Cubulating Small Cancellation Free Products

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ABSTRACT. We give a simplified approach to the cubulation of small-cancellation quotients of free products of cubulated groups. We construct fundamental groups of compact nonpositively curved cube complexes that do not virtually split.

1. INTRODUCTION

Martin and Steenbock recently showed that a small-cancellation quotient of a free product of cubulated groups is cubulated [MS17]. In this paper we revisit their theorem in a slightly weaker form, and reprove it in a manner that capitalizes on the available technology. Combined with an idea of Pride's about small-cancellation groups that do not split, we answer a question posed to us by Indira Chatterji by constructing an example of a compact nonpositively curved cube complex X such that $\pi_1 X$ is nontrivial but does not virtually split.

Section 2 recalls the definitions and theorems that we will use from cubical small-cancellation theory. Section 3 recalls properties of the dual cube complex in the relatively hyperbolic setting. Section 4 recalls the definition of small-cancellation over free products, and describes associated cubical presentations. Section 5 re-proves Pride's result about small-cancellation groups that do not split. Section 6 relates small-cancellation over free products to cubical small-cancellation theory, and proves our main result which is Theorem 6.2. Finally, Section 7 combines Pride's method with Theorem 6.2 to provide cubulated groups that do not virtually split in Example 7.1.

2. BACKGROUND ON CUBICAL SMALL CANCELLATION

2.1. Nonpositively curved cube complexes. We assume the reader is familiar with *CAT(0) cube complexes*, which are CAT(0) spaces having cell structures, where each cell is isometric to a cube (see [BH99, Sag95, Lea22, Wis21]). A *nonpositively curved cube complex* is a cell-complex X whose universal cover $\widetilde{X}$ is a CAT(0) cube

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complex. A *hyperplane* $\widetilde{U}$ in $\widetilde{X}$ is a subspace whose intersection with each n-cube $[0,1]^n$ is either empty or consists of the subspace where exactly one coordinate is restricted to $\frac{1}{2}$. For a hyperplane $\widetilde{U}$ of $\widetilde{X}$, we let $N(\widetilde{U})$ denote its *carrier*, which is the union of all closed cubes intersecting $\widetilde{U}$. The hyperplanes $\widetilde{U}$ and $\widetilde{V}$ *osculate* if $N(\widetilde{U}) \cap N(\widetilde{V}) \neq \emptyset$ but $\widetilde{U} \cap \widetilde{V} = \emptyset$. We will use the *combinatorial metric* on a nonpositively curved cube complex X, so the distance between two points is the length of the shortest combinatorial path connecting them. The *systole* $\|X\|$ is the infimal length of an essential combinatorial closed path in X. A map $\phi : Y \to X$ between nonpositively curved cube complexes is a *local isometry* if ϕ is locally injective, ϕ maps open cubes homeomorphically to open cubes, and whenever a, b are concatenatable edges of Y, if $\phi(a)\phi(b)$ is a subpath of the attaching map of a 2-cube of X, then ab is a subpath of a 2-cube in Y.

2.2. *Cubical presentations and pieces.*

Definition 2.1. A *cubical presentation* $\langle X \mid Y_1, \dots, Y_m \rangle$ consists of a nonpositively curved cube complex X and a set of local isometries $Y_i \looparrowright X$ of nonpositively curved cube complexes. We use the notation X^* for the cubical presentation above. As a topological space, X^* consists of X with a cone on Y_i attached to X for each i.

We often consider the universal cover $\widetilde{X^*}$, whose *cubical part* is the preimage of X under the covering map. The cubical part serves as a "Cayley graph," whose "relators" are the cones.

Definition 2.2. A *cone-piece* of X^* in Y_i is a component of $\widetilde{Y}_i \cap \widetilde{Y}_j$, where $\widetilde{Y}_i$ is a lift of Y_i to the universal cover $\widetilde{X^*}$, excluding the case where $i = j$. A *wall-piece* of X^* in Y_i is a component of $\widetilde{Y}_i \cap N(\widetilde{U})$, where $\widetilde{U}$ is a hyperplane that is disjoint from $\widetilde{Y}_i$. For a constant $\alpha > 0$, we say X^* satisfies the $C'(\alpha)$ *small-cancellation* condition if $\operatorname{diam}(P) < \alpha\|Y_i\|$ for every cone-piece or wall-piece P involving Y_i.

When α is small, the quotient $\pi_1 X^*$ has good behavior. For instance, when X^* is $C'(\frac{1}{12})$ then each immersion $Y_i \looparrowright X$ lifts to an embedding $Y_i \hookrightarrow \widetilde{X^*}$. This is proven in [Wis21, Thm 4.1], and we also refer to [Jan17] for analogous results at $\alpha = \frac{1}{9}$.

2.3. *The $B(8)$ condition.*

We now describe a special case of the B(8) condition within the context of $C'(\alpha)$ metric small-cancellation. A *piece-path* in Y is a path in a piece of Y.

Definition 2.3. A cubical presentation X^* satisfies the $B(8)$ *condition* if there is a wallspace structure on each Y_i as follows.

(1) The collection of hyperplanes of each Y_i are partitioned into classes such that no two hyperplanes in the same class cross or osculate, and the union $U = \cup U_k$ of the hyperplanes in a class forms a *wall* in the sense that $Y_i - U$ is the disjoint union of a left and right halfspace.

(2) If P is a path that is the concatenation of at most 8 piece-paths, and P starts and ends on the carrier $N(U)$ of a wall then P is path-homotopic into $N(U)$.

(3) The wallspace structure is preserved by the group $\mathrm{Aut}(Y_i \to X)$ which consists of automorphisms $\phi : Y_i \to Y_i$ such that $\begin{array}{ccc} Y_i & \longrightarrow & Y_i \\ & \searrow \quad \swarrow & \\ & X & \end{array}$ commutes.

2.4. Properness criterion. A *closed-geodesic* $w \to Y$ in a nonpositively curved cube complex is a combinatorial immersion of a circle whose universal cover $\tilde{w}$ lifts to a combinatorial geodesic $\tilde{w} \to \widetilde{Y}$ in the universal cover of Y.

We quote the following criterion from [FW21, Theorem 3.5].

Theorem 2.4. *Let $X^* = \langle X \mid Y_1, \dots, Y_k \rangle$ be a cubical presentation. Suppose X is compact, and each Y_i is compact and deformation retracts to a closed combinatorial geodesic w_i. Additionally, suppose that for every hyperplane U of Y_i the complement $Y_i \setminus U$ is contractible, and U has an embedded carrier with* $\operatorname{diam} N(U) < \frac{1}{20}\|Y_i\|$. *If X^* is $C'(\frac{1}{20})$, then X^* is $B(8)$ and $\pi_1 X^*$ acts properly and cocompactly on the CAT(0) cube complex dual to the wallspace on $\widetilde{X^*}$.*

Moreover, if each $\pi_1 Y_i \subset \pi_1 X$ is a maximal cyclic subgroup, then $\pi_1 X^*$ acts freely and cocompactly on the associated dual CAT(0) cube complex.

The wallspace that is assigned to each Y_i in the above theorem has a wall for hyperplanes dual to pairs of antipodal edges in w_i. (The complex X is subdivided to ensure that each $|w_i|$ is even.)

2.5. The wallspace structure.

Definition 2.5 (The walls). When X^* satisfies the $B(8)$ condition, $\widetilde{X^*}$ has a *wallspace* structure which we now briefly describe. Two hyperplanes H_1, H_2 of $\widetilde{X^*}$ are *cone-equivalent* if $H_1 \cap Y_i$ and $H_2 \cap Y_i$ lie in the same wall of Y_i for some lift $Y_i \hookrightarrow \widetilde{X^*}$. Cone-equivalence generates an equivalence relation on the collection of hyperplanes of $\widetilde{X^*}$. A *wall* of $\widetilde{X^*}$ is the union of all hyperplanes in an equivalence class. When X^* is $B(8)$, the hyperplanes in an equivalence class are disjoint, and a wall w can be regarded as a wall in the sense that $\widetilde{X^*}$ is the union of two halfspaces meeting along w.

Lemma 2.6. *Let W be a wall of $\widetilde{X^*}$. Let $Y \subset \widetilde{X^*}$ be a lift of some cone Y_i of X^*. Then, either $W \cap Y = \emptyset$ or $W \cap Y$ consists of a single wall of Y.*

The *carrier* $N(W)$ of a wall W of $\widetilde{X^*}$ consists of the union of all carriers of hyperplanes of W together with all cones intersected by hyperplanes of W. The following appears as [Wis21, Corollary 5.30].

Lemma 2.7 (Walls quasi-isometrically embed). *Let X^* be $B(8)$. Suppose that pieces have uniformly bounded diameter. Then, for each wall W, the map $N(W) \to \widetilde{X^*}$ is a quasi-isometric embedding with uniform quasi-isometry constants.*

We will need the following result of Hruska which is proven in [Hru10, Thm 1.5].

Theorem 2.8. *Let G be a finitely generated group that is hyperbolic relative to $\{G_i\}$. Let $H \subset G$ be a finitely generated subgroup that is quasi-isometrically embedded. Then, $H \subset G$ is relatively quasiconvex.*

3. RELATIVE COCOMPACTNESS

The following is a simplified restatement of [HW14, Theorem 7.12] in the case where ♥ = ⋆. We focus it on our application where the wallspace arises from a cubical presentation. We use the notation $\mathcal{N}_d(S)$ for the closed d-neighborhood of S. [1]

Theorem 3.1. *Consider the wallspace $(\widetilde{X^*}, \mathcal{W})$. Suppose G acts properly and cocompactly on the cubical part of $\widetilde{X^*}$ preserving both its metric and wallspace structures, and the action on $\mathcal{W}$ has only finitely many G–orbits of walls. Suppose* Stabilizer(W) *is relatively quasiconvex and acts cocompactly on W for each wall $W \in \mathcal{W}$. Suppose G is hyperbolic relative to $\{G_1, \dots, G_r\}$. For each G_i, let $\widetilde{X}_i \subset \widetilde{X^*}$ be a nonempty G_i–invariant G_i–cocompact subspace. Let $C(\widetilde{X^*})$ be the cube complex dual to $(\widetilde{X^*}, \mathcal{W})$, and for each i let $C(\widetilde{X}_i)$ be the cube complex dual to $(\widetilde{X^*}, \mathcal{W}_i)$ where $\mathcal{W}_i$ consists of all walls W with the property that* diam $(W \cap \mathcal{N}_d(\widetilde{X}_i)) = \infty$ *for some $d = d(W)$.*

Then, there exists a compact subcomplex K such that $C(\widetilde{X^*}) = GK \cup \bigcup_i GC(\widetilde{X}_i)$. Hence, G acts cocompactly on $C(\widetilde{X^*})$ provided that each $C(\widetilde{X}_i)$ is G_i-cocompact.

In our application of Theorem 3.1, X is a "long" wedge of cube complexes $X_1, \dots, X_r$ (see Construction 4.3 for the definition) and $\widetilde{X}_i$ is a lift of the universal cover of X_i to $\widetilde{X^*}$. The wallspace structure of X^* is described in Section 2.5 (see also Lemma 4.4). We will be able to apply Theorem 3.1 because the cube complex $C_\star(\widetilde{X}_i)$ will be G_i-cocompact for the following reason.

Lemma 3.2. *Let G, $(X^*, \mathcal{W})$ be as in Theorem 3.1, and suppose X satisfies $C'(\frac{1}{16})$. Additionally, assume each $\widetilde{X}_i$ has the property that if s is a square with an edge in $\widetilde{X}_i$ then $s \subset \widetilde{X}_i$. Let W be a wall of $\widetilde{X^*}$. Suppose* diam$(W \cap \mathcal{N}_d(\widetilde{X}_i)) = \infty$ *for some i, d. Then, W contains a hyperplane of $\widetilde{X}_i$. Hence, $C_\star(\widetilde{X}_i) = \widetilde{X}_i$ for each i.*

Proof. Suppose diam$(N(W) \cap \mathcal{N}_d(\widetilde{X}_i)) = \infty$. By cocompactness of the action Stabilizer(W) on $N(W)$ and G_i on $\widetilde{X}_i$, there is an infinite-order element g stabilizing both W and $\widetilde{X}_i$.

Each $\widetilde{X}_i \subset \widetilde{X^*}$ is convex by [Wis21, Lemma 3.74], and we may therefore choose a geodesic $\tilde{\gamma}$ in $\widetilde{X}_i$ that is stabilized by g, and let $\tilde{\lambda}$ be a path in $N(W)$ that is stabilized by g. We thus obtain an annular diagram A between closed paths γ and λ which are the quotients of $\tilde{\gamma}$ and $\tilde{\lambda}$ by $\langle g \rangle$. Suppose, moreover, that

[1] There is a small misstatement in [HW14, Thm 7.12], as it requires that $r \geq r_0$ for some constant r_0.

A has minimal complexity among all such choices (A, γ, λ) where $\gamma \to X_i$ has the property that $\tilde{\gamma}$ is a geodesic, and $\lambda \to N(W)$ is a closed path. By [Wis21, Theorem 5.61], A is a square annular diagram, and we may assume it is has no spur. Note that [Wis21, Theorem 5.61] requires "tight innerpaths," which holds at $C'(\frac{1}{16})$ by [Wis21, Lemma 3.70].

Observe that if s is a square with an edge in $\widetilde{X}_i$, then $s \subset \widetilde{X}_i$. Consequently, the minimality of A ensures that A has no square, and so $\gamma = A = \lambda$.

There are now two cases to consider: either $\tilde{\lambda} \subset N(U)$ for some hyperplane U of W; or $\tilde{\lambda}$ has a subpath $u_1 y_j u_2$ traveling along $N(U_1), Y_j, N(U_2)$, where U_1, U_2 are distinct hyperplanes of W, and U_1, U_2 intersect the cone Y_j in antipodal hyperplanes.

In the latter possibility, the $B(8)$ condition is contradicted for Y_j, since $\widetilde{X}_i \cap Y_j$ contains the single piece-path y_j which starts and ends on carriers of distinct hyperplanes of the same wall of Y_j.

In the former possibility, $N(U) \cap \widetilde{X}_i \neq \emptyset$, and so the above square observation ensures that $N(U) \subset \widetilde{X}_i$. Hence, W intersects $\widetilde{X}_i$ as claimed. □

Example 3.3. Consider the quotient $G = \mathbb{Z}^2 * \mathbb{Z}^2 / \langle\langle w_1, w_2 \rangle\rangle$, with the following presentation for some number $m > 0$:

$$\langle a, b \mid aba^{-1}b^{-1} \rangle * \langle c, d \mid cdc^{-1}d^{-1} \rangle \;\Big|\; a^1c^1a^2c^2 \cdots a^mc^m,\; b^1d^1b^2d^2 \cdots b^md^m$$

Note that each piece consists of at most two syllables, whereas the syllable length (see Definition 4.1) of each relator is $2m$. Hence, the $C'_*(\frac{1}{m-1})$ small-cancellation condition over free products is satisfied. (See Definition 4.1.)

The associated space X is the long wedge (see Construction 4.3) of two tori X_1, X_2 corresponding to $\langle a, b \rangle$ and $\langle c, d \rangle$. For $i \in \{1, 2\}$, let Y_i be a square complex built out from an alternating sequence of rectangles and arcs as in Figure 4.1.

The cube complex dual to $\widetilde{X^*}$ has $\frac{m(m+1)}{2}$-dimensional cubes arising from the cone-cells Y_1 and Y_2. More interestingly, the cube complex dual to $(\widetilde{X^*}, \mathcal{W}_1)$ where $\mathcal{W}_1$ consists of the walls intersecting a copy of $\widetilde{X}_1$, has dimension $2m$. This is because all hyperplanes dual to the path a^m cross each other because of Y_1; likewise, all hyperplanes dual to the path b^m cross each other because of Y_2; and every hyperplane dual to the path a^m crosses every hyperplane dual to the path b^m because $\widetilde{X}_1$ is a 2-flat.

4. Small cancellation over free products

Definition 4.1 (The $C'_*(\frac{1}{n})$ small cancellation over a free product). Every element R in the free product $G_1 * \cdots * G_r$ has a unique *normal form* which is a word $h_1 \cdots h_n$ where each h_i lies in a factor of the free product and h_i and

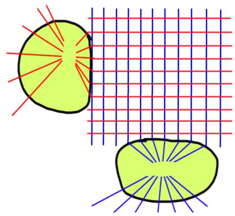

FIGURE 3.1. The walls associated to a 13-cube in the cubulation of a flat.

h_{i+1} lie in different factors for $i = 1, \dots, n-1$. The number n, which we denote by $|R|_*$, is the *syllable length* of R. We say R is *cyclically reduced* if h_1 and h_n also lie in different factors. We say that R is *weakly cyclically reduced* if $h_n^{-1} \neq h_1$ or if $|R|_* \leq 1$. We refer to each h_i as a *syllable*. There is a *cancellation* in the concatenation $P \cdot U$ of two normal forms if the last syllable of P is the inverse of the first syllable of U.

Consider a *presentation over a free product* $\langle G_1 * \cdots * G_r \mid R_1, \dots, R_s \rangle$ where each R_i is a cyclically reduced word in the free product. A word P is a *piece* in R_i, R_j if R_i, R_j have weakly cyclically reduced conjugates R_i', R_j' that can be written as concatenations $P \cdot U_i$ and $P \cdot U_j$, respectively, with no cancellations. The presentation is $C_*'(\frac{1}{n})$ *small cancellation* if $|P|_* < \frac{1}{n}|R_i'|_*$ whenever P is a piece.

If G is a $C_*'(\frac{1}{6})$ small-cancellation quotient of a free product $G_1 * \cdots * G_r$ [LS77, Cor. 9.4], then each factor G_i embeds in G. In particular, G is nontrivial if some G_i is nontrivial. We quote the following result from [Osi06].

Lemma 4.2. *Let G be a quotient of $G_1 * \cdots * G_r$ arising as a $C_*'(\frac{1}{6})$ small-cancellation presentation over a free product. Then, we have that G is hyperbolic relative to $\{G_1, \dots, G_r\}$.*

4.1. Cubical presentation associated with a presentation over a free product.

Construction 4.3. *Let T_r be the union of directed edges $e_1, \dots, e_r$ identified at their initial vertices. The* long wedge *of a collection of spaces $X_1, \dots, X_r$ is obtained from T_r by gluing the basepoint of each X_j to the terminal vertex of e_j. We will later subdivide the edges of T_r. Given groups $G_1, \dots G_r$ such that for each $1 \leq j \leq r$, let $G_j = \pi_1 X_j$ where X_j is a nonpositively curved cube complex, the long wedge X of the collection $X_1, \dots, X_r$ is a cube complex with $\pi_1 X = G_1 * \cdots * G_r$.*

Given an element $R \in G_1 * \cdots * G_r$ with $|R|_* > 1$, there exists a local isometry $Y \to X$ where Y is a compact nonpositively curved cube complex with $\pi_1 Y = \langle R \rangle$. Indeed, let $R = h_1 h_2 \cdots h_t$ where each h_k is an element of some $G_{m(k)}$. For each k let V_k be the compact cube complex that is the combinatorial convex hull of the basepoint p and its translate $h_k p$ in the universal cover $\widetilde{X}_{m(k)}$. We call p the initial

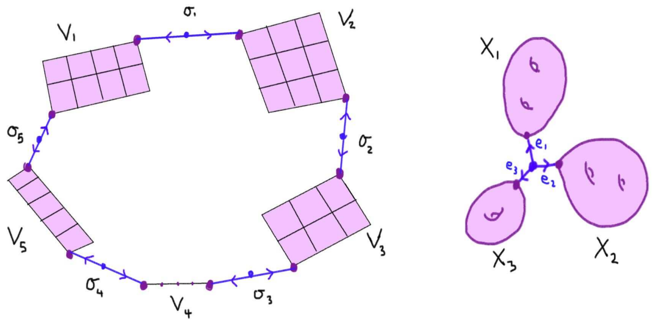

FIGURE 4.1. On the right is a long wedge of surfaces. On the left is a complex Y mapping to X by a local isometry. A relator of syllable length n is represented by such a local isometry having n rectangles.

vertex *of* V_k *and* $h_k p$ *the* terminal vertex *of* V_k. *For each* $1 \le k \le t$ *let* σ_k *be a copy of* $e^{-1}_{m(k)} e_{m(k+1)}$ *where* $m(t+1) = m(1)$. *Finally, we form* Y *from* $\bigsqcup_{k=1}^{t} V_k$ *and* $\bigsqcup_{k=1}^{t} \sigma_k$ *by gluing the terminal vertex of* V_k *to the initial vertex of* σ_k *and the terminal vertex of* σ_k *to the initial vertex of* V_{k+1}. *Note that there is an induced map* $Y \to X$ *which is a local isometry. (See Figure 4.1.)*

Given a presentation $\langle G_1, \dots, G_r \mid R_1, \dots, R_s \rangle$ *over a free product, there is an associated cubical presentation* $X^* = \langle X \mid Y_1, \dots, Y_s \rangle$ *where each* $Y_i \to X$ *is a local isometry associated with* R_i *as above. Finally, any subdivision of the edges* $e_1, \dots, e_r$ *induces a subdivision of* X*, and accordingly a subdivision of each* Y_i. *We thus obtain a new cubical presentation that we continue to denote by* X^*.

Lemma 4.4. *Suppose* $\langle X \mid Y_1, \dots, Y_s \rangle$ *is* $B(8)$ *(after subdividing). Let* $\widetilde{X}_k$ *be the universal cover of* X_k *with the wallspace structure such that each hyperplane is a wall. Then,* $\langle X \mid Y_1, \dots, Y_s, \widetilde{X}_1, \dots, \widetilde{X}_r \rangle$*, where the maps* $\widetilde{X}_j \to X$ *are the local isometries factoring as* $\widetilde{X}_j \to X_j \to X$*, is* $B(8)$. *Moreover, the two wallspace structures can be chosen so that the walls of* $\widetilde{X^*}$ *induced by the two structures are identical.*

Proof. We choose the same wallspace structure on each Y_i as before, and the natural wallspace structure given by the hyperplanes on each $\widetilde{X}_j$. The cone-pieces between $\widetilde{X}_j$ and Y_i are copies of the V_k associated with X_j that appear in Y_i, and hence Condition 2.3(2) holds for each Y_i as before. For each $\widetilde{X}_j$, Conditions 2.3(1) and 2.3(3) hold automatically by our choice of wallspace structure, and Condition 2.3(2) holds since $\widetilde{X}_j$ is contractible. □

Corollary 4.5. *For each wall* W *of* $\widetilde{X^*}$*, the intersection of* $W \cap \widetilde{X}_j$ *is either empty or consists of a single hyperplane.*

Proof. This follows by combining Lemma 4.4 and Lemma 2.6. □

5. CONSTRUCTION OF PRIDE

The following result was proven by Pride in [Pri83]. We give a slightly more geometric version of his proof, which was originally proven only for a $C(n)$ presentation instead of a $C'(\frac{1}{n})$ presentation, which we can obtain as in Remark 5.2.

Lemma 5.1. *Let* $G = \langle x, y \mid R_1, R_2, R_3, R_4, R_5, R_6 \rangle$ *where the relators* R_i *are specified below for associated positive integers* $\alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i$ *for each* $1 \le i \le k$*, and* $k \ge 1$*. Then, G does not split as an amalgamated product or HNN extension.*

$$
\begin{aligned}
R_1(x,y) &= xy^{\alpha_1}xy^{\alpha_2}\cdots xy^{\alpha_k}\\
R_2(x,y) &= yx^{\beta_1}yx^{\beta_2}\cdots yx^{\beta_k}\\
R_3(x,y) &= x^{\gamma_1}y^{-\delta_1}x^{\gamma_2}y^{-\delta_2}\cdots x^{\gamma_k}y^{-\delta_k}\\
R_4(x,y) &= xy^{\rho_1}xy^{-\rho_1}xy^{\rho_2}xy^{-\rho_2}\cdots xy^{\rho_k}xy^{-\rho_k}\\
R_5(x,y) &= yx^{\sigma_1}yx^{-\sigma_1}yx^{\sigma_2}yx^{-\sigma_2}\cdots yx^{\sigma_k}yx^{-\sigma_k}\\
R_6(x,y) &= (xy)^{\tau_1}(x^{-1}y^{-1})^{\theta_1}(xy)^{\tau_2}(x^{-1}y^{-1})^{\theta_2}\cdots(xy)^{\tau_k}(x^{-1}y^{-1})^{\theta_k}
\end{aligned}
$$

Proof. Suppose $G = A *_C B$ or $G = A*_C$, and let T be the associated Bass-Serre tree. Without loss of generality, assume the translation length of y is at least as large as the translation length of x. Choose a vertex $v \in \mathrm{Min}(x)$ for which $\mathsf{d}_T(y\cdot v, v)$ is minimal.

For use in the argument below, given a decomposition of $w \in G$ as a product $w = w_1w_2\cdots w_\ell$, the path

$$[v, w_1\cdot v][w_1\cdot v, w_1w_2\cdot v]\cdots[w_1w_2\cdots w_{\ell-1}\cdot v, w\cdot v]$$

is said to *read* w.

We now show that $v \in \mathrm{Min}(y)$. First suppose that x, and hence y, is a hyperbolic isometry. If $v \notin \mathrm{Min}(y)$, that is, $\mathrm{Min}(x) \cap \mathrm{Min}(y) = \varnothing$, then the axes of x and y in T are disjoint, and v is a vertex in the axis of x minimizing the distance between the two axes. In particular, the concatenation of two nontrivial geodesics $[x^{-1}\cdot v, v][v, y\cdot v]$ would be a geodesic. (See Figure 5.1.) Similarly, we have that $[x\cdot v, v][v, y\cdot v]$, $[x^{-1}\cdot v, v][v, y^{-1}\cdot v]$ and $[x\cdot v, v][v, y^{-1}\cdot v]$ would be geodesics. Consequently, regarding R_6 as a product of elements $\{x^{\pm 1}, y^{\pm 1}\}$, we see that the path reading R_6 would be a geodesic, which contradicts that $R_6 =_G 1$. Now, suppose that x is elliptic and so $x\cdot v = v$. Let e denote the initial edge of $[v, y\cdot v]$, and note that e is also the initial edge of $[v, y^{-1}\cdot v]$ since $v \notin \mathrm{Min}(y)$. The choice of v implies $x\cdot e \neq e$, as otherwise the other endpoint v' of e would satisfy $\mathsf{d}_T(y\cdot v', v') < \mathsf{d}_T(y\cdot v, v)$. Thus, the concatenation of the nontrivial geodesics $[y^{-1}\cdot v, v][v, xy\cdot v]$ is a geodesic, and similarly for $[y^{-1}\cdot v, v][v, x^{-1}y^{-1}v]$, $[y\cdot v, v][v, xy\cdot v]$ and $[y\cdot v, v][v, x^{-1}y^{-1}v]$. It follows that, regarding R_6 as a product of elements

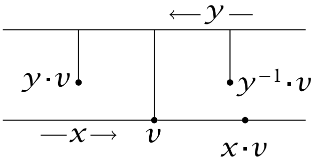

FIGURE 5.1. The case where $\mathrm{Min}(x) \cap \mathrm{Min}(y) = \emptyset$.

$\{xy, x^{-1}y^{-1}\}$, the path reading R_6 is a geodesic, which contradicts that $R_6 =_G 1$. Therefore, $v \in \mathrm{Min}(y)$.

Since $v \in \mathrm{Min}(x) \cap \mathrm{Min}(y)$, the element y is a hyperbolic isometry, because otherwise x, y are elliptic and so v is a global fixed point. Suppose x is also a hyperbolic isometry. At least one of $[y^{-1} \cdot v, v][v, x \cdot v]$ or $[x^{-1} \cdot v, v][v, y \cdot v]$ is not a geodesic, because otherwise the path reading R_1 regarded as a product of $\{x^{\pm 1}, y^{\pm 1}\}$ would be a geodesic. Consequently, both $[x \cdot v, v][v, y \cdot v]$ and $[x^{-1} \cdot v, v][v, y^{-1} \cdot v]$ are geodesics, and hence, regarding R_3 as a product of elements $\{x^{\pm 1}, y^{\pm 1}\}$, the path reading R_3 must be a geodesic, which is a contradiction. Thus, x is an elliptic isometry.

Let e_+ and e_- denote the initial edges of $[v, y \cdot v]$ and $[v, y^{-1} \cdot v]$, respectively. (See Figure 5.2.) Let us explain why $x \cdot e_+ = e_-$. Otherwise, we have that $[y^{-1} \cdot v, v][v, xy \cdot v]$ would be a geodesic since the last edge of $[y^{-1} \cdot v, v]$ is e_- and the first edge of $[v, xy \cdot v]$ is $x \cdot e_+$. Likewise, for $n, m > 0$ the path $[y^{-n} \cdot v, v][v, xy^m \cdot v]$ would be a geodesic, and so too would be its translate $[v, xy^n \cdot v][xy^n \cdot v, xy^n xy^m \cdot v]$. Then, if we regard R_1 as a product $(xy^{\alpha_1})(xy^{\alpha_2}) \cdots (xy^{\alpha_k})$, the path reading R_1 would be a geodesic, contradicting $R_1 =_G 1$.

Since $x \cdot e_+ = e_-$, neither e_- nor e_+ is fixed by x. For any $n, m > 0$ the last edge of $[y^n \cdot v, v]$ is e_+ and the first edge of $[v, xy^m \cdot v]$ is $x \cdot e_+ = e_- \neq e_+$, and so the path $[y^n \cdot v, v][v, xy^m \cdot v]$ is a geodesic, and so is also $[v, y^{-n} \cdot v][y^{-n} \cdot v, y^{-n} xy^m \cdot v]$. Similarly, the last edge of $[y^{-n} \cdot v, v]$ is e_- and the first edge of $[v, xy^{-m} \cdot v]$ is $x \cdot e_- \neq e_-$, and so therefore the path $[y^{-n} \cdot v, v][v, xy^{-m} \cdot v]$ as well as $[v, xy^n \cdot v][xy^n \cdot v, xy^n \cdot xy^{-m} \cdot v]$ are geodesics. Then, regarding R_4 as a product $(xy^{\rho_1})(xy^{-\rho_1}) \cdots (xy^{\rho_k})(xy^{-\rho_k})$, we see that the path reading R_4 is a geodesic, contradicting $R_4 =_G 1$. This completes the proof. ◻

Remark 5.2. In the context of Lemma 5.1, for each n there are choices of k and $\{\alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i \,:\, 1 \leq i \leq k\}$, such that the presentation is $C'(\frac{1}{n})$.

Given $n > 1$, let $k = 3n$ and choose $8k$ numbers $\alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i$ that are all different and lie between $50n$ and $75n$. Then, any piece P in R_i where $i \neq 6$ is of the form $x^l y x^m$ or $y^l x y^m$ for some l, m (possibly 0). Thus, $|P| \leq l + m + 1 \leq 150n + 1$. We also have $|R_i| \geq (k+1)50n = (3n+1)50n$, and so $|P| \leq \frac{1}{n}(150n + 1)n \leq \frac{1}{n}|R_i|$. If P is a piece in R_6, then P is of the from

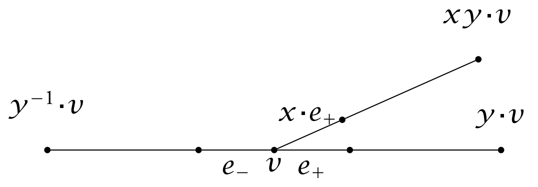

FIGURE 5.2. If $x\cdot e_+ \neq e_-$ then $[y^{-1}\cdot v, v][v, xy\cdot v]$ is a geodesic.

$(xy)^l(x^{-1}y^{-1})^m$, and so $|P| \le 2(l+m) \le 300n$. We also have

$$|R_6| = 2(\tau_1 + \theta_1 + \tau_2 + \cdots + \theta_k) \ge 2(2k)50n = 600n^2.$$

Hence, $|P| \le \frac{1}{n}|R_6|$.

Corollary 5.3. *Let $G_1, \dots, G_r$ be nontrivial groups generated by finite sets of infinite order elements, and suppose $r > 1$. For each $n > 0$, there is a finitely related $C'_*(\frac{1}{n})$ quotient G of $G_1 * \cdots * G_r$ that does not split.*

Proof. Let S_p be the given generating set of G_p for each p, and assume no proper subset of S_p generates G_p. The desired quotient G arises from a presentation $\langle G_1 * \cdots * G_r \mid \mathcal{R}\rangle$, where, if we follow Lemma 5.1, the set of relators is as follows:

$$\mathcal{R} = \Big\{ R_\ell(x,y) \ : \ 1 \le \ell \le 6,\ (x,y) \in S_p \times S_q,\ \text{where}\ \ 1 \le p < q \le r \Big\}$$

where $k(x,y) = 3n$ for each $(x,y) \in S_p \times S_q$, and where the constants $\alpha_i(x,y)$, $\beta_i(x,y)$, $\gamma_i(x,y)$, $\delta_i(x,y)$, $\rho_i(x,y)$, $\sigma_i(x,y)$, $\tau_i(x,y)$, $\theta_i(x,y)$ will be described below. For each $(x,y) \in S_p \times S_q$, let $\alpha_i(x,y)$, $\delta_i(x,y)$ and $\rho_i(x,y)$ be distinct integers > 1 such that $y^m \notin \langle z\rangle$ for $m \in \{\alpha_i(x,y), \delta_i(x,y), \rho_i(x,y)\}$ and $z \in S_q - \{y\}$. This is possible because y has infinite order and $y \notin \langle z\rangle$. Similarly, let $\beta_i(x,y)$, $\gamma_i(x,y)$ and $\sigma_i(x,y)$ be distinct integers > 1 such that $x^m \notin \langle z\rangle$ for $m \in \{\beta_i(x,y), \gamma_i(x,y), \sigma_i(x,y)\}$ and $z \in S_p - \{x\}$. Finally, let $\tau_i(x,y)$ and $\theta_i(x,y)$ be distinct integers between $10n$ and $20n$.

Having chosen the above constants for each $(x,y) \in S_p \times S_q$, we now show that the presentation for G is $C'_*(\frac{1}{n})$. We begin by observing that each $|R_\ell(x,y)|_* \ge 6n$. Let P be a piece in $R^1 = R_{\ell_1}(x_1,y_1)$ and $R^2 = R_{\ell_2}(x_2,y_2)$ where $x_1 \in S_{p_1}$, $y_1 \in S_{q_1}$, $x_2 \in S_{p_2}$, and $y_2 \in S_{q_2}$. If $\{p_1,q_1\} \neq \{p_2,q_2\}$ then $|P|_* \le 1$. Assume that $\{p_1,q_1\} = \{p_2,q_2\}$. First, suppose that $\ell_1 \neq 6$; then, $|P|_* \le 3$. Indeed, if $|P|_* \ge 4$ then two consecutive syllables would appear in distinct cyclically reduced forms of relators, which contradicts our choice of the constants. If $\ell_1 = 6$, then

$$|P|_* \le \max\{\tau_i(x,y)\} + \max\{\theta_i(x,y)\} \le 80n.$$

We also have

$$\begin{aligned}|R_6(x,y)|_* &= 2\left(\tau_1(x,y)+\theta_1(x,y)+\cdots+\tau_k(x,y)+\theta_k(x,y)\right) \geq 2(2k)10n\\ &= 120n^2,\end{aligned}$$

so $|P|_* \leq \frac{1}{n}|R_6(x,y)|_*$.

We now show that G does not split as an amalgamated product. For each $x \in S_p, y \in S_q$ with $p \leq q$ we let $H(x,y) = \langle x, y \mid R_\ell(x,y) : 1 \leq \ell \leq 6\rangle$. By Lemma 5.1, we see that $H(x,y)$ does not split. As there is a homomorphism $H(x,y) \to G$, we deduce that for any splitting of G as an amalgamated free product $G = A *_C B$, the elements x, y are either both in A or both in B. Otherwise, the action of $H(x,y)$ on the Bass-Serre tree of $G = A *_C B$ induces a non-trivial splitting. Considering all such pairs (x,y), we find that the generators of G are either all in A or all in B. Moreover, G cannot split as an HNN extension, since the relators $R_4(x,y)$ and $R_5(x,y)$ show that all generators have finite order in the abelianization of G. ◻

6. Main theorem

The small cancellation over a free product condition $C'_*(\frac{1}{n})$ was defined in Definition 4.1. We start with the following Lemma.

Lemma 6.1. *If $\langle G_1,\dots,G_r \mid R_1,\dots,R_s\rangle$ is $C'_*(\frac{1}{n})$, then for a sufficient subdivision of $e_1,\dots,e_r$ the cubical presentation X^* is $C'(\frac{1}{n})$.*

Proof. Let X' be a subdivision of X induced by a q-fold subdivision of each e_j. We accordingly let Y'_i be the induced subdivision of Y_i, so $Y'_i = \bigsqcup V_k \cup \bigsqcup \sigma_k$ as in Construction 4.3 and with each σ-edge subdivided. We thus obtain a new cubical presentation $\langle X' \mid Y'_1,\dots,Y'_s\rangle$. Since Y_i has $|R_i|_*$ σ-edges, the systole $\|Y'_i\| = \|Y_i\| + 2|R_i|_*(q-1)$. Note that $\|Y'_i\| > \sum_{i=1}^{|R_i|_*}|\sigma_i| = 2q|R_i|_*$, and so $\|Y'_i\| > 2(1+\varepsilon)q|R_i|_*$ for sufficiently small $\varepsilon > 0$. Let $M_i = \max_k\{\mathrm{diam}(V_k)\}$. For a wall-piece P we have $\mathrm{diam}(P) < M_i$. Consider a maximal cone-piece P in Y'_i, and suppose it intersects ℓ different V_k's and contains ℓ' different e_k edges. Note that $2\ell \geq \ell'$, since if P starts or ends with an entire σ_k arc, then it intersects an additional V_k (possibly trivially). We have $\mathrm{diam}(P) \leq \ell M_i + q\ell'$. When $\ell' > 0$, for any $\varepsilon > 0$ we can choose $q \gg 0$ so that $\mathrm{diam}(P) < (1+\varepsilon)q\ell'$. Since P corresponds to a length ℓ syllable piece, the $C'_*(\frac{1}{n})$ hypothesis implies that $\ell < \frac{1}{n}|R_i|_*$, and so $\mathrm{diam}(P) < (1+\varepsilon)q\ell' < 2(1+\varepsilon)q(\frac{1}{n}|R_i|_*) < \frac{1}{n}\|Y'_i\|$. When $\ell' = 0$, then assuming $q > nM_i$ we have $\mathrm{diam}(P) \leq M_i < 2\frac{q}{n}|R_i|_* < \frac{1}{n}\|Y'_i\|$. ◻

Theorem 6.2. *Suppose $G = \langle G_1,\dots,G_r \mid R_1,\dots,R_s\rangle$ satisfies $C'_*(\frac{1}{20})$. If each G_i is the fundamental group of a* [*compact*] *nonpositively curved cube complex, then G acts properly* [*and compactly*] *on a CAT(0) cube complex.*

Moreover, G acts freely if each $\langle R_i\rangle$ is a maximal cyclic subgroup.

Proof. Let X^* be the associated cubical presentation. Lemma 6.1 asserts that X^* is $C'(\frac{1}{20})$ after a sufficient subdivision. For each hyperplane U in Y_i we have $\text{diam}(N(U)) < \frac{1}{20}\|Y_i\|$ if the subdivision is sufficient. Theorem 2.4 asserts that $\pi_1 X^*$ acts freely (or with finite stabilizers if relators are proper powers) on a CAT(0) cube complex C dual to $\widetilde{X^*}$.

Let X'^* be the cubical presentation $\langle X \mid \{Y_i\}, \{\widetilde{X}_j\}\rangle$. By Lemma 4.4, X'^* satisfies $B(8)$ with our previously chosen wallspace structure on each Y_i and the hyperplane wallspace structure on each $\widetilde{X}_j$. Thus, by Lemma 2.6 each $\widetilde{X}_j$ in $\widetilde{X^*} = \widetilde{X'^*}$ intersects the walls of $\widetilde{X^*}$ in hyperplanes of $\widetilde{X}_j$.

Lemma 4.2 asserts that $\pi_1 X^*$ is hyperbolic relative to $\{G_1, \ldots, G_r\}$.

The pieces in $X^* = \langle X \mid \{Y_i\}\rangle$ are uniformly bounded since $\text{diam}(Y_i)$ is uniformly bounded. Thus, $N(W) \to \widetilde{X^*}$ is quasi-isometrically embedded by Lemma 2.7. Hence, Stabilizer$(N(W))$ is relatively quasiconvex with respect to $\{\pi_1 X_j\}$ by Theorem 2.8.

Theorem 3.1 asserts that $\pi_1 X^*$ acts relatively cocompactly on C. Lemma 3.2 asserts that each $C_\star(\widetilde{X}_i) = \widetilde{X}_i$. Hence, if each X_i is compact, we see that C is compact. ◻

7. A CUBULATED GROUP THAT DOES NOT VIRTUALLY SPLIT

Examples were given in [Wis21] of a compact nonpositively curved cube complex X such that X has no finite cover with an embedded hyperplane. It is conceivable that those groups have no (virtual) splitting, but this was not confirmed there.

Example 7.1. There exists a nontrivial group G with the following two properties:

(1) $G = \pi_1 X$ where X is a compact nonpositively curved cube complex.
(2) G does not have a finite index subgroup that splits as an amalgamated product or HNN extension.

Let G_1 be the fundamental group of X_1 which is a compact nonpositively curved cube complex with a nontrivial fundamental group but no nontrivial finite cover. For instance, such complexes were constructed in [Wis96] or [BM97]. By Corollary 5.3, there exists a $C'_*(\frac{1}{20})$ quotient G of the free product $G_1 * \cdots * G_1$ of r copies of G_1, such that G does not split. The group G has no finite index subgroups since $G_1 * \cdots * G_1$ has none. Since $G_1 = \pi_1 X_1$, by Theorem 6.2, G is the fundamental group of a compact nonpositively curved cube complex.

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