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On the isoperimetric profile of the hypercube

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Abstract
We prove that a subset of the hypercube $(0,1)^d$ with volume sufficiently close to $\frac{1}{2}$ has (relative) perimeter greater than or equal to 1, recovering a result by Acerbi, Fusco, and Morini. We also prove that, in contrast with what happens for the high-dimensional sphere $\mathbb{S}^d$, the isoperimetric profile of the hypercube $(0,1)^d$ does not converge to the Gaussian isoperimetric profile as $d \to \infty$.

1 Introduction

For an open set $\Omega \subseteq \mathbb{R}^d$, the relative isoperimetric problem in Ω consists of minimizing the perimeter (in Ω) of a set $E \subseteq \Omega$ with fixed volume. So, given $0 < \lambda < |\Omega|$, one is interested in the minimization problem

$$I_\Omega(\lambda) := \inf \left\{ \mathrm{Per}(E, \Omega) : \ E \subseteq \Omega \text{ so that } |E| = \lambda \right\}, \tag{1.1}$$

where $\mathrm{Per}(E, \Omega)$ denotes the perimeter of E inside Ω; if E has a smooth boundary then $\mathrm{Per}(E, \Omega)$ coincides with $\mathscr{H}^{d-1}(\Omega \cap \partial E)$ (see Sect. 2.2 for the general definition). The (relative) isoperimetric profile of Ω, denoted by $I_\Omega : [0, |\Omega|] \to [0, \infty)$ is the function so that $I_\Omega(\lambda)$ is the value of the infimum appearing in Eq. (1.1) (and $I_\Omega(0) = 0$ and $I_\Omega(|\Omega|) = 0$ if $|\Omega| < \infty$).

The (relative) isoperimetric problem is a classical question with a multitude of applications that has received considerable attention recently. The vast literature on the topic makes it hard to give a complete list of references, so we refer the reader to the three recent works [4, 17, 19] and to the references therein. This paper investigates in particular the isoperimetric profile of the d-dimensional hypercube $(0,1)^d$.

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1.1 The relative isoperimetric problem in $(0, 1)^d$

To frame appropriately our results, let us recall what is known about the relative isoperimetric problem in the hypercube.

Via a symmetrization argument [26, Sect. 1.5], one can show that the isoperimetric profile of the hypercube $(0, 1)^d$ coincides with the isoperimetric profile of the d-dimensional torus $\mathbb{T}^d$.

In the case of the square $(0, 1)^2$, the isoperimetric profile (along with the minimizers of the relative isoperimetric problem) is known [11]. In dimension $d = 3$ (so, for the cube $(0, 1)^3$), it is conjectured [26, pg. 11] (see also [24, Theorem 9]) that the only minimizers are balls, cylinders, and half-spaces intersected with the cube; under this assumption one can determine exactly the isoperimetric profile of $(0, 1)^3$. In any dimension, since $(0, 1)^d$ is a polytope, for small volumes the minimizers of the relative isoperimetric problem are balls centered at the vertices of $(0, 1)^d$ (see [25, Theorem 6.8] or [22, Remark 3.11]). As an immediate consequence, one gets

$$I_{(0,1)^d}(\lambda) = \frac{1}{2}d \left|B_1^{\mathbb{R}^d}\right|^{\frac{1}{d}} \lambda^{\frac{d-1}{d}} \quad \text{for } 0 < \lambda < \lambda_0(d),$$

where $B_1^{\mathbb{R}^d}$ denotes the unit ball in $\mathbb{R}^d$ and $\lambda_0(d)$ is a dimensional constant that goes to 0 as $d \to \infty$.

In every dimension $d \geq 1$, it was proven by Hadwiger [18] that $I_{(0,1)^d}(\frac{1}{2}) = 1$, or equivalently that if $E \subseteq (0, 1)^d$ has measure $|E| = \frac{1}{2}$ then its perimeter is at least 1 (i.e., splitting the cube with a hyperplane parallel to one of its faces is optimal).

We show that the same result holds also if the set E has measure sufficiently close to $\frac{1}{2}$. This statement was part of an open problem stated by Brezis and Bruckstein [12, Open Problem 10.1] (see also [11, Remark 2]). The result is not novel (see [1, Theorem 5.3, Remark 5.4]) but the proof we provide is simpler than the existing one.

Theorem 1.1 *For $d \geq 1$, there exists $\varepsilon_d > 0$ so that $I_{(0,1)^d}(\lambda) = 1$ for all $\lambda \in [\frac{1}{2} - \varepsilon_d, \frac{1}{2} + \varepsilon_d]$. Equivalently, any set of finite perimeter $E \subseteq (0, 1)^d$ with $||E| - \frac{1}{2}| \leq \varepsilon_d$ satisfies* $\operatorname{Per}(E, (0, 1)^d) \geq 1$. *Moreover,* $\operatorname{Per}(E, (0, 1)^d) = 1$ *if and only if (up to a negligible set) $E = \{x \in (0, 1)^d : \hat{v} \cdot x \leq |E|\}$ or $E = \{x \in (0, 1)^d : \hat{v} \cdot x \geq 1 - |E|\}$ for some $\hat{v} \in \{e_1, e_2, \ldots, e_d\}$.*

For values of the volume distinct from $\frac{1}{2}$, the exact value of $I_{(0,1)^d}$ is not known, but a remarkable lower bound with the Gaussian isoperimetric profile was established in [8, Theorem 7] (see also [26, Theorem 7], [3, (2.2)]).

Theorem ([8, Theorem 7]) *Let $I_\gamma := \varphi \circ \Phi^{-1}$ be the Gaussian isoperimetric profile (see Sect. 2.2), where $\varphi(t) := \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}t^2)$ and $\Phi(t) := \int_\infty^t \varphi(s)\, ds$.*

For any $d \geq 1$, it holds that $I_{(0,1)^d} \geq \sqrt{2\pi} I_\gamma$; equivalently

$$\operatorname{Per}(E, (0, 1)^d) \geq \sqrt{2\pi} I_\gamma(|E|) \tag{1.2}$$

for any set $E \subseteq (0, 1)^d$ of finite perimeter.

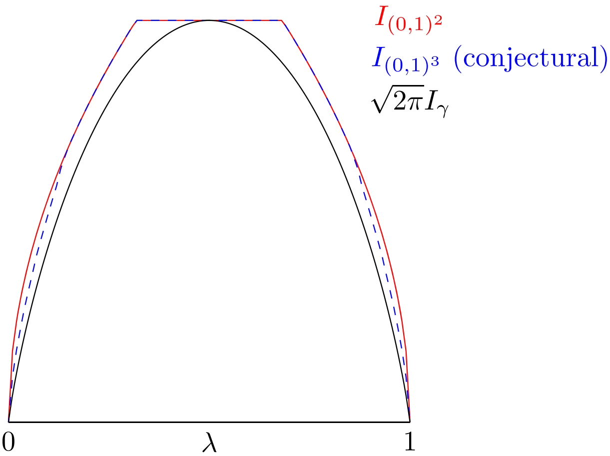

Fig. 1 The blue dashed graph represents the conjectural profile of the cube $(0, 1)^3$; the optimal shapes depend on the volume: balls centered at a vertex, cylinders centered at a side, half-spaces parallel to a face. The figure shows a number of features of the problem: the isoperimetric profile is concave, the lower bound $\sqrt{2\pi}I_\gamma$ is remarkably close to the actual value of $I_{(0,1)^d}$, the profile $I_{(0,1)^d}$ is constant in a neighborhood of $\lambda = \frac{1}{2}$, the profiles $I_{(0,1)^d}$ are decreasing with respect to the dimension $d \geq 1$ (color figure online)

The lower bound shown in this theorem is remarkably precise already in dimensions $d = 2, 3$ (see Fig. 1) and its precision can only improve in higher dimension as $I_{(0,1)^d}$ is decreasing with respect to the dimension d. Furthermore, if instead of the cube $(0, 1)^d$, one considers the case of the sphere $\mathbb{S}^d$ (i.e., one studies the isoperimetric problem in the Riemannian manifold $\mathbb{S}^d$), it turns out that its isoperimetric profile $I_{\mathbb{S}^d}$, appropriately rescaled, converges to I_γ as the dimension $d \to \infty$ (see [7, Theorem 10, Proposition 11] or [26, Theorem 21]).

The facts mentioned in the previous paragraph may lead one to expect that, as the dimension $d \to \infty$, the isoperimetric profile of the cube $I_{(0,1)^d}$ converges to $\sqrt{2\pi}I_\gamma$. This is true when evaluating it at $\lambda \in \{0, \frac{1}{2}, 1\}$. Unexpectedly for the author, we show that this claim is false, i.e., that there is a gap between $\inf_{d\geq 1} I_{(0,1)^d}$ and $\sqrt{2\pi}I_\gamma$.

Theorem 1.2 *For all $d \geq 1$, we have $I_{(0,1)^{d+1}} \leq I_{(0,1)^d}$; let $I_{(0,1)^\infty} = \inf_{d\geq 1} I_{(0,1)^d}$. The function $I_{(0,1)^\infty} : [0, 1] \to [0, 1]$ is a concave function such that*

$$I_{(0,1)^\infty}(\lambda) > \sqrt{2\pi}I_\gamma(\lambda) \quad \textit{forall } \lambda \in (0, 1) \setminus \{\tfrac{1}{2}\}.$$

The proof of Theorem 1.2 is quantitative (i.e., no compactness is used) and thus one could keep track of all the constants and dependences on λ and find an explicit function $g : [0, 1] \to [0, \infty)$ (strictly positive on $(0, 1) \setminus \{\frac{1}{2}\}$ with $0 = g(0) = g(\frac{1}{2}) = g(1)$) such that

$$I_{(0,1)^d}(\lambda) \geq \sqrt{2\pi}I_\gamma(\lambda) + g(\lambda) \quad \text{for all } 0 \leq \lambda \leq 1.$$

We decided not to do this because it would make the proof more cumbersome and the resulting function g would not be optimal in any sense.

Let us remark that Theorem 1.2 may also be interpreted as a dimension-free stability result for the isoperimetric inequality Eq. (1.2).

1.2 Open questions

The results of this paper naturally raise some further questions that we collect here.

Open question 1.1 Does the statement of Theorem 1.1 hold also with an ε independent of the dimension? Equivalently, is there an $\varepsilon > 0$ so that $I_{(0,1)^d}(\lambda) = 1$ for all $\lambda \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ and for all $d \geq 1$?

Open question 1.2 Is it true that for any $\lambda \in [0, 1]$, the sequence $(I_{(0,1)^d}(\lambda))_{d\geq 1}$ is eventually constant? Equivalently, for each λ, does it hold that $I_{(0,1)^d}(\lambda) = I_{(0,1)^\infty}(\lambda)$ for all d sufficiently large (see Theorem 1.2 for the definition of $I_{(0,1)^\infty}$)?

Open question 1.3 Is there an explicit formula for the limiting isoperimetric profile $I_{(0,1)^\infty}$?

1.3 Methods and organization of the paper

The foundation of the proofs of the two main results of this paper (namely Theorems 1.1 and 1.2) is the rigidity of the inequality Eq. (1.2), i.e., if $\mathrm{Per}(E, (0, 1)^d) = \sqrt{2\pi} I_\gamma(|E|)$ then E is a half-cube. We state and prove this rigidity in Sect. 3.

Then, in Sect. 4 we prove Theorem 1.1. The proof is by compactness.

Finally, in Sect. 5 we show Theorem 1.2. The main idea is to reduce the relative isoperimetric problem in $(0, 1)^d$ to the following penalized isoperimetric problem in the Gaussian space $(\mathbb{R}^d, \gamma_d)$:

$$\inf_{F\subseteq\mathbb{R}^d:\,\gamma_d(F)=\lambda} \mathrm{Per}_{\gamma_d}(F) + \int_{\partial F} \sqrt{\sum_{i=1}^{d}(\nu_{\partial F})_i^2 \exp(x_i^2) - 1}\, d\mathscr{H}^{d-1}_{\gamma_d}(x),$$

where $\nu_{\partial F}$ denotes the unit normal to the boundary of F (see Sect. 2.2 for the definitions of γ_d, $\mathscr{H}^{d-1}_{\gamma_d}$, Per_{γ_d}). Then, by using the dimension-free stability of the Gaussian isoperimetric inequality (see [5, 15, 23]), we prove that the two terms of the penalized problem cannot be simultaneously minimized unless $\lambda \in \{0, \frac{1}{2}, 1\}$. The result follows.

Let us remark that (even though our proof does not employ this perspective) the statement of Theorem 1.1 can be interpreted as the fact that for volumes close to $\frac{1}{2}$ the penalized problem is solved by affine half-spaces. This is not the first instance of penalization of the Gaussian isoperimetric problem that preserves the optimality of half-spaces [6].

2 Notation and preliminaries

2.1 Gaussian measure

Let $\varphi : \mathbb{R} \cup \{\pm\infty\} \to (0, \frac{1}{\sqrt{2\pi}}]$ be the function $\varphi(t) := \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}t^2)$. Denote with $\gamma \in \mathcal{P}(\mathbb{R})$ the Gaussian (probability) measure on $\mathbb{R}$, i.e., the measure with density φ.

Let us define the d-dimensional versions of φ and γ as follows. For any $d \geq 1$, let $\varphi_d : \mathbb{R}^d \to (0, (2\pi)^{-d/2}]$ be

$$\varphi_d(x) := \frac{1}{(2\pi)^{d/2}} \exp\left(-\tfrac{1}{2}|x|^2\right) = \varphi(x_1)\varphi(x_2)\cdots\varphi(x_d).$$

Let $\gamma_d \in \mathcal{P}(\mathbb{R}^d)$ be the d-dimensional Gaussian (probability) measure, i.e., the measure with density φ_d or equivalently $\gamma_d = \gamma \otimes \gamma \otimes \cdots \otimes \gamma$ where we are taking the product of d copies of γ.

2.2 Hausdorff measure and perimeter

We denote with $|\cdot|$ the Lebesgue measure in the Euclidean space (of any dimension). We denote with $\mathscr{H}^k$ the k-dimensional Hausdorff measure in the Euclidean space (of any dimension).

For a set $E \subseteq \mathbb{R}^d$ of finite perimeter (for the theory of sets of finite perimeter we suggest the reader to consult [20]), we denote with $\partial^* E$ its reduced boundary [20, Chapter 15] (which coincides with the topological boundary if E is sufficiently regular). Let us recall that the reduced boundary is a $(d-1)$-rectifiable set and thus admits a normal vector $\mathscr{H}^{d-1}$-almost everywhere. The perimeter of E in an open set Ω is defined as[1]

$$\mathrm{Per}(E, \Omega) := \|D\mathbb{1}_E\|(\Omega) = \mathscr{H}^{d-1}(\partial^* E \cap \Omega).$$

Let us now give the analogous definitions in the Gaussian setting. Let us denote with $\mathscr{H}^k_{\gamma_d} := \varphi_d \mathscr{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^d$ weighted by φ_d. For $E \subseteq \mathbb{R}^d$ a set of *locally* finite perimeter, its Gaussian perimeter is defined as

$$\mathrm{Per}_{\gamma_d}(E) = \mathscr{H}^{d-1}_{\gamma_d}(\partial^* E) = \int_{\partial^* E} \varphi_d \, d\mathscr{H}^{d-1}.$$

2.3 The Gaussian isoperimetric inequality

Let $\Phi : \mathbb{R} \cup \{\pm\infty\} \to [0, 1]$ be the function $\Phi(t) := \int_{-\infty}^t \varphi(s)\, ds = \gamma((\infty, t))$ and let $I_\gamma : [0, 1] \to [0, 1]$ be $I_\gamma = \varphi \circ \Phi^{-1}$. The function I_γ is the isoperimetric profile for the Gaussian space in any dimension, that is, for any positive integer $d \geq 1$ and for

[1] Since E is a set of finite perimeter, its indicator function $\mathbb{1}_E$ is a function of bounded variation and thus its distributional derivative is a measure.

any set $E \subseteq \mathbb{R}^d$ of finite perimeter, we have (see [10, 28], and [9, 13] for the equality cases)

$$\mathrm{Per}_{\gamma_d}(E) \geq I_\gamma(\gamma_d(E))$$

and the equality holds if and only if E is an affine half-space.

3 Rigidity of half-cubes

In this section we study the equality cases of Eq. (1.2). We show that if a set $E \subseteq (0,1)^d$ satisfies $\mathrm{Per}(E,(0,1)^d) = \sqrt{2\pi} I_\gamma(|E|)$ then $|E| \in \{0, \frac{1}{2}, 1\}$ and if $|E| = \frac{1}{2}$ then E is a half-cube. This result is already present in [18], but we will need our proof of the rigidity (which is different from and slightly stronger than the one given by Hadwiger) as a building block of the proof of Theorem 1.2. Therefore we report it here.

For the proof, we will need the following simple lemma. This formula for the Jacobian of the restriction to a hyperplane is likely well known, but we could not find any reference.

Lemma 3.1 *Fix $d \geq 2$. Let $A \in GL(d,\mathbb{R})$ be a linear transformation and let $\nu \in \mathbb{R}^d$ be a unit vector. The Jacobian determinant of the restriction of A to the subspace orthogonal to ν is $|\det(A)| \cdot |(A^\intercal)^{-1}\nu|$.*

Proof Take a Borel set $S \subseteq \nu^\perp$. By Fubini's Theorem, we have

$$|S + \{t\nu : 0 < t < 1\}| = \mathscr{H}^{d-1}(S). \tag{3.3}$$

Moreover,

$$|A(S + \{t\nu : 0 < t < 1\})| = |\det(A)| \cdot |S + \{t\nu : 0 < t < 1\}|. \tag{3.4}$$

Write $A\nu = u + \tilde{u}$, where u is the orthogonal projection of $A\nu$ on the hyperspace $A(\nu^\perp)$. Notice that, for all $x \in \mathbb{R}^d$, $\langle (A^\intercal)^{-1}\nu, Ax\rangle = \langle \nu, x\rangle$, so $(A^\intercal)^{-1}\nu$ is orthogonal to the hyperspace $A(\nu^\perp)$. In particular, $\tilde{u}$ is a multiple of $(A^\intercal)^{-1}\nu$. Hence, we have

$$|\tilde{u}| = \frac{|\langle A\nu, (A^\intercal)^{-1}\nu\rangle|}{|(A^\intercal)^{-1}\nu|} = \frac{1}{|(A^\intercal)^{-1}\nu|}. \tag{3.5}$$

Thanks to Fubini's Theorem, we get

$$|A(S + \{t\nu : 0 < t < 1\})| = |A(S) + \{tu + t\tilde{u} : 0 < t < 1\})| = \mathscr{H}^{d-1}(A(S))|\tilde{u}| \tag{3.6}$$

Combining Eqs. (3.3)–(3.6), we obtain $\mathscr{H}^{d-1}(A(S)) = |\det(A)| \cdot |(A^\intercal)^{-1}\nu|\mathscr{H}^{d-1}(S)$ which is equivalent to the desired statement. □

Proposition 3.2 (Rigidity for the Gaussian isoperimetric inequality in the cube) *For $E \subseteq (0,1)^d$ a set of finite perimeter, it holds that*

$$\mathrm{Per}(E,(0,1)^d) \geq \sqrt{2\pi}\, I_\gamma(|E|).$$

This inequality is an equality if and only if (up to a negligible set) $E = \emptyset$ or $E = (0,1)^d$ or $E = \{x \in (0,1)^d : \hat{v}\cdot x \leq \frac{1}{2}\}$ or $E = \{x \in (0,1)^d : \hat{v}\cdot x \geq \frac{1}{2}\}$ for some $\hat{v} \in \{e_1, e_2, \ldots, e_d\}$.

Proof We follow the proof of [26, Theorem 7].

Let $\Phi_d : \mathbb{R}^d \to (0,1)^d$ be the map (see Sect. 2.3 for the definition of Φ)

$$\Phi_d(x_1, x_2, \ldots, x_d) := (\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_d)).$$

Notice that φ_d is the density of the Gaussian measure on $\mathbb{R}^d$ and that Φ_d is a diffeomorphism such that $(\Phi_d)_*(\gamma_d) = \mathscr{L}^d|_{(0,1)^d}$, in particular the Jacobian of Φ_d satisfies $|\det D\Phi_d| = \varphi_d(x)$.

For a finite perimeter set $E \subseteq (0,1)^d$, the area formula [16, Theorem 3.2.3] combined with Lemma 3.1 tells us that

$$\mathrm{Per}(E,(0,1)^d) = \int_{\partial^*(\Phi_d^{-1}(E))} |\det D\Phi_d| \cdot \left| \left((D\Phi_d)^{\mathsf{T}}\right)^{-1} \nu \right| d\mathscr{H}^{d-1}, \tag{3.7}$$

where ν denotes the normal to the reduced boundary $\partial^*(\Phi_d^{-1}(E))$. We have that

$$\left| \left((D\Phi_d)^{\mathsf{T}}\right)^{-1} \nu \right| = \sqrt{2\pi} \sqrt{\sum_{i=1}^{d} \nu_i^2 e^{x_i^2}} \geq \sqrt{2\pi} \tag{3.8}$$

and such inequality holds as an equality if and only if, for all $i = 1, 2, \ldots, d$, we have $\nu_i x_i = 0$. Combining Eqs. (3.7 and 3.8), we obtain

$$\frac{1}{\sqrt{2\pi}} \mathrm{Per}(E,(0,1)^d) \geq \int_{\partial^*(\Phi_d^{-1}(E))} \varphi_d \, d\mathscr{H}^{d-1} = \mathrm{Per}_{\gamma_d}(\Phi_d^{-1}(E)), \tag{3.9}$$

and equality holds if and only if for $\mathscr{H}^{d-1}$-almost every point $x \in \partial^*(\Phi_d^{-1}(E))$, we have $\nu_i(x)x_i = 0$ for all $i = 1, 2, \ldots, d$. The Gaussian isoperimetric inequality (see Sect. 2.3) tells us that

$$\mathrm{Per}_{\gamma_d}(\Phi_d^{-1}(E)) \geq I_\gamma\left(\gamma_d(\Phi_d^{-1}(E))\right) = I_\gamma(|E|), \tag{3.10}$$

and equality holds if and only if $\Phi_d^{-1}(E)$ is an affine half-space (in particular, the normal ν to its boundary is constant) or it is the empty set or it is the whole $\mathbb{R}^d$.

Combining Eqs. (3.9 and (3.10) we obtain the desired inequality. If the inequality of the statement is an equality, then in particular both Eqs. (3.9) and (3.10) must be equalities. So, either $E = \emptyset$ or $E = (0,1)^d$ or $\Phi_d^{-1}(E)$ is an affine half-space and the normal ν to its boundary satisfies $\nu_i x_i = 0$ for all $x \in \partial(\Phi_d^{-1}(E))$ and all $i = 1, 2, \ldots, d$. In the latter case, take $1 \le j \le d$ such that $\nu_j \ne 0$. Then $x_j = 0$ for all $x \in \partial(\Phi_d^{-1}(E))$ and thus $\Phi_d^{-1}(E) = \{x \in \mathbb{R}^d : x \cdot \hat{v} \le 0\}$ with $\hat{v} = e_j$ or $\hat{v} = -e_j$. The sought characterization for E follows. □

4 Proof of Theorem 1.1

As in [1, Theorem 5.3], the proof of Theorem 1.1 is based on a compactness argument. The idea is that a sequence of perimeter-minimizing sets $E_n \subseteq (0, 1)^d$ with $|E_n| \to \frac{1}{2}$ converges in a very strong sense to a minimizer with measure $\frac{1}{2}$, and such a minimizer must be a half-cube thanks to Proposition 3.2.

Proof of Theorem 1.1 Let $(E_n)_{n\in\mathbb{N}} \subseteq (0, 1)^d$ be a sequence of sets of finite perimeter such that

- $|E_n| \to \frac{1}{2}$ as $n \to \infty$.
- The set E_n minimizes $\mathrm{Per}(E_n, (0, 1)^d)$ among the sets with measure equal to $|E_n|$.

The existence of E_n is standard [20, Proposition 12.30]. By [20, Theorem 17.20], we know that its reduced boundary $\partial^* E_n \cap (0, 1)^d$ is a free-boundary integral rectifiable varifold in $(0, 1)^d$ with constant mean curvature.

We will prove that, for n sufficiently large, E_n coincides (up to negligible sets) with $\{x \in (0, 1)^d : x \cdot \hat{v} \le |E_n|\}$ or $\{x \in (0, 1)^d : x \cdot \hat{v} \ge 1 - |E_n|\}$ for some $\hat{v} \in \{e_1, e_2, \ldots, e_d\}$. The desired statement follows immediately.

Let us show that the mean curvature of E_n goes to 0 as $n \to \infty$. The isoperimetric profile $I_{(0,1)^d}$ is concave [21, Corollary 6.11] and satisfies (see Proposition 3.2)

$$\sqrt{2\pi}\, I_\gamma(\lambda) \le I_{(0,1)^d}(\lambda) \le 1$$

for all $0 \le \lambda \le 1$. Notice that the lower bound and the upper bound for $I_{(0,1)^d}$ are both smooth concave functions, they have the same value at $\lambda = \frac{1}{2}$, and the derivatives at $\lambda = \frac{1}{2}$ are equal to 0. Since $I_{(0,1)^d}$ is trapped between two such functions, it follows that $\partial I_{(0,1)^d}(\lambda) \to \{0\}$ as $\lambda \to \frac{1}{2}$, where ∂f denotes the superdifferential[2] of the concave function f. Since E_n is a minimizer for the relative isoperimetric inequality, its mean curvature belongs to $\partial I_{(0,1)^d}(|E_n|)$ as proven in [25, Proposition 4.8] (see also [27, Corollary 2.9.] for the case of ambient spaces with smooth boundary) and therefore we deduce that the mean curvature of E_n goes to 0 as $n \to \infty$.

By compactness [20, Theorem 12.26], up to taking a subsequence, we may assume that E_n converges to E_∞ in the sense that $\mathbb{1}_{E_n} \to \mathbb{1}_{E_\infty}$ in L^1 and $D\mathbb{1}_{E_n} \overset{*}{\rightharpoonup} D\mathbb{1}_{E_\infty}$ in the open set $(0, 1)^d$. Notice that $\mathrm{Per}(E_n, (0, 1)^d) \le 1$ and therefore, by lower semicontinuity of the perimeter, we have $\mathrm{Per}(E_\infty, (0, 1)^d) \le 1$. Moreover $|E_\infty| = \lim |E_n| = \frac{1}{2}$. Thus, by Proposition 3.2, we obtain that, without loss of generality, $E_\infty = \{x \in (0, 1)^d : x_d \le \frac{1}{2}\}$. In particular, we have $\mathrm{Per}(E_n, (0, 1)^d) \to \mathrm{Per}(E_\infty, (0, 1)^d)$ and

[2] The superdifferential $\partial f(\lambda)$ of a concave function f at a point λ is the set of slopes $v \in \mathbb{R}$ so that $f(\lambda + t) \le f(\lambda) + vt$ for all $t \in \mathbb{R}$ so that $\lambda + t$ belongs to the domain of f.

thus, applying [2, Theorem 6.4], the boundaries $\partial^* E_n$ converge to $\partial^* E_\infty$ in the varifold sense.

We have verified all the assumptions necessary to apply [2, Regularity Theorem] in the interior and [14, Theorem 1.1] at the boundary,[3] thus we have that, for n sufficiently large, $\partial^* E_n \cap (0,1)^d$ is a graph over $\partial^* E_\infty \cap (0,1)^d = \{x \in (0,1)^d : x_d = \frac{1}{2}\}$. Since $\partial^* E_\infty \cap (0,1)^d$ is flat, it follows in particular that

$$\mathrm{Per}(E_n, (0,1)^d) = \mathscr{H}^{d-1}(\partial^* E_n \cap (0,1)^d) \geq \mathscr{H}^{d-1}(\partial^* E_\infty \cap (0,1)^d) = 1,$$

with equality if and only if $\partial^* E_n$ is the graph of a *constant* function over $\partial^* E_\infty$, which is exactly the desired statement. □

5 Proof of Theorem 1.2

We will need two simple technical lemmas. Let us emphasize that the theme of this whole section is obtaining estimates that do not depend on the dimension d.

Lemma 5.1 *Let $F \subseteq \mathbb{R}^d$ be a set of locally finite perimeter and let $H \subseteq \mathbb{R}^d$ be an affine half-space. Let $\ell := \mathrm{dist}(0_{\mathbb{R}^d}, \partial H)$ (observe that $\mathscr{H}^{d-1}_{\gamma_d}(\partial H) = \varphi(\ell)$) and let $\pi_{\partial H} : \mathbb{R}^d \to \partial H$ be the projection on the hyperplane ∂H. For any positive real number $r > 0$, we have*

$$\mathscr{H}^{d-1}_{\gamma_d}\Big(\pi_{\partial H}\big(\partial^* F \cap \{x : \mathrm{dist}(x, \partial H) < r\}\big)\Big) \geq \varphi(\ell) - \frac{\varphi(\ell)}{\varphi(\ell + r)} \frac{\gamma_d(F \triangle H)}{r}.$$

Proof Let $V := \pi_{\partial H}\big(\partial^* F \cap \{x : \mathrm{dist}(x, \partial H) < r\}\big)$. Let $\nu_{\partial H}$ be the normal to ∂H, oriented so that $\ell \nu_{\partial H} \in \partial H$. For each $x \in \partial H$, let us consider the 1-dimensional slice $F_x := F \cap \{x + t\nu_{\partial H} : t \in \mathbb{R}\}$; define H_x analogously. For $\mathscr{H}^{d-1}$-almost every $x \in \partial H$, the set F_x is (locally) made of finitely many disjoint intervals and all the extreme points of such intervals belong to $\partial^* F$ (see [20, Remark 18.13]).

For $\mathscr{H}^{d-1}$-almost every $x \in \partial H \setminus V$, we have[4] that $F_x \cap (x + \nu_{\partial H}(-r, r))$ is either empty or equal to $x + \nu_{\partial H}(-r, r)$ (up to negligible sets) because F_x cannot have any boundary point in the interval $x + \nu_{\partial H}(-r, r)$. In both cases, $\mathscr{H}^1_{\gamma_d}(F_x \triangle H_x) \geq r\varphi_d(x + r\nu_{\partial H})$.

Observe that $\varphi_d(x + r\nu_{\partial H}) = \frac{\varphi(\ell+r)}{\varphi(\ell)}\varphi_d(x)$. By Fubini's Theorem, we get

$$\begin{aligned}\gamma_d(F \triangle H) &\geq \int_{\partial H \setminus V} \mathscr{H}^1_{\gamma_d}(F_x \triangle H_x)\, d\mathscr{H}^{d-1}(x) \\ &\geq r \int_{\partial H \setminus V} \varphi_d(x + r\nu_{\partial H})\, d\mathscr{H}^{d-1}(x) = r\frac{\varphi(\ell + r)}{\varphi(\ell)} \mathscr{H}^{d-1}_{\gamma_d}(\partial H \setminus V)\end{aligned}$$

and the desired statement follows. □

[3] One could avoid taking care of the convergence at the boundary by exploiting the equivalence between the isoperimetric profile of the hypercube $(0,1)^d$ and the isoperimetric profile of the torus $\mathbb{T}^d$ mentioned in the introduction.

[4] With $x + \nu_{\partial H}(-r, r)$ we denote the set $\{x + t\nu_{\partial H} : t \in (-r, r)\}$.

We will apply the following lemma only with the function $f(t) = \exp(\frac{1}{2}t^2)$ and it is possible to prove a sharper result in this case, but we decided to prioritize clarity. Informally, the following lemma is a quantitative way to state the fact that a hyperplane with distance ℓ from the origin cannot be a subset of the strip $\{x \in \mathbb{R}^d : |x_i| < \frac{\ell}{2}\}$.

Lemma 5.2 *For any $\ell > 0$ there is a constant $c = c(\ell) > 0$ such that the following statement holds.*

Let $\Sigma \subseteq \mathbb{R}^d$ be an affine hyperplane with $\operatorname{dist}(0_{\mathbb{R}^d}, \Sigma) = \ell$ (which is equivalent to $\mathscr{H}^{d-1}_{\gamma_d}(\Sigma) = \varphi(\ell)$). For any Borel subset $V \subseteq \Sigma$ and any nondecreasing function $f : [0, \infty) \to [0, \infty)$, we have

$$\int_V f(|x_i|)\, d\mathscr{H}^{d-1}_{\gamma_d}(x) \geq \big(c - \mathscr{H}^{d-1}_{\gamma_d}(\Sigma \setminus V)\big) f(\tfrac{\ell}{2})$$

for any $i \in \{1, 2, \dots, d\}$.

Proof Since f is nondecreasing, we have

$$\begin{aligned}
&\int_V f(|x_i|)\, d\mathscr{H}^{d-1}_{\gamma_d}(x) \geq \mathscr{H}^{d-1}_{\gamma_d}(V \cap \{x : |x_i| \geq \tfrac{\ell}{2}\}) f(\tfrac{\ell}{2}) \\
&\geq \big[\varphi(\ell) - \mathscr{H}^{d-1}_{\gamma_d}(\Sigma \cap \{x : |x_i| < \tfrac{\ell}{2}\}) - \mathscr{H}^{d-1}_{\gamma_d}(\Sigma \setminus V)\big] f(\tfrac{\ell}{2}),
\end{aligned}$$

therefore to prove the statement it is sufficient to show that there exists a constant $c = c(\ell)$ so that

$$\mathscr{H}^{d-1}_{\gamma_d}(\Sigma \cap \{x : |x_i| < \tfrac{\ell}{2}\}) \leq \varphi(\ell) - c.$$

Let $\sigma_1, \sigma_2, \dots, \sigma_d \in \mathbb{R}^d$ be an orthonormal basis such that $\Sigma = \ell\sigma_d + \langle\sigma_1, \dots, \sigma_{d-1}\rangle$. In particular, σ_d is the unit normal to Σ. Up to rotation, we may assume that $e_i \in \langle\sigma_1, \sigma_d\rangle$, where e_i is the i-th element of the standard basis of $\mathbb{R}^d$ (so that $e_i \cdot x = x_i$). Then, it must be $e_i = (\sigma_d)_i\sigma_d + \sqrt{1 - (\sigma_d)_i^2}\sigma_1$. Using the parametrization of Σ given by $\mathbb{R}^{d-1} \ni y \mapsto y_1\sigma_1 + \cdots + y_{d-1}\sigma_{d-1} + \ell\sigma_d$, we get

$$\begin{aligned}
&\mathscr{H}^{d-1}_{\gamma_d}(\Sigma \cap \{x : |x_i| < \tfrac{\ell}{2}\}) \\
&= \varphi(\ell)\gamma_{d-1}\Big(\big\{y \in \mathbb{R}^{d-1} : |(\sigma_d)_i\ell + \sqrt{1 - (\sigma_d)_i^2}y_1| < \tfrac{\ell}{2}\big\}\Big) \\
&= \varphi(\ell)\gamma_1\Big(\big\{t \in \mathbb{R} : |(\sigma_d)_i\ell + \sqrt{1 - (\sigma_d)_i^2}t| < \tfrac{\ell}{2}\big\}\Big).
\end{aligned}$$

Let $q{:=}(\sigma_d)_i$. Consider the set appearing at the right-hand side of the last equation. It is a (possibly empty) interval with length $\frac{\ell}{\sqrt{1-q^2}}$ (empty if $|q| = 1$) and it contains 0 if and only if $|q| < \frac{1}{2}$. So, either it does not contain zero and thus its Gaussian measure is less than $\frac{1}{2}$ or its length is bounded by $\frac{2\ell}{\sqrt{3}}$ and thus its Gaussian measure is bounded by $1 - c(\ell)$ for a suitable constant $c(\ell) > 0$. In both cases, the desired statement follows. □

We are now ready to prove Theorem 1.2. The proof *lives* in the Gaussian space $(\mathbb{R}^d, \gamma_d)$ instead of the cube $(0, 1)^d$ (we move everything there with the same map that appeared in the proof of Proposition 3.2). Oversimplifying, the idea of the proof is that if the statement were false, then we would find a set $F \subseteq \mathbb{R}^d$ that is *almost* a half-space and minimizes a boundary integral (see Eq. (5.11)) that is not minimized by half-spaces; this yields a contradiction.

Proof of Theorem 1.2 The inequality $I_{(0,1)^{d+1}} \leq I_{(0,1)^d}$ follows from the fact that the map $E \mapsto E \times (0, 1)$ transforms a subset of $(0, 1)^d$ into a subset of $(0, 1)^{d+1}$ with the same perimeter and the same measure. The concavity of $I_{(0,1)^\infty}$ follows from the concavity of $I_{(0,1)^d}$ [21, Corollary 6.11].

For the second part of the statement, fix a dimension $d \geq 1$ and $0 < \lambda < 1$ different from $\frac{1}{2}$. Let $E \subseteq (0, 1)^d$ be a set of finite perimeter such that $|E| = \lambda$ and $\mathrm{Per}(E, (0, 1)^d) = I_{(0,1)^d}(\lambda)$ (a set E with these properties exists thanks to [20, Proposition 12.30]). By repeating the proof of Proposition 3.2 for the set E, we obtain that

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}} I_{(0,1)^d}(\lambda) - I_\gamma(\lambda)\\ &\geq \mathrm{Per}_{\gamma_d}(F) - I_\gamma(\lambda) + \int_{\partial^* F} \sqrt{\sum_{i=1}^d (\nu_{\partial^* F})_i^2 e^{x_i^2} - 1}\, d\mathscr{H}_{\gamma_d}^{d-1}(x),\end{aligned} \tag{5.11}$$

where $F = \Phi_d^{-1}(E)$ and $\nu_{\partial^* F}$ is the normal to $\partial^* F$. Notice that $\gamma_d(F) = \lambda$. The intuitive idea is that the two terms at the right-hand side cannot be simultaneously small: $\mathrm{Per}_{\gamma_d}(F) - I_\gamma(\lambda)$ is small if $\partial^* F$ is close to an affine hyperplane, while the integral is strictly positive if $\partial^* F$ is an affine half-space not containing the origin (which is guaranteed by the condition $|E| \neq \frac{1}{2}$). There is a crucial difficulty: all our estimates must be uniform in the dimension d, because we want to show that $\frac{1}{\sqrt{2\pi}} I_{(0,1)^d}(\lambda) - I_\gamma(\lambda)$ is bounded away from 0 for all $d \geq 1$.

Let $\delta := \mathrm{Per}_{\gamma_d}(F) - I_\gamma(\lambda)$. We are going to prove that there exist two constants $\delta_1(\lambda)$ and $c_1(\lambda)$ (independent of d) so that

$$\text{if } \delta < \delta_1 \text{ then } \int_{\partial^* F} \sqrt{\sum_{i=1}^d (\nu_{\partial^* F})_i^2 e^{x_i^2} - 1}\, d\mathscr{H}_{\gamma_d}^{d-1}(x) \geq c_1. \tag{5.12}$$

Assuming that Eq. (5.12) holds, thanks to Eq. (5.11), we deduce $\frac{1}{\sqrt{2\pi}} I_{(0,1)^d}(\lambda) - I_\gamma(\lambda) \geq \min\{\delta_1, c_1\}$ and therefore the statement of the Theorem follows.

Let us now prove Eq. (5.12). As will be clear from the proof, one can choose $\delta_1 := \min\{1, \left(\frac{|\Phi^{-1}(\lambda)|}{4}\right)^4\}$ (notice that $\Phi^{-1}(\lambda) \neq 0$ because $\lambda \neq \frac{1}{2}$). By the dimension free stability of the Gaussian isoperimetric inequality [5, Main Theorem and Proposition 4], there is an affine half-space $H \subseteq \mathbb{R}^d$ with $\gamma_d(H) = \gamma_d(F) = \lambda$ so that

$$\gamma_d(F \triangle H) \le C_1 \delta^{\frac{1}{2}}, \tag{5.13}$$

for a constant $C_1 = C_1(\lambda) > 0$ independent of the dimension d. Let $\nu_{\partial H} \in \mathbb{R}^d$ be the outer normal to the affine hyperplane ∂H. Let $\ell > 0$ denote the distance between ∂H and $0_{\mathbb{R}^d}$. Notice that $I_\gamma(\lambda) = \mathscr{H}^{d-1}_{\gamma_d}(\partial H) = \varphi(\ell)$ (therefore if a constant depends only on ℓ then it depends only on λ). We also know [5, Corollary 2 and Proposition 4]

$$\int_{\partial^* F} |\nu_{\partial^* F} - \nu_{\partial H}|^2 \, d\mathscr{H}^{d-1}_{\gamma_d} \le C_2 \delta, \tag{5.14}$$

for a constant $C_2 = C_2(\lambda) > 0$ independent of the dimension d.

By Jensen's inequality applied in the integrand (on the concave function $\sqrt{\cdot} - 1$; observe that $\sum_{i=1}^d (\nu_{\partial^* F})_i^2 = 1$), we have

$$\begin{aligned} &\int_{\partial^* F} \sqrt{\sum_{i=1}^d (\nu_{\partial^* F})_i^2 e^{x_i^2}} - 1 \, d\mathscr{H}^{d-1}_{\gamma_d}(x) \\ &\quad \ge \sum_{i=1}^d \int_{\partial^* F} (\nu_{\partial^* F})_i^2 \Big[\exp\left(\tfrac{1}{2} x_i^2\right) - 1 \Big] \, d\mathscr{H}^{d-1}_{\gamma_d}(x). \end{aligned}$$

Define $U_i := \{x \in \partial^* F \,:\, \nu_{\partial^* F}(x)_i^2 \ge \frac{1}{2}(\nu_{\partial H})_i^2\}$. Using the newly-defined sets U_i, we can continue the chain of inequalities

$$\ge \frac{1}{2} \sum_{i=1}^d (\nu_{\partial H})_i^2 \int_{U_i} \Big[\exp\left(\tfrac{1}{2} x_i^2\right) - 1 \Big] \, d\mathscr{H}^{d-1}_{\gamma_d}(x). \tag{5.15}$$

We show that for many indexes i, the set U_i almost saturates $\partial^* F$. We have

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^d (\nu_{\partial H})_i^2 \mathscr{H}^{d-1}_{\gamma_d}(\partial^* F \setminus U_i) &\le \sum_{i=1}^d \int_{\partial^* F \setminus U_i} |\nu_{\partial^* F}(x)_i^2 - (\nu_{\partial H})_i^2| \, d\mathscr{H}^{d-1}_{\gamma_d}(x) \\ &\le \int_{\partial^* F} \sum_{i=1}^d |\nu_{\partial^* F}(x)_i^2 - (\nu_{\partial H})_i^2| \, d\mathscr{H}^{d-1}_{\gamma_d}(x). \end{aligned}$$

For any two vectors $u, v \in \mathbb{R}^d$, the Cauchy-Schwarz inequality implies $\sum_{i=1}^d |u_i^2 - v_i^2| \leq |u - v||u + v|$, and thus we can continue the chain of inequalities as follows

$$\leq 2 \int_{\partial^* F} |\nu_{\partial^* F}(x) - \nu_{\partial H}| \, d\mathscr{H}^{d-1}_{\gamma_d}(x)$$
$$\leq 2 \left(\int_{\partial^* F} |\nu_{\partial^* F}(x) - \nu_{\partial H}|^2 \, d\mathscr{H}^{d-1}_{\gamma_d}(x) \right)^{\frac{1}{2}} \mathrm{Per}_{\gamma_d}(F)^{\frac{1}{2}} \leq 2\sqrt{C_2 \delta}\, \mathrm{Per}_{\gamma_d}(F)^{\frac{1}{2}}$$

where in the last step we have applied Eq. (5.14). Hence (assuming $\delta \leq 1$, so that $\mathrm{Per}_{\gamma_d}(F)$ is controlled) we deduce

$$\sum_{i=1}^{d} (\nu_{\partial H})_i^2 \mathscr{H}^{d-1}_{\gamma_d}(\partial^* F \setminus U_i) \leq C_3 \delta^{\frac{1}{2}},$$

where $C_3 = C_3(\lambda) > 0$ is a constant. Hence, there is a subset $J \subseteq \{1, 2, \ldots, d\}$ such that

$$\sum_{i \in J} (\nu_{\partial H})_i^2 \geq \frac{1}{2} \quad \text{and} \quad \mathscr{H}^{d-1}_{\gamma_d}(\partial^* F \setminus U_i) \leq 2 C_3 \delta^{\frac{1}{2}} \quad \text{for all } i \in J.$$

Now, fix $i \in J$. We show a lower bound for $\int_{U_i} \exp(\frac{1}{2}x_i^2) - 1 \, d\mathscr{H}^{d-1}_{\gamma_d}(x)$ by projecting onto ∂H. Let $\pi_{\partial H} : \mathbb{R}^d \to \partial H$ be the orthogonal projection on the affine hyperplane ∂H. Denote with $B(\partial H, r)$ the set of points with distance $< r$ from ∂H, i.e., $B(\partial H, r) := \{x \in \mathbb{R}^d : \mathrm{dist}(x, \partial H) < r\}$. Observe that for any subset $U \subseteq B(\partial H, r)$, since $\pi_{\partial H}$ is 1-Lipschitz, the area formula [16, Theorem 3.2.3] gives[5]

$$\begin{aligned}
\mathscr{H}^{d-1}_{\gamma_d}(U) &\geq \int_{\partial H} \sum_{y \in \pi_{\partial H}^{-1}(x) \cap U} \varphi_d(y) \, d\mathscr{H}^{d-1}(x) \\
&\geq \int_{\partial H} \sum_{y \in \pi_{\partial H}^{-1}(x) \cap U} \frac{\varphi(\ell + r)}{\varphi(\ell)} \, d\mathscr{H}^{d-1}_{\gamma_d}(x) \\
&\geq \frac{\varphi(\ell + r)}{\varphi(\ell)} \mathscr{H}^{d-1}_{\gamma_d}(\pi_{\partial H}(U)),
\end{aligned}$$

hence

$$\mathscr{H}^{d-1}_{\gamma_d}\big(\pi_{\partial H}(U)\big) \leq \frac{\varphi(\ell)}{\varphi(\ell + r)} \mathscr{H}^{d-1}_{\gamma_d}(U). \tag{5.16}$$

[5] To justify the second step notice that $y \in \pi_{\partial H}^{-1}(x) \cap B(\partial H, r)$ implies $y = x + t\nu_{\partial H}$ for some $t \in [-r, r]$ and thus $\varphi_d(y) = \varphi_d(x + t\nu_{\partial H}) = \frac{\varphi(\ell+t)}{\varphi(\ell)} \varphi_d(x) \geq \frac{\varphi(\ell+r)}{\varphi(\ell)} \varphi_d(x)$.

Define the subset $V_i \subseteq \partial H$ as

$$V_i := \pi_{\partial H}\Big(U_i \cap B(\partial H, \delta^{\frac14})\Big).$$

Let us show that V_i *saturates* ∂H,

$$\begin{aligned}\mathscr{H}^{d-1}_{\gamma_d}(V_i) &\geq \mathscr{H}^{d-1}_{\gamma_d}\Big(\pi_{\partial H}\big(\partial^* F \cap B(\partial H, \delta^{\frac14})\big)\Big)\\ &\quad - \mathscr{H}^{d-1}_{\gamma_d}\Big(\pi_{\partial H}\big((\partial^* F \setminus U_i) \cap B(\partial H, \delta^{\frac14})\big)\Big)\\ &\geq \varphi(\ell) - \frac{\varphi(\ell)}{\varphi(\ell+\delta^{\frac14})}\frac{\gamma_d(F\triangle H)}{\delta^{\frac14}} - \frac{\varphi(\ell)}{\varphi(\ell+\delta^{\frac14})}\mathscr{H}^{d-1}_{\gamma_d}\Big(\partial^* F \setminus U_i\Big),\end{aligned} \tag{5.17}$$

where we have used Lemma 5.1 and Eq. (5.16). Combining the latter inequality with Eq. (5.13) and with the estimate $\mathscr{H}^{d-1}_{\gamma_d}(\partial^* F \backslash U_i) \leq 2C_3\delta^{\frac12}$, if δ is sufficiently small with respect to ℓ (which depends only on λ), we obtain

$$\mathscr{H}^{d-1}_{\gamma_d}(V_i) \geq \varphi(\ell) - C_4\delta^{\frac14}, \tag{5.18}$$

for a constant $C_4 = C_4(\lambda) > 0$.

For a real parameter $0 < \kappa$, let $L_\kappa : \mathbb{R} \to \mathbb{R}$ be the function

$$L_\kappa(s) := \begin{cases} s+\kappa & \text{if } s \leq -\kappa,\\ 0 & \text{if } -\kappa \leq s \leq \kappa,\\ s-\kappa & \text{if } \kappa \leq s.\end{cases}$$

Observe that, if $x \in \partial H$ and $y = x + t\nu_{\partial H}$, then $\exp(\frac12 y_i^2) - 1 \geq \exp(\frac12 L_{\nu_i t}(x_i)^2) - 1$. Therefore, by repeating the same argument we employed to establish Eq. (5.16), we get

$$\int_{U_i} \exp\left(\tfrac12 x_i^2\right) - 1 \, d\mathscr{H}^{d-1}_{\gamma_d}(x) \geq \frac{\varphi(\ell+\delta^{\frac14})}{\varphi(\ell)} \int_{V_i} \exp\left(\tfrac12 L_{\nu_i\delta^{\frac14}}(x_i)^2\right) - 1 \, d\mathscr{H}^{d-1}_{\gamma_d}(x).$$

Applying Lemma 5.2, if we assume that $\nu_i\delta^{\frac14} < \ell/4$, the last estimate implies

$$\int_{U_i} \exp\left(\tfrac12 x_i^2\right) - 1 \, d\mathscr{H}^{d-1}_{\gamma_d}(x) \geq c_5,$$

for some constant $c_5 = c_5(\ell) > 0$ which does not depend on the dimension d. Combining the latter inequality with Eq. (5.15), we obtain

$$\int_{\partial^* F} \sqrt{\sum_{i=1}^d (\nu_{\partial^* F})_i^2 e^{x_i^2} - 1} \, d\mathscr{H}^{d-1}_{\gamma_d}(x)$$

$$
\geq \frac{1}{2}\sum_{i\in J}(\nu_{\partial H})_i^2 \int_{U_i}\left[\exp\left(\tfrac{1}{2}x_i^2\right)-1\right]d\mathscr{H}^{d-1}_{\gamma_d}(x)
$$
$$
\geq \frac{1}{2}c_5\sum_{i\in J}(\nu_{\partial H})_i^2 \geq \frac{1}{4}c_5,
$$

that concludes the proof of Eq. (5.12) as c_5 is a positive constant that does not depend on the dimension $d \geq 1$. □

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Declarations

Conflict of interest The author states that there is no conflict of interest.

References

1. Acerbi, E., Fusco, N., Morini, M.: Minimality via second variation for a nonlocal isoperimetric problem. Comm. Math. Phys. **322**(2), 515–557 (2013)
2. Allard, W.K.: On the first variation of a varifold. Ann. Math. **2**(95), 417–491 (1972)
3. Ambrosio, L., Bourgain, J., Brezis, H., Figalli, A.: BMO-type norms related to the perimeter of sets. Comm. Pure Appl. Math. **69**(6), 1062–1086 (2016)
4. Antonelli, G., Bruè, E., Fogagnolo, M., Pozzetta, M.: On the existence of isoperimetric regions in manifolds with nonnegative Ricci curvature and Euclidean volume growth. Calc. Var. Partial Differ. Equ. 61.2 **77**, 40 (2022)
5. Barchiesi, M., Brancolini, A., Julin, V.: Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality. Ann. Probab. **45**(2), 668–697 (2017)
6. Barchiesi, M., Julin, V.: Symmetry of minimizers of a Gaussian isoperimetric problem. Probab. Theory Related Fields **177**(1–2), 217–256 (2020)
7. Barthe, F.: Extremal properties of central half-spaces for product measures. J. Funct. Anal. **182**(1), 81–107 (2001)
8. Barthe, F., Maurey, B.: Some remarks on isoperimetry of Gaussian type. Ann. Inst. H. Poincaré Probab. Statist. **36**(4), 419–434 (2000)
9. Bobkov, S.G.: An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space. Ann. Probab. **25**(1), 206–214 (1997)
10. Borell, C.: The Brunn-Minkowski inequality in Gauss space. Invent. Math. **30**(2), 207–216 (1975)
11. Brezis, H., Bruckstein, A.: A sharp relative isoperimetric inequality for the square. C. R. Math. Acad. Sci. Paris **359**, 1191–1199 (2021)
12. Brezis, H.: Some of my favorite open problems. In: Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. (To appear). Available at https://sites.math.rutgers.edu/~brezis/PUBlications/234.pdf
13. Carlen, E.A., Kerce, C.: On the cases of equality in Bobkov's inequality and Gaussian rearrangement. Calc. Var. Partial Differ. Equ. **13**(1), 1–18 (2001)
14. Edelen, N., Li, C.: Regularity of free boundary minimal surfaces in locally polyhedral domains. Comm. Pure Appl. Math. **75**(5), 970–1031 (2022)
15. Eldan, R.: A two-sided estimate for the Gaussian noise stability deficit. Invent. Math. **201**(2), 561–624 (2015)

16. Federer, H.: Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften. Band 153, pp. xiv+676. Springer-Verlag Inc., New York (1969)
17. Fusco, N., Morini, M.: Total positive curvature and the equality case in the relative isoperimetric inequality outside convex domains. Calc. Var. Partial Differ. Equ. 62.3 **102**, 32 (2023)
18. Hadwiger, H.: Gitterperiodische Punktmengen und Isoperimetrie. Monatsh. Math. **76**, 410–418 (1972)
19. Leonardi, G.P., Ritoré, M., Vernadakis, E.: Isoperimetric inequalities in unbounded convex bodies. Mem. Am. Math. Soc. **276**(1354), 1–86 (2022)
20. Maggi, F.: Sets of finite perimeter and geometric variational problems. Vol. 135. Cambridge Studies in Advanced Mathematics. An Introduction to Geometric Measure Theory, pp. xx+454. Cambridge University Press, Cambridge (2012)
21. Milman, E.: On the role of convexity in isoperimetry, spectral gap and concentration. Invent. Math. **177**(1), 1–43 (2009)
22. Morgan, F.: In polytopes, small balls about some vertex minimize perimeter. J. Geom. Anal. **17**(1), 97–106 (2007)
23. Mossel, E., Neeman, J.: Robust dimension free isoperimetry in Gaussian space. Ann. Probab. **43**(3), 971–991 (2015)
24. Ritoré, M.: Examples of constant mean curvature surfaces obtained from harmonic maps to the two sphere. Math. Z. **226**(1), 127–146 (1997)
25. Ritoré, M., Vernadakis, E.: Isoperimetric inequalities in Euclidean convex bodies. Trans. Am. Math. Soc. **367**(7), 4983–5014 (2015)
26. Ros, A.: The isoperimetric problem. In: Global theory of minimal surfaces. Vol. 2. Clay Math. Proc. Amer. Math. Soc., pp. 175-209, Providence (2005)
27. Sternberg, P., Zumbrun, K.: On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint. Comm. Anal. Geom. **7**(1), 199–220 (1999)
28. Sudakov, V. N., Cirel'son, B. S.: "Extremal properties of half-spaces for spherically invariant measures". In: Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 41. Problems in the theory of probability distributions, II, pp. 14–24 (1974)