DIRAC OPERATORS WITH OPERATOR DATA OF WIGNER-VON NEUMANN TYPE

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Abstract. We consider half-line Dirac operators with operator data of Wigner-von Neumann type. If the data is a finite linear combination of Wigner-von Neumann functions, we show absence of singular continuous spectrum and provide an explicit set containing all embedded pure points that depends only on the L^p decay and frequencies of the operator data. For infinite sums of Wigner-von Neumann-like terms, we bound the Hausdorff dimension of the singular part of the spectrum.

1. Introduction

In the spectral theory of Schrödinger operators with slowly decaying potentials, i.e., potentials that are not L^1, an alternative to the classical WKB methods is needed. One approach to studying these so-called 'long-range' operators uses the sum rules of the well-known work by Deift-Killip [9]. Another approach, which we will take in this article, is to study boundedness of eigensolutions. The historical development of spectral analysis via boundedness of eigensolutions involves many authors studying many different species of potentials (see [10] for a more thorough review than we provide here). For instance, for Schrödinger operators with sparse potentials, Pearson [36] and then Kiselev-Last-Simon [21] identified a transition with respect to spectral type at $p = 2$ for L^p potentials. For power-decaying potentials, work of Christ, Kiselev, Molchanov, and Remling [7,18,19,32,38] established conditions for the preservation of the absolutely continuous spectrum. Even in cases where the absolutely continuous spectrum is preserved, constructions have been given which produce singular, and even singular continuous spectrum embedded in the absolutely continuous spectrum [28,33,43].

The first and perhaps most famous construction producing embedded singular spectrum is that of von Neumann and Wigner [35] (see also [41]), who in 1929 introduced a one-dimensional Schrödinger operator H with potential V behaving at infinity as

$$V(x) = -8\frac{\sin(2x)}{x} + O(x^{-2}),$$

which Schrödinger operator has $E = 1$ as an eigenvalue embedded in the absolutely continuous part of the spectrum. Since 1929 many variants of this model have been used to demonstrate and study various forms of 'exotic' singular spectrum (e.g., [1,4,12,15,17,22,29,34,44,49]). Simon's construction in [43] uses the Wigner-von Neumann model as the basic building block in a potential for which the associated Schrödinger operator exhibits dense embedded point spectrum. Two results of Simonov even precisely describe the asymptotics of the spectral density near a critical point for certain Wigner-von Neumann-like perturbations of a periodic potential [45].

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Functions of generalized bounded variation, of which von Neumann and Wigner's potential is a special case, combine slower decay with additional Wigner-von Neumann-like terms with differing frequencies, producing a mixture of bounded variation, decay at infinity, and almost periodicity [50]. This combination is an interesting one for at least three reasons: potentials of bounded variation with decay at infinity preserve the absolutely continuous spectrum [48]; L^1 potentials preserve the purely absolutely continuous spectrum (see, for example, [42]), which underscores the importance of the decay rate for producing an embedded eigenvalue; and there is a generic set of almost periodic potentials producing purely singular spectrum [2].

Historically, most exploration of exotic spectra arising from models based on the Wigner-von Neumann potential restricted to the L^2 case, with a result of Janas-Simonov [16] from the discrete case allowing ℓ^3 decay. Lukić [24–27] used functions of generalized bounded variation to progress to the L^p setting for any integer $2 \leq p < \infty$, showing the absence of singular continuous spectrum and explicitly providing p-dependent finite sets containing all possible instances of embedded pure points. In [27], Lukić extended this work to include potentials with infinitely many summands of generalized bounded variation, in which case the set of possible pure points is in general infinite and singular continuous spectrum is possible, but may be bounded in Hausdorff dimension.

Schrödinger operators and Dirac operators have often been studied in tandem. For example, Naboko [33] demonstrated dense point spectrum in the absolutely continuous spectrum of Dirac-type operators and deduced the same for Schrödinger operators as a special case, and we will below use criteria due to Behncke [3] for the existence of subordinate solutions, which [3] gives in both the Schrödinger and Dirac settings. While Dirac operators with Wigner-von Neumann type operator data have been considered (e.g., [5,31]), the case of decay slower than L^2 remained open. Here we describe this case by adapting for the half-line Dirac operator the work of Lukić on spectral type characterization of models with Wigner-von Neumann type data. The analysis begins in much the same way as in [26], but an important adaptation is required that alters the analysis throughout and the results we obtain.

The Dirac operator commonly appears in at least two unitarily equivalent forms:

$$\Lambda_\varphi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi(x) \\ \overline{\varphi(x)} & 0 \end{pmatrix}, \tag{1.1}$$

$$L_\varphi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} \operatorname{Re}\varphi(x) & \operatorname{Im}\varphi(x) \\ \operatorname{Im}\varphi(x) & -\operatorname{Re}\varphi(x) \end{pmatrix}. \tag{1.2}$$

The form (1.2) is more classical, but the form (1.1) is often more convenient for calculations, and below we opt to work with Λ_φ. Additionally, the form (1.1) appears in the Zakharov-Shabat Lax pair representation of the defocusing nonlinear Schrödinger equation [14]. More details regarding this gauge distinction are available in [6,11,14]. We will study the eigenequation for Λ_φ,

$$\Lambda_\varphi U(x,\eta) = EU(x,\eta), \tag{1.3}$$

where $\eta = 2E$. Our operator data φ will be of almost the same form as the potentials in [27], which form generalizes the famous Wigner-von Neumann potential of [35]. We recall that the variation of a function γ on an interval I is defined as

$$\operatorname{Var}(\gamma, I) = \sup_{k\in\mathbb{N}} \sup_{\substack{x_0,\dots,x_k\in I \\ x_0<\dots<x_k}} \sum_{j=1}^{k} |\gamma(x_j) - \gamma(x_{j-1})|.$$

Definition 1.1. We say φ is of *Wigner-von Neumann type* if it takes the form

$$\varphi(x) = \sum_{j=1}^{N} c_j e^{-i\phi_j x} \gamma_j(x), \tag{1.4}$$

for $N \in \mathbb{N} \cup \{\infty\}$, where $c_j \in \mathbb{C}$, $\phi_j \in \mathbb{R}$, and all of the following conditions hold:

(i) (uniformly bounded variation) the functions $\gamma_j : (0, \infty) \to \mathbb{C}$ obey

$$\sup_j \operatorname{Var}(\gamma_j, (0, \infty)) < \infty. \tag{1.5}$$

(ii) (uniform L^p condition) for some odd $p \in \mathbb{Z}$, $p \geq 3$,

$$\sup_j \int_0^\infty |\gamma_j(t)|^p dt < \infty. \tag{1.6}$$

(iii) (α-type decay of coefficients) for some $\alpha \in (0, \frac{1}{p-2})$,

$$\sum_j |c_j|^\alpha < \infty. \tag{1.7}$$

When p is odd in the definition of Wigner-von Neumann type potentials in [27, Theorem 1.1], that definition coincides with Definition 1.1 (except that the range of allowable α is slightly larger in our case). For reasons to be explained below, it is without loss of generality that we restrict to odd integers p here. If $N < \infty$ in (1.4), we say φ is of *finite Wigner-von Neumann type*.

In this latter case, of course, the condition (1.7) becomes vacuous and the uniform conditions (1.5) and (1.6) simplify to requiring that each function γ_j is of bounded variation and, for some odd $p \geq 3$, each $\gamma_j \in L^p$. We call the ϕ_j *frequencies* and denote the set of all frequencies by $\Phi = \{\phi_j : j \in \mathbb{N}\}$.

The differential expression (1.1) with operator data φ of Wigner-von Neumann type has zero as a regular endpoint and, since φ decays at infinity, is in the limit point case at $+\infty$. Thus, for any $\omega \in \partial\mathbb{D}$, Λ_φ^ω defines an unbounded self-adjoint operator with domain

$$D(\Lambda_\varphi^\omega) = \{f \in H^1((0,\infty), \mathbb{C}^2) : \begin{pmatrix}\omega & \overline{\omega}\end{pmatrix} f(0) = 0\}.$$

The choice of ω is unimportant to the analysis–our results hold regardless of the choice of self-adjoint boundary condition at zero, and we suppress ω hereafter.

Before stating any theorems, let us recall the precise definition of an embedded eigenvalue. First, for any $\psi \in D(\Lambda_\varphi)$ and for $\chi_S(\Lambda_\varphi)$ a spectral projection defined via the functional calculus, the unique finite positive Borel measure μ_ψ satisfying $\langle \psi, \chi_S(\Lambda_\varphi)\psi \rangle = \mu_\psi(S)$ for any $S \subset \mathbb{R}$ is called the spectral measure for ψ. Then, $D(\Lambda_\varphi)$ admits a decomposition into absolutely continuous, singularly continuous, and pure point parts as $D(\Lambda_\varphi) = \mathcal{D}_{ac} \oplus \mathcal{D}_{sc} \oplus \mathcal{D}_{pp}$, for $\mathcal{D}_\bullet = \{\psi \in D(\Lambda_\varphi) : d\mu_\psi \text{ is purely } \bullet\}$. Lastly, we have

$$\sigma(\Lambda_\varphi) = \sigma_{ac}(\Lambda_\varphi) \cup \sigma_{sc}(\Lambda_\varphi) \cup \sigma_{pp}(\Lambda_\varphi),$$

where $\sigma_\bullet(\Lambda_\varphi)$ denotes the spectrum of the restriction of Λ_φ to $\mathcal{D}_\bullet$. An embedded eigenvalue in the absolutely continuous spectrum is an element of $\sigma_{ac}(\Lambda_\varphi) \cap \sigma_{pp}(\Lambda_\varphi)$. When embedded eigenvalues exist, $D(\Lambda_\varphi) \neq \mathcal{D}_{ac}$, even for cases in which $\sigma(\Lambda_\varphi) = \sigma_{ac}(\Lambda_\varphi)$.

The first of our main results shows absence of singular continuous spectrum and provides an explicit set containing all possible embedded pure points in the case where φ is of finite Wigner-von Neumann type. For convenience, we define $\sum_{j=1}^{0} C_j := 0$ for any sequence $\{C_j\}$.

Theorem 1.2. *Let Λ_φ be given by (1.1) with operator data φ of finite Wigner-von Neumann type satisfying the uniform L^p condition (1.6) for some $p = 2n+1$, $n \geq 1$. Then for*

$$\mathfrak{S}_p = \Big\{ \frac{\eta}{2} \big| \eta = \sum_{j=1}^{m} \phi_{k_j} - \sum_{j=1}^{m-1} \phi_{l_j}; \phi_{k_j}, \phi_{l_j} \in \Phi; 1 \leq m \leq n \Big\}, \tag{1.8}$$

which depends only on p and the set Φ of frequencies of φ, on $\mathbb{R} \setminus \mathfrak{S}_p$ the spectral measure μ of Λ_φ is mutually absolutely continuous with Lebesgue measure. Consequently,

(i) $\sigma_{ac}(\Lambda_\varphi) = \mathbb{R}$
(ii) $\sigma_{sc}(\Lambda_\varphi) = \emptyset$
(iii) $\sigma_{pp}(\Lambda_\varphi) \subset \mathfrak{S}_p$ *is a finite set.*

In addition to technical adjustments to the methods in [24,26], the exceptional set $\mathfrak{S}_p$ we obtain differs from the analagous exceptional sets in the settings of Schrödinger operators or orthogonal polynomials on the real line. Namely, not all sums and differences of frequencies from the operator data give rise to possible pure points, but rather only those of the form

$$\sum_{j=1}^{m} \phi_{k_j} - \sum_{j=1}^{m-1} \phi_{l_j}. \tag{1.9}$$

This is a property shared by the exceptional set in the setting of orthogonal polynomials on the unit circle [24], and the reasons for this phenomenon are similar in both settings. On the unit circle, rotating the measure by an angle ψ shifts each of the frequencies ϕ_j by ψ. Thus, from the set of *a priori* possible critical points, only those of the form (1.9) are preserved.

Similarly, to shift the spectral paramater $E = \eta/2$ by ψ in (1.3), we multiply our operator data φ by $e^{i2\psi x}$, which shifts each frequency ϕ_j by 2ψ; again we see that only critical points η of the form (1.9) are preserved. The fact that new elements of the form (1.9) become available only when p increases to an odd integer accounts for the odd p in the L^p condition of the theorem and in Definition 1.1.

Since the set $\mathfrak{S}_p$ grows as $p = 2n+1$ grows, it is natural to ask both whether there exists φ for which $\mathfrak{S}_p$ indeed contains an embedded eigenvalue and whether the growth in the sets $\mathfrak{S}_p$ is necessary or an artifact of our method. To answer these questions, we construct operator data that yields an eigenvalue in $\mathfrak{S}_5 \setminus \mathfrak{S}_3$. A similar argument is available to produce φ with eigenvalue $E \in \mathfrak{S}_p \setminus \mathfrak{S}_{p-2}$ for larger p. Our construction will use operator data of the form

$$\varphi(x) = \sum_{j=1}^{M} c_j x^{-\delta} e^{-i(\phi_j x + \xi_j(x))}, \tag{1.10}$$

where $\delta \in (p^{-1}, (p-2)^{-1}]$, ξ_j are real-valued, and $M < \infty$. We can realize $\gamma_j(x)$ in (1.4) as $c_j x^{-\delta} e^{-i\xi_j(x)}$. Thus defined, φ satisfies the conditions of Definition 1.1.

Theorem 1.3. *Fix $M \geq 3$ and $\delta \in (\frac{1}{5}, \frac{1}{3}]$. Let φ be given by (1.10) with rationally independent ϕ_j. Then there exists $\eta/2 \in \mathfrak{S}_5 \setminus \mathfrak{S}_3$. Moreover, for such η and any c_j, ϕ_j satisfying both*

$$\sum_{j=1}^{M} \frac{|c_j|^2}{\phi_j - \eta} = 0 \tag{1.11}$$

and

$$\sum_{j_1,j_2=1}^{M} |c_{j_1}c_{j_2}|^2 \frac{\phi_{j_1}+\phi_{j_2}-2\eta}{(\phi_{j_1}-\eta)^2(\phi_{j_2}-\eta)^2} = 0, \tag{1.12}$$

there exist functions $\xi_j \in C^1$ such that φ satisfies (1.5) and (1.3) has a solution with asymptotics (4.2) and (4.3). In particular, $\eta/2 \in \sigma_{ac}(\Lambda_\varphi) \cap \sigma_{pp}(\Lambda_\varphi)$.

We will also show that both conditions (1.11) and (1.12) may be simultaneously satisfied.

If φ is of Wigner-von Neumann type with infinitely many nonzero c_j and $p \geq 3$, each of the frequencies ϕ_j is of the form (1.9) with $m = 1$, so that the set $\mathfrak{S}_p$ is in general infinite. In this case we instead bound the Hausdorff dimension of $\mathfrak{S}_p$. Similar to Theorem 1.2, we will have that any maximal spectral measure for Λ_φ is mutually absolutely continuous with Lebesgue measure on $\mathbb{R} \setminus \mathfrak{S}_p$. Recall the decomposition of a maximal spectral measure for Λ_φ, μ, into its absolutely continuous and singular parts:

$$d\mu = d\mu_{\mathrm{ac}} + d\mu_{\mathrm{s}}. \tag{1.13}$$

Theorem 1.4. *Let operator data φ be of Wigner-von Neumann type satisfying the L^p condition (1.6) for some odd $p \geq 3$ and the condition (1.7) for some $\alpha \in (0, \frac{1}{p-2})$. Then the set of energies E for which there exists an unbounded solution to (1.3) has Hausdorff dimension at most $(p-2)\alpha$. In particular, μ_{ac} is mutually absolutely continuous with Lebesgue measure on $\mathbb{R}$ and μ_s is supported on a set of Hausdorff dimension at most $(p-2)\alpha$.*

Results of Remling [39], Christ-Kiselev [8], and, more recently, Liu [23], bound the Hausdorff dimension of embedded singular spectrum for Schrödinger operators with slowly decaying potentials. Those results cannot be directly compared to ours, since theirs concerned general potentials with L^p decay for $1 < p < 2$, whereas Theorem 1.4 considers Wigner-von Neumann potentials with decay slower than L^2 and is sensitive to the value α in Definition 1.1.

In Section 2, we define the Prüfer variables to be used throughout. In Section 3 we prove the form (1.9) of critical points. In Section 4 we prove Theorems 1.2 and 1.3. In Section 5 we prove Theorem 1.4.

2. Subordinacy and Prüfer Variables

We call a solution $U(x,\eta)$ to (1.3) at $E = \eta/2$ a subordinate solution if

$$\lim_{x\to\infty} \frac{\int_0^x \|U(t,\eta)\|^2 dt}{\int_0^x \|V(t,\eta)\|^2 dt} = 0$$

holds for any linearly independent solution $V(x,\eta)$ at E. Subordinate solutions are defined similarly for solutions to the Schrödinger eigenequation. For more on subordinacy theory, see [13].

In [3] it is shown that, for both Schrödinger and Dirac operators, the absence of subordinate solutions on a set A implies purely absolutely continuous spectrum there. We will find an explicit set A of energies E at each of which all solutions are bounded. In [47] it is shown that, in the context of Schrödinger operators, boundedness of all solutions implies absence of subordinate solutions. In the same way, the following lemma, following ideas from [47], completes the desired chain of implications in the Dirac operator setting.

Lemma 2.1. *If all solutions of (1.3) at E are bounded, then there is no subordinate solution at E.*

Proof. Let U and V be linearly independent solutions at E. Boundedness gives $M_U, M_V < \infty$ for

$$M_U := \sup_{x\geq 0} \|U(x,\eta)\|, \; M_V := \sup_{x\geq 0} \|V(x,\eta)\|.$$

Clearly, for $x > 0$ we have

$$\int_0^x \|V(t)\|^2 dt \leq M_V^2 x.$$

The Wronskian of any $f, g \in H^1((0,\infty);\mathbb{C}^2)$, $W[f,g](x)$, is defined as $f(x)^t Jg(x)$ for $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. It is straightforward to show that $W[f,g](x)$ is a nonzero constant when f, g are linearly independent solutions to (1.3). Thus, for any x we have

$$|W[U,V]| = |U(x)^t JV(x)| \leq \|U(x)\|\|JV(x)\| \leq \|U(x)\| M_V.$$

Consequently, for any $x > 0$,

$$\frac{\int_0^x \|U(t)\|^2 dt}{\int_0^x \|V(t)\|^2 dt} \geq \frac{|W[U,V]|^2}{M_V^4} > 0.$$

Since U, V were chosen arbitrarily, taking $x \to \infty$ shows there is no subordinate solution at E. $\square$

In order to prove boundedness of solutions, we perform a Prüfer transformation to an arbitrary solution $U(x,\eta)$. Prüfer variables have been used many times in the spectral theory of Schrödinger and Dirac operators (e.g., [20,21,24–27,29,30,37,40,46]). We set

$$E = \frac{\eta}{2},$$

and for a solution $U(x,\eta)$ of (1.3) at E, we define the Prüfer amplitude r and Prüfer angle θ by

$$U(x,\eta) = r(x,\eta)\begin{pmatrix} (1+i)e^{-i(\frac{\eta}{2}x+\theta(x,\eta))} \\ (1-i)e^{i(\frac{\eta}{2}x+\theta(x,\eta))} \end{pmatrix}. \tag{2.1}$$

The ambiguity in θ is addressed by fixing $\theta(0,\eta) \in [-\pi,\pi)$ and requiring θ be continuous in x. So defined, the variables r and θ satisfy the following system of differential equations:

$$-i\partial_x\theta = i(\operatorname{Re}\varphi(x))\sin(\eta x + 2\theta(x,\eta)) + i(\operatorname{Im}\varphi(x))\cos(\eta x + 2\theta(x,\eta)),$$
$$\partial_x \log r = (\operatorname{Re}\varphi(x))\cos(\eta x + 2\theta(x,\eta)) - (\operatorname{Im}\varphi(x))\sin(\eta x + 2\theta(x,\eta)).$$

Defining the complex Prüfer variable $Z(x,\eta) := r(x,\eta)e^{-i\theta(x,\eta)}$, we have

$$\frac{\partial_x Z(x,\eta)}{Z(x,\eta)} = \partial_x \log r(x,\eta) - i\partial_x\theta(x,\eta) = e^{i(\eta x + 2\theta(x,\eta))}\varphi(x).$$

Thus, $\partial_x \log r(x,\eta) = \operatorname{Re}(e^{i(\eta x+2\theta(x,\eta))}\varphi(x))$, and

$$\log\frac{r(x,\eta)}{r(0,\eta)} = \operatorname{Re}\int_0^x e^{i(\eta t+2\theta(t,\eta))}\varphi(t)dt. \tag{2.2}$$

To prove boundedness of solutions at η, it suffices to bound (2.2). Moreover, we have

$$\partial_x\theta(x,\eta) = -\operatorname{Im} e^{i(\eta x+2\theta(x,\eta))}\varphi(x), \tag{2.3}$$

which identity will prove useful in many of our calculations below due to Lemma 3.1. In later sections, we will often suppress the η- and/or x-dependence of r and θ for conciseness.

3. Nonremovable singularities

Consider the following reindexing of [27, Lemma 2.1]:

Lemma 3.1. *Let $\eta \in \mathbb{R}$ be fixed. Let $P, N \in \mathbb{Z}$ with $P \geq 1$, $I = P + N$, and $K = P - N \geq 0$. Then for $0 \leq a < b < \infty$, $\Gamma(x) = \gamma_{k_1}(x) \cdots \gamma_{k_P}(x)\overline{\gamma_{l_1}} \cdots \overline{\gamma_{l_N}}$, and $\phi = \phi_{k_1} + \cdots + \phi_{k_P} - (\phi_{l_1} + \cdots + \phi_{l_N})$,*

$$\left| \int_a^b \Big((\phi - K\eta) e^{Ki(\eta t + 2\theta(t))} e^{-i\phi t} \Gamma(t) - 2K e^{Ki(\eta t + 2\theta(t))} e^{-i\phi t} \Gamma(t) \frac{\partial \theta}{\partial t} \Big) dt \right| \leq 2\tau^I,$$

where $\tau = \sup_k \operatorname{Var}(\gamma_k, (0, \infty))$.

By (1.5), $\tau < \infty$, and thus by (2.3), each application of Lemma 3.1 appends a new L^p factor to the integrand in (2.2) at a finite cost, $2\tau^I$. Applying Lemma 3.1 repeatedly to (2.2) yields terms of the form

$$f(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^N) \int_0^x e^{Ki(\eta t + 2\theta(t))} \prod_{i=1}^P e^{-i\phi_{k_i} t} \gamma_{k_i}(t) \prod_{j=1}^N e^{i\phi_{l_j} t} \overline{\gamma_{l_j}(t)} dt, \tag{3.1}$$

where $K = P - N \geq 0$ and we've used $[\phi_j]_{j=1}^n$ to denote the ordered n-tuple $(\phi_1, \ldots, \phi_n)$. Once $I = P + N$ grows to p, the L^p condition (1.6) gives a finite $x-$independent upper bound on that term. Going forward, the reader should think of the P ϕ_{k_j} as the *positive* frequencies and the N ϕ_{l_j} as the *negative* frequencies. Thus, I is the total number of frequencies and K is the number of $e^{i(\eta t + 2\theta(t))}$ factors seen in (3.1). Note that P and N depend on I and K via the identities $P = \frac{I+K}{2}$ and $N = \frac{I-K}{2}$. Below we will usually suppress this dependence for conciseness.

We will track the terms (3.1) as I increases to p. Note that such terms appear for any permutation of $[\phi_{k_1}, \ldots, \phi_{k_P}]$ and for any permutation of $[\phi_{l_1}, \ldots, \phi_{l_N}]$, so we can agree to average f over all such terms, by which we mean replacing $f(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^N)$ by

$$f_{I,K}(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^N) = \frac{1}{P!N!} \sum_{\substack{\sigma \in S_P \\ \tau \in S_N}} f(\eta; [\phi_{k_{\sigma(j)}}]_{j=1}^P; [\phi_{l_{\tau(j)}}]_{j=1}^N), \tag{3.2}$$

where S_j denotes the symmetric group on j elements. This averaging is useful both for avoiding counting the distinct permutations of the frequencies ϕ_j and, importantly, for showing that many apparent singularities in η arising in f are, in fact, removable, as we shall see in Section 3.

The symmetrized (3.2) is invariant under permutations of $[\phi_{k_1}, \ldots, \phi_{k_P}]$ and under permutations of $[\phi_{l_1}, \ldots, \phi_{l_N}]$, but not under permutations of $[\phi_{k_1}, \ldots, \phi_{k_P}, \phi_{l_1}, \ldots, \phi_{l_N}]$. Here we see an important difference between the Schrödinger and Dirac settings. The potential of a self-adjoint Schrödinger operator is real-valued. Consequently, applications of Lemma 3.1 yield the terms (3.1) in complex-conjugate pairs–in other words, the P positive and N negative frequencies in (3.1) appear in reversed roles in the conjugate term, with P 'negative' and N 'positive' frequencies. For this reason, the ordering of the appearance of new frequencies via iterated applications of Lemma 3.1 matters not at all, and in [26, Equation 4.7] the associated leading terms f are symmetrically averaged accordingly. In the Dirac setting, since φ is in general complex-valued, we inherit a lesser symmetry and we must distinguish between positive and negative frequencies.

We also see in (3.1) a difference from the setting of orthogonal polynomials on the unit circle. In the Dirac setting, the appearance of $\partial_x \theta(x, \eta) = \frac{i}{2}(e^{i(\eta x + 2\theta)}\varphi - \overline{e^{i(\eta x + 2\theta)}\varphi})$ leads to a change in K, the number of $e^{i(\eta x + 2\theta)}$ factors, by ± 1 in each new term produced by applying Lemma 3.1. In the setting of orthogonal polynomials on the unit circle, on the other hand, the appearance of

$\theta(n+1,\eta)-\theta(n,\eta)$ leads instead to a much more varied effect on K in each of the newly produced terms. In [24], this is dealt with by passing to Taylor expansions of $e^{2ki(\theta_{n+1}-\theta_n)}$, but here we will be able to work directly with $\partial_x\theta(x,\eta)$.

Now, due to the iterative nature of our strategy in implementing Lemma 3.1, the resulting (3.2) admits a recursive relation. For any $I\geq 1$, $0\leq K\leq I$, and permutations $\sigma\in S_P$ and $\tau\in S_N$, $f_{I,K}$ depends only on $f_{I-1,K-1}$ and $f_{I-1,K+1}$ due to Lemma 3.1 and the introduction via $\partial_t\theta$ of both $e^{\pm i(\eta t+2\theta)}$. However, the term (3.1) is not produced in our iterative procedure if either $K<0$ or $K>I$, and we will define $f_{I,K}$ for such K to be zero. The recursion resulting from repeated application of Lemma 3.1 is as follows: if $0\leq K\leq I$ and, as before, $P=\frac{I+K}{2}$, $N=\frac{I-K}{2}\in\mathbb{Z}$,

$$f_{1,1}(\eta;[\phi_{k_1}])=1;$$

$$g_{I,K}(\eta;[\phi_{k_j}]_{j=1}^P;[\phi_{l_j}]_{j=1}^N)=\frac{iK}{\sum_{j=1}^P\phi_{k_j}-\sum_{j=1}^N\phi_{l_j}-K\eta}f_{I,K}(\eta;[\phi_{k_j}]_{j=1}^P;[\phi_{l_j}]_{j=1}^N);\tag{3.3}$$

$$f_{I,K}(\eta;[\phi_{k_j}]_{j=1}^P;[\phi_{l_j}]_{j=1}^N)=\frac{1}{P!N!}\sum_{a=-1}^{1}\sum_{\substack{\sigma\in S_P\\ \tau\in S_N}}\omega_a g_{I-1,K+a}(\eta;[\phi_{k_{\sigma(j)}}]_{j=1}^{\min[P,P+a]};[\phi_{l_{\tau(j)}}]_{j=1}^{\min[N,N-a]}),\tag{3.4}$$

where we shall think of ω_a as a function of $1+1+0$ variables if $a=-1$ or of $1+0+1$ variables if $a=1$, in either case defined as

$$\omega_a=\omega_a(\eta;[\phi_{k_j}]_{j=1}^{\max[0,-a]};[\phi_{l_j}]_{j=1}^{\max[0,a]})=\delta_{a+1}-\delta_{a-1},$$

where δ_j is one if $j=0$ and zero otherwise. By convention, we define $f_{I,K}$ and $g_{I,K}$ to be zero whenever $K>I$, $I-K\notin 2\mathbb{Z}$, or either $I<1$ or $K<0$, regardless of the number of frequencies on which they are made to depend. Of course, ω_a never depends on the frequencies–we only define ω_a in this way to better make sense of the following symmetric product. Let us also note that we suppress the η-dependence of $f_{I,K}$ and $g_{I,K}$ in any argument in which η is fixed.

We define the symmetric product $\odot$ in order to simplify notation: given $\mathfrak{f}$, a function of $1+P_1+N_1$ variables, and $\mathfrak{g}$, a function of $1+P_2+N_2$ variables, we call $P=P_1+P_2$ and $N=N_1+N_2$ and define their symmetric product, $\mathfrak{f}\odot\mathfrak{g}$, as

$$\begin{aligned}(\mathfrak{f}\odot\mathfrak{g})&(\eta;[\phi_{k_j}]_{j=1}^P;[\phi_{l_j}]_{j=1}^N)\\ &=\frac{1}{P!N!}\sum_{\substack{\sigma\in S_P\\ \tau\in S_N}}\mathfrak{f}(\eta;[\phi_{k_{\sigma(j)}}]_{j=1}^{P_1};[\phi_{l_{\tau(j)}}]_{j=1}^{N_1})\mathfrak{g}(\eta;[\phi_{k_{\sigma(j)}}]_{j=P_1+1}^{P};[\phi_{l_{\tau(j)}}]_{j=N_1+1}^{N}).\end{aligned}$$

It is straightforward to check that $\odot$ is commutative and associative. We also define for $0\leq K\leq I$

$$\Xi_{I,K}(\eta;[\phi_{k_j}]_{j=1}^P;[\phi_{l_j}]_{j=1}^N):=\delta_{I-1}\delta_{K-1}.$$

With this notation, we can abbreviate the definition (3.4) as

$$f_{I,K}(\eta;[\phi_{k_j}]_{j=1}^P;[\phi_{l_j}]_{j=1}^N)=\Xi_{I,K}+\sum_{a=-1}^{1}\omega_a\odot g_{I-1,K+a}.\tag{3.5}$$

As noted above, the number K of $e^{i(\eta x+2\theta)}$ factors changes by ±1 with each application of Lemma 3.1. In particular, with enough consecutive decreases, K may shrink to zero. Lemma 3.1 may still be applied so long as the sum $\sum_{j=1}^P\phi_{k_j}-\phi_{l_j}$ is nonzero. When both $K=0$ and $\sum_{j=1}^P\phi_{k_j}-\phi_{l_j}=0$, however, Lemma 3.1 cannot be (usefully) applied, and we depend instead on a cancellation allowed by the following lemma:

Lemma 3.2. *For any $I, p \in \mathbb{Z}$, $\operatorname{Re} f_{I,2p} = 0$. Moreover, if $I = 2P$ and $\sum_{j=1}^{P} \phi_{k_j} - \sum_{j=1}^{P} \phi_{l_j} = 0$, then*

$$f_{I,0}(\eta; [\phi_{k_j}]_{j=1}^{P}; [\phi_{l_j}]_{j=1}^{P}) = f_{I,0}(\eta; [\phi_{l_j}]_{j=1}^{P}; [\phi_{k_j}]_{j=1}^{P}).$$

Proof. We have $\operatorname{Re} f_{I,2p} = 0$ simply by convention when I is odd. We now prove that $\operatorname{Re} f_{I,2p} = 0$ for all $p \in \mathbb{Z}$ by induction on $I = 2P$. The case $P \le 0$ holds trivially. Suppose $\operatorname{Re} f_{2k,2p} = 0$ for all $p \in \mathbb{Z}$ and for all $k < P$. By expanding $f_{I,2n}$ using (3.5) and (3.3), we obtain

$$f_{I,2n} = \sum_{a=-1}^{1} \omega_a \odot \frac{(2n+a)i}{\sum_{j=1}^{q_1} \phi_{k_j} - \sum_{j=1}^{r_1} \phi_{l_j} - (2n+a)\eta} \times \big(\Xi_{I-1,2n+a} + \sum_{b=-1}^{1} \omega_b \odot \frac{(2n+a+b)i}{\sum_{j=1}^{q_2} \phi_{k_j} - \sum_{j=1}^{r_2} \phi_{l_j} - (2n+a+b)\eta} f_{I-2,2n+a+b}\big).$$

In taking the real part, the term with the Ξ factor vanishes due to the i in the first quotient, and the other term vanishes due to the i^2 together with the induction hypothesis.

For the second part, we first prove by induction on n that for $I = 2n$, $n \ge 1$,

$$f_{I,0} = \frac{1}{n!n!} \sum_{\sigma,\tau \in S_n} \sum_{s \in \mathcal{A}(I)} H_{I,s,\sigma,\tau}, \tag{3.6}$$

where $\mathcal{A}(I)$ is the set of $I+1$-tuples $(s_0, s_1, \ldots, s_I)$ with integer components such that $|s_{i+1} - s_i| = 1$, $s_i \ge 1$ for $1 \le i \le I-1$, and $s_0 = s_I = 0$, and

$$H_{I,s,\sigma,\tau}(\eta; [\phi_{k_j}]_{j=1}^{I/2}; [\phi_{l_j}]_{j=1}^{I/2}) = \prod_{m=1}^{I-1} \frac{i(s_m - s_{m-1})s_m}{\sum_{j=1}^{\frac{m+s_m}{2}} \phi_{k_{\sigma(j)}} - \sum_{j=1}^{\frac{m-s_m}{2}} \phi_{l_{\tau(j)}} - s_m\eta}.$$

We obtained $f_{I,0}$ by averaging over permutations before each application of Lemma 3.1. The terms $H_{I,s,\sigma,\tau}$ are obtained by applying Lemma 3.1 $I-1$ times without averaging first. We then return to $f_{I,0}$ by averaging over permutations. We also have that

$$H_{I,s,\sigma,\tau}(\eta; [\phi_{k_j}]_{j=1}^{I/2}; [\phi_{l_j}]_{j=1}^{I/2}) = H_{I,\tilde{s},\tau,\sigma}(\eta; [\phi_{l_j}]_{j=1}^{I/2}; [\phi_{k_j}]_{j=1}^{I/2}),$$

where $\tilde{s}_i = s_{I-i}$. Clearly, $s \in \mathcal{A}(I)$ if and only if $\tilde{s} \in \mathcal{A}(I)$, so summing over $\mathcal{A}(I)$ and averaging in permutations $\sigma, \tau \in S_n$ completes the proof. □

With Lemma 3.2, we see that we may apply Lemma 3.1 until all terms remaining either have p factors γ_j, are bounded by $2\tau^K$ for some finite K, or else are purely imaginary and do not contribute to the Prüfer amplitude. In so doing, we introduce many $g_{I,K}$, each of which appears to introduce a singularity in η at $\frac{1}{K}(\sum_{j=1}^{P} \phi_{k_j} - \sum_{j=1}^{N} \phi_{l_j})$. In fact, such singularities for $K > 1$ are removable, due to the following lemma:

Lemma 3.3. *If $0 < K \le I$ and $0 < k < K$, then*

$$f_{I,K} = \sum_{i=0}^{I} f_{i,k} \odot g_{I-i,K-k}, \tag{3.7}$$

$$g_{I,K} = \sum_{i=0}^{I} g_{i,k} \odot g_{I-i,K-k}. \tag{3.8}$$

Proof. The proof is similar to that of [26, Lemma 5.1 (i)]. We prove (3.7) and (3.8) simultaneously by induction on I. Both statements hold vacuously when $I < 2$ or $K < 2$, so we assume $2 \le K \le I$. Suppose both statements hold for all $\tilde{I} < I$. Then using (3.5) and associativity of $\odot$ yields

$$\begin{aligned}\sum_{i=0}^{I} f_{i,k} \odot g_{I-i,K-k} &= \sum_{i=0}^{I}(\Xi_{i,k} + \sum_{a=-1}^{1} \omega_a \odot g_{i-1,k+a}) \odot g_{I-i,K-k} \\ &= \sum_{i=0}^{I} \Xi_{i,k} \odot g_{I-i,K-k} + \sum_{a=-1}^{1} \omega_a \odot (\sum_{i=0}^{I} g_{i-1,k+a} \odot g_{I-i,K-k}).\end{aligned}$$

Since by our convention $g_{-1,k+a} = 0$ for any k, a, we may reindex in i with no cost as

$$\delta_{k-1}\Xi_{1,1} \odot g_{I-1,K-1} + \sum_{a=-1}^{1} \omega_a \odot (\sum_{i=0}^{I-1} g_{i,k+a} \odot g_{I-1-i,K-k}).$$

At this point, we may apply the induction hypothesis on the inner sum both when $a = -1$ and $a = 1$ so long as $0 < k + a < K + a$ and $0 < K + a \le I$. If $K + a > I$, then for each i, either $g_{i,k+1}$ or $g_{I-1-i,K-k}$ is zero. Since $g_{I-1,K+a}$ is also zero, we may include this term at no cost. That $0 < k < K$ implies $k + 1 < K + a$, but we do have $k + a \le 0$ exactly when $k = 1$ and $a = -1$. This is the only exception we must make to the induction hypothesis, and we're left with

$$\begin{aligned}\sum_{i=0}^{I} f_{i,k} \odot g_{I-i,K-k} &= \delta_{k-1}\Xi_{1,1} \odot g_{I-1,K-1} + \sum_{a=-1}^{1} \omega_a \odot (g_{I-1,K+a} - \delta_{k-1}\delta_{a+1} g_{I-1,K-1}) \\ &= f_{I,K} - \Xi_{I,K} + \delta_{k-1}(\Xi_{1,1} \odot g_{I-1,K-1} - \omega_{-1} \odot g_{I-1,K-1}) = f_{I,K},\end{aligned}$$

where we've used $\Xi_{I,K} = 0$ (since $I > 1$) and $\omega_{-1} = \Xi_{1,1}$.

It remains to show (3.8) at I. As before, let $I = P + N$ and $K = P - N$. Now also set $i = P_{i,k} + N_{i,k}$ and $k = P_{i,k} - N_{i,k}$ for any $1 \le i \le I$ and $0 < k < K$. By (3.3), for any permutations $\sigma \in S_P$ and $\tau \in S_N$ and for any $1 \le i \le I$ and $0 < k < K$, we have

$$\begin{aligned}\frac{K f_{I,K}([\phi_{k_{\sigma(j)}}]_{j=1}^{P}; [\phi_{l_{\tau(j)}}]_{j=1}^{N})}{g_{I,K}([\phi_{k_{\sigma(j)}}]_{j=1}^{P}; [\phi_{l_{\tau(j)}}]_{j=1}^{N})} &= \frac{k f_{i,k}([\phi_{k_{\sigma(j)}}]_{j=1}^{P_{i,k}}; [\phi_{l_{\tau(j)}}]_{j=1}^{N_{i,k}})}{g_{i,k}([\phi_{k_{\sigma(j)}}]_{j=1}^{P_{i,k}}; [\phi_{l_{\tau(j)}}]_{j=1}^{N_{i,k}})} \\ &\quad + \frac{(K-k) f_{I-i,K-k}([\phi_{k_{\sigma(j)}}]_{j=P_{i,k}+1}^{P}; [\phi_{l_{\tau(j)}}]_{j=N_{i,k}+1}^{N})}{g_{I-i,K-k}([\phi_{k_{\sigma(j)}}]_{j=P_{i,k}+1}^{P}; [\phi_{l_{\tau(j)}}]_{j=N_{i,k}+1}^{N})}.\end{aligned}$$

Clearing denominators here and averaging in permutations $(\sigma, \tau) \in S_P \times S_N$, notice that both $f_{I,K}$ and $g_{I,K}$ are each symmetric with respect to (σ, τ), so that we're left with

$$\begin{aligned}\frac{K f_{I,K}}{g_{I,K}} g_{i,k} \odot g_{I-i,K-k} &= \frac{1}{P!N!} \sum_{\substack{\sigma \in S_P \\ \tau \in S_N}} \Big(k f_{i,k}([\phi_{k_{\sigma(j)}}]_{j=1}^{P_{i,k}}; [\phi_{l_{\tau(j)}}]_{j=1}^{N_{i,k}}) g_{I-i,K-k}([\phi_{k_{\sigma(j)}}]_{j=P_{i,k}+1}^{P}; [\phi_{l_{\tau(j)}}]_{j=N_{i,k}+1}^{N}) \\ &\quad + (K-k) f_{I-i,K-k}([\phi_{k_{\sigma(j)}}]_{j=P_{i,k}+1}^{P}; [\phi_{l_{\tau(j)}}]_{j=N_{i,k}+1}^{N}) g_{i,k}([\phi_{k_{\sigma(j)}}]_{j=1}^{P_{i,k}}; [\phi_{l_{\tau(j)}}]_{j=1}^{N_{i,k}})\Big) \\ &= k f_{i,k} \odot g_{I-i,K-k} + (K-k) f_{I-i,K-k} \odot g_{i,k}.\end{aligned}$$

Summing both sides in $0 \le i \le I$ we have

$$\frac{Kf_{I,K}}{g_{I,K}} \sum_{i=0}^{I} g_{i,k} \odot g_{I-i,K-k} = k \sum_{i=0}^{I} f_{i,k} \odot g_{I-i,K-k} + (K-k) \sum_{i=0}^{I} f_{I-i,K-k} \odot g_{i,k}$$

$$= kf_{I,K} + (K-k) \sum_{i=0}^{I} f_{i,l} \odot g_{I-i,K-l} = Kf_{I,K},$$

where $l := K - k$ and we've used (3.7), since $0 < l < K$ also. Dividing both sides by $\frac{Kf_{I,K}}{g_{I,K}}$ completes the proof. □

Lemma 3.3 allows us to reduce $f_{I,K}$ and $g_{I,K}$ to sums of products of only the $f_{i,1}$ and $g_{i,1}$. We may use this reduction and a study of singularities of $g_{i,1}$ to define the correct exceptional set $\mathfrak{S}_p$. Note that $I - K \notin 2\mathbb{Z}$ implies $g_{I,K} = 0$ by our convention and so contributes no singularities.

Lemma 3.4.

(i) For $I = 2n - 1$, $n \ge 1$, if $\eta = \zeta$ is a nonremovable singularity of $g_{I,1}(\eta; [\phi_{k_j}]_{j=1}^{n}; [\phi_{l_j}]_{j=1}^{n-1})$, then ζ may be written in the form (1.9) for $1 \le m \le n$.

(ii) For $1 < K \le I$, $I - K \in 2\mathbb{Z}$, if $\eta = \zeta$ is a nonremovable singularity of $g_{I,K}$, then ζ may be written in the form (1.9) for $1 \le m \le n$, where $n = \lfloor I/2 \rfloor$.

Proof.

(i) The case $n = 1$ follows immediately from (3.3). Suppose the statement holds for $n < N$ and let $I = 2N - 1$. By (3.3), $g_{I,1}$ has $\sum_{j=1}^{N} \phi_{k_j} - \sum_{j=1}^{N-1} \phi_{l_j}$ as a nonremovable singularity and all other nonremovable singularities are those arising from $f_{I,1}$. By (3.5), singularities of $f_{I,1}$ arise from $g_{I-1,2}$, and by (3.8), these are just the singularities arising from $g_{i,1}$ for $0 \le i \le I - 1$. The induction hypothesis completes the proof.

(ii) Fix $2 \le K \le I$. By iteratively applying (3.8) with $k = 1$ to $g_{I,K}$ $K - 1$ times, we see that nonremovable singularities of $g_{I,K}$ arise as those of $g_{i,1}$ for $1 \le i \le I - 1$, since for $i = 0, I$, either $g_{i,\cdot}$ or $g_{I-i,\cdot}$ is zero. By part (i), if $I = 2n$, the nonremovable singularities of $g_{i,1}$ for $1 \le i \le I - 1$ are of the form (1.9) for $1 \le m \le n$. If $I = 2n + 1$, then $g_{I-1,1}$ is zero, so nonremovable singularities of $g_{i,1}$ for $1 \le i \le I - 1$ are still of the form (1.9) for $1 \le m \le n$. In either case, $n = \lfloor I/2 \rfloor$.

□

Corollary 3.5. *For $0 \le K \le I$, $I - K \in 2\mathbb{Z}$, if $\eta = \zeta$ is a nonremovable singularity of $f_{I,K}$, then ζ may be written in the form (1.9) for $1 \le m \le n$, where $n = \lfloor (I-1)/2 \rfloor$.*

Proof. If $I = 2n + 1$ and $K = 0$ or $K = 1$, we have from (3.5)

$$f_{I,K} = \Xi_{I,1} + \omega_1 \odot g_{I-1,K+1},$$

since $g_{I-1,K-1} = 0$. Thus, nonremovable singularities of $f_{I,K}$ are the same as those of $g_{I-1,K+1}$. By Lemma 3.4 (ii), each of these has singularities of the form (1.9) for $1 \le m \le \lfloor (I-1)/2 \rfloor = n$.

On the other hand, if $1 < K \le I$, $I - K \in 2\mathbb{Z}$, we can use (3.7) to write $f_{I,K}$ as

$$f_{I,K} = \sum_{i=0}^{I} f_{i,1} \odot g_{I-i,K-1},$$

from which we see that the nonremovable singularities of $f_{I,K}$ arise from those of $f_{i,1}$ and $g_{i,K-1}$ for $1 \le i \le I$. Singularities of $f_{i,1}$ are of the form (1.9) for $1 \le m \le \lfloor (I-1)/2 \rfloor$ by the first part of the proof (since for I even, $f_{I,1} = 0$). Singularities of $g_{i,K-1}$ are of the form (1.9) for $1 \le m \le \lfloor (I-1)/2 \rfloor$ by Lemma 3.4 (ii) (since $g_{I,K-1} = 0$ by the assumption $I - K \in 2\mathbb{Z}$). □

4. Finitely many summands

If φ is of finite Wigner-von Neumann type, there are finitely many nonremovable singularities arising from applications of Lemma 3.1, all of the form (1.9). We obtain the first of our main results:

Proof of Theorem 1.2. By Lemma 2.1 and subordinacy theory for Dirac operators due to Behncke, it is enough to show that given $\eta/2 \notin \mathfrak{S}_p$, all solutions $U(x, \eta)$ are bounded, regardless of their boundary value at zero. Given such a solution U, we pass to its Prüfer amplitude $r(x, \eta)$ and begin from (2.2). Repeatedly applying Lemma 3.1 to

$$\int_0^x e^{i(\eta t + 2\theta(t,\eta))}\varphi(t)dt,$$

where φ is of the form (1.4), we obtain a finite sum of terms of the form (3.1) with either $I = p$ or $K = 0$ and a finite sum of errors, with each error bounded by $2\tau^K$ for some finite K. The leading functions $f_{I,K}(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^N)$ are meromorphic functions in η where all poles have the form (1.9) with $1 \le m \le n$ by Lemma 3.5. The terms with $I = p$ are uniformly bounded in x by the L^p condition on the β_j. For one of the (finitely many) terms with $K = 0$,

$$f_{2P,0}(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^P) \int_0^x \prod_{j=1}^P e^{-i\phi_{k_j} t} e^{i\phi_{l_j} t} \gamma_{k_j}(t)\overline{\gamma_{l_j}(t)}dt,$$

if $\sum_{j=1}^P \phi_{k_j} - \phi_{l_j} \neq 0$, we may apply Lemma 3.1 once more to give a finite x-independent upper bound. If $\sum_{j=1}^P \phi_{k_j} - \phi_{l_j} = 0$, then there is a corresponding term,

$$f_{2P,0}(\eta; [\phi_{l_j}]_{j=1}^P; [\phi_{k_j}]_{j=1}^P) \int_0^x \prod_{j=1}^P \overline{\gamma_{k_j}(t)}\gamma_{l_j}(t)dt,$$

where the two constants $f_{I,0}(\eta)$ of these two corresponding terms are equal and purely imaginary by Lemma 3.2. The sum of these corresponding terms, then, is purely imaginary, and taking the real part in (2.2) annihilates them both. This shows that $\log r(x, \eta)$ is bounded uniformly in $x \in [0, \infty)$ and completes the proof. □

We now turn to the proof of Theorem 1.3 and show that elements of $\mathfrak{S}_p$ may indeed appear as embedded eigenvalues. We begin with a solution $U(x, \eta)$ to (1.3) for φ of the form (1.10). Consider $r(x, \eta)$ at $\eta_0 = \sum_{j=1}^n \phi_{k_j} - \sum_{j=1}^{n-1} \phi_{l_j}$, where $p = 2n + 1$ and assume that η_0 cannot be rewritten in the form (1.9) using fewer frequencies. We will use the following analog of [26, Lemma 6.1]:

Lemma 4.1. *Let $E \in \mathbb{R}$ and let $r(x, \eta), \theta(x, \eta)$ be the Prüfer variables for a solution $U(x, \eta)$ of (1.3) at $E = \eta/2$. If*

$$\partial_x \log r(x, \eta) = -\frac{B}{x^{(p-2)\gamma}} + b(x, \eta) \tag{4.1}$$

for some $b(x, \eta)$ integrable in x and if $\theta_\infty = \lim_{x\to\infty} \theta(x)$ exists, then for some $A > 0$, we have

$$U(x) = Af(x)\begin{pmatrix}(1+i)e^{-i(\eta/2x+\theta_\infty)} \\ (1-i)e^{i(\eta/2x+\theta_\infty)}\end{pmatrix}(1 + o(1)), \ x \to \infty, \tag{4.2}$$

where

$$f(x) = \begin{cases} x^{-B} & \text{if } \gamma = \frac{1}{p-2} \\ \exp(-\frac{B}{1-(p-2)\gamma} x^{1-(p-2)\gamma}) & \text{if } \gamma \in (\frac{1}{p}, \frac{1}{p-2}). \end{cases} \tag{4.3}$$

Proof. Follows from the proof of [26, Lemma 6.1] and (2.1). □

Beginning from (2.2), we may apply our iterative procedure from the proof of Theorem 1.2 to produce terms with x-independent upper bounds and other terms of the form

$$f_{2n-1,1}(\eta_0; [\phi_{k_j}]_{j=1}^n; [\phi_{l_j}]_{j=1}^{n-1}) x^{-\delta(p-2)} e^{i(\sum_{j=1}^n \xi_{k_j}(x) - \sum_{j=1}^{n-1} \xi_{l_j}(x) + 2\theta)} \prod_{j=1}^n c_{k_j} \prod_{j=1}^{n-1} \overline{c_{l_j}}.$$

We cannot apply Lemma 3.1 to these latter terms due to the nonremovable singularity in $g_{2n-1,1}(\eta)$ at η_0. These terms appear once with each distinct permutation pair (σ, τ) of $(k_1, \dots, k_n) \times (l_1, \dots, l_{n-1})$, and since there may be repeated indices $k_j = k_i$ or $l_j = l_i$, we simply denote the number of such distinct permutations by C. Thus we arrive at

$$\partial_x \log r(x, \eta) = \operatorname{Re}\big(\frac{\Lambda}{x^{\delta(p-2)}} e^{i(\xi(x)+2\theta)} + b(x)\big), \tag{4.4}$$

where $\xi(x) = \sum_{j=1}^n \xi_{k_j}(x) - \sum_{j=1}^{n-1} \xi_{l_j}(x)$, $b \in L^1((0,\infty))$, and

$$\Lambda = C f_{2n-1,1} \prod_{j=1}^n c_{k_j} \prod_{j=1}^{n-1} \overline{c_{l_j}}.$$

Without the exponential term in (4.4), the right hand side is clearly not integrable. Our goal is to choose ξ such that the exponential term above does not oscillate enough to make the right hand side integrable. To that end, we wish to control the behavior of $\partial_x \theta$. Our iterative procedure starting with (2.3) leaves the same terms from before with the addition of terms of the form

$$f_{2P,0}(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^P) \int_0^x \prod_{j=1}^P \gamma_{k_j}(t) \overline{\gamma_{l_j}(t)} dt,$$

where $\sum_{j=1}^P \phi_{k_j} - \phi_{l_j} = 0$, for all $1 \le P \le n$. These terms were eliminated in the proof of Theorem 1.2 by taking the real part since we were after $\partial_x \log r(x,\eta)$, but in controlling $\partial_x \theta$, we take the imaginary part, and these terms must be included. Thus we write

$$\partial_x \theta(x, \eta) = -\operatorname{Im}\big(\Omega(x) + \frac{\Lambda}{x^{\delta(p-2)}} e^{i(\xi(x)+2\theta)} + c(x)\big),$$

where again $c(x) \in L^1$ and now

$$\Omega(x) = \sum_{P=1}^n \sum_{\sum_{j=1}^P \phi_{k_j} - \phi_{l_j} = 0} f_{2P,0}(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^P) \prod_{j=1}^P \gamma_{k_j}(x) \overline{\gamma_{l_j}(x)}.$$

For the moment, let us consider only the case $p = 3$ for convenience. Then Ω simplifies, after appealing to (3.4) to compute $f_{2,0}$, to

$$\Omega(x) = -i \sum_{j=1}^M \frac{|c_j|^2}{\phi_j - \eta} x^{-2\delta}. \tag{4.5}$$

Supposing $\min\{\phi_j : 1 \leq j \leq M\} < \eta < \max\{\phi_j : 1 \leq j \leq M\}$, we may choose the c_j such that the summands in Ω cancel and $\Omega(x) \equiv 0$. Then, using the proof of [26, Lemma 6.3] to instead arrive at a choice of $\xi(x)$ yielding

$$\lim_{x\to\infty} \xi(x) + 2\theta(x) = -\arg \Lambda,$$

rather than $-\frac{\pi}{2} - \arg \Lambda$ (which requires no change to the proof), appealing to (4.4) and Lemma 4.1 shows η_0 is an embedded eigenvalue in the case $p = 3$. This shows that the sets $\mathfrak{S}_p$ indeed contain possible embedded eigenvalues, but does not show that the growth in the sets $\mathfrak{S}_p$ is necessary. For $p = 5$, $\eta_0 \in \mathfrak{S}_5 \setminus \mathfrak{S}_3$, and to prove Theorem 1.3 we show that η_0 in this case can be an embedded eigenvalue.

Proof of Theorem 1.3. When $p = 5$, Ω includes the terms in (4.5) along with the terms

$$\sum_{\sum_{j=1}^{2} \phi_{k_j} - \phi_{l_j} = 0} f_{4,0}(\eta; [\phi_{k_j}]_{j=1}^2; [\phi_{l_j}]_{j=1}^2) \prod_{j=1}^{2} \gamma_{k_j}(x)\overline{\gamma_{l_j}(x)} = \sum_{j_1,j_2=1}^{M} f_{4,0}(\eta; [\phi_{j_1}, \phi_{j_2}]; [\phi_{j_1}, \phi_{j_2}])|c_{j_1}c_{j_2}|^2 x^{-4\delta}.$$

Then from (3.4) we compute

$$f_{4,0}(\eta; [\phi_{j_1}, \phi_{j_2}]; [\phi_{j_1}, \phi_{j_2}]) = -\frac{i}{2}\frac{\phi_{j_1} + \phi_{j_2} - 2\eta}{(\phi_{j_1} - \eta)^2(\phi_{j_2} - \eta)^2}.$$

Thus, if conditions (1.11) and (1.12) both hold, $\Omega(x)$ is identically zero, and the rest of the proof follows exactly as in the $p = 3$ case. □

Lastly, we show that conditions (1.11) and (1.12) can hold simultaneously, so that Theorem 1.3 can hold non-vacuously. We take $M = 3$ in (1.10), so that

$$\varphi(x) = ae^{i\phi x}\gamma_1(x) + be^{i\psi x}\gamma_2(x) + ce^{i\rho x}\gamma_3(x),$$

where $\gamma_j(x) = e^{i\xi_j(x)}x^{-\delta}$ and $a, b, c \in \mathbb{C}$. The conditions (1.11) and (1.12) become

$$\frac{|a|^2}{\phi - \eta} + \frac{|b|^2}{\psi - \eta} + \frac{|c|^2}{\rho - \eta} = 0, \tag{4.6}$$

$$\frac{|a|^4}{(\phi - \eta)^3} + \frac{|b|^4}{(\psi - \eta)^3} + \frac{|c|^4}{(\rho - \eta)^3} + \frac{|ab|^2}{(\phi - \eta)^2(\psi - \eta)^2}(\phi + \psi - 2\eta)$$
$$+ \frac{|ac|^2}{(\phi - \eta)^2(\rho - \eta)^2}(\phi + \rho - 2\eta) + \frac{|bc|^2}{(\psi - \eta)^2(\tau - \eta)^2}(\psi + \rho - 2\eta) = 0. \tag{4.7}$$

Let $E = \eta/2$ for $\eta = \phi + \psi - \rho$. Then ϕ, ψ, ρ rationally independent implies $E \in \mathfrak{S}_5 \setminus \mathfrak{S}_3$. We wish to choose ϕ, ψ, and τ rationally independent such that the above conditions hold. Condition (4.6) is equivalent to $|c|^2 = \frac{|a|^2}{\psi - \rho} + \frac{|b|^2}{\phi - \rho}$, and we assume $\psi < \rho < \phi$ so that this condition may be satisfied for many choices of a and b. Suppose $2\rho - \phi - \psi = 1$. This implies the following simple identities:

$$\begin{aligned} \phi - \eta &= \rho - \psi, & \phi + \psi - 2\eta &= 1, \\ \psi - \eta &= \rho - \phi, & \phi + \rho - 2\eta &= \rho - \psi + 1, \\ \rho - \eta &= 1, & \psi + \rho - 2\eta &= \rho - \phi + 1, \end{aligned}$$

which allow us to rewrite (4.7) as

$$\frac{|a|^4}{(\rho - \psi)^3} + \frac{|b|^4}{(\rho - \phi)^3} + (\frac{|a|^2}{\psi - \rho} + \frac{|b|^2}{\phi - \rho})^2 + \frac{|ab|^2}{(\rho - \psi)^2(\rho - \phi)^2}$$
$$+ \frac{|a|^2}{(\rho - \psi)^2}(\rho - \psi + 1)(\frac{|a|^2}{\psi - \rho} + \frac{|b|^2}{\phi - \rho}) + \frac{|b|^2}{(\rho - \phi)^2}(\rho - \phi + 1)(\frac{|a|^2}{\psi - \rho} + \frac{|b|^2}{\phi - \rho}) = 0,$$

which equality holds identically, independently of the choice of a and b, just by expanding and cancelling. There are many choices of ϕ, ψ, and ρ so that $\psi < \rho < \phi$, $2\rho - \phi - \psi = 1$, and they are rationally independent. We may take, for example, $\phi = \sqrt{5}, \rho = \sqrt{3}$, and $\psi = 2\sqrt{3} - \sqrt{5} - 1$. For any such choice, both conditions (4.6) and (4.7) hold, and Theorem 1.3 implies that $\phi + \psi - \rho$ is an embedded eigenvalue in $\mathfrak{S}_5 \setminus \mathfrak{S}_3$.

5. Infinitely many summands

If φ is not of finite type, our goal is instead to bound the Hausdorff measure of $\mathfrak{S}_p$. Bounding $\dim_H(\mathfrak{S}_p)$ will require careful bookkeeping of the now infinitely many critical points in η. We will summarize the critical point information via the following recursively defined rational functions: let $h_I(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^{P-1})$ be defined for $I = 2P - 1$ by $h_1(\eta; [\phi_{k_1}]) = (\phi_{k_1} - \eta)^{-1}$ and

$$h_I(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^{P-1}) = \frac{1}{\sum_{j=1}^P \phi_{k_j} - \sum_{j=1}^{P-1} \phi_{l_j} - \eta} \sum_{m=0}^{I-1} h_m(\eta; [\phi_{k_j}]_{j=1}^{\frac{m+1}{2}}; [\phi_{l_j}]_{j=1}^{\frac{m-1}{2}}) h_{I-1-m}(\eta; [\phi_{k_j}]_{j=\frac{m+3}{2}}^P; [\phi_{l_j}]_{j=\frac{m+1}{2}}^{P-2}), \tag{5.1}$$

where we again denote by $[\phi_j]_{j=1}^n$ an ordered n-tuple of frequencies of φ. For even I, we define $h_I \equiv 0$. The functions h_I are analogous to the h_J of [27], with additional structure added in order to omit sums of frequencies not of the form (1.9). Our first step is to relate the functions h_I to $g_{I,1}$ in order to identify the singularities of h_I:

Lemma 5.1. *The functions $g_{I,1}$ for $I = 2P - 1$ are just rescaled, symmetrized h_I, namely,*

$$g_{I,1}([\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^{P-1}) = \frac{i}{P!(P-1)!} \sum_{\substack{\sigma \in S_P \\ \tau \in S_{P-1}}} h_I([\phi_{k_{\sigma(j)}}]_{j=1}^P; [\phi_{l_{\tau(j)}}]_{j=1}^{P-1}).$$

Proof. We prove the lemma by induction on P. The case $P = 1$ holds by definition. Suppose the statement holds up to $P - 1$. Let us call $\phi = \sum_{j=1}^P \phi_{k_j} - \sum_{j=1}^{P-1} \phi_{l_j}$. Combining (3.3), (3.4), and (3.8), we have

$$g_{I,1}([\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^{P-1}) = \frac{-i}{\phi - \eta} \sum_{m=0}^{I-1} \frac{1}{P!(P-1)!} \sum_{\substack{\sigma \in S_P \\ \tau \in S_{P-1}}} g_{m,1}([\phi_{k_{\sigma(j)}}]_{j=1}^{\frac{m+1}{2}}; [\phi_{l_{\tau(j)}}]_{j=1}^{\frac{m-1}{2}}) \odot g_{I-1-m,1}([\phi_{k_{\sigma(j)}}]_{j=\frac{m+3}{2}}^P; [\phi_{l_{\tau(j)}}]_{j=\frac{m+1}{2}}^{P-1})$$

Then we use the induction hypothesis to rewrite as

$$\frac{-i}{\phi-\eta}\sum_{m=0}^{I-1}\frac{1}{P!(P-1)!}\sum_{\substack{\sigma\in S_P\\ \tau\in S_{P-1}}}\frac{i}{(\frac{m+1}{2})!(\frac{m-1}{2})!}\sum_{\substack{\mu\in S_{\frac{m+1}{2}}\\ \nu\in S_{\frac{m-1}{2}}}} h_m([\phi_{k_{\mu(\sigma(j))}}]_{j=1}^{\frac{m+1}{2}};[\phi_{l_{\nu(\tau(j))}}]_{j=1}^{\frac{m-1}{2}})$$

$$\odot\frac{i}{(\frac{I-m}{2})!(\frac{I-m-2}{2})!}\sum_{\substack{\alpha\in S_{\frac{I-m}{2}}\\ \beta\in S_{\frac{I-m-2}{2}}}} h_{I-1-m}([\phi_{k_{\alpha(\sigma(j))}}]_{j=\frac{m+3}{2}}^{P};[\phi_{k_{\beta(\tau(j))}}]_{j=\frac{m+1}{2}}^{P-2})$$

$$=\frac{i}{P!(P-1)!}\sum_{\substack{\sigma\in S_P\\ \tau\in S_{P-1}}}\frac{1}{\sum_{j=1}^{P}\phi_{k_{\sigma(j)}}-\sum_{j=1}^{P-1}\phi_{l_{\tau(j)}}-\eta}$$

$$\times\sum_{m=0}^{I-1}h_m([\phi_{k_{\sigma(j)}}]_{j=1}^{\frac{m+1}{2}};[\phi_{l_{\tau(j)}}]_{j=1}^{\frac{m-1}{2}})h_{I-1-m}([\phi_{k_{\sigma(j)}}]_{j=\frac{m+3}{2}}^{P};[\phi_{l_{\tau(j)}}]_{j=\frac{m+1}{2}}^{P-2}),$$

where we've noticed that the permutations μ,ν,α, and β as well as the remaining $\odot$ are redundant since the h_m and h_{I-1-m} have already been symmetrized. The definition of h_I completes the proof. □

Since the first three conditions of the following lemma are satisfied by our operator data of Wigner-von Neumann type by definition, the following lemma gives a sufficient condition in terms of the functions h_I for boundedness of all solutions at E:

Lemma 5.2. *Let operator data φ be given by (1.4), with $c_j\in\mathbb{C}$, $\phi_j\in\mathbb{R}$, and let $\eta\in\mathbb{R}$ such that*

(i) $\sup_j \operatorname{Var}(\gamma_j,(0,\infty)) := \tau < \infty$;
(ii) for some $p=2n+1$, $\sup_j\int_0^\infty|\gamma_j(t)|^p dt := \sigma<\infty$;
(iii) $\sum_{j=1}^\infty |c_j|<\infty$;
(iv) for odd $I=1,\dots,p-2$ *and* $P:=\frac{I+1}{2}$,

$$\sum_{\substack{k_1,\dots,k_P=1\\ l_1,\dots,l_{P-1}=1}}^{\infty}|h_I(\eta;[\phi_{k_j}]_{j=1}^{P};[\phi_{l_j}]_{j=1}^{P-1})\prod_{j=1}^{P}c_{k_j}\prod_{j=1}^{N}\overline{c_{l_j}}|<\infty.$$

Then, for $E=\frac{\eta}{2}$, all solutions of (1.3) are bounded.

We prove Lemma 5.2 in a series of smaller lemmas. Our strategy is to begin with

$$\log\frac{Z(x,\eta)}{Z(0,\eta)}=\int_0^x \mathcal{S}_{1,1}(t)dt,$$

where

$$\mathcal{S}_{I,K}(x)=\sum_{\substack{k_1,\dots,k_P=1\\ l_1,\dots,l_N=1}}^{\infty}f_{I,K}(\eta;[\phi_{k_j}]_{j=1}^{P};[\phi_{l_j}]_{j=1}^{N})\prod_{j=1}^{P}\beta_{k_j}(x)\prod_{j=1}^{N}\overline{\beta_{l_j}(x)}e^{iK(\eta x+2\theta(x))}, \tag{5.2}$$

with $\beta_j(x) = c_j e^{-i\phi_j x}\gamma_j(x)$, and then pass to higher values of I via Lemma 3.1. Note that $\mathcal{S}_{I,K}$ is trivial if $I + K \notin 2\mathbb{Z}$ or $I < K$. We will track errors using

$$E_{I,K} = 2 \sum_{\substack{k_1,\dots,k_{P_K}=1\\ l_1,\dots,l_{N_K}=1}}^{\infty} |g_{I,K} \prod_{j=1}^{P} c_{k_j} \prod_{j=1}^{N} c_{l_j}|.$$

Lemma 5.3. *If φ obeys the conditions of Lemma 5.2, then $E_{I,K}$ is finite for $1 \le K \le I \le p-2$.*

Proof. By Lemma 5.1 and (5.1), since condition (iv) of Lemma 5.2 holds for $1 \le I \le p-2$, $E_{I,1}$ is finite for the same values of I. Then (3.8) gives

$$E_{I,K} \le \sum_{i=0}^{I} E_{i,k} E_{I-i,K-k},$$

for any $0 < k < K$. The lemma follows. □

Lemma 5.4. *If φ obeys the conditions of Lemma 5.2, then the sum $\mathcal{S}_{I,K}(t)$ is absolutely convergent when $0 \le K \le I \le p$. If, moreover, $I = p$, then*

$$\int_0^\infty \sum_{\substack{k_1,\dots,k_P=1\\ l_1,\dots,l_N=1}}^{\infty} |f_{I,K}(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^N) \prod_{j=1}^{P} \beta_{k_j} \prod_{j=1}^{N} \overline{\beta_{l_j}}|dt \le \sum_{a=-1}^{1} |\omega_a| E_{p-1,K+a} \sum_{j=1}^{\infty} |c_j| \sigma^p.$$

Proof. Taking absolute values in (3.5) gives

$$|f_{I,K}| \le |\Xi_{I,K}| + \sum_{a=-1}^{1} |\omega_a| \odot |g_{I-1,K+a}|.$$

Multiplying by

$$|\prod_{j=1}^{P} \beta_{k_j} \prod_{j=1}^{N} \overline{\beta_{l_j}}| \le |\prod_{j=1}^{P} c_{k_j} \prod_{j=1}^{N} c_{l_j}| \tau^I$$

and summing in k_j and l_j proves absolute convergence. If $I = p$, we instead multiply by

$$\int_0^\infty |\prod_{j=1}^{P} \beta_{k_j}(t) \prod_{j=1}^{N} \overline{\beta_{l_j}(t)}|dt \le |\prod_{j=1}^{P} c_{k_j} \prod_{j=1}^{N} c_{l_j}| \sigma^p,$$

and summing again in k_j and l_j completes the proof. □

Lemma 5.5. *For a fixed $I \in \mathbb{N}$, let $P_K := (I+K)/2$, $N_K := (I-K)/2$, and denote $\phi_K = \sum_{j=1}^{P_K} \phi_{k_j} - \sum_{j=1}^{N_K} \phi_{l_j}$, $\Gamma_K = \prod_{j=1}^{P_K} \gamma_{k_j} \prod_{j=1}^{N_K} \overline{\gamma_{l_j}}$, and $C_K = \prod_{j=1}^{P_K} c_{k_j} \prod_{j=1}^{N_K} \overline{c_{l_j}}$. Then,*

$$\sum_{K=0}^{I+1} \mathcal{S}_{I+1,K} = \sum_{K=1}^{I} \Big[\sum_{\substack{k_1,\dots,k_{P_K+1}=1\\ l_1,\dots,l_{N_K}}}^{\infty} C_K c_{k_{P_K+1}} g_{I,K} e^{i(K+1)(\eta t+2\theta)} e^{-i(\phi_K+\phi_{k_{P_K+1}})t} \Gamma_K \gamma_{P_K+1}$$
$$- \sum_{\substack{k_1,\dots,k_{P_K}=1\\ l_1,\dots,l_{N_K+1}=1}}^{\infty} C_K c_{l_{N_K}} g_{I,K} e^{i(K-1)(\eta t+2\theta)} e^{-i(\phi_K-\phi_{l_{N_K}+1})t} \Gamma_K \overline{\gamma_{l_{N_K+1}}} \Big].$$

Proof. Beginning from

$$\sum_{K=0}^{I+1} \mathcal{S}_{I+1,K} = \sum_{K=0}^{I+1} \sum_{\substack{k_1,\dots,k_{P_K}=1 \\ l_1,\dots,l_{N_K}=1}}^{\infty} f_{I+1,K}([\phi_{k_j}]_{j=1}^{P_K};[\phi_{l_j}]_{j=1}^{N_K}) \prod_{j=1}^{P_K} \beta_{k_j}(x) \prod_{j=1}^{N_K} \overline{\beta_{l_j}(x)} e^{iK(\eta x+2\theta(x))},$$

we use (3.4) to rewrite as

$$\sum_{K=0}^{I+1} \sum_{\substack{k_1,\dots,k_{P_K}=1 \\ l_1,\dots,l_{N_K}=1}}^{\infty} \frac{1}{P_K!N_K!} \sum_{a=-1}^{1} \sum_{\substack{\sigma\in S_{P_K} \\ \tau\in S_{N_K}}} \omega_a g_{I,K+a}([\phi_{k_{\sigma(j)}}]_{j=1}^{\min[P_K,P_K+a]};[\phi_{l_{\tau(j)}}]_{j=1}^{\min[N_K,N_K-a]})$$
$$\times \prod_{j=1}^{P_K} \beta_{k_j}(x) \prod_{j=1}^{N_K} \overline{\beta_{l_j}(x)} e^{iK(\eta x+2\theta(x))}.$$

Grouping terms with $g_{I,\tilde{K}}$ of the same indices $(I,\tilde{K})$, each such group has two summands, one from the case where $K=\tilde{K}+1$ and $a=-1$ and another from the case where $K=\tilde{K}-1$ and $a=1$. Each such term has the form

$$\sum_{\substack{k_1,\dots,k_{P_K+1}=1 \\ l_1,\dots,l_{N_K-1}=1}}^{\infty} \frac{1}{(P_K+1)!(N_K-1)!} \sum_{\substack{\sigma\in S_{P_K+1} \\ \tau\in S_{N_K-1}}} g_{I,1}([\phi_{k_{\sigma(j)}}]_{j=1}^{P_K};[\phi_{l_{\tau(j)}}]_{j=1}^{N_K-1}) \prod_{j=1}^{P_K+1} \beta_{k_j} \prod_{j=1}^{N_K-1} \overline{\beta_{l_j}}$$
$$- \sum_{\substack{k_1,\dots,k_{P_K}=1 \\ l_1,\dots,l_{N_K}=1}}^{\infty} \frac{1}{P_K!N_K!} \sum_{\substack{\sigma\in S_{P_K} \\ \tau\in S_{N_K}}} g_{I,1}([\phi_{k_{\sigma(j)}}]_{j=1}^{P_K};[\phi_{l_{\tau(j)}}]_{j=1}^{N_K-1}) \prod_{j=1}^{P_K} \beta_{k_j} \prod_{j=1}^{N_K} \overline{\beta_{l_j}}.$$

By choosing to average over permutations of the frequencies in the first summand of the lemma, and summing in K above, we have equality and complete the proof. ☐

Lemma 5.6. *If φ obeys the conditions of Lemma 5.2, then for $I=1,\dots,p-1$ and $0\le a<b<\infty$,*

$$\Big|\int_a^b \big(\sum_{K=1}^{I} \mathcal{S}_{I,K}(t) - \sum_{K=0}^{I+1} \mathcal{S}_{I+1,K}(t)\big)dt\Big| \le \sum_{K=1}^{I} \frac{1}{K} E_{I,K}\tau^I. \tag{5.3}$$

Proof. For $K\ge 1$, we start from Lemma 3.1 and multiply by $\frac{g_{I,K}}{iK}$ to get

$$\Big|\int_a^b f_{I,K} e^{iK(\eta t+2\theta)} e^{-i\phi t}\Gamma + 2ig_{I,K} e^{iK(\eta t+2\theta)} e^{-i\phi t}\Gamma\theta'\Big| \le \frac{2\tau^I}{K}|g_{I,K}|.$$

Then we multiply through by $\prod_{j=1}^{P} c_{k_j} \prod_{j=1}^{N} \overline{c_{l_j}}$ and sum in all the k_j and l_j from one to infinity, which summation is justified by Fubini's theorem and Lemmas 5.3 and 5.4. Then summing in K from one to I finishes the proof. The term containing the $g_{I,K}$ becomes $\mathcal{S}_{I+1,K}$ using (2.3) and Lemma 5.5. ☐

Proof of Lemma 5.2. By summing over $I=1,\dots,p-1$ in (5.3), we have

$$\Big|\int_a^b \big(\mathcal{S}_{1,1}(t) - \sum_{K=1}^{p} \mathcal{S}_{p,K}(t) - \sum_{I=2}^{p} \mathcal{S}_{I,0}(t)\big)dt\Big| \le \sum_{I=1}^{p-1}\sum_{K=1}^{I} \frac{1}{K} E_{I,K}\tau^I.$$

Then, for $I = p$, we bound the sum in K by

$$
\begin{aligned}
|\sum_{K=1}^{p}\int_a^b \mathcal{S}_{p,K}(t)dt| &\leq \sum_{K=1}^{p}\int_a^b \sum_{\substack{k_1,\ldots,k_{\frac{p+K}{2}}=1 \\ l_1,\ldots,l_{\frac{p-K}{2}}=1}}^{\infty} |f_{p,K}\prod_{j=1}^{\frac{p+K}{2}}\beta_{k_j}\prod_{j=1}^{\frac{p-K}{2}}\overline{\beta_{l_j}}|dt \\
&\leq \sum_{K=1}^{p}\sum_{a=-1}^{1}|\omega_a|E_{p-1,K+a}\sum_{j=1}^{\infty}|c_j|\sigma^p \leq 2\sum_{K=0}^{p-1}E_{p-1,K}\sum_{j=1}^{\infty}|c_j|\sigma^p.
\end{aligned}
$$

We conclude that

$$
\left|\int_0^x \big(\partial_t \log Z_U(t,\eta) - \sum_{I=2}^{p}\mathcal{S}_{I,0}(t)\big)dt\right| \leq \sum_{I=1}^{p-1}\sum_{K=1}^{I}\frac{1}{K}E_{I,K}\tau^I + 2\sum_{K=0}^{p-1}E_{p-1,K}\sum_{j=1}^{\infty}|c_j|\sigma^p. \tag{5.4}
$$

Since the right hand side of (5.4) is independent of the Prüfer variables, the left hand side converges uniformly in solution U. What follows is a continuous analog of the proof of [30, Lemma 8]. By taking the difference of the left hand sides of (5.4) for two linearly independent solutions U and V, and noting that $\sum_{I=2}^{p}\mathcal{S}_{I,0}$ is independent of the choice of solution, we have that

$$
\int_0^x \partial_t(\log Z_U(t,\eta) - \log Z_V(t,\eta))dt
$$

is convergent. Taking real and imaginary parts gives that

$$
\log\frac{r_U(x,\eta)}{r_V(x,\eta)},\ \theta_U(x,\eta) - \theta_V(x,\eta)
$$

both converge as $x \to \infty$. Since $\log\frac{r_U(x,\eta)}{r_V(x,\eta)}$ has a finite limit at infinity, $\frac{r_U(x,\eta)}{r_V(x,\eta)}$ has a finite, nonzero limit at infinity. By the convergence in (5.4) uniform in solution U, there exists $0 < x_0 < \infty$ such that for all solutions U

$$
\left|\int_{x_0}^{\infty} \big(\partial_t \log Z_U(t,\eta) - \sum_{I=2}^{p}\mathcal{S}_{I,0}(t)\big)dt\right| < \frac{\pi}{8}.
$$

Taking the imaginary part then gives that $|\theta_U(x,\eta) - \theta_V(x,\eta) - (\theta_U(x_0,\eta) - \theta_V(x_0,\eta))|$ is bounded by $\pi/4$ for $x \geq x_0$ for any pair of solutions U and V. In particular, if we choose the solution U arbitrarily and then choose V such that $\theta_V(x_0,\eta) = \theta_U(x_0,\eta) - \pi/2 + 2k\pi$ for some $k \in \mathbb{Z}$, then for $x \geq x_0$, we have

$$
\theta_U(x,\eta) - \theta_V(x,\eta) \in (\frac{\pi}{4}, \frac{3\pi}{4}).
$$

In particular, $\sin(\theta_U(x,\eta) - \theta_V(x,\eta)) \in (\frac{\sqrt{2}}{2}, 1]$. Since U and V are linearly independent, their Wronskian is independent of x and is given by some nonzero constant C. We write

$$
W[U,V](x) = 4r_U(x,\eta)r_V(x,\eta)\sin(\theta_U(x,\eta) - \theta_V(x,\eta)) = C \neq 0.
$$

Multiplying through by $\frac{r_U(x,\eta)}{r_V(x,\eta)}$, the boundedness of $\sin(\theta_U(x,\eta) - \theta_V(x,\eta))$ away from zero implies $r_U^2(x,\eta)$, and therefore $r_U(x,\eta)$, has a finite limit at infinity. Since U was chosen arbitrarily, every solution of (1.3) is bounded. □

Remark. It is quicker to merely show absence of subordinate solutions, rather than to show boundedness of solutions. If U were a subordinate solution, then for any linearly independent solution V we would have, by L'Hôpital's rule,

$$0 = \lim_{x\to\infty} \frac{\int_0^x \|U(t,\eta)\|^2 dt}{\int_0^x \|V(t,\eta)\|^2 dt} = \lim_{x\to\infty} \frac{\|U(x,\eta)\|^2}{\|V(x,\eta)\|^2} = \lim_{x\to\infty} \frac{|r_U(x,\eta)|^2}{|r_V(x,\eta)|^2},$$

which contradicts the fact from the first part of the proof that $\frac{r_U(x,\eta)}{r_V(x,\eta)} \to L$ for some nonzero L.

For the following lemma, recall that the Catalan numbers C_n are given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Lemma 5.7. *Let ν be a finite uniformly β-Hölder measure on $\mathbb{R}$.*

(i) If $\alpha \in (0, \beta)$, then for all $\psi \in \mathbb{R}$,

$$\int \frac{1}{|\psi - \eta|^\alpha} d\nu(\eta) \le D_\alpha, \tag{5.5}$$

where D_α is a finite constant that depends only on α.

(ii) For $I \ge 1$, $I = 2P - 1$, and $\alpha \in (0, \frac{\beta}{I})$,

$$\int |h_I(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^{P-1})|^\alpha d\nu(\eta) \le C_I D_{I\alpha}, \tag{5.6}$$

where C_I are Catalan numbers.

Proof.

(i) This is proved in [27, Lemma 4.1].

(ii) We induct on P. The case $P = 1$ holds by (5.5). Suppose the statement holds up to $P - 1$. Integrating one summand in (5.1) and using Hölder's inequality and the induction hypothesis gives

$$\int \left| \frac{1}{\sum_{j=1}^P \phi_{k_j} - \sum_{j=1}^{P-1} \phi_{l_j} - \eta} h_m h_{I-1-m} \right|^\alpha d\nu(\eta) \le D_{I\alpha}^{1/I} (C_m D_{I\alpha})^{m/I} (C_{I-1-m} D_{I\alpha})^{(I-1-m)/I}$$
$$\le C_m C_{I-1-m} D_{I\alpha},$$

and since the Catalan sequence obeys the recursion $C_n = \sum_{j=0}^{n-1} C_j C_{n-1-j}$, summing in $0 \le m \le I - 1$ recovers the Catalan number C_I.

∎

Lemma 5.8. *Suppose (1.7) holds and let $I = 2P - 1$. Then the set of η for which*

$$\sum_{\substack{k_1,\dots,k_P=1 \\ l_1,\dots,l_{P-1}=1}}^{\infty} \left| h_I(\eta; [\phi_{k_j}]_{j=1}^P; [\phi_{l_j}]_{j=1}^{P-1}) \prod_{j=1}^P c_{k_j} \prod_{j=1}^N \overline{c_{l_j}} \right| \tau^I = \infty,$$

has Hausdorff dimension at most $I\alpha$. If $I = 2P$, the same set is empty.

Proof. Let T be the set of η for which condition (iv) of Lemma 5.2 fails. Suppose the Hausdorff dimension of T is greater than $i\alpha$. Then for some $\beta > i\alpha$, $h^\beta(T) = \infty$. This implies the existence of a subset $T' \subset T$ such that $\nu = \chi_{T'} h^\beta$ is a finite uniformly β-Hölder measure with $\nu(T) > 0$. Then Lemma 5.7 implies

$$\int \sum_{\substack{k_1,\dots,k_{\frac{i+1}{2}}=1 \\ l_1,\dots,l_{\frac{i-1}{2}}=1}}^{\infty} \Big| h_i(\eta; [\phi_{k_j}]_{j=1}^{\frac{i+1}{2}}; [\phi_{l_j}]_{j=1}^{\frac{i-1}{2}}) \prod_{j=1}^{\frac{i+1}{2}} c_{k_j} \prod_{j=1}^{\frac{i-1}{2}} \overline{c_{l_j}} \Big|^\alpha d\nu(\eta)$$

$$= \sum_{\substack{k_1,\dots,k_{\frac{i+1}{2}}=1 \\ l_1,\dots,l_{\frac{i-1}{2}}=1}}^{\infty} \Big| \prod_{j=1}^{\frac{i+1}{2}} c_{k_j} \prod_{j=1}^{\frac{i-1}{2}} c_{l_j} \Big|^\alpha \int \Big| h_i(\eta; [\phi_{k_j}]_{j=1}^{\frac{i+1}{2}}; [\phi_{l_j}]_{j=1}^{\frac{i-1}{2}}) \Big|^\alpha d\nu(\eta)$$

$$\leq \sum_{\substack{k_1,\dots,k_{\frac{i+1}{2}}=1 \\ l_1,\dots,l_{\frac{i-1}{2}}=1}}^{\infty} \Big| \prod_{j=1}^{\frac{i+1}{2}} c_{k_j} \prod_{j=1}^{\frac{i-1}{2}} c_{l_j} \Big|^\alpha D_{i\alpha} = D_{i\alpha} \Big(\sum_{j=1}^{\infty} |c_j|^\alpha \Big)^i,$$

which is finite by the α-type decay condition (1.7). Since the integral is finite, the integrand is ν-almost everywhere finite. But since $\alpha \in (0, 1]$, this implies that condition (iv) holds ν-almost everywhere, so that $\nu(T) = 0$, a contradiction. The second statement of the lemma follows from the fact that h_I is zero for even I. □

Proof of Theorem 1.4. Conditions (1.5), (1.6), and (1.7) of Lemma 5.2 are trivially satisfied for every η. Condition (iv) is satisfied away from a set T of Hausdorff dimension at most $(p-2)\alpha$ by Lemma 5.8 (note that condition (iv) of Lemma 5.2 need only be satisfied for odd $I = 1, \dots, p-2$).

By Lemma 5.2 and Lemma 2.1, there are no subordinate solutions for $2E = \eta \in \mathbb{R} \setminus S$, and by the subordinacy theory of Gilbert-Pearson and Behncke, this implies the theorem. □

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