Discrete & Computational Geometry (2023) 69:91–122
https://doi.org/10.1007/s00454-022-00416-6

Combinatorial Properties of Self-Overlapping Curves and Interior Boundaries

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Received: 6 September 2019 / Revised: 18 May 2021 / Accepted: 17 January 2022 /
Published online: 30 September 2022

Abstract

We study the interplay between the recently defined concept of *minimum homotopy area* and the classical topic of *self-overlapping curves*. The latter are plane curves which are the image of the boundary of an immersed disk. Our first contribution is to prove new sufficient combinatorial conditions for a curve to be self-overlapping. We show that a curve γ with Whitney index 1 and without any self-overlapping subcurves is self-overlapping. As a corollary, we obtain sufficient conditions for self-overlappingness solely in terms of the Whitney index of the curve and its subcurves. These results follow from our second contribution, which shows that any plane curve γ, modulo a basepoint condition, is transformed into an *interior boundary* by wrapping around γ with Jordan curves. Equivalently, the minimum homotopy area of γ is reduced to the minimal possible threshold, namely the winding area, through wrapping. In fact, we show that $n + 1$ wraps suffice, where γ has n vertices. Our third contribution is to prove the equivalence of various definitions of self-overlapping curves and interior boundaries, often implicit in the literature. We also introduce and characterize *zero-obstinance curves*, further generalizations of interior boundaries defined by optimality in minimum homotopy area.

Keywords Self-overlapping curves · Interior boundaries · Minimum homotopy area · Immersion

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Editor in Charge: János Pach

An extended abstract of this paper has been published at SoCG [7].

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Mathematics Subject Classification 57K99 · 68U05

1 Introduction

Classically, a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is called **self-overlapping** when there is a continuous map $F : \mathbb{D}^2 \to \mathbb{R}^2$ such that $F|_{\text{int}(\mathbb{D}^2)}$ is an orientation-preserving immersion, and $F|_{\mathbb{S}^1} = \gamma$. One can think of such an immersion as distorting a unit disk that lies flat in the plane and stretching and pulling it continuously without leaving the plane and without twisting or pinching it [17]. A consequence of the non-twisting of an immersion is that any self-overlapping curve γ makes one net counterclockwise turn. Precisely, set $\alpha := \gamma'/\|\gamma'\|$ to be the unit tangent vector and declare the **Whitney index** as $\text{WHIT}(\gamma) = (1/2\pi)\int_0^{2\pi} \alpha\, d\theta$. Then self-overlapping curves have $\text{WHIT}(\gamma) = 1$.

If the disk is painted blue on top and pink on the bottom, then one only sees blue. If we also imagine the disk being semi-transparent, then the blue will appear darker in the regions where it overlaps itself; see Fig. 1. We learn that the interior of a self-overlapping curve always lies locally to the left. This turning condition manifests in γ as $\text{wn}(x, \gamma) \geq 0$ for every $x \in \mathbb{R}^2$. Here, $\text{wn}(x, \gamma)$ is the **winding number** of γ around x, which can be seen as $\theta(1) - \theta(0)$ where $\rho(t) := \gamma(t) - x = (r(t), \theta(t))$ is written in polar coordinates. This non-negative turning condition $\text{wn}(x, \gamma) \geq 0$ is called **positive consistent**. The necessity of Whitney index 1 and positive consistency to be self-overlapping are well known and date back to [21]. Another simple and intuitive view originates from Blank [1]: The curve is self-overlapping when we can cut it along simple paths into simple positively oriented Jordan curves, i.e., a collection of counterclockwise topological disks.

Interior boundaries are generalizations of self-overlapping curves and also have various equivalent definitions. We defer a formal definition to Sect. 3.4. For now, interior boundaries γ can be thought of as composites of self-overlapping curves γ_i (of the same orientation) that have been glued together; see Fig. 2 for an example. In this paper, all curves $\gamma : [0, 1] \to \mathbb{R}^2$ are assumed to be closed, immersed, and generic, i.e., with only finitely many intersection points, each of which are transverse double points. We also assume $\gamma'(t)$ exists and is nonzero for all $t \in [0, 1]$. We show new combinatorial properties of self-overlapping curves and interior boundaries by revealing new connections to the minimum homotopy area of curves.

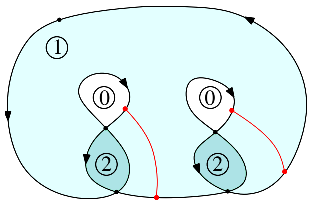

Fig. 1 A self-overlapping curve γ with winding numbers for the faces circled. The Blank cuts, shown in red, slice γ into a collection of simple positively oriented (counterclockwise) Jordan curves

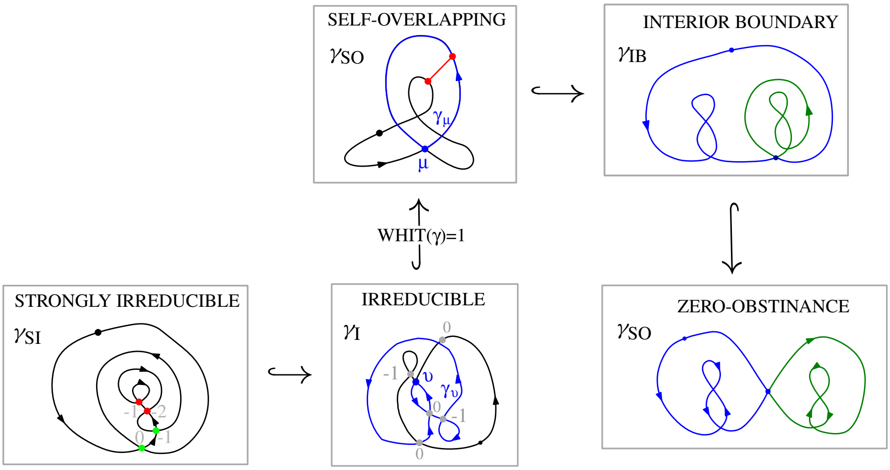

Fig. 2 Example curves of different curve classes and inclusion relationships between the classes. γ_{SO} is self-overlapping as indicated by the Blank cuts in red. γ_{IB} is an interior boundary consisting of two self-overlapping curves (of the same orientation), one in blue the other in green. The bottom row shows curve classes that are introduced in this paper: γ_{SI} is strongly irreducible as can be seen from the non-positive Whitney indices (shown in gray) of its direct split subcurves. Similarly, γ_{I} is irreducible; note that γ_v has Whitney index 1 but is not self-overlapping. Also note that γ_{SO} is not irreducible since γ_u is self-overlapping. γ_{ZO} also consists of two self-overlapping curves but of different orientation and is therefore not an interior boundary, but it has zero obstinance

1.1 Related Work

Self-Overlapping Curves and Interior Boundaries. Self-overlapping curves and interior boundaries have a rather rich history, and have been studied under the lenses of analysis, topology, geometry, combinatorics, and graph theory [1, 4, 10, 14–17, 19, 21]. In the 1960s, Titus [21] provided the first algorithm to test whether a curve is self-overlapping (or an interior boundary), by defining a set of cuts that must cut the curve into smaller subcurves that are self-overlapping (or interior boundaries). In a 1967 PhD thesis [1], Blank proved that a curve is self-overlapping iff there is a sequence of cuts (different from Titus cuts) that completely decompose the curve into simple pieces. He represents plane curves with words and showed that one can determine the existence of a cut decomposition by looking for algebraic decompositions of the word. In the 1970s, Marx [15] extended Blank's work to give an algorithm to test if a curve is an interior boundary. In the 1990s, Shor and Van Wyk [19] expedited Blank's algorithm to run in $O(N^3)$ time for a polygonal curve with N line segments. Their dynamic programming algorithm is currently the fastest algorithm to test for self-overlappingness. It is not known whether this runtime bound is tight or whether a faster runtime might be achievable. In distantly related work, Eppstein and Mumford [4] showed that it is NP-complete to determine whether a fixed self-overlapping curve γ is the 2D projection of an immersed surface $f : S \to \mathbb{R}^3$, where S is a compact two-manifold with boundary. Graver and Cargo [10] instead approached the problem from a graph-theoretical perspective. All of these algorithms also compute the number of 'inequivalent' immersions.

Minimum Homotopy Area. The **minimum homotopy area** $\sigma(\gamma)$ is the infimum of areas swept out by nullhomotopies of a closed plane curve γ. The key link between minimum area homotopies and self-overlapping curves arose in [8, 12], where the authors showed that any curve γ has a minimum area homotopy realized by a sequence of nullhomotopies of special self-overlapping subcurves. The minimum homotopy area was introduced by Chambers and Wang [3] as a more robust metric for curve comparison than homotopy width (i.e., Fréchet distance or one of its variants) or homotopy height [2]. The minimum homotopy area can be computed in $O(N^2 \log N)$ time for consistent curves [3]. For general curves, Nie gave an algorithm to compute $\sigma(\gamma)$ based on an algebraic interpretation of the problem that runs in $O(N^6)$ time, while the self-overlapping decomposition result of [8] yields an exponential-time algorithm. The **winding area** $W(\gamma)$ is the integral over all winding numbers in the plane. A simple argument shows that $\sigma(\gamma) \geq W(\gamma)$; see [3]. Both self-overlapping curves and interior boundaries are characterized by positive consistency and optimality in minimum homotopy area, $\sigma(\gamma) = W(\gamma)$. A curve possessing both of these properties is self-overlapping when $\mathrm{WHIT}(\gamma) = 1$ and an interior boundary when $\mathrm{WHIT}(\gamma) \geq 1$.

1.2 New Results

In this paper, we are interested in sufficient combinatorial conditions for a plane curve to be self-overlapping. Such conditions provide novel mathematical foundations that could pave the way for speeding up algorithms for related problems, such as deciding self-overlappingness or computing the minimum homotopy area of a curve. In the first contribution of this paper (Theorem 4.6 and Corollary 4.8 in Sect. 4), we show that a curve γ with Whitney index 1 and without any self-overlapping subcurves is self-overlapping, and we obtain sufficient conditions for a curve to be self-overlapping solely in terms of the Whitney index of the curve and its subcurves. Here, we only consider **direct split** subcurves γ_v that traverse γ between the first and second appearance of vertex v in the plane graph induced by γ. Our results apply to (strongly) irreducible curves; see Fig. 2: We call γ **irreducible**, if every (proper) direct split is not self-overlapping; if the Whitney index of each such direct split is non-positive, then we call γ **strongly irreducible**.

These results follow from our second contribution (Theorems 4.3 and 4.4 in Sect. 4), which shows that any plane curve γ is transformed into an interior boundary by wrapping around γ with Jordan curves. Equivalently, this means that the minimum homotopy area of γ is reduced to the minimal possible threshold, namely the winding area, through wrapping. See Fig. 3 for an example of wrapping. Of course, we can make a curve positive consistent with repeated wrapping, since a single wrap increases the winding numbers of each face by one. However, our result shows a new and non-trivial connection between wrapping and the minimum homotopy area.

The third contribution of this paper (in Sect. 3) is to unite the various definitions and perspectives on self-overlapping curves and interior boundaries. We prove the equivalence of five definitions of self-overlapping curves and four of interior boundaries

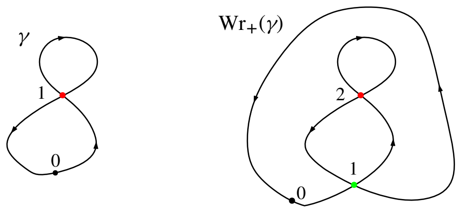

Fig. 3 The curve γ is not self-overlapping, but its wrap $\mathrm{Wr}_+(\gamma)$ is self-overlapping

(Theorems 3.7 and 3.6). To this end, we define the new concept of **obstinance** of a curve γ as $\mathrm{obs}(\gamma) = \sigma(\gamma) - W(\gamma) \geq 0$, and characterize **zero-obstinance** curves (Theorem 3.10), see Fig. 2. Rephrasing our earlier characterization, self-overlapping curves and interior boundaries are positive-consistent curves with zero-obstinance and positive Whitney index.

We conclude by defining a new operation called *balanced loop insertion*, a complementary notion to that of *balanced loop deletion*, the key trick to proving Theorem 4.6. As a parallel to our results on wraps, we show in Theorem 4.11 that careful iteration of balanced loop insertion turns any curve γ with $\mathrm{WHIT}(\gamma) = 1$ (and positive outer basepoint) into a self-overlapping curve.

More supplementary details on the relationship between different curve classes studied in this paper are provided in Appendix B.

2 Preliminaries

We now lay the necessary groundwork on planar curves, homotopies, self-overlapping curves, interior boundaries, and minimum area homotopies that are needed for this paper.

2.1 Regular and Generic Curves

We work with regular, generic, closed plane curves $\gamma : [0, 1] \to \mathbb{R}^2$ with basepoint $\gamma(0) = \gamma(1)$. Let $\mathscr{C}$ denote the set of such curves. A curve γ is **regular** if $\gamma'(t)$ exists and is non-zero for all t; a curve is **generic** if the embedding has only a finite number of intersection points, each of which are transverse crossings. Being generic is a weak restriction, as generic curves are dense in the space of regular curves [22]. Viewing a generic curve γ by its image $\boldsymbol{[\gamma]} \subseteq \mathbb{R}^2$, we can treat γ as a directed plane multigraph $\boldsymbol{G(\gamma)} = (V(\gamma), E(\gamma))$. Here, $\boldsymbol{V(\gamma)} = (p_0, p_1, \ldots, p_n)$ is the set of ordered **vertices** (points) of γ, with basepoint $p_0 = \gamma(0)$ regarded as a vertex as well. An **edge** (p_i, p_j) corresponds to a simple path along γ between p_i and p_j. The **faces** of $G(\gamma)$ are the path-connected components of $\mathbb{R}^2 \setminus [\gamma]$. Each $\gamma \in \mathscr{C}$ has exactly one unbounded face, the exterior face $\boldsymbol{F}_{\mathrm{ext}}$. See Fig. 4. Two curves are combinatorially equivalent when their planar multigraphs are isomorphic. We may therefore define a curve just by its image, orientation, and basepoint. A curve is **simple** if it has no intersection points. We notate $|\gamma| = |V(\gamma) \setminus \{p_0\}|$ as the complexity of γ.

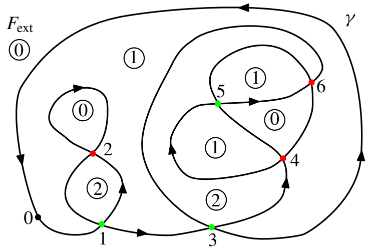

Fig. 4 A curve γ that is self-overlapping. The winding numbers of each face are enclosed by circles. The signed intersection sequence of γ is $0, 1_+, 2_-, 2_+, 1_-, 3_+, 4_-, 5_+, 6_-, 4_+, 5_-, 6_+, 3_-, 0$; vertex labels are shown, and the sign of each vertex is indicated with green (positive) or red (negative). Here, the vertex p_i is labeled as i. The combinatorial relations are: $p_2 \subset p_1$; $p_4, p_5, p_6 \subset p_3$; $p_1, p_2 \mathrm{~S~} p_3, p_4, p_5, p_6$; $p_4, p_5 \mathrm{~L~} p_6$; $p_4 \mathrm{~L~} p_5$

For any $x \in \mathbb{R}^2 \setminus [\gamma]$, the **winding number** $\mathbf{wn}(\boldsymbol{x}, \boldsymbol{\gamma}) = \sum_i a_i$ is defined using a simple path P from x to F_{ext} such that $[P] \cap V(\gamma) = \emptyset$, the image of the path does not touch any vertex. Here, $a_i = 1$ if P crosses γ from left to right at the i-th intersection of P with γ, and $a_i = -1$ otherwise. Since this number is independent of the path chosen and is constant over each face F of $G(\gamma)$, we write $\mathbf{wn}(\boldsymbol{F}, \boldsymbol{\gamma}) = \mathrm{wn}(q, \gamma)$ for any $q \in F$.[1] If $\mathrm{wn}(F, \gamma) \geq 0$ for every face F on $G(\gamma)$, then we call γ **positive consistent**. If $\mathrm{wn}(F, \gamma) \leq 0$ for every face, then γ is **negative consistent**. See Fig. 4 for an example curve illustrating these concepts. The **winding area** of a curve γ is given by $\boldsymbol{W}(\boldsymbol{\gamma}) = \int_{\mathbb{R}^2} |\mathrm{wn}(x, \gamma)|\, dx = \sum_F A(F)|\mathrm{wn}(F, \gamma)|$, where $A(F)$ is the area of the face F and $\mathrm{wn}(x, \gamma) = 0$ for $x \in [\gamma]$. The **depth** $D(F, \gamma)$ of a face F is the minimum number of crossings of any simple path P from F to F_{ext} with γ. The **depth** of γ is the sum $D(\gamma) = \sum_F A(F) D(F, \gamma)$. The **Whitney index WHIT**(γ) of a curve γ is the winding number of the derivative γ' about the origin. The Whitney index measures the number of 'complete counterclockwise turns' made by γ. By [13, Lem. 6.3] and our Lemma 3.3, our definition agrees with the earlier differential-geometric definition $\mathrm{WHIT}(\gamma) = (1/2\pi) \int_0^{2\pi} (\gamma'(t)/\|\gamma'(t)\|)\, d\theta$ from the introduction.[2] A curve γ is **positively oriented** if $\mathrm{WHIT}(\gamma) > 0$ and **negatively oriented** if $\mathrm{WHIT}(\gamma) < 0$.

A basepoint $p_0 = \gamma(0)$ is an **outer basepoint** if p_0 is incident to F_{ext}. Suppose p_0 is an outer basepoint incident to the two faces F and F_{ext}. Then if $\mathrm{wn}(F, \gamma) = 1$, we call p_0 a **positive outer basepoint**. Otherwise, $\mathrm{wn}(F, \gamma) = -1$ and p_0 is a **negative outer basepoint**. Several of our results require γ to have a positive outer basepoint. When discussing self-overlapping curves, it is more natural to treat our closed curves as maps $\mathbb{S}^1 \to \mathbb{R}^2$.[3] In this context, a self-overlapping curve γ is the boundary of an orientation-preserving immersion, i.e., a map of full rank with positive Jacobian, on the open unit disk $\mathsf{int}(\mathbb{D}^2)$. More precisely, γ is **(positive) self-overlapping** when there is a map $F\colon \mathbb{D}^2 \to \mathbb{R}^2$ such that F is continuous on the closed unit disk $\mathbb{D}^2$, the

[1] By Observation 3.2, which gives an equivalent way to view winding numbers, we see this definition is equivalent to the more intuitive one given in the introduction.

[2] In practice, we compute the Whitney index with Lemma 3.3.

[3] Here, we are forced to treat (potentially) self-overlapping curves as maps $\mathbb{S}^1 \to \mathbb{R}^2$ to discuss extensions to $\mathbb{D}^2$. On the other hand, for our constructions of subcurves, our curves are maps $[0, 1] \to \mathbb{R}^2$. This allows each $v \in V(\gamma)$ to have exactly two pre-images.

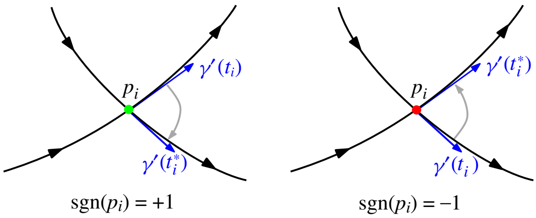

Fig. 5 An intersection point p_i of a curve γ. The vertex p_i is positive if the second tangent vector is a clockwise rotation away from the first tangent vector, negative otherwise

map $F|_{\mathrm{int}(\mathbb{D}^2)}$ is an orientation-preserving immersion, and $F|_{\mathbb{S}^1} = \gamma$.[4] The **reversal** $\overline{\boldsymbol{\gamma}}$ is defined by $\overline{\gamma}(t) = \gamma(1-t)$, viewing the original curve as $\gamma\colon [0,1] \to \mathbb{R}^2$. We call γ **negative self-overlapping** when $\overline{\gamma}\colon \mathbb{S}^1 \to \mathbb{R}^2$ is positive self-overlapping. Unless stated otherwise, the term self-overlapping is used only to mean positive self-overlapping.

2.2 Combinatorial Relations and Intersection Sequences

Following Titus [20], we now describe how the intersection points of a curve $\gamma \in \mathscr{C}$ relate to each other. See Fig. 4 for an illustration of these concepts. Let $p_i, p_j \in V(\gamma)$ be two vertices such that $p_i = \gamma(t_i) = \gamma(t_i^*)$ and $p_j = \gamma(t_j) = \gamma(t_j^*)$ with $t_i < t_i^*$ and $t_j < t_j^*$. Then, one of the following must hold:

- p_i **links** p_j, or $p_i \mathrm{L} p_j$, iff $t_i < t_j < t_i^* < t_j^*$ or $t_j < t_i < t_j^* < t_i^*$,
- p_i is **separate** from p_j, or $p_i \mathrm{S} p_j$, iff $t_i < t_i^* < t_j < t_j^*$ or $t_j < t_j^* < t_i < t_i^*$,
- p_i is **contained in** p_j, or $p_i \subset p_j$, iff $t_j \le t_i < t_i^* \le t_j^*$.

To define the intersection sequence of γ, the vertices are labeled in the order they appear on γ, starting with 0 for the basepoint $\gamma(0)$, and increasing by one each time an unlabeled vertex is encountered. The **signed intersection sequence** consists of the sequence of all vertex labels along γ starting at the basepoint; the first time vertex p_i is visited, the label is augmented with $\mathrm{sgn}(p_i)$, and the second time with $-\mathrm{sgn}(p_i)$. Here, $\mathrm{sgn}(p_i) := \mathrm{sgn}(p_i, \gamma)$ is the **sign** of vertex $p_i = \gamma(t_i) = \gamma(t_i^*)$, and is 1 if the derivative vector γ' rotates clockwise from t_i to t_i^*, and -1 otherwise. More technically, the sign is defined as follows: set $v_1 := \gamma'(t_i)/\|\gamma'(t_i)\| \in \mathbb{S}^1$ and $v_2 = \gamma'(t_i^*)/\|\gamma'(t_i^*)\| \in \mathbb{S}^1$ to be the unit tangents of γ when at v_i. Then there is a unique angle $\theta \in (-\pi, \pi)$, with $\theta \neq \pi$ by regularity, such that v_2 is achieved by rotation by θ from v_1. Then $\mathrm{sgn}(p_i) := -\mathrm{sgn}(\theta)$. Note that $\mathrm{sgn}(p_i)$ depends on the basepoint of the curve. Here, we have only defined the sign of the crossing points. Signs of the basepoint will only be discussed for outer basepoints. In this case, $\mathrm{sgn}(p_0) = +1$ for p_0 a positive outer basepoint and $\mathrm{sgn}(p_0) = -1$ for p_0 a negative outer basepoint. As proved by Titus, interior boundariness is invariant with respect to signed intersection sequences [21]. In other words, any two curves with the same signed intersection

[4] We assume $\mathbb{S}^1 = \partial\mathbb{D}^2$.

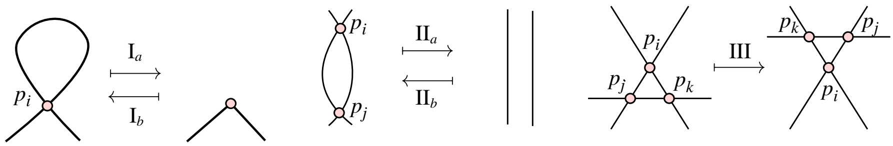

Fig. 6 All three homotopy moves and their reversals. Figure from [8]

sequence (and positive outer basepoint) will either both be interior boundaries or both not be.

2.3 Minimum Homotopies

A **homotopy** between two generic curves γ_0 and γ_1 is a continuous function $H\colon [0,1]^2 \to \mathbb{R}^2$ such that $H(0,\cdot) = \gamma_0$ and $H(1,\cdot) = \gamma_1$. In $\mathbb{R}^2$, any curve is nullhomotopic, i.e., homotopic to a constant map $\gamma_q(t) = q$ for some $q \in \mathbb{R}^2$. Given a sequence of homotopies $(H_i)_{i=1}^k$, we notate the concatenation of these homotopies in order as $\sum_{i=1}^k \boldsymbol{H_i}$. We use the notation $\overline{H}$ for the reversal $\overline{H}(i,t) = H(1-i,t)$ of a homotopy. If $H(0,\cdot) = \gamma_0$ and $H(1,\cdot) = \gamma_1$, we may write $\boldsymbol{\gamma_0} \overset{\boldsymbol{H}}{\rightsquigarrow} \boldsymbol{\gamma_1}$. For both homotopies H and curves γ, we write $\boldsymbol{H^{-1}(q)}$, $\boldsymbol{\gamma^{-1}(q)}$ to denote the standard set-theoretic pre-image of a point $q \in \mathbb{R}^2$.

Homotopy moves are basic local alterations to a curve defined by their action on $G(\gamma)$. These moves come in three pairs [8]; see Fig. 6: The I-moves destroy/create an empty loop, II-moves destroy/create a bigon, and III-moves flip a triangle. We denote the moves that remove vertices as $\mathbf{I}_{\boldsymbol{a}}$ and $\mathbf{II}_{\boldsymbol{a}}$, and moves that create vertices as $\mathbf{I}_{\boldsymbol{b}}$ and $\mathbf{II}_{\boldsymbol{b}}$. See Fig. 6. It is well known that any homotopy such that each intermediate curve is piecewise regular and generic, or almost generic, can be achieved by a sequence of homotopy moves. Thus, without loss of generality, we assume that each time the curve $H(i,\cdot)$ combinatorially changes is through a single homotopy move.

We define the homotopy area, as is standard practice. When discussing area, we need homotopies to be at least piecewise differentiable. We define

$$A(H) := \int_{[0,1]\times[0,1]} \left\| \frac{\partial H}{\partial s} \times \frac{\partial H}{\partial t} \right\| ds\ dt = \int_{[0,1]\times[0,1]} |\det J_H(s,t)|\ ds\ dt.$$

Here, $\overrightarrow{u} \times \overrightarrow{v} = u_1 v_2 - u_2 v_1$ for $\overrightarrow{u} = (u_1,u_2)$, $\overrightarrow{v} = (v_1,v_2)$, and

$$J_H = \begin{pmatrix} \overrightarrow{\dfrac{\partial H}{\partial s}} & \overrightarrow{\dfrac{\partial H}{\partial t}} \end{pmatrix}$$

is the Jacobian matrix of partials. The **minimum homotopy area** is

$$\sigma(\gamma) = \inf_{H \in \mathcal{N}(\gamma)} A(H),$$

where $\mathcal{N}(\gamma)$ is the set of piecewise differentiable nullhomotopies of γ. While the integral for $A(H)$ is improper, Hass' work extending the classical Douglas–Rado solution to *Plateau's problem* tells us that for $\gamma \in \mathscr{C}$, we have $\sigma(\gamma)$ realized by a smooth homotopy, excluding finitely many points [11]. With regularity assumptions on H, such as Lipschitz or piecewise smooth, the homotopy area reduces via the *area formula* of geometric measure theory to

$$A(H) = \int_{x \in \mathbb{R}^2} |H^{-1}(\{x\})| \, dx.$$

The following was shown in [3, 8]:

Lemma 2.1 (homotopy area ≥ winding area) *Let $\gamma \in \mathscr{C}$. Then $\sigma(\gamma) \geq W(\gamma)$.*

A straightforward proof by induction, similar to that of Lemma 2.1 shows the following.

Lemma 2.2 (homotopy area ≤ depth) *Let $\gamma \in \mathscr{C}$. Then $\sigma(\gamma) \leq D(\gamma)$.*

On the directed multigraph $G(\gamma)$, we can define the left and right face of any edge. We call a homotopy **left (right) sense-preserving** if $H(i+\epsilon, t)$ lies on or to the left (right) of the oriented curve $H(i, \cdot)$ for any $i, t \in [0, 1]$ and any $\epsilon > 0$. The following two lemmas provide useful properties about sense-preserving homotopies; the first was proven in [3], the second in [8].

Lemma 2.3 (monotonicity of winding numbers) *Let H be a homotopy. If H is left (right) sense-preserving, then for any $x \in \mathbb{R}^2$ the function $a(i) = \mathrm{wn}(x, H(i, \cdot))$ is monotonically decreasing (increasing).*

Lemma 2.4 (sense-preserving homotopies are optimal) *Let $\gamma \in C$ be consistent. Then a nullhomotopy H of γ is optimal if and only if it is sense-preserving.*

3 Equivalences

In this section, we show the equivalence of different characterizations of interior boundaries (Theorem 3.6) and of self-overlapping curves (Theorem 3.7). Our analysis of curve classes hinges around the concept of obstinance. In Theorem 3.10 we classify zero obstinance curves, which are generalizations of interior boundaries and of self-overlapping curves.

3.1 Direct Splits

Let $\gamma \in \mathscr{C}$ and $p_i \in V(\gamma)$ with $p_i = \gamma(t_i) = \gamma(t_i^*)$ and $t_i < t_i^*$. Then, γ can be split into two subcurves at p_i: The **direct split** is the curve with image $[\gamma|_{[t_i, t_i^*]}]$ with basepoint p_i, and the **indirect split** is the curve with image $[\gamma|_{[t_i^*, 1]}] \cup [\gamma|_{[0, t_i]}]$ with basepoint $\gamma(0)$. We endow both of these curves with the same orientation as γ. Given a direct (or indirect) split $\widetilde{\gamma}$ on a curve γ, we write $\gamma \setminus \widetilde{\gamma}$ for the indirect (or direct)

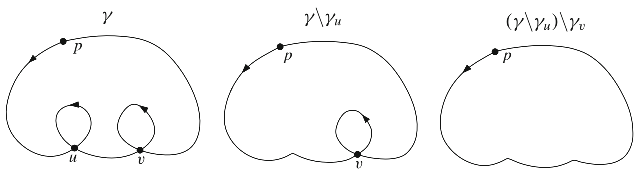

Fig. 7 A curve γ and a subcurve $(\gamma \setminus \gamma_u) \setminus \gamma_v$ that is an indirect split (of an indirect split of γ) yet is not an indirect split itself of γ

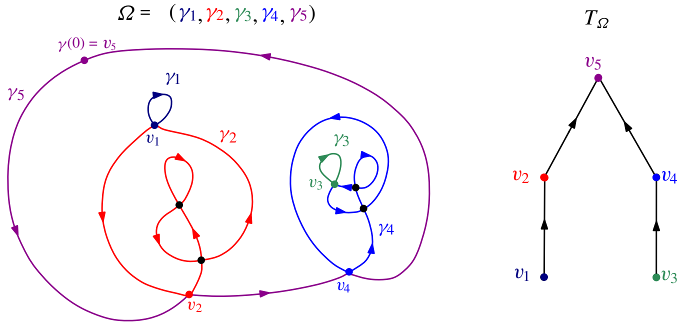

Fig. 8 A self-overlapping decomposition of a self-overlapping curve γ. Here, γ_1 and γ_3 are (proper) direct splits of γ, while γ_2, γ_4, and γ_5 are neither direct nor indirect splits of γ

split complementary to $\widetilde{\gamma}$. We call a direct split **proper** if it is not the entire curve γ. See Fig. 8. If $v = p_i \in V(\gamma)$, we may notate the direct split as γ_i or γ_v. The direct splits carry a great deal of information about the curve. In fact, one can recover the combinatorial relations from the direct splits: $p_i \mathrel{\mathrm{L}} p_j$ iff $p_i \in [\gamma_j]$ and $p_j \in [\gamma_i]$; $p_i \mathrel{\mathrm{S}} p_j$ iff $p_i \notin [\gamma_j]$ and $p_j \notin [\gamma_i]$; and $p_i \subset p_j$ iff p_i is a vertex of γ_j. Being a direct split of a curve is a transitive property. I.e., if $\gamma_i \in \mathscr{C}$ is a direct split on γ, and γ_j is a direct split on γ_i, then γ_j is a direct split on γ. The parallel statement on indirect splits, however, is false. See Fig. 7.

3.2 Decompositions and Loops

A curve $\gamma \in \mathscr{C}$ can be entirely decomposed by iteratively removing direct splits. Given a direct split γ_1 of $\gamma = C_0$, set $C_1 = C_0 \setminus \gamma_1$. Then, inductively take γ_i as a direct split on C_{i-1} and form $C_i = C_{i-1} \setminus \gamma_i$. Iterating until the current curve C_k is simple, we get a decomposition $\Omega = (\gamma_i)_{i=1}^k$ of γ. Note that Ω nearly induces a partition of γ in the sense that $[\gamma] = \bigcup_{i=1}^k [\gamma_i]$ and $\gamma_i \cap \gamma_j \subset V(\gamma)$ for any $i \neq j$. We call Ω a **direct split decomposition** if γ_i is a direct split of C_{i-1}, for all $i \in \{1, 2, \ldots, k\}$. Given a direct split decomposition $\Omega = (\gamma_i)_{i=1}^k$, we write $\boldsymbol{V(\Omega)}$ for the set of basepoints of all $\gamma_i \in \Omega$. See Fig. 8.

Observe that no two vertices $v_i, v_j \in V(\Omega)$ may be linked. Hence, we obtain a partial order $\prec$ on $V(\Omega)$ by declaring $v_i \prec v_j$ whenever $v_i \subset v_j$. We define $\boldsymbol{T_\Omega}$ to be the rooted, directed tree with vertex set $V(T_\Omega) = V(\Omega)$ and edges $e = (v_i, v_j)$ whenever $v_i \subset v_j$ and there is no other vertex $v_k \neq v_i, v_j$ such that $v_i \subset v_k \subset v_j$.

The root of T_Ω corresponds to the basepoint of γ. We consider two direct split decompositions Ω, Γ equivalent, $\Omega \sim \Gamma$, when $T_\Omega = T_\Gamma$. One can easily verify that $\sim$ is an equivalence relation on the set of direct split decompositions of γ. This means that $\Omega = \Gamma$ as sets; the decompositions contain the same elements, just in a different order. If every γ_i is self-overlapping, we call Ω a **self-overlapping decomposition**; it may contain self-overlapping direct splits of positive and negative orientations. The vertex set of a decomposition already determines the direct splits in the decomposition:

Lemma 3.1 *Given a curve $\gamma \in \mathscr{C}$ and a subset $S \subset V(\gamma)$ such that $p_0 \in S$ and no two vertices in S are linked, there exists a unique equivalence class $\mathscr{E}$ of direct split decompositions with $V(\Omega) = S$ for all $\Omega \in \mathscr{E}$.*

Proof Let Ω, Ψ be direct split decompositions of γ with $V(\Omega) = S$. Once we prescribe S, the direct splits appearing in Ω are determined; only the ordering may vary. By definition, we see $T_\Omega = T_\Psi$ as they have the same vertex sets and relations. Hence, $\Omega \sim \Psi$. Thus, we need only prove existence.

We now provide a simple inductive algorithm to build Ω with $V(\Omega) = S$. First, take a minimal element $w \in S$ with respect to $\prec$. Then we set $\gamma_1 := \gamma_w$, the direct split at w. Also, set $C_1 := \gamma \setminus \gamma_1$. Inductively, suppose we have $\gamma_1, \ldots, \gamma_k$ along with basepoints $v_i \in S$ so that γ_i is a direct split on C_{i-1}. Here, $C_i = \gamma \setminus \bigcup_{i=1}^{k} \gamma_i$ tracks the current curve after the first k removals of direct splits. The fact that no $u, v \in S$ are linked guarantees that all of $S - \{v_i\}_{i=1}^{k}$ appear as vertices on C_k $(*)$. Now, take $u \in S - \{v_i\}_{i=1}^{k}$ minimal with respect to $\prec$ again and set $\gamma_{i+1} := (C_i)_u$. It follows by induction that eventually C_l is a simple curve for some index l. Since $p_0 \in S$, we then set $\gamma_{l+1} := C_l$. Thus, $\Omega := (\gamma_i)_{i=1}^{l+1}$ is a direct split decomposition of γ. It is immediate that $V(\Omega) \subset S$. By $(*)$, we see $V(\Omega) \supset S$. □

The observation below follows directly from the definition of winding numbers.

Observation 3.2 *Let Ω be a direct split decomposition of a curve $\gamma \in \mathscr{C}$. Then for any face F in the plane multigraph $G(\gamma)$, $\mathrm{wn}(F, \gamma) = \sum_{\gamma_i \in \Omega} \mathrm{wn}(F, \gamma_i)$.*

We define a **loop** as a simple direct split γ_v of a curve $\gamma \in \mathscr{C}$. Intersection points of γ may lie on γ_v, but none occur as intersections of γ_v with itself. Every non-simple plane curve has a loop; e.g., the direct split γ_w, where w is the highest index vertex on γ in the signed intersection sequence. A loop γ_v is **empty** if v links no vertex $w \in V(\gamma)$. Let $\mathrm{int}(\gamma_v)$ denote its interior, as a set. We call γ_v an **outwards loop** if the edges e_1, e_4, that are incident on v and lie on $\gamma \setminus \gamma_v$, both lie outside $\mathrm{int}(\gamma_v)$. Otherwise γ_v is an **inwards loop**. See Fig. 9. In the case that $\gamma \in \mathscr{C}$ is a simple curve, we regard it as an outwards loop.

The lemma below follows from [9, 18]. Since it requires a digression from our main focus, its proof is given in Appendix A.

Lemma 3.3 (Whitney index through decompositions) *Let $\gamma \in \mathscr{C}$ and Ω be a direct split decomposition of γ. Then* $\mathrm{WHIT}(\gamma) = \sum_{C \in \Omega} \mathrm{WHIT}(C)$.

A consequence of Lemma 3.3 is that iteratively removing loops and summing ± 1 for their signs allows one to quickly compute Whitney indices. Assuming γ is given as a directed plane multigraph, one can adapt a depth-first traversal to compute such a loop decomposition of γ in $O(|\gamma|)$ time, which yields the following corollary:

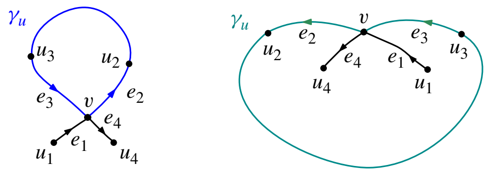

Fig. 9 An outwards loop (left) and an inwards loop (right)

Corollary 3.4 (compute Whitney index) *Let $\gamma \in \mathscr{C}$ be of complexity $n = |\gamma| = |V(\gamma)|$. One can compute a loop decomposition of γ, and $\mathrm{WHIT}(\gamma)$, in $O(n)$ time.*

3.3 Well-Behaved Homotopies

Let H be a nullhomotopy of a curve γ, and consider all the points $A = \{v_i\}_{i=1}^{k}$ of $\mathbb{R}^2$ such that H performs a I_a move to contract a loop to that point. All such points are called **anchor points** of the homotopy H. Following [8] we call a homotopy H **well-behaved** when the anchor points A of H satisfy $A \subseteq V(\gamma)$, i.e., H only contracts loops to vertices of the original curve, not to new vertices created along the way by H. The theorem below from [8] shows that computing minimum homotopy area is reduced to finding an optimal self-overlapping decomposition. The homotopy H guaranteed in the following theorem is well behaved.

Theorem 3.5 (minimum homotopy decompositions) *Let $\gamma \in \mathscr{C}$. Then there is a self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^{k}$ of γ as well as an associated minimum homotopy H_Ω of γ such that $H_\Omega = \sum_{i=1}^{k} H_i$ and each H_i is a nullhomotopy of γ_i. In particular, $\sigma(\gamma) = \min_{\Omega \in \mathscr{D}(\gamma)} \sum_{C \in \Omega} W(C)$, where $\mathscr{D}(\gamma)$ is the set of all self-overlapping decompositions of γ.*

3.4 Equivalence of Interior Boundaries

In this section, we unify different definitions and characterizations of interior boundaries by showing their equivalence. We prefer to begin with interior boundaries, so that self-overlapping curves can just be thought of as 1-interior boundaries in Sect. 3.5. We call a curve γ a ***k*-interior boundary** when (1) $\mathrm{obs}(\gamma) = \sigma(\gamma) - W(\gamma) = 0$, (2) $\mathrm{WHIT}(\gamma) = k > 0$, and (3) γ is positive consistent. We call γ a **(−*k*)-interior boundary** when its reversal $\overline{\gamma}$ is a k-interior boundary. In accordance with Titus [21], we call a curve $\zeta \colon [0, 1] \to \mathbb{R}^2$ a **Titus interior boundary** if there exists a map $F \colon \mathbb{D}^2 \to \mathbb{R}^2$ such that F is continuous on $\mathbb{D}^2$, **properly interior** on $\mathrm{int}(\mathbb{D}^2)$, and $F|_{\mathbb{S}^1} = \zeta$. Here, properly interior means that pre-images are totally disconnected, and that the map is open and orientation-preserving.

We prove the equivalence of these definitions and of two further characterizations below.

Theorem 3.6 (equivalence of interior boundaries) *Let $\gamma \in \mathscr{C}$ have $\mathrm{WHIT}(\gamma) = k > 0$. Then, the following are equivalent:*

(i) *γ is an interior boundary.*

(ii) γ *is a Titus interior boundary.*
(iii) γ *admits a self-overlapping decomposition* $\Omega = (\gamma_i)_{i=1}^k$, *where each* γ_i *is positive self-overlapping.*
(iv) γ *admits a well-behaved left sense-preserving nullhomotopy* H *with exactly* k I_a*-moves.*

Proof (iii) ⇒ (ii): We proceed by induction on k. Observe that the base case is merely that γ self-overlapping implies γ is a Titus interior boundary. We prove this now. Let F be continuous on $\mathbb{D}^2$, an immersion of $\mathrm{int}(\mathbb{D}^2)$, with $\overline{F}|_{\mathbb{S}^1} = \gamma$. Since F is a local diffeomorphism, it is open. An immersion is orientation-preserving by definition. Finally, F a local diffeomorphism with continuous boundary values implies $F^{-1}(x)$ is finite for any $x \in \mathbb{R}^2$, so F is totally disconnected. We conclude that F is properly interior on $\mathrm{int}(\mathbb{D}^2)$ so that γ is a Titus interior boundary.

Now let $k \geq 1$ and let our inductive assumption be that for all $c \in \{1, 2, \ldots, k\}$, if $\gamma \in \mathscr{C}$, $\mathrm{WHIT}(\gamma) = +c$, and γ has a decomposition Ω into c positive self-overlapping curves, then γ is a Titus interior boundary. Now, let $\gamma \in \mathscr{C}$ such that $\mathrm{WHIT}(\gamma) = k + 1$, and γ is an interior boundary. Here, we consider the curve γ as coming in two pieces, $\gamma_1 :=$ the first *self-overlapping* direct split from Ω and its complement $\gamma_2 := \gamma \setminus \gamma_1$. By Lemma 3.3, $\mathrm{WHIT}(\gamma) = \mathrm{WHIT}(\gamma_1) + \mathrm{WHIT}(\gamma_2)$, so $\mathrm{WHIT}(\gamma_2) = k$. Since $\Omega - \{\gamma_1\}$ is a self-overlapping decomposition of C, by inductive hypothesis, C is a Titus interior boundary. Similarly, by the base case γ_1 is a Titus interior boundary. Hence, there exist continuous maps $F_1, F_2 \colon \mathbb{D}^2 \to \mathbb{R}^2$ properly interior on $\mathrm{int}(\mathbb{D}^2)$ such that $F_1|_{\mathbb{S}^1} = \gamma_1$ and $F_2|_{\mathbb{S}^1} = \gamma_2$. Here, Titus provides the final trick—we can glue these two curves together at v by finding an arc interior to both. (We refer the reader to Titus' paper [21] to see a comprehensive explanation of his trick of gluing two properly interior mappings together along an interior arc.) The resulting map $F \colon \mathbb{D}^2 \# \mathbb{D}^2 \to \mathbb{R}^2$, where # denotes the connected sum, extends both F_1 and F_2 and represents the curve γ, i.e., $F|_{\mathbb{S}^1} = \gamma$. Moreover, F is a properly interior map on $\mathrm{int}(\mathbb{D}^2)$. Thus, we have completed the inductive step.

(ii) ⇒ (i): Let γ be a Titus interior boundary. By [21], we have $|\gamma^{-1}(x)| = \mathrm{wn}(x, \gamma)$ for all x. Thus, γ is positive-consistent. Select a continuous map F, properly interior on $\mathrm{int}(\mathbb{D}^2)$ with boundary values γ. Then we see a linear retraction of the disk $r_t \colon \mathbb{D}^2 \to \mathbb{D}^2$ induces a homotopy H with $A(H) = W(\gamma)$, via $H_t(\,\cdot\,) = F(r_t(\,\cdot\,))$. By Lemma 2.1, we have $\sigma(\gamma) = W(\gamma)$. Thus, γ is an interior boundary, and so (ii) ⇒ (i).

(i) ⇒ (iii): Let γ be an interior boundary. By Theorem 3.5, we have an optimal self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^j$ of γ. Suppose, by contradiction, that there exists an $l \leq j$ such that γ_l is negative self-overlapping. Let F be any face contained in the interior $\mathrm{int}(\gamma_l)$. We know by Observation 3.2 that $\mathrm{wn}(F, \gamma) = \sum_{i=1}^j \mathrm{wn}(F, \gamma_i)$, and since γ is positive consistent $\mathrm{wn}(F, \gamma) \geq 0$. Thus there must exist a positive self-overlapping curve $\gamma_i \in \Omega$ with $F \subseteq \mathrm{int}(\gamma_i)$. Consider the nullhomotopies H_l and H_i that are part of the canonical optimal homotopy H_Ω. Then H_l contracts γ_l and is right sense-preserving, while H_i contracts H_i and is left sense-preserving. Thus by Lemma 2.3, H_l increases the winding number on F and H_i decreases the winding number, which means F is swept more than $W(F)$ times, a contradiction. Thus, no negative self-overlapping subcurve γ_l may exist in Ω. Since $\mathrm{WHIT}(C) = 1$ for any positive self-overlapping subcurve and $\mathrm{WHIT}(\gamma) = \sum_{i=1}^{k} \mathrm{WHIT}(\gamma_i)$ by Lemma 3.3, we we must have $k = j$.

(iii) ⇒ (i): Suppose γ has a decomposition Ω into k positive self-overlapping subcurves. Then, the canonical homotopy H_Ω associated to Ω is left sense-preserving, since each minimum nullhomotopy of the subcurves is left sense-preserving. Since sense-preserving homotopies are optimal, see Lemma 2.4, we have $\sigma(\gamma) = W(\gamma)$. For any self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^{k}$, we may conclude that $\mathrm{wn}(x, \gamma) = \sum_{i=1}^{k} \mathrm{wn}(x, \gamma_i)$ by Observation 3.2. Thus, $\mathrm{wn}(x, \gamma) = \sum_{i=1}^{k} \mathrm{wn}(x, \gamma_i) \geq 0$ since each γ_i is positive consistent, as a positive self-overlapping curve.

(i) ⇔ (iv): If γ has a well-behaved left sense-preserving nullhomotopy H with exactly k I_a-moves, then H comes naturally with an associated self-overlapping decomposition Ω of γ with $|\Omega| = k$, and $\mathrm{WHIT}(\gamma) = k > 0$ by Lemma 3.3. We now show that $\sigma(\gamma) = W(\gamma)$. Consider the reversal $\overline{H}$ from the constant curve $\gamma_{p_0}(t) = p_0$ to γ. Then, $\overline{H}$ is right sense-preserving and by Lemma 2.3 the function $a(i) = \mathrm{wn}(x, \overline{H}(i, \cdot))$ is monotonically increasing for any $x \in \mathbb{R}^2$. Since $\mathrm{wn}(x, \gamma_{p_0}) = 0$ for all $x \in \mathbb{R}^2$, we have that $\mathrm{wn}(x, \gamma) \geq 0$ for all $x \in \mathbb{R}^2$. Thus, γ is an interior boundary. Conversely, if γ is a positive interior boundary, then $\mathrm{obs}(\gamma) = 0$ and by Lemma 2.4, and since γ is positive, H is left sense-preserving. Again, by Lemma 3.3, $\mathrm{WHIT}(\gamma) = j$, where j is the number of I_a-moves in any well-behaved nullhomotopy H of γ. Hence, we must have $j = k$, as desired. □

3.5 Equivalences of Self-Overlapping Curves

In this section, we study different characterizations of self-overlapping curves and show their equivalence in Theorem 3.7, which also shows that self-overlapping curves are 1-interior boundaries.

First we describe a geometric formulation of self-overlappingness, inspired by the work of Blank and Marx [1, 15]. Let $\gamma \in \mathscr{C}$ be self-overlapping. Let $P\colon [0, 1] \to \mathbb{R}^2$ be a smooth path between $P(0) = q = \gamma(t_q)$ and $P(1) = p = \gamma(t_p)$, where $p, q \in [\gamma] \setminus V(\gamma)$ are not vertices. Without loss of generality, assume $t_q < t_p$. Let $\widetilde{P} := \gamma|_{[t_q, t_p]}$, and suppose that

- $P \cap \widetilde{P} = \{p, q\}$,
- $C = \widetilde{P} * \overline{P}$ is a simple closed curve,
- C is positively oriented, and
- P crosses γ at p from left to right and at q from right to left (or it crosses at p from right to left and at q from left to right).

See Fig. 10 as well as Fig. 1. Then we call P a **Blank cut** of γ. Any such cut splits γ into two pieces: γ_1 and C. We imagine the Blank cut as removing $C = \widetilde{P} * \overline{P}$ from γ, leaving only γ_1; this can be done with a left sense-preserving homotopy that sweeps the arc $\widetilde{P}$ to the arc P. We are interested in iteratively performing Blank cuts. Set $\gamma_0 := \gamma$ and let P_1 be a Blank cut on γ_0. We obtain γ_1 as above by removing the simple closed curve $\widetilde{P_1} * \overline{P_1}$. Then, inductively take a smooth path P_i as a Blank cut on γ_{i-1}, and let γ_i be the curve after removing $\widetilde{P_i} * \overline{P_i}$. We call $(P_i)_{i=1}^{k}$ a **Blank cut decomposition** of γ when the final curve γ_k is a simple positively oriented curve.

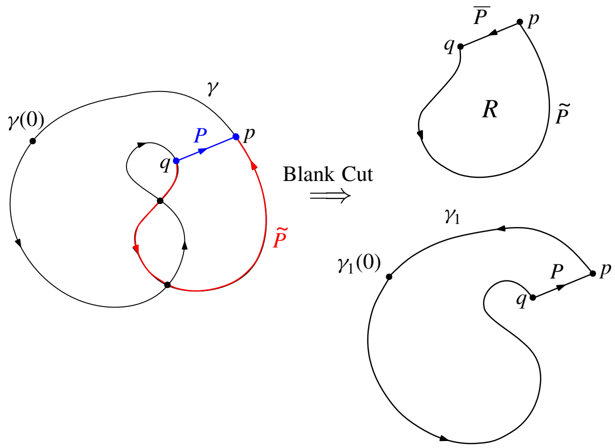

Fig. 10 A Blank cut on a small self-overlapping curve

Theorem 3.7 (equivalent characterizations of self-overlapping curves) *Let* $\gamma \in \mathscr{C}$ *have* $\mathrm{WHIT}(\gamma) = 1$. *Then the following are equivalent:*

(i) (Analysis) γ *is self-overlapping.*
(ii) (Geometry) γ *admits a Blank cut decomposition.*
(iii) (Geometry/Topology) γ *is a* 1*-interior boundary.*
(iv) (Topology) γ *admits a left-sense preserving nullhomotopy* H *with exactly one* I_a*-move.*
(v) (Analysis) γ *is a Titus interior boundary.*

Proof By property (iii) in Theorem 3.6, self-overlapping curves are 1-interior boundaries, since any self-overlapping curve γ has the trivial self-overlapping decomposition $\Omega = (\gamma)$. Thus, we have already established (i) ⇔ (iii) ⇔ (iv) ⇔ (v) in Theorem 3.6. We now prove (ii) ⇔ (iii).

(ii) ⇒ (iii): Any Blank cut P amounts to a left sense-preserving homotopy that deforms $\widetilde{P}$ to P. Hence, iterating these homotopies, the Blank cut decomposition corresponds to a left sense-preserving homotopy from γ to a simple, positively oriented curve. Finally we perform a single I_a-move to complete a left sense-preserving nullhomotopy of γ. By Lemma 2.4, we have $\sigma(\gamma) = W(\gamma)$. Hence, $\mathrm{obs}(\gamma) = 0$. We now examine the reversal $\bar{H}$, from the constant curve $\gamma_{p_0}(t) = p_0$ to γ. Then, $\overline{H}$ is right sense-preserving and by Lemma 2.3 the function $a(i) = \mathrm{wn}(x, \overline{H}(i, \cdot))$ is monotonically increasing for any $x \in \mathbb{R}^2$. Since $\mathrm{wn}(x, \gamma_{p_0}) = 0$ for all $x \in \mathbb{R}^2$, we have that $\mathrm{wn}(x, \gamma) \geq 0$ for all $x \in \mathbb{R}^2$.

(iii) ⇒ (ii): Conversely, let γ have a left sense-preserving nullhomotopy H with one I_a move. As H ends with a I_a move, we may select a subhomotopy H' such that $\gamma \overset{H'}{\rightsquigarrow} C$, where C is a simple self-overlapping curve. Moreover, we may demand $H = H' + H''$, where the unique I_a-move of H occurs during H''. Thus, H' is regular, i.e., consists of a sequence of homotopy moves only of types II_a, II_b, or III,

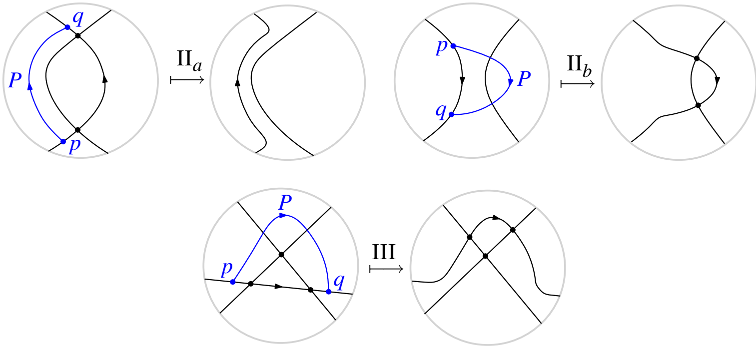

Fig. 11 Homotopy moves II_a, II_b, and III each correspond to a Blank cut (shown in blue)

which deform γ to C. Each of these homotopy moves is equivalent to a Blank cut, as shown in Fig. 11. Thus, γ admits a Blank cut decomposition. □

The following two lemmas provide useful properties of self-overlapping curves, the first of which was proved in [21, Thm. 5].

Lemma 3.8 (empty positively oriented loop) *Let $\gamma \in \mathscr{C}$ have a positive outer basepoint and an empty positively oriented loop. Then, γ is not self-overlapping.*

We conclude this section with a simple yet powerful lemma.

Lemma 3.9 (sense-preserving homotopies) *Let H be a regular homotopy with $\gamma \overset{H}{\rightsquigarrow} \gamma'$.*

(i) *If H is right sense-preserving and γ is self-overlapping, then γ' is self-overlapping.*
(ii) *If H is left sense-preserving and γ is not self-overlapping, then γ' is not self-overlapping.*

Proof To prove (i), assume γ is self-overlapping. Then it has a left sense-preserving nullhomotopy H' by Theorem 3.7. Let us reverse our given homotopy H to obtain $\overline{H}$ by $\overline{H}(s,t) = H(1-s,t)$. Then we note that the concatenation $H'' = \overline{H} + H'$ is a left sense-preserving nullhomotopy for γ'. Since sense-preserving homotopies are optimal, $\sigma(\gamma') = W(\gamma')$. Also, as H'' is regular, $W(\gamma') = W(\gamma) = 1$. Applying Theorem 3.7 again, we conclude that γ' is self-overlapping. Part (ii) follows by contrapositive with a single application of (i): if γ' were self-overlapping then γ must be self-overlapping as well. □

3.6 Zero Obstinance Curves

In this section, we classify curves $\gamma \in \mathscr{C}$ with **zero obstinance**, $\text{obs}(\gamma) := \sigma(\gamma) - W(\gamma) = 0$. See Figs. 2 and 12 for examples of zero-obstinance curves. We show that

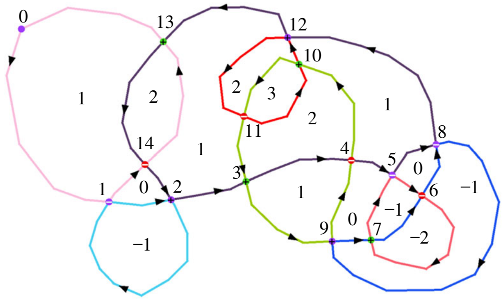

Fig. 12 A zero obstinance curve, with its minimum homotopy decomposition, and winding numbers shown. Each curve in the decomposition is self-overlapping and shown in a different color. The vertices with labels 1, 2, 5, 8, 9 are sign-changing

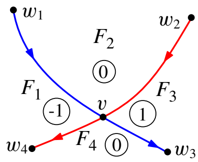

Fig. 13 A sign-changing vertex v. The winding numbers of the faces incident to v, are up to cyclic reordering, $-1, 0, 1, 0$

just as interior boundaries can be decomposed into self-overlapping curves, so too can zero-obstinance curves be decomposed into interior boundaries.

If a curve γ has zero obstinance, then there is a nullhomotopy H which sweeps each face F on $G(\gamma)$ exactly $\mathrm{wn}(F, \gamma)$ times. Note that such a homotopy H is necessarily minimal by Lemma 2.1. Intuitively, this implies that the homotopy H should be locally sense-preserving. We expect it to sweep leftwards on positive consistent regions and rightwards on negative consistent regions. Hence, we might expect regions of the curve where the winding numbers change from positive to negative to be especially problematic. Indeed, let $v \in V(\gamma)$ be incident to the faces $\{F_1, F_2, F_3, F_4\}$. We call v **sign-changing** when, as a multiset, $\{\mathrm{wn}(\gamma, F_1), \mathrm{wn}(\gamma, F_2), \mathrm{wn}(\gamma, F_3), \mathrm{wn}(\gamma, F_4)\} = \{-1, 0, 0, 1\}$; see Figs. 12 and 13.

Theorem 3.10 (zero obstinance characterization) *Let $\gamma \in \mathscr{C}$ and let $\mathscr{S}$ be the sign-changing vertices of γ. Then* $\mathrm{obs}(\gamma) = 0$ *iff no two vertices in $\mathscr{S}$ are linked and any direct split decomposition Ω with vertex set $V(\Omega) = \mathscr{S} \cup \{p_0\}$ contains only interior boundaries.*

Proof Suppose $\mathrm{obs}(\gamma) = 0$. By definition, any zero obstinance curve with consistent winding numbers must be an interior boundary and $\mathscr{S} = \emptyset$. Hence, suppose γ is inconsistent so that $\mathscr{S} \neq \emptyset$. We claim any sign-changing vertex v is an anchor point of every well-behaved minimum homotopy H of γ of the form guaranteed by Theorem 3.5. Let us now proceed by contradiction. Suppose $v \in V(\gamma)$ is a sign-changing vertex with incident faces labeled as in Fig. 13 such that v is not an anchor point of a minimum homotopy H for γ. Write $\Gamma(H)$ as the self-overlapping decomposition of H. As γ has $\mathrm{obs}(\gamma) = 0$, we know that $W(\gamma) = \sigma(\gamma) = A(H)$. In particular, the homotopy H sweeps each face $F \in G(\gamma)$ precisely $\mathrm{wn}(\gamma, F)$ times. Of course, since our homotopy H consists of a sequence of nullhomotopies of self-overlapping subcurves, this means each face F must lie in the interior of $\mathrm{wn}(\gamma, F)$ distinct self-overlapping subcurves $C \in \Gamma(H)$. In particular, if either face F_2, F_4 incident to v is contained in the interior of any curve $C \in \Gamma(H)$, we have a contradiction. Let us now examine the edge $e = (v, w_3)$. This edge must lay on precisely one subcurve $C \in \Gamma(H)$ by our definition of a direct split decomposition. We now have two cases.

Case 1: C is positively self-overlapping. We now recall that for a positive self-overlapping curve, the interior of the curve always lies, locally at each edge, to the left. Since v is not an anchor point of H, it must be the case that C also contains the edge $e_1 = (w_1, v)$. As the face F_2 lies to the left of e_1, this implies $F_2 \subset \mathrm{int}(C)$.

Case 2: C is negative self-overlapping. Here, we use that the interior of a negative self-overlapping curve lies locally to the right. In this case, we see that F_4, lying to the right of edge e_1, satisfies $F_4 \subset \mathrm{int}(C)$.

We conclude that all sign-changing vertices are anchor points of H. This is only possible if none of the sign-changing vertices link each other. Now, let $\Theta = (\gamma_i)_{i=1}^k$ be any direct split decomposition with $V(\Theta) = \mathscr{S} \cup \{p_0\}$. We claim that each curve $\gamma_i \in \Theta$ is an interior boundary. It suffices to prove this claim for any decomposition $\Omega \sim \Theta$. So, we now choose an Ω with ordering compatible with the ordering of $\Gamma(H)$. Note here that each sign-changing vertex is an anchor point of H, so $V(\Omega) \subset V(\Gamma(H))$. Set $\gamma_i \in \Omega$, $\psi_i \in \Gamma(H)$ as the subcurves with basepoint $v_i \in V(\Omega)$. Then Ω is prescribed by demanding that γ_i appear in the same order on Ω as the subcurves ψ_i on $\Gamma(H)$.

Now, by Theorem 3.5, there is a subhomotopy H_i of H which is a nullhomotopy of γ_i. As subhomotopies of a minimum homotopy H, each H_i must be minimum as well, $A(H_i) = \sigma(\gamma_i)$. Observe that

$$\sum_{i=1}^{k} W(\gamma_i) = W(\gamma) = A(H) = \sum_{i=1}^{k} A(H_i) = \sum_{i=1}^{k} \sigma(\gamma_i).$$

So, we must have equality, $\sigma(\gamma_i) = W(\gamma_i)$, for every curve in the decomposition. By Lemma 2.1, this means each $\gamma_i \in \Omega$ has $\mathrm{obs}(\gamma_i) = 0$. Hence, if each γ_i is consistent, they are all interior boundaries, by definition. Of course, if some γ_i were inconsistent, then the winding numbers would change somewhere along the curve. Wherever the winding numbers of γ_i change, we will see a sign-changing vertex $u \in V(\gamma_i)$. But since u is not the basepoint of γ_i, this is a contradiction. Indeed, by the fact that $\mathscr{S} \cup \{p_0\} = V(\Omega)$, no sign-changing vertex can be a crossing point on a curve $\gamma_i \in \Omega$.

Conversely, suppose no sign-changing vertices link each other and that each decomposition $\Omega = (\gamma_i)_{i=1}^k$ of γ with vertex set $V(\Omega) = \mathscr{S} \cup \{p_0\}$ contains only interior boundaries. Then let H_Ω be the homotopy associated to Ω. Thus, we have $W(\gamma) = \sum_{i=1}^k W(\gamma_i) = \sum_{i=1}^k \sigma(\gamma_i) = A(H)$. We conclude H is optimal and $\sigma(\gamma) = W(\gamma)$. Thus, $\mathrm{obs}(\gamma) = 0$. ◻

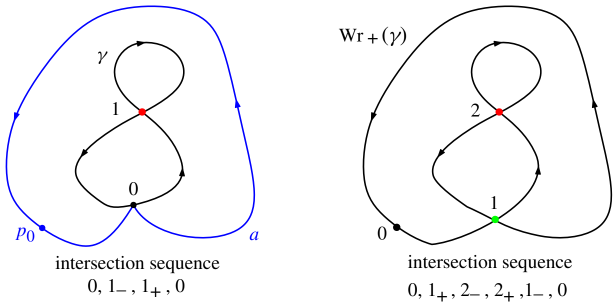

Fig. 14 Left: A curve γ (shown in black) with positive outer basepoint, and the curve α (shown in blue). Right: The *positive* wrap $\mathrm{Wr}_{+}(\gamma)$. While γ is not self-overlapping, its wrap $\mathrm{Wr}_{+}(\gamma)$ is self-overlapping

4 Wraps and Irreducability

In this section, we show (Theorems 4.3 and 4.4) that wrapping around a curve γ until its obstinance is reduced to zero results in an interior boundary. This key result is used to prove sufficient combinatorial conditions for a curve to be self-overlapping based on the Whitney index of the curve and its direct splits (Theorem 4.6 and Corollary 4.8).

4.1 Wraps

Let us now define the construction of the wrap of a curve. Let $\gamma \in \mathscr{C}$, and let I be its signed intersection sequence. Form I' by removing the occurrences of 0 (corresponding to the basepoint) and incrementing each label by one. If γ has a positive outer basepoint $\gamma(0)$, then its (**positive**) **wrap** $\mathrm{Wr}_{+}(\gamma)$ is the unique (class) of curves with signed intersection sequence $0, 1_+, I', 1_-, 0$. This corresponds to gluing a simple positively oriented curve α to γ at $\gamma(0)$, where the interior $\mathrm{int}(\alpha) \supseteq [\gamma]$; the new basepoint $p_0 = \mathrm{Wr}_{+}(\gamma)(0)$ is on α. See Fig. 14. If γ has a negative outer basepoint, the **negative wrap** $\mathrm{Wr}_{-}(\gamma)$ is defined by reversing orientations: $\mathrm{Wr}_{-}(\gamma) = \overline{\mathrm{Wr}_{+}(\overline{\gamma})}$; this corresponds to gluing a simple negatively oriented curve to γ at $\gamma(0)$. We write $\mathrm{Wr}_{+}^{k}(\gamma)$ for the curve achieved from γ by wrapping k times.

To wrap a curve in the direction opposed to the sign of the basepoint, we must be more careful. Without loss of generality, we describe the construction of $\mathrm{Wr}_{-}(\gamma)$ when γ has a positive outer basepoint. Perform a I_b-move to add a simple loop $\tilde{\gamma}$ of the opposite orientation tangent to the basepoint $\gamma(0)$. Let γ' be the curve after the I_b-move, with a basepoint chosen to lie on $\tilde{\gamma}$. We then define $\mathrm{Wr}_{-}(\gamma) = \mathrm{Wr}_{-}(\gamma')$. See Fig. 15.

Clearly one can always wrap any curve $\gamma \in \mathscr{C}$ a sufficient number of times to make $\mathrm{Wr}_{+}^{k}(\gamma)$ positive consistent. Indeed, setting k to be the maximum depth across all faces in $G(\gamma)$ suffices. On the other hand, it is not at all obvious that wrapping always turns a curve into an interior boundary. We prove in Theorem 4.3 that, in fact, positively wrapping always transforms a curve $\gamma \in \mathscr{C}$ with positive outer basepoint into a positive interior boundary. Thus one can think of wrapping as a rectifying

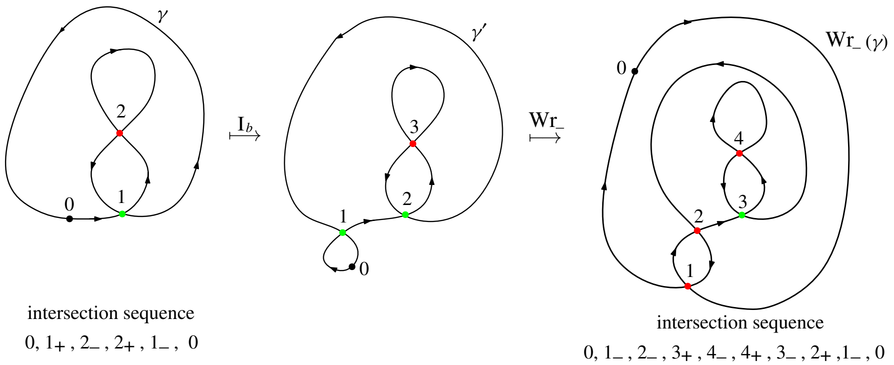

Fig. 15 A curve γ with positive outer basepoint and its transformation into its *negative* wrap $\mathrm{Wr}_-(\gamma)$. First, we perform a I_b-move and then wrap normally on γ'

operation with respect to minimum homotopy, as it always eventually removes all obstinance.

4.2 Simple Path Decompositions

We now describe another type of decomposition for $\gamma \in \mathscr{C}$ that we will need for proving Theorem 4.4. First we prove a simple lemma which states that a curve $\gamma \in \mathscr{C}$ with an outer basepoint has an outwards loop.

Lemma 4.1 (existence of an outwards loop) *Let $\gamma \in \mathscr{C}$ be non-simple and have an outer basepoint. Then γ has an outwards loop.*

Proof Let v be the first self-intersection of γ. Note that $v \neq \gamma(0)$ since γ is not simple. Then γ_v is a loop. Write $\gamma^{-1}(v) = \{t, t^*\}$, where $t < t_*$. Since $\gamma(0)$ lies outside of $\mathrm{int}(\gamma_v)$, as an outer basepoint, we note that if γ_v were inwards, the path $P = \gamma_{[0,t]}$ would cross $[\gamma_v]$ to get from outside the simple curve to inside it. This is then a contradiction, for if the crossing occurred at a point q on $[\gamma_v]$, then q would be the first self-intersection of γ. Indeed, we would reach q a second time before we reach v a second time. Thus, γ_v is outwards. □

We now introduce another decomposition. Here, we imagine walking along γ until it self-intersects. To this end, set $t_0^* := 0$ and inductively put

$$t_i^* := \sup\{t \in [t_{i-1}^*, 1] : \gamma|_{[t_{i-1}^*, t]} \text{ is injective}\}.$$

If $t_i^* = 1$, then we terminate. By definition of t_i^*, it follows that $p_i = \gamma(t_i^*)$ is a vertex of γ. Write the pre-image as $\gamma^{-1}(p_i) = \{t_i, t_i^*\}$, where $t_i < t_i^*$. The intersection sequence from $r = \gamma(t_{i-1}^*)$ until the second occurrence of $p_i = \gamma(t_i) = \gamma(t_i^*)$ is of the following form: $r \cdots p_i \cdots p_i$, where no intersection point occurs twice, except for p_i. Thus, $\gamma_i = \gamma_{[t_i, t_i^*]}$ is a loop. We call the loop γ_1 the **first loop** of γ. By the proof of Lemma 4.1, we know that γ_1 is an outwards loop.

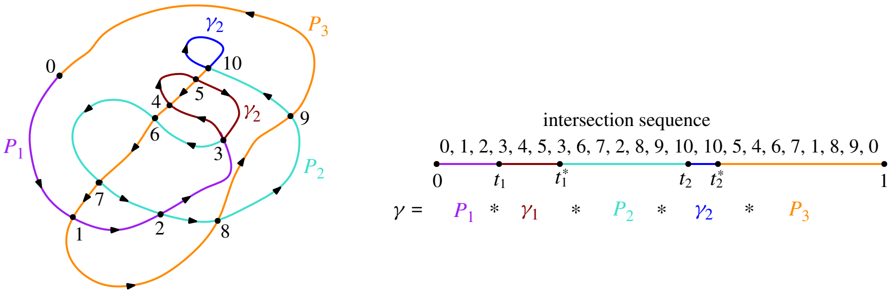

Fig. 16 A strongly irreducible curve with its simple path decomposition shown

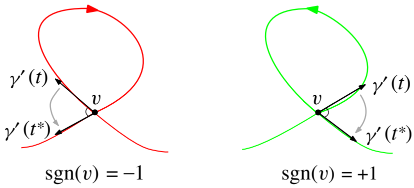

Fig. 17 Two outwards loops; negatively oriented (left) and positively oriented (right). In this setting, $\mathrm{sgn}(v) = \mathrm{WHIT}(\gamma_v)$

Let k be the largest value such that $t_k^* < 1$. Then we set $P_{k+1} = \gamma_{[t_k^*,1]}$. Observe that we have (nearly) partitioned the curve γ into a sequence of paths $\gamma = P_1 * \gamma_1 * \cdots * P_k * \gamma_k * P_{k+1}$, where $*$ denotes concatenation. We call the sequence $(P_1, \gamma_1, \ldots, P_{k+1})$ the **simple path decomposition** of γ; see Fig. 16.

4.3 Wrapping Resolves Obstinance

We are now equipped to prove our second main result on wraps, which shows that repeated wrapping can be used to reduce the obstinance of a curve to zero. This reveals a non-trivial connection between minimum homotopy area and wrapping, and is a core ingredient in the proof of Theorem 4.6. Before the proof, we make a small observation which will be useful.

Observation 4.2 (loop orientations equal basepoint signs) *Let $\gamma \in \mathscr{C}$ and $v \in V(\gamma)$ be the basepoint of an outwards loop γ_v. Then* $\mathrm{sgn}(v) = \mathrm{WHIT}(\gamma_v)$. *This means that* $\mathrm{sgn}(v) = 1$ *iff γ_v is positively oriented; see Fig. 17. Similarly, if γ is simple, then* $\mathrm{sgn}(\gamma(0)) = \mathrm{WHIT}(\gamma)$.

Theorem 4.3 (wrapping resolves obstinance) *Let $\gamma \in \mathscr{C}$ have positive outer basepoint and set $n = |\gamma|$. Then there is a positive integer $k \le n$ so that* $\mathrm{obs}(\mathrm{Wr}_+^k(\gamma)) = 0$. *Moreover,* $\mathrm{Wr}_+^k(\gamma)$ *is a positive interior boundary.*

Proof Let k be the number of negative vertices in $V(\gamma)$. We claim that $\mathrm{Wr}_+^k(\gamma)$ is an interior boundary. We will show this by iteratively constructing a left sense-preserving nullhomotopy H for γ. By property (iv) of Theorem 3.6 it then follows that γ is a positive interior boundary and $\mathrm{obs}(\gamma) = 0$.

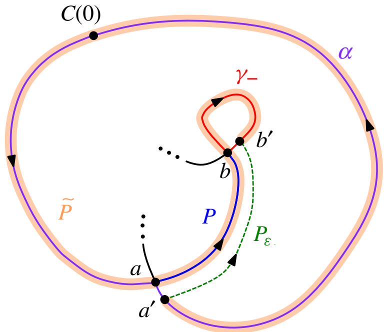

Fig. 18 The combinatorial structure necessary to apply balanced loop deletion: a wrapped curve, with outer wrap α and a negatively oriented loop γ_- as first loop in the simple path decomposition

We first introduce a trick that we call **balanced loop deletion**. See Fig. 18, where all of the following objects are shown. Suppose that $C \in \mathscr{C}$ is positively wrapped, that is $C = \mathrm{Wr}_+(C')$ for some curve $C' \in \mathscr{C}$. And suppose that the first loop γ_- (shown in red) in the simple path decomposition of C is negatively oriented. Let $b = C(t_b) = C(t_b^*)$, with $t_b < t_b^*$, be the basepoint of γ_-. Balanced loop deletion performs a left sense-preserving homotopy H so that $C \overset{H}{\rightsquigarrow} C \setminus (\alpha \cup \gamma_-)$, where α (shown in purple) is the positive outer wrap on C.

Let P (shown in blue) be the simple subpath of C from $a = C(t_a) = C(t_a^*)$ to b, where a is the unique outer intersection point on $[C]$, i.e., the basepoint of the wrap α, and $t_a < t_a^*$. For $\varepsilon > 0$ sufficiently small, let $a' = C(t_a^* + \varepsilon)$ and $b' = C(t_b^* - \varepsilon)$ and let P_ε (shown in dashed green) be a simple path between a' and b' that is ε-close to P in the Hausdorff distance. Let $\widetilde{P} = C|_{[t_a^*+\varepsilon, t_b^*-\varepsilon]}$ (shown in thick beige) be the simple subpath of C from a' to b'. Then $\widetilde{P}$ is the concatenation of (i) the path from a' to a along α, (ii) the path P from a to b, and (iii) the path from b to b' along γ_-. The path $\widetilde{P}$ is simple because each of these subpaths are simple and none of them intersect each other since b is the first self-intersection point of the curve. We now make a crucial observation: $\widetilde{P} * \overline{P_\varepsilon}$ is a simple, positively oriented, closed curve. By definition, this means P_ε is a Blank cut. Thus, there is a left sense-preserving homotopy which sweeps $\widetilde{P}$ to P_ε. The effect of this homotopy on C is that both the outer wrap α and the negatively oriented loop γ_- are deleted. Thus, we have established the existence of left sense-preserving balanced loop deletion.

Now we construct a left sense-preserving nullhomotopy H of $\mathrm{Wr}_+^k(\gamma) = \gamma_1$ by iteratively concatenating several left sense-preserving subhomotopies, so $H = \sum_i H_i$. We proceed inductively as follows. Suppose $H_1, \ldots, H_{i-1}$ have been defined and γ_i is the current curve. Consider the first loop C_i, which is outwards by Lemma 4.1, in the simple path decomposition of γ_i. If C_i is positively oriented we let H_i be the left sense-preserving nullhomotopy that contracts this loop. Otherwise C_i is negatively oriented and we let H_i be the homotopy performing balanced loop deletion. We claim that there is a wrap available to perform this balanced loop deletion. Indeed, for $j = 1, \ldots, i-1$, each homotopy H_j deletes one direct split and at most one indirect split of γ_j, including their basepoints as well as any additional intersections points

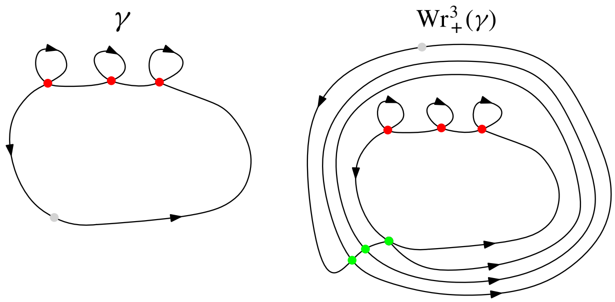

Fig. 19 An example of a family of curves that require $k = n$ wraps to resolve obstinance, where k is the number of negative vertices in $V(\gamma)$. Here, $k = n = 3$. In general, γ_n has positive outer basepoint and signed intersection sequence $0, 1_-, 1_+, 2_-, 2_+, \ldots, n_-, n_+, 0$. Since $\mathrm{WHIT}(\mathrm{Wr}_+^j(\gamma_n)) = 1 - n + j$, it follows that $\mathrm{Wr}_+^j(\gamma_n)$ is not a positive interior boundary until $j = n$. This shows that our bound $k \le n$ cannot be improved

of γ_j that appear on these splits.[5] The signs of the remaining intersection points are not affected. Since there are k negative vertices on $\mathrm{Wr}_+^k(\gamma)$, there can be at most k distinct integers $i_1, \ldots, i_l$ such that the first loop on γ_{i_ν} is negatively oriented. Thus, there are a sufficient number of wraps available on $\mathrm{Wr}_+^k(\gamma)$. We conclude here that all negative vertices will be removed by the algorithm, either during balanced loop deletion or contraction of a positive loop, before the curve becomes simple.

The process of constructing homotopies H_i never gets stuck, and $|\gamma_{i+1}| < |\gamma_i|$. Therefore, at some index m, the current curve γ_m has $|\gamma_m| = 0$, i.e., is simple. Of course, $\gamma_m(0)$ is a vertex from $\mathrm{Wr}_+^k(\gamma)$. Since all negative vertices were removed, γ_m must have a positive basepoint. Therefore, by Observation 4.2, γ_m is positively oriented. Hence, set H_m to contract γ_m with a final left sense-preserving nullhomotopy and $H = \sum_{i=1}^m H_i$ is a left sense-preserving nullhomotopy of $\mathrm{Wr}_+^k(\gamma)$. □

The bound $k \le n$ is tight, as shown in Fig. 19. We now show that wrapping resolves obstinance in either direction of wrapping.

Theorem 4.4 (wrapping resolves obstinance (general)) *Let $\gamma \in \mathscr{C}$ with outer basepoint and set $n = |\gamma|$. Then there are constants $k_-, k_+ \le n + 1$ such that $\mathrm{obs}(\mathrm{Wr}_-^{k_-}(\gamma)) = \mathrm{obs}(\mathrm{Wr}_+^{k_+}(\gamma)) = 0$, $\mathrm{Wr}_-^{k_-}(\gamma)$ is a negative interior boundary, and $\mathrm{Wr}_+^{k_+}(\gamma)$ is a positive interior boundary.*

Proof Without loss of generality suppose $\gamma(0)$ is a positive basepoint, or we may work with the reversal $\overline{\gamma}$. Then $k_+ \le n$ exists directly by Theorem 4.3. To prove the existence of k_-, consider the intermediary curve γ' obtained from γ by performing a I_b-move on the outer edge to create a negatively oriented loop that is entirely outer to the curve. The basepoint of γ' is set on this new negatively oriented loop; therefore γ' has a negative outer basepoint, and the additional vertex on γ' created by the I_b-move is positive. By definition we have $\mathrm{Wr}_-(\gamma) = \mathrm{Wr}_-(\gamma') = \overline{\mathrm{Wr}_+(\overline{\gamma'})}$. By Theorem 4.3

[5] Note: if $q \in V(\gamma)$ lies on the image $[\gamma_k]$, then removing γ_k from γ removes a pre-image of q, so it is no longer a vertex.

there exists $\tilde{k}_+$ for $\overline{\gamma'}$. Due to the additional positive vertex on γ', which is an additional negative vertex on $\overline{\gamma'}$, we have $k_- = \tilde{k}_+ \le k_+ + 1$.

If $\gamma(0)$ is a negative outer basepoint, then the constants $\overline{k_-}$ and $\overline{k_+}$ exist for the reversal $\overline{\gamma}$. And thus we are done by setting $k_+ = \overline{k_-}$ and $k_- = \overline{k_+}$ since

$$\mathrm{Wr}_-^{k_-}(\gamma) = \mathrm{Wr}_-^{\overline{k_+}}(\gamma) = \overline{\mathrm{Wr}_+^{\overline{k_+}}(\overline{\gamma})} \quad \text{and} \quad \mathrm{Wr}_+^{k_+}(\gamma) = \mathrm{Wr}_+^{\overline{k_-}}(\gamma) = \overline{\mathrm{Wr}_-^{\overline{k_-}}(\overline{\gamma})}.$$

□

Let us make a simple observation: once $\mathrm{Wr}_\pm^k(\gamma)$ is an interior boundary, so too is $\mathrm{Wr}_\pm^j(\gamma)$ for any integer $j \ge k$. This holds because we can simply add the extra $j-k$ wraps to the self-overlapping decomposition Ω of $\mathrm{Wr}_+^k(\gamma)$. Consequently, Theorem 4.4 implies that interior boundaries are the equilibrium point for plane curves with respect to the action of wrapping. No matter where we begin, we will always eventually land and stay within the set of interior boundaries.

4.4 Irreducible and Strongly Irreducible Curves

We are now ready to apply Theorem 4.4 to prove sufficient combinatorial conditions for a curve γ to be self-overlapping based on $\mathrm{WHIT}(\gamma)$ and properties of its direct splits. If $\gamma \in \mathscr{C}$ has no proper positive self-overlapping direct splits, we call γ **irreducible**. A special case of irreducibility is of particular interest to us: If $\mathrm{WHIT}(\gamma_v) \le 0$ for all proper direct splits, we call γ **strongly irreducible**. See Figs. 4 and 8, and γ_{SO} in Fig. 25 for examples of strongly irreducible curves. Note that a strongly irreducible curve is irreducible since any positive self-overlapping curve γ has $\mathrm{WHIT}(\gamma) = 1$. We need one simple lemma before proving irreducible curves are self-overlapping.

Lemma 4.5 (existence of a direct split) *Let $\gamma \in \mathscr{C}$ and Ω be a direct split decomposition of γ, with $|\Omega| \ge 2$. Then Ω contains a proper direct split.*

Proof A leaf v_i in the tree T_Ω necessarily corresponds to the basepoint of a direct split γ_i in the decomposition Ω. Since $|\Omega| \ge 2$, this direct split γ_i must be proper. □

Theorem 4.6 (irreducible curves are self-overlapping) *Assume γ has* $\mathrm{WHIT}(\gamma) = 1$ *and positive outer basepoint. If γ is irreducible, then it is self-overlapping.*

Proof Apply Theorem 4.3 to find a $k \in \mathbb{Z}$ such that $\mathrm{Wr}_+^k(\gamma)$ is a positive interior boundary. We know from property (iii) of Theorem 3.6 that there is a self-overlapping decomposition Ω of $\mathrm{Wr}_+^k(\gamma)$ into positive self-overlapping subcurves. By Lemma 4.5, we know that Ω must have a self-overlapping direct split of $\mathrm{Wr}_+^k(\gamma)$, and we will show that γ is the only direct split of $\mathrm{Wr}_+^k(\gamma)$ that can be self-overlapping.

Let w_i be the vertex created by the i-th wrap. The intersection sequence of $\mathrm{Wr}_+^k(\gamma)$ therefore has the prefix $w_k, w_{k-1}, \ldots, w_1$. Then the direct split $\mathrm{Wr}_+^k(\gamma)_{w_i}$ at w_i on $\mathrm{Wr}_+^k(\gamma)$ has $\mathrm{WHIT}(\mathrm{Wr}_+^k(\gamma)_{w_i}) = 1 + (i-1) = i$ by Lemma 3.3, and is therefore not self-overlapping for $i \ge 2$. And any direct split $\mathrm{Wr}_+^k(\gamma)$ at a vertex of γ which is also a proper direct split on γ cannot be self-overlapping since γ is irreducible. Note that

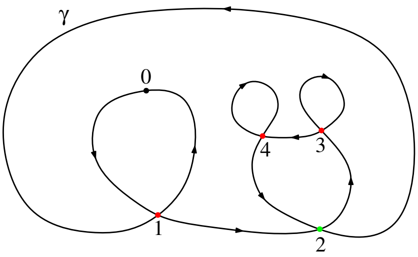

Fig. 20 This curve γ does not have an outer basepoint. It is not self-overlapping, yet γ is strongly irreducible due to the empty positively oriented loop on the indirect split γ_{1*}

by our notation w_1 is the vertex corresponding to the original basepoint $\gamma(0)$. And this is the only vertex at which the direct split $\mathrm{Wr}_+^k(\gamma)_{w_1} = \gamma$ could potentially be self-overlapping. Thus, it follows with Lemma 4.5 that γ is self-overlapping. □

Corollary 4.7 *Let γ have* $\mathrm{WHIT}(\gamma) = 1$ *and positive outer basepoint. Then if γ is not self-overlapping, it has a positive self-overlapping direct split γ_v.*

Again, since strongly irreducible curves are irreducible, Theorem 4.6 implies

Corollary 4.8 (strongly irreducible curves are self-overlapping) *Assume γ has* $\mathrm{WHIT}(\gamma) = 1$ *and positive outer basepoint. If γ is strongly irreducible, then it is self-overlapping.*

Corollary 4.8 tell us something remarkable—conditions on the Whitney indices of a curve and its subcurves alone can be sufficient for self-overlappingness. Note that strongly irreducible curves are a *proper* subset of irreducible curves, see γ_I in Fig. 2. Also, Corollary 4.8 is false without the basepoint assumption, see Fig. 20.

One can decide whether a piecewise linear curve γ is (strongly) irreducible by checking the required condition for each direct split. Let N be the number of line segments of γ and $n = |\gamma| = |V(\gamma)| \in O(N^2)$. Then irreducibility can be tested in $O(nN^3)$ time, using Shor and Van Wyk's algorithm to test for self-overlappingness in $O(N^3)$ time [19]. Strong irreducibility can be decided in $O(n^2)$ time by applying Corollary 3.4 to each direct split of γ.

4.5 Global Balanced Loop Insertion

We now introduce an operation called balanced loop insertion, which is complementary to the balanced loop deletion applied in the proof of Theorem 4.3. We show in Theorem 4.11 that any curve γ with positive outer basepoint and $\mathrm{WHIT}(\gamma) = 1$ can be transformed into a self-overlapping curve, more specifically, a strongly irreducible self-overlapping curve, through a sequence of balanced loop insertions. This result is a nice parallel to Theorem 4.3.

Let $\gamma \in \mathscr{C}$ have a positive outer basepoint. Then given any edge e from $G(\gamma)$, we define **balanced loop insertion** on γ with respect to e as follows: First, perform a I_b-move to insert a negatively oriented loop, on the right side of the edge e, and

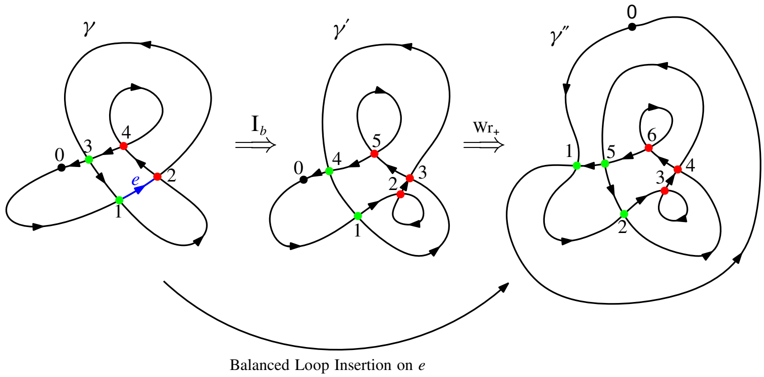

Fig. 21 Balanced loop insertion on a self-overlapping curve γ with respect to an edge e. Note that the curve produced, γ'', is also self-overlapping

smooth the resulting curve γ' until it is generic and regular. Then, wrap around γ' to create $\gamma'' = \mathrm{Wr}_+(\gamma')$. See Fig. 21. This operation is 'balanced' in the sense that we have added both a positive and a negative loop. Hence, by Lemma 3.3, we have

Observation 4.9 *Let γ'' be obtained from γ through balanced loop insertion. Then* $\mathrm{WHIT}(\gamma'') = \mathrm{WHIT}(\gamma)$.

The following is an interesting property of balanced loop insertion.

Lemma 4.10 *Let γ have $\mathrm{WHIT}(\gamma) = 1$ and positive outer basepoint. If γ is strongly irreducible, and γ' is obtained from γ by balanced loop insertion, then γ is strongly irreducible as well.*

Proof Let $i\colon V(\gamma) \to V(\gamma')$ be the inclusion map, sending vertices on γ to the corresponding vertices on γ'. Then the only possible change to the direct split was that we added a negatively oriented loop, so $\mathrm{WHIT}(\gamma'_{i(v)}) \le \mathrm{WHIT}(\gamma_v) \le 0$. The only other direct splits we need to check are those of the vertices u, v created by balanced loop insertion. We note immediately that $\mathrm{WHIT}(\gamma'_u) = -1$ where u is the basepoint of the new negatively oriented loop. Meanwhile, if v is the vertex created by the wrap, then $\mathrm{WHIT}(\gamma'_v) = \mathrm{WHIT}(\gamma) - 1 = 0$. Hence, γ' is strongly irreducible. □

Strong irreducibility is preserved by balanced loop insertion, and consequently, so too is self-overlappingness. If γ has a positive outer basepoint and $\mathrm{WHIT}(\gamma) = 1$, we need a stronger operation to transform a non-self-overlapping curve into a self-overlapping curve.

Let $\gamma \in \mathscr{C}$. **Global balanced loop insertion**, denoted by $M\colon \mathscr{C} \to \mathscr{C}$, applies balanced loop insertion simultaneously once on every edge of γ. More precisely, M applies balanced loop insertion with respect to every edge of the multigraph $G(\gamma)$, in any order. Up to signed intersection sequence, the resulting curve is independent of ordering. By Observation 4.9, it follows that $\mathrm{WHIT}(M(\gamma)) = \mathrm{WHIT}(\gamma)$. Since there are $2|\gamma| + 1$ edges on $G(\gamma)$, the operator $M(\cdot)$ applies balanced loop insertion $2|\gamma| + 1$ times. Equivalently, $M(\gamma)$ can be obtained by performing a I_b-move to the

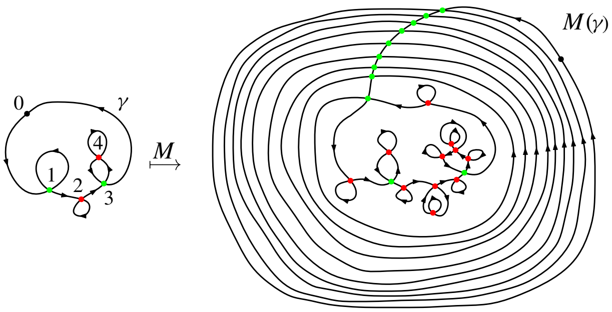

Fig. 22 Global balanced loop insertion applied to a curve γ. Since γ has empty positively oriented loops it is not self-overlapping (by Lemma 3.8). The curve $M(\gamma)$ is strongly irreducible and self-overlapping by Theorem 4.11

right of every edge of γ, adding a new negatively oriented loop, and then wrapping the curve $2|\gamma|+1$ times. See Fig. 22 for an example of global balanced loop insertion.

We now show that this operation transforms any curve γ with positive outer basepoint and $\mathrm{WHIT}(\gamma) = 1$ into a strongly irreducible curve and hence a self-overlapping curve by Corollary 4.8.

Theorem 4.11 *Let γ have positive outer basepoint and* $\mathrm{WHIT}(\gamma) = 1$. *Then $M(\gamma)$ is strongly irreducible and self-overlapping.*

Proof Let γ be a curve with positive outer basepoint. We will utilize the identity $\mathrm{WHIT}(\gamma) = \sum_{v \in V(\gamma)} \mathrm{sgn}(v)$ shown by Titus and Whitney [20, 22]; note the inclusion of the basepoint p_0 with $\mathrm{sgn}(p_0) = +1$ in the vertex set.[6] It follows immediately that $\mathrm{WHIT}(\gamma) \le |\gamma| + 1$ for $\gamma \in \mathscr{C}$ with outer basepoint. While a direct split γ_v may not have an outer basepoint, we can instead consider a curve γ'_v with the same image and orientation as γ_v, and hence the same number of intersection points, and an outer basepoint. We then have $\mathrm{WHIT}(\gamma_v) = \mathrm{WHIT}(\gamma'_v) \le |\gamma'_v|+1 = |\gamma_v|+1$. On the other hand, since every edge of γ_v received at least one negatively oriented loop, as edges of γ_v may be further subdivided on γ, we note that we inserted at least $2|\gamma_v|+1$ negatively oriented loops on the direct split γ_v. Thus, if we write $i\colon V(\gamma) \to V(M(\gamma))$ for the natural inclusion map and set $\tilde{v} = i(v)$, then we see $\mathrm{WHIT}(M(\gamma)_{\tilde{v}}) \le -|\gamma_v| \le 0$. Hence, all the vertices on $M(\gamma)$ that came from γ will not yield direct splits of Whitney index 1 or greater.

Now, the only other vertices to consider are the basepoints of the new negatively oriented loops and the basepoints of the wraps. Clearly, for any vertex u of the former kind, we have $\mathrm{WHIT}(M(\gamma)_u) = -1$ for the direct split at u. We now address the basepoints of the wraps. Let $M(\gamma)_i$ be the direct split and $M(\gamma)_{i^*}$ be the indirect split at the i-th vertex of $M(\gamma)$. By definition $M(\gamma)$ contains $2|\gamma|+1$ outer wraps, which implies $\mathrm{WHIT}(M(\gamma)_{i^*}) = i$ for all $i \in \{1, \dots, 2|\gamma|+1\}$. And since direct splits and indirect splits are complementary, it follows from Lemma 3.3 that $\mathrm{WHIT}(M(\gamma)) = \mathrm{WHIT}(M(\gamma)_i) + \mathrm{WHIT}(M(\gamma)_{i^*})$ and hence $\mathrm{WHIT}(M(\gamma)_i) = 1 - i \le 0$ for any $i \in \{1, \dots, 2|\gamma|+1\}$. Thus, $M(\gamma)$ is indeed strongly irreducible. □

6 Here, we differ from convention. Typically $V(\gamma)$ contains only the crossing points, but for our purposes the basepoint belongs.

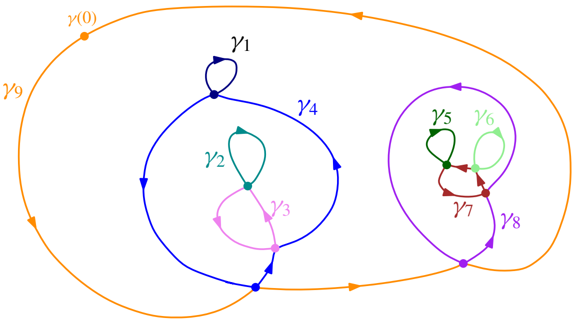

Fig. 23 A loop decomposition of a curve γ

5 Discussion

We introduced new curve classes (zero-obstinance, irreducible, and strongly irreducible curves; see Fig. 2), which help us understand self-overlapping curves and interior boundaries. We proved combinatorial results and showed that wrapping a curve resolves obstinance. These new mathematical foundations for self-overlapping curves and interior boundaries could pave the way for related algorithmic questions. For example, is it possible to decide whether a curve is self-overlapping in $o(N^3)$ time? How fast can one decide self-overlappingness of a curve on the sphere? Can one decide irreducability in $o(n^2)$ time, even in the presence of a large number of linked subcurves?

Acknowledgements We thank Brittany Terese Fasy for fruitful discussions, in particular in the early stages of the research, as well as for proof-reading. Both authors were supported by NSF grant CCF-1618469. Parker Evans was supported by a Goldwater Scholarship from the Goldwater Foundation. This paper is based on the Honors thesis of the first author [5], and an extended abstract of this paper has been published at SoCG [7]. Much of this research was enabled by the use of a computer program [6] which can determine whether a plane curve is self-overlapping, compute its minimum homotopy area, and display the self-overlapping decomposition associated with a minimum homotopy. Figure 12 was created with this program.

Appendix A: Whitney Indices & Loop Decompositions

In this section are interested in a refinement of self-overlapping decompositions. Let $\Omega = (\gamma_i)_{i=1}^n$ be a self-overlapping decomposition. If each γ_i is simple, i.e., a loop, then we call Ω a *loop decomposition*. See Fig. 23 for an example.

We now recall a construction of Seifert [9, 18]. He introduced a decomposition of plane curves by performing so-called "uncrossing moves", which essentially split a curve γ at a vertex v into the two (nearly) disjoint pieces γ_v and γ_{v^*}. See Fig. 24. If we cut the two curves at the vertex v, and smooth them, we obtain two completely disjoint plane curves. Iterating this process across many vertices of γ, one can achieve a decomposition of the original curve γ into a set of Jordan curves.

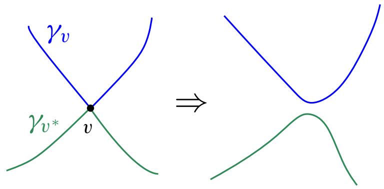

Fig. 24 An uncrossing move applied to a vertex v, splitting the curve into two pieces

Suppose $\gamma \in \mathscr{C}$ is decomposed into Jordan curves $\{C_i\}_{i=1}^k$ using these uncrossing moves. Then Seifert and Gauss [9, 18] showed that $\mathrm{WHIT}(\gamma) = \sum_{i=1}^k \mathrm{WHIT}(C_i)$.[7] Using our terminology and replacing the Jordan curves with loops in a loop decomposition, we obtain the equivalent fact:

Lemma 5.1 (Whitney index through loops) *Let $\gamma \in \mathscr{C}$, let Ω be a loop decomposition of γ, and let n_+ be the number of positively oriented loops and n_- be the number of negatively oriented loops in Ω. Then* $\mathrm{WHIT}(\gamma) = n_+ - n_-$.

As a consequence of this lemma, we can prove linearity of Whitney indices across a direct split decomposition.

Lemma 3.3 (Whitney index through decompositions) *Let $\gamma \in \mathscr{C}$ and Ω be a direct split decomposition of γ. Then* $\mathrm{WHIT}(\gamma) = \sum_{C \in \Omega} \mathrm{WHIT}(C)$.

Proof The key is to piece together loop decomposition's of each subcurve $C \in \Omega$. Write $\Omega = (C_i)_{i=1}^k$ and take Ψ_i to be a loop decomposition of C_i. Then $\Psi = (\Psi_1, \ldots, \Psi_k)$ is a loop decomposition of γ. Since $\mathrm{WHIT}(C_i) = p_i - n_i$ where p_i is the number of positively oriented subcurves in Ψ_i and n_i the number of negatively oriented subcurves, we conclude that

$$\mathrm{WHIT}(\gamma) = \sum_{i=1}^k p_i - \sum_{i=1}^k n_i = \sum_{i=1}^k (p_i - n_i) = \sum_{i=1}^k \mathrm{WHIT}(C_i). \qquad \square$$

Appendix B: Lattice

We introduce two more classes of curves. We say a face F is **good** when its depth is equal to its winding number. If a curve $\gamma \in \mathscr{C}$ is positive consistent and all faces on $G(\gamma)$ are good then we call the curve **good**. We call a curve **basic** if all of its self-overlapping decompositions are loop decompositions. That is, the only self-overlapping decompositions are decompositions into loops. By Theorem 3.10, basic curves with zero obstinance can be decomposed into good curves.

We define SIMPLE, BASIC-ZERO-OBSTINANCE, ZERO-OBSTINANCE as classes of those curves that have the property described by the class name. The classes SO^+, INTERIOR-BOUNDARY$^+$, CONSISTENT$^+$, GOOD$^+$ consist of the curves with the *positive* property described by the class name (positive self-overlapping, positive interior

[7] It is crucial that the crossing points of γ are all transverse double points for this to work.

boundary, positive consistent, and good curves that are positive consistent). Figure 25 and Theorem 5.3 show the relationship between these curve classes. The curves in Fig. 25 show that the inclusions in parts (i), (ii), and (iii) of Theorem 5.3 are proper. We first state a lemma that we need in the proof of the theorem.

Lemma 5.2 (negatively oriented loop) *Let $\gamma \in \mathscr{C}$ be a non-simple, positive self-overlapping curve with positive outer basepoint. Then γ has a negatively oriented loop γ_v.*

Proof Let $\mathcal{V} = \{v \in V(\gamma) : \gamma_v \text{ is a loop}\}$. By Lemma 4.1, we know $\mathcal{V} \neq \emptyset$. Define the relation $v \prec w$ for $v, w \in \mathcal{V}$ whenever $\operatorname{int}(v) \subseteq \operatorname{int}(w)$. It is straightforward to verify that $(\mathcal{V}, \prec)$ is a poset. Of course, since $|\mathcal{V}|$ is finite, we can choose $v_0 \in \mathcal{V}$ minimal with respect to $\prec$. Now, suppose that γ_{v_0} were a positive loop. We show this is a contradiction to complete the proof. Indeed, by minimality of v_0, there are no loops of γ completely contained inside int γ_{v_0}. This means every time a strand of γ crosses from outside to inside γ_w, the strand does not cross itself inside of γ. Topologically, then, $\overline{\operatorname{int}(\gamma_{v_0})}$ looks like a disk with finitely many simple arcs traveling from boundary to boundary. By way of a (regular) right sense-preserving homotopy, we can sweep each such arc until it no longer intersects γ_{v_0}. As we need only sweep finitely many such arcs, let us denote γ' as the result of this process. By Lemma 3.9, we know γ' is self-overlapping. On the other hand, γ' has an empty positive loop, namely the one we just emptied, which contradicts Lemma 3.8. □

Theorem 5.3 (curve classes) *If $\gamma \in \mathscr{C}$ has a positive outer basepoint, then:*

(i) Simple $\subset$ SO$^+$ $\subset$ Interior-Boundary$^+$ $\subset$ Consistent$^+$,
(ii) Simple $\subset$ Good$^+$ $\subset$ Basic-Zero-Obstinance $\subset$ Zero-Obstinance,
(iii) Good$^+$ $\subset$ Interior-Boundary$^+$ $\subset$ Zero-Obstinance,
(iv) Consistent$^+$ $\cap$ Zero-Obstinance $=$ Interior-Boundary$^+$,
(v) Consistent$^+$ $\cap$ Basic-Zero-Obstinance $=$ Good$^+$,
(vi) SO$^+$ $\cap$ Good$^+$ $=$ Simple.

Proof By definition, a simple curve with a positive outer basepoint is positive self-overlapping, and $+k$-boundaries are positive consistent. By Theorem 3.7, a positive self-overlapping curve is a 1-boundary, which proves (i). A simple curve is trivially good. By Lemmas 2.1 and 2.2, good curves have zero obstinance. We now show that good curves are basic by contradiction. Suppose γ admitted a self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^k$ with a non-simple self-overlapping curve γ_j. Then γ_j must contain a negatively oriented loop by Lemma 5.2. But we could then create a finer decomposition of γ by decomposing γ_j into loops. Precisely, let Ψ be a loop decomposition of γ_j and consider $\Gamma = (\gamma_1, \ldots, \gamma_{j-1}, \Psi, \gamma_{j+1}, \ldots, \gamma_k)$. Then Γ is a direct split decomposition of γ refining Ω. Let C be a negatively oriented loop in Ψ and take any face F contained in the interior of C. Now, take a path P from F to the exterior face on $G(\gamma)$ such that the depth is monotonically decreasing along the path P. Since F is contained inside C, the path P must cross C to reach F_{ext}. However, when P crosses past C, we see the depth either decrease by 1 or remain unchanged, while the winding number increases by 1, since C is negatively oriented. We learn that $\operatorname{wn}(F, \gamma) < D(F, \gamma)$, which is a contradiction. This proves (ii), with

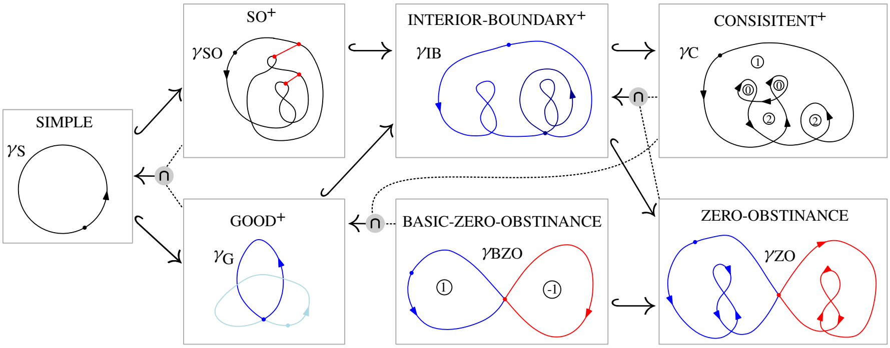

Fig. 25 Lattice of curve classes. Solid arrows are inclusions. Two dashed lines meet to form an inclusion. A member curve of each class is displayed, along with appropriate information to justify membership in its curve class: A set of Blank cuts of γ_{SO} are shown in red, a self-overlapping decomposition of γ_{IB} in blue, the winding numbers of γ_C, the loop decompositions of γ_G and γ_{BZO}, and the self-overlapping decomposition of γ_{ZO}. The inclusions are proper: γ_{BZO} is not consistent and consequently not good; γ_{IB} is not good, and since $\mathrm{WHIT}(\gamma_{IB}) = +2$ it is not self-overlapping; γ_C is not an interior boundary because $\mathrm{WHIT}(\gamma_C) = 1$ but since it has an empty positively oriented loop it is not self-overlapping; γ_{ZO} is not an interior boundary because it is not consistent. See Theorem 5.3

the last inclusion being trivial. Since a good curve is basic and positive consistent, it is a positive interior boundary by property (iii) of Theorem 3.6. And interior boundaries have zero obstinance by definition, which proves (iii).

By definition, interior boundaries are consistent and have zero obstinance, which proves (iv). If a curve γ is basic and positive consistent, it follows that γ admits a loop decomposition Ω with only positively oriented subcurves. Since by Observation 3.2, $\mathrm{wn}(F, \gamma) = \sum_{\gamma_i \in \Omega} \mathrm{wn}(F, \gamma_i)$ and each $\mathrm{wn}(F, \gamma_i) \in \{0, 1\}$, we must have $\mathrm{wn}(F, \gamma_i) \geq D(F, \gamma_i)$. The bound $\mathrm{wn}(F, \gamma) \leq D(F, \gamma)$ holds generally. Thus γ is good, which together with (i) and (ii) proves (v). By Lemma 5.2, a good curve that is self-overlapping may not have a negatively oriented loop and hence must be simple, which together with (i) and (ii) proves (vi). □

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