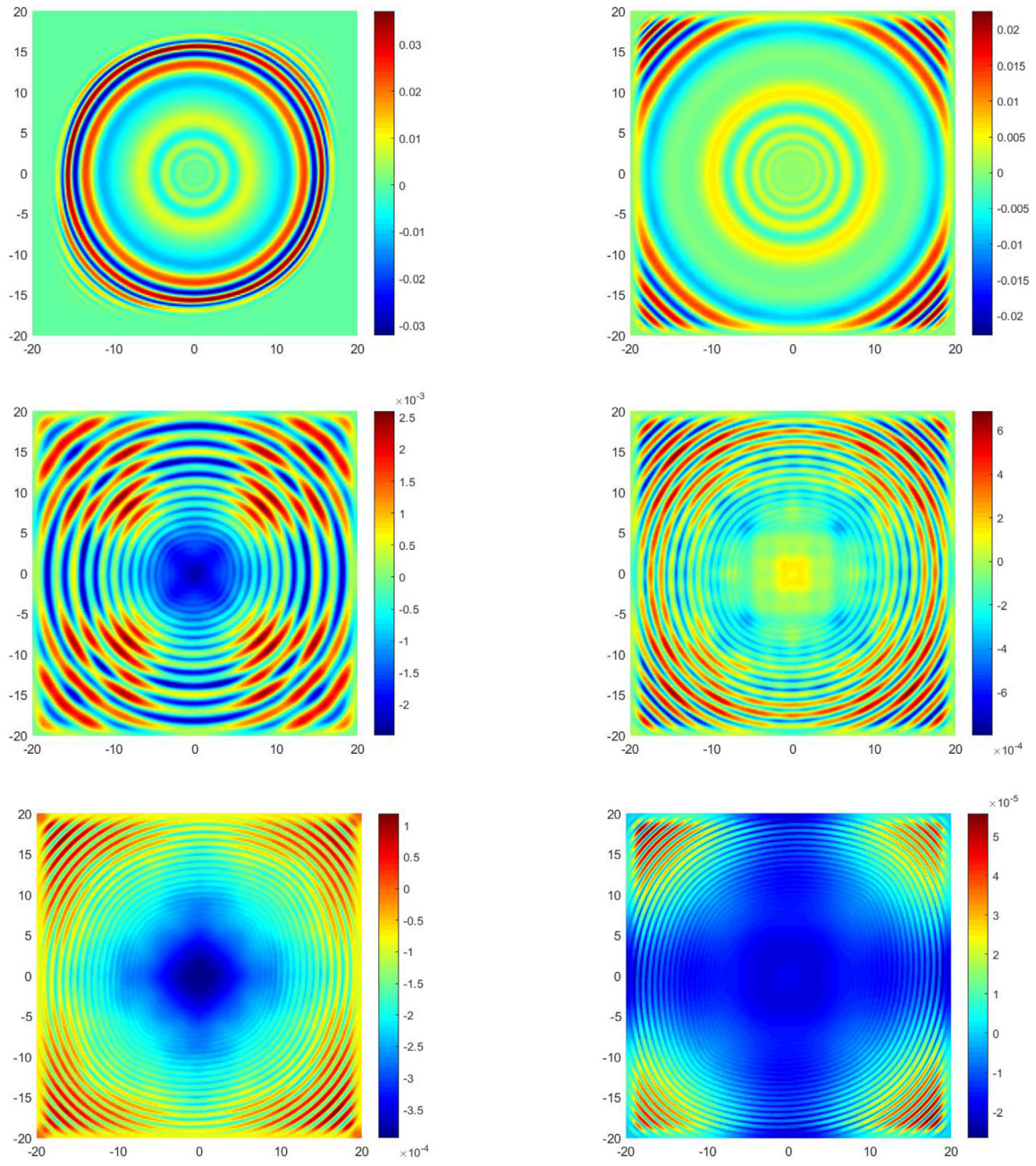

Fig. 1. Snapshots of the magnetic field H obtained at 2000, 3000, 8000, 15 000, 25 000, and 40 000 time steps.

where $g(x, y) = e^{-5(x^2+y^2)}$ and $h(t) = -20(t-1)e^{-10(t-1)^2}$. In our simulation, we use $\omega_e = \omega_m = 2$, $\epsilon_0 = \mu_0 = 1$, $h = 0.2$ and $\tau = 0.01$. The damping function σ_x is a fourth-order polynomial given as:

$$\sigma_x(x) = \begin{cases} \sigma_{max}\left(\frac{x-17}{d}\right)^m, & \text{for } x \geq 17, \\ \sigma_{max}\left(\frac{|x+17|}{d}\right)^m, & \text{for } x \leq -17, \\ 0, & \text{else}, \end{cases}$$

with $\sigma_{max} = -(m+1)\log(R)/(2d)$, $R = 10^{-6}$, $m = 4$. The damping function $\sigma_y(y)$ has exactly the same form as $\sigma_x(x)$.

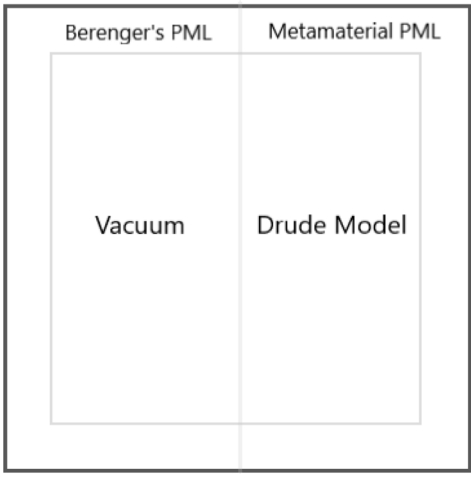

Fig. 2. The setup of the refocusing simulation.

Snapshots of the magnetic field $H = H^x + H^y$ are presented in Fig. 1, which shows that both forward and backward outgoing waves are absorbed well by the PML. This result is similar to what was obtained by the FDTD method in our previous work [24, Fig. 2].

4.3. Example 3. Simulating the refocusing phenomenon

In this example, we simulate a transmission problem between vacuum and a Drude metamaterial surrounded by *Berenger's* PML and the metamaterial PML (cf. Fig. 3), respectively. This transmission problem was originally proposed in [11] and simulated by the FDTD method in our previous work [24] (see Fig. 2).

In this simulation, we put a point wave source $h(t) = \sin(\omega_0 t)$ in the center of vacuum region, while the computational domain is $\Omega = [-20, 20]^2$ (including the PML region with thickness $d = 15h$ on all sides). To create a refocusing phenomenon, we choose parameters $\epsilon_0 = \mu_0 = 1$, $\omega_e = \omega_m = \sqrt{2}\omega_0$, and $\omega_0 = \sqrt{2}$, which leads to $\epsilon(\omega_0) = \mu(\omega_0) = -1$ by the Drude model, i.e., the effective refractive index is equal to -1. The damping functions σ_x and σ_y are the same as Example 2.

For this simulation, we need to solve a multi-physics problem with different sets of governing equations in different subdomains: on the left subdomain Ω_1, the equations are governed by the 2D Berenger PML model; on the right subdomain Ω_2, the governing equations are the metamaterial PML model (1a)–(1f). To simplify the implementation, we unify these models together and rewrite the refocusing modeling equations as:

$$
\begin{aligned}
&\partial_t \boldsymbol{E} + D_J \boldsymbol{J} + \epsilon_0^{-1}\boldsymbol{\sigma}\boldsymbol{E} = \epsilon_0^{-1}\nabla \times H,\\
&D(\partial_t \boldsymbol{J} - \boldsymbol{E}) = 0,\\
&\partial_t H^x + D_K K_x + D_{H1} H^x = D_{E1}\partial_y E_x - D_{E2}\partial_x E_y,\\
&D(\partial_t K_x - H^x) = 0,\\
&\partial_t H^y + D_K K_y + D_{H2} H^y = D_{E2}\partial_y E_x - D_{E1}\partial_x E_y,\\
&D(\partial_t K_y - H^y) = 0,
\end{aligned}
$$

where we denote $\boldsymbol{E} = (E_x, E_y)^T, \boldsymbol{J} = (J_x, J_y)^T, H = H^x + H^y, \boldsymbol{\sigma} = diag(\sigma_y, \sigma_x)$. Furthermore, the coefficients are defined as follows:

$$
D = \begin{cases} 0, & (x,y) \in \Omega_1,\\ 1, & (x,y) \in \Omega_2, \end{cases}\quad
D_J = \begin{cases} 0, & (x,y) \in \Omega_1,\\ \omega_e^2, & (x,y) \in \Omega_2, \end{cases}\quad
D_K = \begin{cases} 0, & (x,y) \in \Omega_1,\\ \omega_m^2, & (x,y) \in \Omega_2, \end{cases}
$$

$$
D_{H1} = \begin{cases} \epsilon_0^{-1}\sigma_x, & (x,y) \in \Omega_1,\\ \mu_0^{-1}\sigma_y, & (x,y) \in \Omega_2, \end{cases}\quad
D_{H2} = \begin{cases} \epsilon_0^{-1}\sigma_y, & (x,y) \in \Omega_1,\\ \mu_0^{-1}\sigma_x, & (x,y) \in \Omega_2, \end{cases}
$$

$$
D_{E1} = \begin{cases} 0, & (x,y) \in \Omega_1,\\ \mu_0^{-1}, & (x,y) \in \Omega_2, \end{cases}\quad
D_{E2} = \begin{cases} \mu_0^{-1}, & (x,y) \in \Omega_1,\\ 0, & (x,y) \in \Omega_2. \end{cases}
$$

The original scheme (5a)–(5f) is revised as follows:

$$
(\delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}, \phi_h) + (D\boldsymbol{J}_h^{n+\frac{1}{2}}, \phi_h) + (\epsilon_0^{-1}\boldsymbol{\sigma}\bar{\boldsymbol{E}}_h^{n+\frac{1}{2}}, \phi_h) = (\epsilon_0^{-1} H_h^{n+\frac{1}{2}}, \nabla \times \phi_h), \quad \forall \phi_h \in U_h^0, \tag{1.6a}
$$

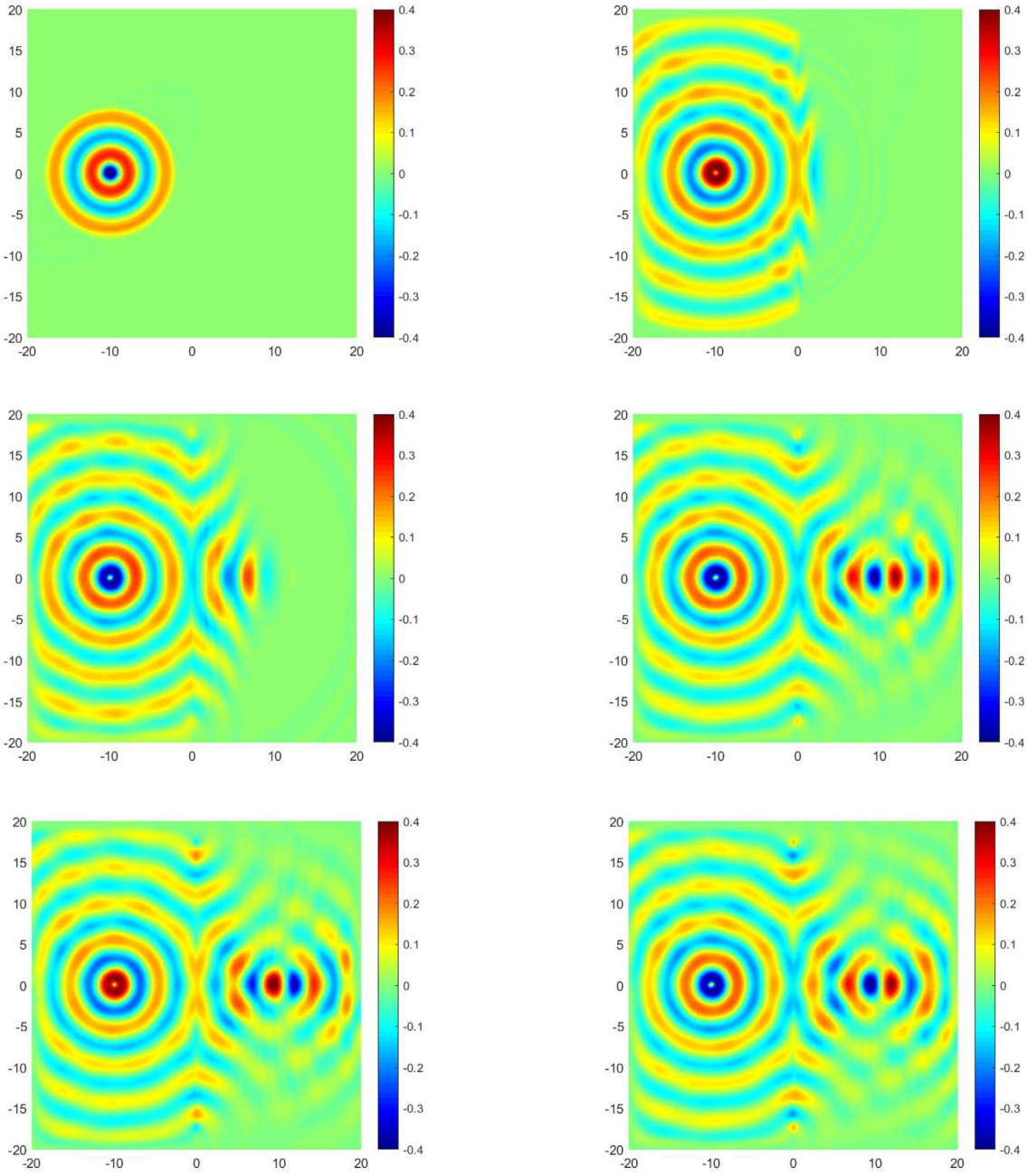

Fig. 3. Snapshots of the magnetic field H obtained at 800, 2000, 4000, 8000, 10 000, and 12 000 time steps.

$$(D\delta_\tau \boldsymbol{J}_h^n, \tilde{\phi}_h) - (D\boldsymbol{E}_h^n, \tilde{\phi}_h) = 0, \quad \forall \tilde{\phi}_h \in U_h^0, \tag{1.6b}$$

$$(\delta_\tau H_h^{x,n+1}, \psi_h) + (D_K K_h^{x,n+1}, \psi_h) + (D_{H1} \bar{H}_h^{x,n+1}, \psi_h) = (D_{E1} \partial_y E_{xh}^{n+1}, \psi_h) - (D_{E2} \partial_x E_{yh}^{n+1}, \psi_h), \quad \forall \psi_h \in V_h, \tag{1.6c}$$

$$(D\delta_\tau K_h^{x,n+\frac{1}{2}}, \psi_{1h}) - (DH_h^{x,n+\frac{1}{2}}, \psi_{1h}) = 0, \quad \forall \psi_{1h} \in V_h, \tag{1.6d}$$

$$(\delta_\tau H_h^{y,n+1}, \tilde{\psi}_h) + (D_K K_h^{y,n+1}, \tilde{\psi}_h) + (D_{H2} \bar{H}_h^{y,n+1}, \tilde{\psi}_h) = (D_{E2} \partial_y E_{xh}^{n+1}) - (D_{E1} \partial_x E_{yh}^{n+1}, \quad \tilde{\psi}_h), \forall \tilde{\psi}_h \in V_h, \tag{1.6e}$$

$$(D\delta_\tau K_h^{y,n+\frac{1}{2}}, \tilde{\psi}_{1h}) - (DH_h^{y,n+\frac{1}{2}}, \tilde{\psi}_{1h}) = 0, \quad \forall \tilde{\psi}_{1h} \in V_h. \tag{1.6f}$$

In our simulation, the source wave $\sin(\omega_0 t)$ is imposed on the H^y field, and we use $h = 0.2$ and $\tau = 0.01$. Snapshots of the obtained H field are presented in Fig. 3, which is similar to what we obtained by the FDTD method in our previous work [24, Fig. 4].

5. Conclusion

In this paper, we propose a fully-discrete finite element method to solve the Drude metamaterial perfectly matched layer model by edge elements. We provide the stability analysis and error estimate for the scheme. Finally, we implement the scheme to justify the theoretical error estimate and demonstrate the wave absorbing effectiveness of this perfectly matched layer model. How to establish the stability and error analysis for the variable damping functions is open and challenging (same issue exists for the second-order wave equation [10]), and we will continue our investigation in the future.

Data availability

Data will be made available on request.

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