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Enumerating numerical sets associated to a numerical semigroup

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ABSTRACT

A numerical set T is a subset of $\mathbb{N}_0$ that contains 0 and has finite complement. The atom monoid of T is the set of $x \in \mathbb{N}_0$ such that $x + T \subseteq T$. Marzuola and Miller introduced the anti-atom problem: how many numerical sets have a given atom monoid? This is equivalent to asking for the number of integer partitions with a given set of hook lengths. We introduce the void poset of a numerical semigroup S and show that numerical sets with atom monoid S are in bijection with certain order ideals of this poset. We use this characterization to answer the anti-atom problem when S has small type.

1. Introduction

1.1. The anti-atom problem

A *numerical set* T is a subset of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ that contains 0 and has a finite complement. The elements of $\mathbb{N}_0 \setminus T$ are called *gaps*. The set of gaps of T is denoted by $\mathcal{H}(T)$ and the number of gaps is the *genus* of T, denoted by $g(T)$. The largest gap is the *Frobenius number* of T, denoted by $F(T)$. A numerical set S that is closed under addition is a *numerical semigroup*. We say that $n_1, \ldots, n_t \in S$ is a *set of generators* of S if S is the set of all linear combinations of these elements with nonnegative integer coefficients. That is,

$$S = \langle n_1, \ldots, n_t \rangle = \{a_1 n_1 + \cdots + a_t n_t \mid a_1, \ldots, a_t \in \mathbb{N}_0\}.$$

A numerical semigroup S has a unique minimal set of generators and the cardinality of this set is the *embedding dimension* of S, denoted by $e(S)$. The smallest nonzero element of S is called its *multiplicity*, denoted by $m(S)$.

The set of *pseudo-Frobenius* numbers of a numerical semigroup S is defined by

$$PF(S) = \{P \in \mathcal{H}(S) \mid P + S \setminus \{0\} \subseteq S\}.$$

Clearly $F(S)$ is one of the pseudo-Frobenius numbers of S. The number of pseudo-Frobenius numbers of S is the *type* of S, denoted by $t(S)$.

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Antokoletz and Miller defined the *atom monoid* of a numerical set T in [1] as

$$A(T) = \{x \in \mathbb{N}_0 \mid x + T \subseteq T\}.$$

It is easy to see that $A(T)$ is always a numerical semigroup contained in T. It is also referred to as the *associated semigroup* of T. Marzuola and Miller raised the *Anti-Atom problem* [11].

Problem 1 (*Anti-Atom Problem*)**.** Let S be a numerical semigroup. How many numerical sets T have $A(T) = S$?

This number is denoted by $P(S)$. The numerical sets T for which $A(T) = S$ are called *numerical sets associated to* S. In this paper we focus on computing $P(S)$.

Motivation for studying $P(S)$ comes from the theory of integer partitions. A partition λ has an associated multiset of hook lengths, denoted $\mathbb{H}(\lambda)$. Let $H(\lambda)$ denote the underlying set of hook lengths of λ. For detailed definitions and for a bijection between numerical sets and integer partitions, see [3, Section 2]. Keith and Nath show that $H(\lambda)$ is the set of gaps of a numerical semigroup S and that numerical sets T associated to S are in bijection with partitions λ with $H(\lambda) = \mathcal{H}(S)$ [10, Corollary 1]. (See also [3, Proposition 3].) Therefore, Problem 1 is equivalent to the following.

Problem 2. Let S be a numerical semigroup. How many partitions λ have $H(\lambda) = \mathcal{H}(S)$?

This correspondence between partitions and numerical semigroups has been studied in several recent papers [3,7–9,14].

1.2. Previous results on $P(S)$

It is easy to see that T and $A(T)$ have the same Frobenius number. Since there are 2^{F-1} numerical sets with Frobenius number F, it follows that

$$\sum_{S\,:\,F(S)=F} P(S) = 2^{F-1}.$$

Marzuola and Miller consider numerical semigroups of the form $N_F = \{0, F + 1 \rightarrow\}$, where the $\rightarrow$ indicates that all positive integers greater than $F + 1$ are in N_F [11]. They prove that there is a positive constant $\gamma \approx 0.4844$ such that

$$\lim_{F\to\infty} \frac{P(N_F)}{2^{F-1}} = \gamma.$$

This means that nearly half of numerical sets T with Frobenius number F have $A(T) = N_F$. In [13], Singhal and Lin consider similar families of numerical semigroups and prove a conjecture from [2]. Let D be a finite set of positive integers, take $F > 2\max(D)$ and define

$$N(D, F) = \{0\} \cup \{F - l \mid l \in D\} \cup \{F + 1 \rightarrow\}.$$

They show that for each D, there is a positive constant γ_D such that

$$\lim_{F\to\infty} \frac{P(N(D, F))}{2^{F-1}} = \gamma_D.$$

Antokoletz and Miller define the *dual* of a numerical set T as

$$T^* := \{x \in \mathbb{Z} \mid F(T) - x \notin T\}$$

and show that $A(T) = A(T^*)$ [1]. Constantin, Houston-Edwards, and Kaplan interpret this construction in terms of partitions. They show that if T corresponds to the partition λ, then T^* corresponds to the conjugate partition $\tilde{\lambda}$. Since $\mathbb{H}(\lambda) = \mathbb{H}(\tilde{\lambda})$, we see that $\mathcal{H}(A(T)) = \mathcal{H}(A(T^*))$, which implies $A(T) = A(T^*)$ [3, Proposition 2].

Theorem 1.1 (*[11, Proposition 1]*)**.** *Let S be a numerical semigroup. If T is a numerical set with $A(T) = S$, then $S \subseteq T \subseteq S^*$. Moreover, $A(S) = A(S^*) = S$.*

Constantin, Houston-Edwards, and Kaplan define a *missing pair* of S to be a pair of gaps of S that sums to $F(S)$ [3, Section 7]. The set

$$M(S) := \{a : a \notin S,\ F(S) - a \notin S\}$$

of gaps in missing pairs is the *void* of S. They show that $S^* = S \cup M(S)$ [3, Lemma 3]. It is easy to check that $|M(S)| = 2g(S) - F(S) - 1$.

Theorem 1.1 implies that $P(S) = 1$ if and only if $M(S) = \emptyset$. A numerical semigroup S for which $x \in S$ if and only if $F(S) - x \notin S$ is called *symmetric*. Therefore, $P(S) = 1$ if and only if S is symmetric [11, Corollary 2]. Fröberg, Gottlieb, and Häggkvist prove that S is symmetric if and only if $t(S) = 1$ [5, Proposition 2].

If S is symmetric, then $F(S)$ is odd. There is a corresponding family of numerical semigroups with even Frobenius numbers. A numerical semigroup S is called *pseudo-symmetric* if $F(S)$ is even and $M(S) = \left\{\frac{1}{2}F(S)\right\}$. If S is pseudo-symmetric, then $t(S) = 2$ [5]. Theorem 1.1 implies that if S is pseudo-symmetric, then $P(S) = 2$ [11, Corollary 2].

These two families could lead one to guess a close relationship between the size of the void $|M(S)|$ and $P(S)$. However this relationship is subtle. A numerical semigroup with $P(S) = 2$ can have arbitrarily large $|M(S)|$ [3, Proposition 16]. We do have an upper bound on $P(S)$ obtained from Theorem 1.1.

Corollary 1.2 (*[3, Corollary 3]*). *For any numerical semigroup S,*

$$P(S) \leq 2^{|M(S)|} = 2^{2g(S)-F(S)-1}.$$

1.3. The void poset and our main results

We define a partial ordering on the void of a numerical semigroup S. Given $x, y \in M(S)$, we say $x \preccurlyeq y$ when $y - x \in S$. This poset is called the *void poset* of S and is denoted by $(\mathcal{M}(S), \preccurlyeq)$. In Proposition 2.2, we show that the maximal elements of $(\mathcal{M}(S), \preccurlyeq)$ are precisely the pseudo-Frobenius numbers of S other than $F(S)$.

It is known that every numerical semigroup S satisfies $t(S) \leq 2g(S) - F(S)$ [12, Proposition 2.2]. If equality holds, then S is called *almost symmetric*. Both symmetric and pseudo-symmetric numerical semigroups are almost symmetric. In Proposition 2.9, we show that almost symmetric numerical semigroups are those for which the void poset has no nontrivial relations.

A subset $I \subseteq M(S)$ is an *order ideal* if for any $x, y \in M(S)$ satisfying $x \preccurlyeq y$ and $x \in I$, we have $y \in I$. If $A(T) = S$, then $T \setminus S \subseteq M(S)$. For an arbitrary subset $I \subseteq M(S)$, $I \cup S$ is not necessarily a numerical set associated to S. In Proposition 2.4, we show that the $I \subseteq M(S)$ for which $S \subseteq A(I \cup S)$ are precisely the order ideals of $(\mathcal{M}(S), \preccurlyeq)$.

In Section 3, we introduce the notion of *Frobenius triangles* of S and use this concept to characterize the order ideals I of $(\mathcal{M}(S), \preccurlyeq)$ for which $T = I \cup S$ is a numerical set associated to S (Theorem 3.9). This is one of the main results of this paper, as it yields an algorithm for computing $P(S)$ (Algorithm 5.1). We also define the *pseudo-Frobenius graph* of a numerical semigroup S and use it to give a lower bound for $P(S)$ in Corollary 3.7.

We use these tools to analyze $P(S)$ for numerical semigroups of small type. As mentioned above, $t(S) = 1$ if and only if $P(S) = 1$. In Theorem 2.5 we show that if $t(S) = 2$, then $P(S) = 2$. In Section 6 we solve the much more difficult problem of characterizing $P(S)$ for numerical semigroups of type 3. In Theorem 6.5 we prove that if $t(S) = 3$, then $P(S) \in \{2, 3, 4\}$, and we characterize in terms of the pseudo-Frobenius numbers of S when each value occurs.

The relationship between $t(S)$ and $P(S)$ is not as straightforward for numerical semigroups of larger type. In Proposition 7.1, we show that a numerical semigroup of type 4 can have $P(S)$ arbitrarily large. In Proposition 8.1 we show that given $t \geq 2$, there is a numerical semigroup with $t(S) = t$ and $P(S) = 2$.

A numerical semigroup S is said to be of *maximal embedding dimension* if $e(S) = m(S)$. It is known that $t(S) \leq m(S) - 1$ [6, Corollary1.23], and that S has maximal embedding dimension if and only if $t(S) = m(S) - 1$ [6, Corollary 2.2]. Therefore, semigroups of maximal embedding dimension have a natural characterization in terms of their type. In Section 8, we compute $P(S)$ for a certain class of these semigroups.

2. The void poset

Recall that the void poset $(\mathcal{M}(S), \preccurlyeq)$ of S is defined by $x \preccurlyeq y$ if and only if $y - x \in S$. For $x, y \in M(S)$, we write $x \prec y$ if $y - x \in S \setminus \{0\}$. In this section we use the structure of this poset to classify the numerical sets T for which $S \subseteq A(T)$. In the next section we determine when such a numerical set also satisfies $A(T) \subseteq S$, thus classifying the numerical sets associated to S.

A poset $(\mathcal{P}, \preccurlyeq)$ is *self-dual* if there exists a bijection $\phi : \mathcal{P} \to \mathcal{P}$ such that $a \preccurlyeq b$ if and only if $\phi(b) \preccurlyeq \phi(a)$.

Lemma 2.1. *For any numerical semigroup S, $(\mathcal{M}(S), \preccurlyeq)$ is self-dual.*

Proof. Define $\phi : M(S) \to M(S)$ by $\phi(x) = F(S) - x$. By the definition of $M(S)$, ϕ is well-defined. The result follows by noting that $y - x = \phi(x) - \phi(y)$. □

We refer to this map as *conjugation* on $(\mathcal{M}(S), \preccurlyeq)$. For $x \in M(S)$, define $\overline{x} = F(S) - x$.

Proposition 2.2. *The set of maximal elements of $(\mathcal{M}(S), \preccurlyeq)$ is*

$$\max(\mathcal{M}(S), \preccurlyeq) = PF(S) \setminus \{F(S)\}.$$

Proof. Let P be a maximal element of $(\mathcal{M}(S), \preccurlyeq)$ and let $F = F(S)$. Given $s \in S \setminus \{0\}$, we know that $P + s \notin M(S)$, since otherwise $P \prec P + s$. Thus, either $P + s \in S$ or $P + s \in \mathcal{H}(S) \setminus M(S)$. In the latter case, $F - P - s \in S$, but this implies $F - P = (F - P - s) + s \in S$, which contradicts the fact that $P \in M(S)$. We conclude that $P + s \in S$ and so $P \in PF(S)$. Moreover, since $P \in M(S)$ we know that $P \neq F$.

For the other direction, suppose $P \in PF(S) \setminus \{F\}$. If $F - P \in S$, then by the definition of $PF(S)$ we see that $F = P + (F - P) \in S$, which is a contradiction. Therefore, $F - P \notin S$ and $P \in M(S)$. Next, if there is some $x \in M(S)$ such that $P \preccurlyeq x$, then $x - P \in S$ and $x \notin S$. Since $P \in PF(S)$, the only way this could happen is if $x - P = 0$, that is, $x = P$. We conclude that P is a maximal element of $(\mathcal{M}(S), \preccurlyeq)$. $\square$

Since $(\mathcal{M}(S), \preccurlyeq)$ is self-dual it follows that its minimal elements are

$$\min(\mathcal{M}(S), \preccurlyeq) = \left\{\overline{P} \mid P \in PF(S) \setminus \{F(S)\}\right\}.$$

Example 2.3. The void posets of $S_1 = \{0, 4, 8, 10 \rightarrow\}$ and $S_2 = \langle 6, 25, 29\rangle$ are as follows:

Recall that an order ideal of a poset is a subset I such that if $x \in I$ and $x \preccurlyeq y$, then $y \in I$.

Proposition 2.4. *Let S be a numerical semigroup with $M(S) = M$ and $I \subseteq M$. We have $S \subseteq A(I \cup S)$ if and only if I is an order ideal of $(\mathcal{M}(S), \preccurlyeq)$.*

Proof. First suppose that I is an order ideal. We show that for each $s \in S$, $s + I \subseteq S \cup I$. Suppose $x \in I$.

- Case 1: If $s + x \in S$, there is nothing else to check.
- Case 2: If $s + x \in \mathcal{H}(S) \setminus M(S)$, then $F - s - x \in S$. Therefore,

  $$F - x = (F - s - x) + s \in S.$$

  This contradicts the fact that $x \in M(S)$, so this case does not occur.
- Case 3: If $s + x \in M(S)$, then $x \preccurlyeq s + x$ in $(\mathcal{M}(S), \preccurlyeq)$, so $s + x \in I$.

Conversely, suppose $S \subseteq A(I \cup S)$. Consider $x, y \in M(S)$ with $x \in I$ and $x \preccurlyeq y$. Then $y - x \in S$ and therefore $y - x \in A(S \cup I)$. That is,

$$(y - x) + (S \cup I) \subseteq (S \cup I).$$

In particular this implies that $y = (y - x) + x \in S \cup I$. Since $y \in M(S)$, we know that $y \notin S$ and therefore, $y \in I$. We conclude that I is an order ideal of $(\mathcal{M}(S), \preccurlyeq)$. $\square$

Theorem 2.5. *Let S be a numerical semigroup of type 2. We have $P(S) = 2$.*

Proof. Let T be a numerical set associated to S. By Proposition 2.4, $T = S \cup I$ for some order ideal I of $(\mathcal{M}(S), \preccurlyeq)$. We prove that either $I = M(S)$ and $T = S \cup M(S)$ or $I = \emptyset$ and $T = S$.

Suppose $PF(S) = \{P, F\}$ with $P < F$. We have $PF(S) \setminus \{F\} = \{P\}$, which means that P is the unique maximal element of $(\mathcal{M}(S), \preccurlyeq)$. This also implies that $P = \max(M(S))$. Moreover, since $(\mathcal{M}(S), \preccurlyeq)$ is self-dual we also see that it has a unique minimal element $\overline{P} = F - P$. We prove that if $I \neq \emptyset$, then $\overline{P} \in I$, which implies that $I = M(S)$.

Since $t(S) \neq 1$, we know that $P(S) \geq 2$. Suppose $T \neq S$. We know that $I = T \setminus S$ is a nonempty order ideal of $(\mathcal{M}(S), \preccurlyeq)$. Since $(\mathcal{M}(S), \preccurlyeq)$ has a unique maximal element, we must have $P \in I$. Since $P \notin A(T)$, there is some $x \in T$ for which $P + x \notin T$. Since $S = A(T)$, we know that $x \notin S$. This means that $x \in I \subseteq M(S)$. We have seen that $F - P$ is the unique minimal element of $(\mathcal{M}(S), \preccurlyeq)$, so $F - P \leq x$.

Now if $F - P < x$, then $F < P + x$, which would contradict the fact that $P + x \notin T$. Therefore $F - P = x$ and hence $F - P \in I$. Finally, since the unique minimal element of $(\mathcal{M}(S), \preccurlyeq)$ is in I, we conclude that $I = M(S)$. $\square$

Example 2.6. Consider $S = \langle 19, 21, 24\rangle$, which has $t(S) = 2$ and $PF(S) = \{98, 113\}$. Note that $(\mathcal{M}(S), \preccurlyeq)$ has a unique maximal element and a unique minimal element. If $I \cup S$ is a numerical set associated to S, then either $I = \emptyset$ or $I = M(S)$ (see Fig. 1).

Proposition 2.7. *Let S be a numerical semigroup.*

(1) We have $M(S) = \emptyset$ if and only if S is symmetric.
(2) We have $|M(S)| = 1$ if and only if S is pseudo-symmetric.

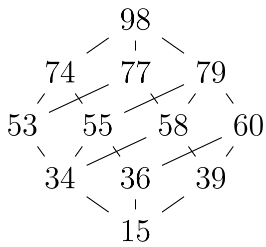


Fig. 1. Void poset of $S = \langle 19, 21, 24\rangle$.

Proof. Recall that $|M(S)| = 2g(S) - F(S) - 1$. A numerical semigroup is symmetric if and only if $F(S) = 2g(S) - 1$ [5, Lemma 1]. This occurs if and only if $M(S) = \emptyset$. A numerical semigroup is pseudo-symmetric if and only if $F(S) = 2g(S) - 2$ [5, Lemma 3]. This happens if and only if $|M(S)| = 1$. □

Proposition 2.8 (*[12, Proposition 2.2]*)**.** *We have*

$$t(S) \leq 2g(S) - F(S) = |M(S)| + 1.$$

Proof. We know that $PF(S) \setminus \{F\} \subseteq M(S)$. Therefore, $t(S) - 1 \leq 2g(S) - F(S) - 1$. □

Recall that S is almost symmetric if and only if $t(S) = 2g(S) - F(S)$.

Proposition 2.9. *A numerical semigroup S is* almost symmetric *if and only if $(\mathcal{M}(S), \preccurlyeq)$ has no nontrivial relations, that is, $x \preccurlyeq y$ implies $x = y$.*

Proof. Since $|PF(S) \setminus \{F(S)\}| = t(S) - 1$ and $PF(S) \setminus \{F(S)\} \subseteq M(S)$, we see that S is almost symmetric if and only if $PF(S) \setminus \{F\} = M(S)$. Proposition 2.2 says that $PF(S) \setminus \{F(S)\}$ is the set of maximal elements of $(\mathcal{M}(S), \preccurlyeq)$. Thus, S is almost symmetric if and only if all elements of $(\mathcal{M}(S), \preccurlyeq)$ are maximal. This is equivalent to $(\mathcal{M}(S), \preccurlyeq)$ having no nontrivial relations. □

We end this section with a quick observation about the parity of $P(S)$. Theorem 1.1 and the concept of the dual of a numerical set leads directly to the following result.

Proposition 2.10. *If $F(S)$ is even, then $P(S)$ is even.*

Proof. Given a numerical set T with $F(T) = F(S)$, note that $\frac{F(S)}{2} \in T$ if and only if $\frac{F(S)}{2} \notin T^*$. This means $T \neq T^*$. Therefore, we can divide the numerical sets associated to S into pairs. □

3. Classifying associated numerical sets

We have seen that if T is a numerical set associated to S, it is necessary that $T \setminus S \subseteq M(S)$ and that $T \setminus S$ is an order ideal of $(\mathcal{M}(S), \preccurlyeq)$. In this section we study a class of order ideals that always give rise to numerical sets associated to S. This gives a lower bound for $P(S)$. We also obtain a complete characterization of the order ideals that lead to numerical sets associated to S.

Definition 3.1. We say a triple $(P, x, y) \in PF(S) \times M(S)^2$ satisfying $P + x + y = F(S)$ is a *Frobenius triangle* of S, and let

$$Tr(S) = \{(P, x, y) \mid P \in PF(S),\ x, y \in M(S),\ P + x + y = F(S)\}$$

denote the set of Frobenius triangles of S. Given an order ideal $I \subseteq M(S)$ and a Frobenius triangle (P, x, y), we say that *I satisfies the Frobenius triangle* (P, x, y) if $P, x \in I$ and $F(S) - y \notin I$. The *pseudo-Frobenius graph* of S, denoted by $GPF(S)$, is the graph with vertices $PF(S) \setminus \{F(S)\}$ that has an edge between P, Q if and only if $P + Q - F(S) \in S$. We denote the number of connected components of $GPF(S)$ by $\kappa(S)$.

An order ideal I of $(\mathcal{M}(S), \preccurlyeq)$ is self-dual if $x \in I$ implies that $\bar{x} \in I$.

Proposition 3.2. *If S is a numerical semigroup and I is a self-dual order ideal of $(\mathcal{M}(S), \preccurlyeq)$, then $I \cup S$ is a numerical set associated to S.*

Proof. By Proposition 2.4, $S \subseteq A(I \cup S)$. Since I is self-dual, $x \in I$ implies $F(S) - x \in I$. However,

$$x + (F(S) - x) = F(S) \notin I \cup S.$$

So $x + (I \cup S) \not\subseteq (I \cup S)$ and hence $x \notin A(I \cup S)$. We conclude that $A(I \cup S) = S$. □

Since self-dual order ideals give numerical sets associated to S, the number of self-dual order ideals of $(\mathcal{M}(S), \preccurlyeq)$ is a lower bound for $P(S)$. We now give a simpler description of self-dual order ideals.

Lemma 3.3. *Let I be an order ideal of $(\mathcal{M}(S), \preccurlyeq)$. If for each $P \in PF(S) \cap I$ we have $\overline{P} \in I$, then I is self-dual.*

Proof. Consider $x \in I$. Pick a maximal element above x, say $P \in PF(S) \setminus \{F(S)\}$ satisfies $x \preccurlyeq P$. Since I is an order ideal, we have $P \in I$. We are given that $\overline{P} \in I$. Since $x \preccurlyeq P$, we see that $\overline{P} \preccurlyeq \overline{x}$. This implies $\overline{x} \in I$. We conclude that I is self-dual. □

Lemma 3.4. *Let I be a self-dual order ideal of $(\mathcal{M}(S), \preccurlyeq)$. If $x \in I,\ y \in M(S)$, and $y \preccurlyeq x$, then $y \in I$.*

Proof. Since I is self-dual, we have $\overline{x} \in I$. Next, since $\overline{x} \preccurlyeq \overline{y}$ and I is an order ideal, we have $\overline{y} \in I$. Finally, since I is self-dual, we conclude that $y \in I$. □

Lemma 3.5. *If I_1, I_2 are self-dual order ideals of $(\mathcal{M}(S), \preccurlyeq)$ and*

$$I_1 \cap PF(S) = I_2 \cap PF(S),$$

then $I_1 = I_2$.

Proof. Given $x \in I_1$, pick a maximal element P above it, say $P \in PF(S) \setminus \{F(S)\}$ satisfies $x \preccurlyeq P$. Since I_1 is an order ideal, $P \in I_1$. Therefore P is in $I_1 \cap PF(S)$ and so $P \in I_2$ also. By Lemma 3.4, $x \in I_2$. This shows that $I_1 \subseteq I_2$. By symmetry $I_1 = I_2$. □

Recall that the pseudo-Frobenius graph of S has vertex set $PF(S) \setminus \{F(S)\}$ and has an edge between P and Q when $P + Q - F(S) \in S$. Note that there is an edge between P and Q if and only if $\overline{P} \preccurlyeq Q$. This is equivalent to $\overline{Q} \preccurlyeq P$. Also note that $GPF(S)$ may possibly have loops. See Example 3.8.

Theorem 3.6. *Let S be a numerical semigroup. If I is a self-dual order ideal of $(\mathcal{M}(S), \preccurlyeq)$ then $I \cap PF(S)$ is a union of connected components of $GPF(S)$.*

Conversely, for any union of connected components of $GPF(S)$ there is a unique self-dual order ideal that contains precisely this set of pseudo-Frobenius numbers.

Proof. Suppose $C_1, \dots, C_\kappa$ are the connected components of $GPF(S)$. Let I be a self-dual order ideal of $(\mathcal{M}(S), \preccurlyeq)$. Suppose $P \in I \cap PF(S)$ and $P, Q \in C_i$ for some i. There is a path $P = P_0, P_1, P_2, \dots, P_{n-1}, P_n = Q$ in $GPF(S)$ from P to Q. If $P_i \in I$, then since I is self-dual, we have $\overline{P_i} \in I$. Moreover, we know that $\overline{P_i} \preccurlyeq P_{i+1}$ since there is an edge between P_i and P_{i+1} in $GPF(S)$. This implies $P_{i+1} \in I$. By induction, we conclude that $Q \in I$. We have shown that if $I \cap C_i \neq \emptyset$, then $C_i \subseteq I$. This means that $I \cap PF(S)$ is a union of connected components of $GPF(S)$.

Conversely, let J be a subset of $\{1, 2, \dots, \kappa\}$. Let

$$I = \{x \in M(S) \mid \exists\, i \in J \text{ and } \ P \in C_i \text{ with } x \preccurlyeq P\}.$$

It is clear that $I \cap PF(S) = \bigcup_{i \in J} C_i$. We now show that I is an order ideal. Suppose $a \in I$, $b \in M(S)$ and $a \preccurlyeq b$. Since $a \in I$, we know that $\exists\, i \in J$, $P \in C_i$ such that $a \preccurlyeq P$. Also consider a minimal element below a. Such an element is of the form $\overline{Q}$ where $Q \in PF(S)$. Since $\overline{Q} \preccurlyeq a \preccurlyeq P$, there is an edge between Q and P in $GPF(S)$. In particular, $Q \in C_i$. Next, consider a maximal element above b, say $R \in PF(S) \setminus \{F(S)\}$ satisfies $b \preccurlyeq R$. Note that $\overline{Q} \preccurlyeq R$, which implies that there is an edge between Q and R in $GPF(S)$. Therefore, $R \in C_i$, and since $b \preccurlyeq R$ we conclude that $b \in I$ and I is an order ideal.

We now show that I is self-dual. Let a, P, Q be as above. Since $\overline{Q} \preccurlyeq a \preccurlyeq P$ we have $\overline{P} \preccurlyeq \overline{a} \preccurlyeq Q$. Since $Q \in C_i$ it follows that $\overline{a} \in I$. We conclude that I is a self-dual order ideal with

$$I \cap PF(S) = \bigcup_{i \in J} C_i.$$

Lemma 3.5 implies that I is the unique self-dual order ideal with this set of pseudo-Frobenius numbers. □

Corollary 3.7. *We have $P(S) \geq 2^{\kappa(S)}$.*

Example 3.8. Consider $S = \{0, 5 \rightarrow\}$, so $F(S) = 4$ and $M(S) = \{1, 2, 3\}$. The poset $(\mathcal{M}(S), \preccurlyeq)$ has no nontrivial relations. Therefore $PF(S) = \{1, 2, 3, 4\}$ and S is almost symmetric. See Fig. 2 for the poset $(\mathcal{M}(S), \preccurlyeq)$ and the graph $GPF(S)$. The graph $GPF(S)$ has 2 connected components. The self-dual order ideals of $(\mathcal{M}(S), \preccurlyeq)$ give 4 numerical sets associated to S: $T_1 = S$, $T_2 = \{1, 3\} \cup S$, $T_3 = \{2\} \cup S$ and $T_4 = \{1, 2, 3\} \cup S = S^*$. There are two more numerical sets associated to S, which come from order ideals that are not self-dual: $T_5 = \{1\} \cup S$, $T_6 = \{1, 2\} \cup S$. We have $P(S) = 6$.

Next we give a complete characterization of the order ideals that lead to numerical sets associated to S. Given $P \in PF(S) \setminus \{F(S)\}$, define

$$Tr_P(S) = \{(P, x, y) \mid x, y \in M(S) \text{ with } P + x + y = F(S)\} \subset Tr(S).$$

(a) Void poset (b) GPF graph

Fig. 2. Void poset and *GPF* graph of $\{0, 5 \rightarrow\}$.

Note that given $P \in PF(S) \setminus \{F(S)\}$ and $x, y \in M(S)$, (P, x, y) is a Frobenius triangle if and only if $P + x = \bar{y}$. Recall that given an order ideal I and a Frobenius triangle (P, x, y), I satisfies the Frobenius triangle (P, x, y) if $P, x \in I$ and $\bar{y} \notin I$.

Theorem 3.9. *Let S be a numerical semigroup, $I \subseteq M(S)$, and $T = I \cup S$. Then T is a numerical set associated to S if and only if*

(1) I is an order ideal of $(\mathcal{M}(S), \preccurlyeq)$, and
(2) for each $P \in I \cap PF(S)$, one of the following conditions is satisfied: $2P \notin S$, $F(S) - P \in I$, or there is a Frobenius triangle (P, x, y) that is satisfied by I.

Moreover, if $I \cup S$ is a numerical set associated to S, then for all $P \in I \cap PF(S)$ either $\overline{P} \in I$ or there is a Frobenius triangle (P, x, y) that is satisfied by I.

Proof. If T is a numerical set associated to S, then Proposition 2.4 implies that I is an order ideal of $(\mathcal{M}(S), \preccurlyeq)$. Suppose $P \in I \cap PF(S)$. We know that $P \notin A(T) = S$, that is, $P + T \nsubseteq T$. This means that there is some $x \in T$ for which $P + x \notin T$. Since $S = A(T)$, we know that $x \notin S$ and hence $x \in I$.

- Case 1: Suppose $P + x \in M(S) \setminus T$. Then $\left(P, x, \overline{P + x}\right)$ is a Frobenius triangle and I satisfies it.
- Case 2: Suppose $P + x \in \mathcal{H}(S) \setminus M(S)$. This means that $F - P - x \in S$. Now suppose $F - P - x \neq 0$. Then since $P \in PF(S)$, we see that

$$F - x = P + (F - P - x) \in S.$$

This contradicts the fact that $x \in I \subseteq M(S)$. Therefore $F - P - x = 0$, that is, $\overline{P} = x \in I$.

Next we prove the other direction. Suppose that I satisfies (1) and (2) in the theorem statement. By Proposition 2.4, $S \subseteq A(T)$. Assume for the sake of contradiction that $S \subsetneq A(T)$. Let $P = \max(A(T) \setminus S)$. Since $F(S) \notin T$, we know that $P \neq F(S)$. Given $s \in S \setminus \{0\}$, we know that $P + s \in A(T)$ as $A(T)$ is closed under addition. Since $P = \max(A(T) \setminus S)$ it follows that $P + s \in S$. This means that $P \in I \cap PF(S)$. Similarly $2P \in A(T)$, and since $2P > P$, we conclude that $2P \in S$. By (2) we know that either $\overline{P} \in I$ or there is some Frobenius triangle that I satisfies.

- Case 1: Suppose $\overline{P} \in I$. Then $P + \overline{P} = F \notin T$. This contradicts the fact that $P \in A(T)$.
- Case 2: Suppose that I satisfies the Frobenius triangle (P, x, y). This means that $x \in I$ and $P + x = \bar{y} \notin I$. This again contradicts the fact that $P \in A(T)$.

We get a contradiction in both cases. Therefore, $S = A(T)$, that is, T is a numerical set associated to S. □

A numerical semigroup S is called *P-minimal* if $P(S) = 2^{\kappa(S)}$.

Corollary 3.10. *If S is a numerical semigroup for which $Tr(S) = \emptyset$, then S is P-minimal.*

Note that the converse of Corollary 3.10 is not true. For example, for $S = \langle 8, 12, 13, 23, 30\rangle$ we have $\kappa(S) = 1$ and $P(S) = 2$, so S is P-minimal. However, $(17, 5, 5) \in Tr(S)$, so $Tr(S) \neq \emptyset$.

4. Structure among frobenius triangles

This section consists of some technical lemmas concerning Frobenius triangles. These will be useful in computing $P(S)$ for certain classes of numerical semigroups. If $P, Q \in PF(S) \setminus \{F(S)\}$ and $P - Q \in M(S)$, then $(Q, P - Q, F(S) - P)$ is a Frobenius triangle in $Tr_Q(S)$. Our main result in this section is the following.

Proposition 4.1. *Let $Q \in PF(S) \setminus \{F(S)\}$. We have $Tr_Q(S) \neq \emptyset$ if and only if $\exists\, P \in PF(S) \setminus \{F(S)\}$ such that $P - Q \in M(S)$.*

Before proving this statement we need some preliminary results.

Lemma 4.2. *Let $P = \max_{<}(M(S))$. We have $P \in PF(S) \setminus \{F(S)\}$ and $Tr_P(S) = \emptyset$.*

Proof. If $P = \max_{<}(M(S))$, then P is a maximal element of $(\mathcal{M}(S), \preccurlyeq)$. Therefore, $P \in PF(S) \setminus \{F(S)\}$. Moreover, $\min_{<}(M(S)) = F(S) - P$, so $Tr_P(S) = \emptyset$. □

Lemma 4.3. *Suppose $Q \in PF(S) \setminus \{F(S)\}$ and $(Q, x, y) \in Tr_Q(S)$.*

(1) If $a \in M(S)$ satisfies $a \prec x$, then $a \preccurlyeq \overline{y}$.
(2) If $b \in M(S)$ satisfies $\overline{y} \prec b$, then $x \preccurlyeq b$.

Proof. We know that $Q + x + y = F(S)$, that is, $\overline{y} - x = Q$. If $x - a \in S \setminus \{0\}$, we have

$$\overline{y} - a = (\overline{y} - x) + (x - a) = Q + (x - a) \in S,$$

since $Q \in PF(S)$. If $b - \overline{y} \in S \setminus \{0\}$, we have

$$b - x = (b - \overline{y}) + (\overline{y} - x) = Q + (b - \overline{y}) \in S. \quad \square$$

Corollary 4.4. *If $P, Q \in PF(S) \setminus \{F(S)\}$, $a \in M(S)$, and $P - Q \in M(S)$, then $a \prec P - Q$ implies $a \prec P$.*

Proof. Note that $(Q, P - Q, F(S) - P) \in Tr_Q(S)$. □

Lemma 4.5. *Suppose I is an order ideal of $(\mathcal{M}(S), \preccurlyeq)$ and $(Q, x, y) \in Tr_Q(S)$ is satisfied by I. Then x is a minimal element of I and $\overline{y}$ is a maximal element of $M(S) \setminus I$.*

Proof. By assumption, $Q, x \in I$ and $\overline{y} \notin I$. If $a \in M(S)$ satisfies $a \prec x$, then Lemma 4.3 implies that $a \preccurlyeq \overline{y}$. Therefore, $a \notin I$. If $b \in M(S)$ satisfies $\overline{y} \prec b$, then Lemma 4.3 implies that $x \preccurlyeq b$. Therefore, $b \in I$. □

Lemma 4.6. *Suppose $Q \in PF(S) \setminus \{F(S)\}$ and $(Q, x, y) \in Tr_Q(S)$. If $\overline{P} \preccurlyeq y$ for some $P \in PF(S) \setminus \{F(S)\}$, then $P - Q \in M(S)$ and $x \preccurlyeq P - Q$.*

Proof. We want to show that $P - Q \in M(S)$. If this is not the case then either $P - Q \in S$ or $F(S) - (P - Q) \in S$.

- Suppose $P - Q \in S$. Then $P \preccurlyeq Q$ in $(\mathcal{M}(S), \preccurlyeq)$. Since P is a maximal element of $(\mathcal{M}(S), \preccurlyeq)$ we see that $P = Q$. Since $\overline{P} \preccurlyeq y$ we have $y - (F(S) - P) \in S$. But $y + P - F(S) = x$, which contradicts the fact that $x \in M(S)$.
- Suppose $F(S) - (P - Q) \in S$. Since $y - \overline{P} = y - F(S) + P \in S$, we have

$$(F(S) - P + Q) + (y - F(S) + P) = Q + y \in S.$$

But $Q + y = F(S) - x$, which contradicts the fact that $x \in M(S)$.

We conclude that $P - Q \in M(S)$. Finally,

$$(P - Q) - x = P - (Q + x) = P - (F(S) - y) = y - \overline{P} \in S,$$

and so $x \preccurlyeq P - Q$. □

Proof of Proposition 4.1. If $(Q, x, y) \in Tr_Q(S)$ then y would be above some minimal element of $(\mathcal{M}(S), \preccurlyeq)$, say $P \in PF(S) \setminus \{F(S)\}$ satisfies $\overline{P} \preccurlyeq y$. Lemma 4.6 implies that $P - Q \in M(S)$.

Conversely, if there is a $P \in PF(S) \setminus \{F(S)\}$ such that $P - Q \in M(S)$, then $(Q, P - Q, F(S) - P) \in Tr_Q(S)$. □

Definition 4.7. A numerical semigroup S is *triangle-free* if whenever $P_1, P_2 \in PF(S)$ satisfy $P_1 - P_2 \in M(S)$, then $P_1 = F(S)$.

Proposition 4.1 implies that $Tr(S) = \emptyset$ if and only if S is triangle-free.

Proposition 4.8. *If S is triangle-free, then it is P-minimal.*

Proof. If S is triangle-free then by Proposition 4.1, $Tr(S) = \emptyset$. By Corollary 3.10, S is P-minimal. □

5. Algorithm to determine $P(S)$

In this section, we present an algorithm to compute $P(S)$ given a numerical semigroup S. The algorithm essentially works by computing, for each Frobenius triangle (P, x, y), the list of order ideals that satisfy (P, x, y) (Theorem 3.9), as well as using Lemma 4.5 to further restrict the list of elements considered.

Table 1
Runtimes for $P(S)$ computations, each using GAP and the package numericalsgps [4].

S	e(S)	t(S)	$P(S)$	Algorithm 5.1	Recursive
$\langle 271, 309, 352, 422\rangle$	4	4	2	0.63 s	0.61 s
$\langle 871, 909, 952, 1022\rangle$	4	4	2	2.3 s	2.3 s
$\langle 603, 608, 613, \ldots, 653\rangle$	11	2	2	2.1 s	2.1 s
$\langle 49, 342, 349, 350\rangle$	4	13	2	725.0 s	0.4 s
$\langle 10, 101, 102, \ldots, 109\rangle$	10	9	126 905	60 s	14.7 s

Algorithm 5.1. Computes $P(S)$ for a numerical semigroup S using Frobenius triangles.

function AssociatedNumericalSets(S)
  $C_P \leftarrow \{(P, F(S) - P, 0)\} \cup Tr_P(S)$ for each $P \in PF(S) \setminus \{F(S)\}$
  $R \leftarrow \emptyset$
  for all $A \subseteq PF(S) \setminus \{F(S)\}$ and $a \in \prod_{P\in A} C_P$ **do**
    $G \leftarrow \{z \in M(S) \mid z - x \in S$ for some $a_P = (P, x, y)\}$
    $B_1 \leftarrow \{z \in M(S) \mid F(S) - y - z \in S$ for some $a_P = (P, x, y)\}$
    $B_2 \leftarrow \{z \in M(S) \mid P - z \in S$ for some $P \in PF(S) \setminus A\}$
    if $G \cap B_1 = \emptyset$ and $G \cap B_2 = \emptyset$ **then**
      $Z \leftarrow M(S) \setminus (G \cup B_1 \cup B_2)$
      Add to R the numerical set $A \cup G \cup Y \cup S$ for each order ideal Y of $(Z, \preccurlyeq)$
    end if
  end for
  return R
end function

Algorithm 5.1 works as follows. Each numerical set T associated to S is enumerated by first choosing the set $A = PF(S) \cap T$ of pseudo-Frobenius numbers in T, followed by choosing, for each $P \in A$, either a Frobenius triangle (P, x, y) satisfied by the order ideal $I = T \setminus S$ or having $\overline{P} \in I$ as in Theorem 3.9. Since I is an order ideal, every element of the set G must lie in I. Moreover, every element of B_1 must lie outside of I by Lemma 4.5, and any element of B_2 must lie outside of I since A contains all maximal elements of I. By Theorem 3.9, these are the only conditions on T, so any such order ideal I of $(M(S), \preccurlyeq)$ containing G and avoiding B_1 and B_2 yields a numerical set $I \cup S$.

We may obtain a slight improvement on the main loop in Algorithm 5.1 by using a recursive implementation. Rather than trying all possible collections a of Frobenius triangles, recursively add Frobenius numbers to a one at a time, each time growing G, B_1, and B_2 appropriately. This allows one to break out whenever a newly added Frobenius triangle renders $G \cap B_1$ or $G \cap B_2$ nonempty, thereby saving iterations of the main loop. Table 1 contains sample runtimes for a Sage implementation of Algorithm 5.1, both with and without utilizing this recursive enhancement.

6. Numerical semigroups of type 3

Recall that a numerical semigroup S has type 1 if and only if $P(S) = 1$. In Theorem 2.5, we proved that $t(S) = 2$ implies $P(S) = 2$. In this section we show that $t(S) = 3$ implies $P(S) \in \{2, 3, 4\}$ and give conditions on S that characterize when each possibility occurs. Throughout this section, S is a numerical semigroup of type 3 with pseudo-Frobenius numbers $P < Q < F$.

Lemma 6.1. *If S is a numerical semigroup with $t(S) = 3$ and $GPF(S)$ has two connected components, then $P(S) = 4$.*

Proof. By Theorem 3.6 we know that there are 4 self-dual order ideals of $(\mathcal{M}(S), \preccurlyeq)$, which lead to 4 numerical sets associated to S. Assume for the sake of contradiction that there is an order ideal I of $(\mathcal{M}(S), \preccurlyeq)$, that is not self-dual, for which $I \cup S$ is a numerical set associated to S. If $GPF(S)$ has two connected components, then $P + Q - F \notin S$. This is equivalent to $\overline{P} \not\preccurlyeq Q$ and also to $\overline{Q} \not\preccurlyeq P$. The maximal element above $\overline{P}$ is not Q, so $\overline{P} \preccurlyeq P$. Similarly $\overline{Q} \preccurlyeq Q$. By Lemma 4.2, $Tr_Q(S) = \emptyset$. Theorem 3.9 implies that if $Q \in I$, then $\overline{Q} \in I$.

Since I is not self-dual, Lemma 3.3 implies that $P \in I$ and $\overline{P} \notin I$. By Theorem 3.9, I must satisfy a Frobenius triangle $(P, x, y) \in Tr_P(S)$. We have $P + x + y = F$, so $x, y < x + y = \overline{P}$. This means that $\overline{P} \not\preccurlyeq x, y$ and so $\overline{Q} \preccurlyeq x, y \preccurlyeq Q$. Since I satisfies the Frobenius triangle (P, x, y), we know that $x \in I$ and $\overline{y} \notin I$. Therefore $Q \in I$ and so $\overline{Q} \in I$. Since $y \preccurlyeq Q$, we see that $\overline{Q} \preccurlyeq \overline{y}$ and therefore $\overline{y} \in I$. This contradicts the fact that I satisfies the Frobenius triangle (P, x, y). □

Lemma 6.2. *If $GPF(S)$ is connected and $Q - P \notin M(S)$, then $P(S) = 2$.*

Proof. In this case S is triangle-free, so Proposition 4.8 implies that $P(S) = 2$. □

Lemma 6.3. *Suppose $Q - P \in M(S)$. If $a \in M(S)$ satisfies $a \not\preccurlyeq Q$, then $Q - P \preccurlyeq a$.*

Proof. Since $a \not\preccurlyeq Q$ we know that $a \preccurlyeq P$, that is, $P - a \in S$. Now $Q - a \notin S$, so either $Q - a \in M(S)$ or $Q - a \in \mathcal{H}(S) \setminus M(S)$.

- Case 1: Suppose $Q - a \in M(S)$. Now $Q - (Q - a) = a \notin S$, which means $Q - a \not\preccurlyeq Q$, and so $Q - a \preccurlyeq P$. We have $P + a - Q \in S$, which implies $Q - P \preccurlyeq a$.
- Case 2: Suppose $Q - a \in \mathcal{H}(S) \setminus M(S)$. This means that $F + a - Q \in S$. But then,

$$F - (Q - P) = (F + a - Q) + (P - a) \in S.$$

This contradicts the fact that $Q - P \in M(S)$. $\square$

Lemma 6.4. *Suppose $GPF(S)$ is connected and $Q - P \in M(S)$. If $F + P = 2Q$ then $P(S) = 3$. If $F + P \neq 2Q$ then $P(S) = 4$.*

Proof. Since $GPF(S)$ is connected there are 2 self-dual order ideals of $(\mathcal{M}(S), \preccurlyeq)$. Proposition 4.1 implies that $Tr_P(S) \neq \emptyset$ and $Tr_Q(S) = \emptyset$. Suppose I is an order ideal for which $I \cup S$ is a numerical set associated to S and I is not self-dual. By Theorem 3.9, if $Q \in I$ then $\overline{Q} \in I$. Since I is not self-dual, Lemma 3.3 implies that $P \in I$ and $\overline{P} \notin I$. Therefore I must satisfy a Frobenius triangle $(P, x, y) \in Tr_P(S)$. This means that $P, x \in I$ and $\overline{y} \notin I$.

The minimal elements of $(\mathcal{M}(S), \preccurlyeq)$ are $\overline{P}$ and $\overline{Q}$, so we must have either $\overline{P} \preccurlyeq y$ or $\overline{Q} \preccurlyeq y$. Lemma 4.6 implies that if $\overline{P} \preccurlyeq y$, then $P - P = 0 \in M(S)$, which is not true. Therefore, $\overline{P} \not\preccurlyeq y$, so $\overline{Q} \preccurlyeq y$. Lemma 4.6 implies that $x \preccurlyeq Q - P$. Since $x \in I$, we see that $Q - P \in I$.

- Case 1: Suppose $x = Q - P$. In this case $\overline{y} = P + x = Q$, so $Q \notin I$. Let

$$I_1 = \{a \in M(S) \mid Q - P \preccurlyeq a\}.$$

Since $Q - P \in I$ we know that $I_1 \subseteq I$. On the other hand given $a \in I$, we know that $a \not\preccurlyeq Q$. Lemma 6.3 implies that $Q - P \preccurlyeq a$, that is, $a \in I_1$. We conclude that $I = I_1$.
- Case 2: Suppose $x \prec Q - P$. Corollary 4.4 implies that $x \prec Q$, and so $Q \in I$. Since $Tr_Q(S) = \emptyset$, Theorem 3.9 implies that $\overline{Q} \in I$.

Since $P - P \notin M(S)$ and $Q - P \in M(S)$, Lemma 4.6 implies that $\overline{P} \not\preccurlyeq x$ and $\overline{Q} \preccurlyeq x$. If $\overline{Q} \prec x$, then Lemma 4.3 implies $\overline{Q} \preccurlyeq \overline{y}$. However, this is impossible since $\overline{Q} \in I$ and $\overline{y} \notin I$. Therefore $\overline{Q} = x$. We have $\overline{y} = P + x = P + F - Q = \overline{Q - P}$.

Let

$$I_2 = \{a \in M(S) \mid \overline{Q} \preccurlyeq a\}.$$

Since $\overline{Q} \in I$ we see that $I_2 \subseteq I$. On the other hand if $a \in M(S) \setminus I_2$, then $\overline{Q} \not\preccurlyeq a$. This means that $\overline{a} \not\preccurlyeq Q$. Lemma 6.3 implies that $Q - P \preccurlyeq \overline{a}$ and so $a \preccurlyeq \overline{Q - P}$. Since $\overline{Q - P} = \overline{y} \notin I$, we see that $a \notin I$. We conclude that $I = I_2$.

We have seen that if $I \cup S$ is a numerical set associated to S then $I \in \{\emptyset, M(S), I_1, I_2\}$. We see that $I_1 = I_2$ if and only if $\overline{Q} = Q - P$, or equivalently, $F + P = 2Q$. Therefore $F + P = 2Q$ implies $P(S) = 3$, and $F + P \neq 2Q$ implies $P(S) = 4$. $\square$

Note that $S^* = S \cup M(S)$ and $(S \cup I_1)^* = S \cup I_2$. We summarize the results of this section.

Theorem 6.5. *Let S be a numerical semigroup of type 3. Suppose $PF(S) = \{P, Q, F\}$ with $P < Q < F$.*

- *If $P + Q - F \notin S$, then $P(S) = 4$.*
- *If $P + Q - F \in S$ and $Q - P \notin M(S)$, then $P(S) = 2$.*
- *If $P + Q - F \in S$, $Q - P \in M(S)$ and $F + P = 2Q$, then $P(S) = 3$.*
- *If $P + Q - F \in S$, $Q - P \in M(S)$ and $F + P \neq 2Q$, then $P(S) = 4$.*

Corollary 6.6. *Let S be a numerical semigroup of type 3. Suppose $PF(S) = \{P, Q, F\}$ with $P < Q < F$. Then S is P-minimal if and only if $Q - P \notin M(S)$.*

Proof. If $Q - P \notin M(S)$, then S is triangle-free and Proposition 4.8 implies that S is P-minimal.

Conversely, suppose S is P-minimal. By Theorem 6.5 either $GPF(S)$ is connected and $Q - P \notin M(S)$ or $GPF(S)$ has two connected components. In the first case we have nothing to prove.

Suppose $GPF(S)$ has two connected components, so $P(S) = 4$. Assume for the sake of contradiction that $Q - P \in M(S)$. Then $Q - (Q - P) = P \notin S$, that is, $Q - P \not\preccurlyeq Q$, and so $Q - P \preccurlyeq P$. Since $GPF(S)$ is not connected $P + Q - F \notin S$, which means $\overline{Q} \not\preccurlyeq P$. This implies $\overline{Q} \not\preccurlyeq Q - P$, and so $\overline{P} \preccurlyeq Q - P$. However this is impossible since $(Q - P) - \overline{P} = Q - F < 0$. Therefore $Q - P \notin M(S)$. $\square$

Example 6.7.

(1) Consider $S_1 = \langle 10, 19, 21, 36, 47 \rangle$, which has $PF(S_1) = \{37, 53, 64\}$. Since $37 + 53 - 64 \notin S_1$, this belongs to the first case of Theorem 6.5 and $P(S_1) = 4$. We see that $GPF(S_1)$ is not connected.
(2) Consider $S_2 = \langle 8, 9, 15, 21, 28 \rangle$, which has $PF(S_2) = \{19, 20, 22\}$. Since $19 + 20 - 22 \in S_2$ and $20 - 19 \notin M(S_2)$, this belongs to the second case of Theorem 6.5 and $P(S_2) = 2$. We see that $GPF(S_2)$ is connected and $Tr(S_2) = \emptyset$.

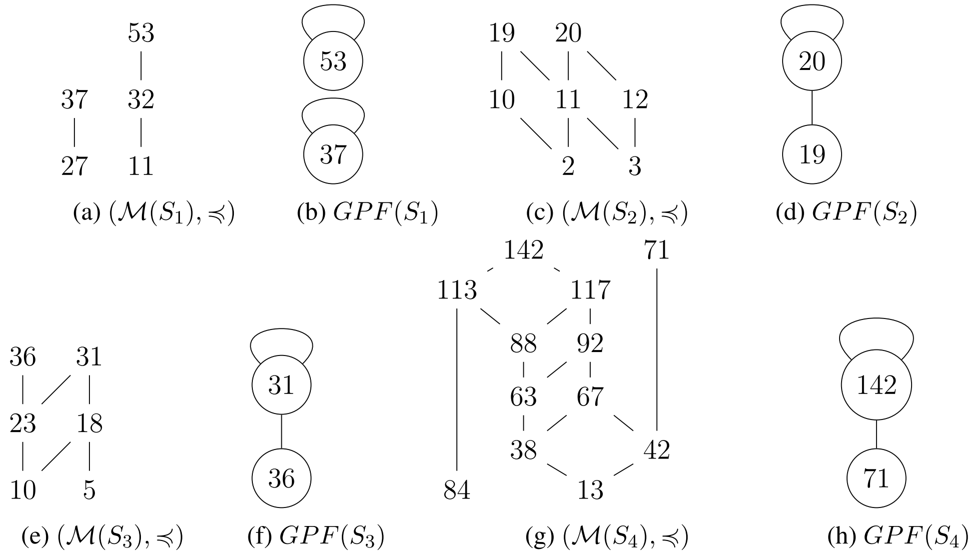


Fig. 3. Void poset and *GPF* graph of $S_1 = \langle 10, 19, 21, 36, 47\rangle$, $S_2 = \langle 8, 9, 15, 21, 28\rangle$, $S_3 = \langle 8, 13, 22, 27\rangle$ and $S_4 = \langle 25, 29, 32, 45\rangle$.

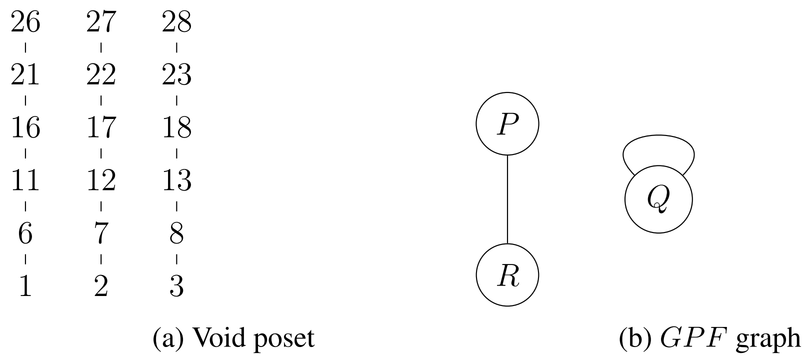


Fig. 4. Void poset and *GPF* graph of $S_6 = \langle 5, 31, 32, 33, 34\rangle$.

(3) Consider $S_3 = \langle 8, 13, 22, 27\rangle$, which has $PF(S_3) = \{31, 36, 41\}$. Since $31 + 36 - 41 \in S_3$, $36 - 31 \in M(S_3)$ and $41 + 31 = 2 \cdot 36$, it belongs to the third case of Theorem 6.5 and $P(S_3) = 3$.
(4) Consider $S_4 = \langle 25, 29, 32, 45\rangle$, which has $PF(S_4) = \{71, 142, 155\}$. Since $71 + 142 - 155 \in S_4$, $142 - 71 \in M(S_4)$ and $71 + 155 \neq 2 \cdot 142$, it belongs to the fourth case of Theorem 6.5 and $P(S_4) = 4$.

Fig. 3 shows the void poset and the GPF graph for S_1, S_2, S_3 and S_4.

7. Numerical semigroups of type 4

In this section we show that there is a significant change in the behavior of $P(S)$ as we move from numerical semigroups of type 3 to numerical semigroups of type 4. We show that there exist numerical semigroups S with $t(S) = 4$ and $P(S)$ arbitrarily large.

Proposition 7.1. *Let*

$$S_n = \{0, 5, 10, 15, \dots, 5n \rightarrow\}.$$

We have $t(S_n) = 4$ *and* $P(S_n) = 2n + 4$.

Proof. We see that $F(S_n) = 5n - 1$ and

$$M(S_n) = \{1, 6, \dots, 5n - 4\} \cup \{2, 7, \dots, 5n - 3\} \cup \{3, 8, \dots, 5n - 2\}.$$

The void poset $(\mathcal{M}(S_n), \preccurlyeq)$ is the union of 3 chains $1 \preccurlyeq 6 \preccurlyeq \cdots \preccurlyeq 5n-4$, $2 \preccurlyeq 7 \preccurlyeq \cdots \preccurlyeq 5n-3$ and $3 \preccurlyeq 8 \preccurlyeq \cdots \preccurlyeq 5n-2$. See Fig. 4 for a depiction of $(\mathcal{M}(S_6), \preccurlyeq)$.

Since $x \in PF(S_n)$ implies that $5 + x \in S$, it is easy to check that $PF(S_n) = \{5n-4, 5n-3, 5n-2, 5n-1\}$, and so $t(S_n) = 4$. Let $P = 5n-4$, $Q = 5n-3$, $R = 5n-2$, and $F = F(S_n) = 5n-1$. It is easy to check that $GPF(S_n)$ has an edge between P and R and a loop on Q. So, $GPF(S_n)$ has two connected components, and so there are 4 self-dual order ideals of $(\mathcal{M}(S_n), \preccurlyeq)$.

Lemma 4.2 implies that $Tr_R(S_n) = \emptyset$. From the description of $M(S_n)$, we can check that

$$Tr(S_n) = \{(5n-3, 1, 1),\ (5n-4, 1, 2),\ (5n-4, 2, 1)\}.$$

Suppose I is an order ideal of $(\mathcal{M}(S_n), \preccurlyeq)$ that is not self-dual and $I \cup S$ is a numerical set associated to S. Lemma 3.3 implies that either $P \in I$ and $\overline{P} \notin I$, or $Q \in I$ and $\overline{Q} \notin I$.

- Case 1: Suppose $Q = 5n-3 \in I$ and $\overline{Q} = 2 \notin I$. Theorem 3.9 implies that I has to satisfy a Frobenius triangle in $Tr_Q(S_n)$, and therefore I must satisfy $(5n-3, 1, 1)$. This means $1 \in I$ and $\overline{1} = 5n-2 \notin I$. Since $1 \in I$, we see that $\{1, 6, \ldots, 5n-4\} \subseteq I$. Since $5n-2 \notin I$, we see that $\{3, 8, \ldots, 5n-2\} \cap I = \emptyset$.

  Since $P \in I \cap PF(S_n)$ and $\overline{P} \notin I$, Theorem 3.9 implies that I satisfies a Frobenius triangle in $Tr_P(S_n)$. We see that I does not satisfy $(5n-4, 1, 2)$ because $\overline{2} = 5n-3 \in I$. We see that I does not satisfy $(5n-4, 2, 1)$ because $2 \notin I$. This is a contradiction.
- Case 2: Suppose $P = 5n-4 \in I$ and $\overline{P} = 3 \notin I$. Theorem 3.9 implies that I satisfies a Frobenius triangle in $Tr_P(S_n)$. First consider the case when I satisfies $(5n-4, 1, 2)$. This means $1 \in I$ and $\overline{2} = 5n-3 \notin I$. This implies $\{1, 6, \ldots, 5n-4\} \subseteq I$ and $\{2, 7, \ldots, 5n-3\} \cap I = \emptyset$. The remaining elements of $M(S_n)$ are $\{8, 11, \ldots, 5n-2\}$. Once we decide the first element in this list to include in I then all larger elements must also be in I. This gives n choices including the choice to not include any elements from this list. Since $\overline{5n-2} = 1 \in I$, Theorem 3.9 implies that each of these choices leads to a numerical set associated to S.

  Next consider the case when I satisfies $(5n-4, 2, 1)$. This means $2 \in I$ and $\overline{1} = 5n-2 \notin I$. This implies $\{2, 7, \ldots, 5n-3\} \subseteq I$ and $\{3, 8, \ldots, 5n-2\} \cap I = \emptyset$. Note that $\overline{Q} = 2 \in I$. The remaining elements of $M(S_n)$ are $\{1, 6, \ldots, 5n-9\}$. As above, once we decide the first element in this list to include in I then all larger elements must also be in I. Each of these n choices leads to a numerical set associated to S.

We have seen that $(\mathcal{M}(S_n), \preccurlyeq)$ has 4 self-dual order ideals and $2n$ order ideals I that are not self dual for which $I \cup S$ is a numerical set associated to S. Therefore, $P(S_n) = 2n + 4$. $\square$

8. Numerical semigroups of large type

In this section we focus on two families of numerical semigroups of large type.

8.1. Numerical semigroups with $P(S) = 2$ and large type

We first describe a family of numerical semigroups where every member has $P(S) = 2$ that includes semigroups of arbitrarily large type.

Proposition 8.1. *Let $n \geq 1$, $m = 2n + 1$, and*

$$S_n = \{0, 2m \rightarrow\} \cup \{m + 2k \mid 0 \leq k \leq n-1\}.$$

Then S_n is a numerical semigroup with $t(S_n) = n + 1$ and $P(S_n) = 2$.

The family we consider is a subset of the family in [3, Proposition16]. They show that each member of the family has $P(S) = 2$ and $|M(S)|$ can be arbitrarily large. They do not discuss the pseudo-Frobenius numbers or the type of these semigroups.

Proof. Note that $F = F(S_n) = 2m - 1 = 4n + 1$. We see that S_n is a numerical semigroup since all nonzero elements of S_n are larger that $\frac{F}{2}$. The void of S_n is

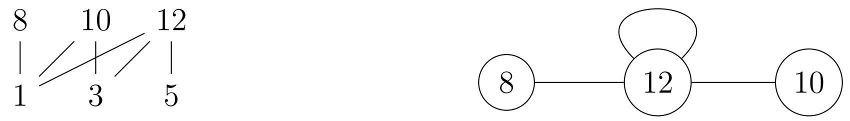
$$M(S_n) = \{2i + 1 \mid 0 \leq i \leq n-1\} \cup \{2i \mid n+1 \leq i \leq 2n\}.$$

For $k \in [0, n-1]$, $2k+1$, $2(k+n+1) \in M(S_n)$ and

$$2(k+n+1) - (2k+1) = 2n + 1 = m \in S_n.$$

This means that $2k + 1 \preccurlyeq 2(k+n+1)$ in $(\mathcal{M}(S_n), \preccurlyeq)$, and so $2k + 1 \notin PF(S_n)$. On the other hand, for $k \in [n+1, 2n]$,

$$2k + m \geq 2(n+1) + 2n + 1 = 4n + 3 > F.$$



(a) Void poset (b) GPF graph

Fig. 5. $(\mathcal{M}(S_3), \preccurlyeq)$ and $GPF(S_3)$ where $S_3 = \{0, 7, 9, 11, 14 \rightarrow\}$.

Since m is the smallest nonzero element of S_n, we see that $2k + S_n \setminus \{0\} \subseteq S_n$, and so $2k \in PF(S_n)$. Therefore,

$$PF(S_n) \setminus \{F\} = \{2k \mid n+1 \leq k \leq 2n\},$$

which means $t(S_n) = n + 1$.

A main idea for the rest of the proof is to show that each S_n is triangle-free and that $GPF(S_n)$ is connected. Proposition 4.8 then implies that $P(S_n) = 2$. See Fig. 5 for the poset $(\mathcal{M}(S_3), \preccurlyeq)$ and $GPF(S_3)$.

Let $P, Q \in PF(S) \setminus \{F\}$ with $P < Q$. Since P and Q are both even so is $P - Q$, and $0 < P - Q \leq 2n$. Therefore, $P - Q \notin M(S_n)$. Proposition 4.1 implies that $Tr(S_n) = \emptyset$, or equivalently, that S_n is triangle-free. By Proposition 4.8, $P(S_n) = 2^{\kappa}$ where κ is the number of connected components of $GPF(S_n)$. To complete the proof, we need only show that $GPF(S_n)$ is connected.

Suppose that $k \in [n+1, 2n]$. We have that $2k,\ 4n \in PF(S_n)$ and

$$2k + 4n - F = 2k - 1 = 2n + 1 + 2(k - n - 1) \in S_n.$$

We see that there is an edge between $2k$ and $4n$ in $GPF(S_n)$. We conclude that $GPF(S_n)$ is connected, completing the proof. $\square$

Propositions 7.1 and 8.1 make it clear that for semigroups of type at least 4, the connection between the set of pseudo-Frobenius numbers $PF(S)$ and $P(S)$ is not so clear.

8.2. Maximal embedding dimension numerical semigroups

Recall that S has maximal embedding dimension if and only if its number of minimal generators is equal to its multiplicity, that is, $e(S) = m(S)$. We characterize $P(S)$ for numerical semigroups S that have maximal embedding dimension and are triangle-free.

Theorem 8.2. *Suppose S is a triangle-free numerical semigroup with maximal embedding dimension. Let $m = m(S)$ and $F = F(S)$.*

(1) Suppose $x, y \in M(S)$ with $x \leq y$. Then $x \preccurlyeq y$ in $(\mathcal{M}(S), \preccurlyeq)$ if and only if $m \mid (y - x)$.
(2) If m is odd then $P(S) = 2^{\frac{m-1}{2}}$.
(3) If m is even and F is odd then $P(S) = 2^{\frac{m-2}{2}}$.
(4) If m and F are both even then $P(S) = 2^{\frac{m}{2}}$.

Proof. Since S is of maximal embedding dimension, $|PF(S)| = m - 1$. For distinct $P, Q \in PF(S)$, we know that $P - Q \notin S$. This implies that $P \not\equiv Q \pmod{m}$. Moreover since pseudo-Frobenius numbers are gaps, none of them is 0 modulo m. Therefore we can label $PF(S) = \{P_1, \ldots, P_{m-1}\}$ with $P_i \equiv i \pmod{m}$.

If $x, y \in M(S)$ satisfy $x \leq y$ and $m \mid (y - x)$, then clearly $x \preccurlyeq y$. Assume for the sake of contradiction that there are $x, y \in M(S)$ satisfying $x \leq y,\ m \nmid (y - x)$, and $x \preccurlyeq y$. Note that none of $y,\ F - x$, and $y - x$ is divisible by m. Suppose that $P_i \equiv y \pmod{m}$, $P_j \equiv F - x \pmod{m}$, and $P_k \equiv y - x \pmod{m}$. Since $y, F - x \notin S$, we know that $y \leq P_i$ and $F - x \leq P_j$. Since $y - x \in S$, we know that $y - x > P_k$. Now $P_j \equiv F - (P_i - P_k) \pmod{m}$ and

$$F - (P_i - P_k) < (P_j + x) - y + (y - x) = P_j,$$

so $F - (P_i - P_k) \notin S$. Since $P_i, P_j \in PF(S)$, we have $P_i - P_j \notin S$. This means that $P_i - P_j \in M(S)$, but this contradicts the fact that S is triangle-free. This completes the proof of (1).

Since S is triangle-free, Proposition 4.8 implies that $P(S) = 2^{\kappa(S)}$, where $\kappa(S)$ is the number of connected components of $GPF(S)$. Suppose $P_i, P_j \in PF(S) \setminus \{F\}$. We see that $F - P_i \preccurlyeq P_j$ if and only if $P_i + P_j \equiv F \pmod{m}$. This means that the graph $GPF(S)$ mostly consists of components of size 2 except when $2P_i \equiv F \pmod{m}$, in which case P_i is its own component has a loop on it. The description of $PF(S)$ given above shows that $GPF(S)$ has $m - 2$ vertices.

- If m is odd then there is exactly one i for which $2P_i \equiv F \pmod{m}$. Therefore, $\kappa(S) = 1 + \frac{m-3}{2}$.
- If m is even and F is odd then there is no i for which $2P_i \equiv F \pmod{m}$. Therefore, $\kappa(S) = \frac{m-2}{2}$.
- If m and F are both even then there are two i for which $2P_i \equiv F \pmod{m}$. Therefore, $\kappa(S) = 2 + \frac{m-4}{2}$. $\square$

Example 8.3. For $m \geq 2$, let S_m be the numerical semigroup generated by

$$\{m\} \cup \{m(m+k-1)+k \mid 1 \leq k \leq m-2\} \cup \{m(2m-1)+m-1\}.$$

We can check that $F(S_m) = m(2m-2)+m-1$ and

$$PF(S_m) = \{m(m+k-2)+k \mid 1 \leq k \leq m-2\} \cup \{m(2m-2)+m-1\}.$$

We see that S_m has maximal embedding dimension since $t(S_m) = m-1$. Next we check that S_m is triangle-free. Suppose $P_j, P_i \in PF(S_m) \setminus \{F(S_m)\}$ with $P_j \equiv j \pmod{m}$, $P_i \equiv i \pmod{m}$ and $1 \leq i < j \leq m-2$. Then

$$P_j - P_i = m(m+j-2)+j-(m(m+i-2)+i) = m(j-i)+j-i.$$

Let $k = m-1-(j-i)$. We have

$$F(S_m) - (P_j - P_i) = m(m+k-1)+k \in S_m.$$

This means that $P_j - P_i \notin M(S_m)$ and so S_m is triangle-free. When m is odd we have $P(S_m) = 2^{\frac{m-1}{2}}$. When m is even, we see that $F(S_m)$ is odd and $P(S_m) = 2^{\frac{m-2}{2}}$.

When S has maximal embedding dimension and is triangle-free, $(\mathcal{M}(S), \preccurlyeq)$ is a union of $m-2$ chains, one for each nonzero residue class modulo m except the one containing $F(S)$. When S has maximal embedding dimension and is not triangle-free, for example when

$$S = \langle 15, 34, 38, 57, 61, 80, 84, 103, 107, 126, 130, 149, 153, 172, 176 \rangle,$$

the structure of $(\mathcal{M}(S), \preccurlyeq)$ can be much more complicated.

Data availability

No data was used for the research described in the article.

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