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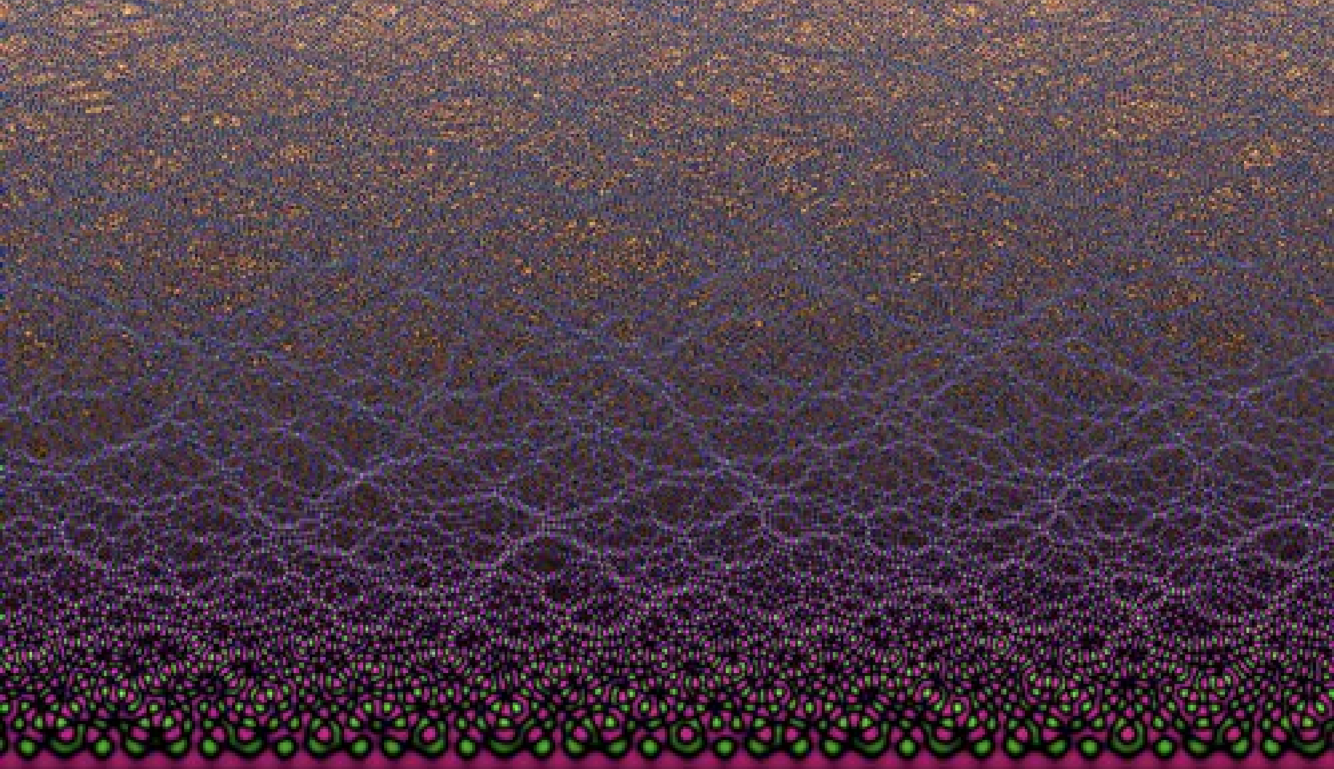

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EXTREME TEMPORAL INTERMITTENCY IN THE LINEAR SOBOLEV TRANSPORT ALMOST SMOOTH NONUNIQUE SOLUTIONS

ALEXEY CHESKIDOV AND XIAOYUTAO LUO

We revisit the notion of temporal intermittency to obtain sharp nonuniqueness results for linear transport equations. We construct divergence-free vector fields with sharp Sobolev regularity $L^1_t W^{1,p}$ for all $p < \infty$ in space dimensions $d \geq 2$ whose transport equations admit nonunique weak solutions belonging to $L^p_t C^k$ for all $p < \infty$ and $k \in \mathbb{N}$. In particular, our result shows that the time-integrability assumption in the uniqueness of the DiPerna–Lions theory is essential. The same result also holds for transport-diffusion equations with diffusion operators of arbitrarily large order in any dimensions $d \geq 2$.

1. Introduction

We consider the linear transport equation on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ with $d \geq 2$:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{1-1}$$

where $\rho : \mathbb{T}^d \times [0, T] \to \mathbb{R}$ is a scalar density function and $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d$ is a given incompressible vector field, i.e., $\operatorname{div} u = 0$ and $\rho_0 : \mathbb{T}^d \to \mathbb{R}$ is a given initial datum. The linearity of the equation allows us to prove the existence of weak solutions — even for very rough vector fields — that satisfy the equation in the sense of distributions

$$\int_{\mathbb{T}^d} \rho_0 \varphi(\,\cdot\,, 0)\, dx = \int_0^T \int_{\mathbb{T}^d} \rho(\partial_t \varphi + u \cdot \nabla \varphi)\, dx\, dt \quad \text{for all } \varphi \in C^\infty_c(\mathbb{T}^d \times [0, T)). \tag{1-2}$$

In this paper, we focus on the issue of the uniqueness/nonuniqueness of weak solutions satisfying (1-2) with $\rho \in L^1_{t,x}$ and $\rho u \in L^1_{t,x}$, for vector fields with Sobolev regularity. The celebrated DiPerna–Lions theory provides natural criteria for the uniqueness of the weak solutions for Sobolev vector fields:

Theorem 1.1 [DiPerna and Lions 1989]. *Let $p, q \in [1, \infty]$, and let $u \in L^1(0, T; W^{1,q}(\mathbb{T}^d))$ be a divergence-free vector field. For any $\rho_0 \in L^p(\mathbb{T}^d)$, there exists a unique renormalized solution $\rho \in C([0, T]; L^p(\mathbb{T}^d))$ to (1-1). Moreover, if*

$$\frac{1}{p} + \frac{1}{q} \leq 1, \tag{1-3}$$

then this solution ρ is unique among all weak solutions in the class $L^\infty(0, T; L^p(\mathbb{T}^d))$.

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In recent years, there has been a growing interest [Brué et al. 2021; Cheskidov and Luo 2021; Modena and Sattig 2020; Modena and Székelyhidi 2018] in showing the (possible) sharpness of the DiPerna–Lions condition (1-3), but so far the nonuniqueness constructions have not reached the full complement of (1-3) in the class of $L^\infty_t L^p$ solutions. In this paper, we show that the time-integrability assumption in the DiPerna–Lions uniqueness theorem is essential. More precisely, we show the following.

Theorem 1.2. *For any dimension $d \geq 2$, there exists $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d$, a divergence-free velocity vector field, satisfying $u \in L^1(0, T; W^{1,p}(\mathbb{T}^d))$ for all $p < \infty$ such that the uniqueness of (1-1) fails in the class*

$$\rho \in \bigcap_{\substack{p<\infty \\ k\in\mathbb{N}}} L^p(0, T; C^k(\mathbb{T}^d)) \quad \textit{and} \quad \rho u \in L^1(\mathbb{T}^d \times [0, T]).$$

This result is proved by the convex integration technique, which was brought to fluid dynamics by the pioneering work [De Lellis and Székelyhidi 2009] and has seen applications to the transport equation in [Brué et al. 2021; Cheskidov and Luo 2021; Modena and Sattig 2020; Modena and Székelyhidi 2018]. More details on the background and historical development will be discussed shortly. The key ingredient in the proof of Theorem 1.2 is the use of temporal intermittency, introduced in our previous works [Cheskidov and Luo 2021; 2022; 2023]. In particular, it improves our previous result [Cheskidov and Luo 2021] in terms of the integrability in time of the solution ρ and the spatial regularity of u and ρ. Moreover, Theorem 1.2 is sharp in the following two ways:

(1) The vector field cannot be $L^1_t W^{1,\infty}$ for which any $L^1_{t,x}$ solution of (1-1) with $\rho u \in L^1_{t,x}$ must coincide[1] a.e. with the Lagrangian solution.

(2) The density class cannot have any $L^\infty_t C^k$ regularity for $k \in \mathbb{N}$ due to the DiPerna–Lions condition (1-3).

Background and comparison. While the classical method of characteristics implies the well-posedness of (1-1) for Lipschitz vector fields, for non-Lipschitz vector fields, the method of characteristics no longer applies, and the well-posedness of (1-1) becomes challenging. The renormalization theory of [DiPerna and Lions 1989] provides powerful well-posedness of (1-1) under suitable Sobolev regularity assumptions on the vector field, and the renormalized solutions are shown to be unique in the regime (1-3).

Since Aizenman's example [1978], there have been examples of nonuniqueness at the Lagrangian level [Alberti et al. 2019; Colombini et al. 2003; Depauw 2003; Drivas et al. 2022; Yao and Zlatoš 2017], that is, constructions of vector fields whose flow maps exhibit degeneration. However, for a long time, the existence of nonunique (Eulerian) weak solutions of (1-1) for divergence-free Sobolev vector fields $u \in L^1_t W^{1,p}$ was unknown. To our knowledge, the first Eulerian construction of nonuniqueness was obtained in [Crippa et al. 2015] using the framework of [De Lellis and Székelyhidi 2009] for bounded vector fields.

Inspired by the spatially intermittent construction in [Buckmaster and Vicol 2019], the breakthrough result [Modena and Székelyhidi 2018] gave the first example of a Sobolev vector field with nonunique weak solutions to (1-1) and led to a lot of interest in improving nonuniqueness constructions to larger functional classes. Below we list the regimes where the nonuniqueness has been achieved:

[1] For instance, by a duality argument using estimates of the flow as in [Ambrosio et al. 2005, Proposition 8.1.7].

(1) [Modena and Székelyhidi 2018; 2019]: $\rho \in C_t L^p$ when $u \in C_t W^{1,q}$ for $1/p + 1/q > 1 + 1/(d-1)$ and $d \geq 3$.

(2) [Modena and Sattig 2020]: $\rho \in C_t L^p$ when $u \in C_t W^{1,q}$ for $1/p + 1/q > 1 + 1/d$.

(3) Bruè, Colombo, and De Lellis [Brué et al. 2021]: positive[2] $\rho \in C_t L^p$ when $u \in C_t W^{1,q}$ for $1/p + 1/q > 1 + 1/d$.

(4) [Cheskidov and Luo 2021]: $\rho \in L^1_t L^p$ when $u \in L^1_t W^{1,q}$ for $1/p + 1/q > 1$ and $d \geq 3$.

In summary, in the class of $L^\infty_t L^p$ densities, the nonuniqueness has been achieved in the regime $1/p + 1/q > 1 + 1/d$, while nonuniqueness in the regime $1/p + 1/q > 1$ is possible if one settles for $L^1_t L^p$ densities. However, it was not known whether $1/p + 1/q = 1$ is still the critical threshold for $L^1_t L^p$ densities.

Our main goal here is to show that the DiPerna–Lions scaling $1/p + 1/q = 1$ becomes irrelevant once the time integrability of ρ is slightly weakened. In particular, Theorem 1.2 follows from the following convex integration construction.

Theorem 1.3. *Let $d \geq 2$, $\varepsilon > 0$, and $N \in \mathbb{N}$. Let $\tilde{\rho} \in C^\infty(\mathbb{T}^d \times \mathbb{R})$ be such that $\operatorname{supp}_t \tilde{\rho} \subset (0, T)$ and $⨍_{\mathbb{T}^d} \rho(x, t)\, dx = 0$ for all $t \in \mathbb{R}$.*

Then there exist a divergence-free vector field $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d$ and a density $\rho : \mathbb{T}^d \times [0, T] \to \mathbb{R}$ such that all of the following hold:

(1) *$u \in L^1(0, T; W^{1,p}(\mathbb{T}^d))$ and $\rho \in L^p(0, T; C^k(\mathbb{T}^d))$ for all $1 \leq p < \infty$ and $k \in \mathbb{N}$.*

(2) *$\rho u \in L^1(\mathbb{T}^d \times [0, T])$ and (ρ, u) is a weak solution to (1-1) in the sense of (1-2).*

(3) *The deviation of ρ in $C^N(\mathbb{T}^d)$ norm is small*: $\|\rho - \tilde{\rho}\|_{L^N_t C^N} \leq \varepsilon$.

(4) *ρ has a compact temporal support*: $\operatorname{supp}_t \rho \subset \operatorname{supp}_t \tilde{\rho}$.

Remarks. (1) Here our initial data is always zero and attained in the classical sense. It is also easy to show that the obtained solution ρ is continuous in time in the sense of distributions (see Lemma 7.7 in [Cheskidov and Luo 2021] for details).

(2) Theorem 1.3 continues to hold for the transport-diffusion equation with a parabolic regularization $\Delta^m \rho$ of arbitrary order in the same regularity classes $(\rho, u) \in L^p_t C^k \times L^1_t W^{1,p}$, see Theorem 6.1. To our knowledge, this is the first example of a PDE where parabolic regularization does not provide any additional rigidity for the uniqueness of a class of weak solutions.

(3) The nonunique solutions ρ must change their signs — it is known by [Caravenna and Crippa 2021, Corollary 5.4] that any sign-definite solution $\rho \in L^1_{t,x}$ of $L^1_t W^{1,d+}$ vector fields is Lagrangian, see also [Brué et al. 2021, Section 8.2].

(4) By the linearity of (1-1), for any initial data $\rho_0 \in L^p(\mathbb{T}^d)$, the constructed vector field gives nonunique solutions in the class $\rho \in L^q_t L^p$ for any $q < \infty$. Indeed, one can add the constructed solution on top of the renormalized solution associated to ρ_0.

[2] Well-posedness for positive ρ can go beyond the DiPerna–Lions range, see [Brué et al. 2021, Theorem 1.5].

Strategy of the proof. We conclude with some final remarks on the proof. As said, we used the convex integration technique brought to fluid dynamics by the pioneering work of [De Lellis and Székelyhidi 2009]. The groundbreaking technique of that work resulted in breakthroughs in the fluids community over the last decade, and we refer readers to [Buckmaster et al. 2019; Buckmaster and Vicol 2019; De Lellis and Székelyhidi 2009; 2013; Isett 2018; Modena and Székelyhidi 2018] for a complete account.

The construction follows the same framework of temporal intermittency in our previous work [Cheskidov and Luo 2021]. A key difference is the regularity $L^p_t C^k$ for the density, which requires extreme intermittency in time when progressing to high frequencies. Since the density does not enjoy any "reasonable" L^∞_t regularity, from the duality $\rho u \in L^1_{t,x}$ we can gain a surprising regularity of almost L^1_t Lipschitz of the vector field. As in that previous work, this extreme temporal intermittency necessitates the use of stationary building blocks, as otherwise, the error produced by the large acceleration of the density becomes insurmountable with the non-Lipschitzness of the vector field, see Lemma 4.1 below. Once extreme intermittency in time is achieved, a little deduction of the time regularity of the density from L^∞_t to L^p_t allows us to gain essentially infinitely many derivatives in space for the density.

Finally, since the density enjoys essentially infinite many derivatives in space, the same construction also holds for transport-diffusion equations with diffusion operators of arbitrarily large order in any dimension $d \geq 2$. Surprisingly, even in dimension $d = 2$ a diffusion of an arbitrarily high order is not able to provide uniqueness for this class of weak solutions.

Organization. The rest of the paper is organized as follows.

- We prove the main theorem stated in the introduction in Section 2 by assuming Proposition 2.1, whose proof is the main content of this paper.
- In Section 3, we first introduce temporal intermittency into the construction, which is essential for our scheme. Next, we recall Mikado densities and Mikado flows as spatial building blocks. Finally, we use these temporal and spatial building blocks to define the density and velocity perturbations.
- In Section 4, we first specify the oscillation and concentration parameters and obtain estimates on the velocity and density perturbations claimed in Proposition 2.1.
- Section 5 is devoted to deriving the new defect field and its estimates, finishing the proof of Proposition 2.1.
- In Section 6, we show that the same nonuniqueness holds for transport-diffusion equations with arbitrarily high order of diffusion as well.
- In the Appendix, we recall some (now standard) technical tools in convex integration, namely the improved Hölder inequalities and antidivergence operators.

2. The main proposition and proof of Theorem 1.3

Notations. Throughout the paper, we fix the spatial domain $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, identified with a periodic box $[0, 1]^d$. Average over $\mathbb{T}^d$ is denoted by $\fint f = \int_{\mathbb{T}^d} f$. Functions on $\mathbb{T}^d$ are identified as periodic ones in $\mathbb{R}^d$, and we say f is $\sigma^{-1}\mathbb{T}^d$-periodic if

$$f(x+\sigma^{-1}k) = f(x) \quad \text{for any } k \in \mathbb{Z}^d.$$

Spatial Lebesgue norms are denoted by $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathbb{T}^d)}$, while we write $\|\cdot\|_{L^p_{t,x}}$ for Lebesgue norms taken in the space-time domain $\mathbb{T}^d \times [0,T]$. If a function f is time-dependent, we write $\|f(t)\|_{L^p}$ to indicate that the spatial norm is taken at a time slice $t \in [0,T]$. For a Banach space X, we use the notation $\|\cdot\|_{L^p_t X}$ to denote the norm on Bochner spaces $L^p([0,T];X)$, such as $\|\cdot\|_{L^1_t W^{k,p}}$ and $\|\cdot\|_{L^p_t C^k}$.

The differentiation operations such as ∇, Δ, and div are meant for differentiation in space only.

We use the notation $X \lesssim Y$, which means $X \le CY$ for some constant $C > 0$. The notation $X \sim Y$ means both $X \lesssim Y$ and $Y \lesssim X$ at the same time.

Continuity-defect equation. As in [Modena and Székelyhidi 2018], we consider the continuity-defect equation to obtain approximate solutions to the transport equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = \operatorname{div} R, \\ \operatorname{div} u = 0, \end{cases} \tag{2-1}$$

where $R : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$ is called the defect field. In what follows, a triple (ρ, u, R) will denote a smooth solution to (2-1). Recall that for a function $f \in L^1_{t,x}$, its temporal support $\operatorname{supp}_t f$ is the closure of the set

$$\{t \in [0,T] : \|f(\cdot,t)\|_{L^1(\mathbb{T}^d)} > 0\}.$$

We now state the main proposition of the paper and use it to prove Theorem 1.3.

Proposition 2.1. *Let $d \ge 2$. There exists a universal constant $M > 0$ such that the following holds.*

Suppose (ρ, u, R) is a smooth solution of (2-1) on $[0,1]$ such that $\operatorname{supp}_t R \subset (0,1)$. Then, for any $1 \le p \in \mathbb{N}$ and any $0 < \delta < \frac{1}{2}$, there exists another smooth solution (ρ_1, u_1, R_1) of (2-1) on $[0,1]$ such that the density perturbation $\theta := \rho_1 - \rho$ and the vector field perturbation $w = u_1 - u$ satisfy the following:

(1) *Both θ and w have zero spacial mean and*

$$\operatorname{supp}_t \theta \subset \operatorname{supp}_t R. \tag{2-2}$$

(2) *θ and w satisfy the estimates*

$$\|\theta\|_{L^p_t C^p} \le \delta, \tag{2-3}$$

$$\|w\|_{L^1_t W^{1,p}} \le \delta, \tag{2-4}$$

$$\|\theta w + \theta u + \rho w\|_{L^1_{t,x}} \le M \|R\|_{L^1_{t,x}}. \tag{2-5}$$

(3) *The new defect field R_1 satisfies*

$$\operatorname{supp}_t R_1 \subset \operatorname{supp}_t R \tag{2-6}$$

and the estimate

$$\|R_1\|_{L^1_{t,x}} \le \delta. \tag{2-7}$$

Proof of Theorem 1.3. We assume $T = 1$ without loss of generality. We will construct a sequence (ρ_n, u_n, R_n), $n = 1, 2, \dots$ of solutions to (2-1) as follows. For $n = 1$, we set

$$\begin{aligned} \rho_1(t) &:= \tilde{\rho}, \\ u_1(t) &:= 0, \\ R_1(t) &:= \mathcal{R}(\partial_t \tilde{\rho}), \end{aligned}$$

where $\mathcal{R} = \Delta^{-1}\nabla$ is the inverse divergence in the Appendix. Then (ρ_1, u_1, R_1) solves (2-1) trivially by the constant mean assumption on $\tilde{\rho}$.

Next, we apply Proposition 2.1 inductively to obtain (ρ_n, u_n, R_n) for $n = 2, 3, \dots$ as follows. Given (ρ_n, u_n, R_n), we apply Proposition 2.1 with parameters

$$p_n = N2^n, \quad \delta_n = \varepsilon 2^{-n},$$

to obtain a new triple $(\rho_{n+1}, u_{n+1}, R_{n+1})$. Then the perturbations $\theta_n := \rho_{n+1} - \rho_n$ and $w_n := u_{n+1} - u_n$ and the defect field R_n satisfy

$$\|\theta_n\|_{L_t^{p_n} C^{p_n}} \le \delta_n, \quad \|w_n\|_{L_t^1 W^{1,p_n}} \le \delta_n, \tag{2-8a}$$

$$\|R_{n+1}\|_{L^1_{t,x}} \le \delta_n, \tag{2-8b}$$

$$\|\theta_n w_n + \theta_n u_n + \rho_n w_n\|_{L^1_{t,x}} \le M \|R_n\|_{L^1_{t,x}}, \tag{2-8c}$$

for all $n = 1, 2, \dots$. In addition, due to (2-6) and (2-2), we have

$$\operatorname{supp}_t \theta_n \subset \operatorname{supp}_t \tilde{\rho} \quad \text{for all } n \in \mathbb{N}. \tag{2-9}$$

Hence by (2-8a) there exists $(\rho, u) \in L_t^p C^p \times L_t^1 W^{1,p}$ for all $p \in \mathbb{N}$ such that

$$\rho_n \to \rho \quad \text{in } L_t^p C^p \quad \text{and} \quad u_n \to u \quad \text{in } L_t^1 W^{1,p} \quad \text{for all } p \in \mathbb{N}. \tag{2-10}$$

Moreover, $\operatorname{supp}_t \rho \subset \operatorname{supp}_t \tilde{\rho}$ due to (2-9). Since $p_n \ge N$ and the time interval is of length 1,

$$\|\rho - \tilde{\rho}\|_{L_t^N C^N} \le \sum_{n \ge 1} \|\theta_n\|_{L_t^N C^N} \le \sum_{n \ge 1} \|\theta_n\|_{L_t^{p_n} C^{p_n}} \le \varepsilon.$$

It remains to show (ρ, u) is a weak solution. We first prove that $\rho u \in L^1_{t,x}$ and $\rho_n u_n \to \rho u$ in $L^1_{t,x}$. Using (2-8c),

$$\|\rho_{n+1} u_{n+1} - \rho_n u_n\|_{L^1_{t,x}} \le M \delta_{n-1} \quad \text{for } n \ge 2. \tag{2-11}$$

Thus the sequence $\rho_n u_n$ is Cauchy in $L^1_{t,x}$, and consequently there is $G \in L^1_{t,x}$ such that $\rho_n u_n \to G$ in $L^1_{t,x}$. Now we claim that $G = \rho u$. Thanks to (2-11), passing to subsequences and dropping subindices, we get $\rho_n \to \rho$ and $u_n \to u$ a.e. in $\mathbb{T}^d \times [0, 1]$. So $\rho_n u_n \to G$ a.e. in $\mathbb{T}^d \times [0, 1]$, and hence $\rho u = G$ and $\rho_n u_n \to \rho u$ in $L^1_{t,x}$. Since in addition $R_n \to 0$ in $L^1_{t,x}$ by (2-8b), it is standard to show that (ρ, u) is a weak solution to (1-1). □

3. Temporal intermittency, building blocks, and perturbations

The rest of the paper is devoted to the proof of Proposition 2.1. In this section, we introduce the temporal and spatial building blocks and use them to define the density and velocity perturbations.

Summary of parameters. Given arbitrarily large $p \in \mathbb{N}$ as in the statement of Proposition 2.1, we will fix three exponents in Lemma 4.1 below: $r > 1$ very close to 1, $0 < \alpha \ll 1$, and $0 < \gamma \ll 1$. These exponents are used to define three large parameters: the concentrations $\kappa, \mu \geq 1$ and oscillation $\sigma \in \mathbb{N}$. These three large parameters satisfy the hierarchy $1 \ll \sigma \ll \mu \ll \kappa$ — whose meaning will be made precise in Section 4 — but their exact values will be fixed at the end depending on the given solution (ρ, u, R).

Temporal functions $\tilde{g}_k$ and g_k. We start with defining the intermittent oscillatory functions $\tilde{g}_k$ and g_k that lie at the heart of our scheme. First, we fix a profile function $\widetilde{G} \in C_c^\infty((0,1))$ such that

$$\int_{[0,1]} \widetilde{G}^2 \, dt = 1, \qquad \int_{[0,1]} \widetilde{G} \, dt = 0, \qquad \|\widetilde{G}\|_{L^\infty} \leq 2, \tag{3-1}$$

and, for $k = 1, \dots, d$, define G_k to be the 1-periodic extension of $\widetilde{G}(\kappa(t - t_k))$, where $t_k \in [0,1]$ are chosen such that G_k have disjoint supports for different k. In other words, $G_k(t) = \sum_{n \in \mathbb{Z}} \widetilde{G}(n + \kappa(t - t_k))$. We will refer to $\kappa \geq 1$ as the temporal concentration parameter.

Next, for a large oscillation parameter $\sigma \in \mathbb{N}$ and a small exponent $0 < \alpha < 1$ to be fixed later, we define σ^{-1}-periodic functions

$$\tilde{g}_k(t) = \kappa^\alpha G_k(\sigma t), \qquad g_k(t) = \kappa^{1-\alpha} G_k(\sigma t). \tag{3-2}$$

We will use $\tilde{g}_\kappa$ to oscillate the densities Φ_k, and g_κ to oscillate the vectors $\boldsymbol{W}_k$, defined in the following section. Note that, by (3-1),

$$\int_{[0,1]} \tilde{g}_k g_k \, dt = 1, \tag{3-3}$$

and by definitions of $\tilde{g}_k$ and g_k,

$$\|\tilde{g}_k\|_{L^q([0,1])} \sim \kappa^{\alpha - 1/q}, \qquad \|\tilde{g}_k'\|_{L^q([0,1])} \sim (\kappa\sigma)\kappa^{\alpha - 1/q}, \qquad \|g_k\|_{L^q([0,1])} \sim \kappa^{1-\alpha-1/q}. \tag{3-4}$$

Temporal correction function h_k. Now we define a σ^{-1}-periodic function $h_k : \mathbb{R} \to \mathbb{R}$ by

$$h_k(t) := \sigma \int_0^t (\tilde{g}_k g_k - 1) \, d\tau, \tag{3-5}$$

so that

$$\sigma^{-1} \partial_t h_k = \tilde{g}_k g_k - 1. \tag{3-6}$$

Thanks to (3-3), we have $\int_{[0,\sigma^{-1}]} \tilde{g}_k g_k \, dt = \sigma^{-1}$. Since $\tilde{g}_k g_k \geq 0$ by their definitions, it follows that h_k is well-defined and satisfies the estimate

$$\|h_k\|_{L^\infty[0,1]} \leq 1. \tag{3-7}$$

The function h_k will be used to design the temporal corrector θ_o in (3-17).

Mikado densities and flows. Here we recall the spatial building blocks for our convex integration construction; the Mikado densities and Mikado flows introduced in [Daneri and Székelyhidi 2017] and [Modena and Székelyhidi 2018]. These are periodic objects supported on pipes with a small radius. Note that we do not require them to have disjoint supports in space — each Mikado object will be coupled with a temporal function $\tilde{g}_k$ or g_k to achieve disjoint supports in space-time.

For $k = 1, \dots, d$, we denote each standard Euclidean basis vector by $\boldsymbol{e}_k = (0, \dots, 1, \dots, 0)$. For any $x \in \mathbb{R}^d$ and $k = 1, \dots, d$, we write $x'_k \in \mathbb{R}^{d-1}$ for the vector $x'_k = (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_d)$.

Let $d \geq 2$ be the spatial dimension. We fix a vector field $\Omega \in C_c^\infty(\mathbb{R}^{d-1})$ and a scalar density $\phi \in C_c^\infty(\mathbb{R}^{d-1})$ such that

$$\operatorname{supp} \Omega \subset (0,1)^{d-1}, \quad \operatorname{div} \Omega = \phi, \quad \int_{\mathbb{R}^{d-1}} \phi^2 = 1. \tag{3-8}$$

For each $k = 1, \dots, d$, we define the nonperiodic Mikado objects

$$\begin{aligned} \widetilde{\Phi}_k(x) &= \phi(\mu x'_k), \\ \widetilde{\Omega}_k(x) &= \mu^{-1} \Omega(\mu x'_k), \\ \widetilde{W}_k(x) &= \mu^{d-1} \phi(\mu x'_k) \boldsymbol{e}_k, \end{aligned} \tag{3-9}$$

define the 1-periodic objects $\Omega_k : \mathbb{T}^d \to \mathbb{R}^d$, $\Phi_k : \mathbb{T}^d \to \mathbb{R}$, and $W_k : \mathbb{T}^d \to \mathbb{R}^d$ as the 1-periodic extensions of (3-9), and then rescale them by a large oscillation factor $\sigma \in \mathbb{N}$:

$$\Phi_k(x) = \Phi_k(\sigma x), \quad \Omega_k(x) = \Omega_k(\sigma x), \quad \boldsymbol{W}_k(x) = W_k(\sigma x). \tag{3-10}$$

We now summarize the properties of the constructed building blocks Ω_k, Φ_k, and $\boldsymbol{W}_k$ in the following theorem.

Theorem 3.1. *For all $\sigma \in \mathbb{N}$ and $\mu \geq 1$, the density Φ_k, potential Ω_k, and vector field $\boldsymbol{W}_k$ defined by (3-10) satisfy the following for every $k = 1, \dots, d$:*

(1) $\boldsymbol{W}_k : \mathbb{T}^d \to \mathbb{R}^d$, $\Phi_k : \mathbb{T}^d \to \mathbb{R}$, *and* $\Omega_k : \mathbb{T}^d \to \mathbb{R}^d$ *are smooth $\sigma^{-1}\mathbb{T}^d$-periodic functions, and $\boldsymbol{W}_k$, Φ_k have zero mean on $\mathbb{T}^d$.*

(2) $\operatorname{div} \boldsymbol{W}_k = \operatorname{div}(\Phi_k \boldsymbol{W}_k) = 0$, *and the density Φ_k is the divergence of the potential $\sigma^{-1}\Omega_k$:*

$$\operatorname{div} \Omega_k = \sigma \Phi_k. \tag{3-11}$$

(3) *For any $1 \leq p \leq \infty$ and $s \geq 0$,*

$$\|\Omega_k\|_{L^p} \lesssim \mu^{-1-(d-1)/p}, \tag{3-12a}$$

$$\|\Phi_k\|_{W^{s,p}} \lesssim (\sigma\mu)^s \mu^{-d-1/p}, \tag{3-12b}$$

$$\|\boldsymbol{W}_k\|_{W^{s,p}} \lesssim (\sigma\mu)^s \mu^{(d-1)(1-1/p)}. \tag{3-12c}$$

(4) *The following identity holds:*

$$\fint_{\mathbb{T}^d} \Phi_k(x) \boldsymbol{W}_k(x)\, dx = \boldsymbol{e}_k. \tag{3-13}$$

Proof. The first two points are direct consequences of the definitions while the last point follows from (3-8).

When $s = 0$, the bounds (3-12a)–(3-12c) follow from the small supports of the nonperiodic objects $\widetilde{\Phi}_k$, $\widetilde{\Omega}_k$, $\widetilde{W}_k$: the support set is a cylinder of radius $\sim \mu^{-1}$ and length 1. The general case $s > 0$ can be obtained by interpolation between the cases $s \in \mathbb{N}$. □

Density and velocity perturbations. Here we define perturbations (θ, w) given a defect field R as in Proposition 2.1.

Recall that the concentration parameters $\mu, \kappa \geq 1$ and the oscillation parameter $\sigma \in \mathbb{N}$ introduced so far will be specified in Lemma 4.1 below. The velocity perturbation is defined by

$$w := \sum_{1 \leq k \leq d} g_k \boldsymbol{W}_k. \tag{3-14}$$

For the density perturbation, first we decompose the defect field

$$R(x,t) = \sum_{1 \leq k \leq d} R_k(x,t) \boldsymbol{e}_k, \tag{3-15}$$

where the $\boldsymbol{e}_k$ are the standard Euclidean basis as before. We define the density perturbation as the sum of the zero-mean projection of the principal part and a small oscillation correction:

$$\theta = \mathbb{P}_{\neq 0} \theta_p + \theta_o,$$

where $\mathbb{P}_{\neq 0} f = f - \fint f$ is the projection removing the spatial mean, and

$$\theta_p := - \sum_{1 \leq k \leq d} \tilde{g}_k R_k \Phi_k, \tag{3-16}$$

$$\theta_o = \sigma^{-1} \operatorname{div} \sum_{1 \leq k \leq d} h_k R_k \boldsymbol{e}_k. \tag{3-17}$$

Note that $\operatorname{div} w = 0$ for all t since $\boldsymbol{W}_k$ is divergence-free, which also implies that

$$\operatorname{div}([\mathbb{P}_{\neq 0} \theta_p] w) = \operatorname{div}(\theta_p w).$$

By definitions, $\operatorname{supp}_t \theta \subset \operatorname{supp}_t R$ as required in (2-2) of Proposition 2.1.

4. Estimates of the density and velocity perturbations

The goal of this section is to obtain estimates (2-3), (2-4), and (2-5) on θ and w claimed in Proposition 2.1.

Choice of parameters. Now we specify all the oscillation and concentration parameters in the perturbation as explicit powers of a large frequency number $\lambda > 0$ that will be fixed in the end.

(1) Oscillation $\sigma \in \mathbb{N}$:

$$\sigma = \lceil \lambda^{2\gamma} \rceil.$$

Without loss of generality, we only consider values of λ such that $\sigma = \lambda^{2\gamma} \in \mathbb{N}$ in what follows.

(2) Concentration $\kappa, \mu \geq 1$:

$$\mu = \lambda, \quad \kappa = \lambda^{(d-2\gamma)/\alpha}.$$

Lemma 4.1. *For any $p \in \mathbb{N}$, there exist constants $\alpha > 0$, $0 < \gamma < \frac{1}{4}$, and $r > 1$ such that the following holds*:

$$(\sigma\mu)^p \kappa^{\alpha - 1/p} \le \lambda^{-\gamma} \quad (\theta_p \in L^p_t C^p), \tag{4-1}$$

$$\kappa^{-\alpha}(\sigma\mu)^1 \mu^{(d-1)(1-1/p)} \le \lambda^{-\gamma} \quad (w \in L^1_t W^{1,p}), \tag{4-2}$$

$$\kappa^{\alpha}\mu^{-1-(d-1)/r} \le \lambda^{-\gamma} \quad (\textit{acceleration error}). \tag{4-3}$$

Proof. We first fix $\gamma > 0$. Condition (4-2) in terms of power of λ reads

$$\frac{d-1}{p} \ge 5\gamma.$$

Since $p < \infty$, this condition is satisfied for $0 < \gamma < \frac{1}{4}$ sufficiently small. Expressing (4-1) in terms of power of λ gives

$$\alpha \le \frac{1}{p}\frac{d - 2\gamma}{(2p\gamma + p + d - \gamma)}.$$

Since $0 < \gamma < \frac{1}{4}$, this condition on α is automatically satisfied when

$$\alpha < \frac{d - \frac{1}{2}}{2p^2 + 2dp}.$$

We then fix $\alpha > 0$ according to this condition.

For condition (4-3), taking $r = 1$, the left-hand side becomes

$$\lambda^{d-2\gamma-1-d+1} = \lambda^{-2\gamma}.$$

Therefore, by continuity, (4-3) holds for $r > 1$ close enough to 1. □

We remark that Lemma 4.1 cannot hold for $p = \infty$ from its proof — the conditions (4-2) and (4-3) become incompatible when $p = \infty$. This is consistent with the $L^1_{t,x}$ unconditional uniqueness of $L^1_t W^{1,\infty}$ vector fields as in [Ambrosio et al. 2005, Proposition 8.1.7].

Estimates for the perturbations. In what follows, C_R will stand for a large constant that only depends on the triple (ρ, u, R) provided as the input by Proposition 2.1. It is important that C_R can *never* depend on the free parameters σ, μ, and κ in the building blocks that we used to define θ and w.

Lemma 4.2 (estimate on θ). *The density perturbation θ satisfies*

$$\|\theta\|_{L^p_t C^p} \le C_R \lambda^{-\gamma}.$$

Proof. For the principle part θ_p, since the space $C^p(\mathbb{T}^d)$ is an algebra, using Hölder's inequality, (3-4), and (3-12b), we obtain

$$\|\theta_p\|_{L^p_t C^p} \le \sum_{1 \le k \le d} \|\tilde{g}_k\|_{L^p} \|R_k\|_{L^\infty_t C^p} \|\Phi_k\|_{L^\infty_t C^p} \le C_R (\sigma\mu)^p \kappa^{\alpha - 1/p} \le C_R \lambda^{-\gamma},$$

where the last inequality holds due to condition (4-1).

For the temporal corrector θ_0 defined in (3-17), by Hölder's inequality and (3-7), we have

$$\|\theta_o\|_{L_t^\infty C^p} \le \sigma^{-1} \sum_{1\le k\le d} \|h_k\|_{L^\infty([0,1])} \|\operatorname{div}(R_k \boldsymbol{e}_k)\|_{L_t^\infty C^p} \le C_R \sigma^{-1}, \tag{4-4}$$

and the final bound holds by the definition of σ. □

Lemma 4.3 (estimate on w). *The velocity perturbation w satisfies*

$$\|w\|_{L_t^1 W^{1,p}} \lesssim \lambda^{-\gamma}.$$

Proof. Using Hölder's inequality, (3-4), and (3-12c), we obtain

$$\|w\|_{L_t^1 W^{1,p}} \le \sum_{1\le k\le d} \|g_k\|_{L^1} \|\boldsymbol{W}_k\|_{W^{1,p}} \lesssim \kappa^{-\alpha} (\sigma\mu) \mu^{(d-1)(1-1/p)}.$$

The conclusion holds thanks to (4-2). □

Lemma 4.4 (estimate on θw). *The following estimate holds*:

$$\|\theta w + \theta u + \rho w\|_{L_{t,x}^1} \lesssim \|R\|_{L_{t,x}^1} + C_R \lambda^{-\gamma}.$$

Proof. Taking the L^1 norm in space and using Lemma A.1 and the fact that $\Phi_k \boldsymbol{W}_k$ is $\sigma^{-1}\mathbb{T}^d$-periodic in space, we obtain

$$\begin{aligned}
\|\theta(t) w(t)\|_{L^1} &\le \sum_{1\le k\le d} |\tilde{g}_k(t) g_k(t)| \|R_k(t) \Phi_k \boldsymbol{W}_k\|_{L^1} \\
&\lesssim \sum_{1\le k\le d} |\tilde{g}_k(t) g_k(t)| \|\Phi_k \boldsymbol{W}_k\|_{L^1} (\|R_k(t)\|_{L^1} + \sigma^{-1} \|R_k(t)\|_{C^1}) \\
&\lesssim \sum_{1\le k\le d} |\tilde{g}_k(t) g_k(t)| (\|R_k(t)\|_{L^1} + \sigma^{-1} \|R_k\|_{C_{t,x}^1}),
\end{aligned}$$

where we used $\|\Phi_k \boldsymbol{W}_k\|_{L_x^1} = 1$ by (3-12b) and (3-12c) in the last step. Now taking the L^1 norm in time, using Lemma A.1 together with σ-periodicity of $g_k(t) g_k(t)$ and the smoothness of $t \mapsto \|R_k(t)\|_{L^1}$, and recalling that $\|g_k g_k\|_{L^1} = 1$, we arrive at

$$\|\theta w\|_{L_{t,x}^1} \lesssim \sum_{1\le k\le d} \|\tilde{g}_k g_k\|_{L^1} (\|R_k\|_{L_{t,x}^1} + \sigma^{-1} C_R) \lesssim \sum_{1\le k\le d} (\|R_k\|_{L_{t,x}^1} + \sigma^{-1} C_R) \lesssim \|R\|_{L_{t,x}^1} + C_R \sigma^{-1},$$

where the implicit constant does not depend on the parameter λ or the given solution (ρ, u, R).

The estimates for the other two terms θu and ρw follow from Lemmas 4.2 and 4.3. Indeed, Hölder's inequality gives

$$\|\theta u\|_{L_{t,x}^1} \le \|\theta\|_{L_{t,x}^1} \|u\|_{L_{t,x}^\infty} \le C_R \lambda^{-\gamma}$$

and

$$\|\rho w\|_{L_{t,x}^1} \le \|w\|_{L_{t,x}^1} \|\rho\|_{L_{t,x}^\infty} \le C_R \lambda^{-\gamma}.$$

□

5. The new defect field R_1 and its estimates

We continue with the proof of Proposition 2.1. Our next goal is to define a suitable defect field R_1 such that the new density ρ_1 and vector field u_1,

$$\rho_1 := \rho + \theta, \quad u_1 := u + w,$$

solve the continuity-defect equation

$$\partial_t \rho_1 + u_1 \cdot \nabla \rho_1 = \operatorname{div} R_1. \tag{5-1}$$

The defect field R_1 will consist of three parts,

$$R_1 = R_{\text{osc}} + R_{\text{lin}} + R_{\text{cor}},$$

each solving the corresponding divergence equation

$$\begin{aligned} \operatorname{div} R_{\text{osc}} &= \partial_t \theta + \operatorname{div}(\theta_p w + R), \\ \operatorname{div} R_{\text{lin}} &= \operatorname{div}(\theta u + \rho w), \\ \operatorname{div} R_{\text{cor}} &= \operatorname{div}(\theta_o w). \end{aligned}$$

So we define the linear error $R_{\text{lin}} = \theta u + \rho w$ and the correction error $R_{\text{cor}} = \theta_o w$ in the usual way and the oscillation error R_{osc} in the following important lemma. Recall that $\mathcal{R}$ and $\mathcal{B}$ are the antidivergence operators defined in the Appendix.

Definition of the oscillation error.

Lemma 5.1 (space-time oscillations). *The following identity holds*:

$$\partial_t \theta + \operatorname{div}(\theta_p w + R) = \operatorname{div}(R_{\text{osc,x}} + R_{\text{osc,t}} + R_{\text{acc}}),$$

where $R_{\text{osc,x}}$ *is the spatial oscillation error*

$$R_{\text{osc,x}} = -\sum_{1 \le k \le d} \tilde{g}_k g_k \mathcal{B}\Big(\nabla R_k, \Big(\Phi_k \boldsymbol{W}_k - \fint_{\mathbb{T}^d} \Phi_k \boldsymbol{W}_k\Big)\Big),$$

$R_{\text{osc,t}}$ *is the temporal oscillation error*

$$R_{\text{osc,t}} = \sigma^{-1} \sum_{1 \le k \le d} h_k \partial_t R_k \boldsymbol{e}_k,$$

and R_{acc} *is the acceleration error*

$$R_{\text{acc}} = -\sum_{1 \le k \le d} \mathcal{B}(\partial_t(\tilde{g}_k R_k), \Phi_k).$$

Proof. By definition of θ_p and w, using the disjointedness of supports of $\tilde{g}_k$ and $g_{k'}$ for $k \neq k'$, we obtain

$$\operatorname{div}(\theta_p w) = -\sum_{1 \le k \le d} \tilde{g}_k g_k \operatorname{div}(R_k \Phi_k \boldsymbol{W}_k). \tag{5-2}$$

Thanks to $\operatorname{div}(\Phi_k \boldsymbol{W}_k) = 0$, for each k,

$$\operatorname{div}(R_k \Phi_k \boldsymbol{W}_k) = \nabla R_k \cdot \mathbb{P}_{\neq 0}(\Phi_k \boldsymbol{W}_k) + \operatorname{div}(R_k \boldsymbol{e}_k)$$

such that from (5-2) we have the decomposition

$$\partial_t \theta + \operatorname{div}(\theta_p w + R) = O_1 + O_2 + O_3, \tag{5-3}$$

with

$$\begin{aligned} O_1 &:= \partial_t \mathbb{P}_{\neq 0} \theta_p, \\ O_2 &:= -\sum_{1 \le k \le d} \tilde{g}_k g_k \nabla R_k \cdot \mathbb{P}_{\neq 0}(\Phi_k \boldsymbol{W}_k), \\ O_3 &:= \partial_t \theta_o - \sum_{1 \le k \le d} \tilde{g}_k g_k \operatorname{div}(R_k \boldsymbol{e}_k) + \operatorname{div} R. \end{aligned}$$

By the definitions of R_{acc} and $\mathcal{R}$, the first term $O_1 = \operatorname{div} R_{\text{acc}}$ since Φ_k has zero mean.

For the second term O_2, by definition of $\mathcal{B}$ and (A-2), we observe that

$$\fint_{\mathbb{T}^d} R_k \operatorname{div}(\Phi_k \boldsymbol{W}_k) + \nabla R_k \cdot \Big(\Phi_k \boldsymbol{W}_k - \fint_{\mathbb{T}^d} \Phi_k \boldsymbol{W}_k \Big) = \operatorname{div} \mathcal{B}\Big(\nabla R_k, \Big(\Phi_k \boldsymbol{W}_k - \fint_{\mathbb{T}^d} \Phi_k \boldsymbol{W}_k \Big)\Big), \tag{5-4}$$

where the meaning of the vector-valued argument is that $\mathcal{B}$ is applied to each of its components. So (5-4) implies $O_2 = \operatorname{div} R_{\text{osc,x}}$.

Finally, for the last term O_3, by the definition of θ_o (3-17), (3-6), and (3-3),

$$\begin{aligned} \partial_t \theta_o &= \sigma^{-1} \sum_{1 \le k \le d} h'_k \operatorname{div}(R_k \boldsymbol{e}_k) + \sigma^{-1} \sum_{1 \le k \le d} h_k \operatorname{div}(\partial_t R_k \boldsymbol{e}_k) \\ &= (\tilde{g}_\kappa g_\kappa - 1) \sum_{1 \le k \le d} \operatorname{div}(R_k \boldsymbol{e}_k) + \sigma^{-1} \sum_{1 \le k \le d} h_k \operatorname{div}(\partial_t R_k \boldsymbol{e}_k), \end{aligned}$$

which implies $O_3 = \operatorname{div} R_{\text{osc,t}}$. □

Estimates of the new defect error. In the remainder of this section, we finish the proof of Proposition 2.1. Given $\delta > 0$, we will show that the sum of $L^1_{t,x}$ norms of each error is less than $C_R \lambda^{-\gamma}$. This concludes the proof provided λ is chosen large enough.

R_{acc} estimate. Taking advantage of the potential Ω_k as in (3-11), we obtain

$$\begin{aligned} \|R_{\text{acc}}\|_{L^1_{t,x}} &= \sigma^{-1} \|\mathcal{B}(\partial_t(\tilde{g}_k R_k), \operatorname{div} \Omega_k)\|_{L^1_{t,x}} \\ &\lesssim C_R \sigma^{-1} \|\tilde{g}_k\|_{W^{1,1}} \|\mathcal{R} \operatorname{div} \Omega_k\|_{L^1} && \text{(by Lemma A.2)} \\ &\lesssim C_R \sigma^{-1} \|\tilde{g}_k\|_{W^{1,1}} \|\Omega_k\|_{L^r} && \text{(by boundedness of } \mathcal{R} \text{ in } L^r) \\ &\lesssim C_R \kappa^\alpha \mu^{-1-(d-1)/r} && \text{(by (3-4) and (3-12a))} \\ &\lesssim C_R \lambda^{-\gamma} && \text{(by (4-3)).} \end{aligned} \tag{5-5}$$

$R_{\mathrm{osc,x}}$ *estimate.* By Hölder's inequality, Lemma A.2, and the bounds $\|\tilde{g}_k g_k\|_{L^1_t} = 1$ and $\|\Phi_k \boldsymbol{W}_k\|_{L^1} = 1$, we obtain

$$\|R_{\mathrm{osc,x}}\|_{L^1_{t,x}} \le \sum_{1\le k\le d} \|\tilde{g}_k g_k\|_{L^1} \|\mathcal{B}(\nabla R_k, \mathbb{P}_{\neq 0}(\Phi_k \boldsymbol{W}_k))\|_{L^\infty_t L^1} \lesssim C_R \sum_{1\le k\le d} \|\mathcal{R}\mathbb{P}_{\neq 0}(\Phi_k \boldsymbol{W}_k)\|_{L^1} \le C_R \lambda^{-\gamma}.$$

$R_{\mathrm{osc,t}}$ *estimate.* By (3-7),

$$\|R_{\mathrm{osc},t}\|_{L^1_{t,x}} = \left\| \sigma^{-1} \sum_{1\le k\le d} h_k \partial_t R_k \boldsymbol{e}_k \right\|_{L^1_{t,x}} \le C_R \sigma^{-1} \sum_{1\le k\le d} \|h_k\|_{L^1} \le C_R \lambda^{-\gamma}.$$

R_{lin} *estimate.* We start with Hölder's inequality

$$\|R_{\mathrm{lin}}\|_{L^1_{t,x}} \le \|\theta\|_{L^1_{t,x}} \|u\|_{L^\infty_{t,x}} + \|\rho\|_{L^\infty_{t,x}} \|w\|_{L^1_{t,x}}.$$

It suffices to show $\|\theta\|_{L^1_{t,x}} \le C_R \lambda^{-\gamma}$ and $\|w\|_{L^1_{t,x}} \le C_R \lambda^{-\gamma}$. These follow from Lemmas 4.2 and 4.3 since $p \ge 1$.

R_{cor} *estimate.* By Hölder's inequality,

$$\|R_{\mathrm{cor}}\|_{L^1_{t,x}} \le \|\theta_o\|_{L^\infty_{t,x}} \|w\|_{L^1_{t,x}}.$$

Since $\|\theta_o\|_{L^\infty_{t,x}} \le C_R \lambda^{-\gamma}$ from its definition (or by (4-4) from Lemma 4.2), by Lemma 4.3 we also have $\|R_{\mathrm{cor}}\|_{L^1_{t,x}} \le C_R \lambda^{-\gamma}$.

Conclusion of the proof of Proposition 2.1. The first point is proved in Section 3 while the second point is proved in Section 4 provided λ is sufficiently large. For the last point, (2-6) follows from the definition of the new defect error R_1, and the estimate follows from the ones in the subsection above by choosing λ sufficiently large once again. Hence Proposition 2.1 is proved.

6. Extension to transport-diffusion equations

In this section, we extend the main results to general transport-diffusion equations

$$\begin{cases} \partial_t \rho - L\rho + u \cdot \nabla \rho = 0, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{6-1}$$

where L is a given constant coefficient differential operator of order $k \in \mathbb{N}$. Weak solutions to (6-1) can be defined analogously to (1-2) by the adjoint of L, and we impose the minimum regularity $\rho \in L^1_{t,x}$ and $\rho u \in L^1_{t,x}$ as before.

The following nonuniqueness result holds for (6-1).

Theorem 6.1. *Let $d \ge 2$ and L be any constant coefficient differential operator of order $k \ge 1$. There exists a divergence-free velocity vector field $u \in L^1(0, T; W^{1,p}(\mathbb{T}^d))$ for all $p < \infty$ such that the uniqueness of (6-1) fails in the class*

$$\rho \in \bigcap_{\substack{p<\infty \\ k\in\mathbb{N}}} L^p(0, T; C^k(\mathbb{T}^d)) \quad \textit{and} \quad \rho u \in L^1(\mathbb{T}^d \times [0, T]).$$

Proof. We only need to check that Proposition 2.1 holds for (6-1). It suffices to check that the linear term $L\rho$ results in a small error, which is defined as

$$R_L := \mathcal{R} L \sum_{1\le k\le d} \tilde{g}_k R_k \Phi_k.$$

Indeed, by L^1 boundedness of $\mathcal{R}$,

$$\|R_L\|_{L^1_{t,x}} \lesssim C_R \sum_{1\le k\le d} \|\tilde{g}_k\|_{L^1} \|\Phi_k\|_{W^{k,1}},$$

where $k \ge 1$ is the order of the linear operator L. Since we only need to prove the results for p large, we can assume $k \le p$, so that, as in the proof of Lemma 4.2,

$$\|R_L\|_{L^1_{t,x}} \lesssim C_R \kappa^{\alpha-1} (\sigma\mu)^p \le C_R \lambda^{-\gamma}.$$

Hence there is no additional constraint coming from the diffusion. □

Appendix: Standard tools in convex integration

In this section, we recall several technical results that are now standard in convex integration.

Improved Hölder's inequality on $\mathbb{T}^d$. We recall the following result due to [Modena and Székelyhidi 2018, Lemma 2.1], which was inspired by [Buckmaster and Vicol 2019, Lemma 3.7].

Lemma A.1. *Let $d \ge 2$, $r \in [1,\infty]$, and $a, f : \mathbb{T}^d \to \mathbb{R}$ be smooth functions. Then, for every $\sigma \in \mathbb{N}$,*

$$\Big| \|a f(\sigma\,\cdot)\|_{L^r(\mathbb{T}^d)} - \|a\|_{L^r(\mathbb{T}^d)} \|f\|_{L^r(\mathbb{T}^d)} \Big| \lesssim \sigma^{-1/r} \|a\|_{C^1(\mathbb{T}^d)} \|f\|_{L^r(\mathbb{T}^d)}. \tag{A-1}$$

Note that the error term on the right-hand side can be made arbitrarily small by increasing the oscillation factor σ.

Antidivergence operators $\mathcal{R}$ and $\mathcal{B}$. We will use the standard antidivergence operator $\Delta^{-1}\nabla$ on $\mathbb{T}^d$, which will be denoted by $\mathcal{R}$. We write $C_0^\infty(\mathbb{T}^d)$ for the space of smooth functions with zero mean on $\mathbb{T}^d$.

It is well known that, for any $f \in C^\infty(\mathbb{T}^d)$, there exists a unique $u \in C_0^\infty(\mathbb{T}^d)$ such that

$$\Delta u = f - \fint f.$$

For any smooth scalar function $f \in C^\infty(\mathbb{T}^d)$, the standard antidivergence operator $\mathcal{R} : C^\infty(\mathbb{T}^d) \to C_0^\infty(\mathbb{T}^d, \mathbb{R}^d)$ can be defined as

$$\mathcal{R} f := \Delta^{-1} \nabla f,$$

which satisfies

$$\operatorname{div}(\mathcal{R} f) = f - \fint_{\mathbb{T}^d} f \quad \text{for all } f \in C^\infty(\mathbb{T}^d).$$

It is well known (see for instance [Modena and Székelyhidi 2018, Lemma 2.2]) that $\mathcal{R}$ is bounded on Sobolev spaces $W^{k,p}(\mathbb{T}^d)$ for all $k \in \mathbb{N}$ and that $\mathcal{R}\operatorname{div}$ is a Calderón–Zygmund operator:

$$\|\mathcal{R}(\operatorname{div} u)\|_{L^r(\mathbb{T}^d)} \lesssim \|u\|_{L^r(\mathbb{T}^d)} \quad \text{for all } u \in C^\infty(\mathbb{T}^d, \mathbb{R}^d) \text{ and } 1 < r < \infty.$$

Recall the following useful fact about $\mathcal{R}$:

$$\mathcal{R}f(\sigma\,\cdot) = \sigma^{-1}\mathcal{R}f \quad \text{for any } f \in C_0^\infty(\mathbb{T}^d) \text{ and any positive } \sigma \in \mathbb{N}.$$

We will also use its bilinear counterpart $\mathcal{B} : C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d, \mathbb{R}^d)$, defined by

$$\mathcal{B}(a, f) := a\mathcal{R}f - \mathcal{R}(\nabla a \cdot \mathcal{R}f).$$

It is easy to see that $\mathcal{B}$ is a left-inverse of the divergence:

$$\operatorname{div}(\mathcal{B}(a, f)) = af - \fint_{\mathbb{T}^d} af\,dx \quad \text{provided that } f \in C_0^\infty(\mathbb{T}^d), \tag{A-2}$$

which can be proved easily using integration by parts. The following estimate is a direct consequence of the boundedness of $\mathcal{R}$ on Sobolev spaces $W^{k,p}(\mathbb{T}^d)$.

Lemma A.2. *Let $d \geq 2$ and $1 \leq r \leq \infty$. Then, for any $a, f \in C^\infty(\mathbb{T}^d)$,*

$$\|\mathcal{B}(a, f)\|_{L^r(\mathbb{T}^d)} \lesssim \|a\|_{C^1(\mathbb{T}^d)} \|\mathcal{R}f\|_{L^r(\mathbb{T}^d)}.$$

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