Journal of Scientific Computing (2024) 100:26
https://doi.org/10.1007/s10915-024-02585-3

Developing and Analyzing Some Novel Finite Element Schemes for the Electromagnetic Rotation Cloak Model

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Received: 18 January 2024 / Revised: 15 April 2024 / Accepted: 26 May 2024

Abstract
One potential application of metamaterials is for designing invisibility cloaks. In this paper, we are interested in a rotation cloak model. Here we carry out the mathematical analysis of this model for the first time. Through a careful analysis, we reformulate a new system of governing partial differential equations by reducing one unknown variable from the originally developed modeling equations in Yang et al. (Commun Comput Phys 25:135–154, 2019). Then some novel finite element schemes are proposed and their stability and optimal error estimate are proved. Numerical simulations are presented to demonstrate that the new schemes for the reduced modeling equations can effectively reproduce the rotation cloaking phenomenon.

Keywords Maxwell's equations · Finite element method · Metamaterials · Rotation cloak

Mathematics Subject Classification 78M10 · 65N30 · 65F10 · 78-08

1 Introduction

The discovery of the electromagnetic (EM) metamaterials in 2000 stimulated a growing interest in developing and analyzing various numerical methods for solving the Maxwell's equations in metamaterials (cf., [5, 6, 10, 21, 28, 30], since metamaterials have many potential revolutionary applications across different areas, such as sensing, nanolithography with light,

Submitted to the editors on January 18, 2024.

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Published online: 08 June 2024

subwavelength imaging with super-resolution, invisibility cloaks (e.g., [3, 15, 24, 25, 29, 33]), optical black hole [31], EM concentrator [18], rotator, and splitter etc [7, 8].

Due to its advantage in dealing with complex geometries and algorithmic robustness, the finite element method (FEM) plays an important role in solving Maxwell's equations. Over the years, many FEMs have been developed and implemented to solve Maxwell's equations in both frequency domain (e.g., [14, 17, 19, 34]) and time domain (e.g, [9, 11, 12, 23]). More details can be found in books on FEMs for Maxwell's equations (e.g., [13, 22, 26]) and review papers [1, 16].

In 2007, Chen and Chan [7] used the transformation media technique to design a real rotation cloak, which is an invisible field rotator that rotates the EM fields so that the source wave from inside/outside the cloak appears as if it comes from a different angle θ_0. In 2009, Chen, Chan and their collaborators [8] made a sample rotator and experimentally demonstrated the field rotation effect as well as the broadband functionality at microwave frequencies. In 2019, Yang et al. [32] derived a set of time-domain Maxwell's equations and proposed a finite element scheme to successfully model the EM field rotation effect. However, up to now, no any mathematical analysis has been done for the modeling equations. Furthermore, the finite element scheme proposed in [32] involves all the unknown variables, including a 2D electric field $\boldsymbol{E}$, a 2D electric flux density $\boldsymbol{D}$, and the scalar magnetic field H. In this paper, we fill the theoretical analysis gap by carrying out some mathematical analysis for this rotator model. More specifically, through a careful observation, we first reformulate the original model into a new set of governing equations involving only $\boldsymbol{E}$ and $\boldsymbol{D}$. Then we establish the existence and uniqueness result for the new modeling equations. Some novel finite element schemes (including both unconditonally and conditionally stable) are developed and analyzed for the new model. Compared to the previous work [32], the new schemes are more efficient and use less memory storage. To our best knowledge, this is the first mathematical analysis paper devoted to the rotation cloak model. The newly proposed schemes and the theoretical analysis are original.

The rest of the paper is organized as follows. In Sect. 2, we first present the original time-domain governing equations for the EM rotation cloak model, then we reformulate it with less unkowns. The existence and uniqueness of the solution for this model are established. In Sect. 3, we propose an unconditionally stable finite element scheme for solving the rotator model, then we establish the discrete stability and the error estimate of the scheme. In Sect. 4, we extend the idea to three similar schemes, and present a conditional stability analysis. In Sect. 5, we present some numerical results to demonstrate the rotation cloaking effect achieved by the model. Finally, we conclude the paper in Sect. 6.

2 The Model Problem and Its Analysis

The 2D electromagnetic rotator cloak modelling equations were originally derived in our previous work (cf. [32, Sec.2.2]). For the sake of completeness, we reiterate some important steps from [32]. This cloaking device consists of two regions: the cloaking region $\Omega_2 = \{(r, \theta) : R_1 \leq r \leq R_2, 0 \leq \theta \leq 2\pi\}$ changes the direction of wave propagation, and the inner core region $\Omega_1 = \{(r, \theta) : 0 \leq r \leq R_1, 0 \leq \theta \leq 2\pi\}$ rotates the incoming wave by an angle θ_0 (cf., Fig. 1). To realize these functions, the exact permittivity and permeability can be derived by the coordinate transformation technique (cf. [22, Sec.9.2]). For this rotator cloak, the coordinate transformation is given as follows (cf. [7]):

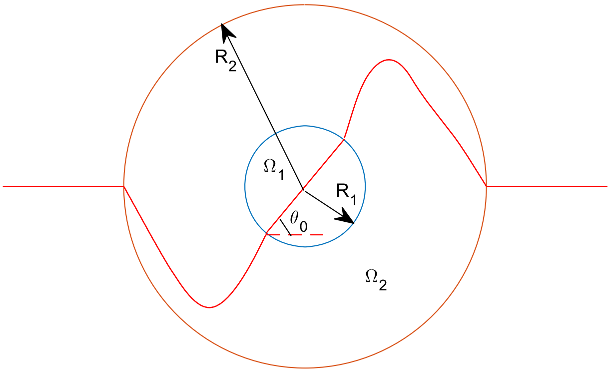

Fig. 1 Illustration of the coordinate transformation of a cylindrical rotator cloak

$$
\begin{aligned}
r' &= r, \quad 0 \le r \le R_2, \\
\theta' &= \begin{cases} \theta + \theta_0, & 0 \le r \le R_1, \\ \theta + \frac{R_2 - r}{R_2 - R_1}\theta_0, & R_1 \le r \le R_2. \end{cases}
\end{aligned} \tag{2.1}
$$

Using the form invariant property of Maxwell's equations (cf. [22, Sec.9.2]), the relative permittivity and permeability for the rotator cloak can be obtained (cf. [32, Sec.2.2]):

$$
\begin{aligned}
\mu_r^{'}(x^{'}, y^{'}) &= 1, \quad 0 \le r' \le R_2, \\
\epsilon_r'(x', y') &= \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & 0 \le r' \le R_1, \\ \begin{pmatrix} a(x', y') & b(x', y') \\ b(x', y') & c(x', y') \end{pmatrix}, & R_1 \le r' \le R_2, \end{cases}
\end{aligned} \tag{2.2}
$$

where we denote

$$
\begin{aligned}
&a(x', y') = 1 + 2m_* \frac{x'y'}{r'^2} + \frac{m_*^2 y'^2}{r'^2}, \quad b(x', y') = -\frac{m_*(x'^2 - y'^2)}{r'^2} - \frac{m_*^2 x'y'}{r'^2}, \\
&c(x', y') = 1 + \frac{x'^2 m_*^2}{r'^2} - \frac{2m_* x'y'}{r'^2}, \quad r' = \sqrt{x'^2 + y'^2}, \quad m_* = \frac{\theta_0 r'}{R_2 - R_1}.
\end{aligned}
$$

Note that $\epsilon_r'(x', y')$ on the cloaking region ($R_1 \le r' \le R_2$) is symmetric and can be diagonlized as

$$
\epsilon_r' = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^T, \quad P = \begin{bmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{bmatrix}, \tag{2.3}
$$

where λ_1 and λ_2 are the two eigenvalues of $\epsilon_r'(x', y')$ given as [32, Sec.2.2]:

$$
\lambda_1 = \frac{2 + m_*^2 - \sqrt{(2 + m_*^2)^2 - 4}}{2} \in (0, 1), \quad \lambda_2 = \frac{2 + m_*^2 + \sqrt{(2 + m_*^2)^2 - 4}}{2} \in (1, \infty).
$$

Furthermore, we denote

$$
p_1 = \sqrt{\frac{\lambda_2 - a(x', y')}{\lambda_2 - \lambda_1}}, \quad p_2 = sgn(b(x', y'))\sqrt{\frac{a(x', y') - \lambda_1}{\lambda_2 - \lambda_1}},
$$

where sgn is the standard sign function.

Since $\lambda_1 < 1$, which is nonphysical and we map it by the lossless Drude dispersive medium model

$$\lambda_1 = 1 - \frac{\omega_e^2}{\omega^2}, \tag{2.4}$$

where ω_e and ω denote the plasma frequency and the general wave frequency, respectively. Substituting (2.4) into (2.3), using the constitutive equation $\hat{\boldsymbol{D}} = \epsilon_0 \epsilon_r' \hat{\boldsymbol{E}}$ and the inverse of matrix ϵ_r', we obtain

$$\frac{1}{\lambda_2(1-\frac{\omega_e^2}{\omega^2})} \begin{bmatrix} \lambda_2 p_1^2 + p_2^2 - \frac{\omega_e^2 p_2^2}{\omega^2} & p_1 p_2(1-\lambda_2) - \frac{p_1 p_2 \omega_e^2}{\omega^2} \\ p_1 p_2(1-\lambda_2) - \frac{p_1 p_2 \omega_e^2}{\omega^2} & p_1^2 + \lambda_2 p_2^2 - \frac{\omega_e^2 p_1^2}{\omega^2} \end{bmatrix} \hat{\boldsymbol{D}} = \epsilon_0 \hat{\boldsymbol{E}}, \tag{2.5}$$

where $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{D}}$ denote the eletric field and electric flux density in the frequency domain, respectively.

Applying the time-harmonic relation $u(\boldsymbol{x}, t) = Re(e^{\sqrt{-1}\omega t} \hat{u}(\boldsymbol{x}, t))$ to (2.5), we can obtain the constitutive equation in the time-domain given below in (2.6b), which along with the Faraday's law and Ampere's law leads to the electromagnetic rotator model (cf. [32, Sec.2.2]): For any $(\boldsymbol{x}, t) \in \Omega \times (0, T]$,

$$\partial_t \boldsymbol{D}(\boldsymbol{x}, t) = \nabla \times H(\boldsymbol{x}, t), \tag{2.6a}$$

$$\epsilon_0 \lambda_2 \left(\partial_{tt} \boldsymbol{E}(\boldsymbol{x}, t) + \omega_e^2 \boldsymbol{E}(\boldsymbol{x}, t)\right) = M_a \partial_{tt} \boldsymbol{D}(\boldsymbol{x}, t) + M_b \boldsymbol{D}(\boldsymbol{x}, t), \tag{2.6b}$$

$$\mu_0 \partial_t H(\boldsymbol{x}, t) = -\nabla \times \boldsymbol{E}(\boldsymbol{x}, t), \tag{2.6c}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, H denotes the magnetic field, $\boldsymbol{E} = (E_x, E_y)'$ is the electric field, $\boldsymbol{D} = (D_x, D_y)'$ is the electric flux density, and ϵ_0 and μ_0 are the free space permittivity and permeability, respectively. Here we adopt the 2D curl operators $\nabla \times H = (\partial_y H, -\partial_x H)'$ and $\nabla \times \boldsymbol{E} = \partial_x E_y - \partial_y E_x$. Finally, matrices M_a and M_b are given as follows:

$$M_a = \begin{bmatrix} p_1^2 \lambda_2 + p_2^2 & p_1 p_2 (1-\lambda_2) \\ p_1 p_2 (1-\lambda_2) & p_1^2 + p_2^2 \lambda_2 \end{bmatrix}, \quad M_b = \omega_e^2 \begin{bmatrix} p_2^2 & p_1 p_2 \\ p_1 p_2 & p_1^2 \end{bmatrix}.$$

To complete the rotator model, we assume that (2.6a)–(2.6c) satisfy the initial conditions

$$\begin{aligned} &\boldsymbol{D}(\boldsymbol{x}, 0) = \boldsymbol{D}_0(\boldsymbol{x}), \quad \partial_t \boldsymbol{D}(\boldsymbol{x}, 0) = \boldsymbol{D}_1(\boldsymbol{x}), \\ &\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \partial_t \boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_1(\boldsymbol{x}), \quad H(\boldsymbol{x}, 0) = H_0(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega, \end{aligned} \tag{2.7}$$

and the perfect conducting (PEC) boundary condition:

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{2.8}$$

where $\boldsymbol{D}_0, \boldsymbol{D}_1, \boldsymbol{E}_0, \boldsymbol{E}_1, H_0$ are some given functions, and $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$.

First, we like to prove the following properties for matrices M_a, M_b, and $M_c := M_a^{-1} M_b$. We need the positive definiteness to invert matrix M_a later to benefit the mathematical analysis of the model, and we also need M_b to be symmetric nonnegative definite in order to define a norm in Theorem 2 below.

Lemma 1 *(I) The matrix M_a is symmetric positive definite.*
(II) The matrix M_b is symmetric nonnegative definite.
(III) For $M_c := M_a^{-1} M_b$, we have $M_c = M_b$.

Proof (I) For any vector (u, v), it is easy to see that

$$(u, v)M_a \begin{bmatrix} u \\ v \end{bmatrix} = u^2(p_1^2\lambda_2 + p_2^2) + 2p_1p_2(1-\lambda_2)uv + (p_1^2 + p_2^2\lambda_2)v^2$$
$$= (up_2 + vp_1)^2 + \lambda_2(up_1 - vp_2)^2 \geq 0, \tag{2.9}$$

and (2.9) equals zero if and only if when $u = v = 0$. This proves that M_a is symmetric positive definite.

(II) It is easy to see that

$$(u, v)M_b \begin{bmatrix} u \\ v \end{bmatrix} = \omega_e^2(p_2^2u^2 + 2p_1p_2uv + p_1^2v^2) = \omega_e^2(p_2u + p_1v)^2 \geq 0, \tag{2.10}$$

which shows that M_b is nonnegative definite.

(III) Since M_a is positive definite, its inverse M_a exists and is given by

$$M_a^{-1} = \frac{1}{\lambda_2}\begin{bmatrix} p_1^2 + p_2^2\lambda_2 & -p_1p_2(1-\lambda_2) \\ -p_1p_2(1-\lambda_2) & p_1^2\lambda_2 + p_2^2 \end{bmatrix}. \tag{2.11}$$

Through some algebraic calculations, and using the fact that $p_1^2 + p_2^2 = 1$, we easily obtain that $M_a^{-1}M_b = M_b$, which completes the proof. □

Taking the time derivative of (2.6a) and using (2.6c), we have

$$\partial_{tt}\boldsymbol{D} = \nabla \times \partial_t H = -\mu_0^{-1}\nabla \times \nabla \times \boldsymbol{E}. \tag{2.12}$$

Using (2.12), we can reduce the original model problem (2.6a)–(2.6c) with three unknowns $(\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H})$ to a problem involving only two unknowns $(\boldsymbol{E}, \boldsymbol{D})$:

$$\partial_{tt}\boldsymbol{D} = -\mu_0^{-1}\nabla \times \nabla \times \boldsymbol{E}, \tag{2.13a}$$
$$\epsilon_0\lambda_2(M_a^{-1}\partial_{tt}\boldsymbol{E} + \omega_e^2 M_a^{-1}\boldsymbol{E}) = \partial_{tt}\boldsymbol{D} + M_b\boldsymbol{D}, \tag{2.13b}$$

where (2.13b) is obtained by multiplying (2.6b) with M_a^{-1} and using Lemma 1 (III).

From now on, we consider the following weak formulation problem of (2.13a)–(2.13b): Find $\boldsymbol{E} \in H_0(curl; \Omega)$ and $\boldsymbol{D} \in H(curl; \Omega)$ such that

$$(\partial_{tt}\boldsymbol{D}, \boldsymbol{\phi}) = -\mu_0^{-1}(\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi} \in H_0(curl; \Omega), \tag{2.14a}$$
$$\epsilon_0\lambda_2(M_a^{-1}\partial_{tt}\boldsymbol{E} + \omega_e^2 M_a^{-1}\boldsymbol{E}, \boldsymbol{\psi}) = (\partial_{tt}\boldsymbol{D} + M_b\boldsymbol{D}, \boldsymbol{\psi}), \quad \forall\, \boldsymbol{\psi} \in H_0(curl; \Omega), \tag{2.14b}$$

subject to the same initial conditions (2.7) (except for H) and the PEC boundary condition (2.8).

Theorem 1 *There exists a unique solution* $(\boldsymbol{E}, \boldsymbol{D}) \in H_0(curl; \Omega) \times H(curl; \Omega)$ *for the problem (2.14a)–(2.14b).*

Proof Denote the Laplace transform of a function $f(t)$ for $t \geq 0$ by $\widehat{f}(s) = \int_0^\infty e^{-st} f(t)dt$. Taking the Laplace transform of (2.14a) and using the initial conditions (2.7), we obtain

$$(s^2\widehat{\boldsymbol{D}} - s\boldsymbol{D}_0 - \boldsymbol{D}_1, \boldsymbol{\phi}) = -\mu_0^{-1}(\nabla \times \widehat{\boldsymbol{E}}, \nabla \times \boldsymbol{\phi}). \tag{2.15}$$

Taking the Laplace transform of (2.14b) and using (2.15), we have

$$\epsilon_0\lambda_2\left(M_a^{-1}(s^2\widehat{\boldsymbol{E}} - s\boldsymbol{E}_0 - \boldsymbol{E}_1) + \omega_e^2 M_a^{-1}\widehat{\boldsymbol{E}}, \boldsymbol{\psi}\right)$$
$$= -\mu_0^{-1}(\nabla \times \widehat{\boldsymbol{E}}, \nabla \times \boldsymbol{\psi}) + (M_b\widehat{\boldsymbol{D}}, \boldsymbol{\psi}). \tag{2.16}$$

Multiplying (2.16) by s^2, then replacing $s^2\widehat{\boldsymbol{D}}$ by (2.15), and collecting like terms, we obtain

$$\mu_0^{-1}\left((s^2 I_2 + M_b)\nabla \times \widehat{\boldsymbol{E}}, \nabla \times \boldsymbol{\psi}\right) + \epsilon_0\lambda_2 s^2(s^2 + \omega_e^2)(M_a^{-1}\widehat{\boldsymbol{E}}, \boldsymbol{\psi})$$
$$= \epsilon_0\lambda_2 s^2\left(M_a^{-1}(s\boldsymbol{E}_0 + \boldsymbol{E}_1), \boldsymbol{\psi}\right) + (M_b(s\boldsymbol{D}_0 + \boldsymbol{D}_1), \boldsymbol{\psi}), \tag{2.17}$$

where we denote the 2 by 2 identity matrix $I_2 = diag(1, 1)$.

Using the Lax-Milgram lemma, we know that (2.17) exists a unique solution $\widehat{\boldsymbol{E}} \in H_0(curl; \Omega)$. The uniqueness of $\boldsymbol{E}$ follows from the uniqueness of the inverse Laplace transform of $\widehat{\boldsymbol{E}}$. The existence and uniqueness of the solution $\boldsymbol{D}$ is guaranteed by (2.14a). □

Denote the $L^2(\Omega)$ norm as $||\cdot||$. We can establish the following stability for the solution of (2.14a) and (2.14b).

Theorem 2 *For the solution $(\boldsymbol{E}, \boldsymbol{D})$ of (2.14a)–(2.14b), the following energy identity holds true for any $t \in (0, T]$:*

$$ENG(t) - ENG(0)$$
$$= \int_0^t 2\left[(M_b\boldsymbol{D}, \partial_t\boldsymbol{E}) + \omega_e^{-2}(M_b\partial_t\boldsymbol{D}, \partial_{tt}\boldsymbol{E}) + \epsilon_0\lambda_2(M_a^{-1}\partial_{tt}\boldsymbol{E} + \omega_e^2 M_a^{-1}\boldsymbol{E}, \partial_t\boldsymbol{D})\right] ds, \tag{2.18}$$

where the energy $ENG(t)$ is defined as

$$ENG(t) := \epsilon_0\lambda_2(||\omega_e^{-1}M_a^{-\frac{1}{2}}\partial_{tt}\boldsymbol{E}||^2 + 2||M_a^{-\frac{1}{2}}\partial_t\boldsymbol{E}||^2 + ||\omega_e M_a^{-\frac{1}{2}}\boldsymbol{E}||^2)$$
$$+\mu_0^{-1}(||\nabla \times \boldsymbol{E}||^2 + ||\omega_e^{-1}\nabla \times \partial_t\boldsymbol{E}||^2) + ||\partial_t\boldsymbol{D}||^2 + ||M_b^{\frac{1}{2}}\boldsymbol{D}||^2. \tag{2.19}$$

Moreover, the following stability holds:

$$ENG(t) \le ENG(0)\cdot \exp(C_* t), \quad \forall\, t \in [0, T], \tag{2.20}$$

where the positive constant C_* depends on those physical parameters of the problem (2.14a) and (2.14b).

Proof To make our proof easy to follow, we divide it into two major parts.

(I) Choosing $\boldsymbol{\psi} = \partial_t\boldsymbol{E}$ in (2.14b), and using (2.14a) with $\boldsymbol{\phi} = \partial_t\boldsymbol{E}$, we have

$$\frac{\epsilon_0\lambda_2}{2}\frac{d}{dt}(||M_a^{-\frac{1}{2}}\partial_t\boldsymbol{E}||^2 + ||\omega_e M_a^{-\frac{1}{2}}\boldsymbol{E}||^2) = -\mu_0^{-1}(\nabla \times \boldsymbol{E}, \nabla \times \partial_t\boldsymbol{E}) + (M_b\boldsymbol{D}, \partial_t\boldsymbol{E})$$
$$= -\frac{\mu_0^{-1}}{2}\frac{d}{dt}||\nabla \times \boldsymbol{E}||^2 + (M_b\boldsymbol{D}, \partial_t\boldsymbol{E}), \tag{2.21}$$

i.e.,

$$\frac{\epsilon_0\lambda_2}{2}\frac{d}{dt}(||M_a^{-\frac{1}{2}}\partial_t\boldsymbol{E}||^2 + ||\omega_e M_a^{-\frac{1}{2}}\boldsymbol{E}||^2) + \frac{\mu_0^{-1}}{2}\frac{d}{dt}||\nabla \times \boldsymbol{E}||^2 = (M_b\boldsymbol{D}, \partial_t\boldsymbol{E}). \tag{2.22}$$

Similarly, taking the time derivative of (2.14b), then using (2.14a) and choosing $\boldsymbol{\psi} = \omega_e^{-2}\partial_{tt}\boldsymbol{E}$, we obtain

$$\frac{\epsilon_0\lambda_2}{2}\frac{d}{dt}(||\omega_e^{-1}M_a^{-\frac{1}{2}}\partial_{tt}\boldsymbol{E}||^2 + ||M_a^{-\frac{1}{2}}\partial_t\boldsymbol{E}||^2) + \frac{\mu_0^{-1}\omega_e^{-2}}{2}\frac{d}{dt}||\nabla\times\partial_t\boldsymbol{E}||^2 = \omega_e^{-2}(M_b\partial_t\boldsymbol{D}, \partial_{tt}\boldsymbol{E}). \tag{2.23}$$

To bound $\boldsymbol{D}$ and $\partial_t\boldsymbol{D}$, choosing $\boldsymbol{\psi} = \partial_t\boldsymbol{D}$ in (2.14b), we have

$$\frac{1}{2}\frac{d}{dt}(||\partial_t\boldsymbol{D}||^2 + ||M_b^{\frac{1}{2}}\boldsymbol{D}||^2) = \epsilon_0\lambda_2(M_a^{-1}\partial_{tt}\boldsymbol{E} + \omega_e^2 M_a^{-1}\boldsymbol{E}, \partial_t\boldsymbol{D}). \tag{2.24}$$

Adding (2.22), (2.23) and (2.24) together, we obtain

$$\frac{d}{dt}ENG(t) = 2\left[(M_b\boldsymbol{D}, \partial_t\boldsymbol{E}) + \omega_e^{-2}(M_b\partial_t\boldsymbol{D}, \partial_{tt}\boldsymbol{E}) + \epsilon_0\lambda_2(M_a^{-1}\partial_{tt}\boldsymbol{E} + \omega_e^2 M_a^{-1}\boldsymbol{E}, \partial_t\boldsymbol{D})\right], \tag{2.25}$$

integrating which with respect to t from 0 to t completes the proof of (2.18).

(II) Using the Cauchy-Schwarz inequality, we can bound those four terms on the right hand side of (2.25) as follows:

$$\begin{aligned}
&2(M_b\boldsymbol{D}, \partial_t\boldsymbol{E}) \le ||M_b^{\frac{1}{2}}M_a^{\frac{1}{2}}||_\infty(||M_b^{\frac{1}{2}}\boldsymbol{D}||^2 + ||M_a^{-\frac{1}{2}}\partial_t\boldsymbol{E}||^2),\\
&2\omega_e^{-2}(M_b\partial_t\boldsymbol{D}, \partial_{tt}\boldsymbol{E}) \le ||\omega_e^{-1}M_bM_a^{\frac{1}{2}}||_\infty(||\partial_t\boldsymbol{D}||^2 + ||\omega_e^{-1}M_a^{-\frac{1}{2}}\partial_{tt}\boldsymbol{E}||^2),\\
&2\epsilon_0\lambda_2(M_a^{-1}\partial_{tt}\boldsymbol{E}, \partial_t\boldsymbol{D}) \le \epsilon_0\lambda_2||\omega_e M_a^{-\frac{1}{2}}||_\infty(||\omega_e^{-1}M_a^{-\frac{1}{2}}\partial_{tt}\boldsymbol{E}||^2 + ||\partial_t\boldsymbol{D}||^2),\\
&2\epsilon_0\lambda_2(\omega_e^2M_a^{-1}\boldsymbol{E}, \partial_t\boldsymbol{D}) \le \epsilon_0\lambda_2||\omega_e M_a^{-\frac{1}{2}}||_\infty(||\omega_e M_a^{-\frac{1}{2}}\boldsymbol{E}||^2 + ||\partial_t\boldsymbol{D}||^2).
\end{aligned}$$

Substituting the above estimates into (2.25) and using the Gronwall inequality (e.g., [20, Lemma 2.1]), we complete the proof of (2.20). □

3 An Unconditionally Stable Scheme and Its Analysis

To solve the problem (2.14a)-(2.14b) by a finite element method, we partition the physical domain Ω by a family of regular triangular mesh T_h with maximum mesh size h, and adopt the r-th order Nédélec edge element space $\boldsymbol{U}_h$ [26, 27]: For any $r \ge 1$,

$$\boldsymbol{U}_h = \{\boldsymbol{u}_h \in H(\text{curl}; \Omega) : \boldsymbol{u}_h|_K \in (p_{r-1})^2 \oplus S_r, \quad \forall K \in T_h\}, \tag{3.1}$$

where $S_r = \{\vec{p} \in (\tilde{p}_r)^2, \boldsymbol{x}\cdot\vec{p} = 0\}$, $\tilde{p}_r$ denotes the space of homogeneous polynomials of degree r, and p_r denotes the space of polynomials of degree less than or equal to r in variables x, y, respectively. To impose the PEC boundary condition (2.8), we denote the subspace $\boldsymbol{U}_h^0 = \{\boldsymbol{u} \in \boldsymbol{U}_h : \boldsymbol{u}\times\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

To construct a fully-discrete scheme, we assume that the time domain $[0, T]$ is discretized uniformly by points $t_i = i\tau, i = 0, 1, \cdots, N_t$, where the time step size $\tau = \frac{T}{N_t}$. For any time function u^n, we introduce the following time difference and averaging operators:

$$\delta_\tau u^{n+\frac{1}{2}} = \frac{u^{n+1} - u^n}{\tau}, \quad \delta_\tau^2 u^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2},$$
$$\delta_{2\tau} u^n = \frac{u^{n+1} - u^{n-1}}{2\tau}, \quad \widetilde{u}^n = \frac{u^{n+1} + u^{n-1}}{2}.$$

Now we consider the following scheme: For any $n \geq 0$, find $\boldsymbol{D}_h^{n+1} \in \boldsymbol{U}_h$, $\boldsymbol{E}_h^{n+1} \in \boldsymbol{U}_h^0$ such that

$$(\delta_\tau^2 \boldsymbol{D}_h^n, \boldsymbol{\phi}_h) = -\mu_0^{-1}(\nabla \times \widetilde{\boldsymbol{E}}_h^n, \nabla \times \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{U}_h^0, \tag{3.2a}$$
$$\epsilon_0 \lambda_2 (M_a^{-1} \delta_\tau^2 \boldsymbol{E}_h^n + \omega_e^2 M_a^{-1} \widetilde{\boldsymbol{E}}_h^n, \boldsymbol{\psi}_h) = (\delta_\tau^2 \boldsymbol{D}_h^n + M_b \boldsymbol{D}_h^n, \boldsymbol{\psi}_h), \quad \forall \boldsymbol{\psi}_h \in \boldsymbol{U}_h^0, \tag{3.2b}$$

where the needed initial approximations $\boldsymbol{D}_h^0$, $\boldsymbol{D}_h^1$, $\boldsymbol{E}_h^0$, $\boldsymbol{E}_h^1$ can be obtained from (2.7) as follows:

$$\boldsymbol{D}_h^0 = \Pi_h^c \boldsymbol{D}_0(\boldsymbol{x}), \quad \boldsymbol{D}_h^1 - \boldsymbol{D}_h^{-1} = 2\tau \Pi_h^c \boldsymbol{D}_1(\boldsymbol{x}), \tag{3.3a}$$
$$\boldsymbol{E}_h^0 = \Pi_h^c \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{E}_h^1 - \boldsymbol{E}_h^{-1} = 2\tau \Pi_h^c \boldsymbol{E}_1(\boldsymbol{x}), \tag{3.3b}$$

where Π_h^c denotes the Nédélec interpolation operator.

We like to remark that this scheme is very easy in practical implementation: At each time step,

Step 1: Substitute (3.2a) into (3.2b) to solve for $\boldsymbol{E}_h^{n+1}$ from the following equation:

$$\epsilon_0 \lambda_2 (M_a^{-1} \delta_\tau^2 \boldsymbol{E}_h^n + \omega_e^2 M_a^{-1} \widetilde{\boldsymbol{E}}_h^n, \boldsymbol{\psi}_h) + \mu_0^{-1}(\nabla \times \widetilde{\boldsymbol{E}}_h^n, \nabla \times \boldsymbol{\psi}_h) = (M_b \boldsymbol{D}_h^n, \boldsymbol{\psi}_h). \tag{3.4}$$

Step 2: Solve (3.2a) for $\boldsymbol{D}_h^{n+1}$.

Note that when $n = 0$, we have to use the initial approximations (3.3a) and (3.3b) to replace $\boldsymbol{D}_h^{-1}$ and $\boldsymbol{E}_h^{-1}$.

Below we will establish both the stability and convergence analysis for our scheme (3.2a)-(3.2b).

3.1 Stability Analysis

In this subsection, we prove the unconditional stability for the scheme (3.2a)-(3.2b). For the solution of (3.2a)-(3.2b), we denote the following discrete energy

$$\begin{aligned} ENG_{ep}^{n+\frac{1}{2}} = \epsilon_0 \lambda_2 \Bigg(& \frac{||\omega_e^{-1} M_a^{-\frac{1}{2}} \delta_\tau^2 \boldsymbol{E}_h^{n+1}||^2 + ||\omega_e^{-1} M_a^{-\frac{1}{2}} \delta_\tau^2 \boldsymbol{E}_h^n||^2}{2} + ||M_a^{-\frac{1}{2}} \delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}||^2 \\ & + ||M_a^{-\frac{1}{2}} \delta_\tau \widetilde{\boldsymbol{E}}_h^{n+\frac{1}{2}}||^2 + \frac{||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_h^{n+1}||^2 + ||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_h^n||^2}{2} \Bigg) \\ & + \mu_0^{-1} \left(\frac{||\nabla \times \boldsymbol{E}_h^{n+1}||^2 + ||\nabla \times \boldsymbol{E}_h^n||^2}{2} + ||\omega_e^{-1} \nabla \times \delta_\tau \widetilde{\boldsymbol{E}}_h^{n+\frac{1}{2}}||^2 \right) \\ & + ||\delta_\tau \boldsymbol{D}_h^{n+\frac{1}{2}}||^2 + \frac{||M_b^{\frac{1}{2}} \boldsymbol{D}_h^{n+1}||^2 + ||M_b^{\frac{1}{2}} \boldsymbol{D}_h^n||^2}{2}. \end{aligned} \tag{3.5}$$

Theorem 3 *For the solution of (3.2a)-(3.2b), the following energy identity holds true: For any $m \in [1, N_t - 2]$,*

$$
\begin{aligned}
&ENG_{ep}^{m+\frac{1}{2}} - ENG_{ep}^{\frac{1}{2}} - \frac{\tau^2}{2}(||\delta_\tau \boldsymbol{D}_h^{m+\frac{1}{2}}||^2 - ||\delta_\tau \boldsymbol{D}_h^{\frac{1}{2}}||^2) \\
&= 2\tau \sum_{n=1}^{m} \Big[(M_b \boldsymbol{D}_h^n, \delta_{2\tau} \boldsymbol{E}_h^n) + \omega_e^{-2}(M_b \delta_{2\tau} \boldsymbol{D}_h^n, \delta_\tau^2 \widetilde{\boldsymbol{E}}_h^n) \\
&\quad + \epsilon_0 \lambda_2 (M_a^{-1} \delta_\tau^2 \boldsymbol{E}_h^n + \omega_e^2 M_a^{-1} \widetilde{\boldsymbol{E}}_h^n, \delta_{2\tau} \boldsymbol{D}_h^n) \Big].
\end{aligned} \tag{3.6}
$$

Furthermore, under the time step constraint:

$$
\tau \le \min \left\{ \frac{1}{2(||\omega_e^{-1} M_b M_a^{\frac{1}{2}}||_\infty + 2\epsilon_0 \lambda_2 ||\omega_e M_a^{-\frac{1}{2}}||_\infty)}, \frac{1}{2||M_b^{\frac{1}{2}} M_a^{\frac{1}{2}}||_\infty}, \frac{1}{\sqrt{2}} \right\}, \tag{3.7}
$$

we have the following discrete stability:

$$
ENG_{ep}^{m+\frac{1}{2}} \le 2 \cdot ENG_{ep}^{\frac{1}{2}} \cdot \exp(C_{**} m\tau), \tag{3.8}
$$

where the constant C_{**} only depends on the physical parameters of the model.

Remark 1 We like to remark that the time step constraint (3.7) only depends on those physical parameters of the model, and is independent of the finite element mesh size h. Hence the scheme (3.2a) and (3.2b) is an unconditionally stable scheme. Moreover, (3.6) is a discrete form of the continuous energy identity (2.18) with an extra small perturbed term.

Proof The proof follows those similar technues developed for the proof of continuous stability given in Theorem 2. To make our proof easy to follow, we divide it into several major parts.

(I) Choosing $\boldsymbol{\psi}_h = \tau \delta_{2\tau} \boldsymbol{E}_h^n$ in (3.4), then using the following identities:

$$
\begin{aligned}
(\delta_\tau^2 u^n, \delta_{2\tau} u^n) &= \left(\frac{\delta_\tau u^{n+\frac{1}{2}} - \delta_\tau u^{n-\frac{1}{2}}}{\tau}, \frac{\delta_\tau u^{n+\frac{1}{2}} + \delta_\tau u^{n-\frac{1}{2}}}{2} \right) \\
&= \frac{1}{2\tau} \left(\left| |\delta_\tau u^{n+\frac{1}{2}} \right| |^2 - \left| |\delta_\tau u^{n-\frac{1}{2}} \right| |^2 \right),
\end{aligned} \tag{3.9}
$$

and

$$
(\widetilde{u}^n, \delta_{2\tau} u^n) = \left(\frac{u^{n+1} + u^{n-1}}{2}, \frac{u^{n+1} - u^{n-1}}{2\tau} \right) = \frac{1}{4\tau} \left(||u^{n+1}||^2 - ||u^{n-1}||^2 \right), \tag{3.10}
$$

with $u = \boldsymbol{E}_h$, we have

$$
\begin{aligned}
&\frac{\epsilon_0 \lambda_2}{2} (||M_a^{-\frac{1}{2}} \delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}||^2 - ||M_a^{-\frac{1}{2}} \delta_\tau \boldsymbol{E}_h^{n-\frac{1}{2}}||^2) \\
&+ \frac{\epsilon_0 \lambda_2}{4} (||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_h^{n+1}||^2 - ||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_h^{n-1}||^2) \\
&+ \frac{\mu_0^{-1}}{4} (||\nabla \times \boldsymbol{E}_h^{n+1}||^2 - ||\nabla \times \boldsymbol{E}_h^{n-1}||^2) = \tau (M_b \boldsymbol{D}_h^n, \delta_{2\tau} \boldsymbol{E}_h^n).
\end{aligned} \tag{3.11}
$$

Dividing (3.11) by τ really leads to a discretized form of (2.22).

(II) Using (3.2b) with $n=n+1$ to subtract itself with $n=n-1$, then dividing the result by 2τ, and using (3.2a), we obtain

$$\begin{aligned}&\epsilon_0\lambda_2(M_a^{-1}\frac{\delta_\tau^2\boldsymbol{E}_h^{n+1}-\delta_\tau^2\boldsymbol{E}_h^{n-1}}{2\tau}+\omega_e^2M_a^{-1}\delta_{2\tau}\widetilde{\boldsymbol{E}}_h^n,\boldsymbol{\psi}_h)\\&+\mu_0^{-1}(\nabla\times\delta_{2\tau}\widetilde{\boldsymbol{E}}_h^n,\nabla\times\boldsymbol{\psi}_h)=(M_b\delta_{2\tau}\boldsymbol{D}_h^n,\boldsymbol{\psi}_h).\end{aligned}\tag{3.12}$$

Choosing $\boldsymbol{\psi}_h=\omega_e^{-2}\cdot\tau\delta_\tau^2\widetilde{\boldsymbol{E}}_h^n$ in (3.12), and using the identities (3.9) and (3.10) with with $u=\widetilde{\boldsymbol{E}}_h$, we obtain

$$\begin{aligned}&\frac{\epsilon_0\lambda_2}{4}\left(||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^{n+1}||^2-||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^{n-1}||^2\right)\\&+\frac{\epsilon_0\lambda_2}{2}(||M_a^{-\frac{1}{2}}\delta_\tau\widetilde{\boldsymbol{E}}_h^{n+\frac{1}{2}}||^2-||M_a^{-\frac{1}{2}}\delta_\tau\widetilde{\boldsymbol{E}}_h^{n-\frac{1}{2}}||^2)\\&+\frac{\omega_e^{-2}\mu_0^{-1}}{2}(||\nabla\times\delta_\tau\widetilde{\boldsymbol{E}}_h^{n+\frac{1}{2}}||^2-||\nabla\times\delta_\tau\widetilde{\boldsymbol{E}}_h^{n-\frac{1}{2}}||^2)\\&=\tau\omega_e^{-2}(M_b\delta_{2\tau}\boldsymbol{D}_h^n,\delta_\tau^2\widetilde{\boldsymbol{E}}_h^n).\end{aligned}\tag{3.13}$$

We like to remark that dividing (3.13) by τ leads to a discretized form of (2.23).

(III) Choosing $\boldsymbol{\psi}_h=\tau\delta_{2\tau}\boldsymbol{D}_h^n$ in (3.2b), and using the following identities

$$\tau(\delta_\tau^2u^n,\delta_{2\tau}u^n)=\frac{1}{2}(||\delta_\tau u^{n+\frac{1}{2}}||^2-||\delta_\tau u^{n-\frac{1}{2}}||^2),\tag{3.14}$$

and

$$\begin{aligned}\tau(u^n,\delta_{2\tau}u^n)&=\left(\frac{2u^n-u^{n+1}-u^{n-1}+(u^{n+1}+u^{n-1})}{2},\frac{u^{n+1}-u^{n-1}}{2}\right)\\&=\frac{1}{4}(||u^{n+1}||^2-||u^{n-1}||^2)-\frac{\tau^3}{2}(\delta_\tau^2u^n,\delta_{2\tau}u^n),\\&=\frac{1}{4}(||u^{n+1}||^2-||u^{n-1}||^2)-\frac{\tau^2}{4}(||\delta_\tau u^{n+\frac{1}{2}}||^2-||\delta_\tau u^{n-\frac{1}{2}}||^2)\end{aligned}\tag{3.15}$$

with $u=\boldsymbol{D}_h$, we obtain

$$\begin{aligned}&\frac{1}{2}(||\delta_\tau\boldsymbol{D}_h^{n+\frac{1}{2}}||^2-||\delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}}||^2)+\frac{1}{4}(||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{n+1}||^2-||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{n-1}||^2)\\&-\frac{\tau^2}{4}(||\delta_\tau\boldsymbol{D}_h^{n+\frac{1}{2}}||^2-||\delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}}||^2)=\tau\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n\\&+\omega_e^2M_a^{-1}\widetilde{\boldsymbol{E}}_h^n,\delta_{2\tau}\boldsymbol{D}_h^n).\end{aligned}\tag{3.16}$$

Again, dividing (3.16) by τ yields a discretized form of (2.24).

Adding (3.11), (3.13) and (3.16), then using the definition of $ENG_{cn}^{n+\frac{1}{2}}$, and summing up the result from $n=1$ to any $m\le N_t-2$, we obtain

$$\begin{aligned}&ENG_{ep}^{m+\frac{1}{2}}-ENG_{ep}^{\frac{1}{2}}-\frac{\tau^2}{2}(||\delta_\tau\boldsymbol{D}_h^{m+\frac{1}{2}}||^2-||\delta_\tau\boldsymbol{D}_h^{\frac{1}{2}}||^2)\\&=2\tau\sum_{n=1}^m\Big[(M_b\boldsymbol{D}_h^n,\delta_{2\tau}\boldsymbol{E}_h^n)+\omega_e^{-2}(M_b\delta_{2\tau}\boldsymbol{D}_h^n,\delta_\tau^2\widetilde{\boldsymbol{E}}_h^n)\\&\quad+\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n+\omega_e^2M_a^{-1}\widetilde{\boldsymbol{E}}_h^n,\delta_{2\tau}\boldsymbol{D}_h^n)\Big].\end{aligned}\tag{3.17}$$

(IV) By absorbing the last left hand side term into the energy term, we immediately have

$$
\begin{aligned}
(1-\frac{\tau^2}{2})ENG_{ep}^{m+\frac{1}{2}} \ &\leq (1+\frac{\tau^2}{2})ENG_{ep}^{\frac{1}{2}} \\
&\quad +2\tau\sum_{n=1}^{m}\Big[(M_b\boldsymbol{D}_h^n,\delta_{2\tau}\boldsymbol{E}_h^n)+\omega_e^{-2}(M_b\delta_{2\tau}\boldsymbol{D}_h^n,\delta_\tau^2\widetilde{\boldsymbol{E}}_h^n) \\
&\quad +\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n+\omega_e^2M_a^{-1}\widetilde{\boldsymbol{E}}_h^n,\delta_{2\tau}\boldsymbol{D}_h^n)\Big]. \qquad (3.18)
\end{aligned}
$$

Now we just need to estimate the last four right hand side terms. First, by the Cauchy–Schwarz inequlaity and the inequality $||\frac{a+b}{2}||^2\leq\frac{1}{2}(||a||^2+||b||^2)$, we have

$$
\begin{aligned}
2\tau\sum_{n=1}^{m}(M_b\boldsymbol{D}_h^n,\delta_{2\tau}\boldsymbol{E}_h^n) &\leq \tau||M_b^{\frac{1}{2}}M_a^{\frac{1}{2}}||_\infty\sum_{n=1}^{m}\left(||M_b^{\frac{1}{2}}\boldsymbol{D}_h^n||^2+||M_a^{-\frac{1}{2}}\delta_{2\tau}\boldsymbol{E}_h^n||^2\right) \\
&\leq \tau||M_b^{\frac{1}{2}}M_a^{\frac{1}{2}}||_\infty\sum_{n=1}^{m}\left(||M_b^{\frac{1}{2}}\boldsymbol{D}_h^n||^2+\frac{||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}}||^2+||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n-\frac{1}{2}}||^2}{2}\right) \\
&\leq \tau||M_b^{\frac{1}{2}}M_a^{\frac{1}{2}}||_\infty\left(\frac{||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{m+1}||^2+||M_b^{\frac{1}{2}}\boldsymbol{D}_h^m||^2}{2}+||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{m+\frac{1}{2}}||^2\right) \\
&\quad +\tau||M_b^{\frac{1}{2}}M_a^{\frac{1}{2}}||_\infty\sum_{n=0}^{m-1}\left(\frac{||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{n+1}||^2+||M_b^{\frac{1}{2}}\boldsymbol{D}_h^n||^2}{2}+||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}}||^2\right).(3.19)
\end{aligned}
$$

By the same arguments, we have

$$
\begin{aligned}
2\tau\sum_{n=1}^{m}\omega_e^{-2}(M_b\delta_{2\tau}\boldsymbol{D}_h^n,\delta_\tau^2\widetilde{\boldsymbol{E}}_h^n) &\leq \tau||\omega_e^{-1}M_bM_a^{\frac{1}{2}}||_\infty\sum_{n=1}^{m}(||\delta_{2\tau}\boldsymbol{D}_h^n||^2+||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\widetilde{\boldsymbol{E}}_h^n||^2) \\
&\leq \tau||\omega_e^{-1}M_bM_a^{\frac{1}{2}}||_\infty\left(\frac{1}{2}||\delta_\tau\boldsymbol{D}_h^{m+\frac{1}{2}}||^2+\frac{||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^{m+1}||^2+||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^m||^2}{2}\right) \\
&\quad +\tau||\omega_e^{-1}M_bM_a^{\frac{1}{2}}||_\infty\sum_{n=0}^{m-1}\left(||\delta_\tau\boldsymbol{D}_h^{n+\frac{1}{2}}||^2+||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^n||^2\right), \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
2\tau\epsilon_0\lambda_2\sum_{n=1}^{m}(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n,\delta_{2\tau}\boldsymbol{D}_h^n) &\leq \tau\epsilon_0\lambda_2||\omega_eM_a^{-\frac{1}{2}}||_\infty\sum_{n=1}^{m}\left(||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^n||^2+||\delta_{2\tau}\boldsymbol{D}_h^n||^2\right) \\
&\leq \tau\epsilon_0\lambda_2||\omega_eM_a^{-\frac{1}{2}}||_\infty\left(||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^m||^2+\frac{1}{2}||\delta_\tau\boldsymbol{D}_h^{m+\frac{1}{2}}||^2\right) \\
&\quad +\tau\epsilon_0\lambda_2||\omega_eM_a^{-\frac{1}{2}}||_\infty\sum_{n=0}^{m-1}\left(||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^n||^2+||\delta_\tau\boldsymbol{D}_h^{n+\frac{1}{2}}||^2\right), \qquad (3.21)
\end{aligned}
$$

and

$$
2\tau\epsilon_0\lambda_2\sum_{n=1}^{m}(\omega_e^2M_a^{-1}\widetilde{\boldsymbol{E}}_h^n,\delta_{2\tau}\boldsymbol{D}_h^n)\leq\tau\epsilon_0\lambda_2||\omega_eM_a^{-\frac{1}{2}}||_\infty\sum_{n=1}^{m}(||\omega_eM_a^{-\frac{1}{2}}\widetilde{\boldsymbol{E}}_h^n||^2+||\delta_{2\tau}\boldsymbol{D}_h^n||^2)
$$

$$\leq \tau\epsilon_0\lambda_2||\omega_e M_a^{-\frac{1}{2}}||_\infty \left(\frac{||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_h^{m+1}||^2 + ||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_h^{m}||^2}{2} + \frac{1}{2}||\delta_\tau \boldsymbol{D}_h^{m+\frac{1}{2}}||^2 \right)$$
$$+\tau\epsilon_0\lambda_2||\omega_e M_a^{-\frac{1}{2}}||_\infty \sum_{n=0}^{m-1} \left(||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_h^{n}||^2 + ||\delta_\tau \boldsymbol{D}_h^{n+\frac{1}{2}}||^2 \right). \tag{3.22}$$

Substituting the estimates of (3.19)-(3.22) into (3.18), then choosing τ small enough, e.g.,

$$\tau(||\omega_e^{-1} M_b M_a^{\frac{1}{2}}||_\infty + 2\epsilon_0\lambda_2||\omega_e M_a^{-\frac{1}{2}}||_\infty) \leq \frac{1}{2},\ \tau||M_b^{\frac{1}{2}} M_a^{\frac{1}{2}}||_\infty \leq \frac{1}{2},\ \tau \leq \frac{1}{\sqrt{2}}, \tag{3.23}$$

which are equivalent to (3.7), and using the discrete Gronwall inequality (e.g., [20, Lemma 3.1]), we complete the proof. □

3.2 Optimal Error Estimate

To establish the error estimate, we need the following interpolation error estimate

$$||\boldsymbol{u} - \Pi_h^c \boldsymbol{u}||_{H(curl;\Omega)} \leq ch^r ||\boldsymbol{u}||_{H^r(curl;\Omega)}, \quad \forall\, \boldsymbol{u} \in H^r(curl;\Omega), \tag{3.24}$$

where we denote the norm $||\boldsymbol{u}||_{H^r(curl;\Omega)} := (||\boldsymbol{u}||_{H^r(\Omega)} + ||\nabla\times\boldsymbol{u}||_{H^r(\Omega)})^{1/2}$ for the Sobolev space

$$H^r(curl;\Omega) = \left\{ \boldsymbol{v} \in H^r(\Omega) \mid\ \nabla\times\boldsymbol{u} \in H^r(\Omega) \right\}.$$

To carry out the convergence analysis, we split the solution errors into two parts: one is the error between the finite element solution and the corrsponding interpolation; the other one is the interpolation error, i.e.,

$$\mathcal{E}_{\mathbf{h}}^{\mathbf{n}} = \boldsymbol{E}_{\mathbf{h}}^{\mathbf{n}} - \boldsymbol{E}(\boldsymbol{x}, \mathbf{t_n}) = (\boldsymbol{E}_{\mathbf{h}}^{\mathbf{n}} - \Pi_{\mathbf{h}}^{\mathbf{c}}\boldsymbol{E}^{\mathbf{n}}) - (\boldsymbol{E}^{\mathbf{n}} - \Pi_{\mathbf{h}}^{\mathbf{c}}\boldsymbol{E}^{\mathbf{n}}) := \boldsymbol{E}_{\mathbf{h}\xi}^{\mathbf{n}} - \boldsymbol{E}_{\mathbf{h}\eta}^{\mathbf{n}},$$
$$\mathcal{D}_{\mathbf{h}}^{\mathbf{n}} = \boldsymbol{D}_{\mathbf{h}}^{\mathbf{n}} - \boldsymbol{D}(\boldsymbol{x}, \mathbf{t_n}) = (\boldsymbol{D}_{\mathbf{h}}^{\mathbf{n}} - \Pi_{\mathbf{h}}^{\mathbf{c}}\boldsymbol{D}^{\mathbf{n}}) - (\boldsymbol{D}^{\mathbf{n}} - \Pi_{\mathbf{h}}^{\mathbf{c}}\boldsymbol{D}^{\mathbf{n}}) := \boldsymbol{D}_{\mathbf{h}\xi}^{\mathbf{n}} - \boldsymbol{D}_{\mathbf{h}\eta}^{\mathbf{n}},$$

here and below we simply denote $\boldsymbol{E}^n := \boldsymbol{E}(\boldsymbol{x}, t_n)$ and $\boldsymbol{D}^n := \boldsymbol{D}(\boldsymbol{x}, t_n)$.

Now we can present the following optimal error estimate for the scheme (3.2a)-(3.2b). The idea is to first derive the error equations, which have exactly the same form as the numerical scheme (3.2a)-(3.2b) plus extra error terms caused by the spatial interpolation and time approximations. Hence, the proof of error estimate follows the stabilty proof closely.

Theorem 4 *Suppose that the analytical solutions $(\boldsymbol{E}, \boldsymbol{D})$ of (2.14a)-(2.14b) are smooth enough, then for any $n \geq 1$ we have*

$$\epsilon_0\lambda_2 \Bigg(\frac{||\omega_e^{-1} M_a^{-\frac{1}{2}}(\delta_\tau^2 \boldsymbol{E}_h^{n+1} - \partial_t^2 \boldsymbol{E}^{n+1})||^2 + ||\omega_e^{-1} M_a^{-\frac{1}{2}}(\delta_\tau^2 \boldsymbol{E}_h^{n} - \partial_t^2 \boldsymbol{E}^{n})||^2}{2}$$
$$+||M_a^{-\frac{1}{2}}(\delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}} - \partial_t \boldsymbol{E}^{n+\frac{1}{2}})||^2 + ||M_a^{-\frac{1}{2}}(\delta_\tau \widetilde{\boldsymbol{E}}_h^{n+\frac{1}{2}} - \partial_t \widetilde{\boldsymbol{E}}^{n+\frac{1}{2}})||^2$$
$$+\frac{||\omega_e M_a^{-\frac{1}{2}}(\boldsymbol{E}_h^{n+1} - \boldsymbol{E}^{n+1})||^2 + ||\omega_e M_a^{-\frac{1}{2}}(\boldsymbol{E}_h^{n} - \boldsymbol{E}^{n})||^2}{2} \Bigg)$$
$$+\mu_0^{-1} \left(\frac{||\nabla\times(\boldsymbol{E}_h^{n+1} - \boldsymbol{E}^{n+1})||^2 + ||\nabla\times(\boldsymbol{E}_h^{n} - \boldsymbol{E}^{n})||^2}{2} + ||\omega_e^{-1}\nabla\times(\delta_\tau \widetilde{\boldsymbol{E}}_h^{n+\frac{1}{2}} - \partial_t \widetilde{\boldsymbol{E}}^{n+\frac{1}{2}})||^2 \right)$$

$$+||\delta_\tau \boldsymbol{D}_h^{n+\frac{1}{2}} - \partial_t \boldsymbol{D}^{n+\frac{1}{2}}||^2 + \frac{||M_b^{\frac{1}{2}}(\boldsymbol{D}_h^{n+1} - \boldsymbol{D}^{n+1})||^2 + ||M_b^{\frac{1}{2}}(\boldsymbol{D}_h^n - \boldsymbol{D}^n)||^2}{2}$$
$$\leq C(\tau^2 + h^r)^2, \tag{3.25}$$

where constant $C > 0$ is independent of h and τ, and r is the degree of the finite element basis functions.

Proof First, we need to derive the error equations. Integrating (2.14b) from $t = t_{n-1}$ to $t = t_{n+1}$, then dividing by 2τ, we obtain

$$\left(\delta_{2\tau}\partial_t \boldsymbol{D}^n + \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} M_b \boldsymbol{D}\, dt, \boldsymbol{\psi}_h\right) = \epsilon_0\lambda_2\left(M_a^{-1}\delta_{2\tau}\partial_t \boldsymbol{E}^n + \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} \omega_e^2 M_a^{-1}\boldsymbol{E}\, dt, \boldsymbol{\psi}_h\right). \tag{3.26}$$

Subtracting (3.26) from (3.2b) and using the error notations, we have the first error equation:

$$\begin{aligned}
&(\delta_\tau^2 \boldsymbol{D}_{h\xi}^n + M_b \boldsymbol{D}_{h\xi}^n, \boldsymbol{\psi}_h) - \epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2 \boldsymbol{E}_{h\xi}^n + \omega_e^2 M_a^{-1}\widetilde{\boldsymbol{E}}_{h\xi}^n, \boldsymbol{\psi}_h)\\
&= (\delta_\tau^2 \boldsymbol{D}_{h\eta}^n + M_b \boldsymbol{D}_{h\eta}^n, \boldsymbol{\psi}_h) - \epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2 \boldsymbol{E}_{h\eta}^n + \omega_e^2 M_a^{-1}\widetilde{\boldsymbol{E}}_{h\eta}^n, \boldsymbol{\psi}_h)\\
&\quad + \left(\delta_{2\tau}\partial_t \boldsymbol{D}^n - \delta_\tau^2 \boldsymbol{D}^n + \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} M_b(\boldsymbol{D} - \boldsymbol{D}^n)\, dt, \boldsymbol{\psi}_h\right)\\
&\quad -\epsilon_0\lambda_2\left(M_a^{-1}(\delta_{2\tau}\partial_t \boldsymbol{E}^n - \delta_\tau^2 \boldsymbol{E}^n) + \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} \omega_e^2 M_a^{-1}(\boldsymbol{E} - \widetilde{\boldsymbol{E}}^n)\, dt, \boldsymbol{\psi}_h\right).
\end{aligned} \tag{3.27}$$

Similarly, integrating (2.14b) (with the substition of (2.14a)) from $t = t_{n-1}$ to $t = t_{n+1}$, then dividing by 2τ, we obtain

$$\begin{aligned}
&\epsilon_0\lambda_2\left(M_a^{-1}\delta_{2\tau}\partial_t \boldsymbol{E}^n + \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} \omega_e^2 M_a^{-1}\boldsymbol{E} dt, \boldsymbol{\psi}\right)\\
&\quad + \mu_0^{-1}\left(\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} \nabla\times\boldsymbol{E} dt, \nabla\times\boldsymbol{\psi}\right) = \left(\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} M_b \boldsymbol{D}\, dt, \boldsymbol{\psi}\right).
\end{aligned} \tag{3.28}$$

Subtracting (3.28) from (3.2b) (with the use of (3.2a)) and using the error notations, we have the second error equation:

$$\begin{aligned}
&\epsilon_0\lambda_2\left(M_a^{-1}\delta_\tau^2 \boldsymbol{E}_{h\xi}^n + \omega_e^2 M_a^{-1}\widetilde{\boldsymbol{E}}_{h\xi}^n, \boldsymbol{\psi}_h\right) + \mu_0^{-1}\left(\nabla\times\widetilde{\boldsymbol{E}}_{h\xi}^n, \nabla\times\boldsymbol{\psi}_h\right) - \left(M_b \boldsymbol{D}_{h\xi}^n, \boldsymbol{\psi}_h\right)\\
&= \epsilon_0\lambda_2\left(M_a^{-1}\delta_\tau^2 \boldsymbol{E}_{h\eta}^n + \omega_e^2 M_a^{-1}\widetilde{\boldsymbol{E}}_{h\eta}^n, \boldsymbol{\psi}_h\right) + \mu_0^{-1}\left(\nabla\times\widetilde{\boldsymbol{E}}_{h\eta}^n, \nabla\times\boldsymbol{\psi}_h\right) - \left(M_b \boldsymbol{D}_{h\eta}^n, \boldsymbol{\psi}_h\right)\\
&\quad + \epsilon_0\lambda_2\left(M_a^{-1}\left(\delta_{2\tau}\partial_t \boldsymbol{E}^n - \delta_\tau^2 \boldsymbol{E}^n\right), \boldsymbol{\psi}_h\right) + \epsilon_0\lambda_2\left(\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} \omega_e^2 M_a^{-1}(\boldsymbol{E} - \widetilde{\boldsymbol{E}}^n) dt, \boldsymbol{\psi}_h\right)\\
&\quad + \mu_0^{-1}\left(\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} \nabla\times(\boldsymbol{E} - \widetilde{\boldsymbol{E}}^n) dt, \nabla\times\boldsymbol{\psi}_h\right) - \left(\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} M_b(\boldsymbol{D} - \boldsymbol{D}^n)\, dt, \boldsymbol{\psi}_h\right).
\end{aligned} \tag{3.29}$$

Note that the left hand sides of (3.27) and (3.29) have exactly the same forms as the equations (3.2b) and (3.4) used in the stability proof of Theorem 3, and the right hand sides (RHS) are extra terms caused by the spatial interpolation and time approximations. By the

Cauchy-Schwarz inequality, the interpolation error estimate (3.24) and Lemma 3 given in Appendix, all RHS terms of (3.27) and (3.29) can be bounded by $O(\tau^2 + h^r)$. Due to their similarities, here we just illustrate how to estimate a few typical terms.

By Lemma 3 (IV) and the interpolation error estimate (3.24), we have

$$\begin{aligned}
&||\delta_\tau^2 \boldsymbol{D}_{h\eta}^n + M_b \boldsymbol{D}_{h\eta}^n||^2 \le 2\left[\frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}} ||\partial_t^2 \boldsymbol{D}_{h\eta}||^2 dt + ||M_b||_\infty^2 ||\boldsymbol{D}_{h\eta}^n||^2\right] \\
&\le 2\left[\frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}} ch^{2r}||\partial_t^2 \boldsymbol{D}||^2_{H^r(\text{curl};\Omega)} dt + ||M_b||_\infty^2 ch^{2r}||\boldsymbol{D}||^2_{L^\infty(0,T;H^r(curl;\Omega))}\right] \\
&\le ch^{2r}\left[||\partial_t^2 \boldsymbol{D}||^2_{L^\infty(0,T;H^r(curl;\Omega))} + ||\boldsymbol{D}||^2_{L^\infty(0,T;H^r(curl;\Omega))}\right].
\end{aligned} \tag{3.30}$$

Similarly, by Lemma 3 (II) and (VII), we have

$$\begin{aligned}
&||\delta_{2\tau}\partial_t \boldsymbol{D}^n - \delta_\tau^2 \boldsymbol{D}^n + \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} M_b(\boldsymbol{D} - \boldsymbol{D}^n)\, dt||^2 \\
&\le c\tau^3 \int_{t_{n-1}}^{t_{n+1}} (||\partial_\tau^4 \boldsymbol{D}||^2 + ||M_b||_\infty^2 ||\partial_\tau^2 \boldsymbol{D}||^2) dt.
\end{aligned} \tag{3.31}$$

Using the same technique developed for the stability proof in Theorem 3 and those RHS estimates, we can obtain: For any $n \ge 1$,

$$\begin{aligned}
&\epsilon_0\lambda_2 \left(\frac{||\omega_e^{-1} M_a^{-\frac{1}{2}} \delta_\tau^2 \boldsymbol{E}_{h\xi}^{n+1}||^2 + ||\omega_e^{-1} M_a^{-\frac{1}{2}} \delta_\tau^2 \boldsymbol{E}_{h\xi}^{n}||^2}{2} + ||M_a^{-\frac{1}{2}} \delta_\tau \boldsymbol{E}_{h\xi}^{n+\frac{1}{2}}||^2 \right. \\
&\left. + ||M_a^{-\frac{1}{2}} \delta_\tau \widetilde{\boldsymbol{E}}_{h\xi}^{n+\frac{1}{2}}||^2 + \frac{||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_{h\xi}^{n+1}||^2 + ||\omega_e M_a^{-\frac{1}{2}} \boldsymbol{E}_{h\xi}^{n}||^2}{2} \right) \\
&+ \mu_0^{-1} \left(\frac{||\nabla \times \boldsymbol{E}_{h\xi}^{n+1}||^2 + ||\nabla \times \boldsymbol{E}_{h\xi}^{n}||^2}{2} + ||\omega_e^{-1} \nabla \times \delta_\tau \widetilde{\boldsymbol{E}}_{h\xi}^{n+\frac{1}{2}}||^2 \right) \\
&+ ||\delta_\tau \boldsymbol{D}_{h\xi}^{n+\frac{1}{2}}||^2 + \frac{||M_b^{\frac{1}{2}} \boldsymbol{D}_{h\xi}^{n+1}||^2 + ||M_b^{\frac{1}{2}} \boldsymbol{D}_{h\xi}^{n}||^2}{2} \le C(\tau^2 + h^r)^2.
\end{aligned} \tag{3.32}$$

Finally, applying the triangle inequality, Lemma 3 and the interpolation error estimate (3.24) to all terms in (3.32), we can complete the proof of (3.25). Due to its technicality, here we just present one estimate as an illustration:

$$\begin{aligned}
&||\delta_\tau^2 \boldsymbol{E}_h^n - \partial_t^2 \boldsymbol{E}^n||^2 = ||\delta_\tau^2 \boldsymbol{E}_{h\xi}^n - \delta_\tau^2 \boldsymbol{E}_{h\eta}^n + (\delta_\tau^2 \boldsymbol{E}^n - \partial_t^2 \boldsymbol{E}^n)||^2 \\
&\le 3\left[||\delta_\tau^2 \boldsymbol{E}_{h\xi}^n||^2 + ||\delta_\tau^2 \boldsymbol{E}_{h\eta}^n||^2 + c\tau^4 ||\partial_t^4 \boldsymbol{E}||^2_{L^\infty(0,T;(L^2(\Omega))^2)}\right] \\
&\le C(\tau^2 + h^r)^2 + Ch^{2r}||\partial_t^2 \boldsymbol{E}||^2_{L^\infty(0,T;H^r(curl;\Omega))} + C\tau^4 ||\partial_t^4 \boldsymbol{E}||^2_{L^\infty(0,T;(L^2(\Omega))^2)}.
\end{aligned} \tag{3.33}$$

□

4 Extensions to Other Similar Schemes

We like to remark that we can construct some other schemes similar to (3.2a)-(3.2b), such as

$$(\delta_\tau^2 \boldsymbol{D}_h^n, \boldsymbol{\phi}_h) = -\mu_0^{-1}(\nabla \times \widetilde{\boldsymbol{E}}_h^n, \nabla \times \boldsymbol{\phi}_h), \quad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{U}_h^0, \tag{4.1a}$$
$$\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2 \boldsymbol{E}_h^n + \omega_e^2 M_a^{-1}\boldsymbol{E}_h^n, \boldsymbol{\psi}_h) = (\delta_\tau^2 \boldsymbol{D}_h^n + M_b \boldsymbol{D}_h^n, \boldsymbol{\psi}_h), \quad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{U}_h^0, \tag{4.1b}$$
$$(\delta_\tau^2 \boldsymbol{D}_h^n, \boldsymbol{\phi}_h) = -\mu_0^{-1}(\nabla \times \boldsymbol{E}_h^n, \nabla \times \boldsymbol{\phi}_h), \quad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{U}_h^0, \tag{4.2a}$$
$$\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2 \boldsymbol{E}_h^n + \omega_e^2 M_a^{-1}\boldsymbol{E}_h^n, \boldsymbol{\psi}_h) = (\delta_\tau^2 \boldsymbol{D}_h^n + M_b \boldsymbol{D}_h^n, \boldsymbol{\psi}_h), \quad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{U}_h^0, \tag{4.2b}$$

and

$$(\delta_\tau^2 \boldsymbol{D}_h^n, \boldsymbol{\phi}_h) = -\mu_0^{-1}(\nabla \times \boldsymbol{E}_h^n, \nabla \times \boldsymbol{\phi}_h), \quad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{U}_h^0, \tag{4.3a}$$
$$\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2 \boldsymbol{E}_h^n + \omega_e^2 M_a^{-1}\widetilde{\boldsymbol{E}}_h^n, \boldsymbol{\psi}_h) = (\delta_\tau^2 \boldsymbol{D}_h^n + M_b \boldsymbol{D}_h^n, \boldsymbol{\psi}_h), \quad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{U}_h^0. \tag{4.3b}$$

It is easy to see that the scheme (4.1a)-(4.1b) can be implemented similarly as (3.2a) and (3.2b) by substituting (4.1a) into (4.1b) to solve for $\boldsymbol{E}_h^{n+1}$ from the following equation:

$$\begin{aligned}\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2 \boldsymbol{E}_h^n, \boldsymbol{\psi}_h) + \mu_0^{-1}(\nabla \times \widetilde{\boldsymbol{E}}_h^n, \nabla \times \boldsymbol{\psi}_h)\\ = (M_b \boldsymbol{D}_h^n, \boldsymbol{\psi}_h) - \epsilon_0\lambda_2(\omega_e^2 M_a^{-1}\boldsymbol{E}_h^n, \boldsymbol{\psi}_h).\end{aligned} \tag{4.4}$$

Then solve (4.1a) for $\boldsymbol{D}_h^{n+1}$.

While the implementation of schemes (4.2a) and (4.2b), (4.3a) and (4.3b) are straightforward by first solving (4.2a) and (4.3a) respectively for $\boldsymbol{D}_h^{n+1}$, then solve (4.2b) and (4.3b) for $\boldsymbol{E}_h^{n+1}$. In this sense, we can think that both schemes (4.2a) and (4.2b), (4.3a) and (4.3b) are explicit.

Furthermore, we like to mention that stability and error estimate for these schemes can be carried out similarly, but their analyses are quite delicate. Since schemes (4.2a) and (4.2b), (4.3a) and (4.3b) are similar, below we just present the stability analysis for (4.2a) and (4.2b).

The same strategy developed in the stabilty proof for the scheme (3.2a) and (3.2b) does not working for the scheme (4.2a) and (4.2b). The stability proof is much more complicated than the previous proof for scheme (3.2a) and (3.2b). Moreover, we need the following standard inverse estimate

$$||\nabla \times \boldsymbol{u}_h|| \le C_{inv} h^{-1}||\boldsymbol{u}_h||, \quad \forall\, \boldsymbol{u}_h \in \boldsymbol{U}_h, \tag{4.5}$$

where the constant $C_{inv} > 0$ is independent of the mesh size h. We also need the following identity.

Lemma 2 *For any sequence function u^n, we have*

$$\begin{aligned}&\left(\delta_\tau u^{n-\frac{1}{2}}, \tau\delta_\tau^2(\frac{u^n + u^{n-1}}{2})\right)\\ &= \frac{1}{4}\left[(||\delta_\tau u^{n+\frac{1}{2}}||^2 - ||\delta_\tau u^{n-\frac{3}{2}}||^2) - \tau^2(||\delta_\tau^2 u^n||^2 - ||\delta_\tau^2 u^{n-1}||^2)\right].\end{aligned} \tag{4.6}$$

Proof First, note that

$$\begin{aligned}&\left(\delta_\tau u^{n-\frac{1}{2}}, \tau\delta_\tau^2(\frac{u^n+u^{n-1}}{2})\right)\\&=\left(\delta_\tau u^{n-\frac{1}{2}}, \frac{1}{2}(\delta_\tau u^{n+\frac{1}{2}}-\delta_\tau u^{n-\frac{1}{2}})+\frac{1}{2}(\delta_\tau u^{n-\frac{1}{2}}-\delta_\tau u^{n-\frac{3}{2}})\right)\\&=\frac{1}{2}(\delta_\tau u^{n-\frac{1}{2}}, \delta_\tau u^{n+\frac{1}{2}}-\delta_\tau u^{n-\frac{1}{2}})+\frac{1}{2}(\delta_\tau u^{n-\frac{1}{2}}, \delta_\tau u^{n-\frac{1}{2}}-\delta_\tau u^{n-\frac{3}{2}}).\end{aligned}\tag{4.7}$$

Using the following identity

$$(a_{n+1}, a_{n+1}-a_n)=\frac{1}{2}\left[(a_{n+1}^2-a_n^2)+(a_{n+1}-a_n)^2\right]\tag{4.8}$$

for $a_n=\delta_\tau u^{n-\frac{1}{2}}$, we obtain

$$\begin{aligned}&\frac{1}{2}(\delta_\tau u^{n+\frac{1}{2}}, \delta_\tau u^{n+\frac{1}{2}}-\delta_\tau u^{n-\frac{1}{2}})\\&=\frac{1}{4}\left[(||\delta_\tau u^{n+\frac{1}{2}}||^2-||\delta_\tau u^{n-\frac{1}{2}}||^2)+||\delta_\tau u^{n+\frac{1}{2}}-\delta_\tau u^{n-\frac{1}{2}}||^2\right]\\&=\frac{1}{4}\left[(||\delta_\tau u^{n+\frac{1}{2}}||^2-||\delta_\tau u^{n-\frac{1}{2}}||^2)+\tau^2||\delta_\tau^2 u^n||^2\right].\end{aligned}\tag{4.9}$$

Using (4.9), we have

$$\begin{aligned}&\frac{1}{2}(\delta_\tau u^{n-\frac{1}{2}}, \delta_\tau u^{n+\frac{1}{2}}-\delta_\tau u^{n-\frac{1}{2}})\\&=-\frac{1}{2}||\delta_\tau u^{n+\frac{1}{2}}-\delta_\tau u^{n-\frac{1}{2}}||^2+\frac{1}{2}(\delta_\tau u^{n+\frac{1}{2}}, \delta_\tau u^{n+\frac{1}{2}}-\delta_\tau u^{n-\frac{1}{2}})\\&=\frac{1}{4}\left[(||\delta_\tau u^{n+\frac{1}{2}}||^2-||\delta_\tau u^{n-\frac{1}{2}}||^2)-\tau^2||\delta_\tau^2 u^n||^2\right].\end{aligned}\tag{4.10}$$

Substituting (4.9) with n replaced by $n-1$ and (4.10) into (4.7) leads to (4.6). □

Now we present the stability analysis for the scheme (4.2a)-(4.2b).

Theorem 5 *For the solution of (4.2a)-(4.2b), we denote the discrete energy*

$$\begin{aligned}ENG_{ex}^{n+\frac{1}{2}}&=\epsilon_0\lambda_2\Big[||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^n||^2+\frac{1}{2}(3||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}}||^2+||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n-\frac{1}{2}}||^2)\\&+\frac{1}{2}(||\omega_e M_a^{-\frac{1}{2}}\boldsymbol{E}_h^{n+1}||^2+||\omega_e M_a^{-\frac{1}{2}}\boldsymbol{E}_h^n||^2)\Big]\\&+\frac{\mu_0^{-1}}{2}\left[(||\nabla\times\boldsymbol{E}_h^{n+1}||^2+||\nabla\times\boldsymbol{E}_h^n||^2)+(||\omega_e^{-1}\nabla\times\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}}||^2+||\omega_e^{-1}\nabla\times\delta_\tau\boldsymbol{E}_h^{n-\frac{1}{2}}||^2)\right]\\&+||\delta_\tau\boldsymbol{D}_h^{n+\frac{1}{2}}||^2+\frac{1}{2}(||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{n+1}||^2+||M_b^{\frac{1}{2}}\boldsymbol{D}_h^n||^2).\end{aligned}\tag{4.11}$$

First, the following energy identity holds true for the solution of (4.2a)-(4.2b): For any $m\in[1, N_t-2]$,

$$\begin{aligned}&ENG_{ex}^{m+\frac{1}{2}}-ENG_{ex}^{\frac{1}{2}}-\frac{\tau^2}{2}(||\delta_\tau\boldsymbol{D}_h^{m+\frac{1}{2}}||^2-||\delta_\tau\boldsymbol{D}_h^{\frac{1}{2}}||^2)\\&-\frac{\epsilon_0\lambda_2\tau^2}{2}\left(||\omega_e M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{m+\frac{1}{2}}||^2-||\omega_e M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{\frac{1}{2}}||^2\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{\mu_0^{-1}\omega_e^2\tau^2}{2}\left(||\omega_e^{-1}\nabla\times\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2-||\omega_e^{-1}\nabla\times\delta_\tau \boldsymbol{E}_h^{\frac{1}{2}}||^2\right)\\
&-\frac{\epsilon_0\lambda_2\omega_e^2\tau^2}{2}\left(||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^m||^2-||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^0||^2\right)\\
&-\frac{\omega_e^{-2}\mu_0^{-1}\tau^2}{2}\left[(||\nabla\times\delta_\tau^2\boldsymbol{E}_h^m||^2+||\nabla\times\delta_\tau^2\boldsymbol{E}_h^{m-1}||^2)-(||\nabla\times\delta_\tau^2\boldsymbol{E}_h^0||^2+||\nabla\times\delta_\tau^2\boldsymbol{E}_h^1||^2)\right]\\
&=2\tau\sum_{n=1}^{m}\left[(M_b\boldsymbol{D}_h^n,\delta_{2\tau}\boldsymbol{E}_h^n)+\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n+\omega_e^2M_a^{-1}\boldsymbol{E}_h^n,\delta_{2\tau}\boldsymbol{D}_h^n)\right]\\
&\quad+\tau\sum_{n=1}^{m}\omega_e^{-2}\left(M_b\delta_{2\tau}\boldsymbol{D}_h^{n-\frac{1}{2}},\delta_\tau^2(\boldsymbol{E}_h^n+\boldsymbol{E}_h^{n-1})\right).
\end{aligned}
\tag{4.12}
$$

Furthermore, under the time step constraint:

$$
\begin{aligned}
\tau\leq\min\Bigg\{&\frac{\epsilon_0\lambda_2}{4(||\omega_e^{-1}M_bM_a^{\frac{1}{2}}||_\infty+8\epsilon_0\lambda_2||\omega_eM_a^{-\frac{1}{2}}||_\infty)},\frac{1}{2||M_b^{\frac{1}{2}}M_a^{\frac{1}{2}}||_\infty},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}\omega_e},\\
&\frac{1}{4\epsilon_0\lambda_2||\omega_eM_a^{-\frac{1}{2}}||_\infty},\frac{\sqrt{\epsilon_0\mu_0\lambda_2}h}{2C_{inv}||M_a^{\frac{1}{2}}||_\infty}\Bigg\},
\end{aligned}
\tag{4.13}
$$

we have the following discrete stability:

$$
ENG_{ex}^{m+\frac{1}{2}}\leq 2\cdot ENG_{ex}^{\frac{1}{2}}\cdot\exp(C^*m\tau),\tag{4.14}
$$

where the constant C^* only depends on the physical parameters of the model.

Remark 2 We like to remark that the time step constraint (4.13) not only depends on the physical parameters of the model, but also on the mesh size h. Moreover, (4.12) is another discrete form of the continuous energy identity (2.18), and has more extra small perturbed terms compared to the energy identity (3.6) established for the scheme (3.2a)-(3.2b).

Proof The proof follows similarly to the stability proof given in Theorem 3, but much more involved. To make our proof easy to follow, we divide it into several major parts.

(I) Choosing $\boldsymbol{\psi}_h=\tau\delta_{2\tau}\boldsymbol{E}_h^n$ in (4.2b), then substituting (4.2a) into (4.2b), and using identities (3.9) and (3.15) with $u^n=\boldsymbol{E}_h^n$, we have

$$
\begin{aligned}
&\frac{\epsilon_0\lambda_2}{2}(||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}}||^2-||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n-\frac{1}{2}}||^2)\\
&\quad+\frac{\epsilon_0\lambda_2}{4}\Big[(||\omega_eM_a^{-\frac{1}{2}}\boldsymbol{E}_h^{n+1}||^2-||\omega_eM_a^{-\frac{1}{2}}\boldsymbol{E}_h^{n-1}||^2)\\
&\quad-\tau^2(||\omega_eM_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}}||^2-||\omega_eM_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n-\frac{1}{2}}||^2)\Big]\\
&\quad+\frac{\mu_0^{-1}}{4}\left[(||\nabla\times\boldsymbol{E}_h^{n+1}||^2-||\nabla\times\boldsymbol{E}_h^{n-1}||^2)-\tau^2(||\delta_\tau\nabla\times\boldsymbol{E}_h^{n+\frac{1}{2}}||^2-||\delta_\tau\nabla\times\boldsymbol{E}_h^{n-\frac{1}{2}}||^2)\right]\\
&=\tau(M_b\boldsymbol{D}_h^n,\delta_{2\tau}\boldsymbol{E}_h^n).
\end{aligned}
\tag{4.15}
$$

Choosing $\boldsymbol{\psi}_h=\tau\delta_{2\tau}\boldsymbol{D}_h^n$ in (4.2b), and using identities (3.14) and (3.15) with $u^n=\boldsymbol{D}_h^n$, we obtain

$$
\frac{1}{2}(||\delta_\tau\boldsymbol{D}_h^{n+\frac{1}{2}}||^2-||\delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}}||^2)
$$

$$+\frac{1}{4}\left[(||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{n+1}||^2-||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{n-1}||^2)-\tau^2(||\delta_\tau\boldsymbol{D}_h^{n+\frac{1}{2}}||^2-||\delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}}||^2)\right]$$
$$=\tau\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n+\omega_e^2M_a^{-1}\boldsymbol{E}_h^n,\delta_{2\tau}\boldsymbol{D}_h^n). \tag{4.16}$$

Adding (4.15) and (4.16) together, then summing up the result from $n=1$ to any $m\leq N_t-2$, we have

$$\begin{aligned}
&\frac{\epsilon_0\lambda_2}{2}(||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{m+\frac{1}{2}}||^2-||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{\frac{1}{2}}||^2)\\
&+\frac{\epsilon_0\lambda_2}{4}\left[(||\omega_eM_a^{-\frac{1}{2}}\boldsymbol{E}_h^{m+1}||^2+||\omega_eM_a^{-\frac{1}{2}}\boldsymbol{E}_h^{m}||^2)-(||\omega_eM_a^{-\frac{1}{2}}\boldsymbol{E}_h^{0}||^2+||\omega_eM_a^{-\frac{1}{2}}\boldsymbol{E}_h^{1}||^2)\right]\\
&-\frac{\epsilon_0\lambda_2\tau^2}{4}\left(||\omega_eM_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{m+\frac{1}{2}}||^2-||\omega_eM_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{\frac{1}{2}}||^2\right)\\
&+\frac{\mu_0^{-1}}{4}\left[||\nabla\times\boldsymbol{E}_h^{m+1}||^2+||\nabla\times\boldsymbol{E}_h^{m}||^2-||\nabla\times\boldsymbol{E}_h^{0}||^2-||\nabla\times\boldsymbol{E}_h^{1}||^2\right]\\
&-\frac{\mu_0^{-1}\tau^2}{4}\left(||\delta_\tau\nabla\times\boldsymbol{E}_h^{m+\frac{1}{2}}||^2-||\delta_\tau\nabla\times\boldsymbol{E}_h^{\frac{1}{2}}||^2\right)+\frac{1}{2}(1-\frac{\tau^2}{2})\left(||\delta_\tau\boldsymbol{D}_h^{m+\frac{1}{2}}||^2-||\delta_\tau\boldsymbol{D}_h^{\frac{1}{2}}||^2\right)\\
&+\frac{1}{4}\left[||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{m+1}||^2+||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{m}||^2-||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{0}||^2-||M_b^{\frac{1}{2}}\boldsymbol{D}_h^{1}||^2\right]\\
&=\tau\sum_{n=1}^{m}(M_b\boldsymbol{D}_h^n,\delta_{2\tau}\boldsymbol{E}_h^n)+\tau\sum_{n=1}^{m}\epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n+\omega_e^2M_a^{-1}\boldsymbol{E}_h^n,\delta_{2\tau}\boldsymbol{D}_h^n).
\end{aligned} \tag{4.17}$$

(II) Using (4.2b) to subtract itself with n reduced to $n-1$, then dividing the result by τ, and using (4.2a), we obtain

$$\begin{aligned}
&\epsilon_0\lambda_2(M_a^{-1}\frac{\delta_\tau^2\boldsymbol{E}_h^n-\delta_\tau^2\boldsymbol{E}_h^{n-1}}{\tau}+\omega_e^2M_a^{-1}\delta_\tau\boldsymbol{E}_h^{n-\frac{1}{2}},\boldsymbol{\psi}_h)\\
&\quad+\mu_0^{-1}(\nabla\times\delta_\tau\boldsymbol{E}_h^{n-\frac{1}{2}},\nabla\times\boldsymbol{\psi}_h)=(M_b\delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}},\boldsymbol{\psi}_h).
\end{aligned} \tag{4.18}$$

Choosing $\boldsymbol{\psi}_h=\omega_e^{-2}\cdot\tau\delta_\tau^2(\frac{\boldsymbol{E}_h^n+\boldsymbol{E}_h^{n-1}}{2})$ in (4.18), and using Lemma 2 with $u^n=\boldsymbol{E}_h^n$, we obtain

$$\begin{aligned}
&\frac{\epsilon_0\lambda_2}{2}(||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^n||^2-||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^{n-1}||^2)\\
&+\frac{\epsilon_0\lambda_2}{4}\left[(||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}}||^2-||M_a^{-\frac{1}{2}}\delta_\tau\boldsymbol{E}_h^{n-\frac{3}{2}}||^2)-\tau^2(||M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^{n}||^2-||M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^{n-1}||^2)\right]\\
&+\frac{\omega_e^{-2}\mu_0^{-1}}{4}\left[(||\nabla\times\delta_\tau\boldsymbol{E}_h^{n+\frac{1}{2}}||^2-||\nabla\times\delta_\tau\boldsymbol{E}_h^{n-\frac{3}{2}}||^2)-\tau^2(||\nabla\times\delta_\tau^2\boldsymbol{E}_h^{n}||^2-||\nabla\times\delta_\tau^2\boldsymbol{E}_h^{n-1}||^2)\right]\\
&=\tau\omega_e^{-2}\left(M_b\delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}},\delta_\tau^2(\frac{\boldsymbol{E}_h^n+\boldsymbol{E}_h^{n-1}}{2})\right).
\end{aligned} \tag{4.19}$$

(III) Summing up (4.19) from $n=1$ to any $m\leq N_t-2$, then adding the result to (4.17), and using the definition of $ENG_{ex}^{n+\frac{1}{2}}$, we obtain

$$
\begin{aligned}
&ENG_{ex}^{m+\frac{1}{2}} - ENG_{ex}^{\frac{1}{2}} - \frac{\tau^2}{2}(||\delta_\tau \boldsymbol{D}_h^{m+\frac{1}{2}}||^2 - ||\delta_\tau \boldsymbol{D}_h^{\frac{1}{2}}||^2) \\
&-\frac{\epsilon_0\lambda_2\tau^2}{2}\left(||\omega_e M_a^{-\frac{1}{2}}\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 - ||\omega_e M_a^{-\frac{1}{2}}\delta_\tau \boldsymbol{E}_h^{\frac{1}{2}}||^2\right) \\
&-\frac{\mu_0^{-1}\omega_e^2\tau^2}{2}\left(||\omega_e^{-1}\nabla\times\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 - ||\omega_e^{-1}\nabla\times\delta_\tau \boldsymbol{E}_h^{\frac{1}{2}}||^2\right) \\
&-\frac{\epsilon_0\lambda_2\omega_e^2\tau^2}{2}\left(||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2 \boldsymbol{E}_h^{m}||^2 - ||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2 \boldsymbol{E}_h^{0}||^2\right) \\
&-\frac{\omega_e^{-2}\mu_0^{-1}\tau^2}{2}\left[(||\nabla\times\delta_\tau^2\boldsymbol{E}_h^m||^2 + ||\nabla\times\delta_\tau^2\boldsymbol{E}_h^{m-1}||^2) - (||\nabla\times\delta_\tau^2\boldsymbol{E}_h^0||^2 + ||\nabla\times\delta_\tau^2\boldsymbol{E}_h^1||^2)\right] \\
&= 2\tau\sum_{n=1}^{m}\left[(M_b\boldsymbol{D}_h^n, \delta_{2\tau}\boldsymbol{E}_h^n) + \epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n + \omega_e^2 M_a^{-1}\boldsymbol{E}_h^n, \delta_{2\tau}\boldsymbol{D}_h^n)\right] \\
&+\tau\sum_{n=1}^{m}\omega_e^{-2}\left(M_b\delta_{2\tau}\boldsymbol{D}_h^{n-\frac{1}{2}}, \delta_\tau^2(\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1})\right).
\end{aligned}
\tag{4.20}
$$

(IV) Dropping those nonnegative terms on the left hand side of (4.20), we have

$$
\begin{aligned}
&ENG_{ex}^{m+\frac{1}{2}} \le ENG_{ex}^{\frac{1}{2}} + \frac{\tau^2}{2}||\delta_\tau \boldsymbol{D}_h^{m+\frac{1}{2}}||^2 \\
&+\frac{\epsilon_0\lambda_2\tau^2}{2}||\omega_e M_a^{-\frac{1}{2}}\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 + \frac{\mu_0^{-1}\omega_e^2\tau^2}{2}||\omega_e^{-1}\nabla\times\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 \\
&+\frac{\epsilon_0\lambda_2\omega_e^2\tau^2}{2}||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2 \boldsymbol{E}_h^{m}||^2 + \frac{\omega_e^{-2}\mu_0^{-1}\tau^2}{2}(||\nabla\times\delta_\tau^2\boldsymbol{E}_h^m||^2 + ||\nabla\times\delta_\tau^2\boldsymbol{E}_h^{m-1}||^2) \\
&+2\tau\sum_{n=1}^{m}\left[(M_b\boldsymbol{D}_h^n, \delta_{2\tau}\boldsymbol{E}_h^n) + \epsilon_0\lambda_2(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n + \omega_e^2 M_a^{-1}\boldsymbol{E}_h^n, \delta_{2\tau}\boldsymbol{D}_h^n)\right] \\
&+\tau\sum_{n=1}^{m}\omega_e^{-2}\left(M_b\delta_{2\tau}\boldsymbol{D}_h^{n-\frac{1}{2}}, \delta_\tau^2(\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1})\right).
\end{aligned}
\tag{4.21}
$$

Now we just need to bound those right hand side terms of (4.21). First, by using the inverse estimate (4.5), we have

$$
\begin{aligned}
\frac{\omega_e^{-2}\mu_0^{-1}\tau^2}{2}||\nabla\times\delta_\tau^2\boldsymbol{E}_h^m||^2 &\le \frac{\omega_e^{-2}\mu_0^{-1}\tau^2}{2}\cdot C_{inv}^2 h^{-2}||\delta_\tau^2\boldsymbol{E}_h^m||^2 \\
&\le \frac{1}{2}\cdot\tau^2\mu_0^{-1}C_{inv}^2 h^{-2}||M_a^{\frac{1}{2}}||_\infty^2||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^m||^2.
\end{aligned}
\tag{4.22}
$$

The term $2\tau\sum_{n=1}^{m}(M_b\boldsymbol{D}_h^n, \delta_{2\tau}\boldsymbol{E}_h^n)$ can be bounded as derived in (3.19). By the Cauchy-Schwarz inequlaity and the inequality $||\frac{a+b}{2}||^2 \le \frac{1}{2}(||a||^2 + ||b||^2)$, we have

$$
\begin{aligned}
&2\tau\epsilon_0\lambda_2\sum_{n=1}^{m}(M_a^{-1}\delta_\tau^2\boldsymbol{E}_h^n, \delta_{2\tau}\boldsymbol{D}_h^n) \le \tau\epsilon_0\lambda_2\omega_e||M_a^{-\frac{1}{2}}||_\infty \\
&\sum_{n=1}^{m}\left[||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^n||^2 + \frac{1}{4}||\delta_\tau\boldsymbol{D}_h^{n+\frac{1}{2}} + \delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}}||^2\right] \\
&\le \tau\epsilon_0\lambda_2\omega_e||M_a^{-\frac{1}{2}}||_\infty\left[||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^m||^2 + \sum_{n=1}^{m-1}||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^n||^2\right.
\end{aligned}
$$

$$+\frac{1}{2}||\delta_\tau \boldsymbol{D}_h^{m+\frac{1}{2}}||^2 + \sum_{n=0}^{m-1}||\delta_\tau \boldsymbol{D}_h^{n+\frac{1}{2}}||^2\Bigg]. \tag{4.23}$$

By the same arguments, we have

$$\begin{aligned}&2\tau\epsilon_0\lambda_2\sum_{n=1}^{m}(\omega_e^2 M_a^{-1}\boldsymbol{E}_h^n, \delta_{2\tau}\boldsymbol{D}_h^n)\\ &\leq \tau\epsilon_0\lambda_2\omega_e||M_a^{-\frac{1}{2}}||_\infty\\ &\left[||\omega_e M_a^{-\frac{1}{2}}\boldsymbol{E}_h^m||^2 + \sum_{n=1}^{m-1}||\omega_e M_a^{-\frac{1}{2}}\boldsymbol{E}_h^n||^2 + \frac{1}{2}||\delta_\tau \boldsymbol{D}_h^{m+\frac{1}{2}}||^2 + \sum_{n=0}^{m-1}||\delta_\tau \boldsymbol{D}_h^{n+\frac{1}{2}}||^2\right],\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}&\tau\sum_{n=1}^{m}\omega_e^{-2}\left(M_b\delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}}, \delta_\tau^2(\boldsymbol{E}_h^n+\boldsymbol{E}_h^{n-1})\right)\\ &\leq \tau||\omega_e^{-1}M_b M_a^{\frac{1}{2}}||_\infty\\ &\left[\sum_{n=1}^{m}||\delta_\tau\boldsymbol{D}_h^{n-\frac{1}{2}}||^2 + \frac{1}{2}||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^m||^2 + \sum_{n=0}^{m-1}||\omega_e^{-1}M_a^{-\frac{1}{2}}\delta_\tau^2\boldsymbol{E}_h^n||^2\right].\end{aligned} \tag{4.25}$$

Substituting the above estimates of (4.22)–(4.25) into (4.21), then choosing τ small enough so that the right hand side terms can be controlled by the corresponding left hand terms of (4.21), e.g.,

$$\begin{aligned}&\tau^2 \leq \frac{1}{2},\ \omega_e^2\tau^2 \leq \frac{1}{2},\ \tau\epsilon_0\lambda_2\omega_e||M_a^{-\frac{1}{2}}||_\infty \leq \frac{1}{4},\ \tau\omega_e^{-1}||M_b^{\frac{1}{2}}M_a^{\frac{1}{2}}||_\infty \leq \frac{\epsilon_0\lambda_2}{4},\\ &\tau\omega_e||M_a^{-\frac{1}{2}}||_\infty \leq \frac{1}{2},\ \tau\epsilon_0\lambda_2\omega_e||M_a^{-\frac{1}{2}}||_\infty + \frac{1}{2}\tau\omega_e^{-1}||M_b M_a^{\frac{1}{2}}||_\infty \leq \frac{\epsilon_0\lambda_2}{8},\\ &\tau||M_b^{\frac{1}{2}}M_a^{\frac{1}{2}}||_\infty \leq \frac{1}{2},\ \ \frac{1}{2}\tau^2\mu_0^{-1}C_{inv}^2 h^{-2}||M_a^{\frac{1}{2}}||_\infty^2 \leq \frac{\epsilon_0\lambda_2}{8},\end{aligned} \tag{4.26}$$

which are equivalent to (4.13), and using the discrete Gronwall inequality (e.g., [20, Lemma 3.1]), we complete the proof. □

Similar error estimate can be established for scheme (4.3a) and (4.3b). Due to page limit and more technicality, we skip it.

5 Numerical Results

In this section, we present some numerical tests to justify our analysis and demonstrate that the proposed rotation cloak model can rotate the electromagnetic fields at a specified angle, while the cloak itself is invisible as it causes little scattering.

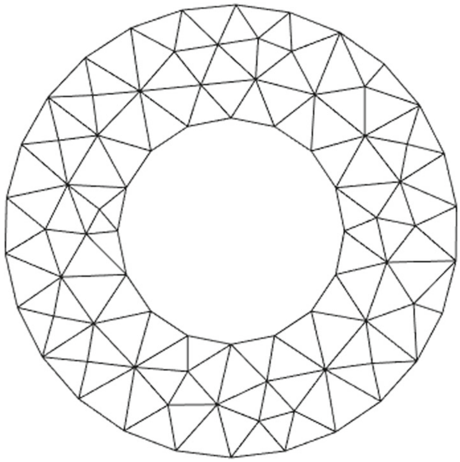

Fig. 2 The coarse grid

5.1 Example 1

This example is developed to test the convergence rate for both schemes (3.2a) and (3.2b), (4.2a) and (4.2b) on an annulus domain with inner radius $R_1 = 0.2$ and outer radius $R_2 = 0.4$.

To construct an analytic solution for testing the convergence rate, we have to add source terms to the original model (2.13a) and (2.13b), i.e., we solve the following problem

$$\partial_{tt} \boldsymbol{D} = -\mu_0^{-1} \nabla \times \nabla \times \boldsymbol{E} + \boldsymbol{g}, \tag{5.1a}$$

$$\epsilon_0 \lambda_2 (M_a^{-1} \partial_{tt} \boldsymbol{E} + \omega_e^2 M_a^{-1} \boldsymbol{E}) = \partial_{tt} \boldsymbol{D} + M_b \boldsymbol{D} + \boldsymbol{f}, \tag{5.1b}$$

where the source functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are calculated by the exact solution

$$\boldsymbol{E} = \begin{pmatrix} e^{-t}(r-0.2)(r-0.4) \\ e^{-t}(r-0.2)(r-0.4) \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} 2ye^{-t} \\ 2xe^{-t} \end{pmatrix}, \quad r = \sqrt{x^2+y^2}.$$

For simplicity, we choose the parameter $\epsilon_0 = 1, \mu_0 = 1, \omega_e = 1, \theta = \frac{\pi}{4}$, and the rest of parameters of modeling equations are calculated by the expressions given in Sec. 2. For our simulation, we fixed the time step $\tau = 2 \times 10^{-5}\, s$, final time $T = 1\, s$, and used a series of continuous refined meshes to test the convergence rate. A sample coarse grid is demonstrated in Fig. 2. Tables 1 and 2 show The obtained convergence rates and computational times (in seconds) by schemes (3.2a) and (3.2b), (4.2a) and (4.2b) are presented in Table 1 and Table 2, respectively. Our results show clearly that both schemes achieved almost the same accuracy, but scheme (4.2a) and (4.2b) is much faster than scheme (3.2a) and (3.2b). The reason is that solving for $\boldsymbol{E}_h^{n+1}$ via (3.4) involves computing and assembling an extra matrix $(\nabla \boldsymbol{\phi}_j, \nabla \boldsymbol{\phi}_i)$ for any basis function $\boldsymbol{\phi}_j$ of $\boldsymbol{U}_h^0$.

5.2 Example 2

This example is used to demonstrate the proposed new modeling equations can rotate the electromagnetic fields at a specified angle, while the cloak itself is invisible as it causes little scattering.

In our simulations, we fix the computational domain $\Omega = [-1, 1]\,m \times [-1, 1]\,m$ discretized by an unstructured triangular mesh with a mesh size $h = 0.01$. We introduce a plane

Table 1 The convergence rate and computational time obtained by scheme (3.2a) and (3.2b)

h_{max}	h_{min}	$\|E - E_h\|$	Rate	$\|D - D_h\|$	Rate	Time (s)
0.12178	0.04961	1.5292887984E-03	–	1.7481064194E-02	–	46.6
0.06131	0.01602	5.1931514285E-04	1.5582	8.1750010134E-03	1.0965	230.3
0.03128	0.01139	2.3532747483E-04	1.1419	3.9835480738E-03	1.0372	571.1
0.01602	0.00512	1.1831112182E-04	0.9921	2.0231599016E-03	0.9774	1743.9
0.00814	0.00219	5.9087403575E-05	1.0017	1.0013051201E-03	1.0147	6610.2

Table 2 The convergence rate and computational time obtained by scheme (4.2a) and (4.2b)

h_{max}	h_{min}	$\|E - E_h\|$	Rate	$\|D - D_h\|$	Rate	Time (s)
0.12178	0.04961	1.5292901919E-03	–	1.7481063283E-02	–	33.7
0.06131	0.01602	5.1931677762E-04	1.5582	8.1750009448E-03	1.0965	86.3
0.03128	0.01139	2.3532772796E-04	1.1419	3.9835482623E-03	1.0372	182.7
0.01602	0.00512	1.1831087464E-04	0.9921	2.0231599967E-03	0.9774	639.2
0.00814	0.00219	5.9087421391E-05	1.0017	1.0013051029E-03	1.0147	4645.3

wave source through a right hand side function S imposed in the Maxwell's equations in the free space region surrounding the cloaking region:

$$\mu_0 \partial_{tt} \boldsymbol{D} = -\nabla \times (\nabla \times \boldsymbol{E}) - \nabla \times S, \quad \text{with } \boldsymbol{D} = \epsilon_0 \boldsymbol{E}, \tag{5.2}$$

where the source function

$$S = \begin{cases} 200\pi \cos(\omega t), & (x, y) \in [-0.8, -0.79] \times [-0.98, 0.98], \\ 0, & \text{elsewhere}, \end{cases} \tag{5.3}$$

and $\omega = 2\pi f$ with an operating frequency $f = 1.0\ GH_z$. To avoid the complicated perfectly matched layer (PML) [2, 4] used in our previous work [32], now we surround the computational domain by an absorption boundary condition given as:

$$\boldsymbol{n} \times (\frac{1}{\mu_0} \nabla \times \boldsymbol{E}) = -\sqrt{\frac{\epsilon_0}{\mu_0}} \partial_t (\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{E})). \tag{5.4}$$

To couple with the rotator model (3.2a) and (3.2b), we implement the free space model (5.2) with the absorption boundary condition (5.4) as follows:

$$\epsilon_0 \mu_0 (\delta_\tau^2 \boldsymbol{E}_h^n, \boldsymbol{\psi}_h) + (\nabla \times \boldsymbol{E}_h^n, \nabla \times \boldsymbol{\psi}_h) - \sqrt{\epsilon_0 \mu_0} < \boldsymbol{n} \times (\boldsymbol{n} \times \delta_{2\tau} \boldsymbol{E}_h^n), \boldsymbol{\psi}_h >= -(\nabla \times S^n, \boldsymbol{\psi}_h), \tag{5.5}$$

where we denote the boundary integral $< \boldsymbol{u}, \boldsymbol{v} > := \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{v}$.

To show that this scheme is conditionally stable, by choosing $\boldsymbol{\psi}_h = \tau \delta_{2\tau} \boldsymbol{E}_h^n$ in (5.5), and using identities (3.14) and (3.15) with $u^n = \boldsymbol{E}_h^n$, we have

$$\frac{\epsilon_0 \mu_0}{2} (||\delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}||^2 - ||\delta_\tau \boldsymbol{E}_h^{n-\frac{1}{2}}||^2) + \sqrt{\epsilon_0 \mu_0} < \boldsymbol{n} \times \delta_{2\tau} \boldsymbol{E}_h^n, \boldsymbol{n} \times \delta_{2\tau} \boldsymbol{E}_h^n >$$
$$+ \frac{1}{4} \left[(||\nabla \times \boldsymbol{E}_h^{n+1}||^2 - ||\nabla \times \boldsymbol{E}_h^{n-1}||^2) - \tau^2 (||\nabla \times \delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}||^2 - ||\nabla \times \delta_\tau \boldsymbol{E}_h^{n-\frac{1}{2}}||^2) \right]$$

$$
\begin{aligned}
&= -\tau \left(\nabla \times S^n, \frac{1}{2}(\delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}} + \delta_\tau \boldsymbol{E}_h^{n-\frac{1}{2}}) \right) \\
&\leq \frac{\tau}{2}||\nabla \times S^n||^2 + \frac{\tau}{4}(||\delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}||^2 + ||\delta_\tau \boldsymbol{E}_h^{n-\frac{1}{2}}||^2),
\end{aligned} \tag{5.6}
$$

where in the last step we used the simple inequality $(a, b) \leq \frac{1}{2}(||a||^2 + ||b||^2)$.

Summing up (5.6) from $n = 1$ to any $m \leq N_t - 2$, and dropping the non-negative boundary integral term on the left side of (5.6), we obtain

$$
\begin{aligned}
&\frac{\epsilon_0\mu_0}{2}(||\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 - ||\delta_\tau \boldsymbol{E}_h^{\frac{1}{2}}||^2) \\
&\quad + \frac{1}{4}\left[(||\nabla \times \boldsymbol{E}_h^{m+1}||^2 + ||\nabla \times \boldsymbol{E}_h^{m}||^2) - (||\nabla \times \boldsymbol{E}_h^{0}||^2 + ||\nabla \times \boldsymbol{E}_h^{1}||^2)\right] \\
&\quad \leq \frac{\tau^2}{4}(||\nabla \times \delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 - ||\nabla \times \delta_\tau \boldsymbol{E}_h^{\frac{1}{2}}||^2) + \frac{\tau}{2}\sum_{n=1}^{m}||\nabla \times S^n||^2 + \frac{\tau}{4}||\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 \\
&\quad\quad + \frac{\tau}{2}\sum_{n=0}^{m-1}||\delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}||^2.
\end{aligned} \tag{5.7}
$$

Using the inverse estimate (4.5), we have

$$
\frac{\tau^2}{4}||\nabla \times \delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 \leq \frac{\tau^2}{4} \cdot C_{inv}^2 h^{-2}||\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2. \tag{5.8}
$$

Substituting (5.8) into (5.7), and under the following time step constraint

$$
\frac{\tau^2 C_{inv}^2 h^{-2}}{4} \leq \frac{\epsilon_0\mu_0}{8} \ \ (\text{or } \tau \leq \frac{\sqrt{\epsilon_0\mu_0}h}{\sqrt{2}C_{inv}}) \ \ \text{and } \frac{\tau}{4} \leq \frac{\epsilon_0\mu_0}{8} \ \ (\text{or } \tau \leq \frac{\sqrt{\epsilon_0\mu_0}}{\sqrt{2}}), \tag{5.9}
$$

we have

$$
\begin{aligned}
&\frac{\epsilon_0\mu_0}{4}||\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 + \frac{1}{4}||\nabla \times \boldsymbol{E}_h^{m+1}||^2 \\
&\quad \leq \frac{\epsilon_0\mu_0}{2}||\delta_\tau \boldsymbol{E}_h^{\frac{1}{2}}||^2 + \frac{1}{4}(||\nabla \times \boldsymbol{E}_h^{0}||^2 + ||\nabla \times \boldsymbol{E}_h^{1}||^2) + \\
&\quad\quad \frac{\tau}{2}\sum_{n=1}^{m}||\nabla \times S^n||^2 + \frac{\tau}{2}\sum_{n=0}^{m-1}||\delta_\tau \boldsymbol{E}_h^{n+\frac{1}{2}}||^2,
\end{aligned} \tag{5.10}
$$

which, by the discrete Gronwall inequality, leads to the following stability

$$
\begin{aligned}
&\epsilon_0\mu_0||\delta_\tau \boldsymbol{E}_h^{m+\frac{1}{2}}||^2 + ||\nabla \times \boldsymbol{E}_h^{m+1}||^2 \\
&\quad \leq C\left[2\epsilon_0\mu_0||\delta_\tau \boldsymbol{E}_h^{\frac{1}{2}}||^2 + ||\nabla \times \boldsymbol{E}_h^{0}||^2 + ||\nabla \times \boldsymbol{E}_h^{1}||^2 + 2\tau\sum_{n=1}^{m}||\nabla \times S^n||^2\right].
\end{aligned} \tag{5.11}
$$

Here we consider a cylindrical electromagnetic rotator with $R_1 = 0.2\,\text{m}, R_2 = 0.4\,\text{m}$. Our computational mesh totally contains 206389 edges and 137326 triangular elements. In our simulation, we choose the time step size $\tau = 2.5 \times 10^{-13}\,s$ and the total number of time steps $N = 70000$. We test two different rotating angles θ_0.

Case 1. $\theta_0 = \frac{\pi}{4}$.

In Figs. 3 and 4, we plot the snapshots of the electric field components E_x and E_y, respectively. Both figures show clearly how the wave gets distorted in the metamaterial region. From

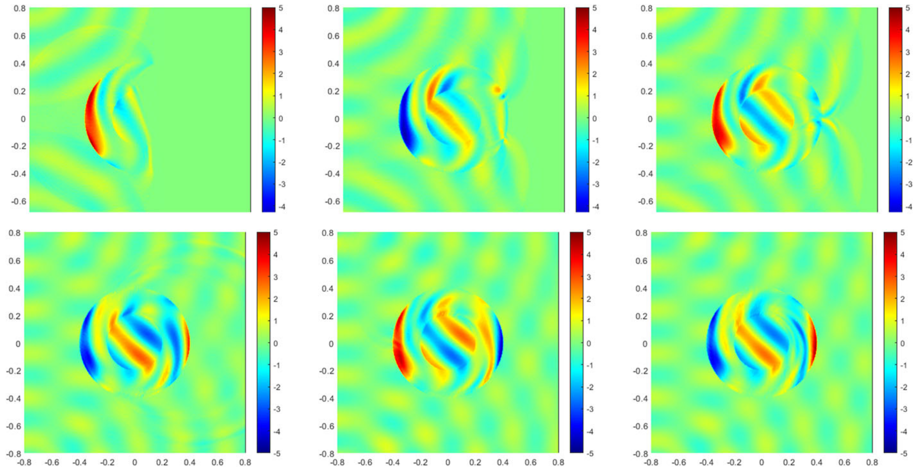

Fig. 3 Snapshots of electric fields E_x at various time steps: (top left) 12000 steps; (top middle) 18000 steps; (top right) 20000 steps; (bottom left) 30000 steps; (bottom middle) 40000 steps; (bottom right) 70000 steps

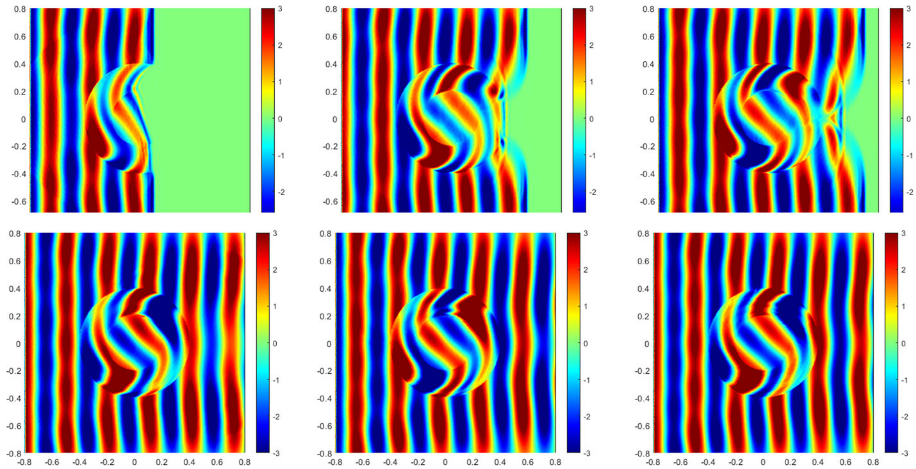

Fig. 4 Snapshots of electric fields E_y at various time steps: (top left) 12000 steps; (top middle) 18000 steps; (top right) 20000 steps; (bottom left) 30000 steps; (bottom middle) 40000 steps; (bottom right) 70000 steps

those pictures, we can see that the structure has very small scattering and obvious rotational effects.

Case 2. $\theta_0 = \frac{\pi}{2}$.

In this example, we take the same physical parameters as Case 1, except that the rotation angle is changed to $\theta_0 = \frac{\pi}{2}$. Some snapshots of of the electric field components $\boldsymbol{E} = (E_x$ and $E_y)$ are presented in Figs. 5 and 6, respectively. Both figures demonstrate that this design indeed rotates the wave $\frac{\pi}{2}$ clockwisely inside the inner region, and in the same time has the invisibility cloaking capability outside the metamaterial region. Figures 5 and 6 are similar to what we obtained by a different algorithm in our previous work [32, Fig.5-6].

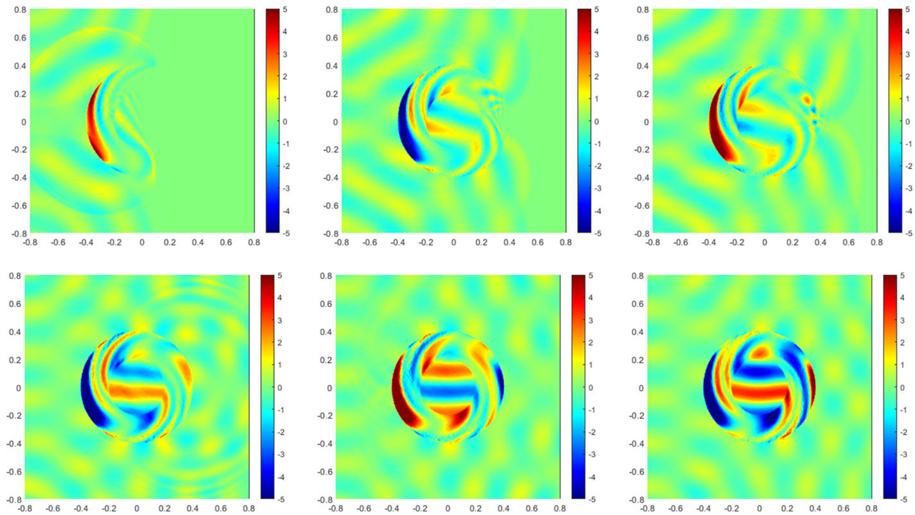

Fig. 5 Snapshots of electric fields E_x at various time steps: (top left) 12000 steps; (top middle) 18000 steps; (top right) 20000 steps; (bottom left) 30000 steps; (bottom middle) 40000 steps; (bottom right) 70000 steps

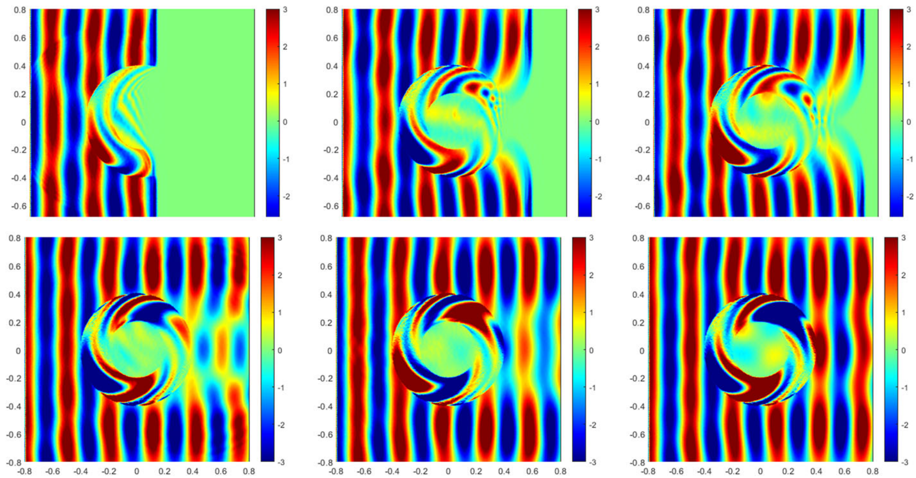

Fig. 6 Snapshots of electric fields E_y at various time steps: (top left) 12000 steps; (top middle) 18000 steps; (top right) 20000 steps; (bottom left) 30000 steps; (bottom middle) 40000 steps; (bottom right) 70000 steps

6 Conclusion

In this paper, we first reformulate a rotation cloak model originally derived in our previous work [32]. The new system of govening equations has one less unknown variable than the old model given in [32]. Existence and uniqueness of the modeling equations are established for the first time. Some novel finite element schemes are proposed and analyzed. Numerical results presented show that the new model can both rotate the wave inside the inner region, and have the invisibility cloaking capability outside the metamaterial region.

Appendix

In this appendix, we present the following lemma, which gives various time difference approximations.

Lemma 3 *The following inequalities hold:*

$$\text{(I)} \quad \left|\left|\frac{u^{n+1}+u^{n-1}}{2} - \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} u(t)\,dt\right|\right|^2 \leq \frac{\tau^3}{15}\int_{t_{n-1}}^{t_{n+1}} \left|\left|\partial_t^2 u(t)\right|\right|^2 dt,\ \forall\, u \in H^2(0,T;L^2(\Omega)),$$

$$\text{(II)} \quad \left|\left|u^n - \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} u(t)\,dt\right|\right|^2 \leq \frac{\tau^3}{8}\int_{t_{n-1}}^{t_{n+1}} ||\partial_t^2 u(t)||^2 dt,\ \forall\, u \in H^2(0,T;L^2(\Omega)),$$

$$\text{(III)} \quad \left|\left|\delta_\tau u^{n+\frac{1}{2}}\right|\right|^2 := \left|\left|\frac{u^{n+1}-u^n}{\tau}\right|\right|^2 \leq \frac{1}{\tau}\int_{t_n}^{t_{n+1}} ||\partial_t u(t)||^2 dt,\ \forall\, u \in H^1(0,T;L^2(\Omega)),$$

$$\text{(IV)} \quad \left|\left|\delta_\tau^2 u^n\right|\right|^2 := \left|\left|\frac{u^{n+1}-2u^n+u^{n-1}}{\tau^2}\right|\right|^2 \leq \frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}} ||\partial_t^2 u(t)||^2 dt,\ \forall\, u \in H^2(0,T;L^2(\Omega)),$$

$$\text{(V)} \quad \left|\left|\delta_\tau^2 u^n - \partial_t^2 u^n\right|\right|^2 \leq \frac{\tau^3}{18}\int_{t_{n-1}}^{t_{n+1}} ||\partial_t^4 u(t)||^2 dt,\ \forall\, u \in H^4(0,T;L^2(\Omega)),$$

$$\text{(VI)} \quad \left|\left|\delta_{2\tau} u^n - \partial_t u^n\right|\right|^2 := \left|\left|\frac{u^{n+1}-u^{n-1}}{2\tau} - \partial_t u^n\right|\right|^2 \leq \frac{\tau^3}{8}\int_{t_{n-1}}^{t_{n+1}} ||\partial_t^3 u(t)||^2 dt,$$

$$\forall\, u \in H^3(0,T;L^2(\Omega)),$$

$$\text{(VII)} \quad \left|\left|\delta_{2\tau}\partial_t u^n - \delta_\tau^2 u^n\right|\right|^2 \leq \frac{13\tau^3}{36}\int_{t_{n-1}}^{t_{n+1}} ||\partial_t^4 u(t)||^2 dt,\ \forall\, u \in H^4(0,T;L^2(\Omega)).$$

Proof The proof of (III) can be found in [22, Lemma 3.19]. Below we give a brief proofs of the rest.

(I) Taking the square of the following integral identity

$$\frac{u^{n+1}+u^{n-1}}{2} - \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} u(t)\,dt \quad = \frac{1}{4\tau}\int_{t_{n-1}}^{t_{n+1}} (t-t_{n-1})(t_{n+1}-t)\partial_t^2 u(t)\,dt$$

and using the inequality

$$\left|\int_{t_{n-1}}^{t_{n+1}} a(t)b(t)\,dt\right|^2 \leq \left(\int_{t_{n-1}}^{t_{n+1}} |a(t)|^2\,dt\right)\left(\int_{t_{n-1}}^{t_{n+1}} |b(t)|^2\,dt\right), \tag{6.1}$$

we have

$$\begin{aligned}
&\left|\frac{u^{n+1}+u^{n-1}}{2} - \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} u(t)\,dt\right|^2 \\
&\leq \frac{1}{16\tau^2}\left(\int_{t_{n-1}}^{t_{n+1}} (t-t_{n-1})^2(t_{n+1}-t)^2\,dt\right)\left(\int_{t_{n-1}}^{t_{n+1}} |\partial_t^2 u(t)|^2\,dt\right) \\
&= \frac{\tau^3}{15}\int_{t_{n-1}}^{t_{n+1}} |\partial_t^2 u(t)|^2\,dt.
\end{aligned}$$

(II) Taking the square of the following integral identity

$$u^n - \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} u(t)\,dt = \frac{1}{2}\Bigg[\int_{t_n}^{t_{n+1}} \left(\frac{t-t_{n-1}}{2\tau} - 1\right)(t_{n+1}-t)\partial_t^2 u(t)\,dt + \int_{t_{n-1}}^{t_n} (\frac{t-t_{n+1}}{2\tau} + 1)(t_{n-1}-t)\partial_t^2 u(t)\,dt\Bigg]$$

and using the inequality (6.1), we have

$$\left|u^n - \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}} u(t)\,dt\right|^2 \le \frac{1}{2}\Bigg[\left(\int_{t_n}^{t_{n+1}} \left(\frac{t-t_{n-1}}{2\tau} - 1\right)^2 (t_{n+1}-t)^2\,dt\right)\left(\int_{t_n}^{t_{n+1}} |\partial_t^2 u(t)|^2\,dt\right) + \left(\int_{t_{n-1}}^{t_n} \left(\frac{t-t_{n+1}}{2\tau} + 1\right)^2 (t_{n-1}-t)^2\,dt\right)\left(\int_{t_{n-1}}^{t_n} |\partial_t^2 u(t)|^2\,dt\right)\Bigg] \le \frac{\tau^3}{8}\int_{t_{n-1}}^{t_{n+1}} |\partial_t^2 u(t)|^2\,dt.$$

(III) Similarly, we have

$$\left|\frac{u^{n+1}-2u^n+u^{n-1}}{\tau^2}\right|^2 = \left|\frac{1}{\tau^2}\left[\int_{t_n}^{t_{n+1}} (t_{n+1}-t)\partial_t^2 u(t)\,dt + \int_{t_{n-1}}^{t_n} (t-t_{n-1})\partial_t^2 u(t)\,dt\right]\right|^2$$
$$\le \frac{2}{\tau^4}\Bigg[\left(\int_{t_n}^{t_{n+1}} (t_{n+1}-t)^2\,dt\right)\left(\int_{t_n}^{t_{n+1}} |\partial_t^2 u(t)|^2\,dt\right) + \left(\int_{t_{n-1}}^{t_n} (t-t_{n-1})^2\,dt\right)\left(\int_{t_{n-1}}^{t_n} |\partial_t^2 u(t)|^2\,dt\right)\Bigg] \le \frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}} |\partial_t^2 u(t)|^2 dt.$$

(IV) Using the same techniques as above, we have

$$|\delta_\tau^2 u^n - \partial_t^2 u^n|^2 = \left|\frac{1}{\tau^2}\left[\int_{t_n}^{t_{n+1}} \frac{(t_{n+1}-t)^3}{3!}\partial_t^4 u(t)\,dt + \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^3}{3!}\partial_t^4 u(t)\,dt|^2\right]\right|^2$$
$$\le \frac{2}{\tau^4}\Bigg[\left(\int_{t_n}^{t_{n+1}} \frac{(t_{n+1}-t)^6}{36}\,dt\right)\left(\int_{t_n}^{t_{n+1}} |\partial_t^4 u(t)|^2\,dt\right) + \left(\int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^6}{36}\,dt\right)\left(\int_{t_{n-1}}^{t_n} |\partial_t^4 u(t)|^2\,dt\right)\Bigg] \le \frac{\tau^3}{18}\int_{t_{n-1}}^{t_{n+1}} |\partial_t^4 u(t)|^2 dt.$$

(V) Similarly, we easily have

$$\left|\frac{u^{n+1}-u^{n-1}}{2\tau} - \partial_t u^n\right|^2 = \left|\frac{1}{2\tau}\left[\int_{t_n}^{t_{n+1}} \frac{(t_{n+1}-t)^2}{2!}\partial_t^3 u(t)\,dt + \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^2}{2!}\partial_t^3 u(t)\,dt|^2\right]\right|^2$$
$$\le \frac{1}{2\tau^2}\Bigg[\left(\int_{t_n}^{t_{n+1}} \frac{(t_{n+1}-t)^4}{4}\,dt\right)\left(\int_{t_n}^{t_{n+1}} |\partial_t^3 u(t)|^2\,dt\right) + \left(\int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^4}{4}\,dt\right)\left(\int_{t_{n-1}}^{t_n} |\partial_t^3 u(t)|^2\,dt\right)\Bigg] \le \frac{\tau^3}{8}\int_{t_{n-1}}^{t_{n+1}} |\partial_t^3 u(t)|^2 dt.$$

(VI) Using the triangle inequality, and (V) and (VI), we obtain

$$|\delta_{2\tau}\partial_t u^n - \delta_\tau^2 u^n|^2 \le 2(|(\delta_{2\tau} - \partial_t)\partial_t u^n|^2 + |\partial_t^2 u^n - \delta_\tau^2 u^n|^2)$$
$$\le 2\left(\frac{\tau^3}{8}\int_{t_{n-1}}^{t_{n+1}} |\partial_t^4 u(t)|^2 dt + \frac{\tau^3}{18}\int_{t_{n-1}}^{t_{n+1}} |\partial_t^4 u(t)|^2 dt\right) = \frac{13\tau^3}{36}\int_{t_{n-1}}^{t_{n+1}} |\partial_t^4 u(t)|^2 dt.$$

□

Acknowledgements The authors are very grateful to two anonymous reviewers for their insightful comments on improving our paper.

Author Contributions Yunqing Huang Funding acquisition, Supervision. Jichun Li Supervision, Methodology, Validation, Writing— review & editing. Bin He Investigation, Software, Writing for numerical tests.

Funding This work was partially supported by National Natural Science Foundation of China (No. 11971410), National Key R&D Program of China (No. 2020YFA0713500), NSF Grant No. DMS-2011943, and NSFC Project 12301511.

Data Availability Data sets generated during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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