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The $K(\pi, 1)$-conjecture implies the center conjecture for Artin groups

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In this note, we prove that the $K(\pi, 1)$-conjecture for Artin groups implies the center conjecture for Artin groups. Specifically, every Artin group without a spherical factor that satisfies the $K(\pi, 1)$-conjecture has a trivial center.

1. Introduction

A Coxeter system (W, S) consists of a group W and a generating set S where W is given by a presentation

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$$W = \langle s \in S | s^2 = (st)^{m_{st}} = 1 \rangle,$$

where $m_{st} \in \{2, 3, \dots\} \cup \{\infty\}$. The associated Artin group A is given by the presentation

$$A = \langle s \in S | \underbrace{sts\cdots}_{m_{st}\, terms} = \underbrace{tst\cdots}_{m_{st}\, terms} \rangle.$$

An Artin group A is *spherical* if the corresponding Coxeter group is finite, and otherwise A is *infinite type*. The Coxeter diagram Γ_S is a graph with vertices corresponding to S and where two vertices are joined by an edge if and only if $m_{st} > 2$. If $m_{st} \geq 4$ we label the edge with m_{st}. A *special subgroup* of A is a subgroup generated by some subset of S. Each special subgroup is itself an Artin group [20]. Each Artin group with standard generating set admits (a possibly trivial) decomposition $A = A_{T_1} \times \cdots \times A_{T_n}$ where each $T_i \subseteq S$ defines a connected component of the Coxeter graph Γ_S. An Artin group A is *irreducible* if its Coxeter diagram is connected. We say A_{T_i} is a *spherical factor* of A if A_{T_i} is spherical. Every irreducible spherical Artin group has an infinite cyclic center [10,3]. Conjecturally, those are the only irreducible Artin groups with nontrivial center.

Conjecture 1 *(The center conjecture). Every Artin group without a spherical factor has trivial center.*

The center conjecture holds for FC-type Artin groups and 2-dimensional Artin groups [11]. Charney and Morris-Wright have shown the center conjecture holds for Artin groups whose defining graphs are not stars of a single vertex [7]. Godelle and Paris further showed that if all Artin groups with $m_{st} \neq \infty$ for all $s, t \in S$ satisfy the center conjecture, then all Artin groups satisfy the center conjecture [11].

The FC-type and 2-dimensional Artin groups also satisfy the $K(\pi, 1)$-conjecture [4].

Conjecture 2 *(The $K(\pi, 1)$-conjecture). The orbit space $\mathcal{H}(W)/W$ of a complexified hyperplane arrangement associated to a Coxeter system (W, S) is a $K(\pi, 1)$ for the Artin group A associated to W.*

For a precise definition of $\mathcal{H}(W)$ and more background, see e.g. the survey paper [17]. It is known that the fundamental group of $\mathcal{H}(W)/W$ is equal to A, so the conjecture is about the asphericity of $\mathcal{H}(W)/W$. In this note, we prove the following:

Theorem 3. *Every Artin group without a spherical factor that satisfies the $K(\pi, 1)$-conjecture has trivial center.*

In fact, we only need the following consequence of the $K(\pi, 1)$-conjecture: an Artin group A which satisfies the $K(\pi, 1)$-conjecture has finite cohomological dimension which is realized by a spherical subgroup, i.e. $\mathrm{cd}(A) = \mathrm{cd}(A_T) = |T|$ for some spherical subset T. See Theorem 12 for the more general statement of our main theorem.

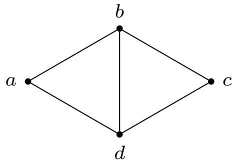

Fig. 1. The Coxeter diagram of the smallest mysterious Artin group according to McCammond.

For example, in [16] McCammond mentions that the center conjecture is unknown for the Artin group A with the Coxeter diagram as in Fig. 1. The group A is given by the presentation

$$\langle a, b, c, d \mid aba = bab, bcb = cbc, cdc = dcd, dad = ada, bdb = dbd, ac = ca \rangle.$$

The $K(\pi, 1)$-conjecture holds for A by a theorem of Charney [6]. That allows us to answer the question of McCammond.

Corollary 4. *The Artin group with the Coxeter diagram as in Fig. 1 has trivial center.*

Another class of Artin groups which satisfy the $K(\pi, 1)$-conjecture are the *locally reducible* Artin groups, where all irreducible spherical subgroups are of rank ≤ 2 [5]. There are many of these with $m_{st} \neq \infty$ for each $s, t \in S$, and as far as we know the center conjecture was open here.

Corollary 5. *Every locally reducible Artin group without spherical factor has trivial center.*

2. Representations of Artin groups in mapping class groups

An Artin group A with standard generating set S has *small type* if $m_{st} \in \{2, 3\}$ for all $s, t \in S$. In this section we recall a representation of small type Artin groups in mapping class groups, due to Crisp-Paris [8], and analyze where certain elements of A are mapped.

Let Σ be a surface. A *multicurve* is a collection of pairwise disjoint simple closed curves on Σ. We say that two multicurves are *disjoint* if their isotopy classes have disjoint representatives. A *multitwist* about a multicurve γ is a product of non-trivial powers of Dehn twists about simple closed curves in γ (the powers can be different for different curves). We shall need the following lemma about commuting multitwists about multicurves.

Lemma 6. *Let γ and γ' be essential multicurves on Σ that do not share a simple closed curve. Let T_γ and $T_{\gamma'}$ be the associated multitwists along γ and γ'. Then T_γ and $T_{\gamma'}$ commute if and only if γ and γ' are disjoint.*

Proof. The if direction is clear, so suppose T_γ and $T_{\gamma'}$ commute. Then T_γ^N and $T_{\gamma'}^N$ commute for all N. By a theorem of Koberda [14], the group generated by large powers

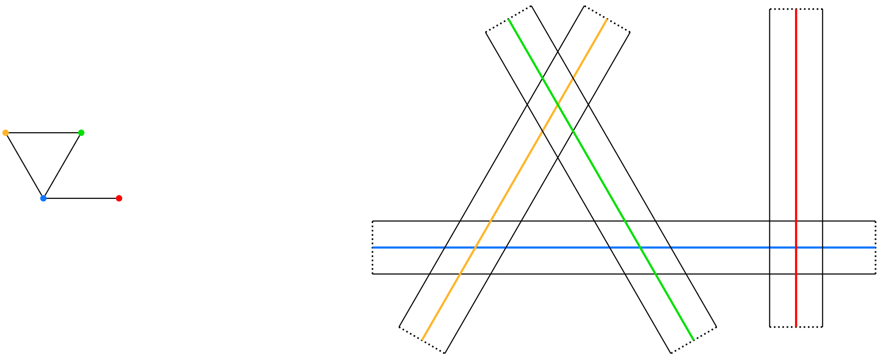

Fig. 2. An example of the surface Σ_S (right) corresponding to the Artin group A with the standard generating set S whose Coxeter graph Γ is illustrated (left). For each vertex s in Γ, there is a corresponding rectangle in the right picture, which is glued along the dotted sides to an annulus An_s. Its meridian γ_s has the same color as s.

of Dehn twists of all curves in $\gamma \cup \gamma'$ is a right-angled Artin group. Therefore, we have two words w and w' in a RAAG, where w and w' are nontrivial powers of commuting generators. It follows from the normal form for RAAG's that these commute exactly when each generator in w commutes with each generator of w' [12]. Therefore, the curves in γ can be isotoped to be disjoint from the curves in γ'. $\square$

Let A_S be a small type Artin group with standard generating set S. We can build an associated surface

$$\Sigma_S = \bigcup_{s \in S} An_s$$

where each An_s is an annulus. We denote the meridian of An_s by γ_s. If $m_{st} = 3$, then we arrange that the annuli An_s, An_t intersect in a single square so that γ_s, γ_t intersect transversely at one point, and any triple intersection of annuli is empty. If $m_{st} = 2$, then An_s, An_t will be disjoint. See Fig. 2 for an example. There is a representation of A_S in the mapping class group of Σ_S where each generator $s \in S$ is mapped to the Dehn twist about the simple closed curve γ_s (see Proposition 9). Full details can be found in [8]. Each subset T of S has an associated subsurface Σ_T of Σ_A. By construction, the subsurface Σ_T for an irreducible spherical subset $T \subseteq S$ and the induced homomorphism $\rho_T : A_T \to \mathrm{Mod}(\Sigma_T)$ are exactly the Perron-Vannier representation of small type spherical Artin groups [18] (see also [13]). For every spherical subset T, the generator of the center of the group A_T is denoted by z_T. By [15, Prop 2.12] $\rho(z_T^4)$ is a multitwist about the boundary of Σ_T. We denote that multicurve by γ_T.

Let K_S be the graph that is the union $\bigcup_{s \in S} \gamma_s \subseteq \Sigma_S$. By construction, there is a deformation retraction $r : \Sigma_S \to K_S$. Thus $H_1(\Sigma_S) = H_1(K_S)$, and in particular if S is spherical then $H_1(\Sigma_S) = \mathbb{Z}^S$. For any $S' \subset S$, the map $H_1(\Sigma_{S'}) \to H_1(\Sigma_S)$ induced by the inclusion $\Sigma_{S'} \hookrightarrow \Sigma_S$ is injective and $H_1(\Sigma_{S'}) \subsetneq H_1(\Sigma_S)$.

Every closed path in a graph is homotopic to a cycle (i.e. a closed path without backtracks). In particular, every homotopy class of a simple closed curve in Σ_S can be realized as a cycle in K_S. We will now view all the simple closed curves in Σ_S as cycles in K_S. In particular, we view components of γ_T for any spherical subset $T \subseteq S$ as cycles in K_S.

Lemma 7. *Let $T \subseteq S$ be an irreducible spherical subset, and $s \in S - T$. Then γ_s intersects γ_T if and only if γ_s intersects γ_t for some $t \in T$.*

Proof. If γ_s does not intersect any γ_t for $t \in T$, then $\rho(s)$ commutes with $\rho(t)$ for all $t \in T$. Then $\rho(s)$ must also commute with $\rho(z_T^2)$, and by Lemma 6 γ_s and γ_T are disjoint.

Now suppose that γ_s intersects γ_t for some $t \in T$. By construction γ_s, γ_t intersect exactly once. Suppose γ_s can be isotoped in Σ_S to be disjoint from $\partial\Sigma_T$. Then $\gamma_s \subseteq \Sigma_T$, since γ_s, γ_t still must intersect. In particular, $[\gamma_s] \in H_1(\Sigma_T) = H_1(K_T)$. This is a contradiction, since $H_1(K_T) \subsetneq H_1(K_{T\cup\{s\}})$. □

Lemma 8. *Let $T_1, T_2 \subseteq S$ be two disjoint, irreducible, spherical subsets. Then any component of γ_{T_1} and any component of γ_{T_2} are non-isotopic and disjoint in Σ_S.*

Proof. Since $T_1, T_2 \subseteq S$ are disjoint, by construction, the subgraphs $K_{T_1}, K_{T_2} \subseteq K_S$ are disjoint. Every connected component of γ_{T_i} can be realized as a cycle contained in K_{T_i}. Any two disjoint cycles in a graph are non-isotopic. The conclusion follows. □

Proposition 9 *([8]). For every small type Artin group A with the standard generating set S, there exists a surface with boundary Σ_S and a homomorphism $\rho : A \to \mathrm{Mod}(\Sigma_S)$ where*

(a) for each $s \in S$, $\rho(s)$ is the Dehn twist about a simple closed curve γ_s,
(b) the simple closed curves γ_s, γ_t are disjoint $\iff m_{st} = 2$,
(c) the simple closed curves γ_s, γ_t intersect exactly once $\iff m_{st} = 3$.

Moreover,

(d) for every irreducible spherical subset $T \subseteq S$, $\rho(z_T^2)$ is the multitwist about a multicurve γ_T which is the boundary of the subsurface Σ_T, and
(e) for every irreducible spherical $T \subseteq S$ and $s \in S - T$, the simple closed curve γ_s and the multicurve γ_T are disjoint if and only if $[s, t] = 1$ for all $t \in T$.

Proof. The fact that ρ is a homomorphism follows from standard relations between Dehn twists, see [8, Prop 4]. The parts (a), (b), (c) follow from [8] as well. Part (d) follows from [15] (see discussion above). Finally part (e) is a consequence of Lemma 7 and Lemma 6. □

Let A be an Artin group with standard generating set S. We say A is *free-of-infinity* if $m_{st} < \infty$ for all $s, t \in S$.

Proposition 10 *([8]). Let A be a free-of-infinity Artin group. Then there exists a small type Artin group $\widetilde{A}$ with standard generating set $\widetilde{S}$ and a homomorphism $\phi : A \to \widetilde{A}$ such that*

- *there exists a partition $\bigsqcup_{s \in S} I(s)$ of $\widetilde{S}$ such that the elements of $I(s)$ pairwise commute and $\phi(s) = \prod_{r \in I(s)} r$,*
- *$m_{st} = 2$ if and only if every element of $I(s)$ and every element of $I(t)$ commute, and*
- *if $m_{st} \geq 3$ then the subgroup generated by $I(s) \cup I(t)$ is a direct product of braid groups on m_{st} strands.*

Let $\rho \circ \phi : A \to \mathrm{Mod}(\Sigma_{\widetilde{A}})$ be the composition of the homomorphism ϕ with the homomorphism $\rho : \widetilde{A} \to \mathrm{Mod}(\Sigma_{\widetilde{A}})$ from Proposition 9. Then

(a) for each $s \in S$, $\rho \circ \phi(s)$ is a multitwist about a multicurve $\gamma_s = \bigcup_{r \in I(s)} \gamma_r$,
(b) $m_{st} = 2$ if and only if every component of γ_s and every component of γ_t are disjoint,
(c) for every spherical subset $T \subset S$, $\rho(z_T^2)$ is the multitwists about a multicurve γ_T, and
(d) for every $T \subseteq S$ and $s \in S - T$, the multicurve γ_s and the multicurve γ_T are disjoint if and only if $[s, t] = 1$ for all $t \in T$.

Proof. The homomorphism ϕ is described in [9] and also in [8]. Parts (a) and (b) follow directly from the construction. Part (c) is proven in [13, Lem 6.1]. Part (d) follows from Lemma 7 and Lemma 6. □

3. The main theorem

We will need the following lemma.

Lemma 11. *Let A_S be an Artin group which splits as a product $A_S = A_U \times A_V$ where A_U is the maximal spherical factor. Suppose that $\mathrm{cd}\, A_S < \infty$. Then $\mathrm{cd}\, A_S = \mathrm{cd}\, A_U + \mathrm{cd}\, A_V$.*

Proof. By [2, Thm 5.5] a group $G = N \times Q$ has $\mathrm{cd}\, G = \mathrm{cd}\, N + \mathrm{cd}\, Q$ provided that

- $\mathrm{cd}\, Q < \infty$, and
- N is of type FP and $H^n(N, \mathbb{Z}N)$ is free for $n = \mathrm{cd}\, N$.

Clearly $\mathrm{cd}\, A_V < \infty$ since $\mathrm{cd}\, A_S < \infty$. Since A_U is a spherical Artin group, A_U has type *FP*. By [19, Thm B] (see also [1]) A_U is a duality group, so $H^n(A_U, \mathbb{Z}A_U)$ is free. The conclusion follows. □

Theorem 12. *Let A_S be an Artin group of infinite type with the standard generating set S such that A_S has no spherical factors. If $\operatorname{cd} A = \operatorname{cd} A_T = |T|$ for some spherical subset $T \subseteq S$, then A_S has trivial center. In particular if A_S satisfies the $K(\pi,1)$-conjecture, then A_S has trivial center.*

Proof for free-of-infinity case. First suppose that A_S is free-of-infinity. Let $T \subseteq S$ be a maximal spherical subset such that $\operatorname{cd} A_S = \operatorname{cd} A_T$. Let $T_1 \sqcup T_2 \sqcup \cdots \sqcup T_n$ be the decomposition of T into irreducible spherical subsets inducing the decomposition $A_T = A_{T_1} \times \cdots \times A_{T_n}$. Since A_S has no spherical factors for each $i = 1, \ldots, n$ there exists $s_i \in S - T$ such that $[s_i, z_{T_i}] \neq 1$ as otherwise A_{T_i} would be a spherical factor of A_S. In particular, for each $i = 1, \ldots, n$, there exists $t_i \in T_i$ such that $[s_i, t_i] \neq 1$.

Consider the representation of $\rho : A_S \to \operatorname{Mod}(\Sigma_S)$ from Proposition 10. By Proposition 10, $\rho(s_i)$ and $\rho(z_{T_i})$ are the Dehn twists about multicurves γ_{s_i} and γ_{T_i} respectively, where γ_{s_i} and γ_{T_i} intersect.

Suppose that A_S has nontrivial center and let $y \in Z(A_S)$ with $y \neq e$. Note that y has infinite order since A_S is torsion-free, as $\operatorname{cd} A_S < \infty$. If $y^k \notin A_T$ for any $k \neq 0$, then $\langle A_T, y\rangle \simeq A_T \times \mathbb{Z}$ is a subgroup of $\operatorname{cd} A + 1$, which is a contradiction. Thus there exists $k \in \mathbb{N}$ such that $y^k \in A_T$. Then $y^k \in Z(A_T)$, i.e. $y^m = \prod_{i=1}^n z_{T_i}^{m_i}$ for some $m > 0$ and at least one of $m_1, \ldots, m_n$, say m_1, is non-zero. By Lemma 8, $\rho(y^m)$ is a multitwist about a multicurve $\gamma = \sqcup \gamma_{T_i}$ in Σ where the union is taken over all i such that $m_i \neq 0$. In particular, γ intersects γ_{s_1}. By Lemma 6, $[\rho(y^m), \rho(s_1)] \neq 1$. Thus $[y, s_1] \neq 1$. This contradicts the fact that y is a central element of A. □

Proof for general case. The general case is induction on the cardinality of S. Suppose $\operatorname{cd} A_S = \operatorname{cd} A_T$ where $T \subseteq S$ is a spherical subset. Suppose there exist generators $v, w \in S$ such that $m_{vw} = \infty$. The group A_S splits as an amalgamated product $A_{S\setminus\{v\}} *_{A_{S\setminus\{v,w\}}} A_{S\setminus\{w\}}$. Since T cannot contain both v and w, we have $T \subseteq S \setminus \{v\}$ or $T \subseteq S \setminus \{w\}$. Without loss of generality we assume that $T \subseteq S\setminus\{v\}$. It follows that $\operatorname{cd} A_{S\setminus\{v\}} = \operatorname{cd} A_T$, as $\operatorname{cd} A_{S\setminus\{v\}} \leq \operatorname{cd} A_S$. If $A_{S\setminus\{v\}}$ has no spherical factor, then by induction $A_{S\setminus\{v\}}$ has trivial center. By [11, Lem 3.2] the center of the amalgamated product A is also trivial.

Now suppose that $A_{S\setminus\{v\}}$ has a nontrivial spherical factor. Let

$$A_{U_1} \times \cdots \times A_{U_p} \times A_{V_1} \times \cdots \times A_{V_q}$$

be the decomposition of $A_{S\setminus\{v\}}$ into irreducible factors where each A_{U_i} is spherical and each A_{V_j} has infinite type. Let $A_V = A_{V_1} \times \cdots \times A_{V_q}$. By maximality $U_i \subseteq T$ for all $i = 1, \ldots, p$. Let $T' = V \cap T$. Then $\operatorname{cd} A_V = \operatorname{cd} A_{T'}$. Indeed by Lemma 11,

$$\operatorname{cd} A_V = \operatorname{cd} A_{S\setminus\{v\}} - \Sigma_{i=1}^p \operatorname{cd} A_{U_i} = \operatorname{cd} A_T - \Sigma_{i=1}^p \operatorname{cd} A_{U_i} = A_{T'}.$$

By the inductive assumption $Z(A_V) = \{1\}$, and thus $Z(A_{S\setminus\{v\}}) \subseteq \langle z_{U_1}\rangle \times \cdots \times \langle z_{U_n}\rangle$.

Since A_S does not have a spherical factor, for every $i = 1, \ldots, n$ we have $[v, z_{U_i}] \neq 1$. In particular, each set U_i contains a standard generator u_i such that $m_{vu_i} \geq 3$. Since

$Z(A_{S\setminus\{v\}}) \subseteq A_T$ and by [11, Lem 3.2] $Z(A) \subseteq Z(A_{S\setminus\{v\}})$, it suffices to prove that v does not commute with any nontrivial element of $Z(A_T) = \langle z_{T_1}, \ldots, z_{T_n}\rangle$. By maximality of T, $A_{T\cup\{v\}}$ is not spherical. By the discussion above, $A_{T\cup\{v\}}$ is irreducible, and in particular it has no spherical factors. If $A_{T\cup\{v\}}$ is free-of-infinity, we are done.

We now assume that $A_{T\cup\{v\}}$ is not free-of-infinity. Consider the quotient homomorphism $\phi : A_{T\cup\{v\}} \to A_{\overline{T}\cup\{\overline{v}\}}$, where for every $t \in T$ such that $m_{tv} = \infty$ the corresponding generators $\overline{t}, \overline{v} \in \overline{T} \cup \{\overline{v}\}$ have $m_{\overline{t}\overline{v}} = 7$. The group $A_{\overline{T}\cup\{\overline{v}\}}$ is irreducible. The only irreducible spherical Artin group containing label 7 is the dihedral Artin group. If $A_{\overline{T}\cup\{\overline{v}\}}$ is the dihedral Artin group, then $A_{T\cup\{v\}} = F_2$ and so $\operatorname{cd} A_S = 1$, i.e. $A_S = F(S)$. Then clearly, A_S has trivial center. Otherwise $A_{\overline{T}\cup\{\overline{v}\}}$ is irreducible and has infinite type. Also $\operatorname{cd} A_{\overline{T}\cup\{\overline{v}\}} = \operatorname{cd} A_{\overline{T}}$. By the free-of-infinity case, $[\overline{v}, \overline{y}] \neq 1$ for any nontrivial $\overline{y} \in \langle z_{\overline{T}_1}, \ldots, z_{\overline{T}_n}\rangle$, as otherwise y would be a central element of $A_{\overline{T}\cup\{\overline{v}\}}$. Thus $[v, y] \neq 1$ for any nontrivial $y \in \langle z_{T_1}, \ldots, z_{T_n}\rangle$. This completes the proof. □

Data availability

No data was used for the research described in the article.

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