Research Article

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Regularity and monotonicity for solutions to a continuum model of epitaxial growth with nonlocal elastic effects

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Abstract: We study the following parabolic nonlocal 4-th order degenerate equation:

$$u_t = -\Big[2\pi H(u_x) + \ln(u_{xx} + a) + \frac{3}{2}(u_{xx} + a)^2\Big]_{xx},$$

arising from the epitaxial growth on crystalline materials. Here H denotes the Hilbert transform, and $a > 0$ is a given parameter. By relying on the theory of gradient flows, we first prove the global existence of a variational inequality solution with a general initial datum. Furthermore, to obtain a global strong solution, the main difficulty is the singularity of the logarithmic term when $u_{xx} + a$ approaches zero. Thus we show that, if the initial datum u_0 is such that $(u_0)_{xx} + a$ is uniformly bounded away from zero, then such property is preserved for all positive times. Finally, we will prove several higher regularity results for this global strong solution. These finer properties provide a rigorous justification for the global-in-time monotone solution to the epitaxial growth model with nonlocal elastic effects on vicinal surface.

Keywords: Fourth order degenerate parabolic equation, global strong solution, regularity, monotonicity

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1 Introduction

One of the most affordable manufacturing processes to produce several key semiconductor materials is the epitaxial growth on crystal surface [17, 18]. It is also used to design experimental materials to show high temperature superconducting properties, or the quantum anomalous hall effect, in magnetic topological insulators [5]. During the growth process, different coherent states are formed due to the balance of competing influences, which is crucial to the study of the various structures of crystal surfaces. The presence of these complicated competing effects usually leads to a high-order, nonlinear, nonlocal model, which requires mathematical validations at both macroscopic and microscopic scales.

The formal derivation of the continuum limit generally starts from a mesoscopic description such as Burton–Cabrera–Frank (BCF) step models [3, 20]; see [7, 19, 21–23]. In these models, several authors con-

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sidered a discrete energy functional

$$E_i = \sum_{i,j} c_1 \ln|x_i - x_j| + c_2 \frac{1}{(x_i - x_j)^2}$$

to incorporate the global elastic interaction between steps x_i and x_j, where c_1, c_2 are proper scaling constants. The resulting epitaxial growth model in terms of the continuum variable $h(x, t)$, which represents the thin film height, is

$$h_t = -[2\pi H(h_x) + (h_x^{-1} + 3h_x)h_{xx}]_{xx}. \tag{1.1}$$

Here

$$H(v)(x) := \frac{1}{|I|} \int_I v(x-y) \cot \frac{\pi y}{|I|} \,\mathrm{d}y$$

denotes the Hilbert transform on a periodic domain $I := (0, 1)$. Under the assumption that the slope h_x of the thin film height h is strictly positive, i.e. $h_x > 0$ for any $t > 0$, Gao, Liu and Lu [10] gave a rigorous proof of the convergence from mesoscopic BCF step models to (1.1). They also obtained the local smooth solution whose monotonicity is preserved up to a (positive) time.

Concerning global solutions, Dal Maso, Fonseca and Leoni [6], and Fonseca, Leoni and Lu [8] showed the existence of a global weak solution by considering another equation for the anti-derivative u of h, which satisfies $h_x = u_{xx} + a$ for some constant $a > 0$, under the assumption that the initial datum is sufficiently regular. That is, the authors considered the parabolic variational equation

$$u_t = -[2\pi H(u_x) + \Phi_a'(u_{xx})]_{xx}, \quad \Phi_a(\xi) := \Phi(\xi + a), \quad \Phi(\xi) := \begin{cases} +\infty, & \xi < 0, \\ 0, & \xi = 0, \\ \xi \ln \xi + \frac{\xi^3}{2}, & \xi > 0, \end{cases} \tag{1.2}$$

on the spatial domain $I = (0, 1)$ with periodic boundary conditions and time $t \geq 0$.

It has been shown in [10, Section 2] that if $u_{xx}(t) + a > 0$ for all $t \geq 0$, then (1.1) can be formally written in the form of the L^2-gradient flow

$$u_t = -\frac{\delta E}{\delta u} = -\Big[2\pi H(u_x) + \ln(u_{xx} + a) + \frac{3}{2}(u_{xx} + a)^2\Big]_{xx}, \tag{1.3}$$

where

$$E(u) := \frac{1}{|I|} \int_I \int_I \ln|\sin(\pi(x-y))|(u_{xx} + a)(u_{yy} + a) \,\mathrm{d}y \,\mathrm{d}x + \int_I \Phi(u_{xx} + a) \,\mathrm{d}x, \tag{1.4}$$

$$\frac{\delta E}{\delta u} := [2\pi H(u_x) + \Phi'(u_{xx} + a)]_{xx}.$$

However, the two equations (1.1) and (1.3) are equivalent under the assumption that $h_x = u_{xx} + a$ is strictly positive for any time; see [6, Theorem 3.1] and also [10, Section 2.5]. To the best of our knowledge, for arbitrarily large times, whether the solution to (1.1) remains strictly monotone is a long-standing question that was never addressed in previous literature. We also refer to [9, 11–16, 24] and the references therein for some other related 4-th order degenerate equations. Instead of the nonlocal term $H(h_x)$ in (1.1) resulting from the global interactions between mesoscopic steps, the 4-th order degenerate equations in [9, 11–16, 24] involve only locally defined terms h_x, h_{xx} due to the nearest-neighbor interactions between steps.

In this paper, we will study finer regularity properties of solutions to (1.2). First, we will prove the existence of a solution in the evolution variational inequality (EVI) sense (see Definition 1.1 below) without the extra regularity assumption [8, (5)] on the initial datum. The second goal is to prove the higher-order regularity and long time behavior of the global strong solution. This is mainly achieved by carefully studying the sub-differential of the total energy E. An important consequence is that the solution to (1.1) remains strictly monotone, which also gives the justification that the hydrodynamic limit proved in [10] from the mesoscopic step models to (1.1) is indeed true for any large time. Another consequence is that the global strong solution converges exponentially to its unique equilibrium.

One of the key issues is that the logarithmic term $\ln(u_{xx}+a)$ has an asymptote as $u_{xx}+a$ approaches zero. This also leads to the issue that the sub-differential of E is not easy to characterize, and can become quite complicated as $u_{xx}+a$ approaches zero. To overcome these issues, we will exploit the gradient flow structure of (1.2) to obtain an important a priori estimate; see Section 3.2. The theory of gradient flows in Hilbert spaces is very well developed. For the corresponding results in metric spaces, we refer the interested readers to [1].

Before introducing our main results, we first clarify that the functional spaces will be $L^2_{\mathrm{per}_0}(I)$, i.e. the space of functions that are square integrable, periodic, with zero average, endowed with the inner product

$$\langle u, v\rangle := \int_I uv\,\mathrm{d}x,$$

and $W^{k,p}_{\mathrm{per}_0}(I)$, defined as the space of functions in $W^{k,p}(I)$ that are periodic and have zero average.

Definition 1.1. Given an initial datum

$$u_0 \in \overline{D(E)}^{\|\cdot\|_{L^2(I)}},$$

we call

$$u : [0,+\infty) \longrightarrow \overline{D(E)}^{\|\cdot\|_{L^2(I)}}$$

a variational inequality solution to (1.3) if $u(t)$ is a locally absolutely continuous curve such that

$$\lim_{t\to 0} u(t) = u_0$$

in $L^2(I)$ and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-v\|^2_{L^2(I)} \le E(v) - E(u(t)) \quad \text{for a.e. } t>0 \text{ and all } v\in D(E).$$

Here, and in the rest of this paper, $D(\cdot)$ will denote the effective domain of a given functional, i.e.

$$D(E) = \{v \in L^2(I) : E(v) < +\infty\},$$

and $\overline{D(E)}^{\|\cdot\|_{L^2(I)}}$ denotes the closure of $D(E)$ with respect to the L^2 distance.

Let us state the main results below.

Theorem 1.2. *Let E be the energy defined in (1.4). Given an initial datum $u_0 \in \overline{D(E)}^{\|\cdot\|_{L^2(I)}}$, equation (1.3) admits a unique EVI solution u, in the sense of Definition 1.1, satisfying*

$$u \in L^\infty_{\mathrm{loc}}(0,+\infty; W^{2,3}_{\mathrm{per}_0}(I)). \tag{1.5}$$

Moreover, if $E(u_0) < +\infty$, we have $u_t \in L^\infty(0,+\infty; L^2(I))$.

Note that (1.5) allows a more general initial datum compared with both [8, Theorem 1] and [10, Theorem 1.1].

Theorem 1.3. *Assume the initial datum*

$$u_0 \in D(\partial E) = \{v \in L^2(I) : \text{the sub-differential } \partial E(v) \neq \emptyset\},$$

and $(u_0)_{xx} + a \ge c > 0$ for some $c > 0$. Then the solution given by Theorem 1.2 is a global strong solution to

$$u_t = -\Big[2\pi H(u_x) + \ln(u_{xx}+a) + \frac{3}{2}(u_{xx}+a)^2\Big]_{xx}$$

and satisfies the following assertions:

(i) *The sub-differential $\partial E(u(t))$ is single-valued for all t, and is given by*

$$\frac{\delta E}{\delta u} := [2\pi H(u_x(t)) + \Phi'(u_{xx}(t)+a)]_{xx}.$$

(ii) *The right metric derivative satisfies*

$$|u'_+|(t) := \lim_{s\downarrow t^+} \frac{\|u(s)-u(t)\|_{L^2(I)}}{s-t} = \Big\|\frac{\delta E}{\delta u}\Big\|_{L^2(I)}$$

for all $t > 0$.

(iii) *The map* $t \mapsto E(u(t))$ *is convex, while*

$$t \mapsto \left\|\frac{\delta E}{\delta u}(t)\right\|_{L^2(I)} \exp(2(\sqrt{3} - 2\ln 2)t)$$

is nonincreasing and right continuous.

(iv) *It holds*

$$u_{xxx}, [\Phi'(u_{xx} + a)]_{xx} \in L^2(0, +\infty; L^2(I)) \cap L^\infty(0, +\infty; L^2(I)), \tag{1.6}$$

$$[\ln(u_{xx} + a)]_x, [(u_{xx} + a)^2]_x \in L^2(0, +\infty; C^0(I)) \cap L^\infty(0, +\infty; C^0(I)), \tag{1.7}$$

$$u_{xxx} \in L^2(0, +\infty; C^0(I)) \cap L^\infty(0, +\infty; C^0(I)), \tag{1.8}$$

$$u_{xxxx}, [\ln(u_{xx}(\cdot, t) + a)]_{xx}, [(u_{xx}(\cdot, t) + a)^2]_{xx} \in L^\infty(0, +\infty; L^2(I)). \tag{1.9}$$

(v) *There exists a lower bound* $c^* > 0$*, defined in* (3.18)*, such that*

$$u_{xx}(t) + a \geq c^* > 0 \quad \text{for any } t > 0.$$

(vi) *The exponential decay to the unique equilibrium* $u^* \equiv 0$*, i.e.*

$$\|u(t) - u^*\|^2_{L^2(I)} \leq \frac{1}{C}(E(u_0) - E(u^*))e^{-4Ct} \quad \text{for all } t > 0,$$

holds, where $C := \sqrt{3} - 2\ln 2 > 0$.

We remark that the assumptions in Theorem 1.3 on the initial datum are equivalent to

$$(u_0)_{xx} + a \geq c > 0, \quad \|\partial E(u_0)\|_{L^2(I)} = \left\|\frac{\delta E}{\delta u}(u_0)\right\|_{L^2(I)} < +\infty$$

due to the calculations for sub-differential ∂E in Lemma 3.1 below.

As an important consequence, since the strong solution u to (1.2) satisfies $u_{xx} + a \geq c^* > 0$, (1.1) and (1.2) are equivalent in a rigorous way, and the function h, whose slope is $h_x = u_{xx} + a \geq c^* > 0$ and which satisfies $\int_I h\,dx = a$, is effectively a solution to (1.1).

Another conclusion is that, for a given a, the steady state solution to (1.1) must be an oblique line with slope a.

This paper is structured as follows. In Section 2, we show that E is λ-convex (see Definition 2.1 below) and lower semi-continuous in $L^2(I)$. This allows us to use the theory of gradient flows of λ-convex energies from [1] to prove Theorem 1.2. In Section 3, we perform crucial a priori estimates and calculations of the sub-differential of E, showing that it is indeed single-valued. This finally allows us to prove the higher regularity results in Theorem 1.3.

2 A gradient flow approach for EVI solution

In this section, we prove the existence of a solution in the EVI sense, by following the gradient flow approach introduced in [1].

We will work almost always in $D(E)$ (i.e. $E(u) < +\infty$), which, as shown in Lemma 2.2 below, is contained in $W^{2,3}_{\mathrm{per}_0}(I)$. It is worth noting that, as our energy requires $u_{xx} + a \geq 0$ a.e., this non-negative condition is preserved when taking the limit. Indeed, let $u_n \subseteq D(E)$ be a sequence with $\sup_n E(u_n) < +\infty$. Then

$$\sup_n \|u_n\|_{W^{2,3}_{\mathrm{per}_0}(I)} < +\infty.$$

Hence, up to a sub-sequence, u_n converges strongly in $L^2_{\mathrm{per}_0}(I)$ to some function $u \in L^2_{\mathrm{per}_0}(I)$, satisfying $u_{xx} + a \geq 0$ a.e.

Before proving the properties for the energy functional E, we recall the definition of λ-convexity from [1] below.

Definition 2.1. Given a functional $\phi : L^2_{\mathrm{per}_0}(I) \to (-\infty, +\infty]$, we say ϕ is λ-convex along curves in the metric space $(L^2_{\mathrm{per}_0}(I), \|\cdot\|_{L^2(I)})$ if

$$\phi((1-t)y_0 + ty_1) \le (1-t)\phi(y_0) + t\phi(y_1) - \frac{1}{2}\lambda t(1-t)\|y_0 - y_1\|^2_{L^2(I)} \quad \text{for all } t \in [0,1],$$

for any $y_0, y_1 \in L^2_{\mathrm{per}_0}(I)$.

Lemma 2.2. *The energy E is $2C$-convex with $C := \sqrt{3} - 2\ln 2 > 0$, and lower semi-continuous with respect to the weak L^2-topology. Moreover, the sub-levels of E are compact in the strong L^2-topology.*

Proof. We prove this lemma in four steps.

(1) Boundedness of E from below. Since for any $u \notin D(E)$ we have $E(u) = +\infty$, we need only to prove $E > -\infty$ on its domain $D(E)$. First, for the second part of the energy $E(u)$ in (1.4), given $u \in L^3(I)$ such that $u_{xx} + a \ge 0$ a.e., we have

$$\begin{aligned}\int_I \Phi(u_{xx}+a)\,dx &= \frac{1}{2}\|u_{xx}+a\|^3_{L^3(I)} + \int_I (u_{xx}+a)\ln(u_{xx}+a)\,dx \\ &\ge \frac{1}{2}\|u_{xx}+a\|^3_{L^3(I)} + |I|\cdot \inf_{\xi>0}\xi\ln\xi.\end{aligned} \tag{2.1}$$

Second, we turn to estimating the first part of the energy $E(u)$ in (1.4). Set

$$g(\xi) := \ln|\sin(\pi\xi)| \le 0.$$

The first term in E becomes

$$\frac{1}{|I|}\int_I (u_{xx}+a)\Big[\underbrace{\int_I g(x-y)(u_{yy}+a)(y)\,dy}_{:=T(x)}\Big]\,dx.$$

Since $g \le 0$ and $u_{yy} + a \ge 0$, we have

$$\begin{aligned}0 \le -T(x) &= -\int_{\mathbb{R}} g(x-y)(u_{yy}+a)\mathbf{1}_I(y)\,dy \\ &= -\int_{\mathbb{R}} [g\cdot\mathbf{1}_{(x-1,x)}](x-y)[(u_{yy}+a)\cdot\mathbf{1}_I](y)\,dy \\ &\le -\int_{\mathbb{R}} [g\cdot\mathbf{1}_{(-1,1)}](x-y)[(u_{yy}+a)\cdot\mathbf{1}_I](y)\,dy \\ &= -\{[g\cdot\mathbf{1}_{(-1,1)}] * [(u_{yy}+a)\cdot\mathbf{1}_I]\}(x),\end{aligned} \tag{2.2}$$

where we used the fact that $x \in (0,1)$ implies $(x-1,x) \subseteq (-1,1)$. Therefore, by Young's inequality, we can estimate the absolute value of the first term in E:

$$\begin{aligned}&\Big|\frac{1}{|I|}\int_I (u_{xx}+a)\Big[\int_I g(x-y)(u_{yy}+a)\,dy\Big]\,dx\Big| \\ &\quad= \frac{1}{|I|}\int_I (u_{xx}+a)\Big[\underbrace{-\int_I g(x-y)(u_{yy}+a)\,dy}_{=-T(x)\ge 0}\Big]\,dx \\ &\quad\le \frac{1}{|I|}\int_I \big|(u_{xx}+a)[(-g\cdot\mathbf{1}_{(-1,1)}) * (\mathbf{1}_I\cdot(u_{yy}+a))]\big|\,dx \quad \text{(by (2.2))} \\ &\quad\le \frac{1}{|I|}\|u_{xx}+a\|_{L^2(I)}\big\|[g\cdot\mathbf{1}_{(-1,1)}] * (\mathbf{1}_I\cdot(u_{yy}+a))\big\|_{L^2(I)} \\ &\quad\le \frac{1}{|I|}\|g\|_{L^1(-1,1)}\|u_{xx}+a\|^2_{L^2(I)},\end{aligned} \tag{2.3}$$

where

$$\|g\|_{L^1(-1,1)} = 2\int_0^1 -\ln|\sin(\pi\xi)|\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^\pi -\ln|\sin(w)|\,\mathrm{d}w = 2\ln 2. \tag{2.4}$$

Combining this with (2.1), we obtain

$$\begin{aligned} E(u) &\geq \frac{1}{2}\|u_{xx}+a\|^3_{L^3(I)} + |I|\cdot(\inf_{\xi>0}\xi\ln\xi) - \|g\|_{L^1(-1,1)}\|u_{xx}+a\|^2_{L^2(I)} \\ &= \frac{1}{2}\|u_{xx}+a\|^3_{L^3(I)} - \frac{1}{e} - (2\ln 2)\|u_{xx}+a\|^2_{L^2(I)}. \end{aligned} \tag{2.5}$$

Thus $-\infty < \inf E \leq E(u) < +\infty$ implies that $u \in W^{2,3}_{\mathrm{per}_2}(I)$. Hence $D(E) \subset W^{2,3}_{\mathrm{per}_0}(I)$.

Moreover, since $\ln\xi \leq \xi$ for any $\xi \geq 1$, we get $\ln(u_{xx}+a) \leq u_{xx}+a$ whenever $u_{xx}+a \geq 1$, and

$$\begin{aligned} \int_I (u_{xx}+a)\ln(u_{xx}+a)\,\mathrm{d}x &= \int_{\{u_{xx}+a\geq 1\}} (u_{xx}+a)\ln(u_{xx}+a)\,\mathrm{d}x + \int_{\{u_{xx}+a<1\}} (u_{xx}+a)\ln(u_{xx}+a)\,\mathrm{d}x \\ &\leq \|u_{xx}+a\|^2_{L^2(I)} + |I|\cdot\sup_{1>\xi>0}\xi\ln\xi. \end{aligned}$$

This, together with estimate (2.3) for the first term in E, shows that

$$E(u) \leq \frac{1}{2}\|u_{xx}+a\|^3_{L^3(I)} + \Big(\frac{2\ln 2}{|I|}+1\Big)\|u_{xx}+a\|^2_{L^2(I)} \leq \|u_{xx}+a\|^3_{L^3(I)} + c. \tag{2.6}$$

(2) λ-convexity in $L^2_{\mathrm{per}_0}(I)$. First, note that if in the λ-convexity inequality

$$E((1-t)u_0+tu_1) \leq (1-t)E(u_0) + tE(u_1) - \frac{1}{2}\lambda t(1-t)\|u_0-u_1\|^2_{L^2(I)}$$

we have either $E(u_0)=+\infty$ or $E(u_1)=+\infty$, then the inequality is trivial. Thus assume both terms are finite. This requires $(u_i)_{xx}+a \geq 0$ a.e., $i=1,2$, and hence $((1-t)u_0+tu_1)_{xx}+a \geq 0$ a.e., too. Thus we can restrict our attention to functions u such that $u_{xx}+a \geq 0$ a.e.

Note that

$$(\Phi(\xi) - \sqrt{3}\xi^2)'' = 3\xi + \xi^{-1} - 2\sqrt{3} \geq 0 \quad \text{for } \xi > 0.$$

Hence

$$u \mapsto \int_I [\Phi(u_{xx}+a) - \sqrt{3}(u_{xx}+a)^2]\,\mathrm{d}x \quad \text{is convex.} \tag{2.7}$$

Rewrite the energy as

$$\begin{aligned} E(u) &= \underbrace{\int_I [\Phi(u_{xx}+a) - \sqrt{3}(u_{xx}+a)^2]\,\mathrm{d}x}_{\text{convex}} \\ &\quad + \sqrt{3}\|u_{xx}+a\|^2_{L^2(I)} + \frac{1}{|I|}\int_I (u_{xx}+a)\Big[\int_I g(x-y)(u_{yy}+a)\,\mathrm{d}y\Big]\,\mathrm{d}x. \end{aligned} \tag{2.8}$$

Next we will prove that the sum of the last two terms in $E(u)$ above is λ-convex.

Given $u, v \in D(E)$, $t \in [0,1]$, notice on the one hand,

$$\begin{aligned} &\int_I\int_I g(x-y)[(1-t)(u_{yy}+a) + t(v_{yy}+a)]\cdot[(1-t)(u_{xx}+a) + t(v_{xx}+a)]\,\mathrm{d}y\,\mathrm{d}x \\ &\quad= \int_I\int_I g(x-y)[(1-t)(u_{xx}+a)(u_{yy}+a) + t(v_{xx}+a)(v_{yy}+a) - t(1-t)(u-v)_{xx}(u-v)_{yy}]\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= (1-t)\int_I\int_I g(x-y)(u_{xx}+a)(u_{yy}+a)\,\mathrm{d}x\,\mathrm{d}y + t\int_I\int_I g(x-y)(v_{xx}+a)(v_{yy}+a)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad - t(1-t)\int_I\int_I g(x-y)(u-v)_{xx}(u-v)_{yy}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

On the other hand,

$$\|(1-t)(u_{xx}+a)+t(v_{xx}+a)\|_{L^2(I)}^2=(1-t)\|u_{xx}+a\|_{L^2(I)}^2+t\|v_{xx}+a\|_{L^2(I)}^2-t(1-t)\|(u-v)_{xx}\|_{L^2(I)}^2.$$

Thus

$$\begin{aligned}\sqrt{3}\|(1-t)(u_{xx}+a)+t(v_{xx}+a)\|_{L^2(I)}^2&+\frac{1}{|I|}\int_I\int_I g(x-y)[(1-t)(u_{yy}+a)+t(v_{yy}+a)]\\&\cdot[(1-t)(u_{xx}+a)+t(v_{xx}+a)]\,dy\,dx\\=(1-t)\Big[\sqrt{3}\|u_{xx}+a\|_{L^2(I)}^2&+\frac{1}{|I|}\int_I\int_I g(x-y)(u_{xx}+a)(u_{yy}+a)\,dx\,dy\Big]\\&+t\Big[\sqrt{3}\|v_{xx}+a\|_{L^2(I)}^2+\frac{1}{|I|}\int_I\int_I g(x-y)(v_{xx}+a)(v_{yy}+a)\,dx\,dy\Big]\\&-t(1-t)\Big[\sqrt{3}\|(u-v)_{xx}\|_{L^2(I)}^2+\frac{1}{|I|}\int_I\int_I g(x-y)(u-v)_{xx}(u-v)_{yy}\,dx\,dy\Big].\end{aligned}\tag{2.9}$$

From (2.3), we know that

$$\begin{aligned}\sqrt{3}\|(u-v)_{xx}\|_{L^2(I)}^2+\frac{1}{|I|}\int_I\int_I g(x-y)(u-v)_{xx}(u-v)_{yy}\,dx\,dy&\geq C\|(u-v)_{xx}\|_{L^2(I)}^2\\&\geq C\|u-v\|_{L^2(I)}^2,\end{aligned}$$

where

$$C:=\sqrt{3}-\|g\|_{L^1(-1,1)}=\sqrt{3}-2\ln 2>0.$$

This, together with (2.9), implies that

$$u\mapsto\sqrt{3}\|u_{xx}+a\|_{L^2(I)}^2+\frac{1}{|I|}\int_I(u_{xx}+a)\Big[\int_I g(x-y)(u_{yy}+a)\,dy\Big]\,dx$$

is λ-convex in $L^2(I)$ with $\lambda=2C$. Thus (2.7) follows, and E is also $2C$-convex in $L^2(I)$.

(3) Lower semi-continuity. Consider a sequence $u_n\to u$ weakly in $L^2(I)$. We need to show

$$\liminf_{n\to+\infty}E(u_n)\geq E(u).$$

Assume, without loss of generality, that lim inf is an actual limit, and that $\sup_n E(u_n)<+\infty$, as otherwise the inequality is trivial. So we know $(u_n)_{xx}+a\geq 0$ and $u_{xx}+a\geq 0$ a.e.

Boundedness of energy $E(u_n)$ implies, by (2.5), that $(u_n)_{xx}+a$ is bounded in $L^3(I)$. Then we know $(u_n)_{xx}\to u_{xx}$ weakly in $L^3(I)$ and $u_n\to u$ strongly in $H^1(I)$. Therefore,

$$\|u_{xx}\|_{L^3(I)}^3\leq\liminf_{n\to+\infty}\|(u_n)_{xx}\|_{L^3(I)}^3<+\infty.$$

This, together with (2.6), implies $E(u)<+\infty$.

We recall the previous (2.8). For the last term, we have

$$\liminf_{n\to+\infty}\frac{1}{|I|}\int_I((u_n)_{xx}+a)\Big[\int_I g(x-y)((u_n)_{yy}+a)\,dy\Big]\,dx=\frac{1}{|I|}\int_I(u_{xx}+a)\Big[\int_I g(x-y)(u_{yy}+a)\,dy\Big]\,dx$$

due to the dominated convergence theorem. The other term

$$\underbrace{\int_I[\Phi(u_{xx}+a)-\sqrt{3}(u_{xx}+a)^2]\,dx+\sqrt{3}\|u_{xx}+a\|_{L^2(I)}^2}_{\text{convex}}$$

is convex and weakly lower semi-continuous. Thus

$$E(u)\leq\liminf_{n\to+\infty}E(u_n),$$

and hence we conclude that E is lower semi-continuous with respect to the weak L^2-topology.

(4) Compactness of sub-levels. Consider a sequence u_n with $E(u_n) \le c$. Boundedness of energy $E(u_n)$ implies, by (2.5), that $(u_n)_{xx} + a$ is bounded in $L^3(I)$. Thus there exists u such that $(u_n)_{xx} \to u_{xx}$ weakly in $L^3(I)$, and $u_n \to u$ strongly in $L^2(I)$. By the lower semi-continuity of E,

$$E(u) \le \liminf_{n\to+\infty} E(u_n) \le c.$$

Thus we complete the proof of this lemma. □

Proof of Theorem 1.2. Notice in Lemma 2.2 we show that all hypotheses of [1, Theorem 4.0.4] are satisfied, with energy E, Hilbert space $L^2_{\mathrm{per}_0}(I)$ and $\lambda = 2C > 0$.

Then by conclusions [1, Theorem 4.0.4 (ii) and (iii)], we know there exists a unique solution u such that $u(t) \in D(E)$, $t > 0$, is a locally absolutely continuous curve with $\lim_{t\to 0^+} u(t) = u_0$ in $L^2(I)$ and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - v\|^2_{L^2} + \frac{1}{2}\lambda\|u(t) - v\|^2_{L^2} + E(u(t)) \le E(v) \quad \text{for a.e. } t > 0 \text{ for all } v \in D(E). \tag{2.10}$$

Then, combining it with the lower bound estimate for E in (2.5), we conclude

$$u \in L^\infty_{\mathrm{loc}}(0, +\infty; W^{2,3}_{\mathrm{per}_0}(I)).$$

Now we turn to proving $u_t \in L^\infty(0, +\infty; L^2(I))$. First, we know that $t \mapsto u(t)$ is locally Lipschitz in $(0, +\infty)$, i.e. for any $t_0 > 0$ there exists $L = L(t_0) > 0$ such that

$$\|u(t_0 + \varepsilon) - u(t_0)\|_{L^2(I)} \le L(t_0)\varepsilon \quad \text{for all } \varepsilon \ge 0.$$

Next, we need to show that such $L(t_0)$ can be essentially taken independent of t_0. For any $t_0 \ge 0$, from (2.10) and $\lambda = 2C > 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - u(t_0)\|^2_{L^2} \le E(u(t_0)) - E(u(t)) \quad \text{for a.e. } t > 0.$$

Then, by conclusion [1, Theorem 4.0.4 (ii)] and the lower bound estimate for E in (2.5), we know

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - u(t_0)\|^2_{L^2} \le E(u_0) + c_0 < \infty,$$

where c_0 is an uninfluential constant. Thus the function $t \mapsto \|u(t_0) - u(t)\|_{L^2(I)}$ is globally Lipschitz with Lipschitz constant less than $E(u_0) + c_0$, which is independent of t_0. From [2, Theorem 1.17], u is differentiable a.e. in $[0, T]$ with respect to $L^2(I)$, and belongs to $W^{1,\infty}([0, T]; L^2(I))$. Hence we know

$$\left\|\frac{u(t_0) - u(t_0 + \varepsilon)}{\varepsilon}\right\|_{L^2(\Omega)} \le E(u_0) + c_0.$$

Thus for a.e. t we have

$$\frac{u(t + \varepsilon) - u(t)}{\varepsilon} \in L^2(I), \qquad \left\|\frac{u(t + \varepsilon) - u(t)}{\varepsilon}\right\|_{L^2(\Omega)} \le E(u_0) + c_0,$$

and the sequence of difference quotients $\frac{u(t+\varepsilon)-u(t)}{\varepsilon}$ is uniformly bounded in $L^2(I)$. Since u is differentiable a.e. in $[0, T]$ and the derivative is unique, we can define

$$\partial_t u(t) := \lim_{\varepsilon\to 0} \frac{u(t + \varepsilon) - u(t)}{\varepsilon},$$

and consequently,

$$\|\partial_t u\|_{L^\infty(0,T;L^2(I))} \le E(u_0) + c_0.$$

The proof is thus complete. □

3 Higher regularity and globally positivity

In this section, we concentrate on proving Theorem 1.3 for the existence and regularity of the strong solution to (1.3), and for the positive lower bound for $u_{xx} + a$. We will first calculate the sub-differential of E for

$u_{xx} + a > 0$ in Section 3.1. Assume $T_{\max}$ is the maximal time (including the case $T_{\max} = +\infty$) such that

$$u_{xx}(t) + a \geq \frac{c^*}{2} > 0, \quad t \in [0, T_{\max}], \tag{3.1}$$

for some positive constant $c^* > 0$. From the local-in-time smooth solution obtained in [10], we know if the initial datum u_0 satisfies $(u_0)_{xx} + a > \frac{c^*}{2}$, then $T_{\max} > 0$. In Section 3.2, we will give the key a priori estimate to show that indeed there is a uniform lower bound c^* such that $u_{xx}(t) + a \geq c^*$ for all times t, and thus $T_{\max} = +\infty$. This significantly simplifies the sub-differential computations since one of the key issues is the singularity given by the logarithmic term $\ln(u_{xx}(t) + a)$. Finally, we will prove Theorem 1.3 in Section 3.3.

3.1 Sub-differential computations when $u_{xx} + a > 0$

In this subsection, we calculate the sub-differential of E when $u_{xx} + a > 0$. The main result is the following.

Lemma 3.1 (sub-differential is single-valued). *For any $u \in D(\partial E)$ such that $u_{xx} + a > 0$, the sub-differential $\partial E(u)$ is single-valued and is given by*

$$\partial E(u) = \{[2\pi H(u_x) + \Phi'(u_{xx} + a)]_{xx}\}. \tag{3.2}$$

Proof. We prove this lemma in two steps.

Step 1. We first prove

$$[2\pi H(u_x) + \Phi'(u_{xx} + a)]_{xx} \in \partial E(u). \tag{3.3}$$

Consider an arbitrary $u \in D(\partial E) \subseteq D(E)$, and let φ be a test function. Without loss of generality, we can assume $u + \varepsilon\varphi \in D(E)$, too, because otherwise we would have

$$\lim_{\varepsilon \to 0} \frac{\Phi_a(u_{xx} + \varepsilon\varphi_{xx}) - \Phi_a(u_{xx})}{\varepsilon} = +\infty,$$

which immediately yields (3.3).

We calculate the elements of $\partial E(u)$ term by term. First, by the convexity of Φ_a, we have

$$\varepsilon \int_I \Phi_a'(u_{xx})\varphi_{xx} \leq \int_I [\Phi_a(u_{xx} + \varepsilon\varphi_{xx}) - \Phi_a(u_{xx})], \tag{3.4}$$

and then the term $[\Phi_a'(u_{xx})]_{xx}$ belongs to the sub-differential of $\int_I \Phi_a(u_{xx})\,\mathrm{d}x$.

Next, we analyze the first part in the energy term:

$$\begin{aligned}
&\int_I\int_I \ln|\sin(\pi(x-y))|(u_{yy} + \varepsilon\varphi_{yy} + a)(u_{xx} + \varepsilon\varphi_{xx} + a)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad - \int_I\int_I \ln|\sin(\pi(x-y))|(u_{yy} + a)(u_{xx} + a)\,\mathrm{d}y\,\mathrm{d}x \\
&= \varepsilon \int_I\int_I \ln|\sin(\pi(x-y))|[\varphi_{yy}(u_{xx} + a) + \varphi_{xx}(u_{yy} + a)]\,\mathrm{d}y\,\mathrm{d}x && (3.5)\\
&\qquad + \varepsilon^2 \int_I\int_I \ln|\sin(\pi(x-y))|\varphi_{xx}\varphi_{yy}\,\mathrm{d}y\,\mathrm{d}x. && (3.6)
\end{aligned}$$

Again, by writing as a convolution, we have

$$\begin{aligned}
\int_I\int_I \ln|\sin(\pi(x-y))|\varphi_{xx}\varphi_{yy}\,\mathrm{d}y\,\mathrm{d}x &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} \ln|\sin(\pi(x-y))|\varphi_{yy}\mathbf{1}_I(y)\,\mathrm{d}y\Big)\varphi_{xx}\mathbf{1}_I(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} (\ln|\sin(\pi \times \cdot)| * (\mathbf{1}_I\varphi'')(x))\varphi_{xx}\mathbf{1}_I(x)\,\mathrm{d}x.
\end{aligned}$$

Hence, noting that $x, y \in I$ implies $x - y \in (-1, 1)$, we obtain

$$
\begin{aligned}
\Big|\int_I\int_I \ln|\sin(\pi(x-y))|\varphi_{xx}\varphi_{yy}\,\mathrm{d}y\,\mathrm{d}x\Big| &\le \int_{\mathbb{R}}\big|((\mathbf{1}_{(-1,1)}\ln|\sin(\pi\times\cdot)|)*(\mathbf{1}_I\varphi'')(x))\varphi_{xx}\mathbf{1}_I(x)\big|\,\mathrm{d}x\\
&\le \|\mathbf{1}_{(-1,1)}\ln|\sin(\pi\times\cdot)|*(\mathbf{1}_I\varphi'')\|_{L^2(\mathbb{R})}\|\mathbf{1}_I\varphi''\|_{L^2(\mathbb{R})}\\
&\le 2\ln 2\|\varphi''\|^2_{L^2(I)},
\end{aligned}
$$

where we use (2.4). Hence the term in (3.6) is of order $O(\varepsilon^2)$.

Now we turn our attention to the term in (3.5). Note first that, by a simple change of variable,

$$
\begin{aligned}
&\int_I\int_I \ln|\sin(\pi(x-y))|[\varphi_{yy}(u_{xx}+a)+\varphi_{xx}(u_{yy}+a)]\,\mathrm{d}y\,\mathrm{d}x\\
&\quad = 2\int_I\Big[\int_I \ln|\sin(\pi(x-y))|(u_{yy}+a)\,\mathrm{d}y\Big]\varphi_{xx}\,\mathrm{d}x\\
&\quad = 2\int_I\Big[\int_I \ln|\sin(\pi(x-y))|u_{yy}\,\mathrm{d}y\Big]\varphi_{xx}\,\mathrm{d}x + 2a\underbrace{\Big[\int_I \ln|\sin(\pi(x-y))|\,\mathrm{d}y\Big]}_{=2\ln 2}\underbrace{\int_I \varphi_{xx}\,\mathrm{d}x}_{=0}.
\end{aligned}
$$

Note $\ln|\sin(\pi(x-y))|$ has an ln-like singularity at $x = y$; hence it belongs to $L^p(I)$ for all p, and u_{yy} belongs to $L^3(I)$. Thus, via integration by parts and the periodicity of I, we have

$$
\begin{aligned}
\int_I \ln|\sin(\pi(x-y))|u_{yy}\,\mathrm{d}y &= -\int_I u_y\frac{\partial}{\partial y}\ln|\sin(\pi(x-y))|\,\mathrm{d}y\\
&= \int_{-1/2}^{1/2} u_y(x-y)\frac{\partial}{\partial y}\ln|\sin(\pi y)|\,\mathrm{d}y\\
&= -\pi\Big[\int_{-1/2}^{0} u_y(x-y)\cot(y)\,\mathrm{d}y - \int_0^{1/2} u_y(x-y)\cot(y)\,\mathrm{d}y\Big].
\end{aligned}
\tag{3.7}
$$

Note both the above integrals have a singularity at $y = 0$, so we need more careful estimates for the last line. Since

$$
\Big|\int_I \ln|\sin(\pi y)|u_{yy}(x-y)\,\mathrm{d}y\Big| < +\infty,
$$

we have

$$
\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \ln|\sin(\pi y)|u_{yy}(x-y)\,\mathrm{d}y = 0.
$$

Therefore, we could rewrite (3.7) as

$$
\begin{aligned}
\int_I \ln|\sin(\pi y)|u_{yy}(x-y)\,\mathrm{d}y &= \lim_{\varepsilon\to 0}\Big[\int_{\varepsilon}^{1/2} \ln|\sin(\pi y)|u_{yy}(x-y)\,\mathrm{d}y + \int_{-1/2}^{-\varepsilon} \ln|\sin(\pi y)|u_{yy}(x-y)\,\mathrm{d}y\Big]\\
&= \pi\lim_{\varepsilon\to 0}\Big[\int_{\varepsilon}^{1/2} \cot(\pi y)u_y(x-y)\,\mathrm{d}y - \int_{-1/2}^{-\varepsilon} \cot(\pi y)u_y(x-y)\,\mathrm{d}y \qquad (3.8)\\
&\qquad - \ln|\sin(\varepsilon\pi)|u_y(x-\varepsilon) + \ln|\sin(\varepsilon\pi)|u_y(x+\varepsilon)\Big]. \qquad (3.9)
\end{aligned}
$$

The limit in (3.8) exists, and it gives the Hilbert transform term $H(u_y)$. For the other term (3.9), we recall that $u_{yy} \in L^3(I)$; hence $u_y \in W^{1,3}(I) \subseteq C^{0,2/3}(I)$. That is, there exists some constant $C_1 > 0$, independent of x, y, such that

$$
|u_y(x+\varepsilon) - u_y(x-\varepsilon)| \le C_1|2\varepsilon|^{2/3},
$$

and (3.9) is now bounded through

$$\lim_{\varepsilon\to 0}\big|\ln|\sin(\varepsilon\pi)|(u_y(x+\varepsilon)-u_y(x-\varepsilon))\big| \le C_1\lim_{\varepsilon\to 0}\big||2\varepsilon|^{2/3}\ln|\sin(\varepsilon\pi)|\big| = 0.$$

Therefore, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\int_I\int_I &\ln|\sin(\pi(x-y))|(u_{yy}+\varepsilon\varphi_{yy}+a)(u_{xx}+\varepsilon\varphi_{xx}+a)\,\mathrm{d}y\,\mathrm{d}x\\ &-\int_I\int_I \ln|\sin(\pi(x-y))|(u_{yy}+a)(u_{xx}+a)\,\mathrm{d}y\,\mathrm{d}x\Big]\\ =&\int_I 2\pi H(u_x)\varphi_{xx}\,\mathrm{d}x.\end{aligned}$$

This, together with the term $[\Phi_a'(u_{xx})]_{xx}$ in (3.4), concludes step 1.

Step 2. We show that the sub-differential $\partial E(u)$ is single-valued. Assume there exists another element $\eta\in\partial E(u)$. To prove that $Au := [2\pi H(u_x)+\Phi'(u_{xx}+a)]_{xx} = \eta$ as elements of $[W^{2,3}_{\mathrm{per}_0}(I)]^*$, we just need to show that

$$\langle Au,\varphi\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)} = \langle \eta,\varphi\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)}$$

for all test functions φ belonging to a suitable dense set $Z(u)\subseteq W^{2,3}_{\mathrm{per}_0}(I)$, which will be constructed below. Here

$$\langle\cdot,\cdot\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)}$$

denotes the duality pairing between $[W^{2,3}_{\mathrm{per}_0}(I)]^*$ and $W^{2,3}_{\mathrm{per}_0}(I)$, induced through the embedding chain

$$W^{2,3}_{\mathrm{per}_0}(I)\hookrightarrow L^2_{\mathrm{per}_0}(I)\hookrightarrow [W^{2,3}_{\mathrm{per}_0}(I)]^*.$$

By the definition of sub-differential, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{E(u+\varepsilon\varphi)-E(u)}{\varepsilon}&\ge\langle Au,\varphi\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)},\\ \lim_{\varepsilon\to 0}\frac{E(u+\varepsilon\varphi)-E(u)}{\varepsilon}&\ge\langle \eta,\varphi\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)},\\ \lim_{\varepsilon\to 0}\frac{E(u-\varepsilon\varphi)-E(u)}{\varepsilon}&\ge\langle Au,-\varphi\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)},\\ \lim_{\varepsilon\to 0}\frac{E(u-\varepsilon\varphi)-E(u)}{\varepsilon}&\ge\langle \eta,-\varphi\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)}.\end{aligned}$$

Therefore, if both the left-hand side terms

$$\lim_{\varepsilon\to 0}\frac{E(u\pm\varepsilon\varphi)-E(u)}{\varepsilon}$$

are finite, then we can infer

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{E(u)-E(u-\varepsilon\varphi)}{\varepsilon}&\le\langle Au,\varphi\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)}\le\lim_{\varepsilon\to 0}\frac{E(u+\varepsilon\varphi)-E(u)}{\varepsilon},\\ \lim_{\varepsilon\to 0}\frac{E(u)-E(u-\varepsilon\varphi)}{\varepsilon}&\le\langle \eta,\varphi\rangle_{[W^{2,3}_{\mathrm{per}_0}(I)]^*,W^{2,3}_{\mathrm{per}_0}(I)}\le\lim_{\varepsilon\to 0}\frac{E(u+\varepsilon\varphi)-E(u)}{\varepsilon}.\end{aligned}$$

So we need to carefully choose φ such that

$$\lim_{\varepsilon\to 0}\frac{E(u+\varepsilon\varphi)-E(u)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{E(u)-E(u-\varepsilon\varphi)}{\varepsilon}\tag{3.10}$$

both exist.

To prove that the Gateaux derivative in (3.10) exists, the only term in $E(u\pm\varepsilon\varphi)$ that might create issues is

$$\int_I\Phi(u_{xx}+a\pm\varepsilon\varphi_{xx})\,\mathrm{d}x$$

since we need $u_{xx} + a \pm \varepsilon\varphi_{xx} > 0$ a.e. to ensure that (3.10) is finite. Let

$$Z_n(u) := \Big\{\varphi \in W^{2,\infty}_{\mathrm{per}_0}(I) : \varphi_{xx} = 0 \text{ on } \Big\{u_{xx} + a < \frac{1}{n}\Big\}\Big\}, \quad Z(u) := \bigcup_{n\geq 1} Z_n(u). \tag{3.11}$$

Therefore, for any $\varphi \in Z(u)$, there exists Z_n such that $\varphi_{xx} = 0$ therein. Then, by construction, for any

$$\varepsilon < \frac{1}{n\|\varphi_{xx}\|_{L^\infty(I)}},$$

we have $u_{xx} + a \pm \varepsilon\varphi > 0$ a.e. It remains to check that $Z(u)$ is dense in $W^{2,3}_{\mathrm{per}_0}(I)$, i.e. for any $\psi \in W^{2,3}_{\mathrm{per}_0}(I)$ there exists a sequence $\psi_n \subseteq Z(u)$ such that $\psi_n \to \psi$ strongly in $W^{2,3}_{\mathrm{per}_0}(I)$. This is done in Lemma 3.2 below. Therefore, we conclude $\eta = Au$ in $[W^{2,3}_{\mathrm{per}_0}(I)]^*$, and thus ∂E is single-valued. □

For brevity, even though the sub-differential $\partial E(u)$ is a set, we will simply write

$$\partial E(u) = [2\pi H(u_x) + \Phi'(u_{xx} + a)]_{xx}$$

instead.

Lemma 3.2. *The set $Z(u)$ constructed in* (3.11) *is dense in $W^{2,3}_{\mathrm{per}_0}(I)$, i.e. for any $v \in W^{2,3}_{\mathrm{per}_0}(I)$ there exists a sequence $v_n \subseteq Z(u)$ such that $v_n \to v$ strongly in $W^{2,3}_{\mathrm{per}_0}(I)$*

Proof. Let $v \in W^{2,3}_{\mathrm{per}_0}(I)$ be given, and we need to approximate v with a sequence $v_n \subseteq Z(u)$. To this aim, we first approximate v_{xx}, and then take anti-derivatives. Let

$$w_n := \min\{v_{xx}\mathbf{1}_{\{u_{xx}+a\geq 1/n\}},\, n\},$$

which, intuitively, plays the role of $(v_n)_{xx}$. That is, w_n is constructed by first setting everything to zero on $\{u_{xx} + a < \frac{1}{n}\}$, and then taking the truncation from above (at height n). Then define

$$z_n(x) := \int_0^x w_n(s)\,\mathrm{d}s - \bar{w}_n, \quad \bar{w}_n := \frac{1}{|I|}\int_0^1 w_n(s)\,\mathrm{d}s, \quad v_n(x) := \int_0^x z_n(s)\,\mathrm{d}s.$$

By construction, $(v_n)_{xx} = w_n$; hence $v_n \in Z(u)$ for any n. Moreover, since $z_n = (v_n)_x$ and v_n have zero average, by Poincaré's inequality, we know that $\|v_n - v\|_{L^3(I)}$ and $\|v_{nx} - v_x\|_{L^3(I)}$ are controlled by $\|(v_n)_{xx} - v_{xx}\|_{L^3(I)}$. By construction,

$$\|(v_n)_{xx} - v_{xx}\|^3_{L^3(I)} \leq \int_{\{u_{xx}+a<1/n\}} |v_{xx}|^3\,\mathrm{d}x + \int_{\{v_{xx}\geq n\}} |v_{xx}|^3\,\mathrm{d}x \to 0$$

since the Lebesgue measures of both $\{u_{xx} + a < \frac{1}{n}\}$ and $\{v_{xx} \geq n\}$ go to zero as $n \to +\infty$. Thus we have shown that $v_n \to v$ strongly in $W^{2,3}_{\mathrm{per}_0}(I)$. □

3.2 The a priori estimate

In this subsection, we show the key a priori estimate which provides the existence of a uniform lower bound $c^* > 0$, defined in (3.18) below, such that the solution satisfies the global-in-time positivity property

$$u_{xx}(t) + c \geq c^* > 0 \quad \text{for all } t.$$

In other words, if the initial datum is uniformly bounded away from zero, so is the solution at all positive times.

Let u be a solution of

$$u_t = -\frac{\delta E}{\delta u} = -[2\pi H(u_x) + \Phi'(u_{xx} + a)]_{xx},$$

satisfying (3.1) for $t \in [0, T_{\max}]$. Note that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_I u_t \frac{\delta E}{\delta u}\,\mathrm{d}t = -\int_I \Big|\frac{\delta E}{\delta u}\Big|^2\,\mathrm{d}t \leq 0$$

and

$$
\begin{aligned}
E(u_0) - \inf E \geq -\int_0^{+\infty} \frac{\mathrm{d}E}{\mathrm{d}t}\,\mathrm{d}t &= \int_0^{+\infty} \Big\|\frac{\delta E}{\delta u}\Big\|_{L^2(I)}^2 \,\mathrm{d}t \\
&= \int_0^{+\infty} \|[2\pi H(u_x(t)) + \Phi'(u_{xx}(t)+a)]_{xx}\|_{L^2(I)}^2 \,\mathrm{d}t \\
&\geq C_I^{-1} \int_0^{+\infty} \|[2\pi H(u_x(t)) + \Phi'(u_{xx}(t)+a)]_{x}\|_{L^2(I)}^2 \,\mathrm{d}t \\
&= C_I^{-1} \int_0^{+\infty} \Big\|2\pi H(u_{xx}(t)) + \Big[\ln(u_{xx}(t)+a) + \frac{3}{2}(u_{xx}(t)+a)^2\Big]_x\Big\|_{L^2(I)}^2 \,\mathrm{d}t,
\end{aligned}
$$

where C_I is the Poincaré constant of I.

We show that the Hilbert transform term is controlled by the singular one. On the one hand,

$$
\begin{aligned}
&\Big\|\Big[\ln(u_{xx}(t)+a) + \frac{3}{2}(u_{xx}(t)+a)^2\Big]_x\Big\|_{L^2(I)}^2 \\
&\quad = \|[\ln(u_{xx}(t)+a)]_x\|_{L^2(I)}^2 + \frac{9}{4}\|[(u_{xx}(t)+a)^2]_x\|_{L^2(I)}^2 + 3\int_I [(u_{xx}(t)+a)^2]_x[\ln(u_{xx}(t)+a)]_x \,\mathrm{d}x \\
&\quad = \|[\ln(u_{xx}(t)+a)]_x\|_{L^2(I)}^2 + \frac{9}{4}\|[(u_{xx}(t)+a)^2]_x\|_{L^2(I)}^2 + 6\|u_{xxx}\|_{L^2(I)}^2.
\end{aligned}
\tag{3.12}
$$

On the other hand, from [4, Proposition 9.1.9],

$$
4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2 = 4\pi^2\|u_{xx}(t)\|_{L^2(I)}^2 = 4\pi^2(\|u_{xx}(t)+a\|_{L^2(I)}^2 - a^2). \tag{3.13}
$$

By the Poincaré inequality,

$$
\|[(u_{xx}(t)+a)^2]_x\|_{L^2(I)}^2 \geq C_I^{-1}\|(u_{xx}(t)+a)^2\|_{L^2(I)}^2 = C_I^{-1}\|u_{xx}(t)+a\|_{L^4(I)}^4. \tag{3.14}
$$

Combining (3.13) and (3.14), there exists a computable constant C_0 such that

$$
4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2 \leq \frac{1}{4}\|[(u_{xx}(t)+a)^2]_x\|_{L^2(I)}^2 \tag{3.15}
$$

whenever

$$
4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2 \geq C_0.
$$

Thus one of the following cases must hold:

(I) The quantity $4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2 \leq C_0$. In this case,

$$
\begin{aligned}
&\Big\|2\pi H(u_{xx}(t)) + \Big[\ln(u_{xx}(t)+a) + \frac{3}{2}(u_{xx}(t)+a)^2\Big]_x\Big\|_{L^2(I)}^2 \\
&\quad \geq 4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2 + \Big\|\Big[\ln(u_{xx}(t)+a) + \frac{3}{2}(u_{xx}(t)+a)^2\Big]_x\Big\|_{L^2(I)} \\
&\qquad \times \Big[\Big\|\Big[\ln(u_{xx}(t)+a) + \frac{3}{2}(u_{xx}(t)+a)^2\Big]_x\Big\|_{L^2(I)} - C_0\Big].
\end{aligned}
\tag{3.16}
$$

So the following dichotomy holds:

(a) Either

$$
\Big\|\Big[\ln(u_{xx}(t)+a) + \frac{3}{2}(u_{xx}(t)+a)^2\Big]_x\Big\|_{L^2(I)} \leq 2C_0,
$$

in which case we get a direct upper bound for

$$
\Big\|\Big[\ln(u_{xx}(t)+a) + \frac{3}{2}(u_{xx}(t)+a)^2\Big]_x\Big\|_{L^2(I)};
$$

(b) or

$$\left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)} \geq 2C_0,$$

i.e. the last term in (3.16) satisfies

$$\left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)} - C_0 \geq \frac{1}{2}\left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)},$$

so (3.16) gives

$$\left\|2\pi H(u_{xx}(t))+\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}^2 \geq \frac{1}{2}\left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}^2.$$

(II) Alternatively, if $4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2 \geq C_0$, then from (3.12) and (3.15) we have the control

$$\begin{aligned}4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2 &\leq \frac{1}{4}\|[(u_{xx}(t)+a)^2]_x\|_{L^2(I)}^2\\ &\leq \frac{1}{9}\left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}^2,\end{aligned}$$

which gives

$$\begin{aligned}&\left\|2\pi H(u_{xx}(t))+\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}^2\\ &\quad\geq 4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2+\left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}^2\\ &\qquad -2\|2\pi H(u_{xx}(t))\|_{L^2(I)}\left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}\\ &\quad\geq 4\pi^2\|H(u_{xx}(t))\|_{L^2(I)}^2+\frac{1}{3}\left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}^2.\end{aligned}$$

Combining all above cases, we have

$$\begin{aligned}\|[\ln(u_{xx}(t)+a)]_x\|_{L^2(I)}^2 &\leq \left\|\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}^2 \quad \text{(by (3.12))}\\ &\leq \max\left\{4C_0^2,\ 3\left\|2\pi H(u_{xx}(t))+\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}^2\right\}.\end{aligned} \tag{3.17}$$

Since, by Lemma 2.2, the energy E is λ-convex, with $\lambda = 2C = 2\sqrt{3} - 4\ln 2 > 0$, it is well known (see, e.g., [1, Theorem 2.4.15]) that

$$t \mapsto e^{\lambda t}\left\|\frac{\delta E(u(t))}{\delta u}\right\|_{L^2(I)} = e^{\lambda t}\left\|2\pi H(u_{xx}(t))+\left[\ln(u_{xx}(t)+a)+\frac{3}{2}(u_{xx}(t)+a)^2\right]_x\right\|_{L^2(I)}$$

is nonincreasing. Therefore, by the assumption $u_{xx}^0 + a > 0$ and by setting

$$H_0 := \|\partial E(u^0)\|_{L^2(I)} = \left\|2\pi H(u_{xx}^0)+\left[\ln(u_{xx}^0+a)+\frac{3}{2}(u_{xx}^0+a)^2\right]_x\right\|_{L^2(I)} < +\infty,$$

we have

$$\left\|2\pi H(u_{xx}(\cdot))+\left[\ln(u_{xx}(\cdot)+a)+\frac{3}{2}(u_{xx}(\cdot)+a)^2\right]_x\right\|_{L^\infty(0,+\infty;L^2(I))} \leq H_0.$$

Combining this with (3.17) finally gives

$$\|\ln(u_{xx}(\cdot)+a)\|_{L^\infty(0,+\infty;L^\infty(I))} \leq C_{\infty,2}\|[\ln(u_{xx}(\cdot)+a)]_x\|_{L^\infty(0,+\infty;L^2(I))} \leq C_{\infty,2}\max\{2C_0, 3H_0\},$$

and hence a uniform bound

$$c^* := e^{-C_{\infty,2}\max\{2C_0,3H_0\}} \tag{3.18}$$

of $u_{xx}(\cdot) + a$ away from zero.

3.3 Proof of higher regularity and Theorem 1.3

Based on the calculations for the sub-differential, and the key a priori estimates from the previous subsections, now we are in the position to prove higher-order regularity results and Theorem 1.3.

Proof of Theorem 1.3 (i)–(iii). From [1, Proposition 1.4.4], we obtain

$$|\partial E|(u(\cdot\,,t)) = \min\{\|\xi\|_{L^2(I)} : \xi \in \partial E(u(\cdot\,,t))\}.$$

From Lemma 3.1, we know that $\partial E(u)$ is single-valued and is given by (3.2), which is statement (i) of Theorem 1.3. Thus statements (ii)–(iii) of Theorem 1.3 follow directly from [1, Theorem 2.4.15] since Lemma 2.2 shows that all its hypotheses are satisfied. □

Proof of Theorem 1.3 (iv) *and* (v). Let $v := u_{xx} + a$. By statement (iii) of Theorem 1.3, the map

$$t \mapsto \Big\|\frac{\delta E}{\delta u}(\cdot\,,t)\Big\|_{L^2(I)} \exp(2Ct)$$

is nonincreasing and right continuous. Since $C > 0$, this implies that

$$t \mapsto \Big\|\frac{\delta E}{\delta u}(\cdot\,,t)\Big\|_{L^2(I)}$$

decreases exponentially in t. Thus u satisfies that, for any $t \geq 0$,

$$-\frac{\delta E}{\delta u} = [2\pi H(u_x) + \Phi'(v)]_{xx} = \Big[2\pi H(u_x) + \ln(u_{xx} + a) + \frac{3}{2}(u_{xx} + a)^2\Big]_{xx}$$

is uniformly bounded in

$$L^2(0,+\infty; L^2(I)) \cap L^\infty(0,+\infty; L^2(I)),$$

which implies

$$[2\pi H(u_x) + \Phi'(v)]_x \in L^\infty(0,+\infty; H^1(I)), \quad 2\pi H(u_x) + \Phi'(v) \in L^\infty(0,+\infty; H^2(I)), \tag{3.19}$$

where $v := u_{xx} + a$ is a shorthand notation. Using that

$$u_{xx} + a \in L^\infty(0,+\infty; L^3(I))$$

implies

$$u_{xx}, H(u_{xx}) \in L^\infty(0,+\infty; L^3(I)),$$

we get

$$[\Phi'(v)]_x = [\ln v]_x + \frac{3}{2}[v^2]_x \in L^\infty(0,+\infty; L^2(I)) \cap L^2(0,+\infty; L^2(I)).$$

Then, by (3.12), we obtain

$$\begin{aligned} +\infty &> \int_0^{+\infty} \Big\|[\ln v(\cdot\,,t)]_x + \frac{3}{2}[v(\cdot\,,t)^2]_x\Big\|^2_{L^2(I)}\,dt \\ &= \int_0^{+\infty} \Big\{\|[\ln v(\cdot\,,t)]_x\|^2_{L^2(I)} + \frac{9}{4}\|[v(\cdot\,,t)^2]_x\|^2 + 6\|u_{xxx}(\cdot\,,t)^2\|^2_{L^2(I)}\Big\}\,dt. \end{aligned} \tag{3.20}$$

Hence

$$[\ln v(\cdot\,,t)]_x, [v(\cdot\,,t)^2]_x, u_{xxx} \in L^2(0,+\infty; L^2(I)).$$

Now that we have $u_{xxx}, H(u_{xxx}) \in L^2(0,+\infty; L^2(I))$, we can use (3.19) to infer

$$[\Phi'(v)]_x = [\ln v]_x + \frac{3}{2}[v^2]_x \in L^2(0,+\infty; H^1(I)),$$

and (1.6) follows. Then, using the embedding $H^1(I) \hookrightarrow C^0(I)$, we obtain

$$L^2(0,+\infty; C^0(I)) \ni |[\Phi'(v)]_x| = |v_x(3v + v^{-1})| \geq 2\sqrt{3}|v_x|,$$

which implies

$$v_x = u_{xxx} \in L^2(0,+\infty; C^0(I)). \tag{3.21}$$

Similarly, since (3.20) and (3.21) also hold for any $t \geq 0$ uniformly, we conclude (1.7) and (1.8).

Statement (v) follows from $\ln(u_{xx} + a) \in L^\infty(0,+\infty; L^\infty(I))$, i.e. $u_{xx} + a \geq c^* > 0$ is bounded away from zero for all $t > 0$. Here the explicit positive lower bound c^* is calculated in (3.18).

Next, by (1.6), for any $t \geq 0$,

$$\begin{aligned}
+\infty &> \Big\|[\ln v(\cdot\,,t)]_{xx} + \frac{3}{2}[v(\cdot\,,t)^2]_{xx}\Big\|^2_{L^2(I)} \\
&= \int_I \Big[|[\ln v(\cdot\,,t)]_{xx}|^2 + \frac{9}{4}|[v(\cdot\,,t)^2]_{xx}|^2\Big]\,dx + 3\int_I [\ln v(\cdot\,,t)]_{xx}\cdot[v(\cdot\,,t)^2]_{xx}\,dx \\
&= \int_I \Big[|[\ln v(\cdot\,,t)]_{xx}|^2 + \frac{9}{4}|[v(\cdot\,,t)^2]_{xx}|^2\Big]\,dx + 6\int_I \Big[v_{xx}(\cdot\,,t)^2 - \frac{v_x(\cdot\,,t)^4}{v(\cdot\,,t)^2}\Big]\,dx.
\end{aligned} \tag{3.22}$$

Note that the only negative term is

$$-\int_I \frac{v_x(\cdot\,,t)^4}{v(\cdot\,,t)^2}\,dx,$$

so we need to bound it from below. From (1.8) and the uniform lower bound $u_{xx} + a \geq c^* > 0$, we know

$$\int_I \frac{v_x(\cdot\,,t)^4}{v(\cdot\,,t)^2}\,dx \leq c\|v_x(\cdot\,,t)\|^2_{C^0(I)} < +\infty$$

uniformly for $t \geq 0$. Hence (3.22) reads

$$\begin{aligned}
+\infty &> \Big\|[\ln v(\cdot\,,t)]_{xx} + \frac{3}{2}[v(\cdot\,,t)^2]_{xx}\Big\|^2_{L^2(I)} \\
&\geq \Big\{\int_I \Big[|[\ln v(\cdot\,,t)]_{xx}|^2 + \frac{9}{4}|[v(\cdot\,,t)^2]_{xx}|^2 + 6v_{xx}(\cdot\,,t)^2\Big]\,dx\Big\} - 6c\|v_x(\cdot\,,t)\|^2_{C^0(I)}
\end{aligned}$$

uniformly for $t \geq 0$, and thus (1.9) follows. This completes the proof of statement (iv). □

Proof of Theorem 1.3 (vi). Since u is periodic with regularity (1.6)–(1.9), the steady state u^* satisfies

$$\frac{\delta E}{\delta u} = \Big[2\pi H(u_x) + \ln(u_{xx} + a) + \frac{3}{2}(u_{xx} + a)^2\Big]_{xx} = 0,$$

which implies

$$2\pi H(u_x) + \ln(u_{xx} + a) + \frac{3}{2}(u_{xx} + a)^2 \equiv \text{const}.$$

This yields $u^* \equiv 0$ is a steady state. From Lemma 2.2, we know that $\frac{\delta E}{\delta u}$ is strictly monotone in L^2, which implies that there is only one steady state $u^* \equiv 0$ such that $\frac{\delta E}{\delta u} = 0$. Thus, combining [1, Theorem 2.4.14] and Lemma 2.2, we conclude the exponential decay of $u(t)$ to its unique equilibrium $u^* = 0$, i.e. statement (vi). □

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