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The linear Lugiato–Lefever equation with forcing and nonzero periodic or nonperiodic boundary conditions

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We consider the linear Lugiato–Lefever equation formulated on a finite interval with *nonzero* boundary conditions. In particular, using the unified transform of Fokas, we obtain *explicit* solution formulae both for the general nonperiodic initial-boundary value problem and for the periodic Cauchy problem. These novel solution formulae involve integrals, as opposed to the infinite series associated with traditional solution techniques, and hence they have analytical as well as computational advantages. Importantly, as the linear Lugiato–Lefever can be related to the linear Schrödinger equation via a simple transformation, our results are directly applicable also to the linear Schrödinger equation posed on a finite interval with nonzero boundary conditions.

1. Introduction

The Lugiato–Lefever equation

$$u_t + i\beta u_{xx} + (1 + i\alpha)u - i|u|^2 u = F \tag{1-1}$$

has recently gained significant attention within the broader applied mathematics community. Here, $u(x, t)$ is a complex-valued function, α and β are real parameters, and F is a positive constant. Equation (1-1) is an envelope model that originates from the Maxwell–Bloch equations and was introduced in [Lugiato and Lefever 1987] as an example for dissipative structure and pattern formation in nonlinear optics. Further information about the derivation and relevance of the Lugiato–Lefever equation as an optical model in various settings, including experimental results that illustrate its physical significance, can be found in [Chembo and Yu 2010; Chembo and Menyuk 2013; Kippenberg et al. 2011; Mandel and Reichel 2017; Qi et al. 2017; Lugiato et al. 2018; Lottes et al. 2021]. Furthermore, several

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works have recently appeared in the literature on the rigorous mathematical study of (1-1), where a direction of particular emphasis concerns the stability of the periodic Cauchy problem — see, for example, the recent works [Delcey and Haragus 2018; Stanislavova and Stefanov 2018; Hakkaev et al. 2019; Haragus et al. 2021; 2023].

The Lugiato–Lefever equation (1-1) arises naturally in the periodic setting and also as a general initial-boundary value problem on a finite interval. In this work, we are concerned with the *forced linear counterpart* of (1-1), namely the equation

$$u_t + i\beta u_{xx} + (1 + i\alpha)u = f(x,t), \tag{1-2}$$

where f is a given forcing function, for which we derive novel, explicit solution formulae in both the periodic and the nonperiodic case with *nonzero* boundary conditions — see expressions (1-7) and (1-5) respectively. Our main motivation behind this study is related to the task of showing well-posedness of the nonlinear equation (1-1) via contraction mapping techniques. In particular, we note that in recent years a new method has been introduced by one of the authors and collaborators for proving the local well-posedness of initial-boundary value problems for nonlinear evolution equations — see, for example, [Fokas et al. 2016; 2017; Himonas et al. 2019; Himonas and Mantzavinos 2020]. This method is based on a contraction mapping argument that crucially relies on the solution formulae obtained for the forced linear counterparts of these problems via the *unified transform* of Fokas [1997; 2008] (also known as the Fokas method). In this respect, the explicit solution formulae (1-5) and (1-7) derived in the present work for the forced linear equation (1-2) will serve as starting points for implementing the method of [Fokas et al. 2016; 2017; Himonas et al. 2019; Himonas and Mantzavinos 2020] in order to establish the local well-posedness of the Lugiato–Lefever equation (1-1) on a finite interval with nonperiodic or periodic nonzero boundary conditions.

Specifically, in this work we explicitly solve the forced linear initial-boundary value problem

$$u_t + i\beta u_{xx} + (1 + i\alpha)u = f(x,t), \quad 0 < x < \ell,\ t > 0, \tag{1-3a}$$

$$u(x,0) = u_0(x), \quad 0 < x < \ell, \tag{1-3b}$$

$$u(0,t) = g_0(t), \quad u(\ell,t) = h_0(t), \quad t > 0, \tag{1-3c}$$

where the initial data u_0 and the boundary data g_0, h_0 are assumed to be sufficiently smooth so that the various computations carried out in this work make sense. In particular, the precise characterization of the optimal regularity of the initial and boundary data is a task which becomes more relevant when aiming to prove the well-posedness of the nonlinear equation (1-1) and hence it is reserved for future work in that direction.

Observe that, in the special case $f(x,t) = F$, equation (1-3a) corresponds to the linearization of the Lugiato–Lefever equation (1-1). Furthermore, note that the initial-boundary value problem (1-3) for the forced linear Lugiato–Lefever equation can be easily related to a corresponding problem for the forced linear Schrödinger equation. Indeed, the transformation $u(x,t) = e^{-(1+i\alpha)t} v(x,t)$ turns problem (1-3) into

$$v_t + i\beta v_{xx} = e^{(1+i\alpha)t} f(x,t), \quad 0 < x < \ell,\ t > 0, \tag{1-4a}$$

$$v(x,0) = u_0(x), \quad 0 < x < \ell, \tag{1-4b}$$

$$v(0,t) = e^{(1+i\alpha)t} g_0(t), \quad v(\ell,t) = e^{(1+i\alpha)t} h_0(t), \quad t > 0, \tag{1-4c}$$

which is an initial-boundary value problem for the familiar forced linear Schrödinger equation with Dirichlet data on the finite interval. *Therefore, the results obtained in this work for the forced linear Lugiato–Lefever equation on the interval are directly applicable also to the solution of the forced linear Schrödinger equation on the interval.* In fact, although the cubic nonlinear Schrödinger equation on the interval has been considered in various works (e.g., [Fokas and Its 2004; Fokas et al. 2005]), to the best of our knowledge the solution to the forced linear problem (1-4) via the unified transform has not been explicitly provided anywhere else in the literature. In particular, here we provide a complete derivation of the solution to this problem, with careful justification of the complex contour deformations that form the core of the unified transform.

It should be noted that the unified transform employed in this work is a universal method for solving initial-boundary value problems that involve evolution equations. The method was introduced in [Fokas 1997] and has since been advanced in multiple settings by many researchers. Indicatively, we mention [Fokas 2001; 2002b; Fokas and Pelloni 2005; Kalimeris and Fokas 2010; Mantzavinos and Fokas 2013], which establish Fokas's method as the initial-boundary value problem analogue of the classical Fourier transform used for the solution of the initial value problem of *linear* evolution equations. Moreover, the unified transform also has a nonlinear component which is applicable to completely integrable equations such as the cubic nonlinear Schrödinger and Korteweg–de Vries equations on the half-line and the finite interval [Fokas and Its 2004; Fokas et al. 2005; Boutet de Monvel et al. 2006; Boutet de Monvel and Shepelsky 2003; 2004; Fokas 2002a] or the Davey–Stewartson and Kadomtsev–Petviashvili equations on the half-plane [Fokas 2009; Mantzavinos and Fokas 2011]. A comprehensive presentation of the unified transform can be found in the monograph [Fokas 2008] as well as the review articles [Fokas and Spence 2012; Deconinck et al. 2014]. Especially relevant to the periodic problem for NLS is the recent work [Deconinck et al. 2021], while a general framework for linear constant-coefficient evolution equations with periodic boundary conditions via the unified transform is laid out in [Trogdon and Deconinck 2012]. Finally, as noted above, the unified transform has recently inspired a new approach for proving well-posedness of general nonlinear evolution equations in the initial-boundary value problem setting; see [Fokas et al. 2016; 2017; Himonas et al. 2019; Himonas and Mantzavinos 2020].

The explicit solution formula for problem (1-3) as derived via the unified transform in Section 3 is

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau\,dk \\
&\quad - \frac{1}{2\pi}\int_{k\in\mathcal{L}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}[e^{ik\ell}\hat{u}_0(k)-e^{-ik\ell}\hat{u}_0(-k)]\,dk \\
&\quad - \frac{\beta}{\pi}\int_{k\in\mathcal{L}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}} k[\tilde{h}_0(\omega,t)-e^{-ik\ell}\tilde{g}_0(\omega,t)]\,dk \\
&\quad - \frac{1}{2\pi}\int_{k\in\mathcal{L}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\int_{\tau=0}^{t} e^{i\omega\tau}[e^{ik\ell}\hat{f}(k,\tau)\,d\tau - e^{-ik\ell}\hat{f}(-k,\tau)]\,d\tau\,dk \\
&\quad + \frac{1}{2\pi}\int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}[\hat{u}_0(k)-\hat{u}_0(-k)]\,dk \\
&\quad + \frac{\beta}{\pi}\int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}} k[e^{ik\ell}\tilde{h}_0(\omega,t)-\tilde{g}_0(\omega,t)]\,dk \\
&\quad + \frac{1}{2\pi}\int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\int_{\tau=0}^{t} e^{i\omega\tau}[\hat{f}(k,\tau)-\hat{f}(-k,\tau)]\,d\tau\,dk, \qquad (1\text{-}5)
\end{aligned}$$

where ω is given by (2-3), $\hat{u}_0$ and $\hat{f}$ denote the finite-interval Fourier transforms of the initial data and the forcing defined by (2-1), $\tilde{g}_0$ and $\tilde{h}_0$ are temporal transforms of the boundary data defined by (2-5), and the complex contours $\mathcal{L}^{\pm}$ correspond to either

(i) the positively oriented boundaries $\partial D^{\pm}$ of the regions $D^{\pm}$ defined by (2-7), i.e., the upper and lower branches of the hyperbola $2\beta\,\mathrm{Re}(k)\,\mathrm{Im}(k) = -1$, or

(ii) the contours $\mathcal{C}^{\pm}$ of Figure 1,

which are equivalent (by Cauchy's theorem).

The general finite interval problem (1-3) is directly related to the periodic Cauchy problem

$$u_t + i\beta u_{xx} + (1+i\alpha)u = f(x,t), \quad 0 < x < \ell,\ t > 0, \qquad (1\text{-}6a)$$

$$u(x,0) = u_0(x), \quad 0 < x < \ell, \qquad (1\text{-}6b)$$

$$u(0,t) = u(\ell,t), \quad u_x(0,t) = u_x(\ell,t), \quad t > 0, \qquad (1\text{-}6c)$$

where the initial data $u_0(x)$ and forcing $f(x,t)$ are periodic functions such that $u_0(x+\ell) = u_0(x)$ and $f(x+\ell,t) = f(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$. Indeed, via a similar approach to the one used for problem (1-3), our analysis yields the following solution formula for the periodic problem (1-6):

$$\begin{aligned}
u(x,t) = {} & \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk \\
& + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau\,dk \\
& - \frac{1}{2\pi}\int_{k\in\mathcal{L}^+} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk \\
& - \frac{1}{2\pi}\int_{k\in\mathcal{L}^+} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau\,dk \\
& + \frac{1}{2\pi}\int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk \\
& + \frac{1}{2\pi}\int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau\,dk,
\end{aligned} \tag{1-7}$$

where, as before, ω is given by (2-3), $\hat{u}_0$ and $\hat{f}$ denote the finite-interval Fourier transforms of the initial data and the forcing defined by (2-1), and the complex contours $\mathcal{L}^{\pm}$ are the ones appearing in formula (1-5).

Structure. The solution formula (1-7) to the periodic problem (1-6) is derived in Section 2. Interesting reductions to the traditional "separation of variables/Fourier series" representation, as well as to the solution of the direct linearization of the Lugiato–Lefever equation (1-1) (i.e., when the forcing f is constant and equal to $F > 0$), are also provided in that section (see (2-13) and (2-11)). The analysis for the general nonperiodic problem (1-3) requires an additional crucial idea and is given in Section 3, leading to the solution formula (1-5). The reduction of this formula to the direct linearization of the Lugiato–Lefever equation (1-1) is also provided (see (3-14)).

2. The periodic problem

Our analysis is done for the linear Lugiato–Lefever equation (1-6a) with general, nonconstant forcing $f(x,t)$. The motivation for this is that the resulting solution formula can be used in the future for studying the nonlinear Lugiato–Lefever equation (1-1). Of course, setting $f(x,t) = F$ reduces (1-6a) to the linearization of (1-1).

2A. ***The global relation and an integral representation for the solution.*** The Fourier transform pair for a function $\phi(x)$ on the interval $0 < x < \ell$ is defined by

$$\begin{aligned}
\hat{\phi}(k) &= \int_{x=0}^{\ell} e^{-ikx}\phi(x)\,dx, \qquad k\in\mathbb{C}, \\
\phi(x) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx}\hat{\phi}(k)\,dk, \quad 0 < x < \ell.
\end{aligned} \tag{2-1}$$

Applying the above Fourier transform to (1-6a) yields

$$\partial_t \hat{u}(k,t)+i\beta\big\{e^{-ik\ell}u_x(\ell,t)-u_x(0,t)+ik[e^{-ik\ell}u(\ell,t)-u(0,t)+ik\hat{u}(k,t)]\big\} +(1+i\alpha)\hat{u}(k,t)=\hat{f}(k,t). \quad (2\text{-}2)$$

In view of the initial condition (1-6b), the periodic boundary conditions (1-6c), and the notation

$$\begin{gathered} \omega(k)=-\beta k^2-i+\alpha, \\ u(0,t)=u(\ell,t)=h(t), \quad u_x(0,t)=u_x(\ell,t)=g(t), \end{gathered} \quad (2\text{-}3)$$

we integrate (2-2) with respect to t to obtain what is known in the unified transform terminology as the *global relation*:

$$e^{i\omega t}\hat{u}(k,t)=\hat{u}_0(k)+i\beta\big\{[\tilde{g}(\omega,t)+ik\tilde{h}(\omega,t)]-e^{-ik\ell}[\tilde{g}(\omega,t)+ik\tilde{h}(\omega,t)]\big\} +\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau, \quad (2\text{-}4)$$

where we have also introduced the notation

$$\tilde{\phi}(\omega,t)=\int_{\tau=0}^{t} e^{i\omega\tau}\phi(\tau)\,d\tau. \quad (2\text{-}5)$$

We remark that the global relation (2-4) is valid for all $k\in\mathbb{C}$ in line with the domain of the interval Fourier transform (2-1).

Inverting the global relation (2-4) for $k\in\mathbb{R}$ by means of (2-1), we find the following *integral representation* for the solution u:

$$\begin{aligned} u(x,t)=\frac{1}{2\pi}\int_{k\in\mathbb{R}} & e^{ikx-i\omega t}\hat{u}_0(k)\,dk \\ &+\frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau\,dk \\ &+\frac{i\beta}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}[\tilde{g}(\omega,t)+ik\tilde{h}(\omega,t)]\,dk \\ &-\frac{i\beta}{2\pi}\int_{k\in\mathbb{R}} e^{ik(x-\ell)-i\omega t}[\tilde{g}(\omega,t)+ik\tilde{h}(\omega,t)]\,dk. \end{aligned} \quad (2\text{-}6)$$

The integral representation (2-6) is not an explicit solution formula since it contains the unknown boundary values $h(t)$ and $g(t)$ through the transforms $\tilde{h}(\omega,t)$ and $\tilde{g}(\omega,t)$. However, it turns out that these unknown transforms can be eliminated from (2-6) by exploiting the analyticity and exponential decay of the relevant integrands in appropriate regions of the complex k-plane.

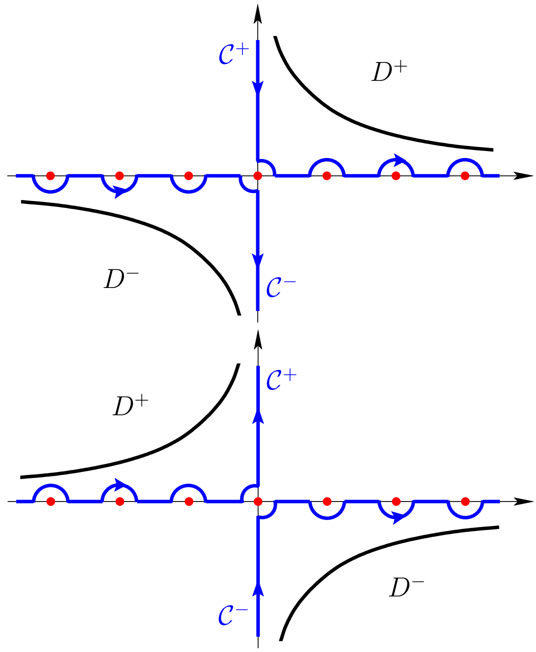

Figure 1. The regions $D^{\pm}$ defined by (2-7) along with their positively oriented boundaries $\partial D^{\pm}$, which correspond to the upper and lower branches of the hyperbola $2\beta\,\mathrm{Re}(k)\,\mathrm{Im}(k) = -1$ (black), and the contours $\mathcal{C}^{\pm}$ (blue) for $\beta < 0$ (top) and $\beta > 0$ (bottom). The red dots along the real axis correspond to the zeros $k_n = \frac{2n\pi}{\ell}$, $n \in \mathbb{Z}$, of the quantity $1 - e^{-ik\ell}$ in the periodic problem, and the zeros $k_n = \frac{n\pi}{\ell}$, $n \in \mathbb{Z}$, of the quantity $e^{ik\ell} - e^{-ik\ell}$ in the finite interval (nonperiodic) problem.

2B. ***Elimination of the unknowns and an explicit solution formula.*** Noting that $|e^{ikx}| = e^{-\mathrm{Im}(k)x}$ and recalling that $x > 0$, we see that e^{ikx} is bounded for $\mathrm{Im}(k) \geqslant 0$ and decays to zero as $|k| \to \infty$ whenever $\mathrm{Im}(k) > 0$. Similarly, the exponential $e^{-i\omega(t-\tau)}$ with $0 < \tau < t$ decays to zero in the region $\mathbb{C} \setminus \overline{D^+ \cup D^-}$, where the regions $D^{\pm}$ are defined by

$$D^{\pm} := \{k \in \mathbb{C} : \mathrm{Im}(k) \gtrless 0 \text{ and } \mathrm{Re}(i\omega) < 0\}. \tag{2-7}$$

More precisely, $D^{\pm} = \{\mathrm{Im}(k) \gtrless 0 \text{ and } 2\beta\,\mathrm{Re}(k)\,\mathrm{Im}(k) + 1 < 0\}$ and $D^+ \cup D^-$ corresponds to the region outside the branches of the hyperbola $2\beta\,\mathrm{Re}(k)\,\mathrm{Im}(k) = -1$ (see Figure 1).

Therefore, $e^{ikx - i\omega(t-\tau)}$ decays to zero as $|k| \to \infty$ inside the region $\{\mathrm{Im}(k) > 0\} \setminus \overline{D^+}$ and is bounded on the closure of that region. Thus, thanks to analyticity in k, Cauchy's integral theorem allows us to deform the path of integration of the third

k-integral in (2-6) from $\mathbb{R}$ to the contour $\mathcal{C}^+$ in the upper half of the complex k-plane (see Figure 1). Similarly, since $|e^{ik(x-\ell)}| = e^{-\operatorname{Im}(k)(x-\ell)}$ and $x < \ell$, $e^{ik(x-\ell)-i\omega(t-\tau)}$ decays to zero as $|k| \to \infty$ inside the region $\{\operatorname{Im}(k) < 0\} \setminus \overline{D^-}$ and is bounded on the closure of that region. Hence, we can deform the contour of integration of the fourth k-integral in (2-6) from $\mathbb{R}$ to the contour $\mathcal{C}^-$ in the lower half of the complex k-plane (see again Figure 1). We remark that both deformations can be rigorously justified along the lines of Proposition 2.1 proved later. Implementing them, we write the integral representation (2-6) as

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk \\ &\quad + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau\,dk \\ &\quad + \frac{i\beta}{2\pi}\int_{k\in\mathcal{C}^+} e^{ikx-i\omega t}[\tilde{g}(\omega,t)+ik\tilde{h}(\omega,t)]\,dk \\ &\quad - \frac{i\beta}{2\pi}\int_{k\in\mathcal{C}^-} e^{ik(x-\ell)-i\omega t}[\tilde{g}(\omega,t)+ik\tilde{h}(\omega,t)]\,dk. \end{aligned} \tag{2-8}$$

Then, substituting for the unknown quantity $\tilde{g}(\omega,t)+ik\tilde{h}(\omega,t)$ via the global relation (2-4), which is valid for all $k \in \mathbb{C}$ and, in particular, for $k \in \mathcal{C}^\pm$, turns (2-8) into the *explicit solution formula* (1-7) with $\mathcal{L}^\pm = \mathcal{C}^\pm$, where we have made crucial use of the following result, which is proved at the end of this section.

Proposition 2.1. *For any* $0 < x < \ell$ *and any* $t > 0$,

$$\int_{k\in\mathcal{C}^+} \frac{e^{ikx}}{1-e^{-ik\ell}}\hat{u}(k,t)\,dk = \int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)}}{1-e^{-ik\ell}}\hat{u}(k,t)\,dk = 0. \tag{2-9}$$

Remark 2.2 (need for deformation). Substituting for the unknown $\tilde{g}(\omega,t) + ik\tilde{h}(\omega,t)$ via (2-4) without first deforming the relevant contours to $\mathcal{C}^\pm$, i.e., at the level of (2-6), yields the tautology $u(x,t) = u(x,t)$. Hence, the deformation to $\mathcal{C}^\pm$ is needed in order to eliminate the unknown.

Remark 2.3 (deformation to the contours $\partial D^\pm$). The zeros of $1-e^{-ik\ell}$ occur at $k = k_n = \frac{2\pi n}{\ell}$, $n \in \mathbb{Z}$, thus they do not introduce any singularities in formula (1-7) since they are avoided by $\mathcal{C}^\pm$ (see Figure 1).

Furthermore, instead of $\mathcal{C}^\pm$, it is possible to deform the contours of integration of the boundary-value integrals in (2-6) from $\mathbb{R}$ to the positively oriented boundaries $\partial D^\pm$ of the regions $D^\pm$, i.e., to the upper and lower branches of the hyperbola $2\beta\operatorname{Re}(k)\operatorname{Im}(k) = -1$ depicted in Figure 1. See Section 3B for a justification of this alternative deformation in the case of the nonperiodic problem and, in particular, for the analogue of Proposition 2.1 in the case of $\partial D^\pm$. Therefore, formula (1-7) provides the explicit solution to the periodic problem (1-6) for the forced linear Lugiato–Lefever equation with *either* of the choices $\mathcal{L}^\pm = \partial D^\pm$ and $\mathcal{L}^\pm = \mathcal{C}^\pm$.

In the special case $f(x,t) = F$, which corresponds to the linearization of the Lugiato–Lefever equation (1-1), formula (1-7) becomes

$$\begin{aligned} u(x,t) = {} & \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk - \frac{F}{2\pi}\int_{k\in\mathbb{R}} e^{ikx}\frac{(1-e^{-ik\ell})(1-e^{-i\omega t})}{k\omega}\,dk \\ & - \frac{1}{2\pi}\int_{k\in\mathcal{L}^+} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk + \frac{F}{2\pi}\int_{k\in\mathcal{L}^+} e^{ikx}\frac{1-e^{-i\omega t}}{k\omega}\,dk \\ & + \frac{1}{2\pi}\int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk - \frac{F}{2\pi}\int_{k\in\mathcal{L}^-} e^{ik(x-\ell)}\frac{1-e^{-i\omega t}}{k\omega}\,dk. \end{aligned} \tag{2-10}$$

This formula can be further simplified by observing that the singularities arising from the zeros of ω are all removable apart from $k = 0$, which is a simple pole due to the presence of k in the denominator of the relevant integrands. Thus, denoting by $C_{\varepsilon,[0,\pi]}(0)$ and $C_{\varepsilon,[\pi,2\pi]}(0)$ the semicircles of radius $\varepsilon > 0$ centered at the origin and oriented counterclockwise from 0 to π and from π to 2π respectively, we employ Cauchy's integral theorem to write

$$\int_{k\in\mathcal{L}^+} e^{ikx}\frac{1-e^{-i\omega t}}{k\omega}\,dk = \left(\int_{k\in\mathbb{R}\setminus[-\varepsilon,\varepsilon]} - \int_{k\in C_{\varepsilon,[0,\pi]}(0)}\right) e^{ikx}\frac{1-e^{-i\omega t}}{k\omega}\,dk$$

and

$$\int_{k\in\mathcal{L}^-} e^{ik(x-\ell)}\frac{1-e^{-i\omega t}}{k\omega}\,dk = \left(\int_{k\in\mathbb{R}\setminus[-\varepsilon,\varepsilon]} + \int_{k\in C_{\varepsilon,[\pi,2\pi]}(0)}\right) e^{ik(x-\ell)}\frac{1-e^{-i\omega t}}{k\omega}\,dk.$$

In turn, we have

$$\begin{aligned} & \int_{k\in\mathcal{L}^+} e^{ikx}\frac{1-e^{-i\omega t}}{k\omega}\,dk - \int_{k\in\mathcal{L}^-} e^{ik(x-\ell)}\frac{1-e^{-i\omega t}}{k\omega}\,dk \\ & \quad = \int_{k\in\mathbb{R}\setminus[-\varepsilon,\varepsilon]} e^{ikx}(1-e^{-ik\ell})\frac{1-e^{-i\omega t}}{k\omega}\,dk - \int_{k\in C_{\varepsilon,[0,\pi]}(0)} e^{ikx}\frac{1-e^{-i\omega t}}{k\omega}\,dk \\ & \qquad - \int_{k\in C_{\varepsilon,[\pi,2\pi]}(0)} e^{ik(x-\ell)}\frac{1-e^{-i\omega t}}{k\omega}\,dk, \end{aligned}$$

and so taking the limit $\varepsilon \to 0^+$ and then using Cauchy's residue theorem for the second and third integrals, as well as the fact that the singularity at $k = 0$ is removable in the first integral, we find

$$\begin{aligned} & \int_{k\in\mathcal{L}^+} e^{ikx}\frac{1-e^{-i\omega t}}{k\omega}\,dk - \int_{k\in\mathcal{L}^-} e^{ik(x-\ell)}\frac{1-e^{-i\omega t}}{k\omega}\,dk \\ & \qquad = \int_{k\in\mathbb{R}} e^{ikx}\frac{(1-e^{-ik\ell})(1-e^{-i\omega t})}{k\omega}\,dk + 2\pi\frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}. \end{aligned}$$

Via the above calculations, expression (2-10) yields the following solution formula for the linearization of the Lugiato–Lefever equation (1-1) with periodic boundary conditions:

$$u(x,t)=\frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk-\frac{1}{2\pi}\int_{k\in\mathcal{L}^+}\frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk \\ +\frac{1}{2\pi}\int_{k\in\mathcal{L}^-}\frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk+\frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}F. \quad (2\text{-}11)$$

2C. ***Reduction to the traditional separation of variables/Fourier series representation.*** We emphasize that the solution formula (2-11) could not have been obtained without deforming from $\mathbb{R}$ to the contours $\mathcal{L}^\pm$. This is because if one were to employ the global relation (2-4) without making these deformations (i.e., directly at the level of (2-6) instead of (2-8)) then one would obtain the tautology $u(x,t)=u(x,t)$. Indeed, without the deformations, the three terms involving the initial datum in (1-7) would cancel one another, and so would the forcing terms, while the last two terms that involve $\hat{u}(k,t)$ would combine to yield $u(x,t)$ via the inverse Fourier transform.

At the same time, if desired, it is possible to collapse the contours $\mathcal{L}^\pm$ involved in formula (2-11) to the real axis. However, in doing so, one must take into account the residue contributions from the poles at $k_n=\frac{2\pi n}{\ell}$, $n\in\mathbb{Z}$, arising from the term $1-e^{-ik\ell}$. In particular, using analyticity (Cauchy's theorem) and exponential decay, we can write the solution formula (2-11) as

$$u(x,t)=\frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk-\frac{1}{2\pi}\int_{k\in\mathcal{L}_\varepsilon^+}\frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk \\ +\frac{1}{2\pi}\int_{k\in\mathcal{L}_\varepsilon^-}\frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk+\frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}F,$$

with

$$\mathcal{L}_\varepsilon^+=\mathbb{R}_\varepsilon\cup\bigcup_{n\in\mathbb{Z}}-C_{\varepsilon,[0,\pi]}(k_n),\quad \mathcal{L}_\varepsilon^-=\mathbb{R}_\varepsilon\cup\bigcup_{n\in\mathbb{Z}}C_{\varepsilon,[\pi,2\pi]}(k_n),$$

where for $0<\varepsilon\leqslant\frac{1}{3}|k_{n+1}-k_n|=\frac{2\pi}{3\ell}$ we define

$$\mathbb{R}_\varepsilon=\bigcup_{n\in\mathbb{Z}}[k_n+\varepsilon,k_{n+1}-\varepsilon],$$
$$C_{\varepsilon,[a,b]}(k_n)=\{|k-k_n|=\varepsilon,\ a\leqslant\arg(k)\leqslant b\},$$

with positive orientation. In fact, breaking down the contours $\mathcal{L}_\varepsilon^\pm$ into their individual components and noting that the integrals along $\mathbb{R}_\varepsilon$ combine to a single integral that does not involve a singular integrand, we take the limit $\varepsilon\to 0^+$ to obtain

$$u(x,t)=\frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{k\in\bigcup_{n\in\mathbb{Z}}C_{\varepsilon,[0,\pi]}(k_n)}\frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk + \frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{k\in\bigcup_{n\in\mathbb{Z}}C_{\varepsilon,[\pi,2\pi]}(k_n)}\frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk + \frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}F. \tag{2-12}$$

Finally, using Cauchy's residue theorem we compute

$$\lim_{\varepsilon\to 0^+}\int_{k\in C_{\varepsilon,[0,\pi]}(k_n)}\frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk = i\pi\lim_{k\to k_n}\left[\frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}(k-k_n)\hat{u}_0(k)\right] = \frac{\pi}{\ell}e^{ik_nx-i\omega(k_n)t}\hat{u}_0(k_n)$$

and, similarly,

$$\lim_{\varepsilon\to 0^+}\int_{k\in C_{\varepsilon,[\pi,2\pi]}(k_n)}\frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\hat{u}_0(k)\,dk = \frac{\pi}{\ell}e^{ik_nx-i\omega(k_n)t}\hat{u}_0(k_n).$$

Therefore, (2-12) becomes

$$u(x,t)=\frac{1}{\ell}\sum_{n\in\mathbb{Z}}e^{ik_nx-i\omega(k_n)t}\hat{u}_0(k_n)+\frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}F,\quad k_n=\frac{2\pi n}{\ell}, \tag{2-13}$$

which is the formula one can expect to obtain via separation of variables and the traditional Fourier series method.

2D. ***Proof of Proposition 2.1.*** We only provide the proof for the integral along $\mathcal{C}^+$, as the argument for the integral along $\mathcal{C}^-$ is entirely analogous. Integrating twice by parts and rearranging, we have

$$\begin{aligned}\int_{k\in\mathcal{C}^+}\frac{e^{ikx}}{1-e^{-ik\ell}}\hat{u}(k,t)\,dk &= \int_{k\in\mathcal{C}^+}e^{ikx}\frac{1}{1-e^{-ik\ell}}\int_{y=0}^{\ell}e^{-iky}u(y,t)\,dy\,dk\\ &= u(0,t)\int_{k\in\mathcal{C}^+}e^{ikx}\frac{1}{ik}\,dk-u_y(0,t)\int_{k\in\mathcal{C}^+}e^{ikx}\frac{1}{k^2}\,dk\\ &\quad -\int_{y=0}^{\ell}u_{yy}(y,t)\int_{k\in\mathcal{C}^+}e^{ikx}\frac{1}{k^2}\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk\,dy.\end{aligned} \tag{2-14}$$

Note that we have used Fubini's theorem in order to interchange the order of integration in the double integral since

$$\left|\int_{y=0}^{\ell}u_{yy}(y,t)\int_{k\in\mathcal{C}^+}e^{ikx}\frac{1}{k^2}\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk\,dy\right| \leqslant \int_{y=0}^{\ell}|u_{yy}(y,t)|\int_{k\in\mathcal{C}^+}\frac{1}{|k|^2}\frac{e^{-\operatorname{Im}(k)(x+\ell-y)}}{|e^{ik\ell}-1|}\,dk\,dy<\infty$$

via steps similar to those in the estimation of I_R and J_R below.

For the first k-integral on the right-hand side of (2-14), by Cauchy's theorem we have

$$\int_{k\in\mathcal{C}^+} e^{ikx}\frac{1}{ik}\,dk = -\lim_{R\to\infty}\int_{k\in C_{R,\theta_0}(0)} e^{ikx}\frac{1}{ik}\,dk,$$

where $C_{R,\theta_0}(0)$ denotes the circular arc

$$C_{R,\theta_0}(0) = \begin{cases} \{\mathrm{R}e^{i\theta} : \theta_0 \leqslant \theta \leqslant \frac{\pi}{2}\}, & \beta < 0, \\ \{\mathrm{R}e^{i\theta} : \frac{\pi}{2} \leqslant \theta \leqslant \pi - \theta_0\}, & \beta > 0, \end{cases}$$

with $\theta_0 = 0$ if $\|R - k_n| \geqslant \frac{2\pi}{3\ell}$ for all $n \in \mathbb{Z}$, and $\theta_0 \in \big(0, \sin^{-1}\big(\frac{2\pi}{3\ell R}\big)\big]$ if there exists $n \in \mathbb{Z}$ such that $|R - k_n| < \frac{2\pi}{3\ell}$. Since $\sin\theta = \sin(\pi-\theta)$, the cases $\beta < 0$ and $\beta > 0$ can be handled in the same way to yield

$$\left|\int_{k\in C_{R,\theta_0}(0)} e^{ikx}\frac{1}{ik}\,dk\right| \leqslant \int_{\theta=0}^{\frac{\pi}{2}} e^{-xR\sin\theta}\,d\theta.$$

Thus, using the well-known inequality

$$\sin\theta \geqslant \frac{2}{\pi}\theta, \quad \theta \in \Big[0, \frac{\pi}{2}\Big], \tag{2-15}$$

we find

$$\left|\int_{k\in C_{R,\theta_0}(0)} e^{ikx}\frac{1}{ik}\,dk\right| \leqslant \int_{\theta=0}^{\frac{\pi}{2}} e^{-xR\cdot\frac{2}{\pi}\theta}\,d\theta = \frac{\pi}{2xR}(1-e^{-xR}) \to 0, \quad R\to\infty,$$

and hence we conclude that

$$\int_{k\in\mathcal{C}^+} e^{ikx}\frac{1}{ik}\,dk = 0. \tag{2-16}$$

Similarly, it follows that the second integral on the right-hand side of (2-14) also equals zero:

$$\int_{k\in\mathcal{C}^+} e^{ikx}\frac{1}{k^2}\,dk = 0. \tag{2-17}$$

Concerning the k-integral inside the double integral of (2-14), Cauchy's theorem implies

$$\int_{k\in\mathcal{C}^+} e^{ikx}\frac{1}{k^2}\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk = -\lim_{R\to\infty}\int_{k\in C_{R,\theta_0}(0)} e^{ikx}\frac{1}{k^2}\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk, \tag{2-18}$$

with $C_{R,\theta_0}(0)$ defined as above. We will estimate this integral by decomposing it into two pieces which we handle separately:

$$\int_{k\in C_{R,\theta_0}(0)} e^{ikx}\frac{1}{k^2}\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk = I_R + J_R, \tag{2-19}$$

where

$$I_R = \int_{k\in C_{R,\theta_0}(0),\operatorname{Im}(k)\geqslant\frac{2\pi}{3\ell}} e^{ikx}\frac{1}{k^2}\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk, \tag{2-20}$$

$$J_R = \int_{k\in C_{R,\theta_0}(0),\operatorname{Im}(k)\leqslant\frac{2\pi}{3\ell}} e^{ikx}\frac{1}{k^2}\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk. \tag{2-21}$$

Estimation of I_R. This term is the easiest of the two. First, observe that, since $y \leqslant \ell$ and $\operatorname{Im}(k) \geqslant 0$,

$$\left|\frac{e^{-iky}}{1-e^{-ik\ell}}\right| = \frac{e^{-\operatorname{Im}(k)(\ell-y)}}{|e^{ik\ell}-1|} \leqslant \frac{1}{|e^{ik\ell}-1|} \tag{2-22}$$

and so

$$|I_R| \leqslant \int_{k\in C_{R,\theta_0}(0),\operatorname{Im}(k)\geqslant\frac{2\pi}{3\ell}} \left|e^{ikx}\frac{1}{k^2}\right|\frac{1}{|e^{ik\ell}-1|}|dk|. \tag{2-23}$$

Therefore, noting that for $k \in C_{R,\theta_0}(0)$ with $\operatorname{Im}(k) \geqslant \frac{2\pi}{3\ell}$ we have $\arg(\theta) \in \left[\sin^{-1}\left(\frac{2\pi}{3\ell R}\right), \frac{\pi}{2}\right]$ when $\beta<0$ and $\arg(\theta) \in \left[\frac{\pi}{2}, \pi-\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)\right]$ when $\beta>0$, we employ the triangle inequality twice to get

$$|I_R| \leqslant \begin{cases} \dfrac{1}{R}\displaystyle\int_{\theta=\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)}^{\frac{\pi}{2}} \frac{e^{-xR\sin\theta}}{|e^{iRe^{i\theta}\ell}-1|}\,d\theta, & \beta<0, \\ \dfrac{1}{R}\displaystyle\int_{\theta=\frac{\pi}{2}}^{\pi-\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)} \frac{e^{-xR\sin\theta}}{|e^{iRe^{i\theta}\ell}-1|}\,d\theta, & \beta>0 \end{cases}$$

$$\leqslant \frac{1}{R}\int_{\theta=\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)}^{\frac{\pi}{2}} \frac{e^{-xR\sin\theta}}{1-e^{-\ell R\sin\theta}}\,d\theta,$$

where we have made the change of variable $\theta \mapsto \pi-\theta$ in the case $\beta > 0$. Thus, noting in addition that $R\sin\theta \geqslant \frac{2\pi}{3\ell}$ and so $1-e^{-\ell R\sin\theta} \geqslant 1-e^{-2\pi/3}$, we find

$$|I_R| \leqslant \frac{1}{R}(1-e^{-\frac{2\pi}{3}})^{-1}\int_{\theta=\theta_0}^{\frac{\pi}{2}} e^{-xR\sin\theta}\,d\theta,$$

and using inequality (2-15) we conclude that $\lim_{R\to\infty} I_R = 0,\ x, y \in (0,\ell)$.

Estimation of J_R. This term is trickier only because it is harder to obtain a positive lower bound for $|e^{ik\ell}-1|$. First, like for I_R, using the triangle inequality and the bound (2-22) we find

$$|J_R| \leqslant \int_{k\in C_{R,\theta_0}(0),\operatorname{Im}(k)\leqslant\frac{2\pi}{3\ell}} \left|e^{ikx}\frac{1}{k^2}\right|\frac{1}{|e^{ik\ell}-1|}|dk|. \tag{2-24}$$

Next, we use the following result.

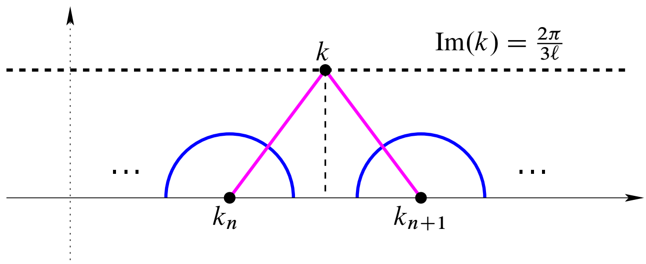

Figure 2. The upper bound for the radius ρ used in the proof of Lemma 2.4.

Lemma 2.4. *For any*

$$k \in \left\{0 \leqslant \operatorname{Im}(k) \leqslant \tfrac{2\pi}{3\ell}\right\} \setminus \bigcup_{n\in\mathbb{Z}} D_{\frac{2\pi}{3\ell}}(k_n),$$

we have $|e^{ik\ell}-1| \geqslant 1-e^{-2\pi/3}$.

Before proving this lemma, we use it to complete the estimation of J_R. From (2-24), we have

$$|J_R| \leqslant \frac{1}{R}(1-e^{-\frac{2\pi}{3}})^{-1} \int_{\theta=\theta_0}^{\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)} e^{-xR\sin\theta}\, d\theta.$$

Hence, employing once again inequality (2-15), we obtain

$$|J_R| \leqslant (1-e^{-\frac{2\pi}{3}})^{-1} \frac{\pi}{2xR^2}\left(e^{-\frac{2xR}{\pi}\theta_0} - e^{-\frac{2xR}{\pi}\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)}\right) \xrightarrow{R\to\infty} 0, \quad x, y \in (0,\ell).$$

Altogether, combining the above with (2-18) and (2-19) we conclude that

$$\int_{k\in\mathcal{C}^+} e^{ikx} \frac{1}{k^2} \frac{e^{-iky}}{1-e^{-ik\ell}}\, dk = 0, \quad x, y \in (0,\ell),$$

and so, in view of (2-14), (2-16) and (2-17),

$$\int_{k\in\mathcal{C}^+} e^{ikx-i\omega t} \frac{e^{i\omega t}}{1-e^{-ik\ell}} \hat{u}(k,t)\, dk = 0, \quad x, y \in (0,\ell),$$

as desired. It remains to establish Lemma 2.4.

Proof of Lemma 2.4. We minimize $|e^{ik\ell}-1|$ as a function of two variables simply by using calculus techniques. For any

$$k \in \left\{0 \leqslant \operatorname{Im}(k) \leqslant \tfrac{2\pi}{3\ell}\right\} \setminus \bigcup_{n\in\mathbb{Z}} D_{\frac{2\pi}{3\ell}}(k_n),$$

there exists $n \in \mathbb{Z}$ such that $k = k_n + \rho e^{i\phi}$, with $\frac{2\pi}{3\ell} \leqslant \rho \leqslant \frac{\sqrt{13}\pi}{3\ell}$ and $0 \leqslant \phi \leqslant \pi$ (see Figure 2 for the bounds on ρ). Then,

$$|e^{ik\ell}-1|^2 = e^{-2\rho\ell\sin\phi} - 2e^{-\rho\ell\sin\phi}\cos(\rho\ell\cos\phi) + 1 =: f(\rho,\phi).$$

First, we look for critical points of $f(\rho,\phi)$ inside $(\rho,\phi)\in\left[\frac{2\pi}{3\ell},\frac{\sqrt{13}\pi}{3\ell}\right]\times[0,\pi]$ by solving the system $f_\rho(\rho,\phi)=0$ and $f_\phi(\rho,\phi)=0$. We note that if $\cos\phi=0$ then $\sin\phi=1$ and $f_\rho=2e^{-\rho\ell}(1-e^{-\rho\ell})\neq 0$, while if $\sin\phi=0$ then $\cos\phi=\pm1$ and $f_\phi=\mp 2\rho\ell[1-\cos(\rho\ell)]\neq 0$. Hence, no critical points arise inside our domain when $\cos\phi\sin\phi=0$. Therefore, assuming $\cos\phi\sin\phi\neq 0$, we multiply the equation $f_\rho(\rho,\phi)=0$ by $\cos\phi$ and the equation $f_\phi(\rho,\phi)=0$ by $\sin\phi/(\rho\ell)$ and then subtract the resulting equations to obtain

$$\sin(\rho\ell\cos\phi)=0 \quad\Longleftrightarrow\quad \rho\ell\cos\phi=\kappa\pi,\ \ \kappa\in\mathbb{Z}.$$

Since $\frac{2\pi}{3\ell}\leqslant\rho\leqslant\frac{\sqrt{13}\pi}{3\ell}$, it follows that $\kappa=0,\pm1$. However, none of these values corresponds to critical points inside our domain since either $f_\rho\neq 0$ (when $\kappa=0$) or $f_\phi\neq 0$ (when $\kappa=\pm1$).

Since f is continuous and there are no critical points inside $\left[\frac{2\pi}{3\ell},\frac{\sqrt{13}\pi}{3\ell}\right]\times[0,\pi]$, the minimum will be attained at the boundary of the domain. If $\rho=\frac{2\pi}{3\ell}$, then

$$f\left(\tfrac{2\pi}{3\ell},\phi\right)=e^{-\frac{4\pi}{3}\sin\phi}-2e^{-\frac{2\pi}{3}\sin\phi}\cos\left(\tfrac{2\pi}{3}\cos\phi\right)+1=:g(\phi),\quad \phi\in[0,\pi].$$

We compute

$$g'(\phi)=-\tfrac{4\pi}{3}e^{-\frac{2\pi}{3}\sin\phi}\left[\cos\phi e^{-\frac{2\pi}{3}\sin\phi}-\cos\left(\phi+\tfrac{2\pi}{3}\cos\phi\right)\right]$$

and observe that $g'\left(\frac{\pi}{2}\right)=0$. To show that $\phi=\frac{\pi}{2}$ is the unique zero of g', we consider the intervals $\left[0,\frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2},\pi\right]$ separately.

If $\phi\in\left[0,\frac{\pi}{2}\right]$, then the function $h(\phi):=\phi+\frac{2\pi}{3}\cos\phi$ has derivative $h'(\phi)=1-\frac{2\pi}{3}\sin\phi$. Thus, the only critical point occurs at $\phi=\sin^{-1}\left(\frac{3}{2\pi}\right)$ with corresponding value

$$\sin^{-1}\left(\tfrac{3}{2\pi}\right)+\tfrac{2\pi}{3}\sqrt{1-\tfrac{9}{4\pi^2}}\simeq 2.34.$$

Moreover, $h(0)=\frac{2\pi}{3}$and $h\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$. Thus, $\frac{\pi}{2}\leqslant h(\phi)<\frac{3\pi}{2}$ and so

$$\cos\left(\phi+\tfrac{2\pi}{3}\cos\phi\right)\leqslant 0$$

for $\phi\in\left[0,\frac{\pi}{2}\right]$. In turn,

$$\cos\phi e^{-\frac{2\pi}{3}\sin\phi}-\cos\left(\phi+\tfrac{2\pi}{3}\cos\phi\right)\geqslant e^{-\frac{2\pi}{3}}\left[\cos\phi-\cos\left(\phi+\tfrac{2\pi}{3}\cos\phi\right)\right]\geqslant 0.$$

Moreover, for this nonnegative lower bound to vanish we must have

$$\cos\phi=3\kappa\quad\text{or}\quad\cos\phi+\tfrac{3}{\pi}\phi+3\kappa=0,\quad \kappa\in\mathbb{Z}.$$

The first of these equations has unique solution $\phi=\frac{\pi}{2}$. The second equation has no solution since $\cos\phi+\frac{3}{\pi}\phi$ has global maximum equal to

$$\sqrt{1-\tfrac{9}{\pi^2}}+\tfrac{3}{\pi}\sin^{-1}\left(\tfrac{3}{\pi}\right)\simeq 1.51$$

(via its unique critical point on $\left[0, \frac{\pi}{2}\right]$ at $\phi = \sin^{-1}\left(\frac{3}{\pi}\right)$) and global minimum equal to 1 (via the end point $\phi = 0$). Thus, we conclude that for $\phi \in \left[0, \frac{\pi}{2}\right]$ the only zero of g' is at $\phi = \frac{\pi}{2}$.

Similarly, if $\phi \in \left[\frac{\pi}{2}, \pi\right]$ then $\cos\left(\phi + \frac{2\pi}{3}\cos\phi\right) \geqslant 0$ and so

$$\cos\phi e^{-\frac{2\pi}{3}\sin\phi} - \cos\left(\phi + \tfrac{2\pi}{3}\cos\phi\right) \leqslant e^{-\frac{2\pi}{3}}\left[\cos\phi - \cos\left(\phi + \tfrac{2\pi}{3}\cos\phi\right)\right] \leqslant 0.$$

As before, we can show that the nonpositive upper bound vanishes only at $\phi = \frac{\pi}{2}$ and is otherwise negative. Thus, for $\phi \in \left[\frac{\pi}{2}, \pi\right]$ the only zero of g' is at $\phi = \frac{\pi}{2}$. Overall, since the only critical point of g on $[0, \pi]$ is at $\phi = \frac{\pi}{2}$, comparing the values $g\left(\frac{\pi}{2}\right) = (1 - e^{-2\pi/3})^2$ and $g(0) = g(\pi) = 3$, we conclude that $g(\phi) = f\left(\frac{2\pi}{3\ell}, \phi\right)$ is minimized at $\phi = \frac{\pi}{2}$ with corresponding value

$$f\left(\tfrac{2\pi}{3\ell}, \tfrac{\pi}{2}\right) = (1 - e^{-\frac{2\pi}{3}})^2.$$

Along the same lines, if $\rho = \frac{\sqrt{13}\pi}{3\ell}$ then we find

$$f\left(\tfrac{\sqrt{13}\pi}{3\ell}, \phi\right) \geqslant f\left(\tfrac{\sqrt{13}\pi}{3\ell}, \tfrac{\pi}{2}\right) = (1 - e^{-\frac{\sqrt{13}\pi}{3}})^2.$$

Finally, if $\phi = 0$ or $\phi = \pi$, then recalling that $\rho\ell \in \left[\frac{2\pi}{3}, \frac{\sqrt{13}\pi}{3}\right]$ we have

$$f(\rho, 0) = f(\rho, \pi) = 2[1 - \cos(\rho\ell)] \geqslant 2\left[1 - \cos\left(\tfrac{2\pi}{3}\right)\right] = 3.$$

Therefore, the global minimum of f is equal to $(1 - e^{-2\pi/3})^2$ as desired, completing the proof of Lemma 2.4. □

3. The finite interval problem

We now consider the initial-boundary value problem (1-3) for the linear Lugiato–Lefever equation with general forcing. While the first part of our derivation is essentially the same as the one for the periodic problem, the elimination of the unknown terms from the integral representation now requires an additional idea, namely the use of a symmetry transformation for the quantity ω. In addition, we provide a justification of the deformation from $\mathbb{R}$ to the contours $\partial D^{\pm}$ and establish Proposition 3.1, which is crucial behind this deformation.

3A. ***The global relation and an integral representation for the solution.*** As in the periodic case, taking the Fourier transform (2-1) of (1-3a), we obtain (2-2). Then, using the boundary conditions (1-3c) and the notation $u_x(0, t) = g_1(t)$, $u_x(\ell, t) = h_1(t)$ for the (unknown) Neumann values, we have

$$\partial_t \hat{u}(k, t) + i\omega(k)\hat{u}(k, t) \\ = i\beta\{g_1(t) - e^{-ik\ell}h_1(t) + ik[g_0(t) - e^{-ik\ell}h_0(t)]\} + \hat{f}(k, t), \quad (3\text{-}1)$$

where ω is defined by (2-3). Hence, integrating with respect to t and using the initial condition (1-6b), as well as the notation (2-5), we obtain the *global relation*

$$e^{i\omega t}\hat{u}(k,t)=\hat{u}_0(k)+i\beta\big\{[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]-e^{-ik\ell}[\tilde{h}_1(\omega,t)+ik\tilde{h}_0(\omega,t)]\big\} + \int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau, \quad (3\text{-}2)$$

where $k\in\mathbb{C}$ and the time transform $\tilde{\phi}(\omega,t)$ of a function $\phi(t)$ is defined by (2-5).

Using the global relation (3-2) for $k\in\mathbb{R}$ together with the inverse Fourier transform (2-1), we obtain the *integral representation*

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk \\ &\quad + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau\,dk \\ &\quad + \frac{i\beta}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk \\ &\quad - \frac{i\beta}{2\pi}\int_{k\in\mathbb{R}} e^{ik(x-\ell)-i\omega t}[\tilde{h}_1(\omega,t)+ik\tilde{h}_0(\omega,t)]\,dk. \end{aligned} \quad (3\text{-}3)$$

The integral representation (3-3) is not an explicit solution formula as it involves the unknown Neumann boundary values $u_x(0,t)$, $u_x(\ell,t)$ through the transforms $\tilde{g}_1(\omega,t)$, $\tilde{h}_1(\omega,t)$. Fortunately, as in the periodic case, these unknowns can be eliminated. However, *unlike the periodic case*, where it sufficed to simply reemploy the global relation after deforming the paths of integration from $\mathbb{R}$ to the complex contours $\mathcal{L}^{\pm}$ (see Figure 1 and the deformed representation (2-8)), here we must additionally exploit a certain *symmetry* of ω.

3B. ***Elimination of the unknowns and an explicit solution formula.*** First, using analyticity in k and Cauchy's theorem from complex analysis, we claim that the integral representation (3-3) can be written in the form

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk \\ &\quad + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(k,\tau)\,d\tau\,dk \\ &\quad + \frac{i\beta}{2\pi}\int_{k\in\mathcal{L}+} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk \\ &\quad - \frac{i\beta}{2\pi}\int_{k\in\mathcal{L}-} e^{ik(x-\ell)-i\omega t}[\tilde{h}_1(\omega,t)+ik\tilde{h}_0(\omega,t)]\,dk, \end{aligned} \quad (3\text{-}4)$$

where the contours $\mathcal{L}^{\pm}$ can be chosen to be either $\partial D^{\pm}$ or $\mathcal{C}^{\pm}$ (see Figure 1). For $\mathcal{L}^{\pm}=\mathcal{C}^{\pm}$, the deformation leading to (3-4) can be established similarly to the periodic case (see Section 2B). Thus, below we justify (3-4) for $\mathcal{L}^\pm = \partial D^\pm$ and, more precisely, for $\mathcal{L}^+ = \partial D^+$ and the third integral in (3-3)-(3-4) since the fourth integral in (3-3)-(3-4) can be handled in the same way, namely we prove that

$$\int_{k\in\mathbb{R}} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk = \int_{k\in\partial D^+} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk. \quad (3\text{-}5)$$

The exponential $e^{ikx-i\omega(t-\tau)}$ decays to zero as $|k|\to\infty$ inside $\{\mathrm{Im}(k)>0\}\setminus\overline{D^+}$ and is bounded on the closure of that region; hence by Cauchy's theorem

$$\begin{aligned}\int_{k\in\mathbb{R}} & e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk \\ &= \lim_{n\to\infty}\int_{k=-\frac{n\pi}{\ell}-\frac{\pi}{2\ell}}^{\frac{n\pi}{\ell}+\frac{\pi}{2\ell}} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk \\ &= \lim_{n\to\infty}\left(\int_{k\in\widetilde{C}_n^+}+\int_{k=i\left(\frac{n\pi}{\ell}+\frac{\pi}{2\ell}\right)}^{i0}+\int_{k=0}^{\frac{n\pi}{\ell}+\frac{\pi}{2\ell}}\right) \\ &\qquad\times e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk, \quad \beta<0, \quad (3\text{-}6)\end{aligned}$$

and

$$\begin{aligned}\int_{k\in\mathbb{R}} & e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk \\ &= \lim_{n\to\infty}\int_{k=-\frac{n\pi}{\ell}-\frac{\pi}{2\ell}}^{\frac{n\pi}{\ell}+\frac{\pi}{2\ell}} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk \\ &= \lim_{n\to\infty}\left(\int_{k=-\frac{n\pi}{\ell}-\frac{\pi}{2\ell}}^{0}+\int_{k=i0}^{i\left(\frac{n\pi}{\ell}+\frac{\pi}{2\ell}\right)}+\int_{k\in\widetilde{C}_n^+}\right) \\ &\qquad\times e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk, \quad \beta>0, \quad (3\text{-}7)\end{aligned}$$

where $\widetilde{C}_n^+$ denotes the circular arc

$$\widetilde{C}_n^+ = \begin{cases}\left\{\rho_n e^{i\theta} : \frac{\pi}{2}\leqslant\theta\leqslant\pi\right\}, & \beta<0,\\ \left\{\rho_n e^{i\theta} : 0\leqslant\theta\leqslant\frac{\pi}{2}\right\}, & \beta>0,\end{cases} \qquad \rho_n := \frac{n\pi}{\ell}+\frac{\pi}{2\ell}, \quad (3\text{-}8)$$

and is depicted in Figure 3. Note that the first equality in (3-6) and (3-7) is justified by the assumption that the improper integral on the left-hand side exists as a double limit, i.e., $\int_{k\in\mathbb{R}} = \lim_{M,N\to\infty}\int_{k=-M}^{N}$, and hence as a sequential limit for any choice of sequences $M = M(n)$ and $N = N(n)$ that tend to ∞.

Proposition 3.1. *For $x>0$, $t>0$ and $\beta\in\mathbb{R}\setminus\{0\}$,*

$$\lim_{n\to\infty}\int_{k\in\widetilde{C}_n^+} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk = 0. \quad (3\text{-}9)$$

Proof. The argument relies on integrating by parts with respect to τ inside the integrals that define the transforms $\tilde{g}_0, \tilde{g}_1$ and then using the well-known inequality

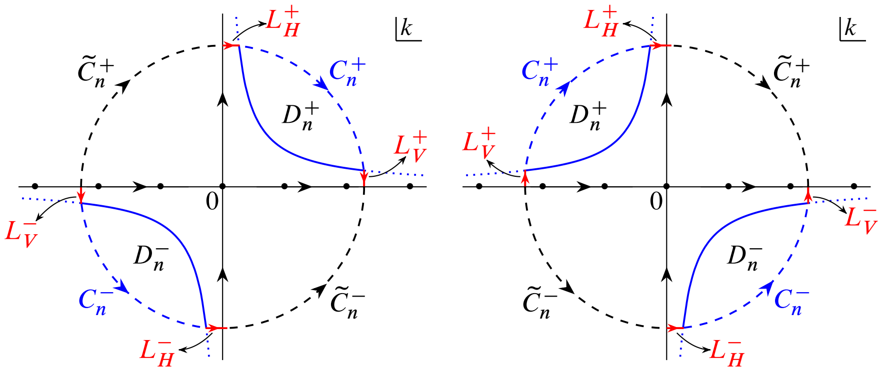

Figure 3. The hyperbola $2\beta\,\mathrm{Re}(k)\,\mathrm{Im}(k) = -1$, whose branches correspond to the contours $\partial D^\pm$, and the circular arcs $\widetilde{C}_n^\pm$, $C_n^\pm$, $L_H^\pm$, $L_V^\pm$ for $\beta < 0$ (left) and $\beta > 0$ (right). The dots along the real axis denote the zeros $k_n = \frac{n\pi}{\ell}$, $n \in \mathbb{Z}$, of $e^{ik\ell} - e^{-ik\ell}$.

(2-15). For this, we assume that g_0, g_1 are locally bounded functions with locally bounded derivatives. We take $\beta < 0$, as the case $\beta > 0$ is entirely analogous. Integrating by parts, we have

$$I_{\widetilde{C}_n^+}(x,t) := \int_{k\in\widetilde{C}_n^+} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}[g_1(\tau)+ikg_0(\tau)]\,d\tau\,dk$$
$$= \int_{k\in\widetilde{C}_n^+} e^{ikx} J(k,t)\,dk, \tag{3-10}$$

where

$$J(k,t) := \frac{1}{i\omega}\biggl(g_1(t)+ikg_0(t)-e^{-i\omega t}[g_1(0)+ikg_0(0)] - \int_{\tau=0}^{t} e^{-i\omega(t-\tau)}[g_1'(\tau)+ikg_0'(\tau)]\,d\tau\biggr).$$

For $k \in \widetilde{C}_n^+$ and n large enough, $|\omega| \geqslant |\beta|\,|k|^2 - \sqrt{1+\alpha^2} > 0$. Thus, parametrizing $k = \rho_n e^{i\theta}$ with $\frac{\pi}{2} \leqslant \theta \leqslant \pi$, which implies $\sin(2\theta) \leqslant 0$, we find

$$|J(\rho_n e^{i\theta},t)| \leqslant \frac{1}{|\beta|\rho_n^2-\sqrt{1+\alpha^2}}\biggl(|g_1(t)|+\rho_n|g_0(t)|+e^{-(1+\beta\rho_n^2\sin(2\theta))t}(|g_1(0)|+\rho_n|g_0(0)|) + \int_{\tau=0}^{t} e^{-(1+\beta\rho_n^2\sin(2\theta))(t-\tau)}(|g_1'(\tau)|+\rho_n|g_0'(\tau)|)\,d\tau\biggr)$$
$$\leqslant \frac{\rho_n}{|\beta|\rho_n^2-\sqrt{1+\alpha^2}}\biggl(|g_1(t)|+|g_0(t)|+|g_1(0)|+|g_0(0)| + \int_{\tau=0}^{t}(|g_1'(\tau)|+|g_0'(\tau)|)\,d\tau\biggr).$$

Thus, $J(\rho_n e^{i\theta}, t)$ decays uniformly in θ as $\rho_n \to \infty$ according to the bound

$$|J(\rho_n e^{i\theta}, t)| \leqslant c_{g_0,g_1,t} \frac{\rho_n}{|\beta|\rho_n^2 - \sqrt{1+\alpha^2}},$$

where

$$c_{g_0,g_1,t} = 2(\|g_0\|_{L^\infty(0,t)} + \|g_1\|_{L^\infty(0,t)}) + T(\|g_0'\|_{L^\infty(0,t)} + \|g_1'\|_{L^\infty(0,t)}) < \infty.$$

Back to (3-10), using this bound together with inequality (2-15) and the fact that $x > 0$, we obtain

$$\begin{aligned} |I_{\widetilde{C}_n^+}(x,t)| &\leqslant \frac{c_{g_0,g_1,t}\,\rho_n^2}{|\beta|\rho_n^2 - \sqrt{1+\alpha^2}} \int_{\theta=0}^{\frac{\pi}{2}} e^{-\frac{2\rho_n x}{\pi}\theta}\, d\theta \\ &= \frac{c_{g_0,g_1,t}\,\rho_n^2}{|\beta|\rho_n^2 - \sqrt{1+\alpha^2}} \cdot \frac{\pi}{2\rho_n x}(1 - e^{-\rho_n x}) \xrightarrow{n\to\infty} 0. \end{aligned}$$ □

In view of (3-9), equations (3-6) and (3-7) simplify to

$$\begin{aligned} &\int_{k\in\mathbb{R}} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t) + ik\tilde{g}_0(\omega,t)]\, dk \\ &\qquad = \lim_{n\to\infty}\left(\int_{k=i\rho_n}^{i0} + \int_{k=0}^{\rho_n}\right) e^{ikx-i\omega t}[\tilde{g}_1(\omega,t) + ik\tilde{g}_0(\omega,t)]\, dk, \\ &\int_{k\in\mathbb{R}} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t) + ik\tilde{g}_0(\omega,t)]\, dk \\ &\qquad = \lim_{n\to\infty}\left(\int_{k=-\rho_n}^{0} + \int_{k=i0}^{i\rho_n}\right) e^{ikx-i\omega t}[\tilde{g}_1(\omega,t) + ik\tilde{g}_0(\omega,t)]\, dk. \end{aligned}$$

Appealing to Cauchy's theorem once again, we can deform the union of the contours on the right-hand side to the union of the circular arcs

$$L_H^+ = \begin{cases} \{\rho_n e^{i\theta} : \frac{\pi}{2} - \theta_n \leqslant \theta \leqslant \frac{\pi}{2}\}, & \beta < 0, \\ \{\rho_n e^{i\theta} : \frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} + \theta_n\}, & \beta > 0, \end{cases}$$

$$L_V^+ = \begin{cases} \{\rho_n e^{i\theta} : 0 \leqslant \theta \leqslant \theta_n\}, & \beta < 0, \\ \{\rho_n e^{i\theta} : \pi - \theta_n \leqslant \theta \leqslant \pi\}, & \beta > 0, \end{cases}$$

where

$$\theta_n := \frac{1}{2}\sin^{-1}\left(-\frac{1}{\beta\rho_n^2}\right),$$

with the contour ∂D_n^+, which denotes the portion of ∂D^+ delimited by L_H^+ and L_V^+ (see Figure 3). Combining this last deformation with the fact that, similarly to (3-9),

$$\begin{aligned} &\lim_{n\to\infty} \int_{k\in L_H^+} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t) + ik\tilde{g}_0(\omega,t)]\, dk \\ &\qquad = 0 = \lim_{n\to\infty} \int_{k\in L_V^+} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t) + ik\tilde{g}_0(\omega,t)]\, dk, \end{aligned}$$

we obtain

$$\int_{k\in\mathbb{R}} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk = \lim_{n\to\infty}\int_{k\in\partial D_n^+} e^{ikx-i\omega t}[\tilde{g}_1(\omega,t)+ik\tilde{g}_0(\omega,t)]\,dk,$$

which, after defining $\int_{k\in\partial D^+} := \lim_{n\to\infty}\int_{k\in\partial D_n^+}$, amounts to the desired equality (3-5).

Having established (3-4), we proceed to the elimination of the transforms $\tilde{g}_1$, $\tilde{h}_1$ of the unknown Neumann boundary values. For this step, we must use an idea which was *not* needed in the periodic case, namely we must exploit the symmetries of ω. In particular, solving the equation $\omega(\nu)=\omega(k)$, we find that the only nontrivial symmetry of ω is $k\mapsto -k$. In turn, since the unknown transforms $\tilde{g}_1(\omega,t)$, $\tilde{h}_1(\omega,t)$ depend on k only through ω, they are invariant under the transformation $k\mapsto -k$, and the global relation (3-2) yields the additional identity

$$e^{i\omega t}\hat{u}(-k,t)=\hat{u}_0(-k)+i\beta\big\{[\tilde{g}_1(\omega,t)-ik\tilde{g}_0(\omega,t)]-e^{ik\ell}[\tilde{h}_1(\omega,t)-ik\tilde{h}_0(\omega,t)]\big\} + \int_{\tau=0}^{t} e^{i\omega\tau}\hat{f}(-k,\tau)\,d\tau, \quad k\in\mathbb{C}. \tag{3-11}$$

Equations (3-2) and (3-11) form a 2×2 system for the unknowns $\tilde{g}_1(\omega,t)$ and $\tilde{h}_1(\omega,t)$. Solving this system for these two quantities and then substituting the resulting expressions into (3-4), we obtain the *explicit solution formula* (1-5), where we have made use of the following analogue of Proposition 2.1.

Proposition 3.2. *For any* $0<x<\ell$ *and any* $t>0$,

$$\int_{k\in\mathcal{L}^+} \frac{e^{ikx}}{e^{ik\ell}-e^{-ik\ell}}[e^{ik\ell}\hat{u}(k,t)-e^{-ik\ell}\hat{u}(-k,t)]\,dk = \int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)}}{e^{ik\ell}-e^{-ik\ell}}[\hat{u}(k,t)-\hat{u}(-k,t)]\,dk=0.$$

For $\mathcal{L}^\pm=\mathcal{C}^\pm$, Proposition 3.2 follows like Proposition 2.1, while for $\mathcal{L}^+=\partial D^+$ its proof is analogous to Proposition 3.1 with $\widetilde{C}_n^+$ replaced by the arc C_n^+ shown in Figure 3 (and with the obvious adjustments in the case of ∂D^-), provided that the following crucial analogue of Lemma 2.4 is established (in order to have an appropriate lower bound for $e^{ik\ell}-e^{-ik\ell}$ along C_n^+).

Lemma 3.3. *For* $k\in C_n^+$ *with* $n\in\mathbb{N}$ *sufficiently large, we have*

$$|e^{ik\ell}-e^{-ik\ell}|\geqslant M_\ell := \min\big\{\sqrt{1+e^{-2\ell}},\, e^{\ell}-e^{-\ell}\big\}>0. \tag{3-12}$$

Therefore, we conclude with some final remarks on the solution formula (1-5) and a proof of Lemma 3.3.

Remark 3.4. The zeros of $e^{ik\ell}-e^{-ik\ell}$ do not introduce any singularities in formula (1-5) since they occur at $k=k_n=\frac{n\pi}{\ell}$, $n\in\mathbb{Z}$, and hence are avoided by the contours $\mathcal{L}^{\pm}$ (see Figure 1).

In the special case $f(x,t)=F$ that corresponds to the linearization of the Lugiato–Lefever equation (1-1), formula (1-5) reduces to

$$\begin{aligned}
u(x,t)=\frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk-\frac{F}{2\pi}\int_{k\in\mathbb{R}} e^{ikx}\frac{(1-e^{-ik\ell})(1-e^{-i\omega t})}{k\omega}\,dk \\
-\frac{1}{2\pi}\int_{k\in\mathcal{L}^+}\frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}[e^{ik\ell}\hat{u}_0(k)-e^{-ik\ell}\hat{u}_0(-k)]\,dk \\
-\frac{\beta}{\pi}\int_{k\in\mathcal{L}^+}\frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}k[\tilde{h}_0(\omega,t)-e^{-ik\ell}\tilde{g}_0(\omega,t)]\,dk \\
+\frac{F}{2\pi}\int_{k\in\mathcal{L}^+}\frac{e^{ikx}(1-e^{-ik\ell})(1-e^{-i\omega t})}{(1+e^{-ik\ell})k\omega}\,dk \\
+\frac{1}{2\pi}\int_{k\in\mathcal{L}^-}\frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}[\hat{u}_0(k)-\hat{u}_0(-k)]\,dk \\
+\frac{\beta}{\pi}\int_{k\in\mathcal{L}^-}\frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}k[e^{ik\ell}\tilde{h}_0(\omega,t)-\tilde{g}_0(\omega,t)]\,dk \\
+\frac{F}{2\pi}\int_{k\in\mathcal{L}^-}\frac{e^{ik(x-\ell)}(1-e^{-ik\ell})(1-e^{-i\omega t})}{(1+e^{-ik\ell})k\omega}\,dk.
\end{aligned}\tag{3-13}$$

This formula can be further simplified by evaluating the integrals multiplied by F via Cauchy's residue theorem. In particular, we note that $k=0$, as well as the zeros of ω in the integrals multiplied by F in (3-13), correspond to removable singularities due to the presence of $1-e^{-ik\ell}$ and $1-e^{-i\omega t}$ respectively. That is, recalling that in the nonperiodic case $k_n=\frac{n\pi}{\ell}$, $n\in\mathbb{Z}$, we see that the only singularities in these integrals arise at $k=k_{2n+1}$, $n\in\mathbb{Z}$, due to the term $1+e^{-ik\ell}$. Hence, introducing the notation $\widetilde{\mathbb{R}}_\varepsilon=\bigcup_{n\in\mathbb{Z}}[k_{2n+1}+\varepsilon,k_{2n+3}-\varepsilon]$, and using Cauchy's residue theorem, we find

$$\begin{aligned}
&\int_{k\in\mathcal{L}^-}\frac{e^{ik(x-\ell)}(1-e^{-ik\ell})(1-e^{-i\omega t})}{(1+e^{-ik\ell})k\omega}\,dk \\
&=\lim_{\varepsilon\to0}\int_{k\in\widetilde{\mathbb{R}}_\varepsilon}\frac{e^{ik(x-\ell)}(1-e^{-ik\ell})(1-e^{-i\omega t})}{(1+e^{-ik\ell})k\omega}\,dk-\frac{2\pi}{\ell}\sum_{n\in\mathbb{Z}}\frac{e^{ik_{2n+1}x}(1-e^{-i\omega(k_{2n+1})t})}{k_{2n+1}\omega(k_{2n+1})}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{k\in\mathcal{L}^+}\frac{e^{ikx}(1-e^{-ik\ell})(1-e^{-i\omega t})}{(1+e^{-ik\ell})k\omega}\,dk \\
&=\lim_{\varepsilon\to0}\int_{k\in\widetilde{\mathbb{R}}_\varepsilon}\frac{e^{ikx}(1-e^{-ik\ell})(1-e^{-i\omega t})}{(1+e^{-ik\ell})k\omega}\,dk-\frac{2\pi}{\ell}\sum_{n\in\mathbb{Z}}\frac{e^{ik_{2n+1}x}(1-e^{-i\omega(k_{2n+1})t})}{k_{2n+1}\omega(k_{2n+1})}.
\end{aligned}$$

Therefore, adding these two expressions and noting that the resulting integral along $\widetilde{\mathbb{R}}_\varepsilon$ no longer contains singularities and hence can be replaced to one along $\mathbb{R}$, we obtain a simplified, final form of formula (3-13) as

$$\begin{aligned}
u(x,t) = {} & \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\hat{u}_0(k)\,dk \\
& - \frac{1}{2\pi}\int_{k\in\mathcal{L}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}[e^{ik\ell}\hat{u}_0(k) - e^{-ik\ell}\hat{u}_0(-k)]\,dk \\
& + \frac{1}{2\pi}\int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}[\hat{u}_0(k) - \hat{u}_0(-k)]\,dk \\
& - \frac{2F}{\ell}\sum_{n\in\mathbb{Z}} \frac{e^{ik_{2n+1}x}(1-e^{-i\omega(k_{2n+1})t})}{k_{2n+1}\omega(k_{2n+1})} \\
& - \frac{\beta}{\pi}\int_{k\in\mathcal{L}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}} k[\tilde{h}_0(\omega,t) - e^{-ik\ell}\tilde{g}_0(\omega,t)]\,dk \\
& + \frac{\beta}{\pi}\int_{k\in\mathcal{L}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}} k[e^{ik\ell}\tilde{h}_0(\omega,t) - \tilde{g}_0(\omega,t)]\,dk.
\end{aligned} \tag{3-14}$$

Proof of Lemma 3.3. If $\mathrm{Im}(k) \geqslant 1$, then by the reverse triangle inequality we get $|e^{ik\ell} - e^{-ik\ell}| \geqslant e^{\mathrm{Im}(k)\ell} - e^{-\mathrm{Im}(k)\ell} \geqslant e^{\ell} - e^{-\ell}$.

If $\mathrm{Im}(k) < 1$, then instead of using the triangle inequality (which will yield a vanishing lower bound as $n \to \infty$), we proceed as follows. Reparametrizing C_n^+ by letting (see also Figure 4) $k = k_n + re^{i\phi}$, we have

$$r = \sqrt{\rho_n^2 - 2\rho_n k_n \cos\theta + k_n^2}, \quad \cot\phi = \frac{\rho_n\cos\theta - k_n}{\rho_n\sin\theta}, \quad \sin\phi = \frac{\rho_n\sin\theta}{r}.$$

Note that $r_n \leqslant r \leqslant R_n$, with

$$\begin{aligned}
r_n &= |\rho_n e^{i\theta_n} - k_n| = \sqrt{\rho_n^2 + k_n^2 - k_n\rho_n\left(\sqrt{1 - \frac{1}{\beta^2\rho_n^4}} + 1\right)}, \\
R_n &= |\sqrt{\rho_n^2 - 1} + i - k_n| = \sqrt{\rho_n^2 + k_n^2 - 2k_n\sqrt{\rho_n^2 - 1}},
\end{aligned}$$

where we have used the fact that

$$\cos\theta_n = \frac{1}{2}\left(\sqrt{1 - \frac{1}{\beta^2\rho_n^4}} + 1\right).$$

Observe that, as $n \to \infty$, $r_n \to \frac{\pi\ell}{2}$ and $R_n \to \sqrt{1 + \left(\frac{\pi\ell}{2}\right)^2}$ as suggested by geometric intuition. Also, for $n \in \mathbb{N}$ large enough we have (see Figure 4)

$$\sqrt{\rho_n^2 - 1} - k_n \leqslant r\cos\phi < \frac{\pi}{2\ell}, \quad 0 < r\sin\phi \leqslant 1. \tag{3-15}$$

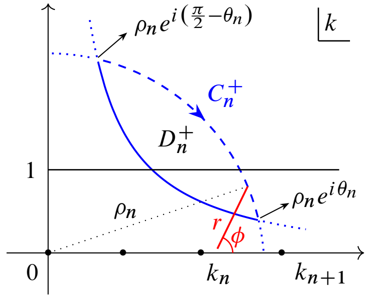

Figure 4. The local parametrization $k = k_n + re^{i\phi}$ used when $\text{Im}(k) \leqslant 1$.

Since $e^{2ik_n\ell} = 1$ and $k_n \in \mathbb{R}$, $|e^{ik\ell} - e^{-ik\ell}| = |e^{i\ell re^{i\phi}} - e^{-i\ell re^{i\phi}}|$ and so

$$|e^{ik\ell} - e^{-ik\ell}|^2 = e^{-2\ell r \sin\phi} - 2\cos(2\ell r\cos\phi) + e^{2\ell r\sin\phi}. \tag{3-16}$$

Note, however, that

$$\sqrt{\rho_n^2 - 1} - k_n = \frac{\pi}{2\ell} - \frac{1}{2\left(\frac{n\pi}{\ell} + \frac{\pi}{2\ell}\right)} - O\left(\frac{1}{2\left(\frac{n\pi}{\ell} + \frac{\pi}{2\ell}\right)^3}\right) \nearrow \frac{\pi}{2\ell}, \quad n \to \infty,$$

which along with inequality (3-15) implies, for n large enough, $\cos(2\ell r\cos\phi) \leqslant 0$. In turn, using again (3-15), (3-16) yields

$$|e^{ik\ell} - e^{-ik\ell}|^2 \geqslant e^{-2\ell r\sin\phi} + e^{2\ell r\sin\phi} \geqslant 1 + e^{-2\ell}. \qquad \square$$

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