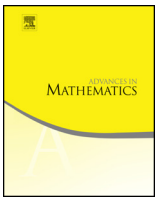
Advances in Mathematics 452 (2024) 109823

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim



Integration on the surreals

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ARTICLE INFO

Article history:
Received 1 September 2021
Received in revised form 10 June 2024
Accepted 19 June 2024
Available online 10 July 2024
Communicated by Slawomir Solecki

MSC:
primary 03H05, 12J15, 34E05, 03E15
secondary 03E25, 03E35, 03E75

Keywords:
Surreal numbers
Surreal integration
Divergent asymptotic series
Transseries

ABSTRACT

Conway's real closed field **No** of surreal numbers is a sweeping generalization of the real numbers and the ordinals to which a number of elementary functions such as log and exponentiation have been shown to extend. The problems of identifying significant classes of functions that can be so extended and of defining integration for them have proven to be formidable. In this paper we address this and related unresolved issues by showing that extensions to **No**, and thereby integrals, exist for most functions arising in practical applications. In particular, we show they exist for a large subclass of the *resurgent functions*, a subclass that contains the functions that at ∞ are semi-algebraic, semi-analytic, analytic, meromorphic, and Borel summable as well as solutions to nonresonant linear and nonlinear meromorphic systems of ODEs or of difference equations. By suitable changes of variables we deal with arbitrarily located singular points. We further establish a sufficient condition for the theory to carry over to ordered exponential subfields of **No** more generally and illustrate the result with structures familiar from the surreal literature. The extensions of functions and integrals that concern us are constructive in nature, which permits us to work in NBG less the Axiom of Choice (for both sets and proper classes). Following the completion of the positive portion of the paper, it is shown that the existence of such constructive extensions and integrals of substantially more general types of functions (e.g.

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https://doi.org/10.1016/j.aim.2024.109823

smooth functions) is obstructed by considerations from the foundations of mathematics.

Contents

1. Introduction

In his seminal work *On Numbers and Games* [18,19], J. H. Conway introduced the system **No** of *surreal numbers*, a strikingly inclusive real closed field containing the reals and the ordinals. In addition to its inclusive structure as an ordered field, **No** has a rich *simplicity hierarchical* or *s-hierarchical* structure, that depends upon its structure as a *lexicographically ordered full binary tree* and arises from the fact that the sums and products of any two members of the tree are the simplest possible elements of the tree consistent with **No**'s structure as an ordered group and an ordered field, respectively, it being understood that x is *simpler than* y (written $x <_s y$) just in case x is a predecessor of y in the tree [39,40,43].

An important subsequent advance in the theory of surreal numbers was the extension from the reals to **No** of the exponential function by Kruskal and Gonshor [19,49]. The Kruskal-Gonshor exponential function exp, like Conway's field operations on **No**, is inductively defined in terms of **No**'s simplicity hierarchical structure making use of the fact that for each pair of subsets L and R of **No** for which every member of L precedes every member of R, there is a *simplest* member of **No**, denoted

$$\{L \,|\, R\},$$

lying between the members of L and the members of R. Conway [19, pages 27, 225, 227] refers to such definitions as *genetic definitions*.[1]

There has been a longstanding program, initiated by Conway, Kruskal and Norton, to develop analysis on **No**, starting with a genetic definition of integration. In the case of Kruskal, it was motivated in large part by the broader goal of providing a new foundation for asymptotic analysis which would include new and more general tools for resumming divergent series and for solving complicated differential equations. However, the initial attempts at defining integration, in particular the genetic definition proposed by Norton [19, page 227], turned out, as Kruskal discovered, to have fundamental flaws [19, page 228]. Despite this disappointment, the search for a theory of surreal integration has continued (see [46] and [67]), but has heretofore remained largely open.[2] In this paper, using a new approach, we construct a theory of integration that is of sufficiently wide applicability for most practical cases, pose questions about possible extensions of the theory, and elucidate the nature of the obstructions to a far more general extension.

[1] At present there is no universally accepted formal theory of Conway's loosely defined conception of a genetic definition in the literature, though [46], [47], [67] and most recently [15] have made contributions toward the development of such a theory. Nevertheless, following Conway, in our informal remarks we freely refer to certain inductive definitions as "genetic".

[2] By contrast, work of Berarducci and Mantova [13,14], Aschenbrenner, van den Dries and van der Hoeven [5], Bagayoko [7], Bagayoko, van der Hoeven and Kaplan [10], Bagayoko and van der Hoeven [8,9] and others (e.g. [11,28,29,69]) has made significant progress toward viewing the surreals as an ordered differential field. This work aims to bring a robust theory of asymptotic differential algebra to all of **No**. Unlike the present work, which is concerned with derivations on surreal functions, the former work is concerned with derivations on surreal numbers.

In real analysis and mathematical physics, the asymptotic expansions at ∞ of solutions to nontrivial equations as well as perturbation expansions with respect to small parameters almost invariably have zero radius of convergence. One of the simplest examples of a divergent series is

$$\sum_{k=0}^{\infty} k! x^{-k-1},\ x \to \infty,$$

a formal solution of the differential equation $y' + y = x^{-1}$, whose general solution is related to the antiderivatives of e^x/x by $y(x) = e^{-x} \int^x e^s s^{-1} ds$. The problem of uniquely assigning functions to divergent expansions in a way that preserves such operations as addition, multiplication, differentiation, integration and composition is a very important and difficult one. A partial solution was provided by Borel summation; however, its domain of applicability is insufficient for many problems of interest in pure and applied analysis. Even for handling relatively common problems in analysis, a satisfactory solution had to wait until the work of Écalle (see [31,33]) which introduced (among other things) the notions of *resurgent functions, resurgent transseries* and *Écalle-Borel summation* for overcoming the limitations of Borel summation. In **No**, on the other hand, for all surreal $x > \infty$, $\sum_{k=0}^{\infty} k! x^{-k-1}$ (and in fact, any formal series in powers of $1/x$ with real coefficients more generally) is *absolutely convergent in the sense of Conway* (see §2.2)[3] and therewith by comparatively simple means defines a unique function for all infinite surreal x. Accordingly, the question naturally arises as to whether building on absolute convergence in the sense of Conway and the ideas of Écalle, we can find a theoretically satisfying way of extending functions and their integrals past ∞ or, more generally, past a singularity at which asymptotic expansions do not exist or are divergent? As we alluded to above, in this paper we provide a qualified affirmative answer to this question.

Making real progress towards solving the above-said integration problem, and more generally in interpreting divergent expansions by means of surreal analysis, requires finding a property-preserving operator (see Definition 11) that extends the members of a wide body of important classical functions from $\mathbb{R}$ to **No**. In turn, the existence of such an extension operator provides a theoretically satisfying and widely applicable definition of integration: in particular, the integral of an extension from $\mathbb{R}$ to **No** of a function on the reals can be defined to be the extension of its integral from $\mathbb{R}$ to **No**.

Any such theory would have to keep in mind that functions whose behavior can be described in terms of exponentials and logarithms are remarkably ubiquitous. Indeed, as G. H. Hardy noted in 1910:

[3] In the context of discussions of **No**, such references to "∞" refer to the gap in **No** separating the positive finite surreals and positive infinite surreals. Similar references to "$-\infty$" are understood analogously. In our discussion, "∞" and "$-\infty$" refer to both gaps and limits depending on context.

> No function has presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmic-exponential terms. [50, 1st Edition, page 35; 2nd Edition, page 32][4,5]

Accordingly, developing a satisfactory theory of integration on the surreals would require building on the exponential ordered field $(\mathbf{No}, \exp)$ of surreal numbers.

Against this backdrop, in the pages that follow, we show that an extension operator E as described above, and thereby extensions of integrals from $\mathbb{R}$ to $\mathbf{No}$, exist for a large subclass $\mathcal{F}_{\mathcal{R}}$ of resurgent functions, which is related via Écalle-Borel summation to a corresponding subclass $\mathbb{T}_{\mathcal{R}}$ of resurgent transseries, which contains all real functions that at ∞ are semi-algebraic, semi-analytic, analytic, and functions with divergent but Borel summable series (see §5), as well as solutions of nonresonant linear or nonlinear meromorphic systems of ODEs or of difference equations. As such, most classical special functions, such as Airy, Bessel, Ei, erf, Gamma, and Painlevé transcendents, are covered by our analysis.[6]

The definitions of the extension operator E and corresponding antidifferentiation and integral operators $\mathsf{A}_{\mathbf{No}}$ and

$$\int_x^y f := \mathsf{A}_{\mathbf{No}}(f)(y) - \mathsf{A}_{\mathbf{No}}(f)(x)$$

given below are *not* genetic in Conway's sense (see Footnote 1). However, unlike Norton's aforementioned definition of integration which was found to be intensional [19, page 228], ours are shown to depend solely on the values of the functions involved. $\mathsf{A}_{\mathbf{No}}$ is defined making use of an antidifferentiation operator A on $\mathbb{T}_{\mathcal{R}}$, which in turn is defined using an antidifferentiation operator $\mathsf{A}_{\mathbb{T}}$ on the exponential ordered field $\mathbb{T}$ of transseries

[4] Note that powers fall in this category since $x^a = e^{a \ln x}$.

[5] The work of Écalle on transseries, and resurgent transseries in particular, sheds important light on Hardy's observation. The system of transseries, which consists of formal series built up from $\mathbb{R}$ and a variable $x > \mathbb{R}$ using powers, exponentiation, logarithms and infinite sums, is the closure of formal power series under a wide range of algebraic and analytical operations [4,21,37]. The subspace of resurgent transseries consists of those transseries which, loosely speaking, have origins in natural problems in analysis (see §6 as well as [31,33]). There is compelling mathematical evidence, albeit thus far no rigorous proof, that resurgent transseries are also closed under the known algebraic and analytical operations. Moreover, they are associated with resurgent functions by means of Écalle-Borel summation. These facts provide a theoretical basis for Hardy's observation that, in practice, functions whose asymptotic behavior can be described in logarithmic-exponential terms are the only ones that arise naturally as solutions of problems in analysis. It should be noted, however, that unlike the asymptotic expansions used at the time of Hardy's cited writings, the infinite sums of logarithmic-exponential terms occurring in Écalle's theory include sums of *countable transfinite length* $> \omega$.

[6] Integration for functions with convergent expansions has been studied in the context of the non-Archimedean ordered field of left-finite power series with real coefficients and rational exponents in [70] and [71]. In addition, for the category of semi-algebraic sets and semi-algebraic functions on arbitrary real-closed fields a full Lebesgue measure and integration theory has been developed in [53] and [54]. See also [55] for integration and measure theory on certain non-Archimedean ordered fields whose value groups have finite Archimedean rank, as well as [16] for various positive and negative results on integration in general non-Archimedean fields.

(Proposition 21). A has the property: for real f the restriction to reals of $\mathsf{A}f$ has limit zero at ∞ whenever the limit exists. This can be viewed as the natural condition for an integral with an endpoint at ∞.

All of the members of $\mathcal{F}_{\mathcal{R}}$ are *resurgent at* ∞ (see Definition 81). Following our treatment of the just-said extension, antidifferentiation and integral operators based on $\mathcal{F}_{\mathcal{R}}$ or $\mathsf{E}(\mathcal{F}_{\mathcal{R}})$ we show by means of a simple change of variables argument that substantial extensions of those operators can be obtained building on a set of functions $\mathcal{F}^*_{\mathcal{R}}$ extending $\mathcal{F}_{\mathcal{R}}$ that contains functions that are resurgent at arbitrary points.

More generally, the original portions of the paper consist of the following. In §3 we introduce the definitions of *extension, antidifferentiation and integral operators* and prove a preliminary result about the existence of integral operators. In §4 we outline the difficulties of defining extensions and integration of functions, and our strategy for overcoming them. Following this, to prepare the way for the proof of the main antidifferentiation theorem, in §7 we establish the requisite results concerned with resurgent functions, resurgent transseries and Écalle-Borel summability. The definitions of the extension and antidifferentiation operators E and $\mathsf{A}_{\mathbf{No}}$, together with proofs of the main antidifferentiation theorem (Theorem 78) are given in §8, along with mention of the uniqueness of E and $\mathsf{A}_{\mathbf{No}}$, the proofs of which are left for a separate paper. This is followed in §9 by the above-mentioned constructions of extensions of E, $\mathsf{A}_{\mathbf{No}}$ and the corresponding integral operator, and in §10 by illustrations of the antidifferentiation and/or extension theorems for exp, the exponential integral Ei, the imaginary error function erfi, the Airy functions Ai and Bi, the log-gamma function, the Gamma function and Jacobi's elliptic function θ_3. In §8(2), a substantially shorter and simplified version of the proof of the main extension theorem is provided for the proper subclass $\mathcal{F}_{conv}$ of $\mathcal{F}_{\mathcal{R}}$ consisting of all functions that, at ∞, have convergent series in integer or fractional powers of $1/x$ or more generally have convergent transseries. By a result of van den Dries [26], these include the semi-analytic functions at ∞. In §11 we generalize our main results by showing that *closure* under absolute convergence in the sense of Conway is a sufficient condition for the theory of extension, antidifferentiation and integral operators outlined above to carry over to ordered exponential subfields of $(\mathbf{No}, \exp)$, and we illustrate the result with substructures of $(\mathbf{No}, \exp)$ that are familiar from the literature. Following this, in §12 we raise a problem and state two open questions that naturally arise from material in the preceding sections.

To help keep the paper self-contained we include three preparatory sections: §2 offers an overview of some basic ingredients of surreal theory; and §5 offers an overview of transseries as well as those aspects of Borel summability theory that provide background for the preliminary discussion of Écalle-Borel summability in §6, which in turn provides background for §7 and §8.

In writings on surreal numbers it is customary to work in NBG (von Neumann-Bernays-Gödel set theory with the Axiom of Global Choice (see, for example, [58]). However, in §3-§11, which constitutes the positive portion of the paper, we need only work in NBG$^-$ (NBG less the Axiom of Choice for both sets and proper classes), since

the extensions of functions and integrals that concern us there have an explicitly "constructive" nature.

Whereas Kruskal hoped to appeal to Conway's notion of absolute convergence to construct new foundations for asymptotic analysis grounded in a robust theory of surreal integration and function extensions more generally, our theory is more modest in its potential scope, limiting its attention to a broad subclass of resurgent functions that arises in most applied settings. In fact, there are reasons to believe that deep hurdles lay in the way of realizing the lofty analytic goals of Kruskal. Indeed, in Section §13 we reverse course and show that the existence of extensions and integrals for substantially more general classes of functions (e.g. the class of smooth functions) cannot be proved in NBG^-, and is in fact obstructed by considerations from the foundations of mathematics.[7]

2. Surreal numbers

This section provides an overview of the basic concepts of the theory of surreal numbers, including the normal forms of surreal numbers, the aforementioned notion of absolute convergence in the sense of Conway and exponentiation. With the exception of Propositions 6 and 7 and Notational Convention 1, which are concerned with absolute convergence in the sense of Conway (see Section 2.2), all of the material in this section is known from the literature.

To avoid possible confusion, we note that here and henceforth we follow the convention of *excluding* 0 from the set $\mathbb{N}$ of natural numbers.

2.1. The algebraico-tree-theoretic structure of **No**

There are a variety of constructions of the surreal numbers (e.g. [19, pages 4-5, 15-16, 65], [2,3,42,38], [39, page 242]), each with its own virtues. For the sake of brevity, here we adopt the construction based on Conway's *sign-expansions* [19, page 65], an approach which has been made popular by Gonshor [49]. In accordance with this approach, a *surreal number* is a function $x : \lambda \to \{-,+\}$ where λ is an ordinal called the *length* of x. The class **No** of surreal numbers so defined carries a canonical linear ordering $<$ as well as a canonical partial ordering $<_s$ defined by the conditions: $x < y$ if and only if x is (*lexicographically*) less than y with respect to the linear ordering on $\{-,+\}$, it being understood that $-<$ *undefined* $<+$; $x <_s y$ (read "x is *simpler than* y") if and only if x is a proper initial segment of y.

By a *tree* $(A, <_A)$ we mean a partially ordered class such that for each $x \in A$, the class $\mathrm{pr}_A = \{y \in A : y <_A x\}$ of *predecessors* of x is a *set* well ordered by $<_A$. The *tree-rank* of $x \in A$, written '$\rho_A(x)$', is the ordinal corresponding to the well-ordered set

[7] Some of the material in §3-§10 of the present paper is a revised and substantially expanded version of material from the positive portion of the arXiv preprint [23]. Further set-theoretic impediments to the realization of Kruskal's program are contained in the negative portion of [23] and remain to be revised and expanded by Harvey Friedman and the first author.

$(\mathrm{pr}_A(x), <_s)$. If $x, y \in A$, then y is said to be an *immediate successor* of x if $x <_s y$ and $\rho_A(y) = \rho_A(x) + 1$; and if $(x_\alpha)_{\alpha<\beta}$ is a chain in A (i.e., a subclass of A totally ordered by $<_s$), then y is said to be an *immediate successor of the chain* if $x_\alpha <_s y$ for all $\alpha < \beta$ and $\rho_A(y)$ is the least ordinal greater than the tree-ranks of the members of the chain. The *length* of a chain $(x_\alpha)_{\alpha<\beta}$ in A is the ordinal β. If each member of A has two immediate successors and every chain in A of limit length (including the empty chain) has one immediate successor, the tree is said to be a *full binary tree*.

Proposition 1. $(\mathbf{No}\ <, <_s)$ *is a lexicographically ordered full binary tree ([40], [42, Theorem 11]).*

Central to the algebraico-tree-theoretic development of the theory of surreal numbers is the following consequence of Proposition 1, where a subclass B of an ordered class $(A, <)$ is said to be *convex*, if $z \in B$ whenever $x, y \in B$ and $x < z < y$.

Proposition 2. *Every nonempty convex subclass of* **No** *has a simplest member. In particular, if L and R are (possibly empty) subsets of* **No** *for which every member of L precedes every member of R (written $L < R$), there is a* simplest *member of* **No** *lying between the members of L and the members of R [40, Theorem 1 and Theorem 4 (i) and (ii)].*

Co-opting notation introduced by Conway, the simplest member of **No** lying between the members of L and the members of R is denoted by

$$\{L|R\}.$$

Following Conway [19, page 4], if $x = \{L|R\}$, we write x^L for a typical member of L and x^R for a typical member of R; $x = \{a, b, c, ...|d, e, f, ...\}$ means that $x = \{L|R\}$ where $a, b, c, ...$ are typical members of L and $d, e, f, ...$ are typical members of R.

Each $x \in \mathbf{No}$ has a *canonical representation* as the simplest member of **No** lying between its predecessors on the left and its predecessors on the right, i.e.

$$x = \{L_{s(x)}|R_{s(x)}\},$$

where

$$L_{s(x)} = \{a \in \mathbf{No} : a <_s x \text{ and } a < x\} \text{ and } R_{s(x)} = \{a \in \mathbf{No} : a <_s x \text{ and } x < a\}.$$

By now letting $x = \{L_{s(x)}|R_{s(x)}\}$ and $y = \{L_{s(y)}|R_{s(y)}\}$, $+, -$ and $\cdot$ are defined by recursion as follows, where x^L, x^R, y^L and y^R are understood to range over the members of $L_{s(x)}, R_{s(x)}, L_{s(y)}$ and $R_{s(y)}$, respectively.

Definition of $x + y$.

$$x + y = \left\{x^L + y, x + y^L | x^R + y, x + y^R\right\}.$$

Definition of $-x$.

$$-x = \left\{-x^R | -x^L\right\}.$$

Definition of xy.

$$xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R | \\ x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}.$$

Despite their cryptic appearance, the definitions of sums and products in **No** have natural interpretations that essentially assert that the sums and products of elements of **No** are the simplest elements of **No** consistent with **No**'s structure as an ordered group and an ordered field respectively (see, for example, [39, page 1236], [40, pages 252-253]). The constraint on additive inverses, which is a consequence of the definition of addition [40, page 1237], ensures that the portion of the surreal number tree less than $0 = \{\varnothing|\varnothing\}$ is (in absolute value) a mirror image of the portion of the surreal number tree greater than 0, 0 being the simplest element of the surreal number tree (see Fig. 1).

A subclass A of **No** is said to be *initial* if $x \in A$ whenever $y \in A$ and $x <_s y$. Although there are many isomorphic copies of the order field of reals in **No**, only one is initial [40, page 1243]. This ordered field, which we denote $\mathbb{R}$, plays the role of the reals in **No**. Similarly, while there are many subclasses A of **No** that are well-ordered proper classes in which for all $x, y \in A$, $x < y$ if and only if $x <_s y$, only one is initial. The latter, which consists of the members of the rightmost branch of (**No** $<, <_s$) (see Fig. 1), is identified as **No**'s class **On** of ordinals.

The nonzero elements of **No** can be partitioned into equivalence classes, called *Archimedean classes*, each consisting of all nonzero members x, y of **No** that satisfy the condition: $m|x| > |y|$ and $n|y| > |x|$ for some positive integers m, n. If a and b are members of distinct Archimedean classes and $|a| < |b|$, then we write $a \ll b$ and a is said to be *infinitesimal (in absolute value) relative to* b.

An element of **No** is said to be a *leader* if it is the simplest member of the positive elements of an Archimedean class of **No**. Since the class of positive elements of an Archimedean class of **No** is convex, by the first part of Proposition 2 the concept of a leader is well defined. There is a unique mapping–*the* ω*-map*–from **No** onto the ordered class of leaders that preserves both $<$ and $<_s$. The image of y under the ω-map is denoted ω^y, and in virtue of its order preserving nature, we have: for all $x, y \in$ **No**,

$$\omega^x \ll \omega^y \text{ if and only if } x < y.$$

Using the ω-map along with other aspects of **No**'s s-hierarchical structure and its structure as a vector space over $\mathbb{R}$, every surreal number can be assigned a canonical "proper name" or *normal form* that is a reflection of its characteristic s–hierarchical properties. These normal forms are expressed as sums of the form

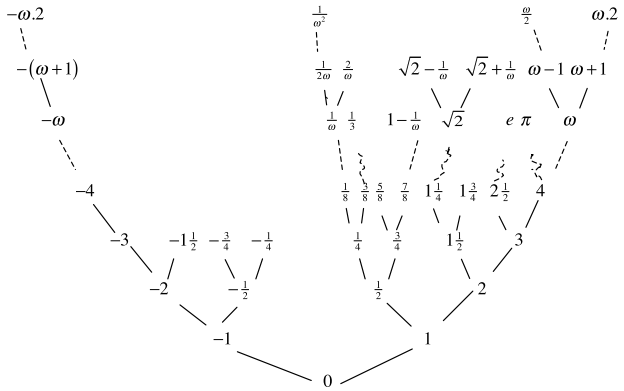

Fig. 1. Early stages of the recursive unfolding of **No**.

$$\sum_{\alpha<\beta} \omega^{y_\alpha}.r_\alpha$$

where β is an ordinal, $(y_\alpha)_{\alpha<\beta}$ is a strictly decreasing sequence of surreals, and $(r_\alpha)_{\alpha<\beta}$ is a sequence of nonzero real numbers. Every such expression is in fact the normal form of some surreal number, the normal form of an ordinal being just its *Cantor normal form* ([19, pages 31-33], [40, §3.1 and §5], [41]).

Making use of these normal forms, Fig. 1 offers a glimpse of the some of the early stages of the recursive unfolding of **No**.

When surreal numbers are represented by their normal forms, order, addition and multiplication in **No** assume more tractable forms with the order defined lexicographically and addition and multiplication defined as for polynomials with $\omega^x\omega^y = \omega^{x+y}$ for all $x, y \in \mathbf{No}$.

Definition 3. An element x of an ordered field is said to be *infinitesimal* if $|x| < 1/n$ for every positive integer n and it is said to be *infinite* if $|x| > n \cdot 1$ for every positive integer n. Thus, in virtue of the lexicographical ordering on normal forms, a surreal number is infinite (infinitesimal) just in case the greatest exponent in its normal form is greater than (less than) 0. As such, each surreal number x has a canonical decomposition into its *purely infinite part*, its *real part*, and its *infinitesimal part*, consisting of the portions of its normal form all of whose exponents are > 0, $= 0$, and < 0, respectively. A surreal number, and a member of an ordered field more generally, will be said to be *finite* if it is not infinite.

2.2. Absolute convergence in the sense of Conway

There is a notion of convergence in **No** for sequences and series of surreals that can be conveniently expressed using normal forms supplemented with dummy terms whose coefficients are zero. Let $x \in \mathbf{No}$ and for each $y \in \mathbf{No}$, let $r_y(x)$ be the coefficient of ω^y

in the normal form of x, it being understood that $r_y(x) = 0$, if ω^y does not occur. Also let $\{x_n : n \in \mathbb{N} \cup \{0\}\}$ be a sequence of surreals so written. Following Siegel [74, page 432], we write

$$x = \lim_{n\to\infty} x_n$$

to mean

$$r_y(x) = \lim_{n\to\infty} r_y(x_n), \text{ for all } y \in \mathbf{No},$$

and say that $\{x_n : n \in \mathbb{N} \cup \{0\}\}$ *converges* to x. We also write

$$x = \sum_{n=0}^{\infty} x_n$$

to mean the partial sums of the series converge to x.

Among the convergent sequences and series of surreals are those whose mode of convergence is quite distinctive. In particular, for each $y \in \mathbf{No}$, there is a nonnegative integer m such that $r_y(x_n) = r_y(x_m)$ for all $n \geq m$. Thus, for each $y \in \mathbf{No}$,

$$r_y(x) = \lim_{n\to\infty} r_y(x_n) = r_y(x_m),$$

where m depends on y. Following Conway, we call this mode of convergence *absolute convergence*.

Notational Convention 1. We will call the normal form to which an absolutely convergent series $\{x_n : n \in \mathbb{N} \cup \{0\}\}$ of normal forms converges the *Limit* of the series and denote it using

$$\operatorname*{Lim}_{n\to\infty} x_n. \tag{1}$$

We use "Limit" as opposed to "limit" and "Lim" as opposed to "lim" to distinguish the surreal notion from its classical counterpart.

Relying on the above and classical combinatorial results of Neumann ([62, pages 206-209], [74, Lemma 3.2], [1, pages 260-266]), one may prove [74, pages 432-434] the following theorem of Conway [19, page 40], which is a straightforward application to **No** of a classical result of Neumann ([62, page 210], [1, page 267]).

Proposition 4. Let f be a formal power series with real coefficients, i.e. let

$$f(x) = \sum_{n=0}^{\infty} r_n x^n \tag{2}$$

where the r_n's are reals. Then $f(\zeta)$ is absolutely convergent for all infinitesimals ζ in **No**, i.e.,

$$f(\zeta) = \operatorname*{Lim}_{n\to\infty} \sum_{m=0}^{n} r_m x^m. \tag{3}$$

Conway's theorem also has the following multivariate formulation [74, page 435].

Proposition 5. Let f be a formal power series in k variables with real coefficients, i.e. let

$$f(x_1, ..., x_k) \in \mathbb{R}[[x_1, ..., x_k]].$$

Then $f(\epsilon_1, ..., \epsilon_k)$ is absolutely convergent for every choice of infinitesimals $\epsilon_1, ..., \epsilon_k$ in **No**.

This can also be written in the following useful form.

Proposition 6. Let $\{c_{\mathbf{k}} : \mathbf{k} \in (\mathbb{N} \cup \{0\})^m\}$ be any multisequence of real numbers and $h_1, ..., h_m$ be infinitesimals. Also let $\mathbf{h}^{\mathbf{k}} = h_1^{k_1} \cdots h_m^{k_m}$. Then

$$\sum_{|\mathbf{k}|\geq 0} c_{\mathbf{k}} \mathbf{h}^{\mathbf{k}} \tag{4}$$

is a well-defined element of **No**.

The following result, in which $\{x_n : n \in \mathbb{N} \cup \{0\}\}$ and $\{y_n : n \in \mathbb{N} \cup \{0\}\}$ are absolutely convergent series of normal forms, collects together some elementary properties of absolute convergence in **No**. Several are very similar to the properties of the usual limits.

Proposition 7. Let $\operatorname{Lim}_{n\to\infty} x_n = x$ and $\operatorname{Lim}_{n\to\infty} y_n = y$, and further let $h \ll 1$, $\tau > 0$ and $a, b \in$ **No**. Then

$$\begin{aligned}
&(a)\ \operatorname{Lim}_{n\to\infty}(ax_n + by_n) = ax + by;\\
&(b)\ \operatorname{Lim}_{n\to\infty} x_n y_n = xy;\\
&(c)\ x \neq 0 \Rightarrow \operatorname{Lim}_{n\to\infty} \frac{1}{x_n} = \frac{1}{x};\\
&(d)\ (\exists k)(\forall n)(|x_n| < k);\\
&(e)\ \operatorname{Lim}_{n\to\infty} h^n = 0;\\
&(f)\ (\forall n)(|x_n| \leq \tau) \Rightarrow |x| \leq \tau.
\end{aligned} \tag{5}$$

Proof of Proposition 7. (a) and (b) are proved in [1, page 271], (d) is evident since no set is cofinal with **No**, (e) follows from Proposition 2 and (f) follows from (e). For (c), since $\mathrm{Lim}_{n\to\infty} x_n \neq 0$, there is a greatest $y \in \mathbf{No}$ such that $r_y(x_n)$ is not eventually zero. Thus, for sufficiently large n, $x_n = r_y\omega^y(1+h_n)$, where h_n is infinitesimal, and, so, it suffices to establish the result for x_n of the form $1/(1+h_n)$. Since $1/(1+h_n) - 1 = -h_n(1+h_n)^{-1}$ and $\mathrm{Lim}_{n\to\infty} h^n = 0$ the coefficients of leaders in h_n eventually vanish, and, as such, eventually vanish for $-h_n(1+h_n)^{-1}$. □

2.3. Surreal exponentiation

As was mentioned above, **No** admits an inductively defined exponential function exp. $(\mathbf{No}, \exp)$ is in fact an elementary extension of the exponential ordered field $(\mathbb{R}, e^x)$ of real numbers [27]. The exponential function on **No** was introduced by Kruskal, and reconstructed and substantially developed by Gonshor [49, Chapter 10]. While the definition of exp is quite complicated for the general case, it reduces to the following simpler and more revealing forms for the three theoretically significant cases.

Proposition 8 *(Gonshor [49]).*

(i) $\exp(x) = e^x$ *for all* $x \in \mathbb{R}$*;*
(ii) $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ *for all infinitesimal* x*;*
(iii) *if* x *is purely infinite, then*

$$\exp(x) \;=\; \left\{0,\ (\exp x^L)(x - x^L)^n \;\middle|\; \frac{\exp x^R}{(x^R - x)^n}\right\},$$

where x^L *and* x^R *range over all the purely infinite predecessors of* x *with* $x^L < x < x^R$*.*

The significance of cases (i)–(iii) emerges from the fact that for an arbitrary surreal x, $\exp(x) = \exp(x_P) \cdot \exp(x_R) \cdot \exp(x_I)$, where x_P, x_R and x_I are the purely infinite, real and infinitesimal components of x, respectively.

Shedding further light on $\exp(x)$ when x is purely infinite is:

Proposition 9 *(Gonshor [49]). The restriction of* exp *to the class of purely infinite surreal numbers is an isomorphism of ordered groups onto* ***No****'s class* $\{\omega^x : x \in \mathbf{No}\}$ *of leaders.*

In subsequent sections of the paper, for the sake of simplicity, we will occasionally write e^x in place of $\exp x$ for the surreal extension of the real function e^x. Readers seeking additional background in the theory of surreal exponentiation may consult [49, 27,44,13,12].

3. Extension, antidifferentiation and integral operators

To introduce the requisite conceptions of extension, antidifferentiation and integral operators, we require some preliminary notions concerning intervals, extensions of functions from the reals to **No** and restrictions of surreal functions to $\mathbb{R}$, where $\mathbb{R}$ is understood to be the canonical copy of the reals in **No** (see §2.1).

By an *interval* I of an ordered class A we mean a convex subclass of A. In addition to the more familiar types of intervals of $\mathbb{R}$ and **No** we will consider are $(a, \infty) := \{x \in \mathbb{R} : x > a\}$ and $(a, \mathbf{On}) := \{x \in \mathbf{No} : x > a\}$, where $a \in \mathbb{R}$. In §8.2 a simple condition is specified under which the forthcoming developments of our theory also apply to the intervals $(-\infty, a) := \{x \in \mathbb{R} : x < a\}$ and $(-\mathbf{On}, a) := \{x \in \mathbf{No} : x < a\}$, for $a \in \mathbb{R}$.

3.1. Derivatives

To formulate the appropriate notions of extension and antidifferentiation operators, we require a generalization of the idea of a *derivative of a function at a point*.

Definition 10 *(Derivative).* Let K be an ordered field. A function f defined on an interval around a is differentiable at a if there is an $f'(a) \in K$ such that $(\forall \epsilon > 0 \in K)(\exists \delta > 0 \in K)$ such that

$$(\forall x \in K)(|x - a| < \delta \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon).$$

As usual, $f'(a)$ is said to be *the derivative of f at a* and f is said to be *differentiable* if the derivative of f exists at each point of its domain. The definition generalizes to higher order derivatives in the usual way.

It is straightforward to check that the derivative so defined on **No** has the same **local** properties (linearity, chain rule, etc.) as its real counterpart. However, because **No** is disconnected, global properties such as Rolle's theorem and its consequences may fail.

3.2. Extension operators

If f is a function, then by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ we mean the domain and range of f respectively. We define λf and $f + g$ for functions f, g as usual, where $\mathrm{dom}(f + g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

Definition 11. Let I be an interval of $\mathbb{R}$ and J be an interval of **No** that contains I.

(1) As usual, we say that $g : J \to \mathbf{No}$ extends $f : I \to \mathbb{R}$ if for every $x \in I$ we have $g(x) = f(x)$, and we denote by $g|I$ the restriction of g to I.

(2) Let $\mathcal{F}$ be a set of real-valued functions defined on intervals of $\mathbb{R}$. By an *extension operator* E on $\mathcal{F}$ we mean a map that associates to each function $f : I \to \mathbb{R}$ in $\mathcal{F}$ a function $E\,f : J \to \mathbf{No}$ in such a manner that

i. for all $f \in \mathcal{F}$, $E\,f$ is an extension of f;
ii. (Linearity) for all $g, h \in \mathcal{F}$ and $C \in \mathbb{R}$, $E(Cg) = CE\,g$ and $E(g + h) = E\,g + E\,h$;
iii. if $\beta, \lambda \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, $g(x) = x^\beta e^{\lambda x}$ and $h(x) = x^n \log(x)$ for all $x \in I$, then $(E\,g)(x) = x^\beta e^{\lambda x}$ and $(E\,h)(x) = x^n \log(x)$ for all $x \in J$.
iv. $E\,f' = (E\,f)'$.

For some important classes of problems we construct extensions that are *multiplicative* or, in other words, that preserve multiplication in the following sense.

Definition 12. An extension operator E is multiplicative on an algebra of functions if for all f and g in the algebra we have $E(fg) = (Ef)(Eg)$.

3.3. Antidifferentiation and integral operators

The following definition provides definitions of both real and surreal antidifferentiation operators.

Definition 13. Let $\mathcal{F}$ be a set of real-valued (surreal-valued) functions whose domains are intervals of $\mathbb{R}$ ($\mathbf{No}$). An *antidifferentiation operator* on $\mathcal{F}_1 \subseteq \mathcal{F}$ is a function $A : \mathcal{F}_1 \to \mathcal{F}$ such that for all $f, g \in \mathcal{F}_1$:

i. $A\,f$ is differentiable and $(A\,f)' = f$;
ii. For any $\lambda \in \mathbb{R}$ ($\lambda \in \mathbf{No}$), $A(\lambda f) = \lambda A\,f$, $A(f + g) = A\,f + A\,g$;
iii. If $y \geq x$ and $f \geq 0$, then $(A\,f)(y) - (A\,f)(x) \geq 0$.
iv. $\forall n \in \mathbb{N}, A\,(x^n) = \frac{1}{n+1} x^{n+1}$ (the right side being the monomial in $\mathcal{F}$).
v. $A\,(\exp)$ equals the real (surreal) exponential.
vi. If $F \in \mathcal{F}_1$ and $F' = f \in \mathcal{F}_1$, then there is a $C \in \mathbb{R}$ ($C \in \mathbf{No}$) such that $A\,f$ exists and equals $F + C$.

For suitable integrals to exist, we need the "second half" of the fundamental theorem of calculus to hold. This is the motivation for the following convention.

Notational Convention 2. Let A be an antidifferentiation operator on $\mathcal{F}_1 \subseteq \mathcal{F}$, and let $f \in \mathcal{F}_1$ and $x, y \in \mathbf{No}$. Define

$$\int_x^y f := A(f)(y) - A(f)(x). \tag{6}$$

The following result demonstrates that the existence of an antidifferentiation operator on $\mathcal{F}_1 \subseteq \mathcal{F}$ implies that $\int_x^y f$ is an operator on $\mathcal{F}_1$ whose properties make it worthy of the appellation "integral operator".

In the following proposition, $\alpha, \beta, a, b, a_1, a_2, a_3 \in \mathbf{No}$, and $f, g, fg, f \circ g, f', g'$ are understood to be elements of $\mathcal{F}_1$ on $[a, b]$, $[a_1, a_2]$, $[a_2, a_3]$ or $[a_1, a_3]$ where applicable. In our constructions we will specify which spaces are closed under the above-said operations.

Proposition 14 *(*Integral operators*).* Let A be an antidifferentiation operator on $\mathcal{F}_1 \subseteq \mathcal{F}$. Then $\int_x^y f$ is an *integral operator* on $\mathcal{F}_1$, meaning a function of three variables, $x, y \in \mathbf{No}$ and $f \in \mathcal{F}_1$, with the properties:

(a) $\left(\int_a^x f\right)' = f$;

(b) $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$;

(c) $\int_a^b f' = f(b) - f(a)$;

(d) $\int_{a_1}^{a_2} f + \int_{a_2}^{a_3} f = \int_{a_1}^{a_3} f$;

(e) $\int_a^b f'g = fg|_a^b - \int_a^b fg'$ if f and g are differentiable on (a, b);

(f) $\int_a^x (f \circ g)g' = \int_{g(a)}^{g(x)} f$ whenever $g \in \mathcal{F}_1$ is differentiable on (a, x).

(g) If f is a positive function and $b > a$, then $\int_a^b f > 0$.

Proof. All these are straightforward. (a) follows from Definition 13 i, and differentiating (6). (b) follows from Definition 13, ii. (e) follows similarly using the chain rule and, taking $g = 1$, it implies (c). (d) follows from Equation (6). For (f), note that since $f \in \mathcal{F}_1$ we have $f = F'$ for some F, and hence $(F \circ g)' = (f \circ g)g'$; the rest is a consequence of (c). And, finally, (g) follows from Definition 13, iii. □

As we alluded to in the introduction, in §8 we construct a wide class of functions defined on intervals of $\mathbb{R}$ of the form (a, ∞), where a may depend on the function, that is closed under antidifferentiation in the sense of Definition 13, and which we extend in the sense of Definition 11 to surreal functions defined on $(a, \mathbf{On})$. By contrast, for our

negative result we only retain some very basic properties of antidifferentiation and work on a space of functions with "very good properties". This is spelled out below in §13.

4. Difficulties of defining extensions and integration of functions, and our strategy for overcoming them

One of the sources of difficulty in extending more general classes of classical functions to **No** and in defining integration for them is the fact that the topology of surreal numbers is totally disconnected, and as such processes *other* than the usual "extensions by continuity" must be employed. A natural class of functions on which extensions and integration can be naturally defined in a way that preserves the expected properties are the analytic functions. This is due to their *unique representations as power series*, which at ∞ take the form

$$f(x) = \sum_{k=0}^{\infty} \frac{c_k}{x^k} \tag{7}$$

where for some positive real R and all $k \in \mathbb{N} \cup \{0\}$ we have $|c_k| \leq R^k$; of course the series in Equation (7) converges for all $x \in \mathbb{R}^+$ such that $x > R$. We can make use of normal forms to define $\mathsf{E}f(x)$ for all surreal numbers greater than R in a way that ensures that E preserves all operations that are preserved by Limits (see §2.2). For this, relying on Proposition 4 and the definition of "Lim" (see §2.2), we simply write

$$\mathsf{E}f(x) = \sum_{k=0}^{\infty} \frac{c_k}{x^k} = \operatorname*{Lim}_{N\to\infty} \sum_{k=0}^{N} \frac{c_k}{x^k}. \tag{8}$$

Similarly, for all $x \in \mathbf{No}$ such that $x > R$ and f as in Equation (7), we let

$$\mathsf{A}_{\mathbf{No}}\, f(x) = c_0 x + c_1 \log x - \sum_{k\geq 2}^{\infty} \frac{c_k}{(k-1)x^{k-1}}. \tag{9}$$

Based on Proposition 7, it is an easy exercise to check that $\mathsf{A}_{\mathbf{No}}$ so defined is an antidifferentiation operator on the class of functions analytic at ∞. In fact, for the class of functions analytic at ∞ and $O(x^{-2})$ for large x ($c_0 = c_1 = 0$ in (7)) this is an antiderivative with "zero constant at ∞" or from ∞. Integration is defined for $R < a \leq x$ by Equation (6).

With obvious adaptations, these definitions, constructions and results extend to functions that are given at ∞ by convergent Puiseux fractional power series or, far more generally, by *convergent transseries* (see §5.2 and e.g. [22, page 143]).

While divergent series and transseries *as formal objects* can be associated in much the same way with actual surreal functions defined on the positive infinite elements of **No**, the difficulty in these cases is to pair them with functions on the *finite* surreals in a

unique way that, additionally, is compatible with common operations in analysis. Indeed, while in classical analysis convergent expansions correspond to a unique function, this is *not* the case for divergent expansions. We overcome this difficulty by using techniques of resurgent functions and Écalle-Borel summation (§8).

The following simple example based on the exponential integral Ei illustrates the non-uniqueness problem in the divergent case. The function $y(x) = e^{-x}\mathrm{Ei}(x)$ is given by

$$y(x) = e^{-x}\,\mathrm{PV}\int_{-\infty}^{x} \frac{e^s}{s} ds \tag{10}$$

where PV stands for the *Cauchy principal value*: for $x > 0$ this is defined as the symmetric limit $\lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{x}\right)$.

This $y(x)$ has the asymptotic series

$$y(x) \sim \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}}, \quad x \to \infty. \tag{11}$$

Since $y(x) = \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}}$ is well defined for all $x \in \mathbf{No}$ via Limits, as in (8), it would be tempting to define the integral $\mathrm{PV}\int_{-\infty}^{x} \frac{e^s}{s} ds$ for all $x > \infty$ as $e^x y(x)$. But here we face a non-uniqueness problem: for any $a \in \mathbb{R}$, the function $y_a(x) = e^{-x}\, PV \int_a^x \frac{e^s}{s} ds$ has the **same** asymptotic series as $y(x)$ given in Equation (11). This is because $y(x) - y_a(x) = Ce^{-x}$ (where the constant C is $PV \int_{-\infty}^{a} \frac{e^s}{s} ds$) and the *power series* asymptotics of e^{-x} for large x is zero. In fact, *classical* asymptotic analysis cannot distinguish between y and the whole family of y_a's. (Contrast this with the fact that two different analytic functions cannot share the same Taylor series).

As a consequence of this type of non-uniqueness, in Section §13 we are able to show that a linear association between functions and **general** divergent series requires a relatively strong consequence of the Axiom of Choice (and as such cannot be instituted based on a specific definition, something which will be the subject of another paper). Accordingly, *the class of divergent series needs to be restricted!* With this in mind, as was mentioned in the introduction, we *limit our analysis to a proper subclass of the resurgent functions*, a subclass that appears to be wide enough to contain those functions which occur commonly in applications. As such, from a practical standpoint, our restriction appears to be relatively mild.

In §6 we introduce the idea of a resurgent function and the closely related idea of a resurgent transseries. The resurgent transseries are of particular importance to us since a unique association can be carried out in a constructive fashion between the class of resurgent divergent transseries, on the one hand, and the class of resurgent functions, on the other. For example, the resurgent function associated with the series in (11) is $e^{-x}\mathrm{Ei}(x)$. Moreover, this association preserves all the local operations with which the summation of convergent Taylor series do. We will use the just-said association to

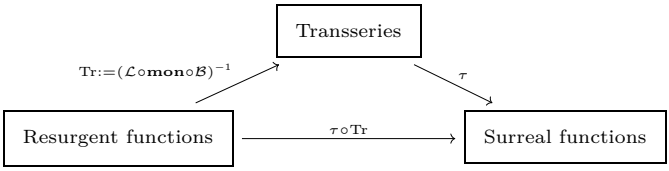

Fig. 2. *The extension operator* E *restricted to the positive infinite case* is the composition of two intermediate isomorphisms: *transseriation*, i.e. $\mathrm{Tr} := (\mathcal{L} \circ \mathbf{mon} \circ \mathcal{B})^{-1}$, from a subspace of resurgent functions to a subspace of transseries, where $\mathcal{L} \circ \mathbf{mon} \circ \mathcal{B}$ is *Écalle-Borel summation*, and a map τ from the just-said subspace of transseries to surreal functions.

define our desired integrals for *the positive infinite case* invoking a pair of isomorphisms– one between a subclass of resurgent functions and a subspace of transseries, and the other between the just-said subspace of transseries and a class of functions on **No.** It is through this pair of isomorphisms (see Fig. 2) that we extend resurgent functions to infinite surreals and define their integrals. Moreover, the integrals so-defined on surreal extensions of resurgent functions (as well as on transseries) have the properties specified in Proposition 14.

We remind the reader that by convention we set the point where our functions have divergent expansions to be at the gap ∞ (see Footnote 3), and as such the only gap past which defining integration is difficult is ∞ itself.

To prepare the way for our discussion of resurgent functions and resurgent transseries, in the following section we will first review some classical results in the theory of Borel summability and the theory of transseries and then prove a new result (Proposition 47) concerning the existence of antiderivatives. Like Proposition 47, most of the material in §5 from subsection 5.5 on is new.

5. Transseries, Borel summation and Borel summable subspaces of transseries

Typically, Borel summability and Écalle-Borel summability deal with series of the form

$$\tilde{f} := \sum_{k=M}^{\infty} c_k x^{-k\beta}, \beta > 0; \ M \in \mathbb{Z} \tag{12}$$

where the coefficients $\{c_k\}_{k \geq M}$ and β are real. The Borel sum of a finite sum is by definition the identity. Hence, we can assume without loss of generality that $M = 1$.

5.1. Classical Borel summation of series

The following definition collects together some of the basic concepts and observations we will employ in this and subsequent sections.

Definition 15 *(Laplace transform, Borel transform, Borel sum and critical time).* For suitable functions F for which the integral exists, the *Laplace transform* $\mathcal{L}F$ of F is defined as:

$$(\mathcal{L}F)(x) = \int_0^\infty e^{-xp} F(p) dp.$$

The *(formal) inverse Laplace transform* of a series $\tilde{f} = \sum_{k=0}^\infty c_k x^{-(k+1)\beta}$ is defined as a term-by-term transform of the series

$$\mathcal{L}^{-1}\tilde{f} = \sum_{k=0}^\infty c_k p^{-k\beta-1}/\Gamma(k\beta),$$

where Γ is the Gamma function; if n is a positive integer, $\Gamma(n) = (n-1)!$.

The *Borel transform* $\mathcal{B}\tilde{f}$ of a formal series $\tilde{f}$ given by Equation (12) with $M = 1$ (see the remarks following Equation (12)) is the series obtained by taking the term-by-term inverse Laplace transform of $\tilde{f}$ in normalized form. If $\beta = 1$, then $\mathcal{B}\tilde{f}$ is analytic at $p = 0$; otherwise it is ramified-analytic and $\mathcal{B}\tilde{f} = p^{-1}A(p^\beta)$ where A is analytic. It is often relatively easy to reduce to the case $\beta = 1$, which we will assume in the following.

The *Borel sum* of $\tilde{f}$ along $\mathbb{R}^+$ exists if after taking the Borel transform $\mathcal{B}\tilde{f}$ of $\tilde{f}$ the following two conditions are satisfied:

(i) The series $\mathcal{B}\tilde{f}$ is convergent, and its sum (by abuse of notation also written $\mathcal{B}\tilde{f}$) is analytic on $\mathbb{R}^+$.[8]

(ii) $\mathcal{B}\tilde{f}$ has exponential bounds on $\mathbb{R}^+$, i.e., $\exists \nu > 0$ such that $\sup_{p>\nu} |e^{-\nu p}(\mathcal{B}\tilde{f})(p)| < \infty$.

When this is the case, the Borel sum of $\tilde{f}$ is by definition $\mathcal{L}\mathcal{B}\tilde{f}$.

For example,

$$\mathcal{L}\mathcal{B}\sum_{k=0}^\infty k!(-1)^k x^{-k-1} = \mathcal{L}F = -e^x \mathrm{Ei}(-x); \quad F(p) := \frac{1}{1+p}. \tag{13}$$

The coefficients c_k of asymptotic series occurring in applications have at most power-of-factorial growth $c_k \sim (k!)^p$ for some (usually integer) p. To apply Borel summation or the more general Écalle-Borel summation to a series of a factorially divergent series, one needs to *normalize* the series by passing to the power of x that ensures that the growth of the coefficient of $x^{-(k+1)\beta}$ is, to leading order, $\Gamma(k\beta)$. The power of the variable with

[8] Mathematically, $\mathcal{B}\tilde{f}$ is a formal series, albeit convergent, and is distinct from its sum–a germ of an analytic function–which in turn is distinct from its analytic continuation on $\mathbb{R}^+$. These distinctions are typically dropped whenever no confusion is possible. For instance, we write with a tacit license that $\mathcal{B}\tilde{f}$ is analytic on $\mathbb{R}^+$.

respect to which this precise factorial growth is achieved is called *Écalle critical time*. An illustration is provided by the asymptotic series of $e^{x^2}\operatorname{erfc}(x)$ as $x \to \infty$,

$$e^{x^2}\operatorname{erfc}(x) \sim \frac{1}{\sqrt{\pi}x} - \frac{1}{2\sqrt{\pi}x^3} + \frac{3}{4\sqrt{\pi}x^3} + \cdots = x^{-1}\sum_{k=1}^{\infty}\frac{c_k}{x^{2k}}, \tag{14}$$

where $\pi c_k = (-1)^k\Gamma(k-1/2)$. To ensure that the growth of the coefficients of the series matches the power of the variable as explained, we need to change the variable to $t = \sqrt{x}$. In this example the critical time is $t = x^{1/2}$.

A calculation shows that

$$\mathcal{B}(\tilde{f}\tilde{g}) = (\mathcal{B}\tilde{f}) * (\mathcal{B}\tilde{g}), \tag{15}$$

where "$*$" is the *Laplace convolution*

$$(F * G)(p) = \int_0^p F(s)G(p-s)ds. \tag{16}$$

Proposition 16 *(The space $S_{\mathcal{B}}$ of Borel summable series).* Let $S_{\mathcal{B}}$ be the space of series which are Borel summable. Then:

(i) $S_{\mathcal{B}}$ is a differential algebra (with respect to formal addition, multiplication, and differentiation of power series), and $\mathcal{LB}$ is an isomorphism of differential algebras.

(ii) If $S_c \subset S_{\mathcal{B}}$ denotes the differential algebra of convergent power series, and we identify a convergent power series with its sum, then $\mathcal{LB}$ is the identity on S_c.

(iii) For $\tilde{f} \in S_B$ and x in the open right half plane, $\mathcal{LB}\tilde{f}$ is asymptotic to $\tilde{f}$ as $|x| \to \infty$.

(iv) The subspace of $S_{\mathcal{B}}$ consisting of series whose Borel transforms are analytic in a disk around the origin and in a nonempty open sector is closed under composition. More precisely, if $\tilde{f}$ and $\tilde{g}$ are elements of this subspace, then so is $\tilde{f} \circ (I + \tilde{g})$, I being the identity map.

(v) Borel summation is a proper extension of the usual summation. More precisely, if $\tilde{f} = \sum_{k\geq 1} c_k x^{-k}$ converges to f in a neighborhood of ∞, then $\mathcal{B}\tilde{f}$ is entire, exponentially bounded and $\mathcal{LB}\tilde{f} = f$.

Proof. Statements (i)–(iii) and (v) are proved in ([22], p. 106); and for the proof of (iv), see ([60] p. 159). □

Note. *Borel sums are analytic for large argument x.* Standard arguments from complex analysis (e.g. combining Morera's theorem with Fubini) show that $\mathcal{LB}\tilde{f}$ is *real analytic* for all sufficiently large $x \in \mathbb{R}$.

Definition 17 *(Borel summation).* The operator of *Borel summation* is defined at any point $x_0 \in \mathbb{R}$ (or $\mathbb{C}$) by moving x_0 to ∞, performing Borel summation at ∞ and moving

the point at ∞ back to x_0. That is, we define $(\mathcal{LB})_{x_0} = M^{-1} \circ \mathcal{LB} \circ M$ where M is the Möbius transformation $x \mapsto x_0 + x^{-1}$ (see also Definition 80).

On Borel summed series that are $O(x^{-2})$, we now define an operator having some of the properties of an antidifferentiation operator in the sense of Definition 13.

Definition 18. Let $S_{\mathcal{B};2}$ be the space of Borel summable series that are $O(x^{-2})$. Further, let $s \in S_{\mathcal{B};2}$, $S = \mathcal{LB}s$, and $\mathsf{A}_{\mathcal{B}}S = -\int_0^\infty p^{-1}e^{-xp}(\mathcal{B}s)(p)dp$. Asymptotic series at infinity are particular cases of transseries at infinity to which $\mathsf{A}_{\mathcal{B}}$ is successively extended in §5.4, §5.8 and §6.

We note that by the general properties of the Laplace transform we have $(A_{\mathcal{B}}f)' = f$ and $A_{\mathcal{B}}f = O(x^{-1})$ for large x. Hence, $\mathsf{A}_{\mathcal{B}}f = \int_\infty^x S(t)dt$.

Proposition 19. $\mathsf{A}_{\mathcal{B}}$, as defined in Definition 18, is well defined on Borel sums of real-valued series and has Properties i–iii and vi from Definition 13.

Proof. If $s = O(x^{-2})$ for large x, then by definition, $(\mathcal{B}s)(p) = O(p)$ for small p. Since $\mathcal{B}s$ is analytic at zero, we have $\mathcal{B}s = pH(p)$ where H is analytic at zero, and hence, $p^{-1}\mathcal{B}s = H(p)$ is analytic at zero as well. Clearly, $\mathcal{B}s$ has analytic continuation on $\mathbb{R}^+$ if and only if $H(p)$ is has analytic continuation on $\mathbb{R}^+$. It is also straightforward to check that $\mathcal{B}s$ is exponentially bounded for large p if and only if H is exponentially bounded for large p. This establishes the existence of $\mathsf{A}_{\mathcal{B}}S$.

Using the exponential bounds and dominated convergence we see that we can differentiate under the integral sign and get $(\mathsf{A}_{\mathcal{B}}S)' = S$, thereby establishing Property i of Definition 13.

Proposition 16 (iii) shows that if s is positive (meaning that the coefficient of the highest power of x is positive), S is a positive function for large x. The positivity of the coefficient of the highest power of x is equivalent to the positivity of $H(0)$, which in turn shows that $\mathsf{A}_{\mathcal{B}}S$ is negative and increasing for large x, establishing Property iii of Definition 13. Property ii of Definition 13, i.e., linearity, follows from the linearity of $\mathcal{B}$ and $\mathcal{L}$, and of multiplication by p^{-1}. Property vi follows from the fact noted above that $\mathsf{A}_{\mathcal{B}}f = \int_\infty^x S(t)dt$ and the fundamental theorem of calculus. In fact, using the remark following Definition 18, we have $C = 0$. □

5.2. Transseries: an overview

As was mentioned in the introduction, a *transseries* over $\mathbb{R}$ is a formal series built up from $\mathbb{R}$ and a variable $x > \mathbb{R}$ using powers, exponentiation, logarithms and infinite sums. Écalle's classical construction of the ordered differential field $\mathbb{T}$ of transseries over

$\mathbb{R}$ is inductive, beginning with log-free transseries [32].[9] There have been a number of alternative constructions since (e.g. [4,30,69,21,51,14]). For a self-contained introduction to transseries, see [37]. Transseries are formal series of the following form in the variables $\mu_1, \mu_2, ..., \mu_n$, called *transmonomials*:

$$\tilde{T} = \sum_{\mathbf{k} > -M} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} := \sum_{k_1, k_2, ..., k_n > -M} c_{k_1, k_2, ..., k_n} \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_n^{k_n}, \tag{17}$$

where the transmonomials are functions of x, the coefficients are members of $\mathbb{R}$ and $M \in \mathbb{Z}$. The set of tuples of integers bounded below used as indices in (17) are well-ordered lexicographically; this indexation, which emphasizes the nature of the generators (transmonomials) is preferable, in the applications we are considering, to one using the corresponding ordinals.

Transseries have (exponential) *heights* and *(logarithmic) depths* that emerge from their inductive construction, but in our discussion we will only be concerned with *log-free, height one* and *height one, depth one* transseries, and these are characterized below in Definitions 24 and 30, respectively. Since context should prevent confusion, we will freely write exponential and logarithmic terms in transseries using e and log, respectively.

In the case of transseries over $\mathbb{R}$ the component terms in $\tilde{T}$ are descendingly well ordered with respect to the *asymptotic order relation* $\gg$; for example, for the transseries $e^x + x + \log x + 1 + x^{-1}$ we have $e^x \gg x \gg \log x \gg 1 \gg x^{-1}$, where $a \gg b$ indicates that a is large (i.e. infinitely large) compared with b.

We say that a transseries $\tilde{T}$ is **positive** if the largest transmonomial of $\tilde{T}$ with respect to $\gg$ has a positive coefficient, negative if $-\tilde{T}$ is positive, and $\tilde{T} = 0$ if all of its coefficients are zero.

There is a striking similarity between transseries over the reals and surreal numbers written in normal form. Aschenbrenner, van den Dries and van der Hoeven [6] have exhibited a canonical elementary embedding ι of the ordered differential field $\mathbb{T}$ of transseries into $(\mathbf{No}, \partial)$ that is the identity on $\mathbb{R}$ and sends x to ω, where ∂ is the derivation on **No** due to Berarducci and Mantova [14]. By appealing to Berarducci and Mantova's construction of $\iota(\mathbb{T}) := \mathbb{R}((\omega))^{LE}$ [14], Ehrlich and Kaplan [44] have shown that $\mathbb{R}((\omega))^{LE}$ is initial. We will have more to say about $\mathbb{R}((\omega))^{LE}$ in §11.

The similarity between transseries over the reals and surreal numbers carries over to the fact that the topology generated by Conway's notion of absolute convergence (see §2.2) is *mutatis mutandis* the same as the following "transseries topology" in the space of transseries.

Definition 20. The *transseries topology* on $\mathbb{T}$ (see [22, p. 131], [37]) is defined by the following convergence notion. Let $\sum_{\mathbf{k} > -M} c_{\mathbf{k}}^{[m]} \boldsymbol{\mu}^{\mathbf{k}}$ be a sequence of transseries, where

[9] Motivated by a problem of Tarski on the model theory of $(\mathbb{R}, e^x)$, Dahn and Göring [24] independently introduced $\mathbb{T}$ as an exponential ordered field.

the superscript $[m]$ designates the mth element of the sequence and $c_{\mathbf{k}}^{[m]}$ designates the sequences of coefficients of the mth element. Then,

$$\lim_{m\to\infty} \sum_{\mathbf{k}>-M} c_{\mathbf{k}}^{[m]} \boldsymbol{\mu}^{\mathbf{k}} = \sum_{\mathbf{k}>-M} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \text{ if and only if } \forall \mathbf{k}\, \exists n \text{ such that } \forall m > n, c_{\mathbf{k}}^{[m]} = c_{\mathbf{k}}, \tag{18}$$

i.e., *if and only if all the coefficients eventually become those of the limit transseries (rather than merely converge to them).*

5.3. Differentiation of transseries

$\mathbb{T}$ is closed under differentiation, where differentiation of transseries is defined by induction on transseries height as termwise differentiation [30,51,22,32]. It is shown in [22] that the termwise differentiation of a transseries is convergent in the transseries topology.

5.4. Integration of transseries

$\mathbb{T}$ is also closed under integration. More specifically, we have:

Proposition 21 *(van den Dries, Macintyre and Marker [30]).* There is an antidifferentiation operator on $\mathbb{T}$, henceforth $\mathsf{A}_{\mathbb{T}}$, that is unique up to an additive real constant.

An independent, alternative proof (in the same spirit) of the existence portion of Proposition 21 was later given in [22, p. 143, Proposition 4.221].[10] In the latter treatment, the operator $\mathsf{A}_{\mathbb{T}}$ is defined as the unique fixed point of a linear inhomogeneous equation whose linear part is contractive in a suitable sense (see [22, Definition 4.186, p. 132]). While the definition is constructive, the expression of the operator is not explicit, in general.

Note 22. Although antidifferentiation in $\mathsf{A}_{\mathbb{T}}$ is unique up to a real constant, there is a natural choice of an antidifferentiation, the one whose values are transseries with zero finite part. The interpretation of this choice is that of integration from ∞, the only point all one-point compactifications of $(1,\infty)$ have in common. However, any other choice of real constant would lead to the same definite integration operator, since the integral is a difference of two antiderivatives, and the constants would cancel.

The following result collects together a number of simple consequences of the above results, taken from [22, p. 143-144, Propositions 4.224-4.225].

[10] The second author wishes to thank Lou van den Dries for helpful remarks on Proposition 21 and the relation between its proof and that of the above-mentioned result in [22].

Proposition 23. The antidifferentiation operator $\mathsf{A}_{\mathbb{T}}$ on $\mathbb{T}$ from Proposition 21 has the following properties for all transseries $\tilde{T}, \tilde{T}_1$ and $\tilde{T}_2$: $\mathsf{A}_{\mathbb{T}}$ is an antiderivative without constant terms, i.e.,

$$\mathsf{A}_{\mathbb{T}}\tilde{T} = L + s,$$

where L is the purely infinite part of $\mathsf{A}_{\mathbb{T}}\tilde{T}$ (i.e. all terms in L are $\gg 1$) and s is the small part of $\mathsf{A}_{\mathbb{T}}\tilde{T}$ (i.e. $1 \gg s$). Here $\mathsf{A}_{\mathbb{T}}\tilde{T}$ is written as a sum as in Equation (17).

Moreover,

$$\begin{aligned}
\mathsf{A}_{\mathbb{T}}(\tilde{T}_1 + \tilde{T}_2) &= \mathsf{A}_{\mathbb{T}}\tilde{T}_1 + \mathsf{A}_{\mathbb{T}}\tilde{T}_2,\\
(\mathsf{A}_{\mathbb{T}}\tilde{T})' = \tilde{T},\ \mathsf{A}_{\mathbb{T}}\tilde{T}' &= \tilde{T}_{\overline{0}},\\
\mathsf{A}_{\mathbb{T}}(\tilde{T}_1\tilde{T}_2') &= (\tilde{T}_1\tilde{T}_2)_{\overline{0}} - \mathsf{A}_{\mathbb{T}}(\tilde{T}_1'\tilde{T}_2),\\
\text{if } \tilde{T}_1 \gg \tilde{T}_2, &\text{ then } \mathsf{A}_{\mathbb{T}}\tilde{T}_1 \gg \mathsf{A}_{\mathbb{T}}\tilde{T}_2,\\
\text{if } \tilde{T} > 0 \text{ and } \tilde{T} \gg 1, &\text{ then } \mathsf{A}_{\mathbb{T}}\tilde{T} > 0,
\end{aligned} \tag{19}$$

where $\tilde{T}_{\overline{0}}$ is the constant-free part of $\tilde{T}$, that is,

$$\text{if } \tilde{T} = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}}\mu^{\mathbf{k}}, \text{ then } \tilde{T}_{\overline{0}} = \sum_{\mathbf{k}\geq\mathbf{k}_0;\mathbf{k}\neq 0} c_{\mathbf{k}}\mu^{\mathbf{k}}$$

and where $(\tilde{T}_1\tilde{T}_2)_{\overline{0}}$ is the transseries $\tilde{T}_1\tilde{T}_2$ with the constant term chosen to be zero.

"Hands-on" constructions of antiderivatives of special transseries that will concern us will be given in Subsection 5.5.1.

5.5. Some subspaces of transseries

In this section we introduce and analyze three spaces of transseries: $\mathbb{T}_-, \mathbb{T}_\ell$ and $\mathbb{T}_+$. Transseries in $\mathbb{T}_-$ occur as solutions of nonlinear ODEs, difference equations and a variety of other nonlinear problems. Transseries in $\mathbb{T}_+$ arise in linear problems and $\mathbb{T}_\ell$ is a space that is generated by repeated antidifferentiation. The minus subscript stands for the fact that all the arguments of the exponentials in the members of the space are nonpositive; the subscript "ℓ" indicates the absence of exponentials, but possible presence of logarithms; and the plus subscript indicates that all the arguments of the exponentials in the members are positive.

The space $\mathbb{T}_-$ is actually a differential algebra. Nonlinear problems rely on the algebraic structure, which we analyze. For the other two spaces we are only interested in their linear properties. The space $\mathbb{T}_\ell \oplus \mathbb{T}_-$ is closed under antidifferentiation.

Definition 24 *(The space $\mathbb{T}_-$ of log-free, height one transseries).* Let $n \in \mathbb{N}$, $\boldsymbol{\beta}, \boldsymbol{\lambda}$ be vectors in $\mathbb{R}^n$, with $\lambda_i > 0$ for $i = 1, ..., n$, and define

$$\tilde{T}_- = \sum_{\mathbf{k}\geq 0, l\geq 0} c_{\mathbf{k},l} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-l} = \sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} \tilde{y}_{\mathbf{k}}(x), \tag{20}$$

where the $\tilde{y}_{\mathbf{k}}$ are formal power series which are $o(1/x)$ for large x. In applications in which $\tilde{y}_{\mathbf{0}}$ starts with a constant, this constant can be subtracted out. To arrange that $\tilde{y}_{\mathbf{k}} = o(1/x)$ for all $\mathbf{k} \neq 0$ we can simply change β_i to $\beta_i + 1$ (with the effect of dividing $\tilde{y}_{\mathbf{k}}$ by $x^{|\mathbf{k}|}$). We denote the space of such $\tilde{T}_-$ by $\mathbb{T}_-$.[11] The parameters n, $\boldsymbol{\lambda}, \boldsymbol{\beta}$ depend on the transseries; when combining two transseries one first needs to embed all of these in a larger parameter space.

The condition in the above definition that the $\tilde{y}_{\mathbf{k}}$ are $o(1/x)$ for large x is a useful convention because it ensures that the only common element of $\mathbb{T}_\ell$ and $\mathbb{T}_-$ is zero, and thereby leads to the uniqueness of decompositions expressed in Proposition 31 and elsewhere. To achieve the same end, $o(x^{-m})$ or, equivalently, $O(x^{-(m+1)})$ could have been used for other values of $m \geq 1$. Our convention explains the choice we adopt in the sequel of writing expressions of the form e^{kx} with $k > 0$ as $(e^{-x})^{-k}$, *as well as the fact that at times we have negative indices in sums* (see, for example, Definition 29).

The condition $\lambda_i > 0$ ensures that there is no infinite ascending chain of terms with respect to the asymptotic order relation. The form expressed by Equation (20) is the most general type of log-free transseries occurring in usual applications.

Note 25. With $\mathbb{R}^n$ replaced by $\mathbb{C}^n$, Equation (20) represents the most general transseries solution of generic, normalized, nonlinear systems of meromorphic ODEs. For such systems, $c_{\mathbf{k},l}$ are vectors, a generalization that can be easily dealt with. On the other hand, allowing for complex coefficients would pose various technical problems in our setting which we prefer to avoid. The term "normalized" refers to the fact that the exponentials are of the form e^{-ax}, that is, the exponents are linear in x. Had we started with e^{-ax^b}, $b \neq 1$, we would *normalize* the transseries by changing the variable to $t = x^b$ (also see Note 49); it can be shown that t thus defined *coincides* with the Écalle critical time introduced in Definition 15.

Proposition 26. *The linear combination and multiplication of two transseries $\tilde{T}^{(1)}$ and $\tilde{T}^{(2)}$ are defined as follows:*

$$a^{(1)}\tilde{T}^{(1)} + a^{(2)}\tilde{T}^{(2)} = \sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} (a^{(1)}\tilde{y}^{(1)}_{\mathbf{k}}(x) + a^{(2)}\tilde{y}^{(2)}_{\mathbf{k}}(x))$$

where $a^{(1)}$ and $a^{(2)}$ are real numbers.

$$\tilde{T}^{(1)}\tilde{T}^{(2)} = \sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} \sum_{\mathbf{j}=0}^{\mathbf{k}} \tilde{y}^{(1)}_{\mathbf{j}}(x)\tilde{y}^{(2)}_{\mathbf{k}-\mathbf{j}}(x). \tag{21}$$

[11] Equation (20) depicts a simple case of a level one transseries, also referred to as a *mixed series*.

With respect to these operations, $\mathbb{T}_-$ is a commutative algebra.

Proof. Straightforward verification. □

Repeated antidifferentiation of elements in $\mathbb{T}_-$ results in polynomials combined with logs which generate the space $\mathbb{T}_\ell$ below, which needs to be adjoined to our construction.

Definition 27 *(The Space $\mathbb{T}_\ell$).* Let $\mathbb{T}_\ell$ be the space of transseries of the form

$$\tilde{T} = \sum_{k=0}^{n} c_k \mathsf{A}_{\mathbb{T}}^k(1/x) + R, \tag{22}$$

where $n \in \mathbb{N} \cup \{0\}$, $c_k \in \mathbb{R}$ $(k = 0, ..., n)$, $\mathsf{A}_{\mathbb{T}}^k(1/x)$ is the kth antiderivative without constant term of $1/x$, and R is a polynomial of $1/x$ without constant term in $1/x$.

Proposition 28. $\mathbb{T}_\ell$ is a space of functions that coincide with their transseries, and is closed under $\mathsf{A}_{\mathbb{T}}$ (see Definition 27). Moreover, each element of $\mathbb{T}_\ell$ can be written uniquely in the form

$$\tilde{T}_\ell = P(x)\log + Q(x) + R(x)$$

where P, Q and R are polynomials and R has no constant term.

Proof. Straightforward: all these are elementary functions. □

Definition 29 *(The Space $\mathbb{T}_+$).* For $j \in \{-M, ..., -1\}$, let the λ_j be a descending sequence of positive reals and let the β_j and the $c_{j,l}$ be arbitrary sequences of reals. Subject to these conditions, further let

$$\tilde{T}_+ = \sum_{-M \le j \le -1;\ l \ge 1} c_{j,l} x^{\beta_j} e^{\lambda_j x} x^{-l} = \sum_{j=-M}^{-1} x^{\beta_j} e^{\lambda_j x} \tilde{y}_j(x), \tag{23}$$

where the $\tilde{y}_j$ are formal power series in powers of $1/x$ satisfying $\tilde{y}_j = O(1/x)$ (as is implied by the expanded form of $\tilde{T}$ in the middle term in Equation (23)). We denote the space of all transseries of type $\tilde{T}_+$ by $\mathbb{T}_+$.

Comment. In the rightmost expression in Equation (23), integer powers of x can be traded between $\tilde{y_j}$ and x^{β_j}, an ambiguity which is immaterial as the middle term in (23) shows.

Definition 30 *(The space $\mathbb{T}_1$ of height one, depth one transseries).* Employing the notations from Definitions 24, 27 and 29, henceforth we denote by $\mathbb{T}_1$ the space $\mathbb{T}_+ \oplus \mathbb{T}_\ell \oplus \mathbb{T}_-$.

Proposition 31. Every $\tilde{T} \in \mathbb{T}_1$ can be written uniquely in the form $\tilde{T} = \tilde{T}_+ + \tilde{T}_\ell + \tilde{T}_-$ where $\tilde{T}_+ \in \mathbb{T}_+$, $\tilde{T}_\ell \in \mathbb{T}_\ell$ and $\tilde{T}_- \in \mathbb{T}_-$.

Proof. This follows from Definition 30, the descendingly well ordering of the component terms of a transseries, and the definitions of the three subspaces, the latter of which collectively imply $\tilde{T}_+ \cap \tilde{T}_\ell = \tilde{T}_\ell \cap \tilde{T}_- = \tilde{T}_+ \cap \tilde{T}_- = \{0\}$ and $\tilde{T}_+ \gg \tilde{T}_\ell \gg \tilde{T}_-$ whenever these component transseries are nonzero. □

Differentiation. It is easy to verify that, if $\tilde{y}$ is a power series, then

$$\left(x^\beta e^{-\lambda x}\tilde{y}\right)' = x^\beta e^{-\lambda x}\left[(\beta x^{-1} - \lambda)\tilde{y} + \tilde{y}'\right] \tag{24}$$

where $\tilde{y}'$ is the termwise differentiation of $\tilde{y}$.

Note 32. The right side of Equation (24) is negative since $\tilde{y}$ is a series with positive coefficients and, as is the case with any asymptotic power series, $\tilde{y}' \ll \tilde{y}$.

Definition 33. Differentiation for the $\mathbb{T}_\ell$ component is termwise differentiation of the constituent monomials; see also Proposition 28. For the other two components, it is defined as termwise differentiation, namely,

$$\begin{aligned}
&\Big(\sum_{j=-M}^{-1} x^{\beta_j} e^{\lambda_j x}\tilde{y}_j(x) + \sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\tilde{y}_{\mathbf{k}}(x)\Big)' \\
&\qquad = \sum_{j=-M}^{-1}\left(x^{\beta_j} e^{\lambda_j x}\tilde{y}_j(x)\right)' + \sum_{\mathbf{k}\geq 0}\left(x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\tilde{y}_{\mathbf{k}}(x)\right)' \\
&\qquad\qquad = \sum_{j=-M}^{-1} x^{\beta_j} e^{\lambda_j x}\left[(\beta_j x^{-1} + \lambda_j)\tilde{y}_j + \tilde{y}_j{}'\right] \\
&\qquad\qquad\qquad + \sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\left[(\boldsymbol{\beta}\cdot\mathbf{k}x^{-1} - \mathbf{k}\cdot\boldsymbol{\lambda})\tilde{y} + \tilde{y}'_{\mathbf{k}}(x)\right].
\end{aligned} \tag{25}$$

Differentiation of an element of $\mathbb{T}_1$ is defined as the sum of the derivatives of its $+, \ell$ and $-$ components.

The infinite sums in Equation (25) converge in the transserries topology; for a proof see [22].

5.5.1. The definition of the operator $\mathsf{A}_\mathbb{T}$ on $\mathbb{T}_-$

We first define $\mathsf{A}_\mathbb{T}$ on the individual components of the transseries, namely on $t_\mathbf{k} = x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\tilde{y}_\mathbf{k}(x)$ and on $t = x^{\beta_j} e^{\lambda_j x}\tilde{y}_j(x), j \in \{-M, ..., -1\}$. To this end, we solve, in transseries, the ODE $\tilde{v}' = t$. The terms $t_\mathbf{k}$ and t are treated very similarly, and we analyze

only $t_{\mathbf{k}}$. If $\mathbf{k}=0$ and $\tilde{y}_0=\sum_{l\geq 2}c_l x^{-l}$, then $\tilde{v}=-\sum_{l\geq 2}(l-1)^{-1}c_l x^{-l+1}$. If $\mathbf{k}\neq 0$ then the substitution $\tilde{v}_{\mathbf{k}}(x)=x^{\boldsymbol{\beta}\cdot\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}w(x)$ in the ODE

$$\tilde{v}'_{\mathbf{k}}=t_{\mathbf{k}} \tag{26}$$

brings it to the form

$$w'-\mathbf{k}\cdot(\boldsymbol{\lambda}-\boldsymbol{\beta}x^{-1})w=y, \tag{27}$$

which has the power series solution $w(x)=\sum_{j\geq 1}w_j x^{-j}$, where the coefficients w_j are uniquely determined by the recurrence relation

$$\mathbf{k}\cdot\boldsymbol{\lambda}c_{j+1}-(\mathbf{k}\cdot\boldsymbol{\beta}-j)c_j=-\tilde{y}_{\mathbf{k},j};\quad c_1=\frac{\tilde{y}_{\mathbf{k},1}}{\mathbf{k}\cdot\boldsymbol{\lambda}}. \tag{28}$$

Next, we define

$$\mathsf{A}_{\mathbb{T}}\left(x^{\boldsymbol{\beta}\cdot\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\tilde{y}_{\mathbf{k}}(x)\right)=x^{\boldsymbol{\beta}\cdot\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}w(x), \tag{29}$$

where $w(x)$ is characterized as above.

5.5.2. The definition of the operator $\mathsf{A}_{\mathbb{T}}$ on $\mathbb{T}_+$

To define $\mathsf{A}_{\mathbb{T}}\left(x^{\beta_j}e^{\lambda_j}\tilde{y}_j(x)\right)$ we proceed as in §5.5.1: we write a differential equation $x^{\beta_j}e^{\lambda_j}\tilde{w}_j(x)=x^{\beta_j}e^{\lambda_j}\tilde{y}_j(x)$, and obtain

$$\left(\frac{\beta_j}{x}+\lambda_j\right)\tilde{w}_j(x)+\frac{d}{dx}\tilde{w}_j(x)-\tilde{y}_j(x)=0.$$

Writing $\tilde{y}_j=\sum_{j=1}^{\infty}d_j x^{-j}$, the coefficients $\{c_m\}_{m\in\mathbb{N}}$ of the power series $\tilde{w}_j$ satisfy the recurrence relation $c_m=\lambda_j^{-1}\left[d_m+(m-1-\beta_j)c_{m-1}\right];\ m\geq 1; c_0=0$. This shows existence and uniqueness of a solution with zero constant term.

Using Proposition 28 and the results in §5.5.1 and §5.5.2 we now extend antidifferentiation to $\mathbb{T}_1$.

Definition 34 *(Definition $\mathsf{A}_{\mathbb{T}}$ on $\mathbb{T}_1$).* $\mathsf{A}_{\mathbb{T}}$ is defined by linearity on $\mathbb{T}_1=\mathbb{T}_+\oplus\mathbb{T}_\ell\oplus\mathbb{T}_-$, by writing

$$\mathsf{A}_{\mathbb{T}}\tilde{T}:=\sum_{j=-M}^{-1}\mathsf{A}_{\mathbb{T}}\left(x^{\beta_j}e^{\lambda_j}\tilde{y}_j(x)\right)+\mathsf{A}_{\mathbb{T}}\tilde{T}_\ell+\sum_{\mathbf{k}\geq 0}\mathsf{A}_{\mathbb{T}}\left(x^{\boldsymbol{\beta}\cdot\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\tilde{y}_{\mathbf{k}}(x)\right), \tag{30}$$

The infinite sum defined above in Equation (30) converges in the transseries topology; a general proof is provided in [22]. The derivative and the antiderivative are inverses of each other.

Proposition 35. Replacing the functions with elements of $\mathbb{T}_1$ everywhere in Definition 13, the operator $\mathsf{A}_{\mathbb{T}}$ restricted to $\mathbb{T}_1$ satisfies the properties i–iv and vi listed there.

Proof. The proof is a straightforward verification. □

5.6. Watson's lemma

The following classical result is essential in determining the asymptotic behavior of Laplace transforms.

Lemma 36 *(Watson's Lemma (see, e.g., [22], p. 31)).* Assume that F is locally integrable and exponentially bounded on $\mathbb{R}^+$, $a, b > 0$ and $F(p) \sim \sum_{k=0}^{\infty} f_k p^{ka+b}$ for small $p > 0$. Then

$$\int_0^{\infty} e^{-xp} F(p) dp \sim \sum_{k\geq 0} \frac{f_k \Gamma(ka+b+1)}{x^{ka+b+1}} \quad \text{as } x \to \infty. \tag{31}$$

5.7. Borel summable subspaces of transseries

Definition 37 *(The Borel summable subspace $\mathbb{T}_{\mathcal{B}}$ of $\mathbb{T}_1$).* We say that a transseries is *Borel-summable* if all power series $\tilde{y}_{\mathbf{k}}$ and $\tilde{y}'_j(x)$ in Equation (20) are Borel summable and there are positive constants c_1, c_2, c_3 (which may depend on $\tilde{T}$) such that for all $\mathbf{k}$ and $p \in \mathbb{R}^+$ we have

$$|(\mathcal{B}\tilde{y}_{\mathbf{k}})(p)| \leq c_1 c_2^{|\mathbf{k}|} e^{c_3 p} \text{ and } |(\mathcal{B}\tilde{y}_j)(p)| \leq c_1 e^{c_3 p}. \tag{32}$$

In view of the summability results we rely on in the sequel, we impose the *nonresonance* condition

$$(\mathbf{k}-\mathbf{k}')\cdot\boldsymbol{\lambda}+\lambda_i-\lambda_j = 0 \text{ if and only if } \mathbf{k}-\mathbf{k}' = 0 \text{ and } i = j \text{ for } i,j \in \{-M,...,-1\}; \tag{33}$$

that is, the condition that linear combinations of the exponents with integer coefficients permitted by our assumptions can only vanish trivially.

Henceforth, by $\mathbb{T}_{\mathcal{B}}$ we mean the subspace of $\mathbb{T}_1$ whose members are Borel summable. We write $\mathbb{T}_{+,\mathcal{B}}, \mathbb{T}_{-,\mathcal{B}}$ for the Borel summable subspaces of $\mathbb{T}_+, \mathbb{T}_-$. By Proposition 28 (a), we may identify $\mathbb{T}_{\ell,\mathcal{B}}$ with $\mathbb{T}_\ell$ and write $\mathcal{L}\mathcal{B}\tilde{T}_\ell = \tilde{T}_\ell$.

For clarity of notation we do not follow the multiindex convention, but, instead, by $|\mathbf{k}|$ we mean $\sqrt{k_1^2 + \cdots + k_n^2}$.

Note 38.

(a) The assumption that all power series $\tilde{y}_{\mathbf{k}}$ in Equation (20) are Borel summable does **not** hold, generically, for nonlinear systems of ODEs. Instead, these series are *resurgent*, a case we study in the next section.
(b) Using (33) we have the linear ordering $\mathbf{k}_1 > \mathbf{k}_2$ if and only if $\mathbf{k}_1\boldsymbol{\lambda} > \mathbf{k}_2\boldsymbol{\lambda}$. By the discussion at the beginning of §5.2, and assuming the formal power series below are nonzero, we have: if $\lambda_1 > \lambda_2$, then $x^{\beta_1}e^{\lambda_1}\tilde{y}_1(x) \gg x^{\beta_2}e^{\lambda_2}\tilde{y}_2(x)$, and if $\mathbf{k}_1 > \mathbf{k}_2$, then $x^{\boldsymbol{\beta}\cdot\mathbf{k}_1}e^{-\mathbf{k}_1\cdot\boldsymbol{\lambda}x}\tilde{y}_{\mathbf{k}_1}(x) \ll x^{\boldsymbol{\beta}\cdot\mathbf{k}_2}e^{-\mathbf{k}_2\cdot\boldsymbol{\lambda}x}\tilde{y}_{\mathbf{k}_2}(x)$.

Proposition 39.

(a) *There exist positive constants c_1, c_2, c_3 such that for all $x > r > c_3$, all $j < 0$, and all $\mathbf{k}$ we have*

$$|(\mathcal{LB}\tilde{y}_{\mathbf{k}})(x)| \le c_1 c_2^{|\mathbf{k}|}(x-c_3)^{-1} \text{ and } |(\mathcal{LB}\tilde{y}_j)(x)| \le c_1(x-c_3)^{-1} \tag{34}$$

Moreover, if $c \neq 0$ and $\tilde{y} = cx^{-m}(1+o(1))$, then $\mathcal{LB}[e^{\lambda x}x^{\beta}\tilde{y}] = cx^{-m}e^{\lambda x}x^{\beta}(1+o(1))$, for some $c \in \mathbb{R}^+$.
(b) *Let $\lambda_1, ..., \lambda_n \in \mathbb{R}^+$, $\beta_1, ..., \beta_n \in \mathbb{R}$, $\lambda = \min\{\lambda_1, ..., \lambda_n\}$ and $\beta = \max\{\beta_1, ..., \beta_n\}$. Also let x_0 be such that for all $x > x_0$ we have $c_3 e^{-\lambda x}x^{\beta} < 1$. Then, for all $x > \max\{c_3, x_0\}$ we have*

$$\sum_{\mathbf{k}>M} x^{\boldsymbol{\beta}\cdot\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\,|\mathcal{LB}\tilde{y}_{\mathbf{k}}| \le c_1(c_2x^{\beta}e^{-\lambda x})^{NM}\frac{1}{(1-c_2x^{\beta}e^{-\lambda x})^N}(x-c_3)^{-1}, \tag{35}$$

where $N \in \mathbb{N}$, $M \in \mathbb{Z}$. In particular, the infinite sum converges uniformly and absolutely (in the analytic sense) for $x > x_0$ if x_0 satisfies $c_2x_0^{\beta}e^{-\lambda x_0} < 1$.
(c) *If, in addition, $M' \le M$ and $\beta = \min\{\beta_0, ..., \beta_{M'}\}$,*

$$\sum_{j=-M'}^{-1} x^{\beta_j}e^{\lambda_j x}|\mathcal{LB}\tilde{y}_j(x)| \le (M'+1)c_1e^{-\lambda_{M'}x}x^{\beta}(x-c_3)^{-1}. \tag{36}$$

(d) *$\mathbb{T}_{-,\mathcal{B}}$ is an algebra, i.e., if $\tilde{T}^{(1)}$ and $\tilde{T}^{(2)}$ are elements of $\mathbb{T}_{-,\mathcal{B}}$ and $a^{(1)}, a^{(2)} \in \mathbb{R}$, then so are $a^{(1)}\tilde{T}^{(1)} + a^{(2)}\tilde{T}^{(2)}$ and $\tilde{T}^{(1)}\tilde{T}^{(2)}$.*

Proof. For the first part of (a) we simply note that, by assumption, $|\mathcal{LB}\tilde{y}_{\mathbf{k}}| \le c_1c_2^{|\mathbf{k}|}\mathcal{L}(e^{c_3p}) = c_1c_2^{|\mathbf{k}|}(x-c_3)^{-1}$, while the second part follows from Proposition 16 and Lemma 36. For (b), by assumption and using (a), we majorize each term in the infinite sum in (35) by $c_1(c_2x^{\beta}e^{-\lambda})^{k_1+...+k_N}(x-c_3)^{-1}$ (the terms of a geometric series) and the result follows. The proof of (c) is similar and, in fact, simpler: we estimate a finite sum in terms of its largest term.

For (d), if the constants in the bounds expressed in (32) for $\tilde{T}^{(i)}, i = 1, 2$ are $c_k^{(i)}$, where $k = 1, 2, 3$ and $i = 1, 2$, and if by c_k we denote $\max\{c_k^{(1)}, c_k^{(2)}\}$, then a bound of the

type (32) for $\tilde{T}^{(1)} + \tilde{T}^{(2)}$ is $(c_1(|a^{(1)}| + |a^{(2)}|)c_2^{|\mathbf{k}|}e^{c_3 p}$. By linearity and the polarization identity $2ab = (a+b)^2 - a^2 - b^2$, for the product it is enough to show that $\mathbb{T}_{\mathcal{B}}$ is closed under squaring. If $\tilde{y}$ satisfies $|\mathcal{B}\tilde{y}| \leq e^{cp}$ for $p \in \mathbb{R}$, then (15) implies $\mathcal{B}|\tilde{y}^2| \leq B|\tilde{y}| * B|\tilde{y}| \leq pe^{cp} \leq e^{(c+1)p}$. Then, using this inequality and estimating the number of terms in the innermost sum in Equation (21) by the rough bound $|\mathbf{k}|^N$ and using $|\mathbf{k}|^N \leq e^{N|\mathbf{k}|}$, we get

$$\left| \mathcal{B} \sum_{\mathbf{j}=0}^{\mathbf{k}} \tilde{y}_{\mathbf{j}}(x)\tilde{y}_{\mathbf{k}-\mathbf{j}}(x) \right| \leq c_1^2 e^{(c_3+1)p}(c_2^2 e^N)^{|\mathbf{k}|},$$

from which the result follows. □

Definition 40. Let $\tilde{T} \in \mathbb{T}_{\mathcal{B}}$ and let λ, β and x_0 be as in Proposition 39. Then the Borel sum of $\tilde{T}$ is defined as

$$\mathcal{L}\mathcal{B}\tilde{T} = \sum_{j=-M}^{-1} x^{\beta_j} e^{\lambda_j x} \mathcal{L}\mathcal{B}\tilde{y}_j + \sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}_j x} \mathcal{L}\mathcal{B}\tilde{y}_{\mathbf{k}} + \tilde{T}_\ell, \tag{37}$$

(see Proposition 28 and Definition 37). We note that, by (35) and the Weierstrass M-test, the infinite series in Equation (37) converges uniformly and absolutely (in the analytic sense of convergence) on the interval $[x_0, \infty)$.

Proposition 41.

(a) If $\tilde{T} \in \mathbb{T}_{\mathcal{B}}$, then $\tilde{T} > 0$ if and only if $\mathcal{L}\mathcal{B}\tilde{T} > 0$ for sufficiently large x.
(b) The kernel of $\mathcal{L}\mathcal{B}$ is zero, i.e., $\{\tilde{T} \in \mathbb{T}_{\mathcal{B}} : \mathcal{L}\mathcal{B}\tilde{T} = 0\} = \{0\}$.

Proof. (a). If $\tilde{y}_j \neq 0$ for some $j < 0$, then we choose the most negative j with this property, and for $\tilde{y}_j$ to be nonzero there must exist an $m \in \mathbb{N}$ and a nonzero $c \in \mathbb{C}$ such that $\tilde{y}_j = cx^{-m}(1+o(1))$. Using Proposition 39 (a), we see that $\mathcal{L}\mathcal{B}\tilde{T} = ce^{\lambda_j x}x^{\beta_j}x^{-m}(1+o(1))$ and the result follows. The proof is very similar if instead $\tilde{y}_j = 0$ for all $j < 0$ and P or Q is nonzero, or, if P and Q are also zero and for some $\mathbf{k}$ we have $\tilde{y}_{\mathbf{k}} \neq 0$.

(b) follows immediately from (a). □

Definition 42. Let $\mathcal{F}_{\mathcal{B}}$ denote the function space $\{\mathcal{L}\mathcal{B}\tilde{T} : \tilde{T} \in \mathbb{T}_{\mathcal{B}}\}$. Mimicking the notation in Definition 30 and Proposition 31, we write: $\mathcal{F}_{\mathcal{B}} = \mathcal{F}_{+,\mathcal{B}} \oplus \mathcal{F}_\ell \oplus \mathcal{F}_{-,\mathcal{B}}$, with $\mathcal{F}_{+,\mathcal{B}}$, $\mathcal{F}_\ell$ and $\mathcal{F}_{-,\mathcal{B}}$ understood in the expected manner.

Note that, by Definition 28, Borel summability is the identity on $\mathcal{F}_\ell$, and as such we could have written equivalently $\mathcal{F}_{\ell,\mathcal{B}}$ in place of $\mathcal{F}_\ell$.

Corollary 43.

(a) *$\mathcal{LB}$ is a bijection between $\mathbb{T}_{\mathcal{B}}$ and $\mathcal{F}_{\mathcal{B}}$.*
(b) *$\mathcal{LB}$ (restricted to $\mathbb{T}_{-,\mathcal{B}}$) is a linear and multiplicative bijection from $\mathbb{T}_{-,\mathcal{B}}$ to $\mathcal{F}_{-,\mathcal{B}}$.*

Proof. (a) is an immediate consequence of Proposition 41 and Definition 42. For (b), bijectivity follows from (a) and the definition of $\mathcal{LB}$; and linearity and multiplicativity follow from the fact that $\mathbb{T}_{-,\mathcal{B}}$ is closed under addition and multiplication and a straightforward calculation using the definition of $\mathcal{LB}$ and the fact that $\mathcal{LB}$ is linear and multiplicative on $\mathcal{S}_{\mathcal{B}}$. □

Note that in virtue of Proposition 31 and Corollary 43 the decomposition in Definition 42 is unique.

5.8. Differentiation and antidifferentiation on $\mathbb{T}_{\mathcal{B}}$

Lemma 44 *(Differentiation). If $\tilde{y}$ is a Borel summable formal power series, then*

(1) $\left(e^{-\lambda x}x^{\beta}\mathcal{LB}\tilde{y}\right)' = e^{-\lambda x}x^{\beta}\mathcal{LB}\dot{\tilde{y}}$, *where*

$$\dot{\tilde{y}} = (\beta x^{-1} - \lambda)\tilde{y} + \tilde{y}' \text{ and } \mathcal{B}\dot{\tilde{y}} = \beta\left(\int_0^p \mathcal{B}\tilde{y}\right) - \lambda\mathcal{B}\tilde{y} - p\mathcal{B}\tilde{y}. \tag{38}$$

(2) *Let $\tilde{y}_{\mathbf{k}}$ be as in Definition 37. Then, for some constants c_1', c_2', c_3' depending only on c_1, c_2, c_3, and all $p \in \mathbb{R}^+$ and $\mathbf{k}$ we have*

$$\left|(\mathcal{B}\dot{\tilde{y}}_{\mathbf{k}})(p)\right| \leq c_1' c_2'^{|\mathbf{k}|} e^{c_3' p}. \tag{39}$$

(3) *The sum*

$$\sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\mathcal{LB}\dot{\tilde{y}}_{\mathbf{k}} \tag{40}$$

converges uniformly and absolutely in the analytic sense, for large x, and with $\tilde{T}$ as in Definition 40 we have

$$(\mathcal{LB}\tilde{T})' = \sum_{j=-M}^{-1} x^{\beta_j} e^{\lambda_j x}\mathcal{LB}\dot{\tilde{y}}_j + \sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\mathcal{LB}\dot{\tilde{y}}_{\mathbf{k}} + \tilde{T}_{\ell}' = \mathcal{LB}(\tilde{T})'. \tag{41}$$

Proof. The fact that $\dot{\tilde{y}}$ is given by the first equation in (38) follows from the isomorphisms induced by Borel summation (see Proposition 16), *viz*:

$$\left(e^{-\lambda x}x^{\beta}\mathcal{L}\mathcal{B}\tilde{y}\right)' = e^{-\lambda x}x^{\beta}\left[\beta x^{-1}\mathcal{L}\mathcal{B}\tilde{y} - \lambda\mathcal{L}\mathcal{B}\tilde{y} + \mathcal{L}\mathcal{B}(\tilde{y}')\right]$$
$$= e^{-x}x^{\beta}\mathcal{L}\left[\beta\left(\int_0^p \mathcal{B}\tilde{y}\right)\lambda\mathcal{B}\tilde{y} - p\mathcal{B}\tilde{y}\right]. \quad (42)$$

For part (2), we note that $\int_0^p |\mathcal{B}y_{\mathbf{k}}| \le e^{c_3 p}\int_0^p 1 = pe^{c_3 p} \le e^{(c_3+1)p}$. The absolute value of the term $-p\mathcal{B}\tilde{y}$ is also bounded by $p|\mathcal{B}y_{\mathbf{k}}| \le e^{(c_3+1)p}$. Next, $|\boldsymbol{\beta}||\mathbf{k}|+|\mathbf{k}||\boldsymbol{\lambda}| \le \exp[|\mathbf{k}|(|\boldsymbol{\beta}|+|\boldsymbol{\lambda}|)]$, and so the result follows. Using (2), uniform and absolute convergence, in the analytic sense, are shown as in the first sentence of the proof of Proposition 39. The rest follows from an elementary theorem about sequences of functions [45, p. 321]. (The estimates above can be improved substantially, but we do not need this here.) □

The Corollary below is an immediate consequence of Lemma 44 and Corollary 43.

Corollary 45 *(Preservation of differentiation).* The space $\mathbb{T}_{\mathcal{B}}$ is closed under differentiation and, for $\tilde{T} \in \mathbb{T}_{\mathcal{B}}$, we have $(\mathcal{L}\mathcal{B}\tilde{T})' = \mathcal{L}\mathcal{B}(\tilde{T})'$; $\mathcal{L}\mathcal{B}$ is a differential space isomorphism. Restricted to $\mathbb{T}_{-,\mathcal{B}}$, $\mathcal{L}\mathcal{B}$ is a differential algebra isomorphism.

In the following definition we extend the operator $\mathsf{A}_{\mathcal{B}}$ of Definition 18 and Proposition 19 to include $\mathcal{F}_{\ell}$ and $\mathcal{F}_{-,\mathcal{B}}$.

Definition 46. For $\tilde{T} \in \mathbb{T}_{\ell} \oplus \mathbb{T}_{-,\mathcal{B}}$, $\mathsf{A}_{\mathbb{T}_{\mathcal{B}}}(\mathcal{L}\mathcal{B}\tilde{T}) := \mathcal{L}\mathcal{B}(\mathsf{A}_{\mathbb{T}}\tilde{T})$.

As the next proposition shows, $\mathsf{A}_{\mathbb{T}_{\mathcal{B}}}$ is well-defined on the image of $\mathbb{T}_{\ell} \oplus \mathbb{T}_{-,\mathcal{B}}$ under $\mathcal{L}\mathcal{B}$, that is, on $\mathcal{F}_{\ell} \oplus \mathcal{F}_{-,\mathcal{B}}$, and takes values in $\mathcal{F}_{\ell} \oplus \mathcal{F}_{-,\mathcal{B}}$.

Proposition 47 *(Antidifferentiation).*

(a) The operator $\mathsf{A}_{\mathbb{T}_{\mathcal{B}}} : \mathcal{F}_{\ell} \oplus \mathcal{F}_{-,\mathcal{B}} \to \mathcal{F}_{\ell} \oplus \mathcal{F}_{-,\mathcal{B}}$ is well defined.
(b) $\mathsf{A}_{\mathbb{T}_{\mathcal{B}}}$ satisfies Properties i–iv and vi of Definition 13.
(c) The space $\mathcal{F}_{\ell} \oplus \mathcal{F}_{-,\mathcal{B}}$ is closed under differentiation and antidifferentiation.

Proof. Clearly, we only need to check the statement on $\mathcal{F}_{-,\mathcal{B}}$. For $\mathbf{k} = 0$, (a) and (b) follow from Definition 18 and Proposition 19. Next, we show (a) for a term of the form $x^{\boldsymbol{\beta}\cdot\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}_j x}\mathcal{L}\mathcal{B}\tilde{y}_{\mathbf{k}}$ with $\mathbf{k} \neq 0$. Using the results in §5.5.1, we need to prove that the solution of the ODE (28) is Borel summable with bounds as in Definition 37. These bounds are needed to prove absolute and uniform convergence of the resulting infinite series, as in Proposition 39.

Taking the Borel transform of (27) and letting $W = \mathcal{B}w$ and $Y = \mathcal{B}y_{\mathbf{k}}$, we get

$$(\mathbf{k}\cdot\boldsymbol{\lambda} + p)W(p) = \mathbf{k}\cdot\boldsymbol{\beta}\int_0^p W(s)ds + Y(p). \quad (43)$$

After differentiation in p, we get a first order linear ODE that can be easily solved by quadratures. However, the estimates we need are more difficult to obtain from the explicit solution, and we use a different approach here. We rewrite Equation (43) in the form

$$W(p) = \frac{\mathbf{k}\cdot\boldsymbol{\beta}}{\mathbf{k}\cdot\boldsymbol{\lambda}+p}\int_0^p W(s)ds + \frac{Y(p)}{\mathbf{k}\cdot\boldsymbol{\lambda}+p}. \tag{44}$$

Choose now

$$c_3' > \sup\left\{\frac{|\mathbf{k}\cdot\boldsymbol{\beta}|}{\mathbf{k}\cdot\boldsymbol{\lambda}+1}, c_3; \mathbf{k}\geq 0\right\}. \tag{45}$$

Let $\mathcal{D}$ be the domain of analyticity of Y. It is easy to check that Equation (44) is contractive in the Banach space

$$\{f \text{ analytic in } \mathcal{D} : \|f\| < \infty, \text{ where } \|f\| = \sup_{p\in\mathcal{D}} e^{-c_3'|p|}\big|f(p)\big|\}.$$

It follows that the $\mathcal{LB}w_{\mathbf{k}}$ exist and satisfy the same estimates as the $\mathcal{LB}\tilde{y}_{\mathbf{k}}$ with the triple (c_1, c_2, c_3) replaced by (c_1, c_2, c_3') with c_3' as in (45), and (a) follows. Part (b) is a consequence of Corollary 45 and of the bounds in terms of (c_1, c_2, c_3') (obtained in (a)) which imply uniform and absolute convergence of the infinite series. And Part (c) is immediate, since $\mathcal{F}_\ell$ is closed under differentiation and antidifferentiation and, by the analysis above, so is $\mathcal{F}_\ell \oplus \mathcal{F}_{-,\mathcal{B}}$ where differentiation and antidifferentiation require switching a term of the form a/x^m between these two spaces.

We now turn to (b) for general $\mathbf{k}$. For Property i, first note that we have already shown that, for $\tilde{T} \in \mathbb{T}_\ell \oplus \mathbb{T}_{-,\mathcal{B}}$ the series through which we defined $\mathsf{A}_{\mathbb{T}_\mathcal{B}}\tilde{T}$ is uniformly and absolutely convergent (in the analytic sense). On the other hand, the series whose terms are the derivatives of the terms of $\mathsf{A}_{\mathbb{T}_\mathcal{B}}\tilde{T}$ converges uniformly and absolutely to $\mathcal{LB}\tilde{T}$, simply because these terms are, by construction, the terms of $\mathcal{LB}\tilde{T}$. The rest follows from the elementary theorem about sequences of functions [45, p. 321] referred to before. Property ii (i.e. linearity) is immediate. For Property iii (positivity), to understand the monotonicity of $\mathsf{A}_{\mathbb{T}_\mathcal{B}}\tilde{T}$, we appeal to Proposition 41 and Lemma 44 to conclude that we only have to examine the dominant term of the *transseries of the derivative* of $\mathsf{A}_{\mathbb{T}_\mathcal{B}}\tilde{T}$. Since by assumption $\tilde{T} > 0$, by the definition of positivity in §5.2, this dominant term is positive and the property follows. Property iv follows from Equation (37) by setting all $\tilde{y}_{\mathbf{k}} = 0$ if $\mathbf{k} \neq 0$, choosing $\beta_0 = n+1$ and $\tilde{y}_0 = 1/x$, and from the definitions of $\mathsf{A}_{\mathbb{T}_\mathcal{B}}$ and of $\mathsf{A}_{\mathbb{T}}$. Finally, for Property vi, let $F = \mathcal{LB}\tilde{T}$ and $f = \mathcal{LB}\tilde{t}$. Since $f = (\mathcal{LB}\tilde{T})'$, the rest follows from Definition 46 and Proposition 47. □

Combining the content of the preceding results in this section, we get:

Theorem 48. $\mathcal{LB}$ is an *isomorphism of commutative differential algebras* between $\mathbb{T}_{-,\mathcal{B}}$ and $\mathcal{F}_{-,\mathcal{B}}$. The space $\mathcal{F}_\ell \oplus \mathcal{F}_{-,\mathcal{B}}$ is closed under differentiation and antidifferentiation.

Note 49. As we mentioned already, to cover generic solutions of nonlinear ODEs we have to allow for more general than Borel summable series, namely *resurgent* ones. Furthermore, $\mathbb{T}_{+,\mathcal{B}}$ is not closed under antidifferentiation; resurgence tools are required to deal with this space.

6. Resurgent functions, resurgent transseries and Écalle-Borel summability: background

6.1. Background: Borel plane singularities along the Laplace transform path and the need to extend Borel summation

So far, antidifferentiation of a transseries $\tilde{T} \in \mathbb{T}_{\mathcal{B}}$ has been defined under the assumption $\tilde{T}_{+,\mathcal{B}} = 0$ (cf. Definition 46). This assumption is needed to ensure Borel summability, as it is manifest in the integral in Equation (56) below, involved in the Borel transform of the terms in $\tilde{T}_{+,\mathcal{B}}$. This condition excludes some very common functions encountered in applications such as $\mathrm{Ei}_a(x) = \int_a^x \frac{e^s}{s} ds$ (where the integral is understood as a Cauchy principal value if $a \in [-\infty, 0)$). Indeed, the transseries of $\mathrm{Ei}_a(x)$ is $e^x \tilde{y}(x) + C_a$ where C_a is a constant depending on the endpoint of integration; clearly, in this case, $\lambda = 1$ (see Definition 24), and classical Borel summation does not apply.

Furthermore, in transseries arising in applications, the points $\mathbf{k} \cdot \boldsymbol{\lambda}$ are singularities of the Borel transforms of $y_{\mathbf{k}}$. Hence, if $\lambda_j \in \mathbb{R}^+$, then a generalization of Borel summability is needed. The condition $\lambda_j \notin \mathbb{R}^+$ may appear to be generic; however, equations arising in applications typically have real coefficients in which case the numbers λ_i come in complex conjugate pairs and, more often than not, are purely real. For instance, for the tronquée solutions of all Painlevé equations $P_{\mathrm{I}} - P_{\mathrm{V}}$ in normalized (Boutroux) coordinates, the values of λ are $\lambda_{1,2} = \pm 1$.

Écalle introduced significant improvements over Borel summation to address such limitations. Among them are the concepts of *critical times* (see Definition 15) and *acceleration/deceleration* to deal with mixed powers of the factorial divergence. Last but not least, and the only additional ingredient we will need, is that of *averaging*.

In linear problems, to avoid the singularities of the integrand on $\mathbb{R}^+$ (when present), one can take the half-half average of the Laplace transforms above and below $\mathbb{R}^+$. On the other hand, in nonlinear equations such as nonlinear ODEs, the average of two solutions is not a solution. Écalle found constructive, universal averages which successfully replace the naive half-half averages mentioned above; however, it is altogether nontrivial to construct them and show that they work. Of these, we use the so-called *organic average*, **mon** [36,34], which is well suited for our general construction. Invoking Borel transforms followed by analytic continuation along paths avoiding singularities, followed by taking

the organic average of these continuations, and finally applying the Laplace transforms, yields a differential field algebra [34,59].

In the remainder of this section we provide an overview of the concepts of averaging and resurgence relevant to the discussion of resurgent functions, resurgent transseries and Écalle-Borel summability below.

6.2. Averages of Borel transforms (and more general functions in the Borel plane)

Assume that one of the singular directions of the Borel transform $Y = \mathcal{B}\tilde{y}$ of a normalized series $\tilde{y}$ is $\mathbb{R}^+$ with a discrete set of singularity locations ω_n, $n \in \mathbb{N}$.

Since in our discussion the only integration axis that comes into play is $\mathbb{R}^+$, henceforth we assume that one of the singular directions of the Borel transform of $\tilde{y}$ is $\mathbb{R}^+$, that $\omega_0 := 0$, and that for each $n \in \mathbb{N}$, ω_n is the nth singularity on $\mathbb{R}^+$, where ω_n increases with increasing n.

Now consider the class of all curves going forward (towards ∞) that circumvent the just-said singularities. One associates with each such curve a unique series $\epsilon_1, \epsilon_2, ..., \epsilon_n, ...$ such that for each $n \in \mathbb{N}$, $\epsilon_n \in \{+, -\}$ where $\epsilon_n = +$ (resp. $-$) indicates that the curve is in the lower (resp. upper) half plane between ω_{n-1} and ω_n. For instance, $+++---...$ describes a curve starting out in the lower half plane and crossing into and remaining in the upper half plane after the third singularity, ω_3.[12] For a point ζ in the open interval (ω_n, ω_{n+1}) along such a curve, the position vector $(\epsilon_1, ..., \epsilon_n)$ is called the *address of the point* ζ on the curve.

A *uniformizing average* or, more simply, an *average* $\mathbf{m} : Y \mapsto \mathbf{m}Y$ of a function Y with singularities at ω_i as described above is defined using a system of *weights* $\mathbf{m}$ which, in turn, is defined via the ω_ns and the ϵ_ns by the following:

$$\mathbf{m}Y(\zeta) = \sum_{\epsilon_1,...,\epsilon_n \in \{+,-\}^n} \mathbf{m}^{\binom{\epsilon_1,...,\epsilon_n}{\omega_1,...,\omega_n}} Y^{\binom{\epsilon_1,...,\epsilon_n}{\omega_1,...,\omega_n}}(\zeta) \quad if \quad \omega_n < \zeta < \omega_{n+1}, \tag{46}$$

where $Y^{\binom{\epsilon_1,...,\epsilon_n}{\omega_1,...,\omega_n}}(\zeta)$ denotes the determination of $Y(\zeta)$ on the open interval (ω_n, ω_{n+1}) along a curve circumventing the singularities as specified by the position vector $(\epsilon_1, ..., \epsilon_n)$ described above, and $\mathbf{m}^{\binom{\epsilon_1,...,\epsilon_n}{\omega_1,...,\omega_n}}$ is its corresponding weight, the definition of which depends on the choice of average in question.

There are many types of averages of Borel transforms to chose from. However, for an average[13] $\mathbf{m}$ to be *well-behaved* with respect to the goals of Écalle-Borel resummation, following Écalle and Menous ([36], [34, page 256]) there are four conditions it should satisfy:

[12] Here we follow Écalle's convention from [34], rather than the convention employed by the first author in [20].

[13] Here we refer to *continuous averages* as opposed to the subclass of discrete ones that assume a fixed (for instance periodic) lattice of singularities. Fixed lattice settings simplify the analysis when this analysis is restricted to one particular equation.

A1. **m** must respect convolution, i.e. $\mathbf{m}(Y * G) = (\mathbf{m}Y)(\mathbf{m}G)$;
A2. **m** must respect real-valuedness;
A3. **m** must respect lateral growth;
A4. **m** should be scale invariant.

A1 ensures that **m** is an algebra homomorphism; **A**2 ensures that $\mathbf{m}Y$ is real whenever Y is real; **A**3 ensures that exponential bounds, which are needed for the Laplace transform to apply, are maintained by averaging; and **A**4, which ensures invariance under homothetic rescalings $\zeta \mapsto const.\zeta$ of $\mathbb{R}^+$, while natural, is not an essential condition. On the other hand, we note that while the functions satisfying **A**4 form an algebra, and functions of "natural origin" possess it, such a condition relies on *resurgence* in some broad sense to hold,[14] and we refer the reader to [34] for an in-depth discussion of these issues.

6.3. Resurgent series and resurgent functions: definitions

Definition 50 *(Resurgent series and resurgent functions).*

(1) In this paper, a normalized series $\tilde{y}$ will be said to be *resurgent* if its Borel transform $Y = \mathcal{B}\tilde{y}$ is:
   (a) analytic or ramified analytic at $p = 0$;
   (b) endlessly continuable (in the sense that the singularities encountered by analytic continuation along any compact curve segment form a discrete set);
   (c) exponentially bounded in every nonsingular direction, while in singular directions Y is in the domain of a well-behaved average.
(2) Sharing the appellation, $Y = \mathcal{B}\tilde{y}$ is called *resurgent* in the Borel plane or simply resurgent, and $y = \mathcal{L}Y$ is called *resurgent* in the physical plane or simply resurgent, if $\tilde{y}$ is resurgent.
(3) A transseries in $\mathbb{T}_+ \oplus \mathbb{T}_\ell \oplus \mathbb{T}_-$ is resurgent if: (a) all component series $\tilde{y}_{\mathbf{k}}(x)$ and $\tilde{y}_j(x)$ (see (20) and (23)) are resurgent; (b) their Borel transforms are exponentially bounded in every nonsingular direction in a $\mathbf{k}, j$-independent way; and, (c) in singular directions, these Borel transforms are in the domain of a well-behaved average.
(4) We denote by $\mathbb{T}_\mathcal{R}$ the space of resurgent transseries in $\mathbb{T}_+ \oplus \mathbb{T}_\ell \oplus \mathbb{T}_-$. Accordingly, $\mathbb{T}_\mathcal{R} = \mathbb{T}_{+,\mathcal{R}} \oplus \mathbb{T}_\ell \oplus \mathbb{T}_{-,\mathcal{R}}$, where $\mathbb{T}_{+,\mathcal{R}}$ denotes the space of resurgent transseries in $\mathbb{T}_+$ and $\mathbb{T}_{-,\mathcal{R}}$ denotes the space of resurgent transseries in $\mathbb{T}_-$. (In virtue of Proposition 28, the members of $\mathbb{T}_\ell$ are necessarily resurgent, and so it would be superfluous to write $\mathbb{T}_{\ell,\mathcal{R}}$ in the decomposition of $\mathbb{T}_\mathcal{R}$.)

[14] Indeed, one can use the uniformization theorem to see that bounds on the "first" Riemann sheet do not constrain growth on other sheets.

6.4. The well-behaved average **mon**

Condition (1)(c) of Definition 50 appeals to the notion of a well-behaved average. There is in fact an entire continuum of such averages [34]. As we mentioned above, for our treatment we adopt the average Écalle has dubbed the "organic average", i.e. the average **mon**, given by

$$\mathbf{mon}^{\binom{\epsilon_1,\dots,\epsilon_n}{\omega_1,\dots,\omega_n}} := 2^{-n}\prod_{i=2}^{n}\left(|\epsilon_{i-1}+\epsilon_i| - \frac{\epsilon_{i-1}\epsilon_i\omega_i}{\omega_1+\cdots+\omega_i}\right),$$

where, on the right side of the equation, it is understood that $|\epsilon_{i-1}+\epsilon_i| = 2$ (resp. 0) if $\epsilon_{i-1} = \epsilon_i$ (resp. $\epsilon_{i-1} \neq \epsilon_i$)[15] and the ω_i's are the members of $\mathbb{R}^+$ defined as above [34, page 272].

Alternatively, **mon** can be defined by recursion ([36, pages 86-87], [34, page 272]), where $\mathbf{mon}^{\binom{\epsilon_1}{\omega_1}} := \frac{1}{2}$, and for each $n > 1$,

$$\mathbf{mon}^{\binom{\epsilon_1,\dots,\epsilon_n}{\omega_1,\dots,\omega_n}} := \mathbf{mon}^{\binom{\epsilon_1,\dots,\epsilon_{n-1}}{\omega_1,\dots,\omega_{n-1}}} P_n,$$

with

$$P_n := 1 - \frac{1}{2}\frac{\omega_n}{\omega_1+\cdots+\omega_n} \quad if \quad \epsilon_{n-1} = \epsilon_n$$

and

$$P_n := \frac{1}{2}\frac{\omega_n}{\omega_1+\cdots+\omega_n} \quad if \quad \epsilon_{n-1} \neq \epsilon_n.$$

In addition to being arguably the simplest of the well-behaved averages, **mon** is distinguished by being the lower limit of such averages [34, page 272] (also see [59]). It is important to note, however, that when restricted to the class of functions that we are concerned with in this paper, because of nonresonance[16] all the well-behaved averages coincide (see Section 8.4).

7. Resurgent functions, resurgent transseries and Écalle-Borel summability

In this section we establish results that we need in §8 to determine the correspondence between the class $\mathcal{F}_{\mathcal{R}}$ of resurgent functions that we referred to in the introduction and their corresponding classes of transseries and surreal functions.

[15] In other words, each + (resp. −) from the left side of the equation may be regarded as being replaced on the right side of the equation by a +1 (resp. −1) or alternatively by a −1 (resp. +1).

[16] Nonresonance (see Equation (33)) is generic: it holds except for a zero measure set in the space of all parameters. As a result of nonresonance there is one active alien derivative per singular direction [34]. We also note that resonance can lead to multiple Écalle critical times and for resummation *Écalle acceleration* is needed. For further details see ([20], §1.1.2).

7.1. The resurgent subspace $\mathbb{T}_{\mathcal{R}}$

The definitions and propositions in the remainder of this subsection, along with some of the proofs, largely mimic their counterparts in the section on ordinary Borel summability.

The operator $\mathbf{mon} \circ \mathcal{B}$ is a proper extension of $\mathcal{B}$. Henceforth, for brevity, we adopt the following conventions.

Definition 51.

$$\hat{\mathcal{B}} := \mathbf{mon} \circ \mathcal{B} \text{ and } \widehat{\mathcal{LB}} := \mathcal{L} \circ \mathbf{mon} \circ \mathcal{B}. \tag{47}$$

$\widehat{\mathcal{LB}}$ is the *Écalle-Borel summation*, a wide generalization of $\mathcal{LB}$.[17]

Recalling that well-behaved averages preserve lateral growth, sufficient growth conditions for a transseries to be Écalle-Borel summable are similar to those for usual Borel summability (see Definition 37): in particular, for any small $\epsilon > 0$ there are positive constants c_1, c_2, c_3 such that for all $\mathbf{k}$

$$\left|(\widehat{\mathcal{B}}\tilde{y}_{\mathbf{k}})(p)\right| \leq c_1 c_2^{|\mathbf{k}|} e^{c_3 p} \text{ and } \left|(\widehat{\mathcal{B}}\tilde{y}_{\mathbf{k}})(p)\right| \leq c_1 c_2^{|\mathbf{k}|} e^{c_3|p|} \text{ if } |\arg p| \in (\epsilon, 2\epsilon). \tag{48}$$

Definition 52. The *Écalle-Borel sum* of $\tilde{T} \in \mathbb{T}_{\mathcal{R}}$ is defined as

$$\widehat{\mathcal{LB}}\tilde{T} = \sum_{\mathbf{k} \geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}+1} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} \widehat{\mathcal{LB}}\tilde{y}_{\mathbf{k}} + \tilde{T}_{\ell} + \sum_{j=-M}^{-1} x^{\beta_j} e^{\lambda_j x} \widehat{\mathcal{LB}}\tilde{y}_j(x). \tag{49}$$

Using the fact that **mon** preserves lateral growth it is easy to extend the results obtained in §5.7 to resurgent transseries. For example, note that in virtue of Definition 52 we have

$$\widehat{\mathcal{LB}}(x^{\beta+n} e^{\lambda x} x^{-n}) = x^n e^{\lambda x} x^{\beta} x^{-n} = x^{\beta} e^{\lambda x}, \tag{50}$$

which implies $\widehat{\mathcal{LB}}(x^{\beta} e^{\lambda x}) = x^{\beta} e^{\lambda x}$.

Proposition 53. Acting on $\mathbb{T}_{\mathcal{B}}$, $\widehat{\mathcal{LB}} = \mathcal{LB}$.

Proof. For any average, the sum of the weights is 1. Hence, if F is real-analytic, then $\mathbf{mon}\, F = F$. □

[17] Strictly speaking, there is a different Écalle-Borel summation for each distinct well-behaved average. The summation we employ is based on **mon**. However, as the uniqueness result referred to above and in §8.4 implies, all Écalle-Borel summations coincide for the restricted class of functions we are concerned with in this paper.

In the statement of the following proposition, the positive constants c_1, c_2 and c_3 are the bounds in (48).

Proposition 54. *Let $\tilde{T} \in \mathbb{T}_{\mathcal{R}}$. Also, let $\lambda = \min\{\lambda_1, ..., \lambda_n\}$ and $\beta = \max\{\beta_1, ..., \beta_n\}$. Further, let x_0 be such that for all $x > x_0$ we have $c_3 e^{-\lambda x} x^{\beta} < 1$. Then:*

(a) *For all $x > \max\{c_3, x_0\}$ we have*

$$\sum_{\mathbf{k}>M} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} \left|\widehat{\mathcal{LB}}\tilde{y}_{\mathbf{k}}\right| \leq c_1 (c_2 x^{\beta} e^{-\lambda x})^{NM} \frac{1}{(1 - c_2 x^{\beta} e^{-\lambda x})^{N}} (x - c_3)^{-1}, \tag{51}$$

where N is the dimension of the vector $\mathbf{k}$.

(b) *The infinite sum in Equation (49) converges uniformly and absolutely on the interval $[r, \infty)$ for any $r \in \mathbb{R}^+$ satisfying the condition $\nu r - |\beta| \log r > |\log c_2|$, as well as in the complex strip $\{x : \Re x \in [r, \infty)\}$.*

(c) *If $M' \leq M$ and $\beta = \min\{\beta_0, ..., \beta_{M'}\}$, then*

$$\sum_{j=-M'}^{-1} x^{\beta_j} e^{\lambda_j x} |\widehat{\mathcal{LB}}\tilde{y}_j(x)| \leq (M' + 1) c_1 e^{-\lambda_{M'} x} x^{\beta} (x - c_3)^{-1}. \tag{52}$$

(d) *$\mathbb{T}_{-,\mathcal{R}}$ is an algebra, i.e., if $\tilde{T}^{(1)}$ and $\tilde{T}^{(2)}$ are elements of $\mathbb{T}_{-,\mathcal{R}}$ and $a^{(1)}, a^{(2)} \in \mathbb{R}$, then $a^{(1)}\tilde{T}^{(1)} + a^{(2)}\tilde{T}^{(2)}$ and $\tilde{T}^{(1)}\tilde{T}^{(2)}$ are elements of $\mathbb{T}_{-,\mathcal{R}}$.*

(e) *If $\tilde{T} \in \mathbb{T}_{\mathcal{R}}$, then $\tilde{T} > 0$ if and only if $\widehat{\mathcal{LB}}\tilde{T} > 0$ for large enough x.*

(f) *The kernel of $\widehat{\mathcal{LB}}$ is zero, i.e., $\{\tilde{T} \in \mathbb{T}_{\mathcal{R}} : \widehat{\mathcal{LB}}\tilde{T} = 0\} = \{0\}$.*

Proof. The proof closely follows the proofs of Proposition 39, Proposition 41 and Corollary 43. □

Definition 55. *Let $\mathcal{F}_{\mathcal{R}} = \{y : \tilde{y} \in \mathbb{T}_{\mathcal{R}}\}$. Further let $\mathcal{F}_{+,\mathcal{R}} \oplus \mathcal{F}_{\ell} \oplus \mathcal{F}_{-,\mathcal{R}}$ be the decomposition of $\mathcal{F}_{\mathcal{R}}$ induced by the decomposition $\mathbb{T}_{+,\mathcal{R}} \oplus \mathbb{T}_{\ell} \oplus \mathbb{T}_{-,\mathcal{R}}$ of $\mathbb{T}_{\mathcal{R}}$ from Definition 50(4).*

If in Propositions 41 through 47 we uniformly replace the subscript $\mathcal{B}$ with the subscript $\mathcal{R}$ and uniformly replace the operator $\mathcal{LB}$ with the operator $\widehat{\mathcal{LB}}$, the resulting propositions remain valid, and the changes in their respective proofs are minor. Indeed, the lateral growth conditions on Borel plane functions and their convolutions are at the crux of the proofs, and they are all respected by well-behaved averages. An important such example is the following analog of Lemma 44(3).

Proposition 56 *(Differentiation on $\mathbb{T}_{\mathcal{R}}$).* Let $\tilde{T} \in \mathbb{T}_{\mathcal{R}}$.

$$(\widehat{\mathcal{LB}}\tilde{T})' = \sum_{j=-M}^{-1} x^{\beta_j} e^{\lambda_j x} \widehat{\mathcal{LB}}\dot{\tilde{y}}_j + \sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} \widehat{\mathcal{LB}}\dot{\tilde{y}}_{\mathbf{k}} + \tilde{T}'_{\ell} = \widehat{\mathcal{LB}}(\tilde{T})'. \tag{53}$$

Moreover, $\sum_{\mathbf{k}\geq 0} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} \widehat{\mathcal{LB}}\dot{y}_{\mathbf{k}}$ converges uniformly and absolutely for large x.

Proof. As indicated above, the proof closely follows the proof of Lemma 44(3). □

Theorem 57. (a) $\widehat{\mathcal{LB}}$ is an *isomorphism* between the spaces $\mathbb{T}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{R}}$ that preserves differentiation and antidifferentiation.

(b) $\widehat{\mathcal{LB}}$ restricted to $\mathbb{T}_{-,\mathcal{R}}$ is an *isomorphism* between the algebras $\mathbb{T}_{-,\mathcal{R}}$ and $\mathcal{F}_{-,\mathcal{R}}$ that preserves differentiation and antidifferentiation.

Proof. The proof of Part (a) follows the same steps as those Corollary 43 through Theorem 48 together with the aforementioned fact established by Écalle and Menous [36] (also see [34]) that **mon** respects lateral growth and convolution (and is clearly linear).

In the same fashion, the proof of the isomorphism of algebras in (b) mirrors, with obvious adaptations, the proof of Proposition 47.

Since the antidifferentiation properties in (b) are important for us, we provide more detail. To begin with,

$$\tilde{T}_+ = \sum_{-M\leq j\leq -1;\ l\geq 1} c_{j,l} x^{\beta_j} e^{\lambda_j x} x^{-l} = \sum_{j=-M}^{-1} x^{\beta_j} e^{\lambda_j x} \tilde{y}_j(x). \tag{54}$$

For each of the terms $x^{\beta_j} e^{\lambda_j x} \tilde{y}_j$ we follow the same calculations as in the proof of Proposition 47 to obtain the integral equation (43). Now the integral operator is not contractive because of the pole of the denominator. Instead, since the sum in $\tilde{T}_+$ is finite, and there are no convergence concerns, rougher estimates suffice. We analyze each term in $\tilde{T}_+$ separately. Differentiating now Equation (54) and proceeding as in the proof of Proposition 47, we get for each term of the sum (setting $\lambda = \lambda_j$ and $\beta = \beta_j$),

$$(\lambda - p)W' + (\beta - 1)W - Y = 0, \tag{55}$$

with the solution (after integration by parts) being

$$W(p) = -\int_0^p \frac{(-\lambda + p)^{\beta-1}}{(-\lambda + s)^{-\beta-1}} Y(s)ds - (-\lambda + p)^{-1} Y(p) - (-\lambda)^{-\beta}(-\lambda + p)^{-1} Y(0). \tag{56}$$

Analyticity of W away from the zeros of the denominators follows from dominated convergence. Preservation of lateral growth is established as in the case of ordinary Borel summability (see Proposition 47 and its proof). The transformations involved in obtaining W from Y are multiplication by $(\lambda - p)^{\alpha}$, where $\alpha_{\pm} = \pm\beta - 1$ and $f \mapsto \int_0^p f(s)ds = f * 1$. By **A**1, convolution preserves growth, while multiplication by $(\lambda - p)^{\alpha}$ preserves growth along any smooth curve. □

Note 58. It is worth noting that (as follows from Definition 52 and the just-proved isomorphism theorem), the transseries of the Écalle-Borel sum of a series is the series itself.

Definition 59. For $\tilde{T} \in \mathbb{T}_{\mathcal{R}}$, let $\mathsf{A}(\widehat{\mathcal{LB}}\tilde{T}) := \widehat{\mathcal{LB}}(\mathsf{A}_{\mathbb{T}}\tilde{T})$.

The following Corollary is an immediate consequence of Definition 59 and Theorem 57.

Corollary 60 *(Antidifferentiation).* A so defined is an antidifferentiation operator (see Definition 13) on $\mathcal{F}_{\mathcal{R}}$ and

$$(\mathsf{A}\tilde{T})' = \mathsf{A}(\tilde{T}') = \widehat{\mathcal{LB}}\tilde{T}.$$

8. Correspondence between resurgent functions, transseries in $\mathbb{T}_1$ and surreal functions: surreal antidifferentiation

In §4 we mentioned that to define integrals on **No** we would invoke a pair of isomorphisms–one between a subclass of resurgent functions and a subspace of transseries, and the other between the just-said subspace of transseries and a class of functions on **No**. These are the maps Tr and τ respectively. We now consider them in turn.

Proposition 61. For each $f \in \mathcal{F}_{\mathcal{R}}$ (see Definition 55) there exists a c such that f is real-analytic on (c, ∞).

Proof. Definition 50 and Theorem 57 imply that, for some positive x_0, the series of analytic functions on the right side of (37) converges uniformly on compact sets in a domain $\mathcal{D} = \{x : |x| > x_0, |\arg x| < c < \pi/2\}$. Hence, $\mathcal{L} \circ \mathbf{mon} \circ \mathcal{B}\tilde{f}$ is analytic in $\mathcal{D}$. In particular it is real-analytic. □

Definition 62. Based on the isomorphism in Theorem 57, we define the operator of *transseriation* Tr to be the inverse of $\widehat{\mathcal{LB}}$.

Example E2: The Écalle-Borel summed transseries of $x^\beta e^{\lambda x}$. In virtue of Definition 52 and the fact that $\widehat{\mathcal{LB}}\,1 = 1$, we have $\mathrm{Tr}(x^\beta e^{\lambda x}) = x^\beta e^{\lambda x}$. The equality is also an immediate consequence of Equation (50).

Proposition 63. If $f \in \mathcal{F}_{\mathcal{R}}$, then there is an $a \in \mathbb{R}^+$ such that $f(x) > 0$ on (a, ∞) if and only if $\mathrm{Tr} f > 0$.

Proof. Since well-behaved averages respect lateral growth, the proof mirrors that of Proposition 41. □

8.1. Differentiation of series that are absolutely convergent in the sense of Conway

In this subsection, we establish a result that will be needed in the sequel about differentiation of series of surreal functions that converge absolutely in the sense of Conway for any value of the variable belonging to some open interval.

Theorem 64. *Let $f_1, ..., f_m$ be twice differentiable infinitesimal functions defined on the positive infinite surreals. For $n \in \mathbb{Z}$, define the function g on each positive infinite surreal x by*

$$g(x) = \sum_{|\mathbf{k}|\geq 0} c_{\mathbf{k}} \mathbf{f}(x)^{\mathbf{k}}, \tag{57}$$

where $\{c_{\mathbf{k}}\}_{k_i \geq n; i \leq m}$ is a sequence of reals. Then g is differentiable for each such x and its derivative is given as follows by termwise differentiation:

$$g'(x) = \sum_{|\mathbf{k}|\geq 0} c_{\mathbf{k}} \left(\sum_{i=1}^{m} k_i \frac{f_i'(x)}{f_i(x)} \right) \mathbf{f}(x)^{\mathbf{k}}, \tag{58}$$

whereby convention we set $f_i'(x)/f_i(x) = 0$ if $f_i(x) = 0$.

Proof. We begin with the following simple observation.

Observation 65. Suppose f is a function such that $f(a+\epsilon) - f(a) = g(a)\epsilon + h(a,\epsilon)\epsilon^2$ where, for some $c > 0$ and sufficiently small ϵ we have $|h(a,\epsilon)| \leq c$. Then f is differentiable at a and $f'(a) = g(a)$. Based on the binomial formula, it is easy to check that, if f is twice differentiable, ϵ is infinitesimal and $k \in \mathbb{N}$, then $f(x+\epsilon)^k - f(x)^k = kf(x)^{k-1}f'(x)\epsilon + k^2 f(x)^{k-2}F(x,k;\epsilon)\epsilon^2$ where F is bounded for $k \in \mathbb{N}$ and infinitesimal ϵ. (Uniform boundedness in k follows from the fact that $0 \leq m^j|\epsilon| \leq 2$ for all $m, j \in \mathbb{N}$).

First note that the sum is absolutely convergent in the sense of Conway by Proposition 6 and the assumption that the f_i are infinitesimals. We will prove the result for $m = 1$; once having done so, the general result follows by induction and the usual decomposition $f_1(x+a)f_2(x+a) - f_1(x)f_2(x) = f_2(x+a)[f_1(x+a) - f_1(x)] + f_1(x)[f_2(x+a) - f_2(x)]$.

For $m = 1$, using Observation 65 and straightforward calculations, it follows that

$$g(x+\epsilon) - g(x) = \epsilon \sum_{k\geq 0} c_k k f(x)^{k-1} f'(x) + \epsilon^2 h(x,\epsilon) \tag{59}$$

where, using another application of Observation 65, we see that $h(x,\epsilon)$ is an absolutely convergent series which is bounded if $\epsilon \ll 1$. □

For the second isomorphism, τ, we require the following definition that trades on the intimate relationship between the members of $\mathbb{T}_1$ and certain surreal functions.

Definition 66. In accordance with Definition 30 and Proposition 28, each element $\tilde{T}$ of $\mathbb{T}_1$ is a transseries of the form

$$\tilde{T} = \sum_{-M\le j\le -1;\ l\ge 1} c_{j,l} x^{\beta_j} e^{\lambda_j x^{-l}} + P(x)\log(x) + Q(x) + R(x) + \sum_{\mathbf{k}\ge 0, l\ge 0} c_{\mathbf{k},l} x^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda} x} x^{-l}, \quad (60)$$

where the first sum is in $\mathbb{T}_+$, the second sum, in which P and Q are polynomials and R is a polynomial without constant term, is in $\mathbb{T}_\ell$, and the last sum belongs to $\mathbb{T}_-$.

With each such $\tilde{T}$ we associate the function $\tilde{T}^f$ consisting of all ordered pairs $(\nu, \tilde{T}^f(\nu))$, where ν is a positive infinite member of **No** and $\tilde{T}^f(\nu)$ is the expression that results from first replacing all occurrences of x on the right side of (60) with occurrences of ν, and then replacing (in the resulting expression) the absolutely convergent sum (with bounds $\mathbf{k} \ge 0, l \ge 0$) with the Lim sum to which it absolutely converges. That is:

$$\tilde{T}^f(\nu) := \sum_{-M\le j\le -1;\ l\ge 1} c_{j,l} \nu^{\beta_j} e^{\lambda_j \nu} \nu^{-l} + P(\nu)\log(\nu) + Q(\nu) + R(\nu) + \operatorname*{Lim}_{m\to\infty} \sum_{|\mathbf{k}|\le m, |l|\le m} c_{\mathbf{k},l} \nu^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}\nu} \nu^{-l}. \quad (61)$$

Let $\tau := \{(\tilde{T}, \tilde{T}^f) : \tilde{T} \in \mathbb{T}_1\}$, $\mathbf{No}^\tau := \{\tilde{T}^f : \tilde{T} \in \mathbb{T}_1\}$ (i.e. the range of the map τ) and let $\mathcal{R}_{\mathbf{No}} := \{\tilde{T}^f : \tilde{T} \in \mathbb{T}_\mathcal{R}\}$. Finally, let $\mathbf{No}^\tau_- := \{\tilde{T}^f : \tilde{T} \in \mathbb{T}_-\}$.

Theorem 67. The map τ in Definition 66 is an isomorphism of vector spaces endowed with differentiation between $\mathbb{T}_1$ and $\mathbf{No}^\tau$ and, when restricted to $\mathbb{T}_-$, an isomorphism of differential algebras between $\mathbb{T}_-$ and $\mathbf{No}^\tau_-$.

Proof. This follows from the fact that the transseries topology

$$\tilde{T} = \lim_{m\to\infty} \sum_{|\mathbf{k}|\le m, |l|\le m} c_{\mathbf{k},l} \nu^{\boldsymbol{\beta}\cdot\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}\nu} \nu^{-l}$$

preserves the differential algebra operations (see §5.2 and, for proofs, [21]), as does Lim (see Proposition 7 and the preceding material in this section). □

To complete the main part of our construction, we need an extension operator E in the sense of Definition 11, and, on restricted domains, a multiplicative extension operator in the sense of Definition 12. It is this to which we now turn.

8.2. The extension operator E

In the following we introduce an extension operator E acting on real-valued functions f on the reals to functions on the finite and positive infinite surreals. Assuming the function f_+ defined by $f_+(x) = f(-x)$ is in $\mathcal{F}_\mathcal{R}$, the extension of f to negative infinite surreal x is simply defined by $(\mathsf{E}f)(x) := (\mathsf{E}f_+)(-x)$, the subscript $+$ indicting that f_+ is defined for $x > 0$. In view of this elementary correspondence and to simplify the exposition we will solely focus on finite and positive infinite surreals.

Definition 68. Let $f \in \mathcal{F}_\mathcal{R}$, and let $c \in \mathbb{R}$ be such that f is real-analytic on (c, ∞) as is assured by Proposition 61. We extend f to $(c, \mathbf{On})$ as follows, whereby a *finite* surreal we mean the leading exponent in its normal form is ≤ 0.

(1) For positive infinite $x \in \mathbf{No}$ we define $(\mathsf{E}f)(x) = (\tau \circ \mathrm{Tr}\ f)(x)$.
(2) For finite $x \in \mathbf{No}$, where x_0 is the real part of x and ζ is the infinitesimal part of x (see Definition 3), we define $(\mathsf{E}f)(x)$ by

$$f(x_0 + \zeta) = f(x_0) + \sum_{k \geq 1} (k!)^{-1} f^{(k)}(x_0) \zeta^k, \tag{62}$$

where the infinite sum is absolutely convergent in the sense of Conway.

Before proceeding further, we offer a couple of observations on the second part of the above definition. To begin with, as above let $f \in \mathcal{F}_\mathcal{R}$ and let $c \in \mathbb{R}$ be such that f is real-analytic on (c, ∞) as in Proposition 61. Also let ϵ be the local radius of convergence of the Taylor series of f at $x_0 \in \mathbb{R}$. For real $|\zeta| < \epsilon$ we have

$$f(x_0 + \zeta) = f(x_0) + \sum_{k \geq 1} (k!)^{-1} f^{(k)}(x_0) \zeta^k. \tag{63}$$

Substituting $x = 1/\zeta$ for the two occurrences of ζ in Equation (63), the right side of the resulting equation is the convergent (*a fortiori* Borel and Écalle-Borel summable) transseries of the left side of the resulting equation about $x = \infty$. In particular $f(x_0 + x^{-1})$ is resurgent, and Definition 68 (2) is a special case of (1). In addition, alternatively and more formally, we can reduce case (2) of the above definition to case (1) by resorting to M, the Möbius transformation $x \mapsto x_0 + x^{-1}$ (see also Definition 80) by defining E by $(\mathsf{E}f)(x_0 + x^{-1}) = [M^{-1} \circ \tau \circ \mathrm{Tr} \circ Mf](x^{-1})$.

Theorem 69. *$\mathsf{E} : \mathcal{F}_\mathcal{R} \to \mathcal{R}_{\mathbf{No}}$ is an isomorphism of linear spaces endowed with differentiation and antidifferentiation. Its restriction to $\mathcal{F}_{-,\mathcal{R}}$ is an isomorphism of differential algebras.*

Proof. In virtue of the preceding remark, the formula for E at an arbitrary point is obtained from the one at ∞, hence it is enough to prove the result in the latter case. But

at ∞ the result follows immediately from Theorems 67 and 57, since E is a composition of isomorphisms. □

Some special cases of extensions are given below.

Corollary 70. *In the following, real functions are assumed to be defined on some interval* $(c,\infty)\subset\mathbb{R}^+$.

(1) *If* $a,b\in\mathbb{R}$ *and* $f:\mathbb{R}^+\to\mathbb{R}$ *is given by* $f(x)=x^a e^{bx}$, *then* $\mathsf{E}f=x^a e^{bx}$ *for all positive* $x\in\mathbf{No}$.
(2) *If* P *is a polynomial and* $f:\mathbb{R}^+\to\mathbb{R}$ *is given by* $f=P(x)\log x$, *then* $\mathsf{E}f=P(x)\log x$ *for all positive* $x\in\mathbf{No}$.
(3) *If* $f\in\mathcal{F}_{\mathcal{R}}$ *and* $f>0$ *on* (c,∞), *then* $(\mathsf{E}f)(x)>0$ *for all* $x\in\mathbf{No}$ *such that* $x>c$.

Proof. (1) and (2) follow immediately from Definitions 52 and 68.

For (3), note that if $f(x)>0$ for all real $x\in(c,\infty)$, then, by Proposition 63 we have $\mathrm{Tr}f>0$ and, plainly, $\tau\circ\mathrm{Tr}f>0$ for all positive infinite $x\in\mathbf{No}$. □

The following result is an immediate consequence of Theorem 69 and of Corollary 70.

Theorem 71. E is an extension operator in the sense of Definition 11. Moreover, E restricted to $\mathcal{F}_{-,\mathcal{R}}$ is a multiplicative extension operator in the sense of Definition 12.

Example: the special case of functions with convergent transseries at ∞.

First note that, if a convergent transseries is of the type expressed in Equation (60) and its sum is f, then $f\in\mathcal{F}_{\mathcal{R}}$. Indeed, in this case the Borel transform of a convergent series $\sum_{l\geq 0}c_{\mathbf{k},l}x^{-l}$ is an entire function, and Écalle-Borel summation coincides with Borel summation (since in the Borel plane there are no singularities), and by Proposition 16 (ii) Borel summation is simply the identity. We denote by $\mathcal{F}_{\mathcal{R},conv}$ the space of the sums (same as Borel sums) of the transseries in $\mathbb{T}_{conv}$. Observe that for $f\in\mathcal{F}_{\mathcal{R},conv}$ we have

$$f=\lim_{L\to\infty}\sum_{L>\mathbf{k}>-M}c_{\mathbf{k}}\boldsymbol{\mu}^{\mathbf{k}}. \tag{64}$$

Definition 72. Let $f\in\mathcal{F}_{\mathcal{R},conv}$. Then $(\mathsf{E}f)(x)$ is defined for positive infinite surreal x as an absolutely convergent series in the sense of Conway, by replacing the exponentials and logarithms in the transseries of f by their surreal counterparts and lim by Lim.

Proposition 73. *The operator* $\mathsf{E}|\mathcal{F}_{\mathcal{R},conv}$ *(i.e.* E *restricted to* $\mathcal{F}_{\mathcal{R},conv}$*) is an isomorphism of algebras between the algebra of convergent transseries* $\mathcal{F}_{\mathcal{R},conv}$ *and its image* $\mathcal{F}_{\mathbf{No},conv}:=\mathsf{E}\mathcal{F}_{\mathcal{R},conv}$.

Proof. This is straightforward since the algebras $\mathcal{F}_{\mathcal{R},conv}$ and $\mathcal{F}_{\mathbf{No},conv}$ consist of limits and Limits, respectively, of finite sums. □

The following result is an immediate consequence of Corollary 70 and Proposition 73.

Proposition 74. The operator $\mathsf{E}|\mathcal{F}_{\mathcal{R},conv}$ is an extension operator in the sense of Definition 11, and $\mathsf{E}|\left(\mathcal{F}_{-,\mathcal{R}} \cap \mathcal{F}_{\mathcal{R},conv}\right)$ is multiplicative in the sense of Definition 12.

8.3. The main theorem on antidifferentiation: the operator $\mathsf{A}_{\mathbf{No}}$

Definition 75. Let $\mathsf{E}(\mathcal{F}_{\mathcal{R}}) := \{\mathsf{E}f : f \in \mathcal{F}_{\mathcal{R}}\}$.

Since $\mathsf{E} : \mathcal{F}_{\mathcal{R}} \to \mathcal{R}_{\mathbf{No}}$ (from Theorem 69) is a surjection, henceforth we write $\mathsf{E}(\mathcal{F}_{\mathcal{R}})$ in place of $\mathcal{R}_{\mathbf{No}}$.

Definition 76. By $\mathsf{A}_{\mathbf{No}}$ we mean the operator defined by the following conditions:

(1) for members of $\mathsf{E}(\mathcal{F}_{\mathcal{R}})$, $\mathsf{A}_{\mathbf{No}} = \mathsf{E}\mathsf{A}\mathsf{E}^{-1}$, where E and A, defined on $\mathcal{F}_{\mathcal{R}}$, are the extension and antidifferentiation operators from Definition 68 and Definition 59;
(2) for $f \in \mathsf{E}(\mathcal{F}_{\mathcal{R}})$ and $\lambda \in \mathbf{No}$, $\mathsf{A}_{\mathbf{No}}(\lambda f) = \lambda \mathsf{A}_{\mathbf{No}} f$.

Example E3: $\mathsf{A}_{\mathbf{No}}(e^x)$**.** Since $\mathsf{A}_{\mathbb{T}} e^x = e^x$, we obtain, for positive infinite surreal x

$$\mathsf{A}_{\mathbf{No}}(e^x) = e^x. \tag{65}$$

It is easy to check that, for any $\lambda \in \mathbf{No}$ and $f, g \in \mathsf{E}(\mathcal{F}_{\mathcal{R}})$ we have $\mathsf{A}_{\mathbf{No}}[\lambda(f+g)] = \lambda \mathsf{A}_{\mathbf{No}} f + \lambda \mathsf{A}_{\mathbf{No}} g$.

We prove in Theorem 78 below that $\mathsf{A}_{\mathbf{No}}$ is an antidifferentiation operator in the sense of Definition 13. To prepare the way, we first prove:

Proposition 77. *In the following we assume that $c \in \mathbb{R}$ and f is defined on $\{x \in \mathbf{No} : x > c\}$.*

(1) *If $f \in \mathsf{E}(\mathcal{F}_{\mathcal{R}})$, then $(\mathsf{A}_{\mathbf{No}} f)' = f$.*
(2) *If $x, y \in (c, \infty) \cap \mathbb{R}$ and $f \in \mathcal{F}_{\mathcal{R}}$, then $(\mathsf{A}_{\mathbf{No}} f)(y) - (\mathsf{A}_{\mathbf{No}} f)(x) = \int_x^y f(s)ds$.*
(3) *If $f \in \mathsf{E}(\mathcal{F}_{\mathcal{R}})$ is nonnegative and $y > x > c$, then $(\mathsf{A}_{\mathbf{No}} f)(y) - (\mathsf{A}_{\mathbf{No}} f)(x) \geq 0$.*

Proof. Assume $f \in \mathsf{E}(\mathcal{F}_{\mathcal{R}})$ and let $f_{\mathbb{R}} := \mathsf{E}^{-1} f$. By the definition of E, $f(x) = f_{\mathbb{R}}(x)$ for any real $x \in (c, \infty)$.

(1) By Theorem 71 and using the construction of $\mathsf{A}_{\mathbf{No}}$, we have

$$(\mathsf{A}_{\mathbf{No}} f)' = (\mathsf{E}\mathsf{A} f_{\mathbb{R}})' = \mathsf{E}(\mathsf{A} f_{\mathbb{R}})' = \mathsf{E} f_{\mathbb{R}} = f.$$

(2) The function $g(x, y) = \int_x^y f_{\mathbb{R}} - [(\mathsf{A} f_{\mathbb{R}})(y) - (\mathsf{A} f_{\mathbb{R}})(x)]$ is real-analytic in (x, y) and $\frac{dg}{dy} = 0$. Hence $g(x, \cdot)$ is a constant. Since $g(x, x) = 0$, g is the zero function.

(3) Since, as noted, $f_{\mathbb{R}}$ coincides with f on $(c,\infty)\subset\mathbb{R}$, we see that $f_{\mathbb{R}}$ is nonnegative. Let $F_{\mathbb{R}}=\mathsf{A}f_{\mathbb{R}}$. By elementary calculus, for $y>x$ in (c,∞) we have $F_{\mathbb{R}}(y)\geq F_{\mathbb{R}}(x)$. Hence, fixing $x\in(c,\infty)$ with y being the variable, and using the properties of E and the definition of $\mathsf{A}_{\mathbf{No}}$ we obtain $\operatorname{sign}(y-x)[\mathsf{A}f_{\mathbb{R}}(y)-\mathsf{A}f_{\mathbb{R}}(x)]\geq 0$ for all $y\in\mathbf{No}$. For finite $y_1\leq y_2\in\mathbf{No}$ for which there is an $x\in(c,\infty)$ such that $y_1\leq x\leq y_2$, we insert an intermediate term to obtain $\mathsf{A}f_{\mathbb{R}}(y_2)-\mathsf{A}f_{\mathbb{R}}(x)+\mathsf{A}f_{\mathbb{R}}(x)-\mathsf{A}f_{\mathbb{R}}(y_1)\geq 0$. If instead $y_1\leq y_2\in\mathbf{No}$ are finite but there is no such real x satisfying the just-said condition, then the standard parts of y_1 and y_2 coincide with some x and the property follows from the Taylor expansions of $\mathsf{A}f_{\mathbb{R}}(x+[y_{1,2}-x])$ around x.

We are left with the analysis of the case when $y_1\leq y_2$ are both positive infinite. Let $f_{\mathbb{R}}=\widehat{\mathcal{LB}}\tilde{T}$. We only analyze the case where the component of $\tilde{T}$ in $\mathbb{T}_+$ is nonzero, say $Cx^{\beta-l}e^{\lambda x}$, as the case where $\tilde{T}\in\mathbb{T}_-\oplus\mathbb{T}_\ell$ is similar. The property that needs to be established is that, if $\tilde{T}\geq 0$, then $F=\mathsf{A}\widehat{\mathcal{LB}}\tilde{T}$ is an increasing function. From the construction of A, for positive infinite ν, $F(\nu)$ is an absolutely convergent Conway power series with dominant term $\lambda^{-1}C\nu^{\beta-l}e^{\lambda\nu}$. We distinguish two cases. If $0<y_2-y_1=\epsilon\ll 1$, then each term in the Conway expansion of F can be reexpanded in ϵ. By this we mean the following. Letting $y_1=\nu$ and $\epsilon_1=\epsilon\nu^{-1}$ we have

$$e^{\lambda_j(\nu+\epsilon)}(\nu+\epsilon)^{\beta_j-l}=e^{\lambda_j\nu}\nu^{\beta_j-l}\mathrm{Lim}_{m\to\infty}\sum_{r=0}^{m}\sum_{k=0}^{m}\frac{\lambda_j^r\epsilon^r}{r!}\binom{\beta-l}{k}\epsilon_1^k, \tag{66}$$

which we insert in the Lim term above in the first sum in Equation (61), and we similarly reexpand the other terms to obtain the Lim as $m\to\infty$ of an $N+2$-dimensional truncated power series. This expansion shows that the dominant term of $F(\nu+\epsilon)-F(\nu)$ is $C\nu^{\beta-l}e^{\lambda\nu}\epsilon>0$. If instead, $0<y_2-\nu=a$ is finite, then, with s being some infinitesimal function and a° being the standard part of a, we have $F(\nu+a)/F(\nu)=(1+a/\nu)^{\beta-l}e^{\lambda a}(1+s(\nu+a))/(1+s(\nu))>e^{\lambda a^\circ}>1$, as it is easy to check. □

It is worth noting that, in virtue of the construction of $\mathsf{A}_{\mathbf{No}}=\mathsf{E}\mathsf{A}\mathsf{E}^{-1}$ we have obtained more than just antidifferentiation; in particular, we have obtained the operator E, which in turn is used in the construction of $\mathsf{A}_{\mathbf{No}}$. $\mathsf{A}_{\mathbf{No}}$ provides the solutions of equations of the form $f'=g$, whereas, by virtue of the fact that E preserves the operations of differential algebra, E can be used to solve classes of nonlinear equations, such as ODEs, and difference equations. This brings us to the main theorem on antidifferentiation.

Theorem 78. $\mathsf{A}_{\mathbf{No}}$ *is an antidifferentiation operator in the sense of Definition 13.*

Proof. The satisfaction of condition i of Definition 13 follows from Proposition 77 (1); the satisfaction of ii follows from Definition 76 (1) and (2) and the linearity of $\widehat{\mathcal{LB}}$; the satisfaction of iii follows from Proposition 77 (2); and the satisfaction of iv and v follows from Proposition 70 (1). For the satisfaction of vi, let $f=Ef_{\mathbb{R}}$, and note that $f'=0$

means $(\mathsf{E}f_{\mathbb{R}})' = 0$, thereby implying $f'_{\mathbb{R}} = 0$. Hence there is a $C \in \mathbb{R}$ such that $f_{\mathbb{R}} = C$, implying $f = C$. □

In virtue of Equation (6), Proposition 14 and Theorem 78, we now have:

Corollary 79. $\int_x^y f = \mathsf{A}_{\mathbf{No}}(f)(y) - \mathsf{A}_{\mathbf{No}}(f)(x)$ *is an integral operator on the domain of* $\mathsf{A}_{\mathbf{No}}$.

8.4. Uniqueness

The existence of a continuum of nonequivalent well-behaved averages induces an apparent nonuniqueness of the operators E and $\mathsf{A}_{\mathbf{No}}$. However, as we mentioned above, when restricted to the class of functions with which we are concerned with in this paper (see the introduction as well as Footnote 16 on nonresonance and the remark it is appended to) all such averages coincide, thereby resulting in unique operators when E and $\mathsf{A}_{\mathbf{No}}$ are thus restricted. A detailed analysis of this will be the subject of a different paper.

9. The extension and antidifferentiation operators E^* and $\mathsf{A}^*_{\mathbf{No}}$

Often singular behavior occurs in other limits than $x \to \infty$. For instance, for a modular form such as the elliptic theta function θ_3, the unit circle in $\mathbb{C}$ is a natural boundary, and the limits of interest on the real line are ± 1 (see §10.2). Here, by changes of variable, we expand the domain of our extension and antidifferentiation operators to handle arbitrary points.

The extension operator E^* is constructed in two stages. We begin by defining E_∞ acting on functions that are resurgent at $x_0 = \infty$, with values in surreal functions defined for positive infinite surreals, namely we define $(\mathsf{E}_\infty f)(x) = (\tau \circ \operatorname{Tr} f)(x)$. For functions that are resurgent at finite x_0, or $x_0 = -\infty$ we simply change variables to bring the case to $x_0 = \infty$. For example, the function $t \mapsto \exp(-1/t)$ is real-analytic on $(0, 1)$ but not at zero; extending it to positive infinitesimal t is done by writing $t = 1/x$ and extending the new function e^{-x} to positive infinite values of x using E_∞. That is, $(\mathsf{E}f)(1/t) := (\mathsf{E}_\infty f)(x)$ with $x = 1/t$. For the sake of completeness we formalize this process in the paragraphs below.

Definition 80. Let $x_0 \in \mathbb{R}, a \in \mathbb{R}^+$ and $f : D(f) \to \mathbb{R}$ be a real-analytic function. If $D(f) = (a, \infty)$, then we let $m(x) = x$, the identity. If $D(f) = (-\infty, -a)$, we let $m(x) = -x$; if $D(f) = (x_0, x_0{+}a)$, then we let $m(x) = x_0{+}1/x$, and if $D(f) = (x_0{-}a, x_0)$, then $m(x) = x_0 - 1/x$. We then define $Mf = f \circ m$. The domain of Mf is (a^{-1}, ∞) in the first two cases and (a, ∞) in the last two.

The class of functions we have heretofore been concerned with that we call "resurgent" are the members of $\mathcal{F}_{\mathcal{R}}$; see Definition 55. The following definition expands the class of functions we subsume under this appellation.

Definition 81. If $Mf \in \mathcal{F}_{\mathcal{R}}$ with M and f as in Definition 80, we say f is *resurgent*. Let $\mathcal{F}^*_{\mathcal{R}}$ be the set of all resurgent functions in the just-said sense. If $y \in \mathcal{F}^*_{\mathcal{R}}$, we say y is *resurgent at* $x_0 = \infty$ (resp. $-\infty$) if $x \mapsto y(x)$ (resp. $x \mapsto y(-x) \in \mathcal{F}^*_{\mathcal{R}}$, and we say y is *resurgent to the right* (resp. *left*) of $x_0 \in \mathbb{R}$ if $x \mapsto y(x_0 + 1/x)$ (resp. $x \mapsto y(x_0 - 1/x) \in \mathcal{F}^*_{\mathcal{R}}$.

Definition 82. Suppose f is resurgent (in the sense of Definition 81). If M is the identity (i.e. if f is resurgent at ∞) we define $\mathsf{E}^*_\infty f$ for $\infty < x \in \mathbf{No}$ by $(\mathsf{E}^*_\infty f)(x) := (\tau \circ \mathrm{Tr}\, f)(x)$ and let $\mathsf{E}^* = \mathsf{E}^*_\infty$. More generally, in all four cases of Definition 80, we set $\mathsf{E}^* := M^{-1}\mathsf{E}^*_\infty M$. Also set $\mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}}) := \{\mathsf{E}^* f : f \in \mathcal{F}^*_{\mathcal{R}}\}$.

Notice that $\mathcal{F}_{\mathcal{R}} \subset \mathcal{F}^*_{\mathcal{R}}$ and, hence, $\mathsf{E}(\mathcal{F}_{\mathcal{R}}) \subset \mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})$.

Theorem 83. E^* is an extension operator in the sense of Definition 11. Moreover, E^* restricted to $\{f : Mf \in \mathcal{F}^*_{-,\mathcal{R}}\}$ is a multiplicative extension operator in the sense of Definition 12.

Proof. Proposition 71 shows that E^*_∞ has the properties stated in the theorem. Conjugation through M, $M^{-1}(\cdot)M$ is an obvious structural isomorphism, ensuring preservation of the required properties. □

9.1. Definition of $\mathsf{A}^*_{\mathbf{No}}$

We now make use of Definition 80 to define integration by changes of variable via M. Recall that $\mathsf{A}_{\mathbf{No}}$ has the intuitive interpretation $(\mathsf{A}_{\mathbf{No}} f)(x) = \int_\infty^x f$. To extend $\mathsf{A}_{\mathbf{No}}$ to other points $x_0 \in \mathbb{R}$ (or $-\infty$), we change the variable of integration to map $x_0 \in \mathbb{R}$ (resp. $-\infty$) to ∞. In the change of variable $Mf = f \circ m$, m is a one-to-one rational function which coincides with its surreal extension.

For example, if $D(f) = (0, \epsilon)$ we have $m(s) = 1/s$ and we note that, heuristically,

$$\int_0^s f(u)du = \int_\infty^{1/s} \left(-\frac{1}{v^2}\right) f\left(\frac{1}{v}\right) dv.$$

More generally, we obtain (still heuristically) $(\mathsf{A}^*_{\mathbf{No}} f)m(x) = (\mathsf{A}_{\mathbf{No}} m' f \circ m)(x)$, which motivates the following definition.

Definition 84. For $Mf \in \mathcal{F}_{\mathcal{R}}$, let $(\mathsf{A}^*_{\mathbf{No}} f) \circ m := \mathsf{A}_{\mathbf{No}}(m' Mf)$.

Theorem 85. $\mathsf{A}^*_{\mathbf{No}}$ *is an antidifferentiation operator on* $\mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})$.

Proof. Using the properties of $\mathsf{A}_{\mathbf{No}}$ and of differentiation it is straightforward to check that the properties listed in Definition 13 hold. We check only i.; the others being similar and in fact simpler.

We rewrite the definition as $[(\mathsf{A}^*_{\mathbf{No}} f)](s) = [\mathsf{A}_{\mathbf{No}}(m'\ f \circ m)](m^{-1}(s))$ and, using the chain rule together with the fact that $\mathsf{A}_{\mathbf{No}}$ is an antidifferentiation operator, we get

$$[(\mathsf{A}^*_{\mathbf{No}} f)]' = \frac{[\mathsf{A}_{\mathbf{No}}(m'\ f \circ m)]'(m^{-1}(s))}{m'(m^{-1}(s))} = \frac{m'(m^{-1}(s))\ (f \circ m)(m^{-1}(s))}{m'(m^{-1}(s))}$$
$$= f(s). \quad \square \tag{67}$$

Note 86. The reader can see that $\mathsf{A}^*_{\mathbf{No}}$ is obtained from $\mathsf{A}_{\mathbf{No}}$ simply by changes of variables, in the same way E^* was obtained from E, with the consequence that the antiderivative of the extension of $f \in \mathcal{F}^*_{\mathcal{R}}$ is the extension of the antiderivative of f.

Corollary 87. $\int_x^y f = \mathsf{A}^*_{\mathbf{No}}(f)(y) - \mathsf{A}^*_{\mathbf{No}}(f)(x)$ *is an integral operator on* $\mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})$.

10. Illustrations of extensions and antidifferentiations

Many of the familiar functions have convergent expansions at points on the real line, where the actions of the extension and antidifferentiation operators are easy to obtain, as in Equation (62) and the cases mentioned in the comments following Definition 68. Accordingly, in the first subsection, we focus on calculating functions for positive infinite surreal values when these functions have a singularity at ∞. In illustrations (i) and (ii), we go over all details of the analysis.

10.1. Functions having a singularity at ∞

(i) First consider the *exponential function* $x \mapsto e^x$. Using Example E3, we have $\mathsf{A}_{\mathbf{No}}(e^x) = e^x$. Hence, by Equation (6), we have

$$\int_0^x e^s ds = e^x - 1, \tag{68}$$

as expected. This stands in contrast to Norton's aforementioned proposed definition of integration which was shown by Kruskal to integrate e^x over the range $[0, \omega]$ to the wrong value e^ω [19, p. 228]. We note that [67] also obtains Equation (68).

(ii) Next consider $t^{-1}e^t$. Its antiderivative is the *exponential integral* Ei.

By changes of variables, we have

$$\mathrm{Ei}(x) := \mathrm{PV}\int_{-\infty}^{x} t^{-1}e^t dt = e^x \mathrm{PV}\int_0^{\infty} \frac{e^{-xp}}{1-p}dp = e^x \mathrm{PV}\mathcal{L}\frac{1}{1-p}. \tag{69}$$

We have $\widehat{\mathcal{L}\mathcal{B}}\tilde{y} = \mathrm{PV}\mathcal{L}(1-p)^{-1}$. Hence, $\mathrm{Tr}(\mathrm{Ei}(x)) = e^x\tilde{y} = \mathsf{A}_{\mathbb{T}}[t^{-1}e^t]$. Using Definition 76, this gives, for all positive infinite surreal x,

$$(\mathsf{A}_{\mathbf{No}}[t^{-1}e^t])(x) = e^x \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}}. \tag{70}$$

The values of Ei for positive finite surreal x are obtained simply from the local Taylor series at the real part of x (see Definition 3), as explained in Equation (62) contained in Definition 68.

The fact that Equation (70) should hold up to an additive constant was known to Conway and Kruskal but the value of this constant resisted their years long effort.[18] As (70) shows, this constant is zero.

(iii) The *imaginary error function* erfi. To calculate

$$f(x) = \int_0^x e^{s^2} ds = \frac{\sqrt{\pi}}{2}\mathrm{erfi}(x) \tag{71}$$

for positive infinite surreal x, we first find the Écalle-Borel summed transseries of f. In this example, the Écalle critical time is not the original variable. By applying integration by parts to the integral in Equation (71) we obtain:

$$\int_0^x e^{s^2} ds \sim e^{s^2}\left(\frac{1}{2x} + \frac{1}{4x^2} + \cdots\right). \tag{72}$$

We notice here that the exponent is $t = s^2$, which is the Écalle critical time (see Definition 15), and therefore we have to pass to the variable t. We then observe that $f'(x) = e^{x^2}$, where $f(0) = 0$. With the substitution $f(x) = x\exp(x^2)g(x^2)$ with $x^2 = t$ we get

$$g' + \left(1 + \frac{1}{2t}\right)g = \frac{1}{2t}. \tag{73}$$

The transseries of g is simply a power series whose Borel transform G satisfies, in accordance with (73),

[18] Oral communication by Martin Kruskal to the first author.

$$(p-1)G' + \frac{1}{2}G = 0, \tag{74}$$

where $G(0) = 1/2$. In this case **mon** gives

$$\frac{1}{4}\left(\int_0^{\infty - 0i} + \int_0^{\infty + 0i}\right) \frac{e^{-tp}}{\sqrt{1-p}}dp = \frac{1}{2}\int_0^1 \frac{e^{-tp}}{\sqrt{1-p}}dp, \tag{75}$$

which leads to

$$f(x) = \frac{1}{2}xe^{x^2}\int_0^1 \frac{e^{-x^2 p}}{\sqrt{1-p}}dp. \tag{76}$$

We note that $f(0) = 0$, and so this expression satisfies both the differential equation and the initial condition. The transseries of f is easily obtained from (75) and Lemma 36 combined with the binomial formula. Ultimately, we arrive at

$$\mathrm{Tr}(f)(x) = \frac{1}{\sqrt{\pi}}e^{x^2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{x^{2n+1}} \tag{77}$$

and, hence,

$$\int_0^x e^{s^2}ds = \frac{1}{\sqrt{\pi}}e^{x^2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{x^{2n+1}} \quad \text{for all surreal } x > \infty. \tag{78}$$

We note that the expression above is **not** valid for $x < \infty$, let alone at $x = 0$, and we don't expect it to satisfy $f(0) = 0$. As is indicated above, for finite x the values of f for Ei are obtained from the local Taylor series.

(iv) The *Airy functions* Ai and Bi. These are two special solutions of the Airy equation

$$y'' = zy.$$

Their asymptotic expansions for large $z \in \mathbb{R}^+$ are

$$\mathrm{Ai}(z) \sim \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}}\sum_{k=0}^{\infty}(-1)^k\frac{u_k}{\zeta^k}; \quad \zeta := \frac{2}{3}z^{3/2} \tag{79}$$

and

$$\mathrm{Bi}(z) \sim \frac{e^{\zeta}}{\sqrt{\pi}z^{1/4}}\sum_{k=0}^{\infty}\frac{u_k}{\zeta^k}, \tag{80}$$

respectively (see [25]). We note that the asymptotic expansions are given in [25] in terms of ζ, which is precisely the Écalle critical time (see Definition 15).

Changing the variable z to ζ, the inverse Laplace transform of $\exp(-\zeta)\mathrm{Ai}(\zeta)\zeta^{-1/6}$ is the hypergeometric function ${}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; -\frac{p}{2}\right)$ which is analytic on $\mathbb{R}^+$, and hence usual Borel summabilty applies.

In the case of Bi, there is just one singularity on $\mathbb{R}^+$ at $p = 2$. A simple calculation, using **mon** for the last case, shows that $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are simply given by these asymptotic series, now interpreted as absolutely convergent series in the sense of Conway.

(v) The *Gamma* and log-*gamma* functions. The Borel summed transseries of the log-gamma function, $\log\Gamma(x)$, is[19]

$$\log\Gamma(x) = x(\log(x) - 1) - \frac{1}{2}\log\frac{x}{2\pi} + \int_0^\infty e^{-xp}\frac{p\coth(p/2) - 2}{2p^2}dp. \tag{81}$$

Using the generating function of the Bernoulli numbers we get, for small p,

$$\frac{p\coth(p/2) - 2}{2p^2} = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}p^n. \tag{82}$$

Hence, the transseries of $\log\Gamma(x)$ is given by

$$\log\Gamma(x) = x(\log(x) - 1) - \frac{1}{2}\log\frac{x}{2\pi} + \sum_{n=0}^{\infty}\frac{B_{2n}n!}{(2n)!x^{n+1}}. \tag{83}$$

Then, for positive infinite surreal x, $\log\Gamma(x)$ is given by the right side of Equation (83) where the infinite series are now interpreted as absolutely convergent series in the sense of Conway.

The Gamma function is simply obtained from the log-gamma function by exponentiation.

10.2. An example of a function whose singularities are at finite points: Jacobi's elliptic function θ_3

Jacobi's elliptic function θ_3 is defined by

$$\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2\sum_{n\in\mathbb{N}} q^{n^2}; \quad |q| < 1 \tag{84}$$

(see e.g. [65]). Clearly θ_3 is analytic in the complex unit disk, in particular, it is real-analytic on $(-1, 1)$. Since the series $\sum_{n\in\mathbb{N}} q^{n^2}$ is *lacunary*, the unit circle is a natural boundary (see e.g. [57]). In particular, the points ± 1 are singular. However, as we will now see $\theta_3 \in \mathcal{F}^*_{\mathcal{R}}$ and therefore it extends to a left (resp. right) surreal neighborhood of

[19] This is a simplification of the form given in [22, p. 96].

1 (resp. -1). Define t by $q = e^{-t}$ and let $q_1 = e^{-\pi^2/t}$. We note that $q \to 1$ is equivalent to $t \to 0$. Jacobi's modular transformations applied to θ_3 give:

$$\theta_3(q) = \sqrt{\frac{\pi}{t}}\theta_3(q_1) = \sqrt{\frac{\pi}{t}}\left(1 + 2\sum_{n\in\mathbb{N}} e^{-n^2\pi^2/t}\right). \tag{85}$$

This is a convergent transseries, and applying the definition of E^* we obtain, for positive infinitesimal ζ,

$$(\mathsf{E}^*\theta_3)(e^{-\zeta}) = \sqrt{\frac{\pi}{\zeta}}\left(1 + 2\sum_{n\in\mathbb{N}} e^{-n^2\pi^2/\zeta}\right). \tag{86}$$

For $q \to -1$ we have similar formulas, and we omit the intermediate steps. Indeed, $\theta_3(q) = \theta_4(-q) = \sqrt{\frac{\pi}{t}}\theta_2(-q_1) = 2q_1^{1/4}\sum_{n\geq 0} q_1^{n(n+1)}$ (see [65]), which implies for q infinitesimally greater than -1,

$$(\mathsf{E}^*\theta_3)(-e^{-\zeta}) = 2\sqrt{\frac{\pi}{\zeta}}e^{-\frac{1}{4}\pi^2/\zeta}\sum_{n\geq 0} e^{-n(n+1)\pi^2/\zeta}. \tag{87}$$

11. The theory of surreal integration: a generalization

It is natural to inquire in which ordered exponential subfields $(K, \exp_K)$ of $(\mathbf{No}, \exp)$ the above theory of surreal integration restricted to K continues to be applicable. In this section we show that a sufficient condition is that K is *closed under absolute convergence in the sense of Conway*, that is, for each formal power series f in $n \geq 0$ variables with coefficients in $\mathbb{R}$, $f(a_1, ..., a_n)$ is absolutely convergent in the sense of Conway in K for every choice of infinitesimals $a_1, ..., a_n$ in K. After having demonstrated this, we will exhibit ordered exponential subfields of $(\mathbf{No}, \exp)$ that are closed in this sense.

Note that, for $n = 0$ the ring of power series in n variables with coefficients in $\mathbb{R}$ is $\mathbb{R}$ itself, and so henceforth we may assume that all references to reals are references to members of $\mathbb{R} \subseteq K \subseteq \mathbf{No}$ and furthermore that $(\mathbb{R}, e^x) \subseteq (K, \exp_K) \subseteq (\mathbf{No}, \exp)$.

Lemma 88. If $(K, \exp_K)$ is an ordered exponential subfield of $(\mathbf{No}, \exp)$ that is closed under absolute convergence in the sense of Conway, then for each $f \in \mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})$ (see Definition 82) and each $x \in \mathrm{dom}(f) \cap K$, $f(x) \in K$.

Proof. Suppose the hypothesis holds and further suppose $f \in \mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})$ and $x \in \mathrm{dom}(f) \cap K$.

Case 1. $f \in \mathsf{E}(\mathcal{F}_{\mathcal{R}})$ and, hence, $\mathsf{E}^{-1}f$ is resurgent at ∞.

If $x = x_0 + \zeta$, where x_0 is in the real-analyticity domain of $\mathsf{E}^{-1}f$ and ζ is infinitesimal, then in virtue of Definitions 80 and 68 and Corollary 70(4),[20]

[20] Since f is an extension of $\mathsf{E}^{-1}f$, $(\mathsf{E}^{-1}f)(x_0) = f(x_0)$ and $(\mathsf{E}^{-1}f)^{(k)}(x_0) = f^{(k)}(x_0)$.

$$\mathsf{E}f(x) = \mathsf{E}f(x_0 + \zeta) = (\mathsf{E}^{-1}f)(x_0) + \sum_{k\geq 1}(k!)^{-1}(\mathsf{E}^{-1}f)^{(k)}(x_0)\zeta^k.$$

As such, since K is closed under absolute convergence in the sense of Conway, $(\mathsf{E}f)(x) \in K$.

If x is positive infinite, then in virtue of Definitions 66 and the first part of Definition 68, $(\mathsf{E}f)(x)$ assumes the form

$$\sum_{-M\leq j\leq -1;\ l\geq 1} c_{j,l}x^{\beta_j}e^{\lambda_j}x^{-l} + P(x)\log(x) + Q(x) + R(x) + \sum_{\mathbf{k}\geq 0, l\geq 0} c_{\mathbf{k},l}x^{\boldsymbol{\beta}\cdot\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}x^{-l},$$

where P, Q and R are polynomials, R being without constant term, M is a natural number, the coefficients and powers are real numbers, and the terms of the form $x^{\beta'_j}$, etc., (or their multiplicative inverses), can be written as exponentials (or multiplicative inverses of exponentials) using the identity $x^a = e^{a\log x}$. Accordingly, since $P(x)\log(x) + Q(x) + R(x)$ and the finite sum over M are clearly both in K, to show the entity denoted by the full expression is in K it remains to note that the Lim term is in K in virtue of K's closure under absolute convergence in the sense of Conway.

Case 2. $f \notin \mathsf{E}(\mathcal{F}_{\mathcal{R}})$. The extension operator E^* is defined as $\mathsf{E}^* := M^{-1}\mathsf{E}^*_\infty M$ where $(Mf)(x) = f(m(x))$. This clearly preserves the range of f: the values of Mf are in K if and only if the values of f are in K. □

For each $f \in \mathcal{F}^*_{\mathcal{R}}$, let $\mathsf{E}^*_K(f) := \mathsf{E}^*(f)|K$, i.e. $\mathsf{E}^*(f)$ restricted to K, and for each $f \in \mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})$, let $\mathsf{A}^{*K}_{\mathbf{No}}(f) := \mathsf{A}^*_{\mathbf{No}}(f)|K$. Also, let $\mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})|K := \{f|K : f \in \mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})\}$.

Theorem 89. *Let $(K, \exp_K)$ be an ordered exponential subfield of $(\mathbf{No}, \exp)$ that is closed under absolute convergence in the sense of Conway, and let $x, y \in K$. Then:*

(1) E^*_K *is an extension operator on $\mathcal{F}^*_{\mathcal{R}}$ in the sense of Definition 11;*
(2) $\mathsf{A}^{*K}_{\mathbf{No}}$ *is an antidifferentiation operator on $\mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})|K$ in the sense of Definition 13;*
(3) $\int_x^y f = \mathsf{A}^{*K}_{\mathbf{No}}(f)(y) - \mathsf{A}^{*K}_{\mathbf{No}}(f)(x)$ *is an integral operator on $\mathsf{E}^*(\mathcal{F}^*_{\mathcal{R}})|K$ in the sense of Proposition 14.*

Proof. (1) follows from Lemma 88 and Theorem 83. In addition, since the antiderivative of the extension of $f \in \mathcal{F}^*_{\mathcal{R}}$ is the extension of the antiderivative of f, to establish (2) we need only further appeal to (1) and Theorem 85. (3) follows from (2) together with Equation (6) and Proposition 14. □

Our first example of a structure that satisfies the hypothesis of Theorem 89 is given by:

Theorem 90. The ordered exponential subfield $\mathbb{R}((\omega))^{LE}$ of $(\mathbf{No}, \exp)$ is closed under absolute convergence in the sense of Conway.

Proof. As was mentioned in §5.2, $\mathbb{R}((\omega))^{LE}$ is an ordered exponential subfield of **No** that is isomorphic to the ordered exponential field $\mathbb{T}$ of transseries. Moreover, $\mathbb{T}$ is equal to the union $\bigcup_{i\in I} H_i$ of a family $\{H_i : i \in I\}$ of Hahn fields having the property: for all $i, j \in I$ there is a $k \in I$ such that $H_i, H_j \subseteq H_k$ (see [4, Appendix A]). Plainly then, for each finite set $a_1, ..., a_n$ of infinitesimals in $\bigcup_{i\in I} H_i$, there is an $m \in I$ such that $a_1, ..., a_n \in H_m$. Moreover, since H_m is a Hahn field, by a classical result of B. H. Neumann [62], for each $f(x_1, ..., x_n) \in \mathbb{R}[[X_1, \dots, X_n]]$, $f(a_1, ..., a_n) \in H_m$. But then $f(a_1, ..., a_n) \in \bigcup_{i\in I} H_i$, which proves the proposition. □

Our next group of examples of structures that satisfy the hypothesis of Theorem 89 comes from work of van den Dries and the second author [27]. The demonstration that these structures do indeed satisfy the said hypothesis rests largely on Propositions 91, 92 and 93 below, the formulations of which require the following definitions.

If Γ is a subgroup of **No** whose universe is a set, then there is a canonical isomorphism f of the Hahn field $\mathbb{R}((\tau^{\Gamma}))$ into **No** for which

$$f(\sum_{\alpha<\beta} \tau^{y_\alpha}.r_\alpha) = \sum_{\alpha<\beta} \omega^{y_\alpha}.r_\alpha.$$

The image of f, denoted $\mathbb{R}((\omega^{\Gamma}))$, is the *Hahn field in* **No** *generated by* Γ. By $\mathbb{R}((\omega^{\Gamma}))_\lambda$ we mean the set consisting of all elements of $\mathbb{R}((\omega^{\Gamma}))$ having supports $(y_\alpha)_{\alpha<\beta<\lambda}$, where λ is a fixed ordinal. Moreover, for each ordinal λ, let $\mathbf{No}(\lambda) := \{x \in \mathbf{No} : \text{tree rank of } x < \lambda\}$ (see §2.1). Finally, as the reader will recall, an *additively indecomposable* ordinal is an ordinal of the form ω^α for some $\alpha \in \mathbf{On}$, and an *ϵ-number* is an ordinal λ such that $\omega^\lambda = \lambda$.

Proposition 91. ([27, Corollary 3.1]) $\mathbf{No}(\lambda)$ (with sums and order inherited from **No**) is an ordered abelian group whenever λ is an additively indecomposable ordinal.

Proposition 92. ([27, Lemma 4.6]) Let λ be an ϵ-number and let Γ be a subgroup of **No**. Then $\mathbb{R}((\omega^{\Gamma}))_\lambda$ (with sums, products and order inherited from **No**) is an ordered field closed under absolute convergence in the sense of Conway.

It should be noted that of the portion of the above result concerned with closure under absolute convergence in the sense of Conway is not explicitly stated as a result in [27], but rather is proved (without the current terminology) in the course of proving the weaker condition of closure under restricted analytic functions (see [27, page 11][21]).

Proposition 93. ([27, Proposition 4.7 (1) and (2)]) Let λ be an ϵ-number. Then:

[21] More specifically, the authors write: "Finally, let $F(X_1, \dots, X_n) \in \mathbb{R}[[X_1, \dots, X_n]]$ be a formal power series in the indeterminates $X_1, \dots, X_n$ with real coefficients. Let $\epsilon_1, \dots, \epsilon_n$ be infinitesimals in $\mathbb{R}((\tau^{\Gamma}))_\lambda$. Since F is not assumed to be a convergent power series, we actually prove more than closure under restricted analytic functions by showing that $F(\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}((\tau^{\Gamma}))_\lambda$.".

(1) $\mathbf{No}(\lambda)$ (with sums, products and order inherited from $\mathbf{No}$) is a real-closed field closed under exponentiation and under taking logarithms of positive elements. Indeed, $\mathbf{No}(\lambda)$ equipped with restricted analytic functions (defined via Taylor expansions) and exponentiation induced by $\mathbf{No}$ is an elementary substructure of $(\mathbf{No}_{\mathrm{an}}, \exp)$ and an elementary extension of $(\mathbb{R}_{\mathrm{an}}, e^x)$.
(2) $\mathbf{No}(\lambda) = \bigcup_\mu \mathbb{R}((\omega^{\mathbf{No}(\mu)}))_\lambda$, where μ ranges over all additively indecomposable ordinals less than λ.

It follows from much the same argument employed in the proof of Theorem 90 that the union of a chain of ordered fields that are closed under absolute convergence in the sense of Conway is itself closed under absolute convergence in the sense of Conway. In virtue of this and Propositions 91, 92 and 93, we now have:

Theorem 94. For each ϵ-number λ, the ordered exponential subfield $\mathbf{No}(\lambda)$ of $(\mathbf{No}, \exp)$ is closed under absolute convergence in the sense of Conway.

Like $\mathbf{No}$ and the extension theory developed in the earlier sections of the paper, Lemma 88 and Theorem 89 as well as the existence of $\mathbb{R}((\omega))^{LE}$ and the $\mathbf{No}(\lambda)$ are provable in NBG$^-$, and are therefore constructive in this sense.[22]

12. Some open questions and a remaining problem

We draw the positive portion of the paper to a close by stating a problem and two open questions that naturally arise from the material in preceding sections.

The mathematical theory of resurgent functions for height one transseries has long been worked out in great detail. In a far ranging recent work [35], however, Écalle has provided what he describes as an "exploratory rather than systematic" presentation of an extension of his theory of resurgent functions, including Écalle-Borel summability, beyond height one transseries to transseries having arbitrary heights and depths. This naturally suggests:

Problem 1. Based on a rigorous theory of arbitrary height and depth transseries, generalize our "constructive" treatment of extension and antidifferentiation operators to all resurgent functions.

A related and perhaps much deeper issue is broached by:

Question 1. Do well-behaved extension operators exist for broad classes of functions that cannot be obtained from the inductive construction yielding transseries? More specifi-

[22] The second author wishes to thank Elliot Kaplan for helpful comments on an earlier version of this section of the paper, and especially for his observation that the proofs of Lemma 88 and Theorem 89 given above continue to hold without the previously stated additional assumption that $(K, \exp_K)$ is initial.

cally, do well-behaved extension operators exist for broad classes of functions defined on surreals of arbitrary length (or at least having lengths larger than ω^ω).

The answer to this question would shed light on the very important but much less understood subject of *formalizability* of functions.[23] We note that Jean Écalle offered the following very interesting observation in response to this question.

> ... under the (reasonable) assumption that the limits for extending operations such as integration on functions of **No** to **No** roughly coincide with the limits for the effective (bi-constructive) formalization for real germs at ∞, one falls back on a subject on which much thought has already been spent, and I think one can confidently predict the broad outline of the answer. The ultimate constructive extensions would:
>
> 1. *include* all formal transfinite iterates of order $\alpha < \omega^\omega$ of the exponential, together with a coherent system of incarnations as real germs (and while the search for *one* privileged system of incarnations is hopeless, comfort may be taken from the fact that all coherent incarnations are isomorphic).
>
> 2. *exclude* the full set of so-called nested expansions (even well-nested ones), for there mutual compatibility conditions would have to be met, which could not possibly be ensured *constructively*, i.e. without massive recourse to AC.

The answer to the following question will shed more light on the deeper structure of the surreal universe.

Question 2. Can the theory of extension, antidifferentiation and integral operators presented in the previous sections of the paper be given a genetic (simplicity-hierarchical) formulation in the *inductive* sense (mentioned in the introduction) that was sort after by Conway, Kruskal and Norton?

The authors do in fact know how to provide a simplicity-hierarchical account for much of the theory in terms of Conway's $\{L|R\}$ notation and hope to present it in a future paper. However, the definitions in terms of Conway's $\{L|R\}$ notation employed in the account are not inductive, and therefore are not genetic in Conway's sense (see Footnote 1).

13. Negative and independence results

With our positive results now at hand, we switch directions by showing that a constructive proof of the existence of analogous extensions and integrals of substantially more general types of functions than those treated above is obstructed by considerations

[23] "[Reducibility] to a properly structured set of real coefficients" [33, p. 75].

from the foundations of mathematics. These considerations apply not only to the surreals, but to any non-Archimedean ordered field $\mathbb{F}$ that extends $\mathbb{R}$ and whose existence can be proved in NBG$^-$. To establish the result, we will direct our attention to a list of very basic properties of antidifferentiation and a space $\mathcal{H}$ of functions with "very good properties".

By a classical result of Mostowski [61], Wang [75], Novack [63], Rosser and Wang [66] and Schoenfield [73], NBG$^-$ is a *conservative extension* of ZF; that is, every theorem of ZF is a theorem of NBG$^-$, and every theorem of NBG$^-$ that can be expressed in ZF (i.e. in the language of sets) is itself a theorem of ZF. Consequently, NBG$^-$ is not only equiconsistent with ZF, but if T is a theory obtained from ZF by supplementing it with a set of axioms A which involve only sets, and T' is obtained from NBG$^-$ by supplementing it with the same set of axioms A, then T is consistent if and only if T' is consistent (see, for example, [48, p. 132]). As a result, the above said relations holding between NBG$^-$ and ZF also hold between NBG$^-$+ C (the Axiom of Choice for sets) and ZFC as well as between NBG$^-$+DC and ZF+DC, where DC is the Axiom of Dependent Choice ([73], [61], [63], [66]). Accordingly, though the main result in this section is concerned with arbitrary non-Archimedean ordered fields that extend $\mathbb{R}$ whose existence can be proved in NBG$^-$, including **No** itself, the preliminary results are about subsets of these structures and as such, when appropriate, to prove these results we freely make use of techniques and results established about or in ZFC, ZF+DC or ZF+DC supplemented with other assertions about sets.

As usual, let ℓ^∞ denote the space of all bounded real-valued sequences, whose members we write as $\{s_n\}$ in place of $\{s_n\}_{n\in\mathbb{N}}$. As the reader will recall, $\phi : \ell^\infty \to \mathbb{R}$ is said to be a *Banach limit* if it is a continuous linear functional satisfying the following conditions: (a) (*positivity*) if $\{s_n\}$ is a nonnegative sequence, then $\phi(\{s_n\}) \geq 0$; (b) (*shift-invariance*) for any sequence $\{s_n\} \in \ell^\infty$, we have $\phi(\{s_n\}) = \phi(\{s_{n+1}\})$; and (c) ($\phi$ *is a limit*) if $\{s_n\}$ is convergent with limit l, then $\phi(\{s_n\}) = l$.

To establish our negative result we will make use of one direction of the following metamathematical result concerning the existence of Banach limits (EBL) which is a simple consequence of results from the literature.[24]

Proposition 95. *EBL is independent of* NBG$^-$*+DC (if* NBG$^-$ *is consistent).*

Proof. Since NBG$^-$+DC is a conservative extension of ZF+DC, it suffices to prove the proposition for ZF+DC. EBL is consistent with ZF+DC, if ZF is consistent, since HB (the Hahn-Banach theorem) implies EBL (e.g. [17, Theorem III.7.1]) and ZF+DC+HB is consistent, if ZF is consistent ([64], [68, p. 516]). Moreover, -EBL is consistent with ZF+DC, if ZF is consistent, since BP (the assertion every set of reals has the Baire

[24] The second author greatfully acknowledges helpful discussions of this matter with Emil Jeřábek [52], Wojowu (Wojtek Wawrów) [76] and others on MathOverflow.

Property) implies -EBL [56, Theorem 44], and there is a model of ZF+DC+BP, if ZF is consistent ([72], [68, p. 516]). □

As we shall see, the following concept is closely related to a Banach limit.

Definition 96. We call L a *sublimit* on ℓ^∞ if:

(1) L is linear, i.e., if $s, t \in \ell^\infty$ and $a, b \in \mathbb{R}$, then $L(as + bt) = aL(s) + bL(t)$;
(2) for every $\{s_n\} \in \ell^\infty$, $L(\{s_n\}) \leq \limsup_{n\to\infty} s_n$.

Using linearity and the fact that $\liminf_{n\to\infty} s_n = -\limsup_{n\to\infty}(-s_n)$, we see that condition (2) of Definition 96 is equivalent to

$$\liminf_{n\to\infty} s_n \leq L(\{s_n\}) \leq \limsup_{n\to\infty} s_n. \tag{88}$$

Lemma 97. ZF proves that a Banach limit exists if and only if a sublimit exists.

Proof. Let B be a Banach limit. We show that B is a sublimit. Indeed, by definition, B is linear. Now assume $\limsup_{n\to\infty} s_n = l$. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that for all $m \geq N$ we have $s_m \leq l + \epsilon$. Let S be the shift operator, $S(\{s_n\}) = \{s_{n+1}\}$. We have, using positivity, $B(\{s_n\}) = BS^N(\{s_n\}) \leq l + \epsilon$ where, as usual, S^N is S applied N times. Since $\epsilon > 0$ is arbitrarily chosen, $B(\{s_n\}) \leq l$.

Now let L be a sublimit. We define the Cesàro summation operator by $C(\{s_n\}) = \{n^{-1}\sum_{j=1}^n s_n\}$. Note that C is a continuous operator on ℓ^∞ of norm 1, and so is L, by (88). We claim that LC is a Banach limit. We just showed that LC is continuous. Moreover, clearly C is a positive operator and so is L by (88). Since $\big[C(\{s_n\}) - CS(\{s_n\})\big]_m = m^{-1}(s_1 - s_{m+1})$, we have $\liminf_{n\to\infty}\big[LC(\{s_n\}) - LCS(\{s_n\})\big] = 0$ and $\limsup_{n\to\infty}\big[LC(\{s_n\}) - LCS(\{s_n\})\big] = 0$, and hence, $LC = LCS$, thereby proving shift-invariance. Finally, if $\{s_n\}$ is convergent with limit l, then plainly $C(\{s_n\})$ is convergent with limit l, and then (88) shows that $LC(\{s_n\})$ is convergent with limit l, completing the proof. □

The extension and antidifferentiation operators introduced in §3 are linear, positive and satisfy a number of other desirable properties. However, the *proto-antidifferentiation* and *proto-extension* operators defined below are only assumed to be linear, positive and satisfy a compatibility condition. By showing that the existence of these proto-operators can neither be proved nor disproved in NBG$^-$+DC, we show that the existence of any extension and antidifferentiation operators having even these minimal properties can neither be proved nor disproved in NBG$^-$+DC, let alone in NBG$^-$. As such, any such operators whose existence can be established in NBG would necessarily be less constructive in nature than E and $\mathsf{A}_{\mathbf{No}}$.

Henceforth, let $\mathbb{F}$ be a non-Archimedean ordered field that extends $\mathbb{R}$ whose existence can be proved in NBG$^-$, and let ρ be an arbitrarily selected and fixed positive infinite

element of $\mathbb{F}$. Also let $\mathcal{H}$ be the space of functions which are real-analytic, that extend to entire functions (holomorphic in $\mathbb{C}$), and decay at least as rapidly as x^{-2} in the sense that $f \in \mathcal{H}$ if and only if $\sup_{x\in\mathbb{R}^+} x^2|f(x)| < \infty$.

Definition 98. Let $A_\rho := \{(x,y) \in (\mathbb{R}^+ \cup \{\rho, \rho^2\})^2 : x < y\}$. A proto-antidifferentiation operator on $\mathcal{H} \times A_\rho$ is an operator λ having the following properties for all $f, g \in \mathcal{H}$, all $(x,y) \in A_\rho$, and all $\alpha, \beta \in \mathbb{R}^+$.

(1) Linearity: $\lambda(\alpha f + \beta g, x, y) = \alpha\lambda(f, x, y) + \beta\lambda(g, x, y)$.
(2) Positivity: If for some $c \in \mathbb{R}^+$ and all $x \in (c, \infty)$ we have $f(x) \geq 0$, then for all $(x,y) \in A_\rho$ with $x \in (c, \infty)$ we have $\lambda(f, x, y) \geq 0$.
(3) Compatibility with the weight of $\mathcal{H}$: $\lambda(x^{-2}, x, y) = x^{-1} - y^{-1}$.

It is easy to see that the operator $(f, x, y) \mapsto \int_x^y f(s)ds$ satisfies the above properties for all real $x > 0$.

Definition 99. Let $E_\rho := \mathbb{R}^+ \cup \{\rho\}$. A proto-extension operator on $\mathcal{H} \times E_\rho$ is an operator Λ having the following properties for all $f \in \mathcal{H}$, all $x \in E_\rho$ and all $\alpha, \beta \in \mathbb{R}^+$.

(1) Linearity: $\Lambda(\alpha f + \beta g, x) = \alpha\Lambda(f, x) + \beta\Lambda(g, x)$.
(2) Positivity: If for some $c \in \mathbb{R}^+$ and all $x \in (c, \infty)$ we have $f(x) \geq 0$, then for all $x \in (c, \infty)$, we have $\Lambda(f, x) \geq 0$.
(3) Compatibility with the weight of $\mathcal{H}$: $\Lambda(x^{-2}, x) = x^{-2}$.

Henceforth, let EPA and EPE be the following statements with $\mathbb{F}$ and ρ understood as above: "There exists a proto-anti-differentiation operator as in Definition 98, and "There exists a proto-extension operator as in Definition 99", respectively. Moreover, henceforth by x° we mean the *standard part* of a finite member x of an ordered field. When x is a finite surreal number, x° is the real part of x (see Definition 3).

Lemma 100. *EPA implies EBL and EPE implies EBL in* NBG^-

Proof. Let $\mathbb{F}$ be an ordered field extending $\mathbb{R}$ that exists in NBG^-, ρ be a positive infinite member of $\mathbb{F}$ and Λ be a proto-extension operator.

For each $s = \{s_n\} \in \ell^\infty$ define $f_s : \mathbb{R} \to \mathbb{R}$ by $f_s(x) = s_1$ for $x \leq 1$, and for $x = (1-t)n + t(n+1)$, where $t \in [0,1]$ and $n \in \mathbb{N}$, by $f_s(x) = (1-t)s_n + ts_{n+1}$. Now let $m \in \mathbb{N}$ and suppose $x > m$. It is easy to check that

$$x^{-2} \inf_{n\geq m} s_n \leq x^{-2} f_s(x) \leq x^{-2} \sup_{n\geq m} s_n. \tag{89}$$

We are now going to approximate f_s by entire functions. For $\epsilon > 0$ let

$$\nu_\epsilon := \nu = 2\|s\|_\infty \pi^{-\frac{1}{2}} \epsilon^{-1} \tag{90}$$

and consider the mollification $f_{s;\epsilon}(x) := \pi^{-\frac{1}{2}}\nu \int_{-\infty}^{\infty} e^{-\nu^2(x-t)^2} f_s(t)dt$. By standard complex analysis, $f_{s;\epsilon}$ is entire, and straightforward estimates show that $\sup_{z\in\mathbb{C}} |e^{-\nu^2|z|^2} f_{s;\epsilon}(z)| < \infty$. Note that, by construction, $|f_s(t) - f_s(x)| \leq 2\|s\|_\infty |t-x|$. Thus, (90) implies

$$|f_{s;\epsilon}(x) - f_s(x)| = \pi^{-\frac{1}{2}}\nu \left| \int_{-\infty}^{\infty} e^{-\nu^2(x-t)^2} (f_s(t) - f_s(x))dt \right|$$
$$\leq 2\|s\|_\infty \pi^{-\frac{1}{2}}\nu \left| \int_{-\infty}^{\infty} e^{-\nu^2 v^2} |v| dv \right| = 2\|s\|_\infty \pi^{-\frac{1}{2}} \nu^{-1} \leq \epsilon. \quad (91)$$

Conditions (2) and (3) of Definition 99 imply that for any $x \in \mathbb{R}^+$ and $\epsilon, \epsilon' > 0$ we have

$$|x^{-2} f_{s;\epsilon}(x) - x^{-2} f_{s;\epsilon'}(x)| \leq x^{-2}(\epsilon + \epsilon'), \quad (92)$$

and hence,

$$|\rho^2 \Lambda(\rho^{-2} f_{s;\epsilon}, \rho) - \rho^2 \Lambda(\rho^{-2} f_{s;\epsilon'}, \rho)| \leq (\epsilon + \epsilon'). \quad (93)$$

Since Equation (91) and the triangle inequality imply that $|\rho^2 \Lambda(\rho^{-2} f_{s;\epsilon}, \rho)| \leq \epsilon + \|s\|_\infty$, $\left(\rho^2 \Lambda(\rho^{-2} f_{s;\epsilon}, \rho)\right)^\circ$ exists, and, by the same argument, so does $\left(\rho^2 \Lambda(\rho^{-2} f_{s;\epsilon'}, \rho)\right)^\circ$. Hence, by taking standard parts in (93), we get

$$\left| \left(\rho^2 \Lambda(\rho^{-2} f_{s;\epsilon}, \rho)\right)^\circ - \left(\rho^2 \Lambda(\rho^{-2} f_{s;\epsilon'}, \rho)\right)^\circ \right| \leq \epsilon + \epsilon', \quad (94)$$

which in turn implies that

$$B_L(s) := \lim_{\epsilon \to 0} \left(\rho^2 \Lambda(\rho^{-2} f_{s;\epsilon}, \rho)\right)^\circ \quad (95)$$

exists. Moreover, it follows from Equation (91) that for any $n \in \mathbb{N}$ and $x \geq n+1$, we have

$$x^{-2} f_{s,\epsilon}(x) \leq x^{-2}(\sup_{n \geq m} s_n + \epsilon). \quad (96)$$

Hence, since $\rho \geq n$ for any $n \in \mathbb{N}$, we have

$$\rho^2 \Lambda(\rho^{-2} f_{s,\epsilon}, \rho) \leq \limsup_{n \to \infty} s_n + \epsilon, \quad (97)$$

which implies

$$B_L(s) \leq \limsup_{n \to \infty} s_n. \quad (98)$$

And so a sublimit exists, and hence, by Lemma 97, a Banach limit exists.

For the portion of the theorem concerned with proto-antidifferentiations (in place of proto-extensions) we change $\Lambda(\rho^{-2}f_{s;\epsilon},\rho)$ to $\lambda(\rho^{-2}f_{s;\epsilon},\rho,\rho^2)$ and the prefactor ρ^2 in front of Λ to a prefactor ρ in front of λ. Since for $a\in\mathbb{R}$, $\rho(a\rho^{-1}-\rho^{-2})^\circ = \rho(a\rho^{-1})^\circ = a$, the proof is, mutatis mutandis, the same. □

Lemma 101. *EBL implies EPA and EPE in* NBG$^-$.

Proof. Let B_L be a Banach limit. To show EPA, define $\lambda(f,x,y)=\int_x^y f(s)ds$ if $(x,y)\in(\mathbb{R}^+)^2$, $\lambda(f,x,\rho)=x^2f(x)(x^{-1}-\rho^{-1})$ for $x\in\mathbb{R}^+$, and $\lambda(f,\rho,\rho^2)=B_L[n^2f(n)](\rho^{-1}-\rho^{-2})$. It is clear that λ is linear and, if $f\geq 0$ is positive and $(x,y)\in A_\rho$, then $\lambda(f,x,y)\geq 0$. For condition (3), we note that if $f(x)=x^{-2}$ then $\lim_{n\to\infty}n^2f(n)=1=B_L(n^2f(n))$ by the definition of a Banach limit, and the property follows.

To show EPE, define $\Lambda(f,x)=f(x)$ if $x\in\mathbb{R}^+$ and $\Lambda(f,\rho)=B_L[n^2f(n)]\rho^{-2}$. It is clear that Λ is linear and, if $f\geq 0$ is positive and $x\in E_\rho$, then $\Lambda(f,x)\geq 0$. For condition (3), we note that if $f(x)=x^{-2}$ then $\lim_{n\to\infty}n^2f(n)=1=B_L(n^2f(n))$ by the definition of a Banach limit, and the property follows. □

Theorem 102.

(1) NBG$^-$ *proves that proto-antidifferentiation operators exist if and only if Banach limits exist.*
(2) NBG$^-$ *proves that proto-extension operators exist if and only if Banach limits exist.*
(3) *EPA and EPE are independent of* NBG$^-$*+DC (if* NBG$^-$*+DC is consistent).*

Proof. (1) and (2) follow from Lemmas 100 and 101, and (3) is an immediate consequence of (1), (2) and Proposition 95. □

Note 103. Other types of negative results can be obtained via Pettis' theorem of automatic continuity, whereby the existence of various other types of desirable extensions would imply the existence of Baire non-measurable sets. This will be explored further in the future paper referred to at the end of the introduction.

Acknowledgments

The authors wish to express their gratitude to Jean Écalle for his important comments and suggestions on an earlier version of this work. We also wish to thank him for granting us permission to quote in §12 his comments regarding the possible extension of our theory.

The work of OC is supported in part by the U.S. National Science Foundation, Division of Mathematical Sciences, Award NSF DMS-2206241.

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