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Residual finiteness of certain 2-dimensional Artin groups

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ABSTRACT

We show that many 2-dimensional Artin groups are residually finite. This includes 3-generator Artin groups with labels ≥ 4 except for $(2m+1, 4, 4)$ for any $m \geq 2$. As a first step towards residual finiteness we show that these Artin groups, and many more, split as free products with amalgamation or HNN extensions of finite rank free groups. Among others, this holds for all large type Artin groups with defining graph admitting an orientation, where each simple cycle is directed.

A group G is *residually finite* if for every $g \in G - \{1\}$ there exists a finite quotient $\phi : G \to \bar{G}$ such that $\phi(g) \neq 1$. The main goal of this paper is to extend the list of Artin groups known to be residually finite. Let Γ be a simple graph where each pair of vertices a, b in Γ is labeled by an integer $M_{ab} \geq 2$. The associated *Artin group*

$$\mathrm{Art}_\Gamma = \langle a \in V(\Gamma) \mid (a,b)_{M_{ab}} = (b,a)_{M_{ab}} \text{ for } a, b \text{ joined by an edge} \rangle.$$

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By $(a,b)_{M_{ab}}$ we denote the alternating word $abab\dots$ of length M_{ab}. The Artin group on two generators with the label M will be denoted by Art_M, and the Artin group with three generators and labels M, N, P will be denoted by Art_{MNP}.

Theorem A. *If $M, N, P \geq 4$ and $(M, N, P) \neq (2m+1, 4, 4)$ (for any permutation), then the Artin group A_{MNP} is residually finite.*

None of the groups in Theorem A with $M, N, P < \infty$ were previously known to be residually finite. We also obtain residual finiteness of many more 2-dimensional Artin groups. For precise statements see Section 6. In a subsequent work [22] we prove the residual finiteness of Artin groups Art_{2MN} where $M, N \geq 4$ and at least one of them is even.

Our proof of Theorem A relies on a splitting of these Artin group as a free product with amalgamation or HNN extension of finite rank free groups. The existence of such splitting depends on the combinatorics of the defining graph. Recall an Artin group Art_Γ with the defining graph Γ has *large type* if all labels in Γ are at least 3. The quotient of an Artin group, obtained by adding the relation $a^2 = 1$ for every $a \in V(\Gamma)$ is a Coxeter group. We say Art_Γ is *spherical* if the corresponding Coxeter quotient is finite, and Art_Γ is 2-*dimensional* if no triple of generators generates a spherical Artin group. In particular, every large type Artin group is 2-dimensional. For the definition of *admissible* partial orientation of Γ, see Definition 4.2. We prove the following.

Theorem B. *If Γ admits an admissible partial orientation, then Art_Γ splits as a free product with amalgamation or an HNN-extension of finite rank free groups.*

The above theorem includes all large type Artin groups whose defining graph Γ admits an orientation such that each cycle is directed.

All linear groups are residually finite by a classical result by Mal'cev [28]. Among Artin group very few classes are known to be residually finite, and even fewer linear. It was once a major open question whether braid groups are linear and it was proved independently by Krammer [25] and Bigelow [7]. Later, the linearity was extended to all spherical Artin groups by Cohen-Wales [12], and independently by Digne [13]. The right-angled Artin groups are also well known to be linear. Since linearity is inherited by subgroups, any virtually special Artin group is linear. Artin groups whose defining graphs are forests are the fundamental groups of graph manifolds with boundary by the work of Brunner [10] and Hermiller-Meier [18], and so they are virtually special by the work of Liu [27] and Przytycki-Wise [30]. Artin groups in certain classes (including 2-dimensional, 3-generators) are not cocompactly cubulated even virtually, unless they are sufficiently similar to RAAGs by Huang-Jankiewicz-Przytycki [17] and independently by Haettel [15]. In particular, if M, N, P are finite, none of the groups in Theorem A, is virtually cocompactly cubulated. Haettel has a conjectural classification of all virtually cocompactly cubulated Artin groups [15]. Haettel also showed that some triangle-free

Artin group act properly but not cocompactly on locally finite, finite dimensional CAT(0) cube complexes [14]. We note that if one of the exponents M, N, P is infinite, then the residual finiteness of Art_{MNP} is well-known.

The list of other known families of residually finite Artin groups is short. An Artin group is of *FC type* if every clique (i.e. a complete induced subgraph) in Γ is the defining graph of a spherical Artin group. (Blasco-Garcia)-(Martinez-Perez)-Paris showed that FC type Artin groups with all labels even are residually finite [4]. (Blasco-Garcia)-Juhasz-Paris showed in [3] the residual finiteness of Artin groups with defining graph Γ where the vertices of Γ admit a partition $\mathcal{P}$ such that

- for each $X \in \mathcal{P}$ the Artin group A_X is residually finite,
- for each distinct $X, Y \in \mathcal{P}$ there is at most one edge in Γ joining a vertex of X with a vertex of Y, and
- the graph $\Gamma/\mathcal{P}$ is either a forest, or a triangle free graph with even labels. The graph $\Gamma/\mathcal{P}$ is defined as follows. The vertices of $\Gamma/\mathcal{P}$ are $\mathcal{P}$, and an edge with label M joins sets $X, Y \in \mathcal{P}$ if there exist $a \in X, b \in Y$ such that $M_{ab} = M$.

In [22] the author proves the residual finiteness of Artin groups Art_{2MN} where $M, N \geq 4$ and at least one of them is even.

The residual finiteness of 3-generator affine Artin groups (i.e. corresponding to affine Coxeter groups), i.e. Art_{244}, Art_{236}, Art_{333} follows from the work of Squier [31]. Squier proved that Art_{244} splits as an HNN extension of F_2 by an automorphism of an index two subgroup, and both Art_{236} and Art_{333} split as $F_3 *_{F_7} F_4$ where F_7 is normal and of finite index in each of the factors. We give a geometric proof of the Squier's splitting of Art_{333} in Example 4.14. The subgroup F_7 has index three and two respectively in the factors F_3 and F_4 in the splitting of Art_{333}. This yields a short exact sequence of groups

$$1 \to F_7 \to \mathrm{Art}_{333} \to \mathbb{Z}/3 * \mathbb{Z}/2 \to 1.$$

In particular Art_{333} is free-by-(virtually free), and therefore virtually free-by-free. Since every split extension of a finitely generated residually finite group by a residually finite group is residually finite [29], we can conclude that Art_{333} is residually finite. Similar arguments yield residual finiteness of Art_{244} and Art_{236}. The residual finiteness of Art_{333} and Art_{244} also follows from the fact that they are commensurable with the quotients of spherical Artin groups modulo their centers, respectively Art_{233}/Z and Art_{234}/Z [11].

Theorem B provides a splitting of Art_Γ as a graph of groups with free vertex groups. In general, the existence of such a splitting does not guarantee residual finiteness. In order to prove Theorem A we carefully analyze the splitting and use a criterion for residual finiteness of certain amalgams of special form. See Theorem 2.8 and Theorem 2.11. The following question is open in general.

Question. *Let A, B, C be finite rank free groups. When is the group $A *_C B$ (or $A*_B$) residually finite?*

One instance where $G = A *_C B$ (or $A*_B$) is residually finite is when C is malnormal in A, B. By the combination theorem of Bestvina-Feighn [2], if A, B are hyperbolic, and C is quasi-convex in both A and B and malnormal in at least one of A, B, then $G = A *_C B$ is hyperbolic. Wise showed that in such a case, G is residually finite [34], and later Hsu-Wise proved that G is in fact virtually special [21]. Another class of examples of residually finite amalgams are doubles of free groups along a finite index subgroup. These groups are virtually direct products of two finite rank free groups [1].

On the other hand there are examples of amalgamated products of free groups that are not residually finite. Bhattacharjee constructed a first example which is an amalgam of two free groups along a common subgroup of finite index in each of the factors [6]. More examples are lattices in the automorphism group of a product of two trees, which split as twisted doubles of free groups along a finite index subgroup, and they were constructed by [33] and [8]. The Burger-Mozes examples are not only non residually finite, but virtually simple.

The paper is organized as follows. In Section 1 we fix notation and recall some geometric group theory tools that we use later. In Section 2 we recall some facts about residual finiteness and prove our criterion for residual finiteness of twisted doubles of free group (Theorem 2.8) and of HNN extensions of free groups (Theorem 2.11). In Section 3 we recall the definition of Artin groups, and describe their non-standard presentations due to Brady-McCammond [9]. In Section 4 we carefully study the presentation complex from the previous section and prove Theorem B (as Theorem 4.3). Finally, in Section 5 we prove Theorem A (as Corollary 5.7 and Corollary 5.12). A proof in the case where at least one label is even, is generalized to a broader family of Artin groups in Section 6.

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1. Graphs

In this section we gather together some standard notions and tools that we use in later sections.

1.1. Basic definitions

A graph is a 1-dimensional CW-complex. All the graphs we consider are finite. The vertex set of a graph X is denoted by $V(X)$, and its edge set is denoted by $E(X)$. Most graphs we consider are multigraphs, i.e. they may have multiple edges with the same endpoints, and *loops*, i.e. edges with the same both endpoints. We refer to graphs without loops and multiple edges with equal endpoints as *simple graphs*.

A map ρ between graphs is *combinatorial* if the image of each vertex is a vertex, and while restricted to an open edge with endpoints v_1, v_2 it is a homeomorphism onto an edge with endpoints $\rho(v_1), \rho(v_2)$. A combinatorial map $\rho : Y \to X$ between graph X, Y is a *combinatorial immersion*, if for every vertex $v \in Y$ and oriented edges e_1, e_2 with terminal vertex v such that $\rho(e_1) = \rho(e_2)$, we have $e_1 = e_2$. A combinatorial immersion $\rho : Y \to X$ induces an injective homomorphism $\pi_1(Y, y) \hookrightarrow \pi_1(X, x)$ [32, Prop 5.3] where x, y are basepoints of X, Y respectively with $\rho(y) = x$. A different basepoint y' in the same connected component of Y as y and such that $\rho(y') = x$ represents a subgroup $\pi_1(Y, y') \hookrightarrow \pi_1(X, x)$ which is conjugate to $\pi_1(Y, y)$.

Let I_n denote a graph with vertex set $\{0, 1, \dots, n\}$ with an edge for every pair of vertices k_1, k_2 such that $|k_2 - k_1| = 1$. Let C_n denote graph I_{n-1} with an additional edge joining $n-1$ and 0. A *path of length* n in a graph X, is a combinatorial immersion $I_n \to X$. A *cycle of length* n in a graph X, is a combinatorial immersion $C_n \to X$. We say a path or cycle is *simple*, if vertices $0, \dots, n-1$ are mapped to distinct vertices in X. We say a path is *closed*, if 0 and n are mapped to the same vertex in X. A *segment* in X is a simple path whose only vertices that are mapped to vertices of valence > 2 in X are its endpoints. We refer to vertices of valence > 2 as *branching vertices*.

Suppose X has a single vertex, i.e. X is a wedge of loops. Let $\rho : Y \to X$ be a combinatorial immersion. If we choose an orientation for each edge of X, then the map $Y \to X$ can be represented by the graph Y with edges oriented and labeled by $E(X)$. Visually, we pick a distinct color for each edge of X and represent $Y \to X$ as Y with edges oriented and colored.

If Γ is a simple graph, we can describe a path as an n-tuple $(a_1, a_2, \dots, a_n)$ of vertices of Γ where $\{a_i, a_i + 1\}$ forms an edge for each $1 \leq i < n$. Similarly we can describe a cycle in Γ as an $n+1$-tuple $(a_1, a_2, \dots, a_n, a_1)$, if $(a_1, a_2, \dots, a_n, a_1)$ is a path.

1.2. Fiber product of graphs

Let $\rho_i : (Y_i, y_i) \to (X, x)$ be a combinatorial immersion of based graphs for $i = 1, 2$. The intersection of subgroups $\pi_1(Y_1, y_1)$ and $\pi_1(Y_2, y_2)$ of $\pi_1(X, x)$ can be computed as the fundamental group of the fiber product of based graphs, by Stallings [32]. The *fiber product of* Y_1 *and* Y_2 *over* X is the pullback in the category of graphs, i.e. it is the graph $Y_1 \otimes_X Y_2$ with the vertex set

$$\{(v_1, v_2) \in V(Y_1) \times V(Y_2) : \rho_1(v_1) = \rho_2(v_2)\}$$

and the edge set

$$\{(e_1, e_2) \in E(Y_1) \times E(Y_2) : \rho_1(e_1) = \rho_2(e_2)\}$$

where $\rho_1(e_1) = \rho_2(e_2)$ is the equality of oriented edges. The graph $Y_1 \otimes_X Y_2$ often has several connected components. The natural combinatorial immersion $Y_1 \otimes_X Y_2 \to X$ induces the embedding $\pi_1(Y_1 \otimes_X Y_2, (y_1, y_2)) \to \pi_1(X, x)$. By [32, Thm 5.5], $\pi_1(Y_1 \otimes_X Y_2, (y_1, y_2))$ is the intersection of $\pi_1(Y_1, y_1)$ and $\pi_1(Y_2, y_2)$ in $\pi_1(X, x)$. See also [24, Section 9].

Suppose X has a unique vertex y. Then $V(Y_1 \otimes_X Y_2) = V(Y_1) \times V(Y_2)$. If $\rho : Y \to X$ is a combinatorial immersion of graphs then connected components of $Y \otimes_X Y$ represent the intersections $H \cap H^g$ where $H := \pi_1(Y, y) < \pi_1(X, x)$ and $g \in \pi_1(X, x)$. In particular, one of the connected components of $Y \otimes_X Y$ is a copy of Y with the vertex set $\{(v, v) : v \in V(Y)\}$. It corresponds to the intersection $H \cap H^g = H$ where $g \in H$. We refer to this connected component of $Y \otimes_X Y$ as *trivial*. All other subgroups of the form $H \cap H^g$ are either $\{e\}$, or their conjugacy classes are represented by nontrivial connected components of $Y \otimes_X Y$.

2. Residual finiteness

A group G is *residually finite* if for every $g \in G - \{e\}$ there exists a finite index subgroup $G' < G$ such that $g \notin G'$. Equivalently, there exists a finite quotient $\phi : G \to \bar{G}$ such that $\phi(g) \neq e$. It is easy to see, that if G has a finite index residually finite subgroup, then G is residually finite.

Let H be a subgroup of G, let $\phi : G \to \bar{G}$ be a (not necessarily finite) quotient and let $\{g_i\}_i \subseteq G - H$ be a collection of elements. We say ϕ *separates* H from $\{g_i\}_i$ if $\phi(g_i) \notin \phi(H)$ for all $i \in I$. A subgroup $H < G$ is *separable* if for every finite collection $\{g_i\}_i \subseteq G - H$, there exists a finite quotient $\phi : G \to \bar{G}$ that separates H from $\{g_i\}_i$. Equivalently, there exists a finite index subgroup $G' <_{f.i} G$ containing H such that $g_i \notin G'$ for all i. To see the equivalence of the two definitions, in one direction take N to be the normal core of G' in G (i.e. the intersection of all conjugates of G' in G) and set $\bar{G} = G/N$. Conversely, take $G' = \phi^{-1}(\phi(H))$.

The main goal of this section is to formulate our criterion for residual finiteness of certain free products of amalgamation and HNN extensions, Theorem 2.8 and Theorem 2.11. We use the following criterion of Wise for residual finiteness of graph of free groups [34]. A graph of groups is *algebraically clean*, if vertex groups are free, and edge groups are free factors in both of their vertex groups.

Theorem 2.1. *[34, Thm 3.4]Let G split as a finite algebraically clean graph of groups where all edge groups are of finite rank. Then G is residually finite.*

2.1. Free factor and separability

Let H, G be finite rank free groups. A famous theorem by Marshall Hall [16] states that every finitely generated subgroup of a free group is virtually a free factor, i.e. if $H < G$ then there exists a finite index subgroup $G' < G$ such that $H < G'$ and H is a free factor of G'. A closely related result states that free groups are *subgroup separable*, i.e. every finitely generated subgroup is separable.

Let X, Y be graphs with basepoint x_0, y_0 respectively. Let $\rho : (Y, y_0) \to (X, x_0)$ be a combinatorial immersion inducing the inclusion of finite rank free group $H := \pi_1(Y, y_0) \hookrightarrow \pi_1(X, x_0) =: G$.

Definition 2.2. Let $\mathcal{A}_\rho \subseteq G$ consist of all $g \in G$ represented by a cycle γ in X such that γ is a concatenation of paths $\gamma_1 \cdot \gamma_2$ where:

- $\gamma_1 = \rho(\mu_1)$ and μ_1 is a non-trivial simple non-closed path in Y going from y_0 to some vertex y_1,
- $\gamma_2 = \rho(\mu_2)$ and μ_2 is either trivial, or is a simple non-closed path in Y going from some vertex y_2 to y_0, where $y_1 \neq y_2 \neq y_0$.

We refer to $\mathcal{A}_\rho$ as the *oppressive set for H in G with respect to ρ*. We say $\mathcal{A}$ is an *oppressive set for H in G*, if there exists a combinatorial immersion ρ with $\mathcal{A} = \mathcal{A}_\rho$.

In Proposition 2.4 we state some properties of the set $\mathcal{A}_\rho$. In particular, we explain the connection between the separation from the set $\mathcal{A}_\rho$ and H being a free factor. In one of the proofs below we use the following easy lemma, due to Karrass-Solitar.

Lemma 2.3 *([26]). Let H be a free factor in G. Then for every finite index subgroup $G' < G$ the intersection $G' \cap H$ is a free factor in G'.*

Proposition 2.4. *Let $\rho : (Y, y_0) \to (X, x_0)$ be a combinatorial immersion of based graphs inducing the inclusion of finite rank free group $H := \pi_1(Y, y_o) \hookrightarrow \pi_1(X, x_o) =: G$, and let $\mathcal{A}_\rho$ be the oppressive set for H in G with respect to ρ.*

(1) $\mathcal{A}_\rho \cap H = \emptyset$.
(2) $\mathcal{A}_\rho = \emptyset$ if and only if ρ is an embedding.
(3) For any based cover $(\hat{X}, \hat{x}_0) \to (X, x_0)$ such that ρ factors through a combinatorial immersion $\hat{\rho} : (Y, y_0) \to (\hat{X}, \hat{x}_0)$, we have $\mathcal{A}_{\hat{\rho}} = \mathcal{A}_\rho \cap \pi_1(\hat{X}, \hat{x}_0)$.
(4) If $\phi : G \to \bar{G}$ is a quotient that separates H from $\mathcal{A}_\rho$, then $H \cap \ker \phi$ is a free factor in $\ker \phi$.

Proof. (1) Suppose that there exists $g \in \mathcal{A}_\rho \cap H$. Then g is represented by a loop γ which can be expressed as a concatenation $\gamma_1 \cdot \gamma_2$ as in Definition 2.2. Since ρ is a combinatorial immersion, there is a unique path μ_1 starting at y_0 such that $\rho(\mu_1) = \gamma_1$, and there is a unique path μ_2 ending at y_0 such that $\rho(\mu_2) = \gamma_2$. Since $g \in H$ the path μ_1 must end at the same vertex as μ_2 starts. This is a contradiction.

(2) Suppose $\mathcal{A}_\rho$ is not empty. That means that there exist a path μ_1 joining vertices y_0 and y_1 in Y and a path μ_2 joining vertices y_2 and y_0 where $y_2 \neq y_0, y_1$ such that

$\rho(\mu_1)\cdot\rho(\mu_2)$ is a closed path. That means that $\rho(y_1)=\rho(y_2)$, i.e. ρ is not an embedding. Conversely, suppose that ρ is not an embedding, and let $y_1, y_2 \in Y$ such that $\rho(y_1) = \rho(y_2)$. Then the image under ρ of a simple path from y_0 to y_1 concatenated with the image of a simple path going y_2 back to y_0 lifts to a closed path in X. That path corresponds to an element of $\mathcal{A}_\rho$.

(3) We first prove that $\mathcal{A}_{\hat\rho} \subseteq \mathcal{A}_\rho \cap \pi_1(\hat X, \hat x_0)$. By definition $\mathcal{A}_{\hat\rho} \subseteq \pi_1(\hat X, \hat x_0)$. Let $g \in \mathcal{A}_{\hat\rho}$ be represented by a cycle $\hat\gamma_1\cdot\hat\gamma_2$ in $\hat X$ as in Definition 2.2. Then $\hat\gamma_1\cdot\hat\gamma_2$ maps to a cycle $\gamma_1\cdot\gamma_2$ in X which still satisfies Definition 2.2 and so, $g \in \mathcal{A}_\rho$. Conversely, let $g \in \mathcal{A}_\rho \cap \pi_1(\hat X, \hat x_0)$ be represented by a cycle $\gamma_1\cdot\gamma_2$ in X. Since $g \in \pi_1(\hat X, \hat x_0)$ the cycle lifts to a cycle $\hat\gamma_1\cdot\hat\gamma_2$ in $\hat X$ based at $\hat x_0$. It follows that $g \in \mathcal{A}_{\hat\rho}$.

(4) Since ϕ separates H from $\mathcal{A}_\rho$, the group $G' := \phi^{-1}(\phi(H))$ contains H but does not contain any element of $\mathcal{A}_\rho$. Let $(\hat X, \hat x_0) \to (X, x_0)$ be a cover corresponding to G'. Since $H \subseteq G'$, the map ρ factors through $\hat\rho : (Y, y_0) \to (\hat X, \hat x_0)$. Since $\mathcal{A}_\rho \cap G' = \emptyset$, by (3) $\mathcal{A}_{\hat\rho} = \emptyset$. By (2) $\hat\rho$ is an embedding. Thus H is a free factor of G'. By Lemma 2.3, $H \cap \ker\phi$ is a free factor in $\ker\phi$. □

The following Lemma will be used to verify that certain quotients separate a subgroup from its oppressive set.

Lemma 2.5. *Let $\rho : (Y, y_0) \to (X, x_0)$ be a combinatorial immersion of graphs where x_0 is the unique vertex of X. Let $Y_\bullet, X_\bullet$ be 2-complexes with the 1-skeletons $Y_\bullet^{(1)} = Y$ and $X_\bullet^{(1)} = X$, and let $\rho_\bullet : (Y_\bullet, y_0) \to (X_\bullet, x_0)$ be a map extending ρ. Let $\phi : \pi_1(X, x_0) \to \pi_1(X_\bullet, x_0)$ be the natural quotient and suppose that $\phi(\pi_1(Y, y_0)) = (\rho_\bullet)_*(\pi_1(Y_\bullet, y_0))$. If the lift to the universal covers $\widetilde{\rho}_\bullet : \widetilde{Y_\bullet} \to \widetilde{X_\bullet}$ of $\rho_\bullet$ is an embedding, then ϕ separates $\pi_1(Y, y_0)$ from $\mathcal{A}_\rho$.*

Proof. The vertex set of $\widetilde{X_\bullet}$ can be identified with $\pi_1 X_\bullet$. By assumption, we can view $\widetilde{Y_\bullet}$ as a subcomplex of $\widetilde{X_\bullet}$ whose vertex set contains vertices corresponding to $\phi(\pi_1(Y, y_0)) = (\rho_\bullet)_*(\pi_1(Y_\bullet, y_0)) \subseteq \pi_1(X_\bullet, x_0)$. Let p the base vertex of $\widetilde{X_\bullet}$ representing the trivial element $e \in \widetilde{X_\bullet}$.

Let $g \in \mathcal{A}_\rho$ be represented by a cycle $\gamma = \gamma_1\cdot\gamma_2$ in X with $\gamma_i = \rho(\mu_i)$ as in Definition 2.2, i.e. μ_1 is a non-trivial simple path in Y starting at y_0 and ending at some $y_1 \neq y_0$, and μ_2 is either trivial or it is a simple path in Y starting at some $y_2 \neq y_0, y_1$ and ending at y_0. The path μ_1 lifts to unique paths $\widetilde{\mu}_1$ starting at p in $\widetilde{Y_\bullet} \subseteq \widetilde{X_\bullet}$. Similarly, the path μ_2 lifts to unique path $\widetilde{\mu}_2$ ending at $g.p$ in $\widetilde{Y_\bullet} \subseteq \widetilde{X_\bullet}$. To prove that ϕ separates $\pi_1(Y, y_0)$ from $\mathcal{A}_\rho$, we need to show that $\phi(g) \notin (\rho_\bullet)_*(\pi_1(Y_\bullet, y_0))$.

Suppose to the contrary, that $\phi(g) \in (\rho_\bullet)_*(\pi_1(Y_\bullet, y_0))$. That means that $\widetilde{\mu}_2$ starts where $\widetilde{\mu}_1$ end, so the concatenation $\widetilde{\mu}_1\cdot\widetilde{\mu}_2$ is a path from p to $g.p$. Since $\widetilde{\mu}_1\cdot\widetilde{\mu}_2$ projects onto $\mu_1\cdot\mu_2$ in $Y \subseteq Y_\bullet$, we conclude that $\mu_1\cdot\mu_2$ is a cycle in Y, which is a contradiction. □

2.2. Residual finiteness of a twisted double

Throughout this section A is a finite rank free group, $C < A$ is a finitely generated subgroup and $\beta : C \to C$ is an automorphism.

Definition 2.6. The *double of A along C twisted by β*, denoted by $D(A, C, \beta)$ is a free product with amalgamation $A *_C A$ where C is mapped to the first factor via the natural inclusion $C \hookrightarrow A$, and to the second factor via the natural inclusion precomposed with β.

Proposition 2.7. *Let $\mathcal{A}$ be an oppressive set for C in A. Suppose there exists a finite quotient $\Psi : D(A, C, \beta) \to K$ such that $\Psi|_A : A \to K$ separates C from $\mathcal{A}$. Then $D(A, C, \beta)$ virtually splits as an algebraically clean graph of finite rank free groups. In particular, $D(A, C, \beta)$ is residually finite.*

Proof. The group $D(A, C, \beta)$ acts on its Bass-Serre tree T with vertex stabilizers conjugate to A, and edge stabilizers conjugate to C. The group $\ker \Psi$ acts on T with a finite fundamental domain, since the index of $\ker \Psi$ in $D(A, C, \beta)$ is finite. The vertex stabilizers are conjugates of $\ker \Psi \cap A = \ker \Psi|_A$, and the edge stabilizers are conjugates of $\ker \Psi \cap C = \ker \Psi|_A \cap C$. By Proposition 2.4(4), $\ker \Psi|_A \cap C$ is a free factor in $\ker \Psi|_A$, i.e. every edge stabilizer is a free factor in each respective vertex stabilizers of the action of $\ker \Psi$ on T. In particular, $\ker \Psi$ splits as a clean graph of free groups, so by Theorem 2.1 $\ker \Psi$ is residually finite. Since $\ker \Psi$ has finite index in $D(A, C, \beta)$ the conclusion follows. □

A subgroup H is *malnormal* in G, if for every $g \in G - H$ we have $H^g \cap H = \{1\}$, where $H^g := g^{-1}Hg$. More generally, a collection $\{H_1, \ldots, H_n\}$ of subgroups of G is *malnormal* in G, if for every $1 \leq i, j \leq n$ and $g \in G$, we have $H_i^g \cap H_j = \{1\}$, unless $i = j$ and $g \in H_i$.

Let $\phi : A \to \bar{A}$ be a quotient and let $\bar{C} := \phi(C)$. The automorphism $\beta : C \to C$ projects to an automorphism $\bar{\beta} : \bar{C} \to \bar{C}$ if and only if $\beta(C \cap \ker \phi) = C \cap \ker \phi$. When that is the case, then ϕ induces a quotient $\Phi : D(A, C, \beta) \to D(\bar{A}, \bar{C}, \bar{\beta})$.

Theorem 2.8. *Suppose there exists a quotient $\phi : A \to \bar{A}$ such that*

(1) $\bar{A}$ is a virtually special hyperbolic group,
(2) $\bar{C} := \phi(C)$ is malnormal and quasiconvex in $\bar{A}$,
(3) ϕ separates C from an oppressive set $\mathcal{A}$ of C in A,
(4) β projects to an automorphism $\bar{\beta} : \bar{C} \to \bar{C}$.

Then $D(A, C, \beta)$ virtually splits as an algebraically clean graph of finite rank free groups. In particular, $D(A, C, \beta)$ is residually finite.

Proof. Condition (4) ensures that ϕ extends to the quotient $\Phi : D(A, C, \beta) \to D(\bar{A}, \bar{C}, \bar{\beta})$. Since ϕ separates C from $\mathcal{A}$, the set $\phi(\mathcal{A}) = \{\phi(a) \mid a \in \mathcal{A}\} \subseteq \bar{A}$ is disjoint from $\bar{C}$. By Bestvina-Feighn [2] $D(\bar{A}, \bar{C}, \bar{\beta})$ is hyperbolic, since it is a free product of two copies of a hyperbolic group $\bar{A}$ amalgamated along a subgroup $\bar{C}$ which is malnormal and quasiconvex in each of the factors (see also [23]). Since $\bar{A}$ is virtually cocompactly special, by Hsu-Wise [21] $D(\bar{A}, \bar{C}, \bar{\beta})$ is cocompactly cubulated. Then by Haglund-Wise [20] $D(\bar{A}, \bar{C}, \bar{\beta})$ is virtually special and in particular QCERF [19]. Thus, $\bar{C}$ is separable in $D(\bar{A}, \bar{C}, \bar{\beta})$. There exists a finite quotient $\Psi : D(\bar{A}, \bar{C}, \bar{\beta}) \to K$ such that $\Psi|_{\bar{A}}$ separates $\bar{C}$ from $\phi(\mathcal{A})$. Thus the composition $\Psi \circ \Phi|_A : A \to K$ separates C from $\mathcal{A}$. The quotient $\Psi \circ \Phi : D(A, C, \beta) \to K$ satisfies the assumptions of Proposition 2.7. Hence $D(A, C, \beta)$ is residually finite. □

In our application of Theorem 2.8, Condition (4) will be verified using the following.

Observation 2.9. *Let Z be a finite graph and let $b : (Z, z_0) \to (Z, z_1)$ be a graph automorphism. Then b together with a choice of a path from z_0 to z_1 induces an automorphism $\beta : \pi_1(Z, z_0) \to \pi_1(Z, z_0)$. If $Z_\bullet$ is a finite 2-complex with the 1-skeleton Z such that b extends to $b_\bullet : Z_\bullet \to Z_\bullet$, then β projects to an automorphism $\beta_\bullet : \pi_1(Z_\bullet, y_0) \to \pi_1(Z_\bullet, y_0)$.*

2.3. Residual finiteness of an HNN extension

Let A, B be finite rank free groups. For $i = 1, 2$ let $\beta_i : B \to A$ denote an injective homomorphism, and denote $B_i = \beta_i(B)$. Let β denote the isomorphism $\beta_2 \cdot \beta_1^{-1} : B_1 \to B_2$. By $A*_B$ we denote the HNN extension of A with respect to $\{\beta_1, \beta_2\}$, i.e.

$$A*_B = \langle A, t \mid t^{-1}bt = \beta(b) \text{ for all } b \in B_1 \rangle$$

Let X be a bouquet of loops, with $\pi_1 X$ identified with A, and for $i = 1, 2$ let $\rho_i : Y_i \to X$ be a combinatorial immersion inducing the inclusion $B_i \hookrightarrow A$. Let $\mathcal{A}_{\rho_1}, \mathcal{A}_{\rho_2}$ be the oppressive sets for B_1, B_2 with respect to ρ_1, ρ_2 respectively.

Proposition 2.10. *Suppose there exists a quotient $\Psi : A*_B \to K$ such that $\Psi|_A : A \to K$ separates B_1 from $\mathcal{A}_1$, and B_2 from $\mathcal{A}_2$. Then $A*_B$ virtually splits as an algebraically clean graph of finite rank free groups. In particular, $A*_B$ is residually finite.*

Proof. The proof is analogous as for Proposition 2.7. □

Theorem 2.11. *Suppose there exists $\phi : A \to \bar{A}$ such that*

(1) $\bar{A}$ is a virtually special hyperbolic group,
(2) $\bar{B}_i := \phi(B_i)$ is quasiconvex in $\bar{A}$ for $i = 1, 2$, and the collection $\{\bar{B}_1, \bar{B}_2\}$ is malnormal in $\bar{A}$,
(3) ϕ separates B_i from $\mathcal{A}_{\rho_i}$ for $i = 1, 2$,

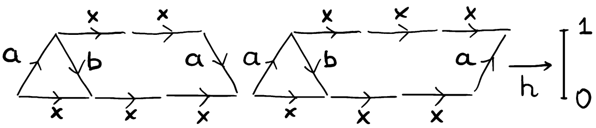

Fig. 1. The 2-cells of the presentation complex of the group presentation of a 2-generator Artin group Art_5 (left) and Art_6 (right).

(4) β *projects to an isomorphism* $\bar{\beta} : \bar{B}_1 \to \bar{B}_2$.

Then $A*_B$ *virtually splits as an algebraically clean graph of finite rank free groups. In particular,* $A*_B$ *is residually finite.*

Proof. The proof is analogous as the proof of Theorem 2.8. It uses Proposition 2.10 in the place of Proposition 2.7. □

3. Artin groups and their Brady-McCammond complex

In this section we describe a complex X_Γ associated to a non-standard presentation of Art_Γ that was introduced and shown to be CAT(0) for many Artin groups by Brady-McCammond in [9]. We then describe certain subspaces of X_Γ that will be used in Section 4 to prove that for certain Γ the group Art_Γ splits as an amalgam of finite rank free groups. We start with the case of 2-generator Artin group.

3.1. Brady-McCammond presentation for a 2-generator Artin group

Consider an Artin group on two generators

$$\mathrm{Art}_M = \langle a, b \mid (a, b)_M = (b, a)_M \rangle$$

where $M < \infty$. By adding an extra generator x and setting $x = ab$ we get another presentation

- if $M = 2m$:

$$\langle a, b, x \mid x = ab, x^m = bx^{m-1}a \rangle$$

- if $M = 2m + 1$:

$$\langle a, b, x \mid x = ab, x^m a = bx^m \rangle$$

See Fig. 1. Let $r_{2m}(a, b, x)$ denote the relation $x^m = bx^{m-1}a$ and let $r_{2m+1}(a, b, x)$ denote the relation $x^m a = bx^m$. Let $X(a, b)$ be the 2-complex corresponding to the above presentation. Denote by $C(a, b)$ the disjoint union of its two 2-cells, and let p :

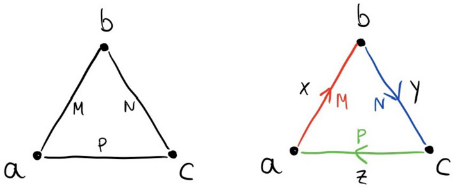

Fig. 2. On the left: the defining graph Γ of a triangle Artin group. On the right: the graph Γ equipped with the cyclic orientation determining a Brady-McCammond presentation with three new generators x, y, z.

$C(a,b) \to X(a,b)$ be the natural projection. There is an embedding of $C(a,b)$ in the plane and a height map h to the interval $[0,1]$ such that h restricted to each edge x is constant, as in Fig. 1. We refer to these edges as *horizontal*, and to the other edges as *non-horizontal*. Note that the map h is not well-defined on $X(a,b)$.

3.2. Brady-McCammond presentation for a general Artin group

A *partial orientation* on a simple graph Γ is a choice of an endpoint $\iota(e)$ for some of the edges e in $E(\Gamma)$. Visually we represent a partial orientation on a simple graph by arrows: an edge e with a choice of vertex $\iota(e)$ is represented as an arrow starting at the vertex $\iota(e)$. We say a cycle (resp. path) γ in a simple graph Γ with a partial orientation ι is *directed*, if for every edge e in the cycle $\iota(e)$ is defined, and $\iota(e) = \iota(e')$ only when $e = e'$. An *orientation* on Γ is a partial orientation where each edge is oriented.

Let Γ be a simple graph with edges labeled by number ≥ 2, with a partial orientation ι where $i\iota(e)$ is defined for an edge e if and only if the label of e is ≥ 3.

Generalizing Section 3.1 we consider the following presentation of Art_Γ with respect to the partial orientation ι:

$$\langle a \in V(\Gamma), x \in E(\Gamma) \mid x = ab,\ r_{M_{ab}}(a,b,x)$$
$$\text{where } x = \{a,b\} \text{ and either } a = \iota(x) \text{ or } M_{ab} = 2\rangle.$$

The partial orientation of the edge $x = \{c,d\}$ determines whether the new generators x equals cd or dc. If $M_{cd} = 2$ we have $x = cd = dc$, which is why we do not need to specify the partial orientation. In the case of a 3-generators Artin group

$$\mathrm{Art}_{MNP} = \langle a,b,c \mid (a,b)_M = (b,a)_M, (b,c)_N = (c,b)_N, (c,a)_P = (a,c)_P\rangle,$$

with $M, N, P < \infty$, the cyclic orientation on the triangle Γ (see Fig. 2) gives the presentation

$$\langle a,b,c,x,y,z \mid x = ab, y = bc, z = ca, r_M(a,b,x), r_N(b,c,y), r_P(c,a,z)\rangle.$$

Let X_Γ be the complex obtained from the union $\bigcup_{(a,b)\in E(\Gamma)} X(a,b)$ by identifying the edges with the same labels. The fundamental group of X_Γ is Art_Γ. Brady-McCammond

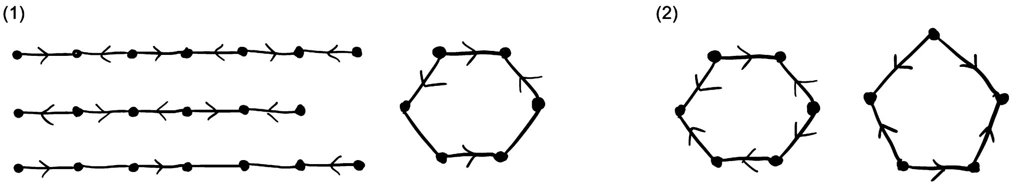

Fig. 3. (1) Examples of misdirected paths and a misdirected cycle. (2) Almost misdirected cycles of even and odd length may contain directed subpaths of length 3 and 2 respectively.

showed in [9] that when all labels are ≥ 3, then X_Γ admits a locally CAT(0) metric provided that there exists an orientation such that

(1) every triangle in Γ is directed,
(2) every 4-cycle in Γ contains a directed path of length at least 2.

Their proof in fact works in a greater generality. Using their methods one can show that for certain graphs with labels 2 the complex X_Γ admits a locally CAT(0) metric. We discuss the condition on Γ in more detail in Section 4.1.

As in the 2-generator case, let C_Γ be the disjoint union of the 2-cells of X_Γ. Again let $p : C_\Gamma \to X_\Gamma$ be the projection map. We also define a height function $h : C_\Gamma \to [0, 1]$ whose restriction to each $C(a, b)$ is the height function defined in Section 3.1.

4. Splittings of Artin groups

4.1. The statement of the Splitting theorem

The main goal of Section 4 is to prove Theorem 4.3, which asserts that under certain assumption on Γ, Art_Γ splits as a free product with amalgamation $A *_C B$ or an HNN-extension $A*_B$ where A, B, C are finite rank free groups. We begin with a precise statement.

Definition 4.1. Let Γ be a simple graph with a partial orientation ι. We say a path γ of length ≥ 2 in Γ is a *misdirected path* if the partial orientation on γ induced by ι can be extended to an orientation such that a maximal directed subpath of γ has length 1. We say an even length cycle γ is a *misdirected cycle* if the induced partial orientation on γ extends to an orientation where maximal directed subpaths of γ have length 1. We say a cycle γ is an *almost misdirected cycle* if γ can be expressed as a cycle $(a_1, \dots, a_n, a_1)$ where the path $(a_1, \dots, a_n)$ is misdirected.

See Fig. 3(1) for examples of misdirected paths and a misdirected cycle. See Fig. 3(2) for examples of almost misdirected cycles. Note that every even length misdirected cycle is almost misdirected, but not vice-versa.

Definition 4.2. Let Γ be a simple graph. Assume edges of Γ are labeled by an integer ≥ 2. We say that a partial orientation ι on Γ is *admissible* if

- $\iota(e)$ for an edge e is defined if and only if the label of e is ≥ 3, and
- no cycle in Γ is almost misdirected.

Theorem 4.3. *Suppose Γ admits an admissible partial orientation. If Γ is a bipartite graph with all labels even, then Art_Γ splits as an HNN-extension $A*_B$, where A, B are finite rank free groups. Otherwise Art_Γ splits as a free product with amalgamation $A*_C B$ where A, B, C are finite rank free groups. Moreover, $\mathrm{rk}\, A = |E(\Gamma)|$, $\mathrm{rk}\, B = 1-|V(\Gamma)|+2|E(\Gamma)|$, and C is an index 2 subgroup of B, so $\mathrm{rk}\, C = 1 - 2|V(\Gamma)| + 4|E(\Gamma)|$.*

We prove Theorem 4.3 in Section 4.9. The condition that Γ has no almost misdirected cycles implies Art_Γ in Theorem 4.3 is 2-dimensional (since no 3-cycle can have an edge labeled by 2). Our condition also implies the other condition given by Brady-McCammond (and included in the end of Section 3.2) ensuring that X_Γ is CAT(0). Therefore all Artin groups satisfying the assumptions of Theorem 4.3 are CAT(0) by [9]. Since a 4-clique does not admit an orientation where each 3-cycle is directed, our condition also implies that the clique number of Γ for is at most 3.

Recall, $\mathrm{Art}(\Gamma)$ has *large type*, if $M_{ab} \geq 3$ for all $\{a, b\} \in E(\Gamma)$. Here are some examples of Artin groups that satisfy the assumptions of Theorem 4.3:

- All large type 3-generator Artin groups.
- More generally, large type Artin group whose defining graph Γ admits an orientation where each simple cycle is directed. This includes Γ that is planar and each vertex has even valence (as observed in [9]).
- Many other Artin groups with the sufficiently small ratio $\frac{\#\text{ labels 2 in }\gamma}{\mathrm{length}(\gamma)}$ in every cycle γ. In particular, this includes Artin groups with Γ where all edges labeled by 2 disconnect the graph and all subgraphs without edges labeled by 2 are as above.

For the rest of this section, we assume that Γ is a fixed connected, labeled, simple graph. We write X for the Brady-McCammond complex X_Γ defined in Section 3.2. The splitting of Art_Γ comes from a decomposition of the 2-complex X into a union of two subspaces where each subspace and the intersection of them all have homotopy type of graphs. We will now describe these subspaces.

4.2. Horizontal graphs in X

We distinguish the following subspaces of X that are the images under p of level sets of the height function h, as defined in Section 3.2.

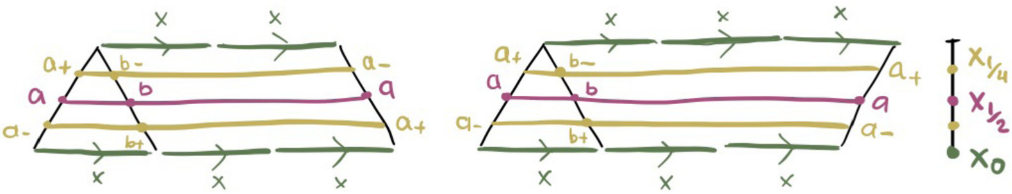

Fig. 4. Horizontal graphs X_0, $X_{1/2}$ and $X_{1/4}$.

(0) The level set $p(h^{-1}(0))$ is denoted by X_0. The intersection of X_0 with every $X(a,b)$ is a single loop labeled by the generator $x = ab$. Thus X_0 is a bouquet of loops, one for each edge in $E(\Gamma)$. See Fig. 4.

(1/2) The level set $p(h^{-1}(\frac{1}{2}))$ is denoted by $X_{1/2}$. We call the points of intersection of $X_{1/2}$ with the non-horizontal edges the *midpoints*. We will abuse the notation, and a will denote the midpoint of the edge labeled by a. The intersection of $X_{1/2}$ with every $X(a,b)$ is a single cycle of length 2 with vertices a, b. The graph $X_{1/2}$ is a union of all these cycles of length two identified along vertices with the same label. Hence $X_{1/2}$ is a copy of the graph Γ with every edge doubled. See Fig. 4.

(1/4) The union of the level set $p(h^{-1}(\frac{1}{4}) \cup h^{-1}(\frac{3}{4}))$ is denoted by $X_{1/4}$. We call the points of intersection of $X_{1/4}$ with the non-horizontal edges the *quarterpoints*, and denote them by a_+, a_-, b_+, b_- where the vertices a_-, a, a_+ are ordered with respect with the orientation of the edge a. Similarly, b_-, b, b_+ are ordered with respect with the orientation of the edge b. See Fig. 4.
If M_{ab} is odd, the intersection of $X_{1/4}$ with $X(a,b)$ is a single cycle of length 4. If M_{ab} is even, the intersection of $X_{1/4}$ with $X(a,b)$ is a disjoint union of two cycles, each of length 2. We describe $X_{1/4}$ in more detail in Section 4.5.

Let us emphasize that $X_{1/2}$ is never a simple graph; it always has double edges. Similarly $X_{1/4}$ does not need to be simple.

4.3. Horizontal tubular neighborhoods in X

Fix $0 < \epsilon < 1/4$. We now define tubular neighborhoods $N_0, N_{1/2}, N_{1/4} \subseteq X$ of graphs $X_0, X_{1/2}, X_{1/4}$.

(0) Let N_0 be an open neighborhood of X_0 of the form $p\left(h^{-1}([0, 1/2-\epsilon) \cup (1/2+\epsilon, 1])\right)$. Note that N_0 deformation retracts onto X_0 with the property that the intersection of N_0 with the 1-skeleton of X is contained in the 1-skeleton of X at all times.

(1/2) Similarly, let $N_{1/2}$ be an open neighborhood of $X_{1/2}$ of the form $p(h^{-1}((\epsilon, 1-\epsilon)))$. Again, $N_{1/2}$ deformation retracts onto $X_{1/2}$ such that $N_{1/2} \cap X^{(1)}$ is contained in $X^{(1)}$ at all times.

(1/4) The intersection $N_0 \cap N_{1/2}$, which we denote by $N_{1/4}$, restricted to $X(a,b)$ is equal to $p\left(h^{-1}((\epsilon, 1/2-\epsilon) \cup (1/2+\epsilon, 1-\epsilon))\right)$. Consequently, $N_{1/4}$ deformation retracts onto $X_{1/4}$ such that $N_{1/4} \cap X^{(1)}$ is contained in $X^{(1)}$ at all times.

We also have $N_0 \cup N_{1/2} = X$ because $[0, 1/2-\epsilon) \cup (1/2+\epsilon, 1] \cup (\epsilon, 1-\epsilon) = [0,1]$.

4.4. Splitting

Let $A = \pi_1 X_0 = \pi_1 N_0$, $B = \pi_1 X_{1/2} = \pi_1 N_{1/2}$ and if $X_{1/4}$ is connected, let $C = \pi_1 X_{1/4} = \pi_1 N_{1/4}$. The group A, B, C are all the fundamental groups of finite graphs, so they are finite rank free groups. The composition $X_{1/4} \hookrightarrow N_0 \to X_0$ of the inclusion $X_{1/4} \hookrightarrow N_0$ with the retraction $N_0 \to X_0$ induces a group homomorphism $C \to A$. Similarly, the composition $X_{1/4} \hookrightarrow N_{1/2} \to X_{1/2}$ induces a group homomorphism $C \to B$.

When $X_{1/4}$ is connected, then so is $N_{1/4}$. Since $N_0 \cup N_{1/2} = X$ and $N_{1/4} = N_0 \cap N_{1/2}$, by the Seifert-Van Kampen theorem we get the following.

Lemma 4.4. *If $X_{1/4}$ is connected and maps $C \to A$ and $C \to B$ are injective, then* $\mathrm{Art}_\Gamma = A *_C B$.

Analogously, we have the following.

Lemma 4.5. *Suppose $X_{1/4}$ has two connected components and $X_{1/4} \to X_{1/2}$ restricted to each connected component is a combinatorial bijection. If $X_{1/4} \to X_0$ restricted to each connected component is π_1-injective, then $\mathrm{Art}_\Gamma = A*_B$, where the two copies of B in A are induced by the two restrictions of $X_{1/4} \to X_0$ to a connected component.*

Proof. Since $X_{1/4}$ has two connected components, X is a graph of spaces with one vertex and one loop, where the vertex space is X_0 and the edge space is $X_{1/2}$ with two maps to X_0 coming from the two restrictions of $X_{1/4} \to X_0$ to a connected component. Since $X_{1/4} \to X_0$ restricted to each connected component is π_1-injective, we get the claimed HNN-extension. □

4.5. The graph $X_{1/4}$

Let us first analyze $X(a,b)_{1/4} := X_{1/4} \cap X(a,b)$. It has four vertices labeled by a_+, a_-, b_+, b_-, and four edges. If M_{ab} is even, then $X(a,b)_{1/4}$ has two edges between a_+, b_- and two edges between a_-, b_+. If M_{ab} is odd, then $X(a,b)_{1/4}$ is a 4-cycle on vertices a_+, b_-, a_-, b_+. We will think of the set of edges of $X(a,b)_{1/4}$ as a disjoint union $E(a,b)' \sqcup E(a,b)''$ where

- The set $E(a,b)'$ is equal $\{\{a_+, b_-\}, \{a_-, b_+\}\}$. Those edges correspond to the segments contained in the 2-cell with the boundary abx^{-1} in the presentation complex (see Fig. 4).

- The set $E(a,b)''$ is equal $\{\{a_+, b_-\}, \{a_-, b_+\}\}$ or $\{\{a_+, b_+\}, \{a_-, b_-\}\}$, depending on the parity of M_{ab}. Those edges correspond to the segments contained in the 2-cell $r_M(a,b,x)$ in the presentation complex (see Fig. 4).

This gives us the following description of $X_{1/4}$ for general Art_Γ.

Description 4.6. The inclusion $X_{1/4} \hookrightarrow N_{1/2}$ composed with the deformation retraction $N_{1/2} \to X_{1/2}$ is a covering map $X_{1/4} \to X_{1/2}$ of degree 2. Consequently, the graph $X_{1/4}$ is a double cover of the graph $X_{1/2}$ and can be described in terms of Γ as follows:

- The vertex set $V(X_{1/4})$ is the disjoint union $V_+ \sqcup V_-$ where each V_+, V_- is in 1-to-1 correspondence with $V(\Gamma)$. For each $a \in V(\Gamma)$ we denote the corresponding vertices by a_+, a_- respectively.
- The set of edges $E(X_{1/4})$ is the disjoint union $E' \sqcup E''$ such that each of the graphs $\Gamma' = (V_+ \sqcup V_-, E')$ and $\Gamma'' = (V_+ \sqcup V_-, E'')$ is a double cover of Γ with $a_\pm \mapsto a$ for every $a \in V(\Gamma)$. In particular, Γ' and Γ'' are simple graphs.
- For each $\{a,b\} \in E(\Gamma)$, E' contains an edge $\{a_+, b_-\}$ and $\{a_-, b_+\}$. In particular, Γ' is a bipartite double cover of Γ.
- For each $\{a,b\} \in E(\Gamma)$ where M_{ab} is even, there is an edge $\{a_+, b_-\}$ and an edge $\{a_-, b_+\}$ in E''. In particular, when M_{ab} is even then $E(X_{1/4})$ contains two copies of the edge $\{a_+, b_-\}$ and two copies of the edge $\{a_-, b_+\}$.
- For each $\{a,b\} \in E(\Gamma)$ where M_{ab} is odd, there is an edge $\{a_+, b_+\}$ and an edge $\{a_-, b_-\}$ in E''.

In particular, every path $\gamma = (a_1, a_2, \dots, a_n)$ in Γ has two lifts in Γ':

- $(a_{1+}, a_{2-}, \dots, a_{n+})$ and $(a_{1-}, a_{2+}, \dots, a_{n-})$, if γ has even length, i.e. n is odd,
- $(a_{1+}, a_{2-} \dots, a_{n-})$ and $(a_{1-}, a_{2+}, \dots, a_{n+})$, if γ has odd length, i.e. n is even.

Every n-cycle $(a_1, a_2, \dots, a_n, a_1)$ in Γ has:

- one lift $(a_{1+}, a_{2-}, \dots a_{n+}, a_{1-}, a_{2+}, \dots, a_{n-}, a_{1+})$ in Γ' of length $2n$, if n is odd,
- two lifts $(a_{1+}, a_{2-}, \dots a_{n-}, a_{1+})$ and $(a_{1-}, a_{2+}, \dots, a_{n+}, a_{1-})$ in Γ', each of length n, if n is even.

See Fig. 5 for $X_{1/4}$ of a 3-generators Artin group Art_{MNP}. As discussed above, if M_{ab} is even, then $X(a,b)_{1/4}$ has two connected components. The following lemma characterizes the graphs Γ for which $X_{1/4}$ is connected.

Lemma 4.7. *The graph $X_{1/4}$ has either one or two connected components. The following are equivalent:*

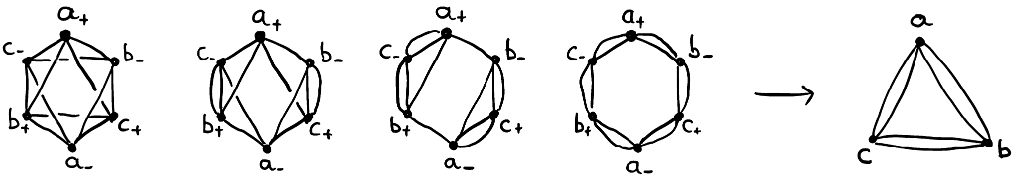

Fig. 5. The graph $X_{1/4}$ if (1) all M, N, P are odd, (2) only N is even, (3) only M is odd, (4) all M, N, P are even. In all cases, $X_{1/4} \to X_{1/2}$ is a double covering map.

- *$X_{1/4}$ has two connected components,*
- *each connected component of $X_{1/4}$ is a copy of $X_{1/2}$,*
- *Γ is a bipartite graph with all labels even.*

Proof. By Description 4.6, $X_{1/4}$ is a double cover of $X_{1/2}$. Since $X_{1/2}$ is connected, $X_{1/4}$ can have at most two connected components. The equivalence of the first two conditions follows directly from that fact. Let us prove that the third condition is equivalent to the first one.

Suppose Γ is a bipartite graph with all labels even. Since all labels are even, $X_{1/4}$ is isomorphic to the graph Γ' with all edges doubled, so it suffices to show that Γ' is not connected. Let $U \sqcup W$ be the two parts of $V(\Gamma)$, i.e. $U \sqcup W = V(\Gamma)$, and each edge of Γ joins a vertex of U with a vertex of W. Denote by $U_\pm, V_\pm$ the preimage in $V_\pm$ of U, W respectively. Then $U_+ \sqcup W_-$ and $U_- \sqcup W_+$ are the vertex sets of the two connected components of $X_{1/4}$. Indeed, this is true by the description of paths in Γ' in Description 4.6.

Now suppose that Γ is not a bipartite graph with all labels even. That means either Γ has an edge with an odd label, or there is an odd length cycle in Γ. We show that in both cases there is a path joining vertices a_+, a_- in $X_{1/4}$ for some $a \in V(\Gamma)$ (equivalently any, again by Description 4.6). If $\{a, b\} \in E(\Gamma)$ with M_{ab} odd, then there is a path with vertices a_+, b_-, a_- in $X_{1/4}$. If $(a_1, \ldots, a_{2n+1})$ is an odd length cycle in Γ, then by Description 4.6 its lift to Γ' contains a path joining a_{1+}, a_{1-} as a subpath. $\square$

4.6. The map $X_{1/4} \to X_{1/2}$

As in Description 4.6, the map $X_{1/4} \to X_{1/2}$ factors as the composition of the inclusion $X_{1/4} \hookrightarrow N_{1/2}$ with the deformation retraction $N_{1/2} \to X_{1/2}$. Under that map, $a_\pm \in V(X_{1/4})$ is mapped to $a \in V(X_{1/2})$ and every edge $\{a_\pm, b_\pm\} \in E(X_{1/4})$ is mapped to $\{a, b\} \in E(X_{1/2})$.

If Γ is a bipartite graph with all labels even, then by Lemma 4.7, $X_{1/4}$ is a disjoint union of two copies of $X_{1/2}$ and the map $X_{1/4} \to X_{1/2}$ is the identity map while restricted to each of the connected components. Otherwise, by Lemma 4.7, $X_{1/4}$ is a connected double cover of $X_{1/2}$. Then $C = \pi_1 X_{1/4} \to B = \pi_1 X_{1/2}$ is an inclusion of an index 2 subgroup. The quotient $B/C = \mathbb{Z}/2\mathbb{Z}$ can be identified with the automorphism group of the covering space $X_{1/4}$ over $X_{1/2}$. In the case of 3-generator Artin group this automorphism can be viewed as a π-rotation of the graph $X_{1/4}$ (with respect to the planar representation as in Fig. 5).

4.7. The map $X_{1/4} \to X_0$

In this section we analyze the map $X_{1/4} \to X_0$ which is obtained by composing the inclusion $X_{1/4} \hookrightarrow N_0$ with the deformation retraction $N_0 \to X_0$. This map is never combinatorial, and it might identify two distinct edges with the same origin. This map, unlike $X_{1/4} \to X_{1/2}$, depends on the partial orientation ι of Γ. In this section we give a description of the map $X_{1/4} \to X_0$, and in the next section, we characterize when this map is π_1-injective, in terms of the combinatorics of Γ and ι.

In order to understand the map $X_{1/4} \to X_0$ we express it as a composition $X_{1/4} \to \overline{X}_{1/4} \to X_0$ where the first map collapses some edges to a point and subdivides some other edges, and the second one is a combinatorial map. Proposition 4.10 gives conditions on Γ for $X_{1/4} \to \overline{X}_{1/4}$ to be a homotopy equivalence, and so to be π_1-injective. Proposition 4.11 gives conditions for $\overline{X}_{1/4} \to X_0$ to be a combinatorial immersion and consequently, π_1-injective.

The graph $\overline{X}_{1/4}$ is obtained from $X_{1/4}$ in two steps:

(1) An edge that is sent to a vertex in X_0 is collapsed to a vertex, which results in identification of its endpoints. The edges that get collapsed are certain edges of E'.
(2) An edge that is sent to a single edge of X_0 via a degree m map, is subdivided into a path of length m. The edges that get subdivided are certain edges of E''.

We know from Section 4.5 that $X(a,b)_{1/4}$ is a 4-cycle $(a_-, b_+, a_+, b_-, a_-)$ if M_{ab} is odd, and a disjoint union of 2-cycles (a_-, b_+, a_-) and (a_+, b_-, a_+) if M_{ab} is even. In both cases $X(a,b)_{1/4}$ is mapped to a single loop in $X_0(a,b)$. In all cases the map $X(a,b)_{1/4} \to X(a,b)_0 \simeq S^1$ is a degree M_{ab} map. If $M_{ab} = 2m$, then the map has degree m restricted to each of the connected components. Since an incoming edge a and an outgoing edge b are adjacent in $X(a,b)$, the edge $\{a_+, b_-\}$ of the graph $X(a,b)_{1/4}$ is collapsed to a point (see Fig. 4). Similarly, looking at the degrees of the map $X(a,b)_{1/4} \to X(a,b)_0$ restricted to other edges we find how to subdivide these edges to ensure that the map becomes combinatorial. Let $\overline{X}(a,b)_{1/4}$ be the image of $X(a,b)_{1/4}$ in $\overline{X}_{1/4}$.

Description 4.8. The graph $\overline{X}(a,b)_{1/4}$ is obtained from $X(a,b)_{1/4}$ by:

- ($M_{ab} = 2m+1$, $\iota(\{a,b\}) = a$): collapsing edge $\{a_+, b_-\}$ from $E(a,b)'$, and subdividing each of edges $\{a_+, b_+\}$ and $\{a_-, b_-\}$ from $E(a,b)''$ into a path of length m. Consequently, $\overline{X}(a,b)_{1/4}$ is a cycle of length $2m+1$.
- ($M_{ab} = 2$): collapsing edges $\{a_+, b_-\}$ and $\{a_-, b_+\}$ from $E(a,b)'$. Consequently, $\overline{X}(a,b)_{1/4}$ is a disjoint union of two length 1 loops.

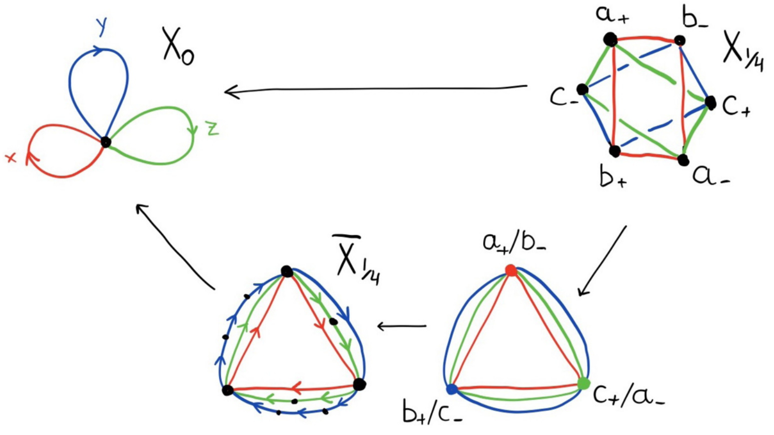

Fig. 6. The graph $X_0, X_{1/4}, \overline{X}_{1/4}$ in Art$_{375}$. The map $\overline{X}_{1/4} \to X_0$ is combinatorial.

- ($M_{ab} = 2m \geq 4$, $\iota(\{a,b\}) = a$): collapsing edge $\{a_+, b_-\}$ from $E(a,b)'$, subdividing the edge $\{a_+, b_-\}$ from $E''_{1/4}$ into a path of length m, and subdividing the edge $\{a_-, b_+\}$ from $E''_{1/4}$ into a path of length $m-1$. Consequently, $\overline{X}(a,b)_{1/4}$ is a disjoint union of cycles of length m each.

The above description of $\overline{X}(a,b)_{1/4}$ gives us the following description of $\overline{X}_{1/4}$. See Fig. 6 for an example of the factorization of the map $X_{1/4} \to X_0$ as a homotopy equivalence $X_{1/4} \to \overline{X}_{1/4}$ and a combinatorial map $\overline{X}_{1/4} \to X_0$. In that example $M_{ab} = 3, M_{bc} = 7, M_{ca} = 5$.

Description 4.9. The graph $\overline{X}_{1/4}$ can be described in terms of Γ and ι as follows.

- There are two kinds of vertices in $\overline{X}_{1/4}$. Let V_{old} denote the set of vertices that are the images of vertices in $X_{1/4}$, and V_{new} consists of all other vertices that are introduced in the subdivision. There are two kinds of edges $\overline{E}', \overline{E}''$.
- The vertices in V_{old} correspond to the equivalence classes of $V(X_{1/4}) = V_+ \sqcup V_-$ where the equivalence relation is generated by $a_+ \sim b_-$ for every $\{a,b\} \in E(\Gamma)$ and every $a = \iota(\{a,b\})$ or $M_{ab} = 2$.
- The edges in $\overline{E}'$ are identified with the set $E' - \{\{a_+, b_-\} \in E' \mid \{a,b\} \in E(\Gamma) \text{ with } a = \iota(\{a,b\}) \text{ or } M_{ab} = 2\}$, i.e. $\overline{E}'$ is the collection of all edges of E' that do not get collapsed.
- For each edge $\{a_\pm, b_\pm\}$ in E'' there is a path of length m or $m-1$ as in Description 4.8, consisting of edges of $\overline{E}''$ and joining appropriate vertices in V_{old}. The vertices inside such paths form the set V_{new}.

Each $\overline{X}(a,b)_{1/4}$ admits a natural combinatorial immersion onto a corresponding loop of X_0. The map $\overline{X}_{1/4} \to X_0$ is defined piecewise using these maps $\overline{X}(a,b)_{1/4} \to X_0$. In the next subsection, we characterize when $\overline{X}_{1/4} \to X_0$ is π_1-injective.

4.8. Conditions for π_1-injectivity of $X_{1/4} \to X_0$

We are now ready to characterize when the map $X_{1/4} \to X_0$ is π_1-injective. The next two propositions ensure that the maps $X_{1/4} \to \overline{X}_{1/4}$ and $\overline{X}_{1/4} \to X_0$ respectively, are π_1-injective. We refer to Definition 4.1 for the definition of an (almost) misdirected cycles. We emphasize that a misdirected even length cycle in the statement of Proposition 4.10 is not assumed to be simple.

Proposition 4.10. *Let Γ be a simple graph with edges labeled by numbers ≥ 2 with a partial orientation ι such that $\iota(e)$ of an edge e is defined if and only if the label of e is ≥ 3. Then $X_{1/4} \to \overline{X}_{1/4}$ is a homotopy equivalence if and only if Γ has no misdirected even length cycles and no cycles with all edges labeled by 2.*

Proof. We refer to Description 4.6 for the structure of $X_{1/4}$ and to Description 4.9 for the structure of $\overline{X}_{1/4}$. By construction, $X_{1/4} \to \overline{X}_{1/4}$ is obtained by collapsing certain edges of $X_{1/4}$ followed by edge subdivision. The edge subdivision never changes the homotopy type of a graph, but edge collapsing might. The map $X_{1/4} \to \overline{X}_{1/4}$ fails to be a homotopy equivalence if and only if there is a cycle in $X_{1/4}$ with all edges collapsed in $\overline{X}_{1/4}$

First let us assume that Γ has a misdirected even length cycle $(a_1, a_2, \dots, a_n, a_1)$. Then each of the edges in one of the cycles $(a_{1+}, a_{2-}, \dots, a_{n-}, a_{1+})$ or $(a_{1-}, a_{2+}, \dots, a_{n+}, a_{1-})$ of $\Gamma' \subseteq X_{1/4}$ gets collapsed. Thus, $X_{1/4} \to \overline{X}_{1/4}$ is not a homotopy equivalence. Now suppose that Γ has an odd length cycle $(a_1, a_2, \dots, a_n, a_1)$ with all edges labeled by 2. Then its lift to Γ' is the cycle $(a_{1+}, a_{2-}, \dots, a_{n+}, a_{1-}, \dots, a_{n-}, a_{1+})$ and all of its edges get collapsed in $\Gamma' \subseteq \overline{X}_{1/4}$.

Conversely, suppose that there exists a cycle γ' in $X_{1/4}$ all of whose edges are collapsed in $\overline{X}_{1/4}$. Only edges from the set E' might be collapsed by Description 4.9, so $\gamma' \subseteq \Gamma'$. Without loss of generality, we can assume that γ' is a simple cycle. Let γ be the image of γ' in Γ.

First suppose γ is not simple. Then there exists a vertex $a \in \Gamma$ such that γ' passes through both a_- and a_+, i.e. γ' can be expressed as $(a_-, b_{1+}, b_{2-}, \dots, b_{k-}, a_+, c_{1-}, c_{2+}, \dots, c_{l+}, a_-)$ for some k, l. Since the plus and minus signs in labels of γ alternate, the numbers k, l must be even. If each edge of γ' is collapsed, that means that the partial orientation on γ induced by ι extends to an orientation on $\gamma = (a, b_1, b_2, \dots, b_k, a, c_1, c_2, \dots, c_l, a)$ as pictured in Fig. 7. In particular, γ is a misdirected even length (non-simple) cycle.

Now suppose that γ is simple. Either $\gamma' \to \gamma$ is two-to-one or one-to-one, depending on the parity of the length of γ. If the length of γ is odd, then $\gamma' \to \gamma$ is two-to-one. For every edge $\{a, b\}$ in γ, both edges $\{a_-, b_+\}$ and $\{a_+, b_-\}$ are contained in γ', by Description 4.6. Moreover, if they both get collapsed, that means that $M_{ab} = 2$. Thus γ

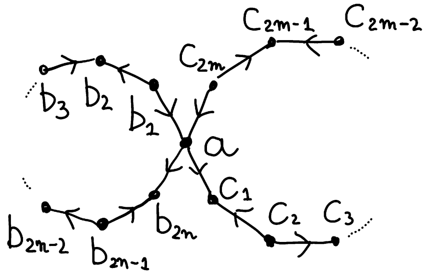

Fig. 7. The misdirected non-simple cycle γ.

is a cycle with all edges labeled by 2. If the length of γ is even, then $\gamma' \to \gamma$ is one-to-one. This necessarily means that γ is a misdirected even length cycle. $\square$

Proposition 4.11. *Let Γ be a simple graph with all labels ≥ 2 with a partial orientation ι such that $\iota(e)$ of an edge e is defined if and only if the label of e is ≥ 3. Suppose Γ has no misdirected even length cycles. Then $\overline{X}_{1/4} \to X_0$ is a combinatorial immersion if and only if Γ has no almost misdirected cycles.*

Proof. By construction $\overline{X}_{1/4} \to X_0$ is always a combinatorial map. It fails to be an immersion precisely when there are more than one oriented edges at some vertex of $\overline{X}_{1/4}$ mapping to the same oriented edge of X_0.

Recall that the edges of X_0 are in one-to-one correspondence with edges of $E(\Gamma)$, and for each edge x of X_0 the edges of $X_{1/4}$ mapping to x are exactly those coming from a single $X(a,b)_{1/4}$ for some $\{a,b\} \in E(\Gamma)$. Recall that every graph $X(a,b)_{1/4}$ is a single cycle if M_{ab} is odd, and a union of two cycles if M_{ab} is even. We claim that the map $\overline{X}_{1/4} \to X_0$ is not an immersion if and only if there exists a path in Γ' joining two vertices v_1, v_2 of $X(a,b)_{1/4}$ such that

- v_1, v_2 are not identified within $\overline{X}(a,b)_{1/4}$ as described in Description 4.8,
- the path gets entirely collapsed in $\overline{X}_{1/4}$.

Indeed, if such path exists then v_1, v_2 project to distinct vertices $\bar{v}_1, \bar{v}_2 \in \overline{X}(a,b)_{1/4}$. Each $\bar{v}_i$ is adjacent to the unique oriented edge e_i that maps onto the oriented edge x in X_0. By the second condition $\bar{v}_1, \bar{v}_2$ become identified in $\overline{X}_{1/4}$. However, the edges e_1, e_2 remain distinct in $\overline{X}_{1/4}$, and therefore $\overline{X}_{1/4} \to X_0$ is not an immersion. Conversely, if there are two oriented edges $\bar{e}_1, \bar{e}_2$ in $\overline{X}_{1/4}$ that maps onto the oriented edge x in X_0, then $\bar{e}_1, \bar{e}_2$ are images of distinct oriented edges e_1, e_2 in $\overline{X}(a,b)_{1/4}$. The initial vertices of e_1, e_2 must be distinct vertices in $\overline{X}(a,b)_{1/4}$ which become identified in $\overline{X}_{1/4}$. Thus their preimages in $X_{1/4}$ must be connected by a path as above.

We first prove that if Γ contains an almost misdirected cycle, then $\overline{X}_{1/4} \to X_0$ is not a combinatorial immersion. An odd length almost misdirected cycle $\gamma = (a_1, a_2, \dots, a_n, a_1)$ in Γ where the path $(a_1, a_2, \dots, a_n)$ is misdirected, yields a path γ' in Γ' joining either

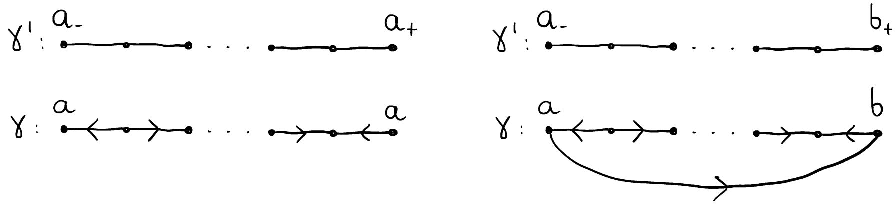

Fig. 8. The orientation on γ extending the partial orientation induced by ι.

a_{1-}, a_{1+} or a_{n-}, a_{n+} that gets entirely collapsed in $\overline{X}_{1/4}$. Neither the pair a_{1-}, a_{1+} or a_{n-}, a_{n+} is identified within any copy of $X(a,b)_{1/4}$. Thus the map $\overline{X}_{1/4} \to X_0$ is not an immersion.

Now suppose that $\gamma = (a_1, \dots, a_n, a_1)$ is an even length almost misdirected cycle where the path $(a_1, a_2, \dots, a_n)$ is misdirected. By the assumption, γ is not misdirected and we can assume (by possibly replacing γ with the cycle $(a_1, a_n, \dots, a_1)$) that there exists an orientation $\bar{\iota}$ extending the partial on γ induced by ι such that $\bar{\iota}(\{a_1, a_2\}) = a_2$, $\bar{\iota}(\{a_2, a_3\}) = a_2, \dots, \bar{\iota}(\{a_{n-1}, a_n\}) = a_n$, and $\bar{\iota}(\{a_1, a_n\}) = a_1$. Then the path $a_{1-}, a_{2+}, \dots a_{n_+}$ gets entirely collapsed. The vertices a_{1-}, a_{n_+} become identified but the edge $\{a_{1-}, a_{n_+}\}$ of Γ' was not collapsed in $\overline{X}(a_1, a_n)_{1/4}$. That means that the map $\overline{X}_{1/4} \to X_0$ not a combinatorial immersion.

Conversely, suppose that there exists a path γ' in $X_{1/4}$ that joins one of a_-, b_+ with one of a_+, b_- with all edges getting collapsed in $\overline{X}_{1/4}$ such that γ' is not a single edge of $X(a,b)_{1/4}$. First consider the case where $M_{ab} = 2$. Since only certain edges from the set E' get collapsed we can assume that $\gamma' \subseteq \Gamma'$. Without loss of generality by possibly extending γ' by extra edges $\{a_-, b_+\}$ or $\{a_+, b_-\}$, we can assume that γ' joins a_- and a_+. Then γ' projects to $\gamma \in \Gamma$, which is an odd length almost misdirected cycle. See Fig. 8 (left).

Now suppose $M_{ab} \neq 2$, and let $\iota(\{a,b\}) = a$, i.e. the edge $\{a_+, b_-\}$ of Γ' is collapsed but the edge $\{a_-, b_+\}$ remains uncollapsed in $\overline{X}_{1/4}$. If there is a path γ' in Γ' joining a_- and a_+, or b_- and b_+ with all edges getting collapsed, then the argument above again gives an odd length almost misdirected cycle in Γ. Otherwise, there must be a path γ' in $X_{1/4}$ joining a_- with b_+. Then γ' projects to $\gamma \in \Gamma$, such that $\gamma \cup \{a,b\}$ is an even length almost misdirected cycle (which is not misdirected). See Fig. 8 (right). □

4.9. Proof of the Splitting theorem

We are finally ready to prove Theorem 4.3.

Proof of Theorem 4.3. By Proposition 4.10 and Proposition 4.11, the map $X_{1/4} \to X_0$ factors as a composition of a homotopy equivalence and a combinatorial immersion, and thus is π_1-injective. By Lemma 4.7, $X_{1/4}$ is connected if and only if Γ is not a bipartite graph with all labels even. In such case, the conclusion follows from Lemma 4.4. If Γ is a bipartite graph with all labels even, then by Lemma 4.7 $X_{1/4}$ has two connected components, and each of them is a copy of $X_{1/2}$. By Lemma 4.5 Art_Γ splits as an HNN-extension $A*_B$ where $A = \pi_1 X_0$ and $B = \pi_1 X_{1/2}$. If Γ is not a bipartite graph with all labels even, then by Lemma 4.4 Art_Γ splits as $A *_C B$ where $C = \pi_1 X_{1/4}$. Since X_0 is a bouquet of $|E(\Gamma)|$ loops, $\mathrm{rk}\, A = |E(\Gamma)|$. The graph $X_{1/2}$ is a copy of Γ with doubled edges, so $\chi(X_{1/2}) = |V(\Gamma)| - 2|E(\Gamma)|$. Hence $\mathrm{rk}\, B = 1 - |V(\Gamma)| + 2|E(\Gamma)|$. In the case of amalgamated product, $\mathrm{rk}\, C = 2\,\mathrm{rk}\, B - 1 = 1 - 2|V(\Gamma)| + 4|E(\Gamma)|$, since the index of C in B is two. □

Remark 4.12 *(Twisted double of free groups as index two subgroup of G).* Let G be any amalgamated product $A *_C B$ of groups such that the index of C in B is two. Let g be a representative of the nontrivial coset of B/C and denote by $\beta : C \to C$ the automorphism given by $\beta(h) = g^{-1}hg$. Since $g^2 \in C$, β^2 is an inner automorphism of C. The group $G = A *_C B$ has an index two subgroup isomorphic to the twisted double $D(A, C, \beta)$, which is the kernel of the homomorphism $G \to B/C$.

In particular, every Art_Γ that splits as an amalgamated product as in Theorem 4.3 has an index two subgroup $D(A, C, \beta)$. Geometrically, β is a nontrivial deck transformation of the graph $X_{1/4}$ as a covering space of $X_{1/2}$. In the case of the three generator Art_Γ, β can be viewed as a rotation by π (with respect to the planar representation in Fig. 5). The choice of the element $g \in B - C$ corresponds to the choice of a path joining a basepoint in $X_{1/4}$ with the opposite vertex (e.g. a_+ with a_-).

4.10. Explicit splittings for 3-generator Artin groups

Let us now explicitly describe the splitting in Theorem 4.3 in the case of large type Artin group where Γ is a triangle.

Corollary 4.13. *Let Art_{MNP} be an Artin group where $M, N, P \geq 3$. Then $\mathrm{Art}_{MNP} = A *_C B$ where $A \simeq F_3$, $B \simeq F_4$ and $C \simeq F_7$, and $[B : C] = 2$. The map $C \to A$ is induced by the maps pictured in Fig. 9. Moreover, Art_{MNP} has an index two subgroup that is isomorphic to the twisted double $D(A, C, \beta)$ where $\beta : C \to C$ is given by $\beta(h) = g^{-1}hg$ for some (equivalently any) $g \in B - C$.*

Proof. Since Γ is a triangle, we have $|V(\Gamma)| = |E(\Gamma)| = 3$. By ordering Γ cyclically, we obtain a graph without misdirected cycles. By Theorem 4.3, Art_Γ splits as $A *_C B$ where $\mathrm{rk}\, A = 3$, $\mathrm{rk}\, B = 1 - 3 + 2 * 3 = 4$ and $\mathrm{rk}\, C = 2 * 4 - 1 = 7$. The maps $X_{1/4} \to X_0$ inducing $A \to C$ in Fig. 9 come directly from the descriptions in Section 4.7. The index two subgroup isomorphic to a twisted double comes from Remark 4.12. □

Example 4.14 *(Art_{333}).* By Corollary 4.13, Art_{333} splits as $F_3 *_{F_7} F_4$ and the map $\overline{X}_{1/4} \to X_0$ is a regular cover of degree 3. See the top of Fig. 9 with $m = n = p = 1$. Thus $C \simeq F_7$ is a normal subgroup in each of the factors and $[C : A] = 3$. This splitting of Art_{333} as $F_3 *_{F_7} F_4$ was first proved in [31]. We have the following short exact sequence

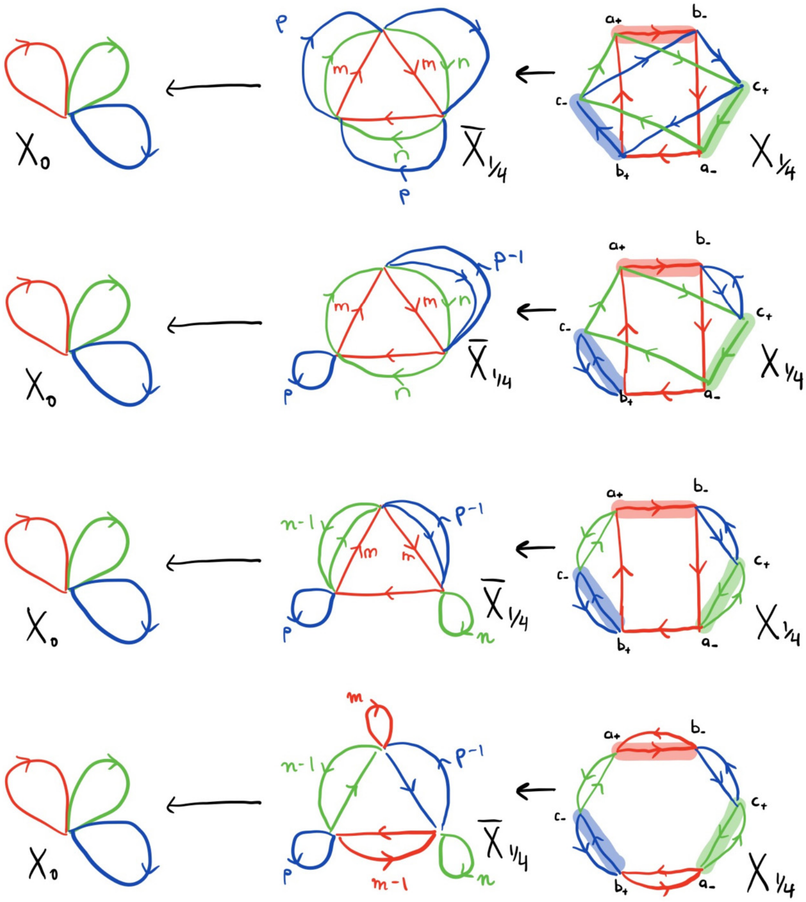

Fig. 9. The map $X_{1/4} \to X_0$ when (1) none, (2) one, (3) two or (4) all of $M_{ab} = M, M_{bc} = N, M_{ca} = P$ are even, respectively. Specifically, $M = 2m$ or $2m + 1$, $N = 2n$ or $2n + 1$, and $P = 2p$ or $2p + 1$. Here we use the convention where the edge labeled by a number k is a concatenation of k edges of the given color. The distinguished edges in $X_{1/4}$ are the ones that get collapsed to a vertex in $\overline{X}_{1/4}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$1 \to F_7 \to \mathrm{Art}_{333} \to \mathbb{Z}/3 * \mathbb{Z}/2 \to 1.$$

We conclude that Art_{333} is (fin. rank free)-by-(virtually fin. rank free), and therefore virtually (fin. rank free)-by-free. In particular, Art_{333} is virtually a split extension of a finite rank free group by a free group. Since every split extension of a finitely generated residually finite group by residually finite group is residually finite [29], Art_{333} is residually finite.

5. Residual finiteness of 3-generator Artin groups

In this section, we prove Theorem A. By Corollary 4.13, Art_{MNP} with $M, N, P \geq 3$ splits as a free product with amalgamation $A *_C B$ of finite rank free groups, and is virtually a twisted double $D(A, C, \beta)$. Throughout this section, A, B, C are the groups from the splitting in Corollary 4.13 and $M, N, P \geq 4$. We begin with computing how far the subgroup C is from being malnormal in A. Then we prove Theorem A (stated as Corollary 5.7 and Corollary 5.12) by applying Theorem 2.8. In Section 5.2 we consider the easier case where at least one of M, N, P is even and then in Section 5.3 we proceed with the case where M, N, P are all odd.

5.1. Failure of malnormality

A twisted double $D(A, C, \beta)$ where A, C are finite rank free groups and C is malnormal in A is hyperbolic by [2]. However, Art_Γ is never hyperbolic, unless Γ is a single point, in which case $\mathrm{Art}_\Gamma = \mathbb{Z}$. Thus the intersection $C^g \cap C$ must be nontrivial for some $g \in A$. Understanding how the edge group C intersects its conjugates plays a crucial role in our proof.

The intersection $C^g \cap C$ can be computed using the fiber product $\overline{X}_{1/4} \otimes_{X_0} \overline{X}_{1/4}$ (see Section 1.2). The map $\overline{X}_{1/4} \to X_0$ is described in Section 4.7 and pictured in Fig. 9.

Let F denote the fiber product $\overline{X}_{1/4} \otimes_{X_0} \overline{X}_{1/4}$. The vertex set $V(F)$ is the product $V(\overline{X}_{1/4}) \times V(\overline{X}_{1/4})$ and the edge set $E(F)$ is a subset of $E(\overline{X}_{1/4}) \times E(\overline{X}_{1/4})$. All the nontrivial connected components of F (i.e. the ones without vertices of the form (v, v) for some $v \in V(\overline{X}_{1/4})$) correspond to some $C \cap C^g$ where $g \notin C$ by [32].

Let Y be either $\overline{X}_{1/4}$ or $\overline{X}_{1/4} \otimes_{X_0} \overline{X}_{1/4}$. We continue to represent the map $Y \to X_0$ by coloring the edges of Y where each color represents one of the edges of X_0. We say a cycle or a path in Y is *monochrome*, if it is mapped onto a single loop in Y.

Note that any two simple monochrome cycles in $\overline{X}_{1/4}$ of the same color, have the same length. Hence all the simple monochrome cycles lift to their copies in F. Thus any connected component of F is a union of simple monochrome cycles whose lengths are the same as in $\overline{X}_{1/4}$. The branching vertices (i.e. of valence > 2) of connected components of F are contained in $V_{old} \times V_{old} \subseteq V(\overline{X}_{1/4}) \times V(\overline{X}_{1/4})$, since V_{old} are the only branching vertices of $\overline{X}_{1/4}$. In particular, all the segments (i.e. paths between branching vertices with all internal vertices of valence 2) in F are monochrome.

Lemma 5.1 *(All odd). Suppose $(M, N, P) = (2m+1, 2n+1, 2p+1)$ where $m, n, p \geq 2$. Then the intersection $C^g \cap C$ for $g \in A - C$ is either trivial, or its conjugacy class is represented by a subgraph of the graph in Fig. 10.*

Proof. This proof is a direct computation of the fiber product of F. Let $\{r_0, \ldots, r_{2m}\}$, $\{g_0, \ldots, g_{2n}\}$ and $\{b_0, \ldots, b_{2p}\}$ be the sets of cyclically ordered (consistently with the orientation of the cycle) vertices in $\overline{X}_{1/4}$ of red, green and blue cycle respectively such that $v_r := r_0 = g_0 = b_0$, $v_g := r_m = g_n = b_1$ and $v_b := r_{m+1} = g_{2n} = b_{p+1}$ are in V_{old}. The vertices v_r, v_g, v_b come from collapsing a red, green, blue edge of $X_{1/4}$ respectively. They are respectively the top, the bottom right and the bottom left vertices in $\overline{X}_{1/4}$ in Fig. 9. The connected component containing vertices $(v_r, v_g), (v_g, v_b), (v_b, v_r)$ is

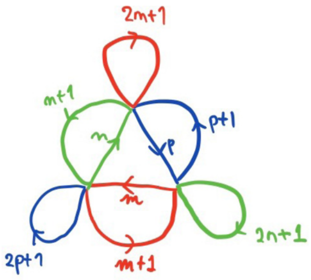

Fig. 10. A non-trivial component of F, when M, N, P are all odd. The vertex (v_r, v_g) is the bottom left.

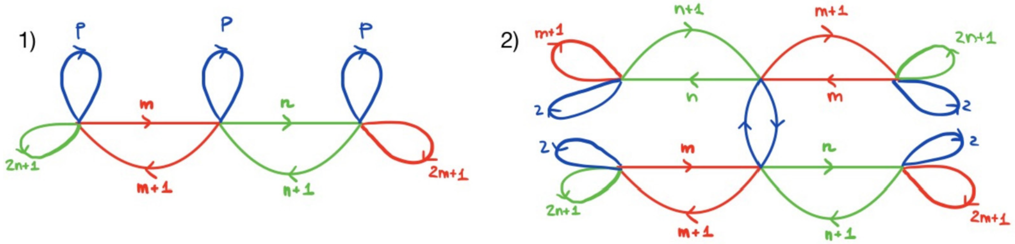

Fig. 11. A non-trivial component of F, when M, N are odd and P is even.

illustrated in Fig. 10. Another copy of that graph is the connected component containing $(v_r, v_b), (v_g, v_r), (v_b, v_g)$. All other nontrivial connected components do not have any branching vertices, and so are single monochrome cycles, or single vertices. □

Lemma 5.2 *(One even). If $(M, N, P) = (2m + 1, 2n + 1, 2p)$ where $m, n, p \geq 2$, then the intersection $C^g \cap C$ for $g \in A$ is either trivial, or its conjugacy class is represented by a subgraph of the graph in Fig. 11.*

Proof. We analyze the fiber product F as in proof of Lemma 5.1. Let $\{r_0, \ldots, r_{2m}\}$ and $\{g_0, \ldots, g_{2n}\}$ be the sets of cyclically ordered vertices of red and green cycle respectively, and $\{b_0, \ldots, b_{p-1}\}$ and $\{b_p \ldots, b_{2p-1}\}$ be the sets of cyclically ordered vertices of the two blue cycles such that $v_r := r_0 = g_0 = b_0$, $v_g := r_m = g_n = b_1$ and $v_b := r_{m+1} = g_{2n} = b_p$. As before the only branching vertices in F are pairs of branching vertices of $\overline{X}_{1/4}$.

If $p > 2$, then F has two connected components, one containing the vertices $(v_r, v_g), (v_g, v_b), (v_b, v_r)$ and one containing the vertices $(v_r, v_b), (v_g, v_r), (v_b, v_g)$. Each of them is a copy of the graph is illustrated in Fig. 11(1). In the first case, the vertex (v_r, v_g) is in the center. All the connected components without branching vertices are simple monochrome cycles, or single vertices.

If $p = 2$, then the vertices (v_r, v_g) and (v_g, v_r) are adjacent. In that case F is connected and is illustrated in Fig. 11(2). □

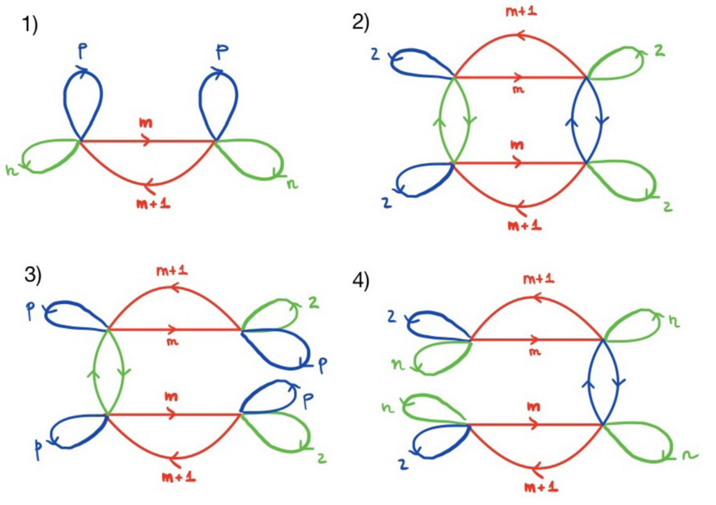

Fig. 12. A non-trivial component of F, when M is odd and N, P are even.

Lemma 5.3 *(Two even). If $(M, N, P) = (2m + 1, 2n, 2p)$ where $m, n, p \geq 2$, then the intersection $C^g \cap C$ for $g \in A$ is either trivial, or its conjugacy class is represented by a subgraph of the graph in Fig. 12.*

Proof. As before, let $\{r_0, \ldots, r_{2m}\}$, $\{g_0, \ldots, g_{n-1}\}$, $\{g_n, \ldots, g_{2n-1}\}$, $\{b_0, \ldots, b_{p-1}\}$ and $\{b_p \ldots, b_{2p-1}\}$ be the sets of cyclically ordered vertices of monochrome cycles such that $v_r := r_0 = g_0 = b_0$, $v_g := r_m = g_n = b_1$ and $v_b := r_{m+1} = g_{n-1} = b_p$. If $n, p > 2$, then there is a connected component of F containing branching vertices (v_b, v_r) and (v_r, v_g), and distinct connected component containing the vertices (v_r, v_b) and (v_g, v_r). Each is a copy of the graph illustrated in Fig. 12(1).

If $n = p = 2$, then there is one connected component of F containing all fours branching vertices. It is illustrated in Fig. 12(2). The cases where exactly one of n, p is equal 2 are illustrated in Fig. 12(3) and 12(4).

All other components are simple monochrome cycles, or single vertices. □

Lemma 5.4 *(All even). If $(M, N, P) = (2m, 2n, 2p)$ where $m, n, p \geq 3$, then the intersection $C^g \cap C$ for $g \in A$ is either trivial, or its conjugacy class is represented by a subgraph of the graph in Fig. 13.*

Proof. As before, let $\{r_0, \ldots, r_{m-1}\}$, $\{r_m, \ldots, r_{2m-1}\}$, $\{g_0, \ldots, g_{n-1}\}$, $\{g_n, \ldots, g_{2n-1}\}$, $\{b_0, \ldots, b_{p-1}\}$ and $\{b_p \ldots, b_{2p-1}\}$ be the sets of cyclically ordered vertices of monochrome cycles such that $v_r := r_0 = g_0 = b_0$, $v_g := r_m = g_n = b_1$ and $v_b := r_{m+1} = g_{n-1} = b_p$. If $m, n, p > 2$, then each connected component of F contains at most one branching vertex. Any such connected component is a copy of the graph illustrated in Fig. 13(1). Otherwise, a connected component of F contains at most two branching vertices. If $m = 2$, then the connected component containing (v_g, v_b) also contains (v_b, v_g) but no

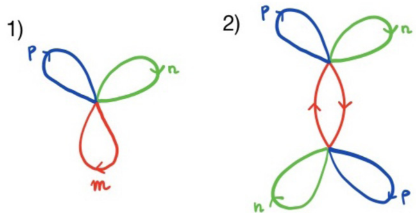

Fig. 13. A non-trivial component of F, when M, N, P are all even.

other branching vertices. Such connected component is illustrated in Fig. 132. There are analogous graphs for $n = 2$, v_r, v_b and for $p = 2$, v_r, v_g. All other components are simple monochrome cycles, or single vertices. □

Remark 5.5. If at least one of M, N, P is even and $(M, N, P) \neq (2m + 1, 4, 4)$ (for any permutation), then all the simple cycles in the fiber product of F are monochrome. It follows immediately from Lemmas 5.2, 5.3, 5.4.

5.2. At least one even exponent

We now will apply Theorem 2.8 to the twisted double $D(A, C, \beta)$ that is an index two subgroup of Art_{MNP} in Corollary 4.13. In this section we consider the case where at least one of M, N, P is even. Let $\mathcal{A}_\rho$ be the oppressive set of C in A with respect to $\rho : \overline{X}_{1/4} \to X_0$.

Proposition 5.6. *Suppose $M, N, P \geq 4$ and at least one of M, N, P is even. Suppose that $(M, N, P) \neq (2m + 1, 4, 4)$ (for any permutation). There exists a quotient $\phi : A \to \bar{A}$ such that*

(1) $\bar{A}$ is virtually free,
(2) $\bar{C} = \phi(C)$ is free and is malnormal in $\bar{A}$,
(3) ϕ separates C from $\mathcal{A}_\rho$,
(4) $\beta : C \to C$ projects to an automorphism $\bar{\beta} : \bar{C} \to \bar{C}$.

Proof. For each number k define

$$\bar{k} = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even,} \\ k & \text{if } k \text{ is odd.} \end{cases}$$

Let

$$\bar{A} = \langle x, y, z \mid x^{\bar{M}}, y^{\bar{N}}, z^{\bar{P}} \rangle = \mathbb{Z}/\bar{M}\mathbb{Z} * \mathbb{Z}/\bar{N}\mathbb{Z} * \mathbb{Z}/\bar{P}\mathbb{Z},$$

and let $\phi : A \to \bar{A}$ be the natural quotient. As a free product of finite groups $\bar{A}$ is virtually free. Geometrically, we obtain $\bar{A}$ as the fundamental group of a 2-complex $X_\bullet$

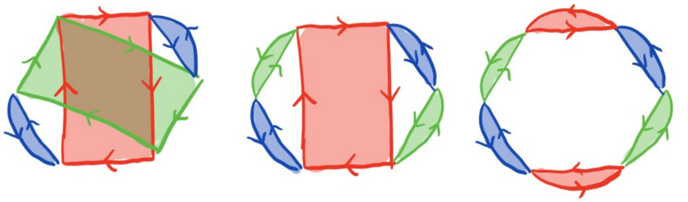

Fig. 14. The 2-cells in the presentation complex of $\bar{A}$ can be pulled back to $X_{1/4}$. These are three cases where at least one of M, N, P is even. These new 2-complexes admit a rotation by π which represents the automorphism β.

obtained from the bouquet of circles X_0 by attaching 2-cells along $x^{\bar{M}}, y^{\bar{N}}$ and $z^{\bar{P}}$. Let $Y_\bullet$ be a 2-complex obtained from $\overline{X}_{1/4}$ by attaching a 2-cell along each of the simple monochrome cycles with labels $x^{\bar{M}}, y^{\bar{N}}$ and $z^{\bar{P}}$. The complex $Y_\bullet$ has the homotopy type of a graph (see Fig. 9), so $\pi_1 Y_\bullet$ is a free group. There is an induced map $\rho_\bullet : Y_\bullet \to X_\bullet$ which lifts to an embedding $\widetilde{Y}_\bullet \to \widetilde{X}_\bullet$ of the universal covers. We have $\pi_1 Y_\bullet = \bar{C}$. By Lemma 2.5, ϕ separates C from $\mathcal{A}_\rho$.

In Lemma 5.2, Lemma 5.3 and Lemma 5.4, we computed the graphs representing the intersections $C \cap C^g$ for $g \in A$. The intersections of $\bar{C} \cap \bar{C}^{\bar{g}}$ for $\bar{g} \in \bar{A}$ can be represented by the graphs obtained in those lemmas with 2-cells added along simple monochrome cycles with labels $x^{\bar{M}}, y^{\bar{N}}$ and $z^{\bar{P}}$. The graphs become contractible after attaching 2-cells to the simple monochrome cycles (see Remark 5.5). It follows that $\bar{C}$ is malnormal in $\bar{A}$.

The 2-cells of $Y_\bullet$ can be pulled back along the homotopy equivalence $X_{1/4} \to \overline{X}_{1/4}$. See Fig. 14. The pulled back 2-cells in Fig. 14 have boundary cycles that are denoted by the same colors as the corresponding boundary cycles of the corresponding 2-cells in $\overline{X}_{1/4}$. By Observation 2.9, β projects to an automorphism $\bar{\beta} : \bar{C} \to \bar{C}$. □

Since free groups are locally quasiconvex, $\bar{C}$ is quasiconvex in $\bar{A}$. By combining Proposition 5.6 with Theorem 2.8 we have the following.

Corollary 5.7. *If at least one M, N, P is even and $(M, N, P) \neq (2m+1, 4, 4)$ (for any permutation), then* Art_{MNP} *splits as an algebraically clean graph of finite rank free groups. In particular,* Art_{MNP} *is residually finite.*

5.3. All exponents odd

We will now apply Theorem 2.8 in the case where M, N, P are all odd. Again, let $\mathcal{A}_\rho$ be the oppressive set of C in A with respect to $\rho : \overline{X}_{1/4} \to X_0$. The main goal of this section is the following.

Proposition 5.8. *Suppose $(M, N, P) = (2m+1, 2n+1, 2p+1)$ where $m, n, p \geq 2$. There exists a quotient $\phi : A \to \bar{A}$ such that*

(1) $\bar{A}$ is a hyperbolic von Dyck group,

(2) $\bar{C} := \phi(C)$ *is a free group of rank* 2 *and is malnormal in* $\bar{A}$*,*
(3) ϕ *separates* C *from* $\mathcal{A}_\rho$*,*
(4) $\beta : C \to C$ *projects to an automorphism* $\bar{\beta} : \bar{C} \to \bar{C}$*.*

Proof. Let $\phi : A \to \bar{A}$ be the natural quotient where $\bar{A}$ is given by the presentation

$$\bar{A} = \langle x, y, z \mid x^M, y^N, z^P, x^m y^n z^p \rangle. \tag{$*$}$$

The group $\bar{A}$ is the *von Dyck group* $D(M, N, P)$. Remind, $D(M, N, P)$ is the index two subgroup of the group of reflection of a triangle in $\mathbb{H}^2$ with angles $\frac{\pi}{M}, \frac{\pi}{N}, \frac{\pi}{P}$, and can be given by the presentation

$$D(M, N, P) = \langle a, b, c \mid a^M, b^N, c^P, abc \rangle. \tag{$**$}$$

In order to see that $\bar{A}$ is isomorphic to $D(M, N, P)$, note that x^m, y^n, z^p are generators of $\bar{A}$. Indeed, since $m(M-2) = m(2m-1) = M(m-1)+1$, we have

$$(x^m)^{M-2} = x^{M(m-1)+1} = x$$

and similarly $(y^n)^{N-2} = y$ and $(z^p)^{P-2} = z$. By setting $a = x^m$, $b = y^n$ and $c = z^p$, and rewriting the presentation in generators a, b, c, we get the presentation ($**$).

Let $X_\bullet$ be the presentation complex of ($*$). The 1-skeleton of $X_\bullet$ can be identified with X_0. Let $Y_\bullet$ be a 2-complex obtained from $\overline{X}_{1/4}$ by attaching the following 2-cells

- one simple monochrome cycle with label x^M, y^N, z^P respectively for each color,
- two copies of a 2-cell with the boundary word $x^m y^n z^p$.

By Lemma 5.11 (stated after this proof), $\pi_1 Y_\bullet = \bar{C}$. In Lemma 5.1, we computed the graph representing an intersection $C \cap C^g$ for $g \in A$. The intersection $\bar{C} \cap \bar{C}^{\bar{g}}$ for $\bar{g} \in \bar{A}$ can be represented by a 2-complex obtained from that graph by attaching the 2-cells as along all cycles with labels $x^M, y^N, z^P, x^m y^n z^p$. After attaching the 2-cells the complex becomes contractible. Thus $\bar{C}$ is malnormal in $\bar{A}$.

We now show that β projects to $\bar{C}$. As in proof of Proposition 5.6, all the 2-cells of $Y_\bullet$ can be pulled back along the homotopy equivalence $X_{1/4} \to \overline{X}_{1/4}$. See Fig. 15 for the five 2-cells that we attach to $X_{1/4}$ and that correspond to the five 2-cells of $Y_\bullet$. Three of the 2-cells pulled back to $X_{1/4}$ in the figure have boundary cycles that are denoted by the same colors as the corresponding boundary cycles of the corresponding 2-cells in $\overline{X}_{1/4}$. The remaining two have boundary cycles of length three and correspond to the two copies of a 2-cell with the boundary $x^m y^n z^p$ in $\overline{X}_{1/4}$. By Observation 2.9, β projects to an automorphism $\bar{\beta} : \bar{C} \to \bar{C}$.

Finally, it remains to prove that ϕ separates C from $\mathcal{A}$. Let $X'_\bullet$ be the presentation complex of ($**$), and let $Y'_\bullet$ be a 2-complex with the 1-skeleton as in Fig. 16, three monochrome 2-cells and two with boundary word abc. There is a natural immersion

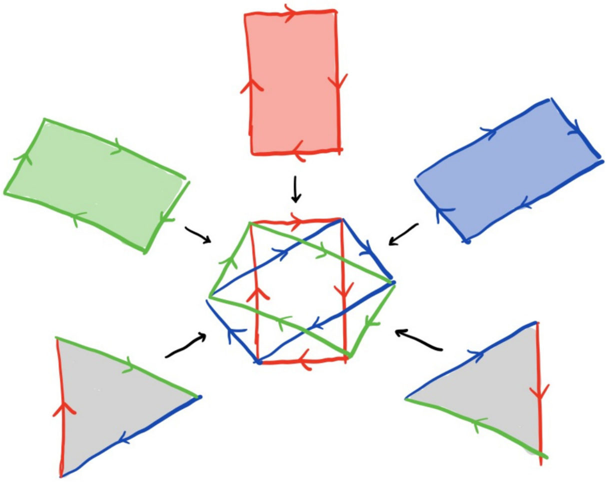

Fig. 15. The 2-cells in the presentation complex of $\bar{A}$ can be pulled back to $X_{1/4}$ in the case where M, N, P are all odd. The new 2-complex admits the π-rotation which represents the automorphism β. The rotation exchanges the two triangular 2-cells and leaves other 2-cells invariant.

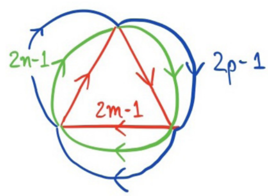

Fig. 16. The graph Y'. Red arrows correspond to generator a, green to b, and blue to c.

$\rho'_\bullet : Y'_\bullet \to X'_\bullet$ inducing the inclusion $\bar{C} \to \bar{A}$. Let Y' and X' be the 1-skeleta of $Y'_\bullet$ and $X'_\bullet$ respectively, and let $\rho' : Y' \to X'$ be the map $\rho'_\bullet$ restricted to the 1-skeleta. The map ρ' is an inclusion of $\pi_1 Y' \simeq F_7$ in $\pi_1 X' \simeq F_3$. In terms of the original generators of A, we have $\pi_1 X' = \langle x^m, y^n, z^p \rangle$, so this is a different inclusion $F_7 \to F_3$ than $C \to A$. However, the image $\phi'(\mathcal{A}'_\rho) \subseteq \bar{A}$ of the oppressive set $\mathcal{A}_{\rho'}$ with respect to ρ' is equal to $\phi(\mathcal{A}_\rho) \subseteq \bar{A}$. Indeed, all the pairs of paths μ_1, μ_2 in Y' that $\rho'(\mu_1) \cdot \rho'(\mu_2)$ is a closed path are in one-to-one correspondence with such pairs of paths in $\overline{X}_{1/4}$ (see Fig. 16 for Y' and Fig. 9 for $\overline{X}_{1/4}$). Thus to show that ϕ separates C from $\mathcal{A}_\rho$, it suffices to show $\phi'(\mathcal{A}_{\rho'})$ is disjoint from $\bar{C}$ in $\bar{A}$.

Let $\widetilde{X}'_\bullet$ denote the universal cover of $X'_\bullet$ with the 2-cells with the same boundary identified (i.e. M copies of the 2-cell whose boundary word is a^M are collapsed to a single 2-cell, and similarly with b^N, c^P). The complex $\widetilde{X}'_\bullet$ admits a metric so that makes it isometric to $\mathbb{H}^2$. In particular, $\widetilde{X}'_\bullet$ is CAT(0). Consider the induced metric on $Y'_\bullet$. Since $\rho'_\bullet$ is an immersion, a lift $\widetilde{Y}'_\bullet \to \widetilde{X}'_\bullet$ is a local isometric embedding (i.e. every point in $\widetilde{Y}'_\bullet$ has a neighborhood such that the restriction of $\widetilde{Y}'_\bullet \to \widetilde{X}'_\bullet$ to that neighborhood is an isometry onto its image), and by [5, Proposition II.4.14], it is an embedding. By Lemma 2.5, ϕ'

separates $\pi_1 Y'$ from $\mathcal{A}_{\rho'}$. This means that $\bar{C}$ is disjoint from $\phi'(\mathcal{A}_{\rho'}) = \phi(\mathcal{A}_\rho)$, and so ϕ separates C from $\mathcal{A}_\rho$. $\square$

We will prove the last missing bit in Lemma 5.11. First, we recall a version of the ping-pong lemma and its application in the hyperbolic plane, which allows us to show that certain convex subsets of $\mathbb{H}^2$ are disjoint.

Lemma 5.9 *(Ping-pong Lemma). Let a group generated by u, v act on a set Ω and let U_+, U_-, V_+, V_- be disjoint subsets of Ω such that*

$$u(\Omega - U_-) = U_+,$$
$$v(\Omega - V_-) = V_+.$$

Then u, v freely generate a free group.

Lemma 5.10. *Let $ABCD$ be a convex quadrangle in $\mathbb{H}^2$ with all internal angles $\leq \frac{\pi}{2}$. Then the lines $\overline{AB}$ and $\overline{CD}$ do not intersect in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$.*

Proof. Two lines in $\mathbb{H}^2$ do not intersect in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ if and only if there exists a common perpendicular line, i.e. a line that intersects each of the two lines at angle $\frac{\pi}{2}$. Consider the shortest geodesic segment p between segments AB and CD. The segment p is contained inside the closed quadrangle $ABCD$, by the assumption on the angles of $ABCD$. Moreover, the angles between p and each of the segments AB, CD are equal $\frac{\pi}{2}$. This proves that the line containing p is perpendicular to the lines $\overline{AB}$ and $\overline{CD}$. $\square$

We are now ready to complete the proof of Proposition 5.8. The group $\bar{C}$ and the complexes $Y_\bullet, X_\bullet$ are as in the proof of Proposition 5.8.

Lemma 5.11. *Let $M, N, P \geq 5$. The group $\bar{C}$ in the proof of Proposition 5.8 is the fundamental group of the 2-complex $Y_\bullet$ and the map $Y_\bullet \to X_\bullet$ induces the inclusion of group $\bar{C} \to \bar{A}$. In particular, $\bar{C}$ is a free group of rank 2.*

Proof. It is clear that the 2-cells in $X_\bullet$ pull back to the five 2-cells of $Y_\bullet$, so $\bar{C}$ is necessarily the image of $\pi_1 Y_\bullet$ in $\bar{A}$. By pushing free edges into the 2-cells, we can show that the wedge based at the a_+/b_- (the top vertex in $\overline{X}_{1/4}$ in Fig. 9) of two loops with boundary words $x^m y^{-n}$ and $z^{-p}x^m$ is a retract of Y. In particular, $\pi_1 Y = F_2$. In order to show that $\pi_1 Y_\bullet = \bar{C}$, we will show that $\pi_1 Y_\bullet$ maps to a free group of rank two in $\bar{A} = \pi_1 X_\bullet$. We will show that the elements $u = x^m y^{-n}$ and $v = z^{-p}x^m$ generate F_2 in $\bar{A}$. In the generators a, b, c of $\bar{A}$ as in presentation (∗∗) given above, we have $u = ab^{-1}$ and $v = c^{-1}a$. The group $\bar{A}$ is an index two subgroup of a reflection group generated by the reflection in the sides of triangle with angles $\frac{\pi}{2m+1}, \frac{\pi}{2n+1}, \frac{\pi}{2p+1} \leq \frac{\pi}{5}$ in $\mathbb{H}^2$. Therefore $\bar{A}$ preserves the tilling of $\mathbb{H}^2$ with triangles with those angles. See Fig. 17. We use the

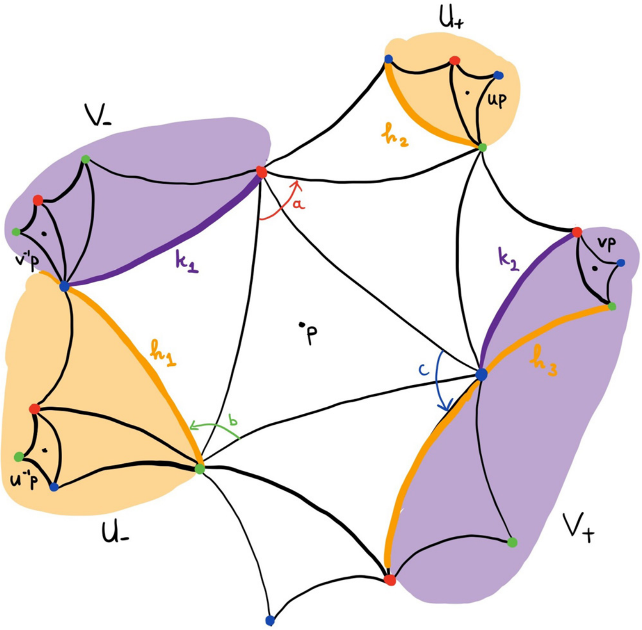

Fig. 17. A portion of the hyperbolic plane tilling with a triangle whose all three angles are $\frac{\pi}{5}$.

hyperplanes from this tiling to define subsets U_+, U_-, V_+, V_- and apply Lemma 5.9. Let P_a, P_b, P_c be three vertices of a triangle in the tiling such that the isometry a fixes P_a, b fixes P_b and c fixes P_c. Let k_1 be the line $a^{-1}(\overline{P_aP_b})$, and let h_1 be the line $b(\overline{P_bP_c})$. The lines k_1 and h_1 intersect, see Fig. 17. Let

$$h_2 := uh_1,$$
$$k_2 := vk_1,$$
$$h_3 := vh_1.$$

Clearly k_2 and h_3 intersect. We claim that no other pairs of lines among h_1, h_2, h_3, k_1, k_2 intersect. Since $M, N, P \geq 5$, all angles in all triangles are $\leq \frac{\pi}{5}$. For each pair of hyperplanes that we claim are disjoint, there exists a geodesic quadrangle with two opposite sides lying in those hyperplanes, and with all angles $\leq \frac{\pi}{2}$. By Lemma 5.10 such hyperplanes are disjoint.

Let U_+ be the closed outward halfplane of h_2, i.e. the halfplane that does not contain any of h_1, h_3, k_1, k_2. Let U_- be the open outward halfplane of h_1. We clearly have $u(\mathbb{H}^2 - U_-) = U_+$. Now, let V_+ be the union of the closed outward halfplanes of k_2 and h_3 (i.e. the halfplanes not containing h_1, k_1 or h_2), and let V_- be the intersection of the open outward halfplanes of k_1 and the open inward halfplane of h_1. We have $v(\mathbb{H}^2 - V_-) = V_+$. The subspaces U_+, U_-, V_+, V_- are pairwise disjoint. By Lemma 5.9, u and v freely generate a free group. □

By Proposition 5.8 and Theorem 2.8 we get the following.

Corollary 5.12. *If $M, N, P \geq 5$ are all odd, then the group* Art_{MNP} *splits as an algebraically clean graph of finite rank free groups. In particular,* Art_{MNP} *is residually finite.*

We finish this section with the analogous construction as in the proof of Proposition 5.8 but in the case of Artin group Art_{333}.

Example 5.13. If $(M, N, P) = (3, 3, 3)$ then group C has index 3 in A. Let $\bar{A}$ be the Euclidean von Dyck group $D(3, 3, 3)$ obtained in the same way as in proof of Proposition 5.8. Then the subgroup $\bar{C} = \mathbb{Z}^2$. Indeed, the complex $Y_\bullet$ has one additional 2-cell whose boundary reads the third copy of the word xyz. This complex is homeomorphic to a closed surface with $\chi(Y_\bullet) = 3 - 9 + 6 = 0$, so $Y_\bullet$ is homeomorphic to a torus. Note that the 2-cells of $Y_\bullet$ still can be pulled back to $X_{1/4}$. The third triangle pulls back to a hexagon, which is invariant under the graph automorphism b. Thus it is still true that β projects to $\bar{C}$.

6. Residual finiteness of more general Artin groups

The proof of residual finiteness of a three generator Artin group where at least one exponent is even, generalizes to other Artin groups. Throughout this section Γ is a graph admitting an admissible partial orientation, so by Theorem 4.3 Art_Γ splits as a free product with amalgamation or an HNN extension of finite rank free groups.

Theorem 6.1. *If all the simple cycles in nontrivial connected components of F are monochrome, then* Art_Γ *is residually finite.*

Proof. This proof is analogous to the proof of Proposition 5.6. The quotient $\bar{A}$ of A is obtained by adding a relation $x^{\bar{M}}$ for each generator x of A corresponding to an edge in Γ with label M and where $\bar{M}$ is either $\frac{M}{2}$ or M, depending on parity of M. Then $\bar{A}$ is virtually free, and $\bar{C}$ is free. The assumption that simple cycles in nontrivial connected components of F are monochrome, ensures that $\bar{C}$ is malnormal. The universal cover $\widetilde{X}_\bullet$ of the Cayley 2-complex of $\bar{A}$ can be homotoped to a tree by replacing each monochrome 2-cycle corresponding to a $x^{\bar{M}}$ with an $\bar{M}$-star graph whose middle vertex corresponds to the 2-cell and other vertices correspond to the original vertices. We note that the presentation 2-complex $Y_\bullet$ of $\bar{C}$ can also be homotoped to a graph in that way. It follows that the map $\widetilde{Y}_\bullet \to \widetilde{X}_\bullet$ is a local isometric embedding, and consequently an embedding, by [5, Proposition II.4.14]. By Lemma 2.5 ϕ separates C from the oppressive set $\mathcal{A}$ of C in A. All the attached 2-cells of $\overline{X}_{1/4}$ can be pulled back to $X_{1/4}$ in a way that β projects to $\bar{\beta}$. Depending on whether $\overline{X}_{1/4}$ is connected or not, the conclusion follows from Theorem 2.8 or Theorem 2.11. □

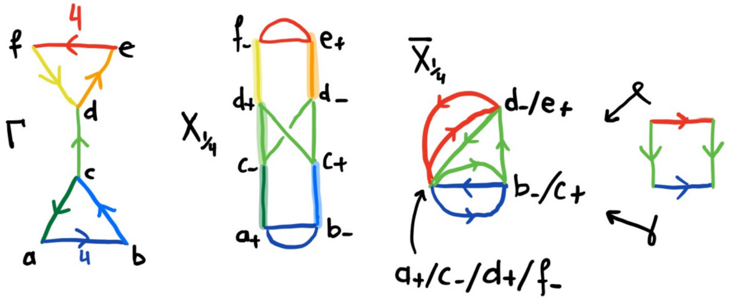

Fig. 18. In the above example $M_{ab} = M_{ef} = 4$ and M_{cd} is odd. For each edge e of Γ. The second graph is a part of $X_{\frac{1}{4}}$ and the third graph is the image of that part of $X_{1/4}$ in $\overline{X}_{1/4}$, which admits two different combinatorial immersions of the cycle on the right.

Corollary 6.2. *Let* Γ *be a graph admitting an admissible partial orientation. If all labels are even and* ≥ 6*, then* Art_Γ *is residually finite.*

Proof. For every color with corresponding label $2m$, there are three segments of that color in $\overline{X}_{1/4}$, which have lengths $1, m-1, m$ respectively. The segments of the length 1 and $m-1$ form one cycle and the other segment forms its own cycle. Since the branching vertices in the fiber product F are pairs of branching vertices, a lift of every monochrome cycle has exactly one branching vertex. It follows that all simple cycles in nontrivial connected components of F are monochrome. By Theorem 6.1, we are done. □

There are many more examples of graphs satisfying the assumption of Theorem 6.1. However, in the following example, Theorem 6.1 cannot be applied to any admissible partial orientation of Γ.

Example 6.3. Let Γ be the graph on the left in Fig. 18. Note that every admissible partial orientation of Γ is the same up to a permutation of the vertex labels. The second picture in Fig. 18 is a part of the graph $X_{1/4}$. Edges that are thickened get collapsed in $\overline{X}_{1/4}$, see the next graph. Finally, on the right we have a cycle that admits two distinct combinatorial immersion to $\overline{X}_{1/4}$. This yields a non monochrome simple cycle in F.

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