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Split-Spectrum based Distributed Estimator for a Continuous-Time Linear System on a Time-Varying Graph

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***Abstract*— A simply structured distributed estimator is described for estimating the state of a continuous-time, jointly observable, input free, multi-channel linear system whose sensed outputs are distributed across a fixed multi-agent network. The estimator is then extended to non-stationary networks whose neighbor graphs switch according to a switching signal with a dwell time, or switch arbitrarily under appropriate assumptions. The estimator is guaranteed to solve the problem, provided a network-widely shared gain is sufficiently large. The lower bound of the gain is derived. This is accomplished by appealing to the "split-spectrum" approach and exploiting several well-known properties of invariant subspace. The proposed estimators are inherently resilient to abrupt changes in the number of agents and communication links in the inter-agent communication graph upon which the algorithms depend, provided the network is redundantly strongly connected and redundantly jointly observable.**

I. INTRODUCTION

The problem of estimating the state of a linear system whose measured outputs are distributed across a network has been under active study for a long time. The so-called "split-spectrum" observer proposed in [1] appears to be the first provably correct distributed estimator applicable to the distributed estimations problem under reasonably non-restrictive assumptions. This particular observer and its generalizations [2], [3] rely on the fact that the unobservable space $\mathcal{S}$ of any given matrix pair $(C_{p\times n}, A_{n\times n})$ is A-invariant, thereby enabling one to split the spectrum of A into two disjoint subsets, one the spectrum of A restricted to $\mathcal{S}$ and the other the compliment of this subset in the spectrum of A. A completely different approach to the distributed estimator problem is articulated in [4]. This type of observer and its generalization [5] rely on correctness proofs that exploit concepts from classical decentralized control [6]. All of these observers [1]–[5] have exponentially stable error systems with controllable convergence rates; because of this, all are able to function correctly in the presence of measurement noise, although not necessarily optimally since noise is not explicitly taken into account. The split-spectrum observers described in [1]–[3] are simpler in structure and easier to construct than the observers described in [4], [5]. On the other hand the split-spectrum observers are applicable only to continuous-time systems and cannot be easily modified to handle discrete-time systems, whereas the observer described in [5] can be. Nonetheless, if one is willing to introduce switching, the idea of a split spectrum observer can in fact extend to the discrete-time case [7]. The addition of switching of course diminishes the main virtue of the split-spectrum observers which is simplicity.

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While all of the distributed observers just discussed are intended primarily for problems in which the associated communication networks are fixed and independent of time, all are in fact also applicable to problems with time-varying communication networks provided the time variations are sufficiently slow. This is a direct consequence of the fact that for each estimator, the associated error system is an exponentially stable linear system. Just how to construct a distributed observer to deal with faster changing networks is a much more challenging problem. One estimator which successfully addresses this challenge is the hybrid distributed observer described in [8], [9]. It has been shown that when operating synchronously, this particular observer provides exponentially convergent state estimates at a preassigned rate no matter how fast the communication graph changes, just so long as it is strongly connected for all time. By expanding on earlier work in [10], the papers [11] provide a procedure for constructing a centralized-designed but distributed observer for time-varying neighbor graphs. It requires the sharing of an index which records the age of the information across the network, and the agents are designed to act in a sequential manner to do state estimation. The resulting algorithm, which is tailored exclusively to discrete-time systems, requires a network-wide initialization step that is to sort the agents in a specific order. Thereby it can deal with state estimation under assumptions which are weaker than strong connectivity.

Another approach to the distributed estimation problem when time varying communication graphs are to be dealt with, is inspired by the split-spectrum observers described in [1]–[3]. In this case, for a given upper bound on how fast the communication graph changes, it is possible to construct several different types of distributed observers which delivers exponential convergence provided that in each case, the time varying communication graph is always strongly connected. The aim of this paper is to describe these observers.

The remainder of this paper is organized as follows. Section III proposes the estimator, and discusses the error dynamics based on the split-spectrum idea. The associated background analysis is provided in Section III-B when the neighbor graph $\mathbb{N}(t)$ is constant. Section III-C discusses the case when the neighbor graph is switching according to a switching signal. Section III-D provides analysis on the convergence of the estimator when the switching signal of the neighbor graph is arbitrary. Section IV provides simulation evidence to validate the estimators. Finally, Section V concludes the paper.

II. Problem Formulation

We are interested in a network of $m > 0$ {possibly mobile} autonomous agents labeled $1, 2, \ldots, m$ which are able to receive information from their "neighbors", where by a *neighbor* of agent i is meant any agent within agent i's reception range. We write $\mathcal{N}_i(t)$ for the labels of agent i's neighbors at time $t \in [0, \infty)$ and always take agent i to be a neighbor of itself. Neighbor relations at time t are characterized by a directed graph $\mathbb{N}(t)$ with m vertices and a set of arcs defined so that there is an arc in $\mathbb{N}(t)$ from vertex j to vertex i whenever agent j is a neighbor of agent i at time t. Since each agent i is always a neighbor of itself, $\mathbb{N}(t)$ has a self-arc at each of its vertices. Each agent i can sense a continuous-time signal $y_i \in \mathbb{R}^{s_i}$ where

$$y_i = C_i x, \quad i \in \mathbf{m} \stackrel{\Delta}{=} \{1, 2, \ldots, m\} \tag{1}$$

$$\dot{x} = Ax \tag{2}$$

and $x \in \mathbb{R}^n$. We assume throughout that $C_i \neq 0,\ i \in \mathbf{m}$, and that the system defined by (1) and (2) is *jointly observable*; i.e., with $C = \begin{bmatrix} C_1' & C_2' & \cdots & C_m' \end{bmatrix}'$, the matrix pair (C, A) is observable. Joint observability is equivalent to the requirement that $\bigcap_{i \in \mathbf{m}} \mathcal{V}_i = 0$where $\mathcal{V}_i$ is the *unobservable space* of (C_i, A); i.e. $\mathcal{V}_i = \ker \begin{bmatrix} C_i' & (C_i A)' & \cdots & (C_i A^{n-1})' \end{bmatrix}'$. As is well known, $\mathcal{V}_i$ is the largest A-invariant subspace contained in the kernel of C_i. Generalizing the results that follow to the case when (C, A) is only detectable is quite straightforward and can be accomplished using well-known ideas. However, the commonly made assumption that each pair $(C_i, A),\ i \in \mathbf{m}$, is observable, or even just detectable, is very restrictive, grossly simplifies the problem and is unnecessary. The assumption $C_i \neq 0$ is not necessary provided the more relaxed problem is properly formulated, but the assumption is made for the sake of simplicity.

The problem of interest is to construct a suitably defined family of linear estimators in such a way so that no matter what the estimators' initial states are, each agent obtains an asymptotically correct estimate x_i of x in the sense that the estimation error $x_i(t) - x(t)$ converges to zero as fast as $e^{-\lambda t}$ does, where λ is an arbitrarily chosen but fixed positive number.

III. The Estimator

The estimator to be considered consists of m private estimators of the form for $i \in \mathbf{m}$

$$\dot{x}_i = (A + K_i C_i) x_i - K_i y_i - g P_i \left(x_i - \frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} x_j \right) \tag{3}$$

where $m_i(t)$ is the number of labels in $\mathcal{N}_i(t)$, g is a suitably defined scalar positive gain, each K_i is a suitably defined matrix, and for each $i \in \mathbf{m}$, P_i is the orthogonal projection on the unobservable space of (C_i, A).

The definitions of K_i and g begin with the specification of a desired convergence rate bound $\lambda > 0$. To begin with, each matrix K_i is defined as follows. For each fixed $i \in \mathbf{m}$, write Q_i for any full rank matrix whose kernel is the unobservable space of (C_i, A) and let $\bar{C}_i$ and $\bar{A}_i$ be the unique solutions to $\bar{C}_i Q_i = C_i$ and $Q_i A = \bar{A}_i Q_i$ respectively. Then the matrix pair $(\bar{C}_i, \bar{A}_i)$ is observable. A matrix $\bar{K}_i$ can be chosen to ensure that $e^{(\bar{A}_i + \bar{K}_i \bar{C}_i)t}$ converges to zero at least as fast as $e^{-\lambda t}$ converges to zero. There are several well-documented ways to do this {e.g, spectrum assignment algorithms or Riccati equation solvers}, since each pair $(\bar{C}_i, \bar{A}_i)$ is observable. Having chosen such $\bar{K}_i$, K_i is then chosen to be $K_i = Q_i^{-1} \bar{K}_i$ where Q_i^{-1} is a right inverse for Q_i. The definition implies that

$$Q_i (A + K_i C_i) = (\bar{A}_i + \bar{K}_i \bar{C}_i) Q_i \tag{4}$$

and that $(A + K_i C_i) \mathcal{V}_i \subset \mathcal{V}_i$. The latter, in turn, implies that there is a unique matrix A_i which satisfies $(A + K_i C_i) V_i = V_i A_i$ where V_i is a basis matrix[1] for $\mathcal{V}_i$. Prior to explaining how to choose g, it will be helpful to explain what defining the K_i in this way accomplishes.

A. The Error Dynamics

First note from (3), that the state estimation error $e_i = x_i - x$ satisfies

$$\dot{e}_i = (A + K_i C_i) e_i - g P_i \Big(e_i - \frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} e_j \Big) \tag{5}$$

Consequently the overall error vector $e = \begin{bmatrix} e_1' & \ldots & e_m' \end{bmatrix}'$ satisfies

$$\dot{e} = \big(\bar{A} - g P (I_{mn} - \bar{S}(t)) \big) e \tag{6}$$

where $\bar{A} = \text{block diag } \{A + K_1 C_1, A + K_2 C_2, \ldots, A + K_m C_m\}$, $P = \text{block diag } \{P_1, P_2, \ldots, P_m\}$, $\bar{S}(t) = S(t) \otimes I_n$ with $S(t) = D_{\mathbb{N}(t)}^{-1} \mathcal{A}_{\mathbb{N}(t)}'$. Here I_k is the $k \times k$ identity matrix, $\mathcal{A}_{\mathbb{N}(t)}$ is the adjacency matrix of $\mathbb{N}(t)$ and $D_{\mathbb{N}(t)}$ is the diagonal matrix whose ith diagonal entry is the in-degree of $\mathbb{N}(t)$'s ith vertex. Note that $\mathbb{N}(t)$ is the graph[2] of $S'(t)$ and that the diagonal entries of $S'(t)$ are all positive because each agent is a neighbor of itself. The matrix $S(t)$ is evidently a stochastic matrix.

Note that the subspace $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_m$ is $\bar{A}$-invariant because $(A + K_i C_i) \mathcal{V}_i \subset \mathcal{V}_i,\ i \in \mathbf{m}$. Next, let $Q = \text{block diag } \{Q_1, Q_2, \ldots, Q_m\}$ and $V = \text{block diag } \{V_1, V_2, \ldots, V_m\}$ where, recall, V_i is a matrix whose columns form an orthonormal basis for $\mathcal{V}_i$. Then Q is a full rank matrix whose kernel is $\mathcal{V}$ and V is a basis matrix for

[1] For simplicity, we assume that the columns of V_i constitute an orthonormal basis for $\mathcal{V}_i$ in which case $P_i = V_i V_i'$.

[2] The *graph* of an $n \times n$ matrix M is that directed graph on n vertices possessing an arc from vertex i to vertex j if $m_{ij} \neq 0$ [12, p. 357].

$\mathcal{V}$ whose columns form an orthonormal set. It follows that $P = VV'$, that $QP = 0$, and that

$$Q\bar{A} = \bar{A}_V Q \tag{7}$$
$$\bar{A}V = V\tilde{A} \tag{8}$$

where

$$\bar{A}_V = \text{block diag}\ \{\bar{A}_1 + \bar{K}_1\bar{C}_1, \ldots, \bar{A}_m + \bar{K}_m\bar{C}_m\} \tag{9}$$

$$\tilde{A} = \text{block diag}\ \{A_1, A_2, \ldots, A_m\}$$

Let V' be any left inverse for V and let Q^{-1} be that right inverse for Q for which $V'Q^{-1} = 0$. Then

$$\bar{A} - gP(I_{mn} - \bar{S}(t)) = T\begin{bmatrix}\bar{A}_V & 0 \\ \hat{A}_V(t) & A_V(t)\end{bmatrix}T^{-1} \tag{10}$$

where $\hat{A}_V(t) = V^{-1}(\bar{A} - g(I_{mn} - \bar{S}(t)))Q^{-1}$ and $A_V(t) = \tilde{A} - gV'(I_{mn} - \bar{S}(t))V$. Here $T = \begin{bmatrix}Q^{-1} & V\end{bmatrix}$. It is easy to check that $T^{-1} = \begin{bmatrix}Q' & V\end{bmatrix}'$.

Recall that $\bar{K}_i$ have been already been chosen so that each matrix exponential $e^{(\bar{A}_i + \bar{K}_i\bar{C}_i)t}$ converges to zero at least as fast as $e^{-\lambda t}$. Because of this and the fact that $\hat{A}_V(t)$ is a bounded matrix, to ensure that for each fixed τ, the state transition matrix $\Phi(t,\tau)$ converges to zero as fast as $e^{-\lambda t}$, it is enough to choose g so that the state transition matrix of $A_V(t)$ converges to zero at least as fast as $e^{-\lambda t}$. The requisite condition on g is provided below for three different neighbor graph connectivity assumptions.

B. Constant Neighbor Graph

This subsection focuses on the case when the neighbor graph $\mathbb{N}(t)$ is a constant graph $\mathbb{N}$. According to (10), the spectrum of $\bar{A} - gP(I_{mn} - \bar{S})$ is equivalent to the union of the spectrum of $\bar{A}_V$ and A_V. Since the spectrum of $\bar{A}_i + \bar{K}_i\bar{C}_i,\ i \in \mathbf{m}$, is assignable with $\bar{K}_i$, to show for g sufficiently large that $\bar{A} - gP(I_{mn} - \bar{S})$ is a continuous-time stability matrix with a prescribed convergence rate as large as λ, it is enough to show that for g sufficiently large, the matrix $A_V = \tilde{A} - gV'(I_{mn} - \bar{S})V$ is a continuous-time stability matrix with a prescribed convergence rate as large as λ. This proves to be a simple consequence of the following proposition whose proof can be found in the proof of Proposition 1 in [3].

Proposition 1: $-V'(I_{mn} - \bar{S})V$ is a continuous-time stability matrix.

Next, we show that $e^{(\tilde{A} - gV'((I_m - S)\otimes I_n)V)t}$ can be made to converge to zero as fast as $e^{-\lambda t}$ does by choosing g sufficiently large. Let $\pi = \begin{bmatrix}\pi_1 & \pi_2 & \ldots & \pi_m\end{bmatrix}'$ be a positive vector such that $S'\pi = \pi$. Without loss of generality, assume π is normalized so that the sum of its entries equals 1. Let Π be the diagonal matrix whose diagonal entries are the entries of π. Then $\Pi\mathbf{1} = \pi$ where $\mathbf{1}$ is the m-vector of all 1s. Let $L = 2\Pi - \Pi S - S'\Pi$.

To proceed, set

$$H = \text{ block diag } \{\pi_1 I_{n_1}, \pi_2 I_{n_2}, \ldots, \pi_m I_{n_m}\} \tag{11}$$

where $n_i = \dim \mathcal{V}_i$ and note that $VH = (\Pi \otimes I_n)V$. Since $((S - I_m)'\otimes I_n)(\Pi\otimes I_n) = ((S - I_m)'\Pi)\otimes I_n$ it must be true that $(V'((S - I_m)\otimes I_n)V)'H = V'(((S - I_m)'\Pi)\otimes I_n)V$ and thus that

$$H(V'(I_{mn} - \bar{S})V) + (V'(I_{mn} - \bar{S})V)'H = V'(L\otimes I_n)V \tag{12}$$

We exploit (12). Note in particular that

$$\begin{aligned} & H(\lambda I + A_V) + (\lambda I + A_V)'H \\ = \ & H(\lambda I + \tilde{A}) + (\lambda I + \tilde{A})'H - gV'(L\otimes I_n)V \end{aligned}$$

Since $V'(L\otimes I_n)V$ is positive definite according to Proposition 1, by picking g sufficiently large, $H(\lambda I + \tilde{A}) + (\lambda I + \tilde{A})'H - gV'(L\otimes I_n)V$ will be negative definite implying that $\lambda I + A_V$ is a stability matrix and thus that $\tilde{A} - gV'(I_{mn} - \bar{S})V$ is a stability matrix for which $e^{(\tilde{A} - gV'(I_{mn} - \bar{S})V)t}$ converges to zero as fast as $e^{-\lambda t}$ does. In other words, any value of g will have the desired property provided

$$g \geq \frac{\lambda_{\max}\left(H(\lambda I + \tilde{A}) + (\lambda I + \tilde{A})'H\right)}{\lambda_{\min}\left(V'(L\otimes I_n)V\right)} \tag{13}$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are the largest eigenvalue and the smallest eigenvalue of a symmetric matrix respectively. We summarize:

Theorem 1: For any given positive number λ, if the neighbor graph $\mathbb{N}$ is fixed and strongly connected, and the system defined by (1) and (2) is jointly observable, then there are matrices $K_i,\ i \in \mathbf{m}$ such that for g sufficiently large, each estimation error $x_i(t) - x(t)$ of the distributed estimator defined by (3), converges to zero as $t \to \infty$ as fast as $e^{-\lambda t}$ converges to zero.

C. Switching Neighbor Graph with Dwell Time Constraint

In the sequel the problem is studied under the assumption that $\mathbb{N}(t)$ changes according to a switching signal with a fixed dwell time, or a variable but with fixed average dwell time. To characterize the assumed time dependence of $\mathbb{N}(t)$, let $\mathcal{G} = \{\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_{|\mathcal{G}|}\}$ denote the set of all directed, strongly connected graphs on m vertices which have self-arcs at all vertices; here $|\mathcal{G}|$ is the number of graphs in $\mathcal{G}$. In some situations, the switching signals always have consecutive discontinuities separated by a value which is no less than a fixed positive real number τ_D, which is the *dwell time* [13]. In certain situations, the switching signals may occasionally have consecutive discontinuities separated by less than τ_D, but for which the average interval between consecutive discontinuities is no less than τ_D. This leads to the concept of average dwell time. With τ_D and δ fixed and positive define $\mathcal{S}_{\text{avg}}$ as the set of all piecewise-constant switching signals $\sigma : [0,\infty) \to \{1, 2, \ldots, |\mathcal{G}|\}$ satisfying $\delta_\sigma(t_0, t) \leq \delta_0 + \frac{t - t_0}{\tau_D}$. Here $\delta_\sigma(t_0, t)$ denotes the number of discontinuities of σ in the open interval (t_0, t). The constant τ_D is called the average dwell-time and δ_0 the chatter bound [14]. By the set of all time-varying neighbor graphs with average dwell-time τ_D is meant the set $\{\mathbb{G}_\sigma : \sigma \in \mathcal{S}_{\text{avg}}\}$. Fig. 1 is used to better explain the concept of switching signals. Suppose τ_D is a fixed positive number.

For the switching signal σ_1, the consecutive discontinuities are separated by an interval larger than τ_D, so the switching signal σ_1 is consistent with a dwell time of at least τ_D. For the switching signal σ_2, the consecutive discontinuities are separated occasionally less than τ_D, but the average interval between consecutive discontinuities is no less than τ_D. So the switching signal σ_2 has an average dwell time of at least τ_D. Note that switching according to a dwell time is a special case of switching according to an average dwell time. In the following, it is assumed that $\mathbb{N} \in \{\mathbb{G}_\sigma : \sigma \in \mathcal{S}_{\text{avg}}\}$.

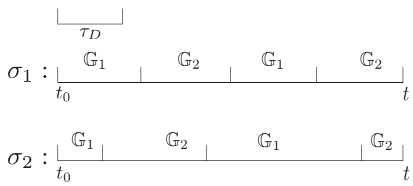

Fig. 1. Examples of switching signals

The problem to which this subsection is addressed is this. For fixed averaged dwell-time τ_D and the chatter bound δ_0, devise a procedure for crafting m local estimators, one for each agent, so that for each neighbor graph $\mathbb{N} \in \{\mathbb{G}_\sigma : \sigma \in \mathcal{S}_{\text{avg}}\}$, all m state estimation errors converge to zero exponentially fast at a prescribed rate.

The estimator to be considered is the same as the estimator described in (3), with the exception that g is chosen differently. According to the argument at the start of the section, it remains to be shown that with g sufficiently large, the state transition matrix of $A_V(t)$ converges to zero as fast as $e^{-\lambda t}$ does. To accomplish this, the following result will be used.

Lemma 1: Let $M_1, M_2, \ldots, M_{|\mathcal{G}|}$ be a set of $n \times n$ exponentially stable real matrices associated with a set $\mathcal{G} = \{\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_{|\mathcal{G}|}\}$ of directed strongly connected graphs with self-arcs at all vertices. Let σ denote the switching signal with avverage dwell time τ_D governing selection of a graph from $\mathcal{G}$. Then for any $n \times n$ real matrix N and positive number λ there is a positive number g^*, depending on τ_D for which, for each $\sigma \in \mathcal{S}_{\text{avg}}$ and $g \geq g^*$, all solutions to

$$\dot{x} = (N + gM_\sigma)x \tag{14}$$

converge to zero as fast as $\exp(-\lambda t)$ does.

Proof of Lemma 1:[3] By hypothesis, each M_i is exponentially stable. Thus there are positive constants $c_i > 1$ and λ_i such that

$$\|e^{M_i t}\| \leq c_i e^{-\lambda_i t} \tag{15}$$

for any $i \in \{1, 2, \ldots, |\mathcal{G}|\}$. Here $\|\cdot\|$ is any given submultiplic ative norm on $\mathbb{R}^{n \times n}$. Let $c = \max_{i \in \{1,2,\ldots,|\mathcal{G}|\}} c_i$, and $\lambda^* = \min_{i \in \{1,2,\ldots,|\mathcal{G}|\}} \lambda_i$.

Fix $\lambda > 0$ and let g be any gain satisfying

$$g \geq \frac{\tau_D(\lambda + \|N\|c) + \ln c}{\tau_D \lambda^*} \tag{16}$$

[3]The symbols used in this proof such as g, c and λ^* are generic and do not have the same meanings as the same symbols do when used elsewhere in this paper.

We claim that for any number τ, and any switching signal $\sigma \in \mathcal{S}_{\text{avg}}$, the transition matrix of gM_σ, namely $\Phi_\sigma(t, \tau)$, converges to zero as fast as $e^{-\alpha t}$ does where

$$\alpha = \lambda + \|N\|c \tag{17}$$

To understand why this is so, by (15),

$$\|\Phi_\sigma(t, \tau)\| \leq c^{\delta_\sigma(\tau,t)} e^{-g\lambda^*(t-\tau)} \tag{18}$$

where $\delta_\sigma(\tau, t)$ is the number of switching between (τ, t). By (16), $e^{g\lambda^*} \geq c^{\frac{1}{\tau_D}} e^{\alpha}$. From this and the fact that $\delta_\sigma(\tau, t) \leq \delta_0 + \frac{t-\tau}{\tau_D}$,

$$\|\Phi_\sigma(t, \tau)\| \leq c^{\delta_0} c^{\frac{t-\tau}{\tau_D}} e^{-g\lambda^*(t-\tau)} \leq c^{\delta_0 - \frac{\tau}{\tau_D}} e^{-\alpha(t-\tau)}.$$

Thus, the claim is true.

In view of (14) and the variation of constants formula,

$$x(t) = \Phi_\sigma(t, 0)x(0) + \int_0^t \Phi_\sigma(t, \mu)Nx(\mu)d\mu \tag{19}$$

As $\|\Phi_\sigma(t, \tau)\| \leq ce^{-\alpha t}$ for all τ and $\sigma \in \mathcal{S}_{\text{avg}}$,

$$\|x(t)\| \leq ce^{-\alpha t}\|x(0)\| + \int_0^t ce^{-\alpha(t-\mu)}\|N\|\|x(\mu)\|d\mu$$

By multiplying by $e^{\alpha t}$ on both sides, one obtains $e^{\alpha t}\|x(t)\| \leq c\|x(0)\| + \int_0^t \|N\|ce^{\alpha\mu}\|x(\mu)\|d\mu$. From this and the Bellman-Gronwall Lemma there follows

$$e^{\alpha t}\|x(t)\| \leq c\|x(0)\|e^{\int_0^t \|N\|cd\mu}$$

Since $\int_0^t \|N\|cd\mu = \|N\|ct$, it follows that $e^{\alpha t}\|x(t)\| \leq c\|x(0)\|e^{\|N\|ct}$ and thus that

$$\|x(t)\| \leq c\|x(0)\|e^{(\|N\|c-\alpha)t}$$

From this and (17) it follows that

$$\|x(t)\| \leq c\|x(0)\|e^{-\lambda t}$$

This completes the proof. ■

Recall that $A_V(t) = \tilde{A} - gV'(I_{mn} - \bar{S}(t))V$. By Proposition 1, for any fixed time τ, $-V'(I_{mn} - \bar{S}(\tau))V$ is exponentially stable if the graph of $S(\tau)'$ is strongly connected. Note $\tilde{A}$ is fixed and bounded. According to Lemma 1, for each $\sigma \in \mathcal{S}_{\text{avg}}$ there is a positive number g, depending on τ_D so that the transition matrix of $A_V(t)$ converges to zero at least as fast as $e^{-\lambda t}$ does. This is accomplished by choosing g sufficiently large. Based on the proof of Lemma 1, it is sufficient to pick g to satisfy

$$g \geq \frac{\ln c + (\lambda + \|\tilde{A}\|c)\tau_D}{\lambda^* \tau_D} \tag{20}$$

where c and λ^* are two positive numbers chosen so that for any fixed τ, $\|e^{-V'(I_{mn}-\bar{S}(\tau))Vt}\| \leq ce^{-\lambda^* t}$, and $c > 1$. We summarize:

Theorem 2: For any fixed positive numbers τ_D and λ, there exists a positive number g^* with the following property. For any value of $g \geq g^*$, any neighbor graph $\mathbb{N} \in \{\mathbb{G}_\sigma : \sigma \in \mathcal{S}_{\text{avg}}\}$, if the system defined by (1) and (2) is jointly observable, then there are matrices K_i, $i \in \mathbf{m}$ such that, each state estimation error $e_i = x_i - x,\ i \in \mathbf{m}$ of the distributed estimator defined by (3) converges to zero as $t \to \infty$ as fast as $e^{-\lambda t}$ does.

It is worth mentioning that when the neighbor graph is fixed, the dwell time τ_D or the average dwell time τ_D is infinity. Then the condition (20) in Theorem 2 degenerates to the condition (13) for exponential stability with the ensured convergence rate λ under a fixed topology.

D. Arbitrary Switching Neighbor Graph

In the sequel the problem is studied under arbitrary switching, but with a more restrictive graph assumption. The estimator to be considered is the same as (3), with the exception that g is chosen differently.

Theorem 3: For any fixed positive number λ, there exists a positive number g^* with the following property. For any value of $g \geq g^*$, any time-varying neighbor graph $\mathbb{N}(t)$, if the system defined by (1) and (2) is jointly observable, the neighbor graph $\mathbb{N}(t)$ is undirected and connected, and the stochastic matrix $S(t)$ of graph $\mathbb{N}(t)$ is doubly stochastic, then there are matrices $K_i,\ i \in \mathbf{m}$ such that each state estimation error $x_i(t) - x(t)$ of the distributed observer defined by (3), converges to zero as $t \to \infty$ as fast as $e^{-\lambda t}$ converges to zero.

Proof of Theorem 3: Recall $\Phi_V(t, \tau)$ is the transition matrix of $A_V(t)$ for any $t \geq \tau \geq 0$. If we can show that there exist a constant c so that

$$\|\Phi_V(t, \tau)\| \leq c e^{-\lambda(t-\tau)}, \quad \forall t \geq \tau \geq 0$$

the remaining proof is the same as the proof of Theorem 1 which is omitted here.

It is left to show that $\|\Phi_V(t, \tau)\| \leq c e^{-\lambda(t-\tau)}, \quad \forall t \geq \tau \geq 0$ by choosing g sufficiently large. We explore matrix $A_V(t)$. Recall that $A_V(t) = \tilde{A} - gV'((I_m - S(t)) \otimes I_n)V$. In particular,

$$(\lambda I + A_V(t)) + (\lambda I + A_V(t))' = (\lambda I + \tilde{A}) + (\lambda I + \tilde{A})' - gV'((2I_m - S(t) - S'(t)) \otimes I_n)V$$

Since each $S(t)$ is doubly stochastic, $2I_m - S(t) - S'(t)$ has row sum 0, all its off-diagonal entries are non-positive, and all its diagonal entries are positive. That is this matrix can be seen as a generalized Laplacian matrix of a connected graph. By Proposition 1, for any t, $-V'((2I_m - S(t) - S'(t)) \otimes I_n)V$ is negative definite. Thus by picking g sufficiently large, $(\lambda I + A_V(t)) + (\lambda I + A_V(t))'$ will be negative definite for any time t.

Consider system $\dot{\bar{z}} = A_V(t)\bar{z}$.

Let $V = \bar{z}'\bar{z}$. Then

$$\dot{V} = \bar{z}'(A_V(t)' + A_V(t))\bar{z} \leq -2\lambda \bar{z}'\bar{z}$$

Therefore, $\Phi_V(t, \tau)$ converges to zero as fast as $e^{-\lambda(t-\tau)}$ does, i.e.,

$$\|\Phi_V(t, \tau)\| \leq c e^{-\lambda(t-\tau)}, \quad \forall t \geq \tau \geq 0$$

This completes the proof. ■

E. Resilience

The concept of a passively resilient algorithm is proposed in [9]. By a passively resilient algorithm for a distributed process is meant an algorithm which, by exploiting built-in network and data redundancies, can continue to function correctly in the face of abrupt changes in the number of vertices and arcs in the inter-agent communication graph upon which the algorithm depends. All the proposed continuous-time distributed estimators are inherently resilient to these abrupt changes provided the network is redundantly strongly connected and redundantly jointly observable, with a careful gain picking before the algorithm starts. Details can be found in Section 5 of [9].

IV. Simulations

This section provides simulations to illustrate the state estimation performance. The neighbor graph will switch back and forth between Fig. 2 (a) and Fig. 2 (b), which can be seen as modeling a connection failure happening between agent 1 and agent 3 randomly.

Consider the three-channel, four-dimensional, continuous-time system described by the equations $\dot{x} = Ax,\ y_i = C_i x,\ i \in \{1,\ 2,\ 3\}$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

and C_i is the ith unit row vector in $\mathbb{R}^{1\times 4}$. Note that A is a matrix with eigenvalues at $\pm 1j$, and $\pm 1.4142j$. While the system is jointly observable, no single pair (C_i, A) is observable. The observer convergence rate is designed to be $\lambda = 1$. The first step is to design K_i as stated in Section III. This is to control the spectrum of the matrix $\bar{A}_V$ (the local observer dynamics) as defined in (10).

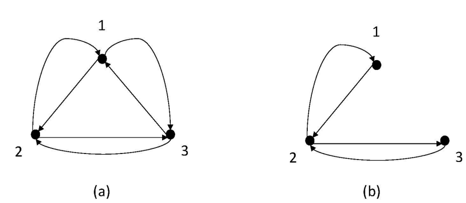

Fig. 2. The neighbor graph

For agent 1:

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ Q_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ V_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}'$$

$$K_1 = \begin{bmatrix} -5 & -5 & 0 & 0 \end{bmatrix}'$$

For agent 2:

$$A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ Q_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ V_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}'$$

$$K_2 = \begin{bmatrix} 5 & -5 & 0 & 0 \end{bmatrix}'$$

For agent 3:

$$A_3 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix},\ Q_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ V_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}'$$

$$K_3 = \begin{bmatrix} 0 & 0 & -5 & -4 \end{bmatrix}'$$

Two cases are considered. First, suppose the neighbor $\mathbb{N}(t)$ is fixed as shown in Figure 2(a). With $g = 10$ obtained using (13), the real part of the right most eigenvalue of A_V is less than -1. With randomly chosen initial state values, traces of this simulation are shown in Fig. 3 where x_i^1 and x^1 denote the first components of x_i and x respectively. .Moreover, the norm of the estimation error is plotted in Fig. 4 from which we can see that it is exponentially convergent with the approximate rate $\lambda = 1$.

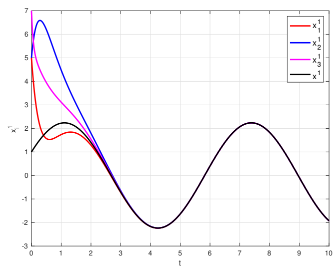

Fig. 3. Trajectory of the performance

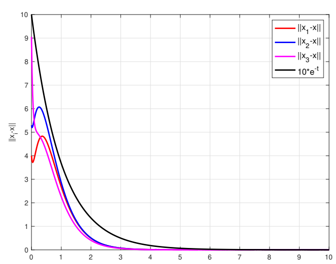

Fig. 4. The trajectory of the norm of the error

Second, suppose the neighbor $\mathbb{N}(t)$ is time-varying and switching back and forth between Figure 2(a) and Figure 2(b) according to the indicator function in Fig. 5. That is when the function value is 1, the neighbor graph is Figure 2(a), and when the function value is 0, the neighbor graph is Figure 2(b). It is arranged that the average dwell time is $\tau_D = 0.0369$ for this simulation. With random chosen initial state values and $g = 10$, the norm of the estimation error is shown in Fig. 6.

V. Conclusion

This paper studies the distributed estimation problem when the neighbor graph is time-varying but always strongly connected. It has been shown that for any switching signal with appropriate constraints, each agent can estimate the state exponentially fast with a pre-assigned convergence rate. Studying the distributed observer problem when the

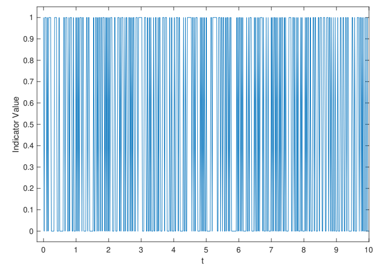

Fig. 5. The indicator function

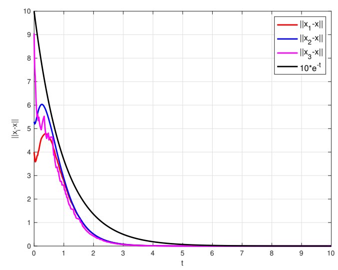

Fig. 6. The trajectory of the norm of the error

neighbor graph is not always strongly connected, but strongly connected over an interval would be future work.

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