Stochastic Control/Stopping Problem with Expectation Constraints

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Abstract

We study a stochastic control/stopping problem with a series of inequality-type and equality-type expectation constraints in a general non-Markovian framework. We demonstrate that the stochastic control/stopping problem with expectation constraints (CSEC) is independent of a specific probability setting and is equivalent to the constrained stochastic control/stopping problem in weak formulation (an optimization over joint laws of Brownian motion, state dynamics, diffusion controls and stopping rules on an enlarged canonical space). Using a martingale-problem formulation of controlled SDEs in spirit of [45], we characterize the probability classes in weak formulation by countably many actions of canonical processes, and thus obtain the upper semi-analyticity of the CSEC value function. Then we employ a measurable selection argument to establish a dynamic programming principle (DPP) in weak formulation for the CSEC value function, in which the conditional expected costs act as additional states for constraint levels at the intermediate horizon.

This article extends [20] to the expectation-constraint case. We extend our previous work [4] to the more complicated setting where the diffusion is controlled. Compared to that paper the topological properties of diffusion-control spaces and the corresponding measurability are more technically involved which complicate the arguments especially for the measurable selection for the super-solution side of DPP in the weak formulation.

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1 Introduction

In this paper, we analyze a continuous-time stochastic control/stopping problem with a series of inequality-type and equality-type expectation constraints in a general non-Markovian framework.

Let a decision maker start from time $t \in [0,\infty)$ with a historical path of state $\mathbf{x}|_{[0,t]}$. She can choose an open-loop control $\mu = \{\mu_s\}_{s\in[t,\infty)}$ to make the state process evolve according to some controlled SDE on a probability space $(\mathcal{Q},\mathcal{F},\mathfrak{p})$ whose drift and diffusion coefficients depend on the past trajectories of the solution. Let $\mathcal{X}^{t,\mathbf{x},\mu} = \big\{\mathcal{X}^{t,\mathbf{x},\mu}_s\big\}_{s\in[t,\infty)}$ denote this controlled state process. A typical example of this non-Markovian setting is the stochastic control with delay: since it takes the system some time to collect/analyze the information, the drift/diffusion term of the dynamic may have a delay in the state variable or control variable. Under continuous drift/diffusion coefficients, the optimal control problem with delay admits a dynamic programming principle (DPP) and its value function satisfies the associated Hamilton-Jacobi-Bellman (HJB) equation, see [22, 21, 23, 40, 39, 17] among others.

In our problem, the decision maker can also select an exercise time τ to maximize the expectation of her accumulative reward $\int_t^\tau f\big(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r\big)\,dr$ plus her terminal reward $\pi\big(\tau,\mathcal{X}^{t,\mathbf{x},\mu}_{\tau\wedge\cdot}\big)$ while she is subject to a series of constraints: for $i\in\mathbb{N}$, the expectation of some accumulative cost $\int_t^\tau g_i\big(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r\big)dr$ should not exceed certain level y_i and the expectation of some other accumulative cost $\int_t^\tau h_i\big(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r\big)dr$ should exactly reach certain level z_i.

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Such a stochastic control/stopping problem with expectation constraints (SCEC for short) has many applications in economy, engineering, finance, management, etc.

Let $V(t,\mathbf{x},y,z)$ be the SCEC value with $(y,z):=\big(\{y_i\},\{z_i\}\big)$. We aim to establish a DPP of this value function without imposing any regularity condition on reward/cost functions in time, state and control variables. With such a DPP, our further research will try to characterize the SCEC value function V as a viscosity solution of the corresponding path-dependent HJB equations. Then we will derive an effective numerical scheme for the value function V and implement it for practical examples including certain SCEC problems with state/control delay in dynamics aforementioned.

A dynamic programming principle of a general stochastic control problem allows one to optimize the problem stage by stage in a backward recursive way. It plays an important role in the study of stochastic control theory as it is crucial for obtaining a viscosity-solution characterization of the value function of the control problem and thus a numerical calculation of the value function (see e.g. [7], [20] and [37] for a synopsis of DPP development). Although it is intuitive, a general DPP is technically subtle to verify. In particular, the wellposedness of the DPP first requires the value function to be measurable so that people can conduct optimization at an intermediate horizon.

To obtain the measurability of the SCEC value function, we first study the topological and measurable properties of the path space $\mathbb{J}$ of diffusion control μ: More precisely, we show that $\mathbb{J}$ is a Borel space under a weak topology and attain a representation of its Borel sigma-field $\mathscr{B}(\mathbb{J})$ (Lemma 1.2 and Lemma 1.3). Inspired by [18, 20], we then embed diffusion control μ and stopping rule τ together with the Brownian and state information into an enlarged canonical space $\overline{\Omega}$ and regard their joint distribution as a new type of controls. The optimization of the total expected reward over constrained diffusion controls/stopping times transforms into a maximal expectation of reward functional over a class $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ of probability measures on $\overline{\Omega}$ under which four canonical coordinates $(\overline{W},\overline{U},\overline{X},\overline{T})$ serve as Brownian motion, diffusion control, state process and stopping rules respectively. We demonstrate that such a transformation is equivalent (Theorem 3.1), namely, the value $V(t,\mathbf{x},y,z)$ of SCEC in strong formulation (i.e., on $\mathcal{Q}$) is equal to the value $\overline{V}(t,\mathbf{x},y,z)$ of SCEC in weak formulation (i.e., over $\overline{\Omega}$). Hence, the SCEC value is a robust value, independent of a specific probability model.

For the measurability of SCEC value functions, we next take advantage of the martingale-problem formulation from [45] to describe the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ as a series of probabilistic tests on stochastic behaviors of the canonical coordinates of $\overline{\Omega}$. With such a countable characterization, we employ a Polish space of diffusion control processes (Lemma 4.1) and a Polish space of stopping times constructed in [4] to deduce that the set-valued mapping $(t,\mathbf{x},y,z)\mapsto\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ has Borel-measurable graph and the SCEC value function $V=\overline{V}$ is thus upper semi-analytic in $(t,\mathbf{x},y,z)$, (Theorem 4.1).

Our main achievement is to derive a DPP for $\overline{V}$ in weak formulation, which takes conditional expectations of the remaining costs as additional states for constraint levels at the intermediate horizon (Theorem 5.1). For the subsolution side of this DPP, we use the regular conditional probability distribution to show that the probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$, $\forall\,(t,\mathbf{x},y,z)$ are stable under conditioning.

For the supersolution side of the DPP, we exploit a measurable selection theorem in the analytic-set theory to paste a class of locally $\varepsilon-$optimal probability measures. We make a delicate analysis to demonstrate that the second canonical coordinate $\overline{U}$ serves as a constrained diffusion control under the pasted probability measure, and we apply the martingale-problem formulation again to indicate that the canonical coordinates $(\overline{W},\overline{X})$ are still Brownian motion and the state process under the pasted probability measure. Similar to the arguments in [4], the fourth canonical coordinate $\overline{T}$ is a constrained stopping time under the pasted probability measure. To wit, the probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$'s are also stable under concatenation.

Relevant Literature.

Kennedy [26] employed a *Lagrange multiplier* method to reformulate a discrete-time optimal stopping problem with first-moment constraint as a minimax problem and showed that the optimal value of the dual problem is equal to that of the primal problem. The Lagrangian technique was later adopted in many economic/financial applications of optimal stopping problems with expectation constraints, see e.g. [35, 27, 24, 2, 48, 28, 33, 32, 47]. Pfeiffer et al. [34] recently took a Lagrange relaxation approach to obtain a duality result for general stochastic control problems with expectation constraints.

In their study of a continuous-time stochastic optimization problem of controlled Markov processes, El Karoui, Huu Nguyen and Jeanblanc-Picqué [18] viewed joint laws of state and control processes as *control rules* on the product space of canonical state space and control space. They utilized a measurable selection theorem in the analytic-set theory to establish a DPP without assuming any regularity on the reward functional. Nutz et al. [31, 30] came up with a similar idea to analyze a superhedging problem under volatility uncertainty. They modeled the "uncertainty" by path-dependent classes of controlled-diffusion laws and explored the analytic measurability of these classes. Using the measurable selection techniques, the authors obtained DPP result in a form of time-consistency of a sub-linear expectation and they thus established a duality formula for the robust superhedging of measurable claims. The approach of [31, 30] was later developed by e.g. [36, 37] to derive DPPs of various non-Markovian control problems. Yu et al. [16] took a similar measurable selection argument to analyze the DPP of a stochastic control problem with certain expectation constraint in which they dynamically relaxed the expectation constraint by a family of auxiliary supermartingales.

El Karoui and Tan [19, 20] used the measurable selection argument to attain the DPP for a general stochastic control/stopping problem by embedding diffusion controls and stopping times into an enlarged canonical space in the spirit of [18]. However, the probability class they considered in weak formulation is not suitable for stochastic control/stopping with expectation constraints, see our discussion in Subsection 3.3. Instead, we additionally require in (D4) of Definition 3.1 that under each $\overline{P}$ of $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ the time canonical coordinate $\overline{T}$ acts as some stopping time (it turns out that such a restriction does not affect the unconstrained stochastic control/stopping problem in weak formulation). By constructing a Polish space of diffusion control processes and utilizing a Polish space of stopping times from [4], we manage to derive the Borel measurability of graph $[\![\,\overline{\mathcal{P}}\,]\!]$ and thus obtain the measurability of the SCEC value functions. Because of condition (D4) and expectation constraints, it is more technically involved to verify the stability of our probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ under conditioning and concatenation and thus establish a DPP for the SCEC value function $\overline{V}$.

As to the optimal stopping problems with expectation constraints, Ankirchner et al. [1] and Miller [29] took different approaches by transforming the constrained optimal stopping problems for diffusion processes to stochastic optimization problems with martingale controls. The former characterizes the value function in terms of a Hamilton-Jacobi-Bellman equation and obtains a verification theorem, while the latter embeds the optimal stopping problem with first-moment constraint into a time-inconsistent (unconstrained) stopping problem. However, the authors only postulate dynamic programming principles for their corresponding problems. In contrast, our previous work [4] exploited a measurable selection method to rigorously establish a dynamic programming principle for the optimal stopping problem with expectation constraints.

An interesting related topic to our research is optimal stopping with constraint on the distribution of stopping times. Bayraktar and Miller [3] studied the problem of optimally stopping a Brownian motion with the restriction that the distribution of the stopping time must equal a given measure with finitely many atoms, and obtained a dynamic programming result which relates each of the sequential optimal control problems. Källblad [25] used measure-valued martingales to transform the distribution-constrained optimal stopping problem to a stochastic control problem and derived a DPP by measurable selection arguments. From the perspective of optimal transport, Beiglböck et al. [6] gave a geometric interpretation of optimal stopping times of a Brownian motion with distribution constraint.

Moreover, for stochastic control problems with state constraints, stochastic target problems with controlled losses and related geometric DPP, see [11, 12, 14, 41, 42, 43, 15, 9, 13, 10].

The rest of the paper is organized as follows: Section 2 introduces the stochastic control/stopping problem with expectation constraints in a generic probabilistic setting. Section 3 shows that the stochastic control/stopping problem with expectation constraints can be equivalently embedded into an enlarged canonical space: i.e., the SCEC in strong formulation has the same value as the SCEC in weak formulation. In Section 4, we use the martingale-problem formulation to make a countable characterization of the probability class in weak formulation. With such a characterization, we employ a Polish space of diffusion control processes and a Polish space of stopping times to demonstrate that the SCEC value function is upper semi-analytic. Then in Section 5, we utilize a measurable selection argument to establish a dynamic programming principle in weak formulation for the SCEC value function. We defer the proofs of our main results and the lengthy proofs of auxiliary results to Section 6 and put some technical lemmata in the appendix.

We close this section by a description of our notation and a review of the martingale-problem formulation of controlled SDEs.

1.1 Notation and Preliminaries

Throughout this paper, let us denote $a^+ := a \vee 0$ and $a^- := (-a) \vee 0$ for any $a \in \mathbb{R}$. We set $\mathbb{Q}_+ := \mathbb{Q} \cap [0,\infty)$, $\mathbb{Q}_+^{2,<} := \big\{(s,r) \in \mathbb{Q}_+ \times \mathbb{Q}_+ : s < r\big\}$ and set $\Re := (-\infty,\infty]^{\mathbb{N}}$ as the product of countably many copies of $(-\infty,\infty]$. On $\mathbb{T} := [0,\infty]$ we define a metric $\rho_+(t_1,t_2) := \big|\arctan(t_1) - \arctan(t_2)\big|$, $\forall\, t_1,t_2 \in \mathbb{T}$ and consider the induced topology by ρ_+.

For a generic topological space $\big(\mathbb{X}, \mathfrak{T}(\mathbb{X})\big)$, denote its Borel sigma-field by $\mathscr{B}(\mathbb{X})$. We let $\mathfrak{P}(\mathbb{X})$ be the set of all probability measures on $\big(\mathbb{X}, \mathscr{B}(\mathbb{X})\big)$ and equip $\mathfrak{P}(\mathbb{X})$ with the topology of weak convergence $\mathfrak{T}_\sharp\big(\mathfrak{P}(\mathbb{X})\big)$. By e.g. Corollary 7.25.1 of [7], $\big(\mathfrak{P}(\mathbb{X}), \mathfrak{T}_\sharp(\mathfrak{P}(\mathbb{X}))\big)$ is a *Borel space* (i.e., homeomorphic to a Borel subset of a complete separable metric space).

Let $n \in \mathbb{N}$. For any $x \in \mathbb{R}^n$ and $\delta \in (0,\infty)$, let $O_\delta(x)$ denote the open ball centered at x with radius δ and let $\overline{O}_\delta(x)$ be its closure. For any $x, \widetilde{x} \in \mathbb{R}^n$ we denote the usual inner product by $x \cdot \widetilde{x} := \sum_{i=1}^n x_i \widetilde{x}_i$, and for any $n \times n-$real matrices $A, \widetilde{A}$ we denote the Frobenius inner product by $A : \widetilde{A} := trace\big(A\widetilde{A}^T\big)$, where $\widetilde{A}^T$ is the transpose of $\widetilde{A}$. Let $\big\{\mathcal{E}_i^n\big\}_{i \in \mathbb{N}}$ be a countable subbase of the Euclidean topology $\mathfrak{T}(\mathbb{R}^n)$ on $\mathbb{R}^n$. Then $\mathscr{O}(\mathbb{R}^n) := \Big\{ \underset{i=1}{\overset{n}{\cap}} \mathcal{E}_{k_i}^n : \{k_i\}_{i=1}^n \subset \mathbb{N} \Big\} \cup \{\emptyset, \mathbb{R}^n\}$ forms a countable base of $\mathfrak{T}(\mathbb{R}^n)$ and thus $\mathscr{B}(\mathbb{R}^n) = \sigma\big(\mathscr{O}(\mathbb{R}^n)\big)$. We also set $\widehat{\mathscr{O}}(\mathbb{R}^n) := \underset{k \in \mathbb{N}}{\cup} \big(\mathbb{Q}_+ \times \mathscr{O}(\mathbb{R}^n)\big)^k$. For any $\varphi \in C^2(\mathbb{R}^n)$, let $D\varphi$ be its gradient, $D^2\varphi$ be its Hessian matrix and denote $D^0\varphi := \varphi$. For $i = 1, \cdots, n$, define $\varphi_i(x) := x_i$, $\forall\, x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We let $\mathfrak{C}(\mathbb{R}^n)$ collect these coordinate functions and their products, i.e., $\mathfrak{C}(\mathbb{R}^n) := \{\varphi_i\}_{i=1}^n \cup \{\varphi_i \varphi_j\}_{i,j=1}^n$.

Let $(\Omega, \mathcal{F}, P)$ be a generic probability space. For subsets A_1, A_2 of Ω, we denote $A_1 \Delta A_2 := (A_1 \cap A_2^c) \cup (A_2 \cap A_1^c)$. For a random variable ξ on Ω with values in a measurable space $(\mathcal{Q}, \mathcal{G})$, we say ξ is $\mathcal{F}/\mathcal{G}-$measurable if its induced sigma-field $\xi^{-1}(\mathcal{G}) := \{\xi^{-1}(\mathcal{A}) : \forall\, \mathcal{A} \in \mathcal{G}\}$ is included in $\mathcal{F}$. For a sub-sigma-field $\mathfrak{F}$ of $\mathcal{F}$, define $\mathscr{N}_P(\mathfrak{F}) := \big\{\mathcal{N} \subset \Omega : \mathcal{N} \subset A$ for some $A \in \mathfrak{F}$ with $P(A) = 0\big\}$, which collects all $P-$null sets with respect to $\mathfrak{F}$. For two sub-sigma-fields $\mathfrak{F}_1, \mathfrak{F}_2$ of $\mathcal{F}$, we denote $\mathfrak{F}_1 \vee \mathfrak{F}_2 := \sigma(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. Let $t \in [0,\infty)$. For a filtration $\mathbf{F} = \{\mathcal{F}_s\}_{s \in [t,\infty)}$ of $\mathcal{F}$, we decree $\mathcal{F}_{t-} := \mathcal{F}_t$ and define $\mathcal{F}_{s-} := \sigma\Big(\underset{r \in [t,s)}{\cup} \mathcal{F}_r\Big)$, $\forall\, s \in (t,\infty)$; we also set $\mathcal{F}_\infty := \sigma\Big(\underset{s \in [t,\infty)}{\cup} \mathcal{F}_s\Big)$ and refer to filtration $\mathbf{F}^P = \big\{\mathcal{F}_s^P := \sigma\big(\mathcal{F}_s \cup \mathscr{N}_P(\mathcal{F}_\infty)\big)\big\}_{s \in [t,\infty)}$ as the $P-$augmentation of $\mathbf{F}$. For a process $X = \{X_s\}_{s \in [t,\infty)}$ on Ω with values in a topological space, its raw filtration is $\mathbf{F}^X = \big\{\mathcal{F}_s^X := \sigma(X_r; r \in [t,s])\big\}_{s \in [t,\infty)}$. We denote the $P-$augmentation of $\mathbf{F}^X$ by $\mathbf{F}^{X,P} = \big\{\mathcal{F}_s^{X,P} := \sigma\big(\mathcal{F}_s^X \cup \mathscr{N}_P(\mathcal{F}_\infty^X)\big)\big\}_{s \in [t,\infty)}$ and let $\mathscr{P}^X$ be the $\mathbf{F}^X-$predictable sigma$-$field of $[t,\infty) \times \Omega$. We call X a continuous process if its paths are all continuous. When the time variable s of X has complicated form, we may write $X(s,\omega)$ as $X_s(\omega)$ for readability. By default, a Brownian motion $\{B_s\}_{s \in [t,\infty)}$ on $(\Omega, \mathcal{F}, P)$ is with respect to its raw filtration $\mathbf{F}^B$ unless stated otherwise.

Fix $d, l \in \mathbb{N}$. Let $\Omega_0 = \big\{\omega \in C([0,\infty); \mathbb{R}^d) : \omega(0) = \mathbf{0}\big\}$ be the space of all $\mathbb{R}^d-$valued continuous paths starting from $\mathbf{0}$, which is a Polish space under the topology of locally uniform convergence. Let P_0 be the Wiener measure on $\big(\Omega_0, \mathscr{B}(\Omega_0)\big)$, under which the canonical process $W = \{W_s\}_{s \in [0,\infty)}$ of Ω_0 is a $d-$dimensional standard Brownian motion. For any $t \in [0,\infty)$, $W_s^t := W_s - W_t$, $s \in [t,\infty)$ is also a Brownian motion on $\big(\Omega_0, \mathscr{B}(\Omega_0), P_0\big)$. Let $\Omega_X = C([0,\infty); \mathbb{R}^l)$ be the space of all $\mathbb{R}^l-$valued continuous paths endowed with the topology of locally uniform convergence. The function $\mathfrak{l}_1(t,\omega_0) := \omega_0(t \wedge \cdot)$ is continuous in $(t,\omega_0) \in [0,\infty) \times \Omega_0$ while the function $\mathfrak{l}_2(t,\omega_X) := \omega_X(t \wedge \cdot)$ is continuous in $(t,\omega_X) \in [0,\infty) \times \Omega_X$.

Let $\mathbb{U}$ be a Polish space with a compatible metric $\rho_{\mathbb{U}}$ and let $u_0 \in \mathbb{U}$. As a Polish space, $\mathbb{U}$ is homeomorphic to a Borel subset $\mathfrak{E}$ of $[0,1]$, we denote this homeomorphism by $\mathscr{I} : \mathbb{U} \mapsto \mathfrak{E}$. Let $C_b\big([0,\infty) \times \mathbb{U}\big)$ $\big($resp. $\widehat{C}_b\big([0,\infty) \times \mathbb{U}\big)\big)$ collect all real-valued bounded continuous (resp. bounded uniformly continuous) functions on $[0,\infty) \times \mathbb{U}$. For any $(\delta, \mathfrak{m}, \phi) \in (0,\infty) \times \mathfrak{P}\big([0,\infty) \times \mathbb{U}\big) \times C_b\big([0,\infty) \times \mathbb{U}\big)$, set $O_\delta(\mathfrak{m}, \phi) := \big\{\mathfrak{m}' \in \mathfrak{P}\big([0,\infty) \times \mathbb{U}\big) : \big|\int_0^\infty \int_{\mathbb{U}} \phi(t,u)\big(\mathfrak{m}'(dt,du) - \mathfrak{m}(dt,du)\big)\big| < \delta\big\}$.

Lemma 1.1. *There exist* $\{\mathfrak{m}_k\}_{k \in \mathbb{N}} \subset \mathfrak{P}\big([0,\infty) \times \mathbb{U}\big)$ *and* $\{\phi_j\}_{j \in \mathbb{N}} \subset \widehat{C}_b\big([0,\infty) \times \mathbb{U}\big)$ *such that* $\big\{O_{\frac{1}{n}}(\mathfrak{m}_k, \phi_j) : n, k, j \in \mathbb{N}\big\}$ *forms a countable subbase of* $\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty) \times \mathbb{U}\big)\big)$.

Proof: Proposition 7.19 of [7] shows that the topology of weak convergence $\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty) \times \mathbb{U}\big)\big)$ on $\mathfrak{P}\big([0,\infty) \times \mathbb{U}\big)$ can be generated by a subbase $\Lambda := \big\{O_\delta(\mathfrak{m}, \phi_j) : \delta \in (0,\infty), \mathfrak{m} \in \mathfrak{P}\big([0,\infty) \times \mathbb{U}\big), j \in \mathbb{N}\big\}$, where $\{\phi_j\}_{j \in \mathbb{N}}$ is a countable dense subset of $\widehat{C}_b\big([0,\infty) \times \mathbb{U}\big)$. As the Borel space $\big(\mathfrak{P}([0,\infty) \times \mathbb{U}), \mathfrak{T}_\sharp\big(\mathfrak{P}([0,\infty) \times \mathbb{U})\big)\big)$ is separable, it has a countable dense subset $\{\mathfrak{m}_k\}_{k \in \mathbb{N}}$.

To show that $\widetilde{\Lambda} := \big\{O_{\frac{1}{n}}(\mathfrak{m}_k, \phi_j) : n, k, j \in \mathbb{N}\big\}$ is another subbase of $\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty) \times \mathbb{U}\big)\big)$, it suffices to verify that any member of Λ is a union of some members in $\widetilde{\Lambda}$: Let $(\delta, \mathfrak{m}, j) \in (0,\infty) \times \mathfrak{P}\big([0,\infty) \times \mathbb{U}\big) \times \mathbb{N}$ and let $\mathfrak{m}' \in O_\delta(\mathfrak{m}, \phi_j)$. There exists $n \in \mathbb{N}$ such that $\frac{2}{n} < \delta - \big|\int_0^\infty \int_{\mathbb{U}} \phi_j(t,u)\big(\mathfrak{m}'(dt,du) - \mathfrak{m}(dt,du)\big)\big|$, and one can find $\mathfrak{m}_k \in O_{\frac{1}{n}}(\mathfrak{m}'; \phi_j)$ for some $k \in \mathbb{N}$.

For any $\mathfrak{m}'' \in O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)$, we can deduce that $\big|\int_0^\infty\int_{\mathbb{U}}\phi_j(t,u)\big(\mathfrak{m}''(dt,du)-\mathfrak{m}(dt,du)\big)\big| \le \big|\int_0^\infty\int_{\mathbb{U}}\phi_j(t,u)\big(\mathfrak{m}''(dt,du)-\mathfrak{m}_k(dt,du)\big)\big| + \big|\int_0^\infty\int_{\mathbb{U}}\phi_j(t,u)\big(\mathfrak{m}_k(dt,du)-\mathfrak{m}'(dt,du)\big)\big| + \big|\int_0^\infty\int_{\mathbb{U}}\phi_j(t,u)\big(\mathfrak{m}'(dt,du)-\mathfrak{m}(dt,du)\big)\big| \le \frac{2}{n} + \big|\int_0^\infty\int_{\mathbb{U}}\phi_j(t,u)\big(\mathfrak{m}'(dt,du)-\mathfrak{m}(dt,du)\big)\big| < \delta$, which implies that $\mathfrak{m}' \in O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j) \subset O_\delta(\mathfrak{m},\phi_j)$. □

Let $\mathbb{J} := L^0\big([0,\infty);\mathbb{U}\big)$ denote the equivalence classes of $\mathbb{U}$-valued Borel measurable functions on $[0,\infty)$ in the sense that $\mathfrak{u}_1,\mathfrak{u}_2\in\mathbb{J}$ are equivalent if $\mathfrak{u}_1(t)=\mathfrak{u}_2(t)$ for a.e. $t\in(0,\infty)$. It will serve as the path space of $\mathbb{U}-$valued diffusion controls. We can embed $\mathbb{J}$ into $\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)$ via a mapping $\mathfrak{i}_{\mathbb{J}}: \mathbb{J}\ni\mathfrak{u}\mapsto e^{-t}\delta_{\mathfrak{u}(t)}(du)dt\in\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)$. Let $\mathfrak{T}_\sharp(\mathbb{J})$ be the topology induced by $\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)\big)$ via $\mathfrak{i}_{\mathbb{J}}$. According to Lemma 1.1, $\mathfrak{T}_\sharp(\mathbb{J})$ is generated by a countable subbase

$$\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big)=\Big\{\mathfrak{u}\in\mathbb{J}:\Big|\int_0^\infty e^{-t}\phi_j(t,\mathfrak{u}(t))dt-\int_0^\infty\int_{\mathbb{U}}\phi_j(t,u)\mathfrak{m}_k(dt,du)\Big|<\frac{1}{n}\Big\},\quad\forall\, n,k,j\in\mathbb{N}. \tag{1.1}$$

The proofs of next two Lemmata are relatively lengthy, see Section 6 for them.

Lemma 1.2. $\big(\mathbb{J},\mathfrak{T}_\sharp(\mathbb{J})\big)$ *is a Borel space.*

Let $L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)$ collect all real−valued Borel-measurable functions on $(0,\infty)\times\mathbb{U}$. For any $\varphi\in L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)$, define $I_\varphi(\mathfrak{u}) := \int_0^\infty\varphi(s,\mathfrak{u}(s))ds=\int_0^\infty\varphi^+(s,\mathfrak{u}(s))ds-\int_0^\infty\varphi^-(s,\mathfrak{u}(s))ds$, $\forall\,\mathfrak{u}\in\mathbb{J}$. The Borel sigma-field $\mathscr{B}(\mathbb{J})$ of $\mathfrak{T}_\sharp(\mathbb{J})$ can be generated by these random variables I_φ on $\mathbb{J}$.

Lemma 1.3. *1) We have* $\mathscr{B}(\mathbb{J})=\sigma\big(I_\varphi;\varphi\in L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)\big)$.
2) Let $\psi:(0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}\mapsto[-\infty,\infty]$ *be a Borel-measurable function. Then the mapping* $\Psi(t,\mathfrak{s},\omega_0,\omega_X,\mathfrak{u}) := \int_t^{t+\mathfrak{s}}\psi\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr=\int_t^{t+\mathfrak{s}}\psi^+\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr-\int_t^{t+\mathfrak{s}}\psi^-\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr$, $(t,\mathfrak{s},\omega_0,\omega_X,\mathfrak{u})\in[0,\infty)\times[0,\infty)\times\Omega_0\times\Omega_X\times\mathbb{J}$ *is* $\mathscr{B}[0,\infty)\otimes\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$*measurable.*

Let $b:(0,\infty)\times\Omega_X\times\mathbb{U}\mapsto\mathbb{R}^l$ and $\sigma:(0,\infty)\times\Omega_X\times\mathbb{U}\mapsto\mathbb{R}^{l\times d}$ be two Borel-measurable functions such that for any $t\in(0,\infty)$

$$\big|b(t,\omega_X,u)-b(t,\omega'_X,u)\big|+\big|\sigma(t,\omega_X,u)-\sigma(t,\omega'_X,u)\big|\le\kappa(t)\big\|\omega_X-\omega'_X\big\|_t,\quad\forall\,\omega_X,\omega'_X\in\Omega_X,\ \forall\,u\in\mathbb{U}, \tag{1.2}$$

$$\text{and}\quad\int_0^t\sup_{u\in\mathbb{U}}\big(|b(r,\mathbf{0},u)|^2+|\sigma(r,\mathbf{0},u)|^2\big)dr<\infty, \tag{1.3}$$

where $\kappa:(0,\infty)\mapsto(0,\infty)$ is some non-decreasing function and $\big\|\omega_X-\omega'_X\big\|_t := \sup\limits_{s\in[0,t]}\big|\omega_X(s)-\omega'_X(s)\big|$. Under conditions (1.2) and (1.3), controlled SDEs with coefficients (b,σ) are well-posed (see e.g. Theorem V.7 of [38]):

Proposition 1.1. *Let $(\Omega,\mathcal{F},P)$ be a probability space. Given $t\in[0,\infty)$, let $\{B^t_s\}_{s\in[t,\infty)}$ be a $d-$dimensional Brownian motion on $(\Omega,\mathcal{F},P)$ and let $\mu=\{\mu_s\}_{s\in[t,\infty)}$ be a $\mathbb{U}-$valued, $\mathbf{F}^{B^t,P}-$progressively measurable process. For any $\mathbf{x}\in\Omega_X$, the SDE with the open-loop control μ*

$$X_s=\mathbf{x}(t)+\int_t^s b(r,X_{r\wedge\cdot},\mu_r)dr+\int_t^s\sigma(r,X_{r\wedge\cdot},\mu_r)dB^t_r,\quad\forall\,s\in[t,\infty)\ \ \textit{with initial condition}\ X\big|_{[0,t]}=\mathbf{x}|_{[0,t]} \tag{1.4}$$

admits a unique strong solution $X^{t,\mathbf{x},\mu}=\{X^{t,\mathbf{x},\mu}_s\}_{s\in[0,\infty)}$ $\big($i.e., $X^{t,\mathbf{x},\mu}$ is an $\{\mathcal{F}^{B^t,P}_{s\vee t}\}_{s\in[0,\infty)}-$adapted continuous process satisfying (1.4) and $P\big\{X^{t,\mathbf{x},\mu}_s=\widetilde{X}^{t,\mathbf{x},\mu}_s,\ \forall\,s\in[0,\infty)\big\}=1$ if $\big\{\widetilde{X}^{t,\mathbf{x},\mu}_s\big\}_{s\in[0,\infty)}$ is another $\{\mathcal{F}^{B^t,P}_{s\vee t}\}_{s\in[0,\infty)}-$adapted continuous process satisfying (1.4)$\big)$.

Let $\mathscr{H}_o$ collect all $(-\infty,\infty]-$valued Borel-measurable functions ψ on $(0,\infty)\times\Omega_X\times\mathbb{U}$ such that for any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$ and any $\mathbb{U}-$valued $\mathbf{F}^{W^t}-$predictable process $\mu^o=\{\mu^o_s\}_{s\in[t,\infty)}$, one has $E_{P_0}\big[\int_t^\infty\psi^-(r,X^{t,\mathbf{x},\mu^o}_{r\wedge\cdot},\mu^o_r)dr\big]<\infty$, where $\{X^{t,\mathbf{x},\mu^o}_s\}_{s\in[0,\infty)}$ is the unique strong solution of (1.4) on $(\Omega,\mathcal{F},P)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0\big)$ with $(B^t,\mu)=\big(W^t,\mu^o\big)$.

Moreover, we take the conventions $\inf\emptyset:=\infty$, $\sup\emptyset:=-\infty$ and $(+\infty)+(-\infty)=-\infty$ (i.e. for any $a_1,a_2\in[-\infty,\infty]$ we set $a_1-a_2:=-\infty$ if $a_1=a_2=\infty$). In particular, on a measure space $(\Omega,\mathcal{F},\mathfrak{m})$, one can define the integral $\int_\Omega\xi\,d\mathfrak{m}:=\int_\Omega\xi^+\,d\mathfrak{m}-\int_\Omega\xi^-\,d\mathfrak{m}$ for any $[-\infty,\infty]-$valued $\mathcal{F}-$measurable random variable ξ on Ω.

1.2 Review of Martingale-Problem Formulation of Controlled SDEs

In this subsection, we consider a general measurable space $(\Omega,\mathcal{F})$. Let $\{B_s\}_{s\in[0,\infty)}$ be an $\mathbb{R}^d-$valued continuous process on Ω with $B_0=\mathbf{0}$ and let $X=\{X_s\}_{s\in[0,\infty)}$ be an $\mathbb{R}^l-$valued continuous process on Ω such that (B_s,X_s) is $\mathcal{F}-$measurable for each $s\in[0,\infty)$.

Given $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, let P be a probability measure on $(\Omega,\mathcal{F})$ such that $P\big\{X_s=\mathbf{x}(s),\ \forall\, s\in[0,t]\big\}=1$. We set $B^t_s:=B_s-B_t,\ \forall\, s\in[t,\infty)$ and let $\mu=\{\mu_s\}_{s\in[t,\infty)}$ be a $\mathbb{U}-$valued, $\mathbf{F}^{B^t,P}-$progressively measurable process. Define filtration $\mathbf{F}^t=\{\mathcal{F}^t_s\}_{s\in[t,\infty)}$ by $\mathcal{F}^t_s:=\mathcal{F}^{B^t}_s\vee\mathcal{F}^X_s=\sigma(B^t_r;r\in[t,s])\vee\sigma(X_r;r\in[0,s])$ and filtration $\mathbf{F}^{t,P}=\{\mathcal{F}^{t,P}_s\}_{s\in[t,\infty)}$ by $\mathcal{F}^{t,P}_s:=\mathcal{F}^{B^t,P}_s\vee\mathcal{F}^X_s=\sigma\big(\mathcal{F}^{B^t}_s\cup\mathscr{N}_P\big(\mathcal{F}^{B^t}_\infty\big)\big)\vee\mathcal{F}^X_s$. For any $\varphi\in C^2(\mathbb{R}^{d+l})$, we define

$$M^{t,\mu}_s(\varphi):=\varphi\big(B^t_s,X_s\big)-\int_t^s\overline{b}(r,X_{r\wedge\cdot},\mu_r)\cdot D\varphi\big(B^t_r,X_r\big)dr-\frac{1}{2}\int_t^s\overline{\sigma}\,\overline{\sigma}^T(r,X_{r\wedge\cdot},\mu_r):D^2\varphi(B^t_r,X_r)dr,\quad \forall\, s\in[t,\infty),$$

where $\overline{b}(r,\omega_X,u):=\begin{pmatrix}0\\ b(r,\omega_X,u)\end{pmatrix}\in\mathbb{R}^{d+l}$, $\overline{\sigma}(r,\omega_X,u):=\begin{pmatrix}I_{d\times d}\\ \sigma(r,\omega_X,u)\end{pmatrix}\in\mathbb{R}^{(d+l)\times d}$, $\forall\,(r,\omega_X,u)\in(0,\infty)\times\Omega_X\times\mathbb{U}$. Clearly, $\big\{M^{t,\mu}_s(\varphi)\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^{t,P}-$adapted continuous process and it is even $\mathbf{F}^t-$adapted if the control process μ is only $\mathbf{F}^{B^t}-$progressively measurable. For any $n\in\mathbb{N}$ and $\mathfrak{a}\in\mathbb{R}^{d+l}$, set $\tau^t_n(\mathfrak{a}):=\inf\big\{s\in[t,\infty):|(B^t_s,X_s)-\mathfrak{a}|\ge n\big\}\wedge(t+n)$, which is an $\mathbf{F}^t-$stopping time. In particular, we denote $\tau^t_n(\mathbf{0})$ by τ^t_n.

In virtue of [45], we have the following martingale-problem formulation of controlled SDEs with coefficients (b,σ) on Ω.

Proposition 1.2. *Under the probabilistic setup of this subsection, the process $\big\{M^{t,\mu}_{s\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}$ is bounded for any $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$ and the following statements are equivalent on $(\Omega,\mathcal{F},P)$:*
(i) The process B^t is a Brownian motion and $P\{X_s=X^{t,\mathbf{x},\mu}_s,\ \forall\, s\in[0,\infty)\}=1$, where $\big\{X^{t,\mathbf{x},\mu}_s\big\}_{s\in[0,\infty)}$ is the unique $\big\{\mathcal{F}^{B^t,P}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process solving SDE (1.4).
(ii) $\big\{M^{t,\mu}_{s\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^{t,P}-$martingale for any $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$.
(iii) $\big\{M^{t,\mu}_{s\wedge\tau^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^{t,P}-$martingale for any $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$.

Moreover, if the control process μ is $\mathbf{F}^{B^t}-$progressively measurable, the $\mathbf{F}^{t,P}-$martingales mentioned in (ii) and (iii) are $\mathbf{F}^t-$martingales.

The proof of this result is an easy extension of [4, Proposition 1.2] to the control case, please see our ArXiv version [5] for details. We also have the following consequence of Proposition 1.2.

Proposition 1.3. *Let $(\Omega,\mathcal{F},P)$ be a probability space. Given $t\in[0,\infty)$, let $\{B_s\}_{s\in[0,\infty)}$ be an $\mathbb{R}^d-$valued continuous process on Ω with $B_0=\mathbf{0}$ such that the process B^t is a Brownian motion on $(\Omega,\mathcal{F},P)$, and let $\mu=\{\mu_s\}_{s\in[t,\infty)}$ be a $\mathbb{U}-$valued, $\mathbf{F}^{B^t,P}-$progressively measurable process.*

Let $(t,\mathbf{w})\in[0,\infty)\times\Omega_0$ and define $B^{t,\mathbf{w}}_s(\omega):=\mathbf{w}(s\wedge t)+B^t_{s\vee t}(\omega),\ \forall\,(s,\omega)\in[0,\infty)\times\Omega$. There exists a $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process $\mu^o=\{\mu^o_s\}_{s\in[t,\infty)}$ on Ω_0 and an $\mathcal{N}_\mu\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$ such that for any $\omega\in\mathcal{N}^c_\mu$, $\mu_s(\omega)=\mu^o_s\big(B^{t,\mathbf{w}}(\omega)\big)$ for a.e. $s\in(t,\infty)$. It also holds for any $\mathbf{x}\in\Omega_X$ and $\psi\in\mathscr{H}_o$ that $P\big\{X^{t,\mathbf{x},\mu}_s=X^{t,\mathbf{x},\mu^o}_s(B^{t,\mathbf{w}}),\ \forall\, s\in[0,\infty)\big\}=1$ and $E_P\big[\int_t^\infty\psi^-(r,X^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r)dr\big]=E_{P_0}\big[\int_t^\infty\psi^-(r,X^{t,\mathbf{x},\mu^o}_{r\wedge\cdot},\mu^o_r)dr\big]<\infty$.

2 Stochastic Control/Stopping Problem with Expectation Constraints

Let $(\mathcal{Q},\mathcal{F},\mathfrak{p})$ be a probability space equipped with a $d-$dimensional standard Brownian motion $\{\mathcal{B}_s\}_{s\in[0,\infty)}$.

Let $t\in[0,\infty)$. We set $\mathcal{B}^t_s=\mathcal{B}_s-\mathcal{B}_t,\ \forall\, s\in[t,\infty)$, which is also a Brownian motion on $(\mathcal{Q},\mathcal{F},\mathfrak{p})$. Let $\mathcal{U}_t$ collect all $\mathbb{U}-$valued, $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$progressively measurable processes $\mu=\{\mu_s\}_{s\in[t,\infty)}$ and let $\mathcal{S}_t$ denote the set of all $[t,\infty]-$valued $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping times. For any $(\mathbf{x},\mu)\in\Omega_X\times\mathcal{U}_t$, Proposition 1.1 shows that the SDE with the open-loop control μ

$$\mathcal{X}_s=\mathbf{x}(t)+\int_t^s b\big(r,\mathcal{X}_{r\wedge\cdot},\mu_r\big)dr+\int_t^s\sigma\big(r,\mathcal{X}_{r\wedge\cdot},\mu_r\big)d\mathcal{B}_r,\ \ \forall\, s\in[t,\infty)\ \text{ with initial condition } \mathcal{X}\big|_{[0,t]}=\mathbf{x}|_{[0,t]}\qquad(2.1)$$

admits a unique strong solution $\mathcal{X}^{t,\mathbf{x},\mu}=\big\{\mathcal{X}^{t,\mathbf{x},\mu}_s\big\}_{s\in[0,\infty)}$ on $\big(\mathcal{Q},\mathcal{F},\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}},\mathfrak{p}\big)$ $\big($i.e., $\mathcal{X}^{t,\mathbf{x},\mu}$ is the unique $\big\{\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process solving SDE (2.1)$\big)$.

Let $f \in \mathscr{H}_o$, $\{g_i, h_i\}_{i\in\mathbb{N}} \subset \mathscr{H}_o$ and let $\pi: [0,\infty)\times\Omega_X \mapsto (-\infty,\infty]$ be a Borel-measurable function bounded from below by some $c_\pi \in (-\infty, 0)$.

Given a historical path $\mathbf{x}|_{[0,t]}$, the dynamic of the state then evolves along process $\big\{\mathcal{X}^{t,\mathbf{x},\mu}\big\}_{s\in[t,\infty)}$ if the decision maker chooses a control process $\mu \in \mathcal{U}_t$. The decision maker also determines an exercise time $\tau \in \mathcal{S}_t$ to cease the game, at which she will receive an accumulative reward $\int_t^\tau f\big(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r\big)\,dr$ plus a terminal reward $\pi\big(\tau, \mathcal{X}^{t,\mathbf{x},\mu}_{\tau\wedge\cdot}\big)$ (both random rewards can take negative values). The investor intends to maximize the expectation of her total wealth, but her choice of (μ,τ) is subject to a series of expectation constraints

$$E_{\mathfrak{p}}\Big[\int_t^\tau g_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\Big] \le y_i, \quad E_{\mathfrak{p}}\Big[\int_t^\tau h_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\Big] = z_i, \quad \forall\, i \in \mathbb{N} \tag{2.2}$$

for some $(y,z) = \big(\{y_i\}_{i\in\mathbb{N}}, \{z_i\}_{i\in\mathbb{N}}\big) \in \Re\times\Re$. One can regard each $\int_t^\tau g_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr$ or $\int_t^\tau h_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr$ as certain accumulative cost. So the value of this stochastic control/stopping problem with expectation constraints (SCEC for short) is

$$V(t,\mathbf{x},y,z) := \sup_{(\mu,\tau)\in\mathcal{C}_{t,\mathbf{x}}(y,z)} E_{\mathfrak{p}}\Big[\int_t^\tau f\big(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r\big)dr + \mathbf{1}_{\{\tau<\infty\}}\pi\big(\tau, \mathcal{X}^{t,\mathbf{x},\mu}_{\tau\wedge\cdot}\big)\Big], \tag{2.3}$$

where $\mathcal{C}_{t,\mathbf{x}}(y,z) := \big\{(\mu,\tau)\in\mathcal{U}_t\times\mathcal{S}_t : E_{\mathfrak{p}}\big[\int_t^\tau g_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] \le y_i,\ E_{\mathfrak{p}}\big[\int_t^\tau h_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] = z_i,\ \forall\, i\in\mathbb{N}\big\}$.

Remark 2.1. *Let* $(t,\mathbf{x}) \in [0,\infty)\times\Omega_X$.
1) (finitely many constraints) For $i\in\mathbb{N}$, *the constraint* $E_{\mathfrak{p}}\big[\int_t^\tau g_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] \le y_i$ *holds for any* $(\mu,\tau)\in\mathcal{U}_t\times\mathcal{S}_t$ *if* $y_i = \infty$, *and the constraint* $E_{\mathfrak{p}}\big[\int_t^\tau h_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] = z_i$ *holds for any* $(\mu,\tau)\in\mathcal{U}_t\times\mathcal{S}_t$ *if* $\big(h_i(\cdot,\cdot,\cdot), z_i\big) = (0,0)$.
1a) If we take $\big(y_i, h_i(\cdot,\cdot,\cdot), z_i\big) = (\infty,0,0)$, $\forall\, i\in\mathbb{N}$, *there is no expectation constraint at all.*
1b) If one takes $y_i = \infty$, $\forall\, i \ge 2$ *and* $\big(h_i(\cdot,\cdot,\cdot), z_i\big) = (0,0)$, $\forall\, i\in\mathbb{N}$, (2.2) *reduces to a single constraint* $E_{\mathfrak{p}}\big[\int_t^\tau g_1(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] \le y_1$. *In addition, if* $y_1 \ge 0$, *then* $(\mu,t)\in\mathcal{C}_{t,\mathbf{x}}(y,\mathbf{0})$ *for any* $\mu\in\mathcal{U}_t$.
1c) If one takes $y_i = \infty$, $\forall\, i\in\mathbb{N}$ *and* $\big(h_i(\cdot,\cdot,\cdot), z_i\big) = (0,0)$, $\forall\, i\ge 2$, (2.2) *degenerates to* $E_{\mathfrak{p}}\big[\int_t^\tau h_1(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] = z_1$.
1d) If we take $\big(y_i, h_i(\cdot,\cdot,\cdot), z_i\big) = (\infty,0,0)$, $\forall\, i\ge 2$, (2.2) *becomes a couple of constraints* $E_{\mathfrak{p}}\big[\int_t^\tau g_1(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] \le y_1$ *and* $E_{\mathfrak{p}}\big[\int_t^\tau h_1(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] = z_1$.
1e) If we take $g_2 = -g_1$, $y_2 \ge -y_1$; $y_i = \infty$, $\forall\, i \ge 3$ *and* $\big(h_i(\cdot,\cdot,\cdot), z_i\big) = (0,0)$, $\forall\, i\in\mathbb{N}$, (2.2) *becomes a range constraint* $-y_2 \le E_{\mathfrak{p}}\big[\int_t^\tau g_1(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] \le y_1$.
2) (moment constraints) Let $i\in\mathbb{N}$, $a\in(0,\infty)$ *and* $q\in[1,\infty)$. *If* $g_i(s,\mathbf{x},u) = aqs^{q-1}$, $\forall\,(s,\mathbf{x},u)\in(0,\infty)\times\Omega_X\times\mathbb{U}$ *(resp.* $h_i(s,\mathbf{x},u) = aqs^{q-1}$, $\forall\,(s,\mathbf{x},u)\in(0,\infty)\times\Omega_X\times\mathbb{U}$*), then the expectation constraint* $E_{\mathfrak{p}}\big[\int_t^\tau g_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] \le y_i$ *(resp.* $E_{\mathfrak{p}}\big[\int_t^\tau h_i(r, \mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] = z_i$*) specifies as a moment constraint* $E_{\mathfrak{p}}\big[a(\tau^q - t^q)\big] \le y_i$ *(resp.* $E_{\mathfrak{p}}\big[a(\tau^q - t^q)\big] = z_i$*).*

To study the measurability of value function V and derive a dynamic programming principle for V without imposing any continuity condition on functions f, π, g_i's and h_i's in time and state variables, we follow [18]'s approach to embed the controls, the stopping rules as well as the Brownian/state information into an enlarged canonical space via a mapping $\omega \mapsto \big(\mathcal{B}_\cdot(\omega), \mu_\cdot(\omega), \mathcal{X}^{t,\mathbf{x},\mu}_\cdot(\omega), \tau(\omega)\big)$ and consider their joint law as a new type of controls.

3 Weak Formulation

In this section, we study the stochastic control/stopping problem with expectation constraints in a weak formulation or over an enlarged canonical space

$$\overline{\Omega} := \Omega_0 \times \mathbb{J} \times \Omega_X \times \mathbb{T}.$$

As $\big(\mathbb{J}, \mathfrak{T}_\sharp(\mathbb{J})\big)$ is a Borel space by Lemma 1.2, $\overline{\Omega}$ is also a Borel space under the product topology. Let $\mathfrak{P}(\overline{\Omega})$ be the space of all probability measures on $\big(\overline{\Omega}, \mathscr{B}(\overline{\Omega})\big)$ equipped with the topology of weak convergence, which is also a Borel space (see e.g. Corollary 7.25.1 of [7]). For any $\overline{P}\in\mathfrak{P}(\overline{\Omega})$, set $\mathscr{B}_{\overline{P}}(\overline{\Omega}) := \sigma\big(\mathscr{B}(\overline{\Omega}) \cup \mathscr{N}_{\overline{P}}(\mathscr{B}(\overline{\Omega}))\big)$.

3.1 Setup

We define the canonical coordinates on $\overline{\Omega}$ by

$$\big(\overline{W}_s(\overline{\omega}),\overline{U}_s(\overline{\omega}),\overline{X}_s(\overline{\omega})\big):=\big(\omega_0(s),\mathfrak{u}(s),\omega_X(s)\big),\quad s\in[0,\infty)\quad\text{and}\quad \overline{T}(\overline{\omega}):=\mathfrak{t},\quad \forall\,\overline{\omega}=\big(\omega_0,\mathfrak{u},\omega_X,\mathfrak{t}\big)\in\overline{\Omega}.$$

Given $t\in[0,\infty)$, we define

$$\overline{W}^t_s(\overline{\omega}):=\overline{W}_s(\overline{\omega})-\overline{W}_t(\overline{\omega})\quad\text{and}\quad \overline{\Upsilon}^t_s(\overline{\omega}):=\int_t^s e^{-r}\mathscr{I}\big(\overline{U}_r(\overline{\omega})\big)dr\in[0,1),\quad \forall\,(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}.$$

For the weak formulation of the SCEC, we need to consider those probabilities of $\mathfrak{P}(\overline{\Omega})$ under which the canonical coordinates $(\overline{W},\overline{U},\overline{X},\overline{T})$ serve as Brownian motion, diffusion control, state process and stopping rules respectively.

Definition 3.1. *For any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, let $\overline{\mathcal{P}}_{t,\mathbf{x}}$ be the collection of all probability measures $\overline{P}\in\mathfrak{P}(\overline{\Omega})$ satisfying:*
(D1) There exists a $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process $\widehat{\mu}=\{\widehat{\mu}_s\}_{s\in[t,\infty)}$ on Ω_0 such that $\overline{P}\big\{\overline{U}_s=\overline{\mu}_s$ for a.e. $s\in(t,\infty)\big\}=1$, where $\overline{\mu}_s:=\widehat{\mu}_s(\overline{W})$, $\forall\,s\in[t,\infty)$.
(D2) The process $\overline{W}^t$ is a $d-$dimensional Brownian motion on $\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}\big)$.
(D3) $\overline{P}\big\{\overline{X}_s=\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s,\ \forall\,s\in[0,\infty)\big\}=1$, where $\big\{\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s\big\}_{s\in[0,\infty)}$ is an $\big\{\mathcal{F}^{\overline{W}^t,\overline{P}}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process that uniquely solves the following SDE with the open-loop control $\overline{\mu}$ on $\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}\big)$:

$$\overline{\mathscr{X}}_s=\mathbf{x}(t)+\int_t^s b\big(r,\overline{\mathscr{X}}_{r\wedge\cdot},\overline{\mu}_r\big)dr+\int_t^s\sigma\big(r,\overline{\mathscr{X}}_{r\wedge\cdot},\overline{\mu}_r\big)d\overline{W}_r,\ \ \forall\,s\in[t,\infty)\ \textit{with initial condition}\ \overline{\mathscr{X}}\big|_{[0,t]}=\mathbf{x}\big|_{[0,t]}.\quad(3.1)$$

(D4) There exists a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}$ on Ω_0 such that $\overline{P}\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}=1$.

Let $t\in[0,\infty)$. For any $s\in[t,\infty)$, define $\overline{\mathcal{F}}^t_s:=\mathcal{F}^{\overline{W}^t}_s\vee\mathcal{F}^{\overline{X}}_s=\sigma\big(\overline{W}^t_r;r\in[t,s]\big)\vee\sigma\big(\overline{X}_r;r\in[0,s]\big)$, which is countably generated by $\big\{\overline{X}_r^{-1}(\mathcal{O}):r\in\mathbb{Q}\cap[0,t],\mathcal{O}\in\mathscr{O}(\mathbb{R}^l)\big\}\cup\big\{(\overline{W}^t_r,\overline{X}_r)^{-1}(\mathcal{O}'):r\in\mathbb{Q}\cap(t,s],\mathcal{O}'\in\mathscr{O}(\mathbb{R}^{d+l})\big\}$. We denote the filtration $\big\{\overline{\mathcal{F}}^t_s\big\}_{s\in[t,\infty)}$ by $\overline{\mathbf{F}}^t$. Let $\widehat{\mu}=\{\widehat{\mu}_s\}_{s\in[t,\infty)}$ be a $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process on Ω_0. Then $\overline{\mu}_s(\overline{\omega}):=\widehat{\mu}_s\big(\overline{W}(\overline{\omega})\big)$, $\forall\,(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$ is a $\mathbb{U}-$valued, $\mathbf{F}^{\overline{W}^t}-$predictable process on $\overline{\Omega}$. For any $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$,

$$\overline{M}^{t,\overline{\mu}}_s(\varphi):=\varphi\big(\overline{W}^t_s,\overline{X}_s\big)-\int_t^s\overline{b}\big(r,\overline{X}_{r\wedge\cdot},\overline{\mu}_r\big)\cdot D\varphi\big(\overline{W}^t_r,\overline{X}_r\big)dr-\frac{1}{2}\int_t^s\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{X}_{r\wedge\cdot},\overline{\mu}_r\big):D^2\varphi\big(\overline{W}^t_r,\overline{X}_r\big)dr,\quad\forall\,s\in[t,\infty)$$

is an $\overline{\mathbf{F}}^t-$adapted continuous process and $\overline{\tau}^t_n(\mathfrak{a}):=\inf\big\{s\in[t,\infty):\big|(\overline{W}^t_s,\overline{X}_s)-\mathfrak{a}\big|\ge n\big\}\wedge(t+n)$ is an $\overline{\mathbf{F}}^t-$stopping time. We will simply denote $\overline{\tau}^t_n(\mathbf{0})$ by $\overline{\tau}^t_n$.

Let us also define a shifted canonical process on $\overline{\Omega}$ by $\overline{\mathscr{W}}^t_{\mathfrak{s}}(\overline{\omega}):=\overline{W}_{t+\mathfrak{s}}(\overline{\omega})-\overline{W}_t(\overline{\omega})=\overline{W}^t_{t+\mathfrak{s}}(\overline{\omega})$, $\forall\,(\mathfrak{s},\overline{\omega})\in[0,\infty)\times\overline{\Omega}$. (Note: the subscript $\mathfrak{s}\in[0,\infty)$ of $\overline{\mathscr{W}}^t$ is the relative time after t while the subscript $s\in[t,\infty)$ of $\overline{W}^t$ is the real time.) Given $\mathfrak{s}\in[0,\infty]$, one has $\mathcal{F}^{\overline{\mathscr{W}}^t}_{\mathfrak{s}}=\sigma\big(\overline{\mathscr{W}}^t_{\mathfrak{r}};\mathfrak{r}\in[0,\mathfrak{s}]\cap\mathbb{R}\big)=\sigma\big(\overline{W}^t_{t+\mathfrak{r}};\mathfrak{r}\in[0,\mathfrak{s}]\cap\mathbb{R}\big)=\sigma\big(\overline{W}^t_r;r\in[t,t+\mathfrak{s}]\cap\mathbb{R}\big)=\mathcal{F}^{\overline{W}^t}_{t+\mathfrak{s}}$. In particular, $\mathcal{F}^{\overline{\mathscr{W}}^t}_\infty=\mathcal{F}^{\overline{W}^t}_\infty$. It then holds for any $\overline{P}\in\mathfrak{P}(\overline{\Omega})$ that $\mathcal{F}^{\overline{\mathscr{W}}^t,\overline{P}}_{\mathfrak{s}}=\sigma\big(\mathcal{F}^{\overline{\mathscr{W}}^t}_{\mathfrak{s}}\cup\mathscr{N}_{\overline{P}}(\mathcal{F}^{\overline{\mathscr{W}}^t}_\infty)\big)=\sigma\big(\mathcal{F}^{\overline{W}^t}_{t+\mathfrak{s}}\cup\mathscr{N}_{\overline{P}}(\mathcal{F}^{\overline{W}^t}_\infty)\big)=\mathcal{F}^{\overline{W}^t,\overline{P}}_{t+\mathfrak{s}}$.

According to the martingale-problem formulation of controlled SDEs (Proposition 1.2), we have an alternative description of the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$:

Remark 3.1. *Let $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$. In definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$,*
(i) (D1) is equivalent to
(D1′) There exists a $\mathbb{U}-$valued, $\mathbf{F}^{W}-$predictable process $\ddot{\mu}=\{\ddot{\mu}_{\mathfrak{s}}\}_{\mathfrak{s}\in[0,\infty)}$ on Ω_0 such that $\overline{P}\big\{\overline{\mathscr{U}}^t_{\mathfrak{s}}=\ddot{\mu}_{\mathfrak{s}}(\overline{\mathscr{W}}^t)$ for a.e. $\mathfrak{s}\in(0,\infty)\big\}=1$, where $\overline{\mathscr{U}}^t_{\mathfrak{s}}:=\overline{U}_{t+\mathfrak{s}}$, $\forall\,\mathfrak{s}\in[0,\infty)$.
(ii) Under (D1)=(D1′), (D2)+(D3) is equivalent to
(D2′) $\overline{P}\{\overline{X}_s=\mathbf{x}(s),\ \forall\,s\in[0,t]\}=1$ and $\big\{\overline{M}^{t,\overline{\mu}}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\overline{\mathbf{F}}^t,\overline{P}\big)-$martingale, $\forall\,(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$.
(iii) (D4) is equivalent to
(D4′) There exists a $[0,\infty]-$valued $\mathbf{F}^{W,P_0}-$stopping time $\ddot{\tau}$ on Ω_0 such that $\overline{P}\big\{\overline{T}=t+\ddot{\tau}\big(\overline{\mathscr{W}}^t\big)\big\}=1$.

Remark 3.2. *Let $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$ and let $\overline{P}\in\mathfrak{P}(\overline{\Omega})$ satisfy (D1)+(D2)+(D3) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.*
(1) For any $\psi\in\mathscr{H}_o$, Proposition 1.3 shows that $E_{\overline{P}}\big[\int_t^\infty\psi^-(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]=E_{\overline{P}}\big[\int_t^\infty\psi^-\big(r,\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_{r\wedge\cdot},\overline{\mu}_r\big)dr\big]<\infty$.

(2) Let $(\varphi, n, \mathfrak{a}) \in C^2(\mathbb{R}^{d+l}) \times \mathbb{N} \times \mathbb{R}^{d+l}$. As $\big\{\overline{M}^{t,\overline{\mu}}_{s\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\overline{\mathbf{F}}^t, \overline{P}\big)-$martingale, the optional sampling theorem implies that for any two $[t,\infty]-$valued $\overline{\mathbf{F}}^t-$stopping times $\overline{\zeta}_1, \overline{\zeta}_2$ with $\overline{\zeta}_1 \le \overline{\zeta}_2$, $\overline{P}-$a.s.,

$$E_{\overline{P}}\Big[\big(\overline{M}^{t,\overline{\mu}}_{\overline{\zeta}_2\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi) - \overline{M}^{t,\overline{\mu}}_{\overline{\zeta}_1\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big] = E_{\overline{P}}\Big[E_{\overline{P}}\Big[\overline{M}^{t,\overline{\mu}}_{\overline{\zeta}_2\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi) - \overline{M}^{t,\overline{\mu}}_{\overline{\zeta}_1\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi)\Big|\overline{\mathcal{F}}^t_{\overline{\zeta}_1}\Big]\mathbf{1}_{\overline{A}}\Big] = 0, \quad \forall\, \overline{A} \in \overline{\mathcal{F}}^t_{\overline{\zeta}_1}. \tag{3.2}$$

Let $(t,\mathbf{x}) \in [0,\infty) \times \Omega_X$, $(y,z) = \big(\{y_i\}_{i\in\mathbb{N}}, \{z_i\}_{i\in\mathbb{N}}\big) \in \Re \times \Re$ and set $\overline{R}(t) := \int_{\overline{T}\wedge t}^{\overline{T}} f(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r)dr + \mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T}, \overline{X}_{\overline{T}\wedge\cdot}\big)$. Given a historical state path $\mathbf{x}|_{[0,t]}$, the value of the stochastic control/stopping problem with expectation constraints

$$E_{\overline{P}}\Big[\int_t^{\overline{T}} g_i(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r)dr\Big] \le y_i, \quad E_{\overline{P}}\Big[\int_t^{\overline{T}} h_i(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r)dr\Big] = z_i, \quad \forall\, i \in \mathbb{N} \tag{3.3}$$

in weak formulation is

$$\overline{V}(t,\mathbf{x},y,z) := \sup_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)} E_{\overline{P}}\big[\overline{R}(t)\big] = \sup_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)} E_{\overline{P}}\Big[\int_t^{\overline{T}} f(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r)dr + \mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T}, \overline{X}_{\overline{T}\wedge\cdot}\big)\Big],$$

where $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z) := \Big\{\overline{P} \in \overline{\mathcal{P}}_{t,\mathbf{x}} : E_{\overline{P}}\big[\int_t^{\overline{T}} g_i(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r)dr\big] \le y_i,\ E_{\overline{P}}\big[\int_t^{\overline{T}} h_i(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r)dr\big] = z_i,\ \forall\, i \in \mathbb{N}\Big\}$. We will simply call $\overline{V}(t,\mathbf{x},y,z)$ the *weak* value of the stochastic control/stopping problem with expectation constraints. In case $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z) = \emptyset$, $\overline{V}(t,\mathbf{x},y,z) = -\infty$ by the convention $\sup\emptyset := -\infty$.

We can consider another weak value function of the SCEC: Let $(\mathbf{w},\mathbf{u}) \in \Omega_0 \times \mathbb{J}$ and define $\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}} := \big\{\overline{P} \in \overline{\mathcal{P}}_{t,\mathbf{x}} : \overline{P}\big\{\overline{W}_s = \mathbf{w}(s),\ \forall\, s \in [0,t];\ \overline{U}_s = \mathbf{u}(s) \text{ for a.e. } s \in (0,t)\big\} = 1\big\}$ as the subclass of $\overline{\mathcal{P}}_{t,\mathbf{x}}$ given the historical Brownian path $\mathbf{w}\big|_{[0,t]}$ and the historical control trajectory $\mathbf{u}\big|_{[0,t]}$. The weak value of the stochastic control/stopping problem with expectation constraints (3.3) given $(\mathbf{w},\mathbf{u},\mathbf{x})\big|_{[0,t]}$ is $\overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z) := \sup\limits_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}(y,z)} E_{\overline{P}}\big[\overline{R}(t)\big]$, where $\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}(y,z) := \big\{\overline{P} \in \overline{\mathcal{P}}_{t,\mathbf{x}}(y,z) : \overline{P}\big\{\overline{W}_s = \mathbf{w}(s),\ \forall\, s \in [0,t];\ \overline{U}_s = \mathbf{u}(s) \text{ for a.e. } s \in (0,t)\big\} = 1\big\}$.

3.2 The Equivalence between Strong and Weak Formulation

One of our main results in the next theorem demonstrates that the value $V(t,\mathbf{x},y,z)$ in (2.3) coincides with the weak value $\overline{V}(t,\mathbf{x},y,z)$, and is even equal to $\overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)$.

Theorem 3.1. *Let $(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z) \in [0,\infty) \times \Omega_0 \times \mathbb{J} \times \Omega_X \times \Re \times \Re$. Then $V(t,\mathbf{x},y,z) = \overline{V}(t,\mathbf{x},y,z) = \overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)$, and $\mathcal{C}_{t,\mathbf{x}}(y,z) \ne \emptyset \Leftrightarrow \overline{\mathcal{P}}_{t,\mathbf{x}}(y,z) \ne \emptyset \Leftrightarrow \overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}(y,z) \ne \emptyset$.*

Theorem 3.1 indicates that the value of the SCEC is independent of a specific probabilistic setup and is also indifferent to the Brownian/control history. This result even allows us to deal with the robust case:

Remark 3.3. *Let $\big\{(\mathcal{Q}_\alpha, \mathcal{F}_\alpha, \mathfrak{p}_\alpha)\big\}_{\alpha\in\mathfrak{A}}$ be a family of probability spaces, where $\mathfrak{A}$ is a countable or uncountable index set $\big($e.g. one can consider a non-dominated class $\{\mathfrak{p}_\alpha\}_{\alpha\in\mathfrak{A}}$ of probability measures on a measurable space $(\mathcal{Q},\mathcal{F})\big)$.*

Let $\alpha \in \mathfrak{A}$ and let $\mathcal{B}^\alpha = \big\{\mathcal{B}^\alpha_s\big\}_{s\in[0,\infty)}$ be a $d-$dimensional standard Brownian motion on $(\mathcal{Q}_\alpha, \mathcal{F}_\alpha, \mathfrak{p}_\alpha)$. Given $t \in [0,\infty)$, set $\mathcal{B}^{\alpha,t}_s := \mathcal{B}^\alpha_s - \mathcal{B}^\alpha_t$, $s \in [t,\infty)$, let $\mathcal{U}^\alpha_t$ collect all $\mathbb{U}-$valued, $\mathbf{F}^{\mathcal{B}^{\alpha,t},\mathfrak{p}_\alpha}-$progressively measurable processes $\mu^\alpha = \{\mu^\alpha_s\}_{s\in[t,\infty)}$ and let $\mathcal{S}^\alpha_t$ denote the set of all $[t,\infty]-$valued $\mathbf{F}^{\mathcal{B}^{\alpha,t},\mathfrak{p}_\alpha}-$stopping times. For any $(\mathbf{x},\mu^\alpha) \in \Omega_X \times \mathcal{U}^\alpha_t$, let $\mathcal{X}^{t,\mathbf{x},\mu^\alpha} = \big\{\mathcal{X}^{t,\mathbf{x},\mu^\alpha}_s\big\}_{s\in[0,\infty)}$ be the unique $\big\{\mathcal{F}^{\mathcal{B}^{\alpha,t},\mathfrak{p}_\alpha}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process solving the SDE with the open-loop control μ^α

$$\mathcal{X}_s = \mathbf{x}(t) + \int_t^s b(r, \mathcal{X}_{r\wedge\cdot}, \mu^\alpha_r)dr + \int_t^s \sigma(r, \mathcal{X}_{r\wedge\cdot}, \mu^\alpha_r)d\mathcal{B}^\alpha_r, \ \ \forall\, s \in [t,\infty) \ \text{with initial condition}\ \mathcal{X}\big|_{[0,t]} = \mathbf{x}\big|_{[0,t]}$$

on $\big(\mathcal{Q}_\alpha, \mathcal{F}_\alpha, \mathbf{F}^{\mathcal{B}^{\alpha,t},\mathfrak{p}_\alpha}, \mathfrak{p}_\alpha\big)$.

Then we know from Theorem 3.1 that for any $(t,\mathbf{x}) \in [0,\infty) \times \Omega_X$ and $(y,z) = \big(\{y_i\}_{i\in\mathbb{N}}, \{z_i\}_{i\in\mathbb{N}}\big) \in \Re \times \Re$

$$\overline{V}(t,\mathbf{x},y,z) = \sup_{\alpha\in\mathfrak{A}} \sup_{(\mu^\alpha,\tau_\alpha)\in\mathcal{C}^\alpha_{t,\mathbf{x}}(y,z)} E_{\mathfrak{p}_\alpha}\Big[\int_t^{\tau_\alpha} f\big(r, \mathcal{X}^{t,\mathbf{x},\mu^\alpha}_{r\wedge\cdot}, \mu^\alpha_r\big)dr + \mathbf{1}_{\{\tau_\alpha<\infty\}}\pi\big(\tau_\alpha, \mathcal{X}^{t,\mathbf{x},\mu^\alpha}_{\tau_\alpha\wedge\cdot}\big)\Big],$$

where $\mathcal{C}^\alpha_{t,\mathbf{x}}(y,z) := \big\{(\mu^\alpha,\tau_\alpha) \in \mathcal{U}^\alpha_t \times \mathcal{S}^\alpha_t : E_{\mathfrak{p}_\alpha}\big[\int_t^{\tau_\alpha} g_i(r, \mathcal{X}^{t,\mathbf{x},\mu^\alpha}_{r\wedge\cdot}, \mu^\alpha_r)dr\big] \le y_i,\ E_{\mathfrak{p}_\alpha}\big[\int_t^{\tau_\alpha} h_i(r, \mathcal{X}^{t,\mathbf{x},\mu^\alpha}_{r\wedge\cdot}, \mu^\alpha_r)dr\big] = z_i,\ \forall\, i \in \mathbb{N}\big\}$. To wit, the weak value $\overline{V}(t,\mathbf{x},y,z)$ is also equal to the robust value of the SCEC under model uncertainty.

3.3 A Comparison with Unconstrained Case

The purpose of this subsection is to demonstrate that Theorem 3.1 is not a simple extension of the equivalence result between strong and weak formulation of an (unconstrained) stochastic control/stopping problem obtained in [20].

When $\big(y_i,h_i(\cdot,\cdot),z_i\big)=(\infty,0,0),\ \forall\, i\in\mathbb{N}$, the unconstrained version of Theorem 3.1 states that for any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, $V(t,\mathbf{x}):=\underset{(\mu,\tau)\in\mathcal{U}_t\times\mathcal{S}_t}{\sup} E_{\mathfrak{p}}\big[\int_t^\tau f\big(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r\big)dr+\mathbf{1}_{\{\tau<\infty\}}\pi\big(\tau,\mathcal{X}^{t,\mathbf{x},\mu}_{\tau\wedge\cdot}\big)\big]$ is equal to $\overline{V}(t,\mathbf{x}):=\underset{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}}{\sup} E_{\overline{P}}\big[\,\overline{R}(t)\big]$. On the other hand, [20] showed that for any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, $V(t,\mathbf{x})$ equals $\overline{\overline{V}}(t,\mathbf{x}):=\underset{\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}}{\sup} E_{\overline{P}}\big[\,\overline{R}(t)\big]$, where $\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}$ collects all $\overline{P}\in\mathfrak{P}(\overline{\Omega})$ satisfying (D1), (D2), (D3) and "$\overline{P}\big\{\overline{T}\ge t\big\}=1$" (We summarize [20]'s result in our terms for an easy comparison with our work). As $\overline{\mathcal{P}}_{t,\mathbf{x}}\subset\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}$, the equality $V(t,\mathbf{x})=\underset{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}}{\sup} E_{\overline{P}}\big[\,\overline{R}(t)\big]=\underset{\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}}{\sup} E_{\overline{P}}\big[\,\overline{R}(t)\big]$ indicates that the probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}$'s are more accurate than $\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}$'s to describe the (unconstrained) stochastic control/stopping problem in weak formulation.

The condition (D4) on $\overline{\mathcal{P}}_{t,\mathbf{x}}$ is necessary for the expectation-constraint case. Without it, the weak value $\overline{\overline{V}}(t,\mathbf{x},y,z):=\underset{\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}(y,z)}{\sup} E_{\overline{P}}\big[\,\overline{R}(t)\big]$ $\Big($with $\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}(y,z):=\Big\{\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}:\ E_{\overline{P}}\big[\int_t^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]\le y_i,\ E_{\overline{P}}\big[\int_t^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]=z_i,\ \forall\, i\in\mathbb{N}\Big\}\Big)$ may not be equal to $V(t,\mathbf{x},y,z)$ for the following reason:

In Proposition 4.3 of [20], the key to show $\overline{\overline{V}}(t,\mathbf{x})\le V(t,\mathbf{x})$ or $E_{\overline{P}}\big[\,\overline{R}(t)\big]\le V(t,\mathbf{x})$ for a given $\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}$, relies on transforming the hitting times of process $\big\{E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\in[t,s]\}}\big|\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty\big]\big\}_{s\in[t,\infty)}$ to a member of $\mathcal{S}_t$. More precisely, the so-called *Property* (K) assures an $\mathbf{F}^{W^t,P_0}-$adapted càdlàg process $\widehat{\vartheta}.$ such that $\widehat{\vartheta}_s(\overline{W})=E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\in[t,s]\}}\big|\mathcal{F}^{\overline{W}^t}_s\big]=E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\in[t,s]\}}\big|\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty\big]$, $\overline{P}-$a.s. for any $s\in[t,\infty)$. It follows that $E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\in[t,s]\}}\mathbf{1}_{\{\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_\cdot\in A\}}\big|\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty\big]=\mathbf{1}_{\{\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_\cdot\in A\}}\widehat{\vartheta}_s(\overline{W})=\int_t^s\mathbf{1}_{\{\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}\in A\}}\widehat{\vartheta}(dr,\overline{W})$, $\overline{P}-$a.s. for any $(s,A)\in[t,\infty)\times\mathscr{B}(\Omega_X)$, where $\overline{\mu}=\big\{\overline{\mu}_s=\widehat{\mu}_s(\overline{W})\big\}_{s\in[t,\infty)}$ is the $\mathbb{U}-$valued, $\mathbf{F}^{\overline{W}^t}-$predictable process in (D1) and $\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}=\big\{\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s\big\}_{s\in[0,\infty)}$ is the unique solution of SDE (3.1). Let Φ be a non-negative Borel-measurable function on $[0,\infty)\times\mathbb{J}\times\Omega_X$. Then a standard approximation argument and the "change-of-variable" formula yield that $E_{\overline{P}}\big[\Phi(\overline{T},\overline{U},\overline{X})\big|\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty\big]=\int_t^\infty\Phi(r,\overline{U},\overline{X})\widehat{\vartheta}(dr,\overline{W})=\int_0^1\Phi(\varrho(\overline{W},\lambda),\widehat{\mu}(\overline{W}),\overline{X})d\lambda$, $\overline{P}-$a.s., where $\varrho(\omega_0,\lambda):=\inf\big\{s\in[t,\infty):\ \widehat{\vartheta}_s(\omega_0)>\lambda\big\},\ \forall\,(\omega_0,\lambda)\in\Omega_0\times(0,1)$. Set $\mu_s:=\widehat{\mu}_s(\mathcal{B}),\ \forall\, s\in[t,\infty)$. Since the joint $\overline{P}-$distribution of $\big(\overline{W},\widehat{\mu}.(\overline{W})\big)$ is equal to the joint $\mathfrak{p}-$distribution of $\big(\mathcal{B},\widehat{\mu}.(\mathcal{B})\big)$, we can deduce that the joint $\overline{P}-$distribution of $\big(\overline{W},\overline{\mu}.,\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}\big)$ is equal to the joint $\mathfrak{p}-$distribution of $(\mathcal{B},\mu.,\mathcal{X}^{t,\mathbf{x},\mu})$ and thus

$$E_{\overline{P}}\big[\Phi(\overline{T},\overline{U},\overline{X})\big]=\int_0^1E_{\overline{P}}\big[\Phi\big(\varrho(\overline{W},\lambda),\widehat{\mu}(\overline{W}),\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}\big)\big]d\lambda=\int_0^1E_{\mathfrak{p}}\big[\Phi\big(\varrho(\mathcal{B},\lambda),\widehat{\mu}(\mathcal{B}),\mathcal{X}^{t,\mathbf{x},\mu}\big)\big]d\lambda. \tag{3.4}$$

As $\tau_\lambda:=\varrho(\mathcal{B},\lambda)\in\mathcal{S}_t$ for each $\lambda\in(0,1)$, taking Φ to be the total reward function implies that

$$E_{\overline{P}}\big[\,\overline{R}(t)\big]=\int_0^1E_{\mathfrak{p}}\Big[\int_t^{\tau_\lambda}f\big(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r\big)dr+\mathbf{1}_{\{\tau_\lambda<\infty\}}\pi\big(\tau_\lambda,\mathcal{X}^{t,\mathbf{x},\mu}_{\tau_\lambda\wedge\cdot}\big)\Big]d\lambda\le\int_0^1V(t,\mathbf{x})d\lambda=V(t,\mathbf{x}). \tag{3.5}$$

However, this argument is not applicable to the expectation-constraint case: Given a $\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}(y,z)$, since (μ,τ_λ) may not belong to $\mathcal{C}_{t,\mathbf{x}}(y,z)$ for a.e. $\lambda\in(0,1)$, one can not get $E_{\overline{P}}\big[\,\overline{R}(t)\big]\le V(t,\mathbf{x},y,z)$ like (3.5). Actually, for each $\lambda\in(0,1)$, (μ,τ_λ) is only of $\mathcal{C}_{t,\mathbf{x}}(y_\lambda,z_\lambda)$ with $(y_\lambda,z_\lambda)=\big(\{y^i_\lambda\}_{i\in\mathbb{N}},\{z^i_\lambda\}_{i\in\mathbb{N}}\big)$ and $(y^i_\lambda,z^i_\lambda):=\Big(E_{\mathfrak{p}}\big[\int_t^{\tau_\lambda}g_i(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r)dr\big],E_{\mathfrak{p}}\big[\int_t^{\tau_\lambda}h_i(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r)dr\big]\Big)$. For $i\in\mathbb{N}$, choosing accumulative cost functions for Φ in (3.4) renders that

$$\int_0^1E_{\mathfrak{p}}\Big[\int_t^{\tau_\lambda}g_i(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r)dr\Big]d\lambda=E_{\overline{P}}\bigg[\int_t^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\bigg]\le y_i$$

and similarly $\int_0^1E_{\mathfrak{p}}\big[\int_t^{\tau_\lambda}h_i(r,\mathcal{X}^{t,\mathbf{x},\mu}_{r\wedge\cdot},\mu_r)dr\big]d\lambda=z_i$, so $V\big(t,\mathbf{x},\{\int_0^1y_\lambda d\lambda\}_{i\in\mathbb{N}},\{\int_0^1z_\lambda d\lambda\}_{i\in\mathbb{N}}\big)\le V(t,\mathbf{x},y,z)$. Then the attempt to show $E_{\overline{P}}\big[\,\overline{R}(t)\big]\le V(t,\mathbf{x},y,z)$ reduces to deriving a Jensen-type inequality:

$$\int_0^1V(t,\mathbf{x},y_\lambda,z_\lambda)d\lambda\le V\Big(t,\mathbf{x},\Big\{\int_0^1y_\lambda d\lambda\Big\},\Big\{\int_0^1z_\lambda d\lambda\Big\}\Big).$$

But this does not hold since the value function V is not concave in level z of equality-type expectation constraints.

4 The Measurability of SCEC Values

In this section, using the martingale-problem formulation of controlled SDEs, we characterize the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$ by countably many stochastic behaviors of the canonical coordinators $(\overline{W},\overline{U},\overline{X},\overline{T})$ of $\overline{\Omega}$. This will enable us to analyze the measurability of value functions of the stochastic control/stopping problem with expectation constraints.

Let $\mathfrak{U}$ be the equivalence classes of all $\mathbb{U}-$valued, $\mathbf{F}^{W,P_0}-$predictable processes on Ω_0 in the sense that $\mu^1,\mu^2\in\mathfrak{U}$ are equivalent if $\big\{(s,\omega_0)\in[0,\infty)\times\Omega_0:\mu^1_s(\omega_0)\neq\mu^2_s(\omega_0)\big\}$ is a $ds\times dP_0-$null set. Given $\mu\in\mathfrak{U}$, Fubini Theorem shows that $\mathcal{N}:=\big\{\omega_0\in\Omega_0:\mu_\cdot(\omega_0)\notin\mathbb{J}\big\}$ is an $\mathcal{F}^{W,P_0}_\infty-$measurable set with zero P_0-measure or $\mathcal{N}\in\mathscr{N}_{P_0}\big(\mathcal{F}^W_\infty\big)$. By modifying μ on $\mathcal{N}$, we obtain an $\mathbf{F}^{W,P_0}-$predictable process with all paths in $\mathbb{J}$: $\{\mu_s\mathbf{1}_{\mathcal{N}^c}+u_0\mathbf{1}_{\mathcal{N}}\}_{s\in[0,\infty)}$, which is in the same equivalence class as μ. To wit, one can assume without loss of generality that for all $\mu\in\mathfrak{U}$, $\mu_\cdot(\omega_0)\in\mathbb{J}$ for any $\omega_0\in\Omega_0$. We equip $\mathfrak{U}$ with the topology of local convergence in measure, i.e., the metric

$$\rho_{\mathfrak{U}}(\mu^1,\mu^2):=E_{P_0}\Big[\int_0^\infty e^{-s}\big(1\wedge\rho_{\mathbb{U}}(\mu^1_s,\mu^2_s)\big)ds\Big],\quad \forall\,\mu^1,\,\mu^2\in\mathfrak{U}.$$

Also, let $\mathfrak{S}$ be the equivalence classes of all $[0,\infty]-$valued, $\mathbf{F}^{W,P_0}-$stopping times on Ω_0 in the sense that $\tau_1,\tau_2\in\mathfrak{S}$ are equivalent if $P_0\{\tau_1=\tau_2\}=1$. We endow $\mathfrak{S}$ with the metric

$$\rho_{\mathfrak{S}}(\tau_1,\tau_2):=E_{P_0}\big[\rho_+(\tau_1,\tau_2)\big],\qquad\forall\,\tau_1,\,\tau_2\in\mathfrak{S}.$$

Lemma 4.1. *$\big(\mathfrak{U},\rho_{\mathfrak{U}}\big)$ and $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$ are two complete separable metric spaces, i.e., Polish spaces.*

Proof: We know from Lemma 4.1 of [4] that $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$ is a complete separable metric space. The verification of the complete separable metric space $\big(\mathfrak{U},\rho_{\mathfrak{U}}\big)$ is similar to our demonstration of the complete separable metric space $\big(\mathbb{J},\rho_{\mathbb{J}}\big)$ in the proof of Lemma 1.2, we refer interested readers to the ArXiv version [5] of the current paper for details. □

For any $(\mu,\tau)\in\mathfrak{U}\times\mathfrak{S}$, define their joint distribution with W under P_0 by $\Gamma(\mu,\tau):=P_0\circ(W,\mu,\tau)^{-1}\in\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$.

Lemma 4.2. *The mapping $\Gamma:\mathfrak{U}\times\mathfrak{S}\mapsto\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$ is a continuous injection from $\mathfrak{U}\times\mathfrak{S}$ into $\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$.*

The proof of Lemma 4.2 is relatively lengthy, see Section 6 for it.

For any $(t,\overline{P})\in[0,\infty)\times\mathfrak{P}\big(\overline{\Omega}\big)$, define a probability measure on $\big(\Omega_0\times\mathbb{J}\times\mathbb{T},\mathscr{B}(\Omega_0\times\mathbb{J}\times\mathbb{T})\big)$ by $\overline{Q}_{t,\overline{P}}(D):=\overline{P}\big\{(\overline{\mathscr{W}}^t,\overline{\mathscr{U}}^t,\overline{T}-t)\in D\big\}$, $\forall\,D\in\mathscr{B}(\Omega_0\times\mathbb{J}\times\mathbb{T})$.

Lemma 4.3. *The mapping $[0,\infty)\times\mathfrak{P}\big(\overline{\Omega}\big)\ni(t,\overline{P})\mapsto\overline{Q}_{t,\overline{P}}\in\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$ is continuous.*

Proof: Lemma A.4 implies that $\overline{\Phi}(s,\overline{\omega}):=\big(\overline{\mathscr{W}}^s(\overline{\omega}),\overline{\mathscr{U}}^s(\overline{\omega}),\overline{T}(\overline{\omega})-s\big)=\big(\mathscr{W}^s(\overline{W}(\overline{\omega})),\mathscr{U}^s(\overline{U}(\overline{\omega})),\overline{T}(\overline{\omega})-s\big)$, $\forall\,(s,\overline{\omega})\in[0,\infty)\times\overline{\Omega}$ is a continuous mapping from $[0,\infty)\times\overline{\Omega}$ to $\Omega_0\times\mathbb{J}\times\mathbb{T}$. Then the proof of the lemma is similar to that of [4, Lemma 4.3]. The key is to utilize the continuity of mapping $\overline{\Phi}$ as well as Prohorov's Theorem. We refer interested readers to the ArXiv version [5] of the current paper for details. □

For $t\in[0,\infty)$ and $\varphi\in C^2(\mathbb{R}^{d+l})$, define process

$$\overline{M}^t_s(\varphi):=\varphi\big(\overline{W}^t_s,\overline{X}_s\big)-\int_t^s\overline{b}\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)\cdot D\varphi\big(\overline{W}^t_r,\overline{X}_r\big)dr-\frac{1}{2}\int_t^s\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big):D^2\varphi\big(\overline{W}^t_r,\overline{X}_r\big)dr,\quad\forall\,s\in[t,\infty).$$

We can use Remark 3.1 and Lemma 4.2 to decompose the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$ as the intersection of countably many action sets of processes $(\overline{W},\overline{U},\overline{X},\overline{T})$:

Proposition 4.1. *For any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$ is the intersection of the following three subsets of $\mathfrak{P}\big(\overline{\Omega}\big)$:*
i) $\overline{\mathcal{P}}^1_{t,\mathbf{x}}:=\big\{\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big):\overline{P}\{\overline{X}_s=\mathbf{x}(s),\,\forall\,s\in[0,t]\}=1\big\}$.
ii) $\overline{\mathcal{P}}^2_t:=\big\{\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big):\overline{Q}_{t,\overline{P}}\in\Gamma(\mathfrak{U}\times\mathfrak{S})\big\}$.
iii) $\overline{\mathcal{P}}^3_t:=\Big\{\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big):E_{\overline{P}}\Big[\Big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{t})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\Big)\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^t_{t+s_i},\overline{X}_{t+s_i})\in\mathcal{O}_i\}}\Big]=0,\ \forall\,(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N},\ \forall\,(\mathfrak{s},\mathfrak{t})\in\mathbb{Q}^{2,<}_+,\ \forall\,\{(s_i,\mathcal{O}_i)\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})\Big\}$.

Based on the countable decomposition of the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$ by Proposition 4.1, the next proposition shows that the graph of probability classes $\{\overline{\mathcal{P}}_{t,\mathbf{x}}\}_{(t,\mathbf{x})\in[0,\infty)\times\Omega_X}$ is a Borel subset of $[0,\infty)\times\Omega_X\times\mathfrak{P}(\overline{\Omega})$, which is crucial for the measurability of the value functions $V=\overline{V}$.

Proposition 4.2. *The graph* $\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle:=\big\{\big(t,\mathbf{x},\overline{P}\big)\in[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big):\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}\big\}$ *is a Borel subset of* $[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big)$.

Set $D_{\overline{\mathcal{P}}}:=\big\{(t,\mathbf{x},y,z)\in[0,\infty)\times\Omega_X\times\Re\times\Re:\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)\neq\emptyset\big\}$ and $\mathcal{D}_{\overline{\mathcal{P}}}:=\big\{(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)\in[0,\infty)\times\Omega_0\times\mathbb{J}\times\Omega_X\times\Re\times\Re:\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}(y,z)\neq\emptyset\big\}$.

Corollary 4.1. *The graph* $[\![\overline{\mathcal{P}}]\!]:=\big\{\big(t,\mathbf{x},y,z,\overline{P}\big)\in D_{\overline{\mathcal{P}}}\times\mathfrak{P}\big(\overline{\Omega}\big):\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)\big\}$ *is a Borel subset of* $D_{\overline{\mathcal{P}}}\times\mathfrak{P}\big(\overline{\Omega}\big)$ *and the graph* $\{\!\{\overline{\mathcal{P}}\}\!\}:=\big\{\big(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z,\overline{P}\big)\in\mathcal{D}_{\overline{\mathcal{P}}}\times\mathfrak{P}\big(\overline{\Omega}\big):\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}(y,z)\big\}$ *is a Borel subset of* $\mathcal{D}_{\overline{\mathcal{P}}}\times\mathfrak{P}\big(\overline{\Omega}\big)$.

Proof: 1) Let $\mathfrak{f}$: $(0,\infty)\times\Omega_X\times\mathbb{U}\mapsto[0,\infty]$ be a Borel-measurable function. Taking $\psi(r,\mathfrak{x},\boxminus,u):=\mathfrak{f}(r,\mathfrak{x},u)$, $\forall\,(r,\mathfrak{x},\boxminus,u)\in(0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$ in Lemma 1.3 (2) shows that the mapping $\Psi_{\mathfrak{f}}(t,\mathfrak{s},\omega_X,\mathfrak{u}):=\Psi(t,\mathfrak{s},\mathbf{0},\omega_X,\mathfrak{u})$ $=\int_t^{t+\mathfrak{s}}\psi\big(r,\mathfrak{l}_2(r,\omega_X),0,\omega_X(r),\mathfrak{u}(r)\big)dr=\int_t^{t+\mathfrak{s}}\mathfrak{f}\big(r,\mathfrak{l}_2(r,\omega_X),\mathfrak{u}(r)\big)dr$, $(t,\mathfrak{s},\omega_X,\mathfrak{u})\in[0,\infty)\times[0,\infty)\times\Omega_X\times\mathbb{J}$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$measurable.

Since the random variables $(\overline{X},\overline{U})$ on $\overline{\Omega}$ are $\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$measurable, it follows that the mapping

$$\begin{aligned}\overline{\Psi}_{\mathfrak{f}}(t,\overline{\omega}) &:= \int_t^{\overline{T}(\overline{\omega})\vee t}\mathfrak{f}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}),\overline{U}_r(\overline{\omega})\big)dr=\lim_{n\to\infty}\int_t^{(\overline{T}(\overline{\omega})\wedge n)\vee t}\mathfrak{f}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}),\overline{U}_r(\overline{\omega})\big)dr\\ &= \lim_{n\to\infty}\int_t^{t+(\overline{T}(\overline{\omega})\wedge n-t)^+}\mathfrak{f}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}),\overline{U}_r(\overline{\omega})\big)dr=\lim_{n\to\infty}\Psi_{\mathfrak{f}}\big(t,(\overline{T}(\overline{\omega})\wedge n-t)^+,\overline{X}(\overline{\omega}),\overline{U}(\overline{\omega})\big),\end{aligned}\tag{4.1}$$

$\forall\,(t,\overline{\omega})\in[0,\infty)\times\overline{\Omega}$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\overline{\Omega})-$measurable, and Lemma A.3 of [4] implies that the mapping

$$\widehat{\Psi}_{\mathfrak{f}}(t,\overline{P}):=\int_{\overline{\omega}\in\overline{\Omega}}\overline{\Psi}_{\mathfrak{f}}(t,\overline{\omega})\overline{P}(d\overline{\omega})=E_{\overline{P}}\Big[\int_{\overline{T}\wedge t}^{\overline{T}}\mathfrak{f}\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\Big],\quad\forall\,(t,\overline{P})\in[0,\infty)\times\mathfrak{P}(\overline{\Omega})\tag{4.2}$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable.

Let $i\in\mathbb{N}$. Taking $\mathfrak{f}=g_i^{\pm}$ and $\mathfrak{f}=h_i^{\pm}$ in (4.2) yields that both $\widehat{\Psi}_{g_i}(t,\overline{P}):=E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]$ and $\widehat{\Psi}_{h_i}(t,\overline{P}):=E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}h_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]$, $\forall\,(t,\overline{P})\in[0,\infty)\times\mathfrak{P}(\overline{\Omega})$ are $\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable. Then the set

$$\mathscr{D}:=\big\{(t,\mathbf{x},y,z,\overline{P})\in[0,\infty)\times\Omega_X\times\Re\times\Re\times\mathfrak{P}(\overline{\Omega}):\widehat{\Psi}_{g_i}(t,\overline{P})\le y_i,\ \widehat{\Psi}_{h_i}(t,\overline{P})=z_i,\ \forall\,i\in\mathbb{N}\big\}$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\Re)\otimes\mathscr{B}(\Re)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable. Since $\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle\in\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\Omega_X\big)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)$ by Proposition 4.1, using the projection $\overline{\Pi}_1(t,\mathbf{x},y,z,\overline{P}):=\big(t,\mathbf{x},\overline{P}\big)$ yields that $[\![\overline{\mathcal{P}}]\!]=\big\{(t,\mathbf{x},y,z,\overline{P})\in[0,\infty)\times\Omega_X\times\Re\times\Re\times\mathfrak{P}\big(\overline{\Omega}\big):\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}};\ E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]\le y_i,\ E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}h_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]=z_i,\ \forall\,i\in\mathbb{N}\big\}=\overline{\Pi}_1^{-1}\big(\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle\big)\cap\mathscr{D}$ is a Borel subset of $D_{\overline{\mathcal{P}}}\times\mathfrak{P}\big(\overline{\Omega}\big)$.

2) Since $\mathfrak{l}_1(t,\omega_0):=\omega_0(t\wedge\cdot)$ is continuous in $(t,\omega_0)\in[0,\infty)\times\Omega_0$, the mapping $\Phi_W(t,\mathbf{w},\overline{\omega}):=\mathbf{1}_{\{\mathfrak{l}_1(t,\overline{W}(\overline{\omega}))-\mathfrak{l}_1(t,\mathbf{w})=0\}}$, $(t,\mathbf{w},\overline{\omega})\in[0,\infty)\times\Omega_0\times\overline{\Omega}$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\overline{\Omega})-$measurable.

Taking $\psi(r,\mathfrak{x},\boxminus,u):=e^{-r}\mathscr{I}(u)$, $\forall\,(r,\mathfrak{x},\boxminus,u)\in(0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$ in Lemma 1.3 (2) shows that the mapping $\Psi_{\mathscr{I}}(t,\mathfrak{s},\mathfrak{u}):=\Psi(t,\mathfrak{s},\mathbf{0},\mathbf{0},\mathfrak{u})=\int_t^{t+\mathfrak{s}}\psi\big(r,\mathfrak{l}_2(r,\mathbf{0}),0,0,\mathfrak{u}(r)\big)dr=\int_t^{t+\mathfrak{s}}e^{-r}\mathscr{I}\big(\mathfrak{u}(r)\big)dr$, $(t,\mathfrak{s},\mathfrak{u})\in[0,\infty)\times[0,\infty)\times\mathbb{J}$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}[0,\infty)\otimes\mathscr{B}(\mathbb{J})-$measurable. As the random variable $\overline{U}$ on $\overline{\Omega}$ is $\mathscr{B}(\overline{\Omega})/\mathscr{B}(\mathbb{J})-$measurable, we can deduce that the mapping $\overline{\Psi}_{\mathscr{I}}\big(t,\mathfrak{u},\overline{\omega}\big):=\prod_{q\in\mathbb{Q}_+}\mathbf{1}_{\{\Psi_{\mathscr{I}}(0,t\wedge q,\overline{U}(\overline{\omega}))-\Psi_{\mathscr{I}}(0,t\wedge q,\mathfrak{u})=0\}}$, $(t,\mathfrak{u},\overline{\omega})\in[0,\infty)\times\mathbb{J}\times\overline{\Omega}$ is $\mathscr{B}[0,\infty)\times\mathscr{B}(\mathbb{J})\times\mathscr{B}(\overline{\Omega})-$measurable. Then an application of Lemma A.3 of [4] again renders that the mapping

$$\overline{\mathfrak{Y}}(t,\mathbf{w},\mathbf{u},\overline{P}):=\int_{\overline{\omega}\in\overline{\Omega}}\Phi_W(t,\mathbf{w},\overline{\omega})\,\overline{\Psi}_{\mathscr{I}}\big(t,\mathbf{u},\overline{\omega}\big)\overline{P}(d\overline{\omega})=\overline{P}\big\{\overline{W}_s=\mathbf{w}(s),\ \forall\,s\in[0,t];\ \overline{U}_s=\mathbf{u}(s)\text{ for a.e. }s\in(0,t)\big\},$$

$\forall\,(t,\mathbf{w},\mathbf{u},\overline{P})\in[0,\infty)\times\Omega_0\times\mathbb{J}\times\mathfrak{P}(\overline{\Omega})$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\mathbb{J})\otimes\mathscr{B}(\mathfrak{P}(\overline{\Omega}))-$measurable. Here we used the fact that $\Psi_{\mathscr{I}}(0,t\wedge q,\overline{U}(\overline{\omega}))=\Psi_{\mathscr{I}}(0,t\wedge q,\mathbf{u})$, $\forall\,q\in\mathbb{Q}_+$ iff $\int_0^s e^{-r}\mathscr{I}\big(\overline{U}_r(\overline{\omega})\big)dr=\int_0^s e^{-r}\mathscr{I}\big(\mathbf{u}(r)\big)dr$, $\forall\,s\in[0,t]$ iff $\overline{U}_s(\overline{\omega})=\mathbf{u}(s)$ for a.e. $s\in(0,t)$. By the projections $\overline{\Pi}_2(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z,\overline{P}):=\big(t,\mathbf{x},\overline{P}\big)$, $\overline{\Pi}_3(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z,\overline{P}):=\big(t,\mathbf{x},y,z,\overline{P}\big)$ and $\overline{\Pi}_4(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z,\overline{P}):=(t,\mathbf{w},\mathbf{u},\overline{P})$, we can derive that $\{\!\{\overline{\mathcal{P}}\}\!\}=\overline{\Pi}_2^{-1}\big(\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle\big)\cap\overline{\Pi}_3^{-1}(\mathscr{D})\cap\overline{\Pi}_4^{-1}\big(\overline{\mathfrak{Y}}^{-1}(1)\big)$ is a Borel subset of $\mathcal{D}_{\overline{\mathcal{P}}}\times\mathfrak{P}\big(\overline{\Omega}\big)$. □

By Corollary 4.1, the value function $\overline{V}$ is upper semi-analytic and is thus universally measurable.

Theorem 4.1. *The value function $\overline{V}(t,\mathbf{x},y,z)$ is upper semi-analytic on $D_{\overline{\mathcal{P}}}$ and the value function $\overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)$ is upper semi-analytic on $\mathcal{D}_{\overline{\mathcal{P}}}$.*

Proof: Since the measurability of functions π and $\mathfrak{l}_2$ shows that $\varpi(s,\omega_X):=\mathbf{1}_{\{s<\infty\}}\pi\big(s,\mathfrak{l}_2(s,\omega_X)\big)$, $(s,\omega_X)\in[0,\infty]\times\Omega_X$ is $\mathscr{B}[0,\infty]\otimes\mathscr{B}(\Omega_X)-$measurable, taking $\mathfrak{f}=f^{\pm}$ in (4.1) renders that the mapping

$$\overline{\Psi}_{f,\pi}(t,\overline{\omega}):=\int_t^{\overline{T}(\overline{\omega})\vee t} f\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}),\overline{U}_r(\overline{\omega})\big)dr+\varpi\big(\overline{T}(\overline{\omega}),\overline{X}(\overline{\omega})\big),\quad \forall\,(t,\overline{\omega})\in[0,\infty)\times\overline{\Omega}$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\overline{\Omega})-$measurable. Lemma A.3 of [4] implies that $\overline{\mathscr{V}}(t,\overline{P}):=\int_{\overline{\omega}\in\overline{\Omega}}\overline{\Psi}_{f,\pi}(t,\overline{\omega})\overline{P}(d\overline{\omega})$, $(t,\overline{P})\in[0,\infty)\times\mathfrak{P}(\overline{\Omega})$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable. Then Corollary 4.1 and Proposition 7.47 of [7] yield that $\overline{V}(t,\mathbf{x},y,z)=\sup\limits_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}\overline{\mathscr{V}}(t,\overline{P})=\sup\limits_{(t,\mathbf{x},y,z,\overline{P})\in[[\overline{\mathcal{P}}]]}\overline{\mathscr{V}}(t,\overline{P})$ is upper semi-analytic on $D_{\overline{\mathcal{P}}}$ and $\overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)=\sup\limits_{(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z,\overline{P})\in\{\{\overline{\mathcal{P}}\}\}}\overline{\mathscr{V}}(t,\overline{P})$ is upper semi-analytic on $\mathcal{D}_{\overline{\mathcal{P}}}$. □

5 Dynamic Programming Principle for $\overline{V}$

In this section, we explore a dynamic programming principles (DPP) for the value function $\overline{V}$ in weak formulation, which takes the conditional expected integrals of constraint functions as additional states.

Given $t\in[0,\infty)$, let $\overline{\gamma}$ be a $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping time and let $\overline{P}\in\mathfrak{P}(\overline{\Omega})$. According to Lemma 1.3.3 and Theorem 1.1.8 of [45], $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ is countably generated and there thus exists a family $\{\overline{P}^t_{\overline{\gamma},\overline{\omega}}\}_{\overline{\omega}\in\overline{\Omega}}$ of probability measures in $\mathfrak{P}(\overline{\Omega})$, called the *regular conditional probability distribution* (r.c.p.d.) of $\overline{P}$ with respect to $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, such that

(R1) for any $\overline{A}\in\mathscr{B}(\overline{\Omega})$, the mapping $\overline{\omega}\mapsto\overline{P}^t_{\overline{\gamma},\overline{\omega}}(\overline{A})$ is $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable;

(R2) for any $(-\infty,\infty]-$valued, $\mathscr{B}_{\overline{P}}(\overline{\Omega})-$measurable random variable $\overline{\xi}$ that is bounded from below under $\overline{P}$, it holds for any $\overline{\omega}\in\overline{\Omega}$ except on a $\overline{\mathcal{N}}_{\overline{\xi}}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ that $\overline{\xi}$ is $\mathscr{B}_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}(\overline{\Omega})-$measurable and $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\,\overline{\xi}\,\big]=E_{\overline{P}}\big[\,\overline{\xi}\,\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})$;

(R3) for some $\overline{\mathcal{N}}_0\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}(\overline{A})=\mathbf{1}_{\{\overline{\omega}\in\overline{A}\}}$, $\forall\,(\overline{\omega},\overline{A})\in\overline{\mathcal{N}}^c_0\times\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$.

Let $\overline{\omega}\in\overline{\Omega}$ and set $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}:=\big\{\overline{\omega}'\in\overline{\Omega}\colon\overline{W}^t_r(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}),\ \forall\,r\in[t,\overline{\gamma}(\overline{\omega})]\big\}$. We know from Galmarino's test that

$$\overline{\gamma}(\overline{\omega}')=\overline{\gamma}(\overline{\omega}),\quad \forall\,\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}},\tag{5.1}$$

and $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}$ is thus $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable. Since $\overline{\omega}\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}$ for any $\overline{\omega}\in\overline{\Omega}$, (R3) shows that

$$\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\big)=\mathbf{1}_{\big\{\overline{\omega}\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\big\}}=1,\quad \forall\,\overline{\omega}\in\overline{\mathcal{N}}^c_0.\tag{5.2}$$

For any $i\in\mathbb{N}$, define $\overline{Y}^i_{\overline{P}}(\overline{\gamma}):=E_{\overline{P}}\Big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big]$ and $\overline{Z}^i_{\overline{P}}(\overline{\gamma}):=E_{\overline{P}}\Big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big]$. So $\big(\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big):=\Big(\big\{\overline{Y}^i_{\overline{P}}(\overline{\gamma})\big\}_{i\in\mathbb{N}},\big\{\overline{Z}^i_{\overline{P}}(\overline{\gamma})\big\}_{i\in\mathbb{N}}\Big)$ is an $\Re\times\Re-$valued $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable random variable.

In terms of the r.c.p.d. $\big\{\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\}_{\overline{\omega}\in\overline{\Omega}}$, the probability class $\big\{\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)\colon(t,\mathbf{x},y,z)\in D_{\overline{\mathcal{P}}}\big\}$ is stable under conditioning as follows. It will play an important role in deriving the sub-solution side of the DPP for $\overline{V}$.

Proposition 5.1. *Given $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, let $\overline{\gamma}$ be a $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping time and let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}$. There exists a $\overline{P}-$null set $\overline{\mathcal{N}}$ such that*

$$\overline{P}^t_{\overline{\gamma},\overline{\omega}}\in\overline{\mathcal{P}}_{\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\Big(\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big),\quad \forall\,\overline{\omega}\in\big\{\overline{T}\ge\overline{\gamma}\big\}\cap\overline{\mathcal{N}}^c.\tag{5.3}$$

Now, we are ready to present a dynamic programming principle in weak formulation for the value function $\overline{V}$, in which $\big(\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)$ act as additional states for constraint levels at the intermediate horizon $\overline{\gamma}$.

Theorem 5.1. *Given $(t,\mathbf{x},y,z)\in D_{\overline{\mathcal{P}}}$, let $\big\{\overline{\gamma}_{\overline{P}}\big\}_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}$ be a family of $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping times. Then*

$$\begin{aligned}\overline{V}(t,\mathbf{x},y,z)=\sup_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}E_{\overline{P}}\bigg[&\mathbf{1}_{\{\overline{T}<\overline{\gamma}_{\overline{P}}\}}\Big(\int_t^{\overline{T}}f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big)\\&+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}_{\overline{P}}\}}\Big(\int_t^{\overline{\gamma}_{\overline{P}}}f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\overline{V}\Big(\overline{\gamma}_{\overline{P}},\overline{X}_{\overline{\gamma}_{\overline{P}}\wedge\cdot},\overline{Y}_{\overline{P}}\big(\overline{\gamma}_{\overline{P}}\big),\overline{Z}_{\overline{P}}\big(\overline{\gamma}_{\overline{P}}\big)\Big)\Big)\bigg].\end{aligned}$$

Based on Theorem 3.1 and Theorem 5.1, we plan to show in our sequel projects that the SCEC value function $V(t,\mathbf{x},y,z)=\overline{V}(t,\mathbf{x},y,z)$ solves the corresponding path-dependent Hamilton-Jacobi-Bellman equations in a viscosity sense. Then we will derive an effective numerical scheme for the SCEC value function $V(t,\mathbf{x},y,z)$ and implement it for practical examples including some SCEC problems with state/control delay in dynamics.

6 Proofs

Proof of Lemma 1.2: 1) *We first show that $\mathbb{J}$ is a complete separable space under the metric*

$$\rho_{\mathbb{J}}(\mathfrak{u},\mathfrak{u}'):=\int_0^\infty e^{-s}\big(1\wedge\rho_{\mathbb{U}}(\mathfrak{u}(s),\mathfrak{u}'(s))\big)ds,\quad \forall\,\mathfrak{u},\mathfrak{u}'\in\mathbb{J}.$$

1a) Let $\{\mathfrak{u}_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\big(\mathbb{J},\rho_{\mathbb{J}}\big)$. *We shall construct in* (6.1) *a limit $\mathfrak{u}_*\in\mathbb{J}$ of the sequence $\{\mathfrak{u}_n\}_{n\in\mathbb{N}}$.*

There exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\rho_{\mathbb{J}}\big(\mathfrak{u}_{n_k},\mathfrak{u}_{n_{k+1}}\big)<2^{-k}$, $\forall\,k\in\mathbb{N}$. Fix $k\in\mathbb{N}$. Since $1\wedge(a+b)\le 1\wedge a+1\wedge b$, $\forall\,a,b\in[0,\infty)$, it holds for any $\ell\in\mathbb{N}$ that $1\wedge\rho_{\mathbb{U}}(\mathfrak{u}_{n_k}(s),\mathfrak{u}_{n_{k+\ell}}(s))\le\sum_{i=1}^{\ell}1\wedge\rho_{\mathbb{U}}(\mathfrak{u}_{n_{k+i-1}}(s),\mathfrak{u}_{n_{k+i}}(s))$, $s\in(0,\infty)$, taking $\sup\limits_{\ell\in\mathbb{N}}$ yields that

$$1\wedge\Big(\sup_{\ell\in\mathbb{N}}\rho_{\mathbb{U}}\big(\mathfrak{u}_{n_k}(s),\mathfrak{u}_{n_{k+\ell}}(s)\big)\Big)=\sup_{\ell\in\mathbb{N}}\Big(1\wedge\rho_{\mathbb{U}}\big(\mathfrak{u}_{n_k}(s),\mathfrak{u}_{n_{k+\ell}}(s)\big)\Big)\le\sum_{i\in\mathbb{N}}1\wedge\rho_{\mathbb{U}}\big(\mathfrak{u}_{n_{k+i-1}}(s),\mathfrak{u}_{n_{k+i}}(s)\big),\quad s\in(0,\infty).$$

The monotone convergence theorem then implies that $\int_0^\infty e^{-s}\Big(1\wedge\Big(\sup\limits_{\ell\in\mathbb{N}}\rho_{\mathbb{U}}(\mathfrak{u}_{n_k}(s),\mathfrak{u}_{n_{k+\ell}}(s))\Big)\Big)ds\le\sum_{i\in\mathbb{N}}\rho_{\mathbb{J}}\big(\mathfrak{u}_{n_{k+i-1}},\mathfrak{u}_{n_{k+i}}\big)\le\sum_{i\in\mathbb{N}}2^{-k-i+1}=2^{-k+1}$. So $\big\{n_k\big\}_{k\in\mathbb{N}}$ has a subsequence $\big\{n_{k_m}\big\}_{m\in\mathbb{N}}$ such that $\lim\limits_{m\to\infty}e^{-s}\Big(1\wedge\Big(\sup\limits_{\ell\in\mathbb{N}}\rho_{\mathbb{U}}(\mathfrak{u}_{n_{k_m}}(s),\mathfrak{u}_{n_{k_m+\ell}}(s))\Big)\Big)=0$ or $\lim\limits_{m\to\infty}\Big(\sup\limits_{\ell\in\mathbb{N}}\rho_{\mathbb{U}}\big(\mathfrak{u}_{n_{k_m}}(s),\mathfrak{u}_{n_{k_m+\ell}}(s)\big)\Big)=0$ for all $s\in(0,\infty)$ except on a $ds-$null set $\mathcal{N}$ of $(0,\infty)$. Given $s\in(0,\infty)\backslash\mathcal{N}$, one has $\lim\limits_{m\to\infty}\bigg(\sup\limits_{j\in\mathbb{N}}\rho_{\mathbb{U}}\big(\mathfrak{u}_{n_{k_m}}(s),\mathfrak{u}_{n_{k_{j+m}}}(s)\big)\bigg)=0$, i.e., $\big\{\mathfrak{u}_{n_{k_m}}(s)\big\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{U}$. Let $\mathfrak{u}_o(s)$ be the limit of $\big\{\mathfrak{u}_{n_{k_m}}(s)\big\}_{m\in\mathbb{N}}$ in $\big(\mathbb{U},\rho_{\mathbb{U}}\big)$.

Define $\underline{\mu}(s):=\varliminf\limits_{m\to\infty}\mathscr{I}\big(\mathfrak{u}_{n_{k_m}}(s)\big)\in[0,1]$, $s\in[0,\infty)$, which is a Borel measurable function on $[0,\infty)$. Then

$$\mathfrak{u}_*(s):=\mathscr{I}^{-1}\big(\underline{\mu}(s)\big)\mathbf{1}_{\{\underline{\mu}(s)\in\mathfrak{E}\}}+u_0\mathbf{1}_{\{\underline{\mu}(s)\notin\mathfrak{E}\}},\quad s\in[0,\infty)\tag{6.1}$$

is a $\mathbb{U}-$valued Borel measurable function on $[0,\infty)$, i.e., $\mathfrak{u}_*\in\mathbb{J}$. For $s\in(0,\infty)\backslash\mathcal{N}$, the continuity of mapping $\mathscr{I}$ shows $\underline{\mu}(s)=\lim\limits_{m\to\infty}\mathscr{I}\big(\mathfrak{u}_{n_{k_m}}(s)\big)=\mathscr{I}\big(\mathfrak{u}_o(s)\big)\in\mathfrak{E}$, so $\mathfrak{u}_*(s)=\mathscr{I}^{-1}\big(\underline{\mu}(s)\big)=\mathfrak{u}_o(s)$. The dominated convergence theorem implies $\lim\limits_{m\to\infty}\rho_{\mathbb{J}}\big(\mathfrak{u}_{n_{k_m}},\mathfrak{u}_*\big)=\lim\limits_{m\to\infty}\int_0^\infty\mathbf{1}_{\{s\in\mathcal{N}^c\}}e^{-s}\big(1\wedge\rho_{\mathbb{U}}(\mathfrak{u}_{n_{k_m}}(s),\mathfrak{u}_o(s))\big)ds=0$.

Let $\varepsilon\in(0,1)$. There exists an $N\in\mathbb{N}$ such that $\rho_{\mathbb{J}}\big(\mathfrak{u}_{\mathfrak{n}},\mathfrak{u}_{\mathfrak{n}'}\big)<\varepsilon/2$ for any $\mathfrak{n},\mathfrak{n}'\ge N$. We can also find a $m\in\mathbb{N}$ such that $n_{k_m}\ge N$ and that $\rho_{\mathbb{J}}\big(\mathfrak{u}_{n_{k_m}},\mathfrak{u}_*\big)<\varepsilon/2$. It then holds for any $n\in\mathbb{N}$ with $n\ge N$ that $\rho_{\mathbb{J}}\big(\mathfrak{u}_n,\mathfrak{u}_*\big)\le\rho_{\mathbb{J}}\big(\mathfrak{u}_n,\mathfrak{u}_{n_{k_m}}\big)+\rho_{\mathbb{J}}\big(\mathfrak{u}_{n_{k_m}},\mathfrak{u}_*\big)<\varepsilon$. Hence, $\mathfrak{u}_*$ is the limit of $\{\mathfrak{u}_n\}_{n\in\mathbb{N}}$ in $\big(\mathbb{J},\rho_{\mathbb{J}}\big)$.

1b) *In this step, we demonstrate that $\mathbb{J}$ is separable under $\rho_{\mathbb{J}}$.*

(i) *We first construct a dense subset $\widehat{\mathbf{C}}$ of $\mathbb{J}$ under $\rho_{\mathbb{J}}$*: Let $\{u_i\}_{i\in\mathbb{N}}$ be a countable dense subset of $\big(\mathbb{U},\rho_{\mathbb{U}}\big)$ and let $\{O_i\}_{i\in\mathbb{N}}$ be a countable base of the Euclidean topology on $[0,\infty)$.

Given $n\in\mathbb{N}$, let us enumerate the 2^n elements of $\big\{\underset{i\in I}{\cup}O_i: I\subset\{1,\cdots,n\}\big\}$ by $\big\{\breve{O}^n_1,\cdots,\breve{O}^n_{2^n}\big\}$ and consider the following countable collections of $\mathscr{B}[0,\infty)/\mathscr{B}[-1,1]-$measurable functions on $[0,\infty)$:

$$\mathscr{C}_n:=\Big\{\mathscr{I}(u_j)\mathbf{1}_{\breve{O}^n_i}-\mathbf{1}_{(\breve{O}^n_i)^c}: i\in\{1,\cdots,2^n\},\ j\in\mathbb{N}\Big\},\quad \widetilde{\mathscr{C}}_n:=\Big\{\max_{i=1,\cdots,k}\mathfrak{x}_i: k\in\mathbb{N},\ \{\mathfrak{x}_1,\cdots,\mathfrak{x}_k\}\subset\mathscr{C}_n\Big\}.$$

Clearly, each $\mathfrak{x}\in\widetilde{\mathscr{C}}_n$ takes values in a finite subset of $\big\{\mathscr{I}(u_i)\big\}_{i\in\mathbb{N}}\cup\{-1\}$. By additionally assigning $\mathscr{I}^{-1}(-1):=u_0$, one has $\widetilde{\mathbf{C}}:=\big\{\mathscr{I}^{-1}(\mathfrak{x}):\mathfrak{x}\in\widetilde{\mathscr{C}}_n$ for some $n\in\mathbb{N}\big\}\subset\big\{\{u_i\}_{i=0}^\infty-$valued $\mathscr{B}[0,\infty)-$measurable functions$\big\}=:\widehat{\mathbf{C}}$.

We claim that $\widehat{\mathbf{C}}$ is a dense subset of $\mathbb{J}$ under $\rho_{\mathbb{J}}$: To see this, for any $i,n\in\mathbb{N}$ we set $o^n_i:=\big\{u\in\mathbb{U}:\rho_{\mathbb{U}}(u,u_i)<2^{-n}\big\}$ as the open ball centered at u_i with radius 2^{-n}. We also set $\widetilde{o}^n_1:=o^n_1$ and $\widetilde{o}^n_i:=o^n_i\Big\backslash\Big(\underset{j<i}{\cup}o^n_j\Big)$ for $i\ge2$. Given $\mathfrak{u}\in\mathbb{J}$ and

$n\in\mathbb{N}$, define a member of $\widehat{\mathbf{C}}$ by $\mathfrak{u}_n(s):=\sum_{i\in\mathbb{N}}\mathbf{1}_{\{s\in\mathcal{E}_i^n\}}u_i$, $s\in[0,\infty)$, where $\mathcal{E}_i^n:=\big\{s\in[0,\infty):\mathfrak{u}(s)\in\widetilde{o}_i^n\big\}\in\mathscr{B}[0,\infty)$. As $\rho_{\mathbb{U}}\big(\mathfrak{u}_n(s),\mathfrak{u}(s)\big)=\sum_{i\in\mathbb{N}}\mathbf{1}_{\{s\in\mathcal{E}_i^n\}}\rho_{\mathbb{U}}\big(u_i,\mathfrak{u}(s)\big)<2^{-n}$ for any $s\in[0,\infty)$, one has $\rho_{\mathbb{J}}\big(\mathfrak{u}_n,\mathfrak{u}\big)=\int_0^\infty e^{-s}\big(1\wedge\rho_{\mathbb{U}}(\mathfrak{u}_n(s),\mathfrak{u}(s))\big)ds\le 2^{-n}$. So $\widehat{\mathbf{C}}$ is a dense subset of $\mathbb{J}$ under $\rho_{\mathbb{J}}$.

(ii) *We then show that the countable collection $\widetilde{\mathbf{C}}$ is dense in $\widehat{\mathbf{C}}$ and is thus dense in $\mathbb{J}$ under $\rho_{\mathbb{J}}$.*

Let $\widehat{\mathfrak{u}}\in\widehat{\mathbf{C}}$ and $\varepsilon\in(0,1)$. We set $\widehat{\mathcal{E}}_i=\big\{s\in[0,\infty):\widehat{\mathfrak{u}}(s)=u_i\big\}\in\mathscr{B}[0,\infty)$, $\forall\, i\in\mathbb{N}$. Since $\lambda(\mathcal{E}):=\int_{s\in\mathcal{E}}e^{-s}ds$ is a probability measure on $\big([0,\infty),\mathscr{B}[0,\infty)\big)$, there exists $N\in\mathbb{N}$ such that $\lambda\Big(\underset{i>N}{\cup}\widehat{\mathcal{E}}_i\Big)<\varepsilon/3$.

Given $i=1,\cdots,N$, Proposition 7.17 of [7] shows that $\lambda(\mathcal{O}_i\backslash\widehat{\mathcal{E}}_i)<\frac{\varepsilon}{3N}$ for some open subset $\mathcal{O}_i$ of $[0,\infty)$ containing $\widehat{\mathcal{E}}_i$. Since $\mathcal{O}_i=\underset{n\in\mathbb{N}}{\cup}O_{\ell_n^i}$ for some subsequence $\big\{\ell_n^i\big\}_{n\in\mathbb{N}}$ of $\mathbb{N}$, one can find $n_i\in\mathbb{N}$ such that $\lambda\big(\mathcal{O}_i\backslash\breve{\mathcal{O}}_i\big)<\frac{\varepsilon}{3N}$, where $\breve{\mathcal{O}}_i:=\overset{n_i}{\underset{n=1}{\cup}}O_{\ell_n^i}\in\mathscr{B}[0,\infty)$. As $\breve{\mathcal{O}}_i\in\big\{\breve{O}_j^{\mathfrak{n}}\big\}_{j=1}^{2^{\mathfrak{n}}}$ for $\mathfrak{n}:=\underset{i=1,\cdots,N}{\max}\ell_{n_i}^i$, we see that $\mathfrak{x}_i:=\mathscr{I}(u_i)\mathbf{1}_{\breve{\mathcal{O}}_i}-\mathbf{1}_{\breve{\mathcal{O}}_i^c}$ belongs to $\mathscr{C}_{\mathfrak{n}}$. Define $\mathfrak{u}:=\mathscr{I}^{-1}\Big(\underset{i=1,\cdots,N}{\max}\mathfrak{x}_i\Big)\in\widetilde{\mathbf{C}}$.

For $i=1,\cdots,N$, if $\mathcal{A}_i:=\big(\widehat{\mathcal{E}}_i\cap\breve{\mathcal{O}}_i\big)\backslash\Big(\underset{j\le N;j\ne i}{\cup}\breve{\mathcal{O}}_j\Big)\in\mathscr{B}[0,\infty)$ is not empty, it holds for any $s\in\mathcal{A}_i$ that $\mathfrak{x}_j(s)=\mathbf{1}_{\{j=i\}}\mathscr{I}(u_i)-\mathbf{1}_{\{j\ne i\}}$ for $j\in\{1,\cdots,N\}$ and thus $\mathfrak{u}(s)=\mathscr{I}^{-1}\big(\mathfrak{x}_i(s)\big)=u_i=\widehat{\mathfrak{u}}(s)$. Also, if $\mathcal{A}_0:=\Big(\underset{i\in\mathbb{N}}{\cup}\widehat{\mathcal{E}}_i\Big)^c\bigcap\Big(\overset{N}{\underset{j=1}{\cup}}\breve{\mathcal{O}}_j\Big)^c$ is not empty, it holds for any $s\in\mathcal{A}_0$ that $\mathfrak{u}(s)=\mathscr{I}^{-1}(-1)=u_0=\widehat{\mathfrak{u}}(s)$. Then $\rho_{\mathbb{J}}\big(\mathfrak{u},\widehat{\mathfrak{u}}\big)=\int_{s\in\mathcal{A}}e^{-s}\big(1\wedge\rho_{\mathbb{U}}(\mathfrak{u}(s),\widehat{\mathfrak{u}}(s))\big)ds$ for $\mathcal{A}:=\Big(\overset{N}{\underset{i=0}{\cup}}\mathcal{A}_i\Big)^c=\Big(\overset{N}{\underset{i=1}{\cup}}(\widehat{\mathcal{E}}_i\backslash\mathcal{A}_i)\Big)\bigcup\Big(\underset{i>N}{\cup}\widehat{\mathcal{E}}_i\Big)\bigcup\bigg(\Big(\underset{i\in\mathbb{N}}{\cup}\widehat{\mathcal{E}}_i\Big)^c\bigcap\Big(\overset{N}{\underset{j=1}{\cup}}\breve{\mathcal{O}}_j\Big)\bigg)\subset[0,\infty)$. Since

$$\widehat{\mathcal{E}}_i\backslash\mathcal{A}_i=\big(\widehat{\mathcal{E}}_i\cap\breve{\mathcal{O}}_i^c\big)\bigcup\Big(\underset{j\le N;j\ne i}{\cup}\big(\widehat{\mathcal{E}}_i\cap\breve{\mathcal{O}}_i\cap\breve{\mathcal{O}}_j\big)\Big)\subset\big(\mathcal{O}_i\backslash\breve{\mathcal{O}}_i\big)\bigcup\Big(\underset{j\le N;j\ne i}{\cup}\breve{\mathcal{O}}_j\backslash\widehat{\mathcal{E}}_j\Big),\quad i=1,\cdots,N$$

and since $\Big(\underset{i\in\mathbb{N}}{\cup}\widehat{\mathcal{E}}_i\Big)^c\bigcap\Big(\overset{N}{\underset{j=1}{\cup}}\breve{\mathcal{O}}_j\Big)=\overset{N}{\underset{j=1}{\cup}}\Big(\breve{\mathcal{O}}_j\cap\Big(\underset{i\in\mathbb{N}}{\cup}\widehat{\mathcal{E}}_i\Big)^c\Big)\subset\overset{N}{\underset{j=1}{\cup}}\big(\breve{\mathcal{O}}_j\cap\widehat{\mathcal{E}}_j^c\big)\subset\overset{N}{\underset{j=1}{\cup}}\big(\mathcal{O}_j\backslash\widehat{\mathcal{E}}_j\big)$, we can deduce that $\mathcal{A}\subset\Big(\overset{N}{\underset{i=1}{\cup}}\mathcal{O}_i\backslash\breve{\mathcal{O}}_i\Big)\bigcup\Big(\overset{N}{\underset{i=1}{\cup}}\mathcal{O}_i\backslash\widehat{\mathcal{E}}_i\Big)\bigcup\Big(\underset{i>N}{\cup}\widehat{\mathcal{E}}_i\Big)$. It follows that $\rho_{\mathbb{J}}\big(\mathfrak{u},\widehat{\mathfrak{u}}\big)\le\lambda(\mathcal{A})<\varepsilon$. Namely, the countable collection $\widetilde{\mathbf{C}}$ is dense in $\widehat{\mathbf{C}}$ under $\rho_{\mathbb{J}}$.

Therefore, $\big(\mathbb{J},\rho_{\mathbb{J}}\big)$ is a complete separable metric space.

2) *Next, we demonstrate that the mapping $\mathfrak{i}_{\mathbb{J}}\colon\big(\mathbb{J},\rho_{\mathbb{J}}\big)\mapsto\Big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big),\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)\big)\Big)$ is a continuous injection.*

Let $\mathfrak{u},\mathfrak{u}'\in\mathbb{J}$ such that $\mathfrak{i}_{\mathbb{J}}(\mathfrak{u})=\mathfrak{i}_{\mathbb{J}}(\mathfrak{u}')$. Since $\int_0^t e^{-s}\mathscr{I}\big(\mathfrak{u}(s)\big)ds=\int_0^\infty\int_{\mathbb{U}}\mathbf{1}_{\{s\le t\}}\mathscr{I}(u)\mathfrak{i}_{\mathbb{J}}(\mathfrak{u})(ds,du)=\int_0^\infty\int_{\mathbb{U}}\mathbf{1}_{\{s\le t\}}\mathscr{I}(u)\mathfrak{i}_{\mathbb{J}}(\mathfrak{u}')(ds,du)=\int_0^t e^{-s}\mathscr{I}\big(\mathfrak{u}'(s)\big)ds$ for any $t\in(0,\infty)$, we see that $e^{-s}\mathscr{I}\big(\mathfrak{u}(s)\big)=e^{-s}\mathscr{I}\big(\mathfrak{u}'(s)\big)$ and thus $\mathfrak{u}(s)=\mathfrak{u}'(s)$ for a.e. $s\in(0,\infty)$, namely, $\mathfrak{u}=\mathfrak{u}'$ in $\mathbb{J}$. So $\mathfrak{i}_{\mathbb{J}}$ is an injection.

Let $\{\mathfrak{u}_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{J}$ converging to a $\mathfrak{u}\in\mathbb{J}$ under $\rho_{\mathbb{J}}$. We show that $\mathfrak{i}_{\mathbb{J}}(\mathfrak{u}_n)$ converges to $\mathfrak{i}_{\mathbb{J}}(\mathfrak{u})$ under $\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)\big)$: By e.g. Lemma 7.6 of [7], this is equivalent to verify that

$$\lim_{n\to\infty}\int_0^\infty\int_{\mathbb{U}}\phi(s,u)\mathfrak{i}_{\mathbb{J}}(\mathfrak{u}_n)(ds,du)=\int_0^\infty\int_{\mathbb{U}}\phi(s,u)\mathfrak{i}_{\mathbb{J}}(\mathfrak{u})(ds,du)\tag{6.2}$$

for any bounded continuous function $\phi\colon[0,\infty)\times\mathbb{U}\mapsto\mathbb{R}$.

Let ϕ be such a continuous function on $[0,\infty)\times\mathbb{U}$. For the limit (6.2), it suffices to show that any subsequence $\big\{\mathfrak{u}_{n_k}\big\}_{k\in\mathbb{N}}$ of $\{\mathfrak{u}_n\}_{n\in\mathbb{N}}$ has in turn a subsequence $\big\{\mathfrak{u}_{n_k'}\big\}_{k\in\mathbb{N}}$ satisfying (6.2): As $\lim\limits_{k\to\infty}\int_0^\infty e^{-s}\big(1\wedge\rho_{\mathbb{U}}(\mathfrak{u}_{n_k}(s),\mathfrak{u}(s))\big)ds=\lim\limits_{k\to\infty}\rho_{\mathbb{J}}(\mathfrak{u}_{n_k},\mathfrak{u})=0$, there exists a subsequence $\big\{\mathfrak{u}_{n_k'}\big\}_{k\in\mathbb{N}}$ of $\big\{\mathfrak{u}_{n_k}\big\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty}\rho_{\mathbb{U}}\big(\mathfrak{u}_{n_k'}(s),\mathfrak{u}(s)\big)=0$ for a.e. $s\in(0,\infty)$. Then the continuity of ϕ and the dominated convergence theorem imply that $\lim\limits_{k\to\infty}\int_0^\infty\int_{\mathbb{U}}\phi(s,u)\mathfrak{i}_{\mathbb{J}}\big(\mathfrak{u}_{n_k'}\big)(ds,du)=\lim\limits_{k\to\infty}\int_0^\infty e^{-s}\phi\big(s,\mathfrak{u}_{n_k'}(s)\big)ds=\int_0^\infty e^{-s}\phi\big(s,\mathfrak{u}(s)\big)ds=\int_0^\infty\int_{\mathbb{U}}\phi(s,u)\mathfrak{i}_{\mathbb{J}}(\mathfrak{u})(ds,du)$.

So $\mathfrak{i}_{\mathbb{J}}$ is a continuous injection from the complete separable metric space $\big(\mathbb{J},\rho_{\mathbb{J}}\big)$ to the Borel space $\Big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big),\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)\big)\Big)$. (In particular, the topology induced by $\rho_{\mathbb{J}}$ is stronger than $\mathfrak{T}_\sharp(\mathbb{J})$.) Then the image $\mathfrak{i}_{\mathbb{J}}(\mathbb{J})$ is a Lusin subset and thus a Borel subset of $\Big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big),\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)\big)\Big)$, see e.g. Theorem A.6 of [46].

As the embedding mapping $\mathfrak{i}_{\mathbb{J}}$ is clearly a homeomorphism between $\big(\mathbb{J},\mathfrak{T}_\sharp(\mathbb{J})\big)$ and $\mathfrak{i}_{\mathbb{J}}(\mathbb{J})$ with the relative topology to $\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)\big)$, we obtain that $\big(\mathbb{J},\mathfrak{T}_\sharp(\mathbb{J})\big)$ is a Borel space. $\square$

Proof of Lemma 1.3: 1) For any $t\in(0,\infty)$ and $\phi\in C_b\big([0,\infty)\times\mathbb{U}\big)$, set $\mathcal{I}^\phi_t(\mathfrak{u}):=\int_0^t\phi(s,\mathfrak{u}(s))ds,\ \forall\,\mathfrak{u}\in\mathbb{J}$. *We first show that* $\mathfrak{F}:=\sigma\big(\mathcal{I}^\phi_t;t\in(0,\infty),\phi\in C_b([0,\infty)\times\mathbb{U})\big)=\mathscr{B}(\mathbb{J})$.

1a) Let $t\in(0,\infty)$, $\phi\in C_b\big([0,\infty)\times\mathbb{U}\big)$ and $a\in\mathbb{R}$. *We claim that the set* $A:=\{\mathfrak{u}\in\mathbb{J}:\mathcal{I}^\phi_t(\mathfrak{u})<a\}$ *belongs to* $\mathfrak{T}_\sharp(\mathbb{J})$.

To see this, we let $\mathfrak{u}\in A$ and set $\varepsilon:=\frac12\big(a-\mathcal{I}^\phi_t(\mathfrak{u})\big)$. There exists a positive bounded continuous function β on $[0,\infty)$ such that $\beta(s)=e^s,\ \forall\,s\in[0,t]$ and $\int_t^\infty e^{-s}\beta(s)ds\le\frac12\varepsilon\big(\|\phi\|_\infty\vee1\big)^{-1}$, where $\|\phi\|_\infty:=\underset{(s,u)\in[0,\infty)\times\mathbb{U}}{\sup}|\phi(s,u)|$. Set $\mathcal{O}:=\mathfrak{i}_\mathbb{J}^{-1}\big(O_\varepsilon(\mathfrak{i}_\mathbb{J}(\mathfrak{u}),\beta\phi)\big)$, which clearly contains $\mathfrak{u}$. Let $\mathfrak{u}'\in\mathcal{O}$. We can deduce that

$$\varepsilon>\Big|\int_0^\infty e^{-s}\beta(s)\big(\phi(s,\mathfrak{u}'(s))-\phi(s,\mathfrak{u}(s))\big)ds\Big|\ge\Big|\int_0^t\big(\phi(s,\mathfrak{u}'(s))-\phi(s,\mathfrak{u}(s))\big)ds\Big|-\Big|\int_t^\infty e^{-s}\beta(s)\big(\phi(s,\mathfrak{u}'(s))-\phi(s,\mathfrak{u}(s))\big)ds\Big|.$$

Since $\big|\int_t^\infty e^{-s}\beta(s)\big(\phi(s,\mathfrak{u}'(s))-\phi(s,\mathfrak{u}(s))\big)ds\big|\le2\|\phi\|_\infty\int_t^\infty e^{-s}\beta(s)ds\le\varepsilon$, one has $\big|\mathcal{I}^\phi_t(\mathfrak{u}')-\mathcal{I}^\phi_t(\mathfrak{u})\big|<2\varepsilon$, which implies that $\mathcal{I}^\phi_t(\mathfrak{u}')<a$ or $\mathfrak{u}'\in A$. So $\mathfrak{u}\in\mathcal{O}\subset A$. As the induced topology $\mathfrak{T}_\sharp(\mathbb{J})$ is generated by the subbase $\Big\{\mathfrak{i}_\mathbb{J}^{-1}\big(O_\delta(\mathfrak{m},\phi)\big)$, $\forall\,(\delta,\mathfrak{m},\phi)\in(0,\infty)\times\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)\times C_b\big([0,\infty)\times\mathbb{U}\big)\Big\}$, we see that $A\in\mathfrak{T}_\sharp(\mathbb{J})\subset\mathscr{B}(\mathbb{J})$.

It follows that $\{\mathfrak{u}\in\mathbb{J}:\mathcal{I}^\phi_t(\mathfrak{u})\in\mathcal{E}\}\in\mathscr{B}(\mathbb{J})$ for any $\mathcal{E}\in\mathscr{B}(\mathbb{R})$ and thus $\mathfrak{F}=\sigma\big(\mathcal{I}^\phi_t;t\in(0,\infty),\phi\in C_b([0,\infty)\times\mathbb{U})\big)\subset\mathscr{B}(\mathbb{J})$.

1b) Let $\big\{\mathfrak{i}_\mathbb{J}^{-1}\big(O_{\frac1n}(\mathfrak{m}_k,\phi_j)\big)\big\}_{n,k,j\in\mathbb{N}}$ be the countable subbase of $\mathfrak{T}_\sharp(\mathbb{J})$ in (1.1). Given $n,k,j\in\mathbb{N}$, let $\{I^j_t\}_{t\in(0,\infty)}$ denote $\{\mathcal{I}^\phi_t\}_{t\in(0,\infty)}$ with $\phi(s,u)=e^{-s}\phi_j(s,u),\ \forall\,(s,u)\in[0,\infty)\times\mathbb{U}$ and set $\kappa_{k,j}:=\int_0^\infty\int_\mathbb{U}\phi_j(t,u)\mathfrak{m}_k(dt,du)\in\mathbb{R}$. Since the function $I^j_*(\mathfrak{u}):=\underset{t\to\infty}{\underline{\lim}}I^j_t(\mathfrak{u})$, $\mathfrak{u}\in\mathbb{J}$ is $\mathfrak{F}-$measurable and since $\int_0^\infty e^{-s}\phi_j(s,\mathfrak{u}(s))ds=\lim_{t\to\infty}\int_0^t e^{-s}\phi_j(s,\mathfrak{u}(s))ds=\lim_{t\to\infty}I^j_t(\mathfrak{u})=I^j_*(\mathfrak{u})$ holds for any $\mathfrak{u}\in\mathbb{J}$, (1.1) shows that $\mathfrak{i}_\mathbb{J}^{-1}\big(O_{\frac1n}(\mathfrak{m}_k,\phi_j)\big)=\big\{\mathfrak{u}\in\mathbb{J}:\big|I^j_*(\mathfrak{u})-\kappa_{k,j}\big|<1/n\big\}\in\mathfrak{F}$.

Then $\mathfrak{T}_\sharp(\mathbb{J})\subset\mathfrak{F}$ and thus $\mathscr{B}(\mathbb{J})=\mathfrak{F}$.

2) *We demonstrate the second statement of Lemma 1.3 in several steps. Then* $\mathscr{B}(\mathbb{J})=\sigma\big(I_\varphi;\varphi\in L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)\big)$ *easily follows.*

Let $\{\omega^i_X\}_{i\in\mathbb{N}}$ be a countable dense subset of Ω_X and let $\big\{\boxminus_j=(w_j,x_j)\big\}_{j\in\mathbb{N}}$ be a countable dense subset of $\mathbb{R}^{d+l}$. Given $n\in\mathbb{N}$, we set $O^n_i:=\big\{\omega_X\in\Omega_X:\rho_{\Omega_X}(\omega_X,\omega^i_X)<1/n\big\}\in\mathscr{B}(\Omega_X)$ and $\widetilde{O}^n_i:=O^n_i\Big\backslash\Big(\underset{i'<i}{\cup}O^n_{i'}\Big)\in\mathscr{B}(\Omega_X)$ for any $i\in\mathbb{N}$. Let us also denote $\mathcal{E}^n_j:=O_{\frac1n}(\boxminus_j)\Big\backslash\Big(\underset{j'<j}{\cup}O_{\frac1n}(\boxminus_{j'})\Big)$ for any $j\in\mathbb{N}$.

2a) *We first show that the second statement holds for a continuous function* $\phi\colon[0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}\mapsto[0,c_\phi]$ *with* $c_\phi\in(1,\infty)$, *i.e. the mapping* $\Omega_0\times\Omega_X\times\mathbb{J}\ni(\omega_0,\omega_X,\mathfrak{u})\mapsto\int_0^T\phi\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr$ *is Borel-measurable.*

Let $T\in(0,\infty)$ and $m\in\mathbb{N}$. We set $t^m_k:=k2^{-m}T,\ \forall\,k\in\{0,1,\cdots,2^m\}$. Then, we let $k\in\{0,1,\cdots,2^m-1\}$.

For any $i,j\in\mathbb{N}$, as the function $\phi_{i,j}(s,u):=\phi(s,\omega^i_X,\boxminus_j,u)$, $(s,u)\in[0,\infty)\times\mathbb{U}$ is of $C_b\big([0,\infty)\times\mathbb{U}\big)$, we know from Part (1) that $I^{m,k}_{i,j}(\mathfrak{u}):=\mathcal{I}^{\phi_{ij}}_{t^m_{k+1}}(\mathfrak{u})-\mathcal{I}^{\phi_{ij}}_{t^m_k}(\mathfrak{u})=\int_{t^m_k}^{t^m_{k+1}}\phi\big(r,\omega^i_X,\boxminus_j,\mathfrak{u}(r)\big)dr,\ \forall\,\mathfrak{u}\in\mathbb{J}$ is $\mathscr{B}(\mathbb{J})-$measurable. Since the function $\mathfrak{l}_2(t^m_k,\omega_X)$ is continuous in $\omega_X\in\Omega_X$ and since the function $\mathcal{W}^m_k(\omega_0,\omega_X):=\big(\omega_0(t^m_k),\omega_X(t^m_k)\big)$ is continuous in $(\omega_0,\omega_X)\in\Omega_0\times\Omega_X$, we can deduce from the continuity of ϕ and the bounded convergence theorem that the mapping

$$\begin{aligned}\Phi^m_k(\omega_0,\omega_X,\mathfrak{u})&:=\int_{t^m_k}^{t^m_{k+1}}\phi\big(r,\mathfrak{l}_2(t^m_k,\omega_X),\omega_0(t^m_k),\omega_X(t^m_k),\mathfrak{u}(r)\big)dr\\&=\lim_{n\to\infty}\int_{t^m_k}^{t^m_{k+1}}\sum_{i,j\in\mathbb{N}}\mathbf{1}_{\{\mathfrak{l}_2(t^m_k,\omega_X)\in\widetilde{O}^n_i\}}\mathbf{1}_{\{\mathcal{W}^m_k(\omega_0,\omega_X)\in\mathcal{E}^n_j\}}\phi\big(r,\omega^i_X,\boxminus_j,\mathfrak{u}(r)\big)dr\\&=\lim_{n\to\infty}\sum_{i,j\in\mathbb{N}}\mathbf{1}_{\{\mathfrak{l}_2(t^m_k,\omega_X)\in\widetilde{O}^n_i\}}\mathbf{1}_{\{\mathcal{W}^m_k(\omega_0,\omega_X)\in\mathcal{E}^n_j\}}I^{m,k}_{i,j}(\mathfrak{u}),\quad\forall\,(\omega_0,\omega_X,\mathfrak{u})\in\Omega_0\times\Omega_X\times\mathbb{J}\end{aligned}$$

is $\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$measurable. Then the continuity of $\mathfrak{l}_2(r,\omega_X)$ in $r\in[0,\infty)$ and the bounded convergence theorem imply that the mapping

$$\begin{aligned}\Phi(\omega_0,\omega_X,\mathfrak{u})&:=\int_0^T\phi\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr=\lim_{m\to\infty}\int_0^T\sum_{k=0}^{2^m-1}\mathbf{1}_{\{r\in[t^m_k,t^m_{k+1})\}}\phi\big(r,\mathfrak{l}_2(t^m_k,\omega_X),\omega_0(t^m_k),\omega_X(t^m_k),\mathfrak{u}(r)\big)dr\\&=\lim_{m\to\infty}\sum_{k=0}^{2^m-1}\Phi^m_k(\omega_0,\omega_X,\mathfrak{u}),\quad\forall\,(\omega_0,\omega_X,\mathfrak{u})\in\Omega_0\times\Omega_X\times\mathbb{J}\end{aligned}$$

is also $\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$measurable.

2b) *We next use the monotone class theory to demonstrate that the second statement holds for all bounded Borel-measurable functions on* $[0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$.

Set $\mathbb{X}_1:=[0,\infty)$, $\mathbb{X}_2:=\Omega_X$, $\mathbb{X}_3:=\mathbb{R}^{d+l}$ and $\mathbb{X}_4:=\mathbb{U}$. They are all metric spaces. Let $i=1,2,3,4$. We denote the corresponding metric of $\mathbb{X}_i$ by $\rho_{\mathbb{X}_i}$ and let C_i be a closed subset of $\mathbb{X}_i$ Given $n\in\mathbb{N}$, we define an open subset O^i_n of $\mathbb{X}_i$ by $O^i_n:=\Big\{x\in\mathbb{X}_i: dist_{\mathbb{X}_i}(x,C_i):=\inf\limits_{x'\in C_i}\rho_{\mathbb{X}_i}(x,x')<1/n\Big\}$. In light of Urysohn's Lemma, there exist continuous function $\phi^i_n:\mathbb{X}_i\to[0,1]$ such that $\phi^i_n(x)=1$, $\forall\,x\in C_i$ and $\phi^i_n(x)=0$, $\forall\,x\in(O^i_n)^c$. So $\widehat{\phi}_n(s,\omega_X,\boxminus,u):=\phi^1_n(s)\phi^2_n(\omega_X)\phi^3_n(\boxminus)\phi^4_n(u)\in[0,1]$, $(s,\omega_X,\boxminus,u)\in[0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$ is a continuous function satisfying $\lim\limits_{n\to\infty}\widehat{\phi}_n(s,\omega_X,\boxminus,u)=\mathbf{1}_{\{(s,\omega_X,\boxminus,u)\in C_1\times C_2\times C_3\times C_4\}}$, $\forall\,(s,\omega_X,\boxminus,u)\in[0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$.

Let $T\in(0,\infty)$. Since the mappings $\int_0^T\widehat{\phi}_n\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr$, $(\omega_0,\omega_X,\mathfrak{u})\in\Omega_0\times\Omega_X\times\mathbb{J}$ are $\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$measurable for all $n\in\mathbb{N}$ by Part (2a), the bounded convergence theorem shows that the function

$$\int_0^T\mathbf{1}_{\{(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r))\in C_1\times C_2\times C_3\times C_4\}}dr=\lim_{n\to\infty}\int_0^T\widehat{\phi}_n\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr,$$

$(\omega_0,\omega_X,\mathfrak{u})\in\Omega_0\times\Omega_X\times\mathbb{J}$ is also $\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$measurable.

Let $\mathcal{H}$ collect all real-valued Borel-measurable functions ψ on $[0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$ such that the mapping

$$\Omega_0\times\Omega_X\times\mathbb{J}\ni(\omega_0,\omega_X,\mathfrak{u})\mapsto\int_0^T\psi\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr$$

is $\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$measurable. Clearly, $\mathcal{H}$ is closed under linear combination. If $\{\psi_n\}_{n\in\mathbb{N}}\subset\mathcal{H}$ is a sequence of non-negative functions that increases to a bounded function ψ on $[0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$, the bounded convergence theorem implies that ψ is also of $\mathcal{H}$.

Since $\{C_1\times C_2\times C_3\times C_4: C_1\subset[0,\infty),\,C_2\subset\Omega_X,\,C_3\subset\mathbb{R}^{d+l}\text{ and }C_4\subset\mathbb{U}\text{ are closed}\}$ is a Pi-system that generates $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{R}^{d+l})\otimes\mathscr{B}(\mathbb{U})$, we know from the monotone class theorem that $\mathcal{H}$ includes all bounded Borel-measurable functions on $[0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$.

2c) *Now, we consider an unbounded Borel-measurable function* $\psi:(0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}\mapsto[-\infty,\infty]$.

Set $s^m_k:=k2^{-m}$ for any $m\in\mathbb{N}$ and $k\in\mathbb{N}\cup\{0\}$. Given $n\in\mathbb{N}$, the bounded convergence theorem renders that the mapping

$$\begin{aligned}[0,\infty)\times[0,\infty)\times\Omega_0\times\Omega_X\times\mathbb{J}\ni(t,\mathfrak{s},\omega_0,\omega_X,\mathfrak{u})&\mapsto\int_t^{t+\mathfrak{s}}n\wedge\psi^{\pm}\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr\\&=\lim_{m\to\infty}\sum_{k,\ell=0}^{\infty}\mathbf{1}_{\{t\in[s^m_k,s^m_{k+1})\}}\mathbf{1}_{\{\mathfrak{s}\in[s^m_\ell,s^m_{\ell+1})\}}\int_{s^m_k}^{s^m_{k+\ell}}n\wedge\psi^{\pm}\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr,\end{aligned}$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})-$measurable. Then it follows from the monotone convergence theorem that the mapping

$$\Psi^{\pm}(t,\mathfrak{s},\omega_0,\omega_X,\mathfrak{u}):=\int_t^{t+\mathfrak{s}}\psi^{\pm}\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr=\lim_{n\to\infty}\int_t^{t+\mathfrak{s}}n\wedge\psi^{\pm}\big(r,\mathfrak{l}_2(r,\omega_X),\omega_0(r),\omega_X(r),\mathfrak{u}(r)\big)dr$$

is Borel-measurable in $(t,\mathfrak{s},\omega_0,\omega_X,\mathfrak{u})\in[0,\infty)\times[0,\infty)\times\Omega_0\times\Omega_X\times\mathbb{J}$, proving the second statement.

Clearly, $\mathfrak{F}\subset\sigma\big(I_\varphi;\varphi\in L^0\big([0,\infty)\times\mathbb{U};\mathbb{R}\big)\big)=\sigma\big(I_\varphi;\varphi\in L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)\big)$, where $L^0\big([0,\infty)\times\mathbb{U};\mathbb{R}\big)$ collect all real$-$valued Borel-measurable functions on $[0,\infty)\times\mathbb{U}$. Let φ be a non-negative function in $L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)$. Taking $\psi(r,\mathfrak{x},\boxminus,u):=\varphi(r,u)$, $\forall\,(r,\mathfrak{x},\boxminus,u)\in(0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$ shows that the mapping

$$I_\varphi(\mathfrak{u})=\lim_{T\to\infty}\int_0^T\varphi\big(r,\mathfrak{u}(r)\big)dr=\lim_{T\to\infty}\int_0^T\psi\big(r,\mathbf{0},0,\mathfrak{u}(r)\big)dr=\lim_{T\to\infty}\Psi(0,T,\mathbf{0},\mathbf{0},\mathfrak{u}),\quad\forall\,\mathfrak{u}\in\mathbb{J}$$

is $\mathscr{B}(\mathbb{J})-$measurable. Hence, we have $\mathscr{B}(\mathbb{J})=\mathfrak{F}=\sigma\big(I_\varphi;\varphi\in L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)\big)$. □

Proof of Proposition 1.3: 1a) *Given a sigma-field $\mathcal{G}$ of Ω_0, we claim that*

$$\mathscr{S}_{t,\mathbf{w}}(\mathcal{G}):=\big\{A\subset\Omega:\exists\,\mathcal{A}\in\mathcal{G}\text{ and }\mathcal{N}\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)\text{ s.t. }\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\mathcal{A}\}}=\mathbf{1}_{\{\omega\in A\}},\ \forall\,\omega\in\mathcal{N}^c\big\}$$

is a sigma-field of Ω.

As $\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\Omega\}}=1=\mathbf{1}_{\{\omega\in\Omega\}}$ for any $\omega\in\Omega$, it is clear that $\Omega\in\mathscr{S}_{t,\mathbf{w}}\big(\mathcal{G}\big)$. When $A\in\mathscr{S}_{t,\mathbf{w}}\big(\mathcal{G}\big)$, there exist $\mathcal{A}\in\mathcal{G}$ and $\mathcal{N}\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$ such that $\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\mathcal{A}\}}=\mathbf{1}_{\{\omega\in A\}}$, $\forall\,\omega\in\mathcal{N}^c$. Then $\mathcal{A}^c\in\mathcal{G}$ satisfies that $\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\mathcal{A}^c\}}=1-\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\mathcal{A}\}}=1-\mathbf{1}_{\{\omega\in A\}}=\mathbf{1}_{\{\omega\in A^c\}}$, $\forall\,\omega\in\mathcal{N}^c$, so A^c also belongs to $\mathscr{S}_{t,\mathbf{w}}\big(\mathcal{G}\big)$. If $\{A_k\}_{k\in\mathbb{N}}\subset\mathscr{S}_{t,\mathbf{w}}\big(\mathcal{G}\big)$, for any $k\in\mathbb{N}$ there exist $\mathcal{A}_k\in\mathcal{G}$ and $\mathcal{N}_k\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$ such that $\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\mathcal{A}_k\}}=\mathbf{1}_{\{\omega\in A_k\}}$, $\forall\,\omega\in\mathcal{N}^c_k$. Then $\underset{k\in\mathbb{N}}{\cap}\mathcal{A}_k\in\mathcal{G}$ satisfies that $\mathbf{1}_{\big\{B^{t,\mathbf{w}}(\omega)\in\underset{k\in\mathbb{N}}{\cap}\mathcal{A}_k\big\}}=\prod\limits_{k\in\mathbb{N}}\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\mathcal{A}_k\}}=\prod\limits_{k\in\mathbb{N}}\mathbf{1}_{\{\omega\in A_k\}}=\mathbf{1}_{\big\{\omega\in\underset{k\in\mathbb{N}}{\cap}A_k\big\}}$, $\forall\,\omega\in\underset{k\in\mathbb{N}}{\cap}\mathcal{N}^c_k$, which shows that $\underset{k\in\mathbb{N}}{\cap}A_k\in\mathscr{S}_{t,\mathbf{w}}\big(\mathcal{G}\big)$ with $(\mathcal{A},\mathcal{N})=\Big(\underset{k\in\mathbb{N}}{\cap}\mathcal{A}_k,\underset{k\in\mathbb{N}}{\cup}\mathcal{N}_k\Big)$. Hence $\mathscr{S}_{t,\mathbf{w}}\big(\mathcal{G}\big)$ is a sigma$-$field of Ω.

Let $s\in[t,\infty)$. For any $r\in[t,s]$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, since $\omega\in(B^t_r)^{-1}(\mathcal{E})$ iff $W^t_r(B^{t,\mathbf{w}}(\omega))=W_r(B^{t,\mathbf{w}}(\omega))-W_t(B^{t,\mathbf{w}}(\omega))=B^t_r(\omega)\in\mathcal{E}$ iff $B^{t,\mathbf{w}}(\omega)\in(W^t_r)^{-1}(\mathcal{E})$, one has $\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in(W^t_r)^{-1}(\mathcal{E})\}}=\mathbf{1}_{\{\omega\in(B^t_r)^{-1}(\mathcal{E})\}}$, $\forall\,\omega\in\Omega$, which means $(B^t_r)^{-1}(\mathcal{E})\in\mathscr{S}_{t,\mathbf{w}}(\mathcal{F}^{W^t}_s)$ with $\mathcal{A}=(W^t_r)^{-1}(\mathcal{E})\in\mathcal{F}^{W^t}_s$ and $\mathcal{N}=\emptyset$. So $\mathcal{F}^{B^t}_s\subset\mathscr{S}_{t,\mathbf{w}}(\mathcal{F}^{W^t}_s)$. When $s>t$, as $\mathcal{F}^{B^t}_r\subset\mathscr{S}_{t,\mathbf{w}}(\mathcal{F}^{W^t}_r)\subset\mathscr{S}_{t,\mathbf{w}}(\mathcal{F}^{W^t}_{s-})$ for any $r\in[t,s)$, one further has $\mathcal{F}^{B^t}_{s-}=\sigma\Big(\underset{r\in[t,s)}{\cup}\mathcal{F}^{B^t}_r\Big)\subset\mathscr{S}_{t,\mathbf{w}}(\mathcal{F}^{W^t}_{s-})$.

1b) *Similar to* $\mathscr{S}_{t,\mathbf{w}}\big(\mathcal{G}\big)$ *in Part (1a),* $\widehat{\mathscr{S}_{t,\mathbf{w}}}:=\big\{D\subset[t,\infty)\times\Omega:\exists\,\mathcal{D}\in\mathscr{P}^{W^t}$ *and* $\mathcal{N}\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$ *s.t.* $\mathbf{1}_{\{(s,B^{t,\mathbf{w}}(\omega))\in\mathcal{D}\}}=\mathbf{1}_{\{(s,\omega)\in D\}},\ \forall\,(s,\omega)\in[t,\infty)\times\mathcal{N}^c\big\}$ *is a sigma$-$field of* $[t,\infty)\times\Omega$:

Set $\Lambda_t:=\big\{\{t\}\times A_o:\,A_o\in\mathcal{F}^{B^t}_t\big\}\cup\big\{(s,\infty)\times A:\,s\in[t,\infty)\cap\mathbb{Q},\,A\in\mathcal{F}^{B^t}_{s-}\big\}$, which generates the $\mathbf{F}^{B^t}-$predictable sigma$-$field $\mathscr{P}^{B^t}$. For any $A_o\in\mathcal{F}^{B^t}_t\subset\mathscr{S}_{t,\mathbf{w}}(\mathcal{F}^{W^t}_t)$, there exist $\mathscr{A}_o\in\mathcal{F}^{W^t}_t$ and $\mathcal{N}\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$ such that $\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\mathscr{A}_o\}}=\mathbf{1}_{\{\omega\in A_o\}}$, $\forall\,\omega\in\mathcal{N}^c$. We can deduce that $\mathbf{1}_{\{(s,B^{t,\mathbf{w}}(\omega))\in\{t\}\times\mathscr{A}_o\}}=\mathbf{1}_{\{(s,\omega)\in\{t\}\times A_o\}}$, $\forall\,(s,\omega)\in[t,\infty)\times\mathcal{N}^c$. So $\{t\}\times A_o$ is of $\widehat{\mathscr{S}_{t,\mathbf{w}}}$ with $\mathcal{D}=\{t\}\times\mathscr{A}_o$. For any $s\in[t,\infty)\cap\mathbb{Q}$ and $A\in\mathcal{F}^{B^t}_{s-}\subset\mathscr{S}_{t,\mathbf{w}}(\mathcal{F}^{W^t}_{s-})$, one can find some $\mathscr{A}\in\mathcal{F}^{W^t}_{s-}$ and $\mathcal{N}\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$ such that $\mathbf{1}_{\{B^{t,\mathbf{w}}(\omega)\in\mathscr{A}\}}=\mathbf{1}_{\{\omega\in A\}}$, $\forall\,\omega\in\mathcal{N}^c$. It then holds for any $(r,\omega)\in[t,\infty)\times\mathcal{N}^c$ that $\mathbf{1}_{\{(r,B^{t,\mathbf{w}}(\omega))\in(s,\infty)\times\mathscr{A}\}}=\mathbf{1}_{\{(r,\omega)\in(s,\infty)\times A\}}$, which implies $(s,\infty)\times A\in\widehat{\mathscr{S}_{t,\mathbf{w}}}$ with $\mathcal{D}=(s,\infty)\times\mathscr{A}$. Hence, $\widehat{\mathscr{S}_{t,\mathbf{w}}}$ contains Λ_t and thus includes $\mathscr{P}^{B^t}$.

1c) Now, let $\{\mu_s\}_{s\in[t,\infty)}$ be a general $\mathbb{U}-$valued, $\mathbf{F}^{B^t,P}-$progressively measurable process on Ω.

Similar to Lemma 2.4 of [44], one can construct a $[0,1]-$valued $\mathbf{F}^{B^t}-$predictable process $\big\{\nu_s\big\}_{s\in[t,\infty)}$ on Ω such that $\nu_s(\omega)=\mathscr{I}\big(\mu_s(\omega)\big)$ for $ds\times dP-$a.s. $(s,\omega)\in[t,\infty)\times\Omega$. Fubini Theorem yields that for all $\omega\in\Omega$ except on a $\mathcal{N}^1_\nu\in\mathscr{N}_P\big(\mathcal{F}^{B^t}_\infty\big)$, $\nu_s(\omega)=\mathscr{I}\big(\mu_s(\omega)\big)$ for a.e. $s\in[t,\infty)$. By Part (1b) and a standard approximation scheme, we can find a $[0,1]-$valued $\mathbf{F}^{W^t}-$predictable process $\big\{\nu^o_s\big\}_{s\in[t,\infty)}$ on Ω_0 and an $\mathcal{N}^2_\nu\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$ such that $\nu_s(\omega)=\nu^o_s(B^{t,\mathbf{w}}(\omega))$ for any $(s,\omega)\in[t,\infty)\times(\mathcal{N}^2_\nu)^c$.

Set $\mathcal{N}_\mu=\mathcal{N}^1_\nu\cup\mathcal{N}^2_\nu\in\mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$ and define $\mu^o_s(\omega_0):=\mathscr{I}^{-1}\big(\nu^o_s(\omega_0)\big)\mathbf{1}_{\{\nu^o_s(\omega_0)\in\mathfrak{E}\}}+u_0\mathbf{1}_{\{\nu^o_s(\omega_0)\notin\mathfrak{E}\}}$, $(s,\omega_0)\in[t,\infty)\times\Omega_0$, which is a $\mathbb{U}-$valued $\mathbf{F}^{W^t}-$predictable process. Given $\omega\in\mathcal{N}^c_\mu$, it holds for a.e. $s\in(t,\infty)$ that $\nu^o_s(B^{t,\mathbf{w}}(\omega))=\nu_s(\omega)=\mathscr{I}\big(\mu_s(\omega)\big)$ and thus $\mu^o_s(B^{t,\mathbf{w}}(\omega))=\mu_s(\omega)$.

2) Let $\mathbf{x}\in\Omega_X$ and let $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. On Ω_0, $M^{t,\mu^o}_s(\varphi):=\varphi\big(W^t_s,\,X^{t,\mathbf{x},\mu^o}_s\big)-\int_t^s\overline{b}\big(r,\,X^{t,\mathbf{x},\mu^o}_{r\wedge\cdot},\mu^o_r\big)\cdot D\varphi\big(W^t_r,\,X^{t,\mathbf{x},\mu^o}_r\big)dr-\frac{1}{2}\int_t^s\overline{\sigma}\,\overline{\sigma}^T\big(r,\,X^{t,\mathbf{x},\mu^o}_{r\wedge\cdot},\mu^o_r\big):D^2\varphi(W^t_r,X^{t,\mathbf{x},\mu^o}_r)dr$, $\forall\,s\in[t,\infty)$ is an $\mathbf{F}^{W^t,P_0}-$adapted continuous process and $\tau^{t,\mu^o}_n:=\inf\big\{s\in[t,\infty):\big|(W^t_s,\,X^{t,\mathbf{x},\mu^o}_s)\big|\ge n\big\}\wedge(t+n)$ is an $\mathbf{F}^{W^t,P_0}-$stopping time. As $X^{t,\mathbf{x},\mu^o}$ is the unique strong solution of SDE (1.4) on $\big(\Omega_0,\mathscr{B}(\Omega_0),P_0\big)$ with $(B^t,\mu)=\big(W^t,\mu^o\big)$, taking $(\Omega,\mathcal{F},P,B,X,\mu)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W,X^{t,\mathbf{x},\mu^o},\mu^o\big)$ in Part (iii) of Proposition 1.2 shows that $\Big\{M^{t,\mu^o}_{s\wedge\tau^{t,\mu^o}_n}(\varphi)\Big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^{W^t,P_0}-$martingale.

We next show that the process $\big(M^{t,\mu^o}_s(\varphi)\big)(B^{t,\mathbf{w}})$, $\forall\,s\in[t,\infty)$ *stopped by* $\tau^{t,\mu^o}_n(B^{t,\mathbf{w}})$ *is a bounded* $\mathbf{F}^{B^t,P}-$*martingale. Then the second statement of Proposition 1.3 easily follows from Proposition 1.2.*

Since $W^t_s(B^{t,\mathbf{w}}(\omega))=B^t_s(\omega)$, $\forall\,(s,\omega)\in[t,\infty)\times\Omega$, applying Lemma A.1 with $t_0=t$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\Omega,\mathcal{F},P,B\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W\big)$ and $\Phi=B^{t,\mathbf{w}}$ implies that $\breve{X}_s:=X^{t,\mathbf{x},\mu^o}_s(B^{t,\mathbf{w}})$, $\breve{M}_s(\varphi):=\big(M^{t,\mu^o}_s(\varphi)\big)(B^{t,\mathbf{w}})$, $s\in[t,\infty)$ are $\mathbf{F}^{B^t,P}-$adapted continuous processes and $\breve{\tau}_n:=\tau^{t,\mu^o}_n(B^{t,\mathbf{w}})$ is an $\mathbf{F}^{B^t,P}-$stopping time.

Let $t\le s<r<\infty$ and $\big\{(s_i,\mathcal{E}_i)\big\}_{i=1}^k\subset[t,s]\times\mathscr{B}(\mathbb{R}^d)$. We can also deduce from Lemma A.1 that

$$
\begin{aligned}
0&=E_{P_0}\Big[\Big(M^{t,\mu^o}_{r\wedge\tau^{t,\mu^o}_n}(\varphi)-M^{t,\mu^o}_{s\wedge\tau^{t,\mu^o}_n}(\varphi)\Big)\mathbf{1}_{\underset{i=1}{\overset{k}{\cap}}(W^t_{s_i})^{-1}(\mathcal{E}_i)}\Big]\\
&=\int_{\omega_0\in\Omega_0}\Big(\big(M^{t,\mu^o}(\varphi)\big)\big(r\wedge\tau^{t,\mu^o}_n(\omega_0),\omega_0\big)-\big(M^{t,\mu^o}(\varphi)\big)\big(s\wedge\tau^{t,\mu^o}_n(\omega_0),\omega_0\big)\Big)\mathbf{1}_{\underset{i=1}{\overset{k}{\cap}}\{W^t_{s_i}(\omega_0)\in\mathcal{E}_i\}}(P\circ(B^{t,\mathbf{w}})^{-1})(d\omega_0)\\
&=\int_{\omega\in\Omega}\Big(\big(M^{t,\mu^o}(\varphi)\big)\big(r\wedge\tau^{t,\mu^o}_n(B^{t,\mathbf{w}}(\omega)),B^{t,\mathbf{w}}(\omega)\big)-\big(M^{t,\mu^o}(\varphi)\big)\big(s\wedge\tau^{t,\mu^o}_n(B^{t,\mathbf{w}}(\omega)),B^{t,\mathbf{w}}(\omega)\big)\Big)\mathbf{1}_{\underset{i=1}{\overset{k}{\cap}}\{W^t_{s_i}(B^{t,\mathbf{w}}(\omega))\in\mathcal{E}_i\}}P(d\omega)
\end{aligned}
$$

$$= \int_{\omega\in\Omega} \Big(\big(M^{t,\mu^o}(\varphi)\big)\big(r\wedge\breve{\tau}_n(\omega), B^{t,\mathbf{w}}(\omega)\big) - \big(M^{t,\mu^o}(\varphi)\big)\big(s\wedge\breve{\tau}_n(\omega), B^{t,\mathbf{w}}(\omega)\big)\Big) \mathbf{1}_{\underset{i=1}{\overset{k}{\cap}}\{B^t_{s_i}(\omega)\in\mathcal{E}_i\}} P(d\omega)$$
$$= E_P\Big[\Big(\breve{M}_{r\wedge\breve{\tau}_n}(\varphi) - \breve{M}_{s\wedge\breve{\tau}_n}(\varphi)\Big)\mathbf{1}_{\underset{i=1}{\overset{k}{\cap}}(B^t_{s_i})^{-1}(\mathcal{E}_i)}\Big].$$

So the Lambda-system $\breve{\Lambda}_{s,r} := \big\{A \in \mathcal{F}^{B^t,P}_\infty : E_P\big[\big(\breve{M}_{r\wedge\breve{\tau}_n}(\varphi) - \breve{M}_{s\wedge\breve{\tau}_n}(\varphi)\big)\mathbf{1}_A\big] = 0\big\}$ includes the Pi-system $\Big\{ \underset{i=1}{\overset{k}{\cap}} (B^t_{s_i})^{-1}(\mathcal{E}_i) : \big\{(s_i,\mathcal{E}_i)\big\}_{i=1}^k \subset [t,s]\times\mathscr{B}(\mathbb{R}^d)\Big\} \cup \mathscr{N}_P(\mathcal{F}^{B^t}_\infty)$. An application of Dynkin's Pi-Lambda Theorem (see e.g Theorem 3.2 of [8]) shows that $\mathcal{F}^{B^t,P}_s \subset \breve{\Lambda}_{s,r}$, i.e., $E_P\big[\big(\breve{M}_{r\wedge\breve{\tau}_n}(\varphi) - \breve{M}_{s\wedge\breve{\tau}_n}(\varphi)\big)\mathbf{1}_A\big] = 0$ for any $A \in \mathcal{F}^{B^t,P}_s$. To wit, the $\mathbf{F}^{B^t,P}-$adapted continuous process $\breve{M}_s(\varphi) = \varphi\big(B^t_s, \breve{X}_s\big) - \int_t^s \overline{b}\big(r, \breve{X}_{r\wedge\cdot}, \mu_r\big)\cdot D\varphi\big(B^t_r, \breve{X}_r\big)dr - \frac{1}{2}\int_t^s \overline{\sigma}\,\overline{\sigma}^T\big(r, \breve{X}_{r\wedge\cdot}, \mu_r\big) : D^2\varphi(B^t_r, \breve{X}_r)dr,\ \forall\, s\in[t,\infty)$ stopped by the $\mathbf{F}^{B^t,P}-$stopping time $\breve{\tau}_n = \inf\big\{s\in[t,\infty) : \big|(B^t_s, \breve{X}_s)\big| \ge n\big\}\wedge(t+n)$ is a bounded $\mathbf{F}^{B^t,P}-$martingale. Using Part (i) of Proposition 1.2 yields that $P\{\breve{X}_s = X^{t,\mathbf{x},\mu}_s,\ \forall\, s\in[0,\infty)\} = 1$.

Moreover, for any $\psi\in\mathscr{H}_o$, one has $E_P\big[\int_t^\infty \psi^-(r, X^{t,\mathbf{x},\mu}_{r\wedge\cdot}, \mu_r)dr\big] = E_P\big[\int_t^\infty \psi^-\big(r, X^{t,\mathbf{x},\mu^o}_{r\wedge\cdot}(B^{t,\mathbf{w}}), \mu^o_r(B^{t,\mathbf{w}})\big)dr\big] = E_{P_0}\big[\int_t^\infty \psi^-(r, X^{t,\mathbf{x},\mu^o}_{r\wedge\cdot}, \mu^o_r)dr\big] < \infty$. □

Proof of Theorem 3.1: Fix $(t,\mathbf{w},\mathbf{u},\mathbf{x}) \in [0,\infty)\times\Omega_0\times\mathbb{J}\times\Omega_X$ and $(y,z) = \big(\{y_i\}_{i\in\mathbb{N}}, \{z_i\}_{i\in\mathbb{N}}\big)\in\Re\times\Re$.

1) *We first show that* $V(t,\mathbf{x},y,z) \le \overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)$: If $\mathcal{C}_{t,\mathbf{x}}(y,z)=\emptyset$, then $V(t,\mathbf{x},y,z) = -\infty \le \overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)$.

So we assume $\mathcal{C}_{t,\mathbf{x}}(y,z)\neq\emptyset$ and let $(\mu,\tau)\in\mathcal{C}_{t,\mathbf{x}}(y,z)$. Define a process $\mathcal{B}^{t,\mathbf{w}}_s(\omega) := \mathbf{w}(s\wedge t) + \mathcal{B}^t_{s\vee t}(\omega)$, $\forall\,(s,\omega)\in[0,\infty)\times\mathcal{Q}$. According to Proposition 1.3, there exists a $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process $\widehat{\mu} = \{\widehat{\mu}_s\}_{s\in[t,\infty)}$ on Ω_0 and an $\mathcal{N}_\mu\in\mathscr{N}_{\mathfrak{p}}(\mathcal{F}^{\mathcal{B}^t}_\infty)$ such that for any $\omega\in\mathcal{N}^c_\mu$, $\mu_s(\omega) = \widehat{\mu}_s\big(\mathcal{B}^{t,\mathbf{w}}(\omega)\big)$ for a.e. $s\in(t,\infty)$. By Fubini Theorem, $\mathcal{N} := \big\{\omega\in\mathcal{Q} : \mu_s(\omega)$ is not Borel-measurable in $s\in[t,\infty)\big\}$ is an $\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_\infty-$measurable set with zero $\mathfrak{p}-$measure or $\mathcal{N}\in\mathscr{N}_{\mathfrak{p}}\big(\mathcal{F}^{\mathcal{B}^t}_\infty\big)$. Then process $\widetilde{\mu}_s(\omega) := \big(\mathbf{1}_{\{s\in[0,t)\}}\mathbf{u}(s) + \mathbf{1}_{\{s\in[t,\infty)\}}\mu_s(\omega)\big)\mathbf{1}_{\{\omega\in\mathcal{N}^c\}} + u_0\mathbf{1}_{\{\omega\in\mathcal{N}\}}$, $\forall\,(s,\omega)\in[0,\infty)\times\mathcal{Q}$ satisfies that $\widetilde{\mu}.(\omega)\in\mathbb{J}$ for any $\omega\in\mathcal{Q}$.

1a) *We define a mapping* $\Psi(\omega) := \big(\mathcal{B}^{t,\mathbf{w}}(\omega), \widetilde{\mu}.(\omega), \mathcal{X}^{t,\mathbf{x},\mu}(\omega), \tau(\omega)\big)\in\overline{\Omega}$, $\forall\,\omega\in\mathcal{Q}$ *and discuss its measurability.*

Let us simply set $\theta = (t,\mathbf{x},\mu)$. Since $\overline{W}^t_s(\Psi(\omega)) = \overline{W}_s(\Psi(\omega)) - \overline{W}_t(\Psi(\omega)) = \mathcal{B}^{t,\mathbf{w}}_s(\omega) - \mathcal{B}^{t,\mathbf{w}}_t(\omega) = \mathcal{B}^t_s(\omega)$, $\forall\,(s,\omega)\in[t,\infty)\times\mathcal{Q}$ and since $\big\{\mathcal{X}^\theta_s\big\}_{s\in[0,\infty)}$ is an $\big\{\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process, we can deduce that the mapping Ψ is $\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_s\big/\overline{\mathcal{F}}^t_s-$measurable for any $s\in[t,\infty)$.

Let $\varphi\in L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)$. The $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$progressive measurability of process $\big\{\widetilde{\mu}_s\big\}_{s\in[t,\infty)}$ implies that process $\big\{\varphi(s,\widetilde{\mu}_s)\big\}_{s\in[t,\infty)}$ is also $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$progressively measurable and the random variable $I_\varphi(\widetilde{\mu}.(\omega)) = \int_0^t \varphi(s,\mathbf{u}(s))ds + \int_t^\infty \varphi(s,\widetilde{\mu}_s(\omega))ds$, $\omega\in\mathcal{Q}$ is thus $\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_\infty-$measurable. Lemma 1.3 (1) then renders that the mapping $\widetilde{\mu}. : \mathcal{Q}\mapsto\mathbb{J}$ is $\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_\infty\big/\mathscr{B}(\mathbb{J})-$measurable, which together with the $\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_\infty-$measurability of $\mathcal{B}^{t,\mathbf{w}}$, $\mathcal{X}^\theta$, τ shows that the mapping Ψ is also $\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_\infty\big/\mathscr{B}(\overline{\Omega})-$measurable.

1b) *Using the martingale-problem formulation* (*Proposition 1.2 and Remark 3.1*), *we demonstrate that the probability measure* $\overline{P}_\Psi$ *induced by* Ψ $\big($*i.e.*, $\overline{P}_\Psi\big(\overline{A}\big) := \mathfrak{p}\big(\Psi^{-1}\big(\overline{A}\big)\big)$, $\forall\,\overline{A}\in\mathscr{B}(\overline{\Omega})\big)$ *belongs to* $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

Set $\overline{\mu}_s := \widehat{\mu}_s(\overline{W})$, $\forall\,s\in[t,\infty)$, which is a $\mathbb{U}-$valued, $\mathbf{F}^{\overline{W}^t}-$predictable process on $\overline{\Omega}$. Since $\mathcal{N}^c\cap\mathcal{N}^c_\mu\subset\mathcal{N}^c\cap\big\{\omega\in\mathcal{Q} : \mu_s(\omega) = \widehat{\mu}_s(\mathcal{B}^{t,\mathbf{w}}(\omega))$ for a.e. $s\in(t,\infty)\big\} = \mathcal{N}^c\cap\big\{\omega\in\mathcal{Q} : \widetilde{\mu}_s(\omega) = \widehat{\mu}_s(\mathcal{B}^{t,\mathbf{w}}(\omega))$ for a.e. $s\in(t,\infty)\big\}$, we can deduce that $\overline{P}_\Psi\big\{\overline{U}_s = \overline{\mu}_s$ for a.e. $s\in(t,\infty)\big\} = \mathfrak{p}\big\{\overline{U}_s(\Psi) = \widehat{\mu}_s(\overline{W}(\Psi))$ for a.e. $s\in(t,\infty)\big\} = \mathfrak{p}\big\{\omega\in\mathcal{Q} : \widetilde{\mu}_s(\omega) = \widehat{\mu}_s(\mathcal{B}^{t,\mathbf{w}}(\omega))$ for a.e. $s\in(t,\infty)\big\} = 1$. Namely, $\overline{P}_\Psi$ satisfies (D1) in the definition of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

Fix $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. We define an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$adapted continuous process $\mathcal{M}^\theta_s(\varphi) := \varphi\big(\mathcal{B}^t_s, \mathcal{X}^\theta_s\big) - \int_t^s \overline{b}\big(r, \mathcal{X}^\theta_{r\wedge\cdot}, \mu_r\big)\cdot D\varphi\big(\mathcal{B}^t_r, \mathcal{X}^\theta_r\big)dr - \frac{1}{2}\int_t^s \overline{\sigma}\,\overline{\sigma}^T\big(r, \mathcal{X}^\theta_{r\wedge\cdot}, \mu_r\big) : D^2\varphi(\mathcal{B}^t_r, \mathcal{X}^\theta_r)dr$, $\forall\,s\in[t,\infty)$ and define an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping time $\mathsf{T}^\theta_n := \inf\big\{s\in[t,\infty) : \big|(\mathcal{B}^t_s, \mathcal{X}^\theta_s)\big| \ge n\big\}\wedge(t+n)$. Applying Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X,\mu) = (\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B},\mathcal{X}^\theta,\mu)$ yields that $\big\{\mathcal{M}^\theta_{s\wedge\mathsf{T}^\theta_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}},\mathfrak{p}\big)-$martingale.

Since $\overline{P}_\Psi\big\{\overline{X}_s = \mathbf{x}(s),\ \forall\,s\in[0,t]\big\} = \mathfrak{p}\big\{\overline{X}_s(\Psi) = \mathbf{x}(s),\ \forall\,s\in[0,t]\big\} = \mathfrak{p}\big\{\mathcal{X}^\theta_s = \mathbf{x}(s),\ \forall\,s\in[0,t]\big\} = 1$, using Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X,\mu) = \big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}_\Psi,\overline{W},\overline{X},\overline{\mu}\big)$ shows that $\big\{\overline{M}^{t,\overline{\mu}}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\overline{\mathbf{F}}^t-$adapted continuous process under $\overline{P}_\Psi$. Given $\omega\in\mathcal{N}^c_\mu$, since $\overline{\mu}_s\big(\Psi(\omega)\big) = \widehat{\mu}_s\big(\overline{W}(\Psi(\omega))\big) = \widehat{\mu}_s\big(\mathcal{B}^{t,\mathbf{w}}(\omega)\big) = \mu_s(\omega)$ for a.e. $s\in(t,\infty)$, we see that $\big(\overline{M}^{t,\overline{\mu}}_s(\varphi)\big)\big(\Psi(\omega)\big) = \big(\mathcal{M}^\theta_s(\varphi)\big)(\omega)$, $\forall\,s\in[t,\infty)$ and $\overline{\tau}^t_n\big(\Psi(\omega)\big) = \mathsf{T}^\theta_n(\omega)$. Then

$$\begin{aligned}\big(\overline{M}^{t,\overline{\mu}}_{s\wedge\overline{\tau}^t_n}(\varphi)\big)\big(\Psi(\omega)\big) &= \big(\overline{M}^{t,\overline{\mu}}(\varphi)\big)\big(s\wedge\overline{\tau}^t_n(\Psi(\omega)),\Psi(\omega)\big) = \big(\overline{M}^{t,\overline{\mu}}(\varphi)\big)\big(s\wedge\mathsf{T}^\theta_n(\omega),\Psi(\omega)\big)\\ &= \big(\mathcal{M}^\theta(\varphi)\big)\big(s\wedge\mathsf{T}^\theta_n(\omega),\omega\big) = \big(\mathcal{M}^\theta_{s\wedge\mathsf{T}^\theta_n}(\varphi)\big)(\omega), \quad \forall\,(s,\omega)\in[t,\infty)\times\mathcal{N}^c_\mu. \end{aligned} \tag{6.3}$$

Let $t_1,t_2\in[t,\infty)$ with $t_1<t_2$ and let $\overline{A}\in\overline{\mathcal{F}}^t_{t_1}$. As $\Psi^{-1}(\overline{A})\in\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_{t_1}$, the $\big(\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}},\mathfrak{p}\big)-$martingality of $\big\{\mathcal{M}^\theta_{s\wedge\mathrm{T}^\theta_n}(\varphi)\big\}_{s\in[t,\infty)}$ and (6.3) imply that $E_{\overline{P}_\Psi}\Big[\big(\overline{M}^{t,\overline{\mu}}_{t_2\wedge\overline{\tau}^t_n}(\varphi)-\overline{M}^{t,\overline{\mu}}_{t_1\wedge\overline{\tau}^t_n}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=E_{\mathfrak{p}}\Big[\Big(\big(\overline{M}^{t,\overline{\mu}}_{t_2\wedge\overline{\tau}^t_n}(\varphi)\big)(\Psi)-\big(\overline{M}^{t,\overline{\mu}}_{t_1\wedge\overline{\tau}^t_n}(\varphi)\big)(\Psi)\Big)\mathbf{1}_{\Psi^{-1}(\overline{A})}\Big]=E_{\mathfrak{p}}\Big[\big(\mathcal{M}^\theta_{t_2\wedge\mathrm{T}^\theta_n}(\varphi)-\mathcal{M}^\theta_{t_1\wedge\mathrm{T}^\theta_n}(\varphi)\big)\mathbf{1}_{\Psi^{-1}(\overline{A})}\Big]=0$. So $\big\{\overline{M}^{t,\overline{\mu}}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\overline{\mathbf{F}}^t,\overline{P}_\Psi\big)-$martingale. By Remark 3.1 (ii), $\overline{P}_\Psi$ satisfies (D2)+(D3) in the definition of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

Since $W^t_s(\mathcal{B}^{t,\mathbf{w}}(\omega))=\mathcal{B}^{t,\mathbf{w}}_s(\omega)-\mathcal{B}^{t,\mathbf{w}}_t(\omega)=\mathcal{B}^t_s(\omega)$ for any $(s,\omega)\in[t,\infty)\times\Omega$, taking $(\Omega,\mathcal{F},P,B,\Phi)=(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B},\mathcal{B}^{t,\mathbf{w}})$ in Lemma A.2 (2) shows that $\mathfrak{p}\big\{\tau=\widehat{\tau}(\mathcal{B}^{t,\mathbf{w}})\big\}=1$ for some $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}$ on Ω_0, it follows that $\overline{P}_\Psi\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}=\mathfrak{p}\big\{\overline{T}(\Psi)=\widehat{\tau}(\overline{W}(\Psi))\big\}=\mathfrak{p}\big\{\tau=\widehat{\tau}(\mathcal{B}^{t,\mathbf{w}})\big\}=1$.

1c) *We further show that $\overline{P}_\Psi$ is of the probability class $\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}(y,z)$.*

Since $\overline{W}_s\big(\Psi(\omega)\big)=\mathcal{B}^{t,\mathbf{w}}_s(\omega)=\mathbf{w}(s),\ \forall\,(s,\omega)\in[0,t]\times\mathcal{Q}$ and since $\overline{U}_s\big(\Psi(\omega)\big)=\widetilde{\mu}_s(\omega)=\mathbf{u}(s),\ \forall\,(s,\omega)\in[0,t)\times\mathcal{N}^c$, it is clear that $\overline{P}_\Psi\big\{\overline{W}_s=\mathbf{w}(s),\forall\,s\in[0,t];\overline{U}_s=\mathbf{u}(s)$ for a.e. $s\in(0,t)\big\}=\mathfrak{p}\big\{\overline{W}_s(\Psi)=\mathbf{w}(s),\forall\,s\in[0,t];\overline{U}_s(\Psi)=\mathbf{u}(s)$ for a.e. $s\in(0,t)\big\}=1$. Thus $\overline{P}_\Psi\in\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}$. For any $i\in\mathbb{N}$,

$$
\begin{aligned}
E_{\overline{P}_\Psi}\bigg[\int_t^{\overline{T}} g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\bigg]&=E_{\mathfrak{p}}\bigg[\int_t^{\overline{T}(\Psi)} g_i\big(r,\overline{X}_{r\wedge\cdot}(\Psi),\overline{U}_r(\Psi)\big)dr\bigg]=E_{\mathfrak{p}}\Big[\int_t^\tau g_i(r,\mathcal{X}^\theta_{r\wedge\cdot},\widetilde{\mu}_r)dr\Big]\\
&=E_{\mathfrak{p}}\bigg[\int_t^\tau g_i(r,\mathcal{X}^\theta_{r\wedge\cdot},\mu_r)dr\bigg]\le y_i
\end{aligned}
\tag{6.4}
$$

and similarly $E_{\overline{P}_\Psi}\big[\int_t^{\overline{T}} h_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]=E_{\mathfrak{p}}\big[\int_t^\tau h_i(r,\mathcal{X}^\theta_{r\wedge\cdot},\mu_r)dr\big]=z_i$, which means that $\overline{P}_\Psi\in\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}(y,z)$. Then an analogy to (6.4) renders that $E_{\mathfrak{p}}\big[\int_t^\tau f(r,\mathcal{X}^\theta_{r\wedge\cdot},\mu_r)dr+\mathbf{1}_{\{\tau<\infty\}}\pi(\tau,\mathcal{X}^\theta_{\tau\wedge\cdot})\big]=E_{\overline{P}_\Psi}\Big[\int_t^{\overline{T}} f\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr+\mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big]\le\overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)$. Taking supremum over $(\mu,\tau)\in\mathcal{C}_{t,\mathbf{x}}(y,z)$ yields $V(t,\mathbf{x},y,z)\le\overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)$.

2) As $\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{u},\mathbf{x}}(y,z)\subset\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$, we automatically have $\overline{V}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)\le\overline{V}(t,\mathbf{x},y,z)$. It remains to demonstrate that $\overline{V}(t,\mathbf{x},y,z)\le V(t,\mathbf{x},y,z)$. If $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)=\emptyset$, then $\overline{V}(t,\mathbf{x},y,z)=-\infty\le V(t,\mathbf{x},y,z)$.

Assume $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)\ne\emptyset$ and let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$. *We use Definition 3.1 to find a $\mathbb{U}-$valued process $\widehat{\nu}.$ and a stopping time $\widehat{\gamma}$ on Ω_0 such that $\overline{P}\big\{(\overline{X},\overline{U},\overline{T})=(\mathcal{X}^{t,\mathbf{x},\nu},\widehat{\nu},\widehat{\gamma})(\overline{W})\big\}=1$. It then follows that $(\nu,\gamma):=(\widehat{\nu},\widehat{\gamma})(\mathcal{B})\in\mathcal{C}_{t,\mathbf{x}}(y,z)$ and $\overline{V}(t,\mathbf{x},y,z)\le V(t,\mathbf{x},y,z)$.*

By (D1) of Definition 3.1, there exists a $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process $\widehat{\nu}=\{\widehat{\nu}_s\}_{s\in[t,\infty)}$ on Ω_0 such that $\overline{P}\big\{\overline{U}_s=\overline{\nu}_s$ for a.e. $s\in(t,\infty)\big\}=1$, where $\overline{\nu}_s(\overline{\omega}):=\widehat{\nu}_s\big(\overline{W}(\overline{\omega})\big),\ \forall\,(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$ is a $\mathbb{U}-$valued, $\mathbf{F}^{\overline{W}^t}-$predictable process on $\overline{\Omega}$.

Set $\vartheta=(t,\mathbf{x},\overline{\nu})$. Given $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$, $\overline{\mathcal{M}}^\vartheta_s(\varphi):=\varphi(\overline{W}^t_s,\overline{\mathscr{X}}^\vartheta_s)-\int_t^s\overline{b}\big(r,\overline{\mathscr{X}}^\vartheta_{r\wedge\cdot},\overline{\nu}_r\big)\cdot D\varphi(\overline{W}^t_r,\overline{\mathscr{X}}^\vartheta_r)dr-\frac12\int_t^s\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{\mathscr{X}}^\vartheta_{r\wedge\cdot},\overline{\nu}_r\big):D^2\varphi(\overline{W}^t_r,\overline{\mathscr{X}}^\vartheta_r)dr,\ s\in[t,\infty)$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$adapted continuous process and $\overline{\tau}^\vartheta_n:=\inf\big\{s\in[t,\infty):\big|(\overline{W}^t_s,\overline{\mathscr{X}}^\vartheta_s)\big|\ge n\big\}\wedge(t+n)$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$stopping time. Since $\overline{W}^t$ is a Brownian motion under $\overline{P}$ by (D2) of Definition 3.1, applying Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X,\mu)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{W},\overline{\mathscr{X}}^\vartheta,\overline{\nu}\big)$ shows that

$$
\big\{\overline{\mathcal{M}}^\vartheta_{s\wedge\overline{\tau}^\vartheta_n}(\varphi)\big\}\text{ is a bounded }\mathbf{F}^{\overline{W}^t,\overline{P}}-\text{martingale.}\tag{6.5}
$$

Let $(\mathfrak{u}_o,\mathfrak{x}_o,\mathfrak{t}_o)$ be an arbitrary triplet in $\mathbb{J}\times\Omega_X\times[t,\infty]$ and define a mapping $\Psi_o:\mathcal{Q}\mapsto\overline{\Omega}$ by $\Psi_o(\omega):=\big(\mathcal{B}(\omega),\mathfrak{u}_o,\mathfrak{x}_o,\mathfrak{t}_o\big)\in\overline{\Omega},\ \forall\,\omega\in\mathcal{Q}$. (Actually, we are indifferent to the second, third and fourth components of $\Psi_o(\omega)$.) Since $\overline{W}^t_s(\Psi_o(\omega))=\overline{W}_s(\Psi_o(\omega))-\overline{W}_t(\Psi_o(\omega))=\mathcal{B}^t_s(\omega)$ for any $(s,\omega)\in[t,\infty)\times\mathcal{Q}$, applying Lemma A.1 with $t_0=t$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B}\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{W}\big)$ and $\Phi=\Psi_o$ yields that

$$
\Psi_o^{-1}\big(\mathcal{F}^{\overline{W}^t}_s\big)=\mathcal{F}^{\mathcal{B}^t}_s,\ \Psi_o^{-1}\big(\mathcal{F}^{\overline{W}^t,\overline{P}}_s\big)\subset\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_s,\ \forall\,s\in[t,\infty]\quad\text{and}\quad\big(\mathfrak{p}\circ\Psi_o^{-1}\big)(\overline{A})=\overline{P}(\overline{A}),\ \ \forall\,\overline{A}\in\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty.\tag{6.6}
$$

Then $\mathscr{X}^\vartheta_s(\omega):=\overline{\mathscr{X}}^\vartheta_s(\Psi_o(\overline{\omega}))$, $s\in[0,\infty)$ defines an $\big\{\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process.

Set $\nu_s(\omega):=\widehat{\nu}_s\big(\mathcal{B}(\omega)\big),\ \forall\,(s,\omega)\in[t,\infty)\times\mathcal{Q}$, which is a $\mathbb{U}-$valued, $\mathbf{F}^{\mathcal{B}^t}-$predictable process on $\mathcal{Q}$. Let $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. We define an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$adapted continuous process $\mathscr{M}^\vartheta_s(\varphi):=\varphi(\mathcal{B}^t_s,\mathscr{X}^\vartheta_s)-\int_t^s\overline{b}\big(r,\mathscr{X}^\vartheta_{r\wedge\cdot},\nu_r\big)\cdot D\varphi(\mathcal{B}^t_r,\mathscr{X}^\vartheta_r)dr-\frac12\int_t^s\overline{\sigma}\,\overline{\sigma}^T\big(r,\mathscr{X}^\vartheta_{r\wedge\cdot},\nu_r\big):D^2\varphi(\mathcal{B}^t_r,\mathscr{X}^\vartheta_r)dr,\ \forall\,s\in[t,\infty)$ and define an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping time $\zeta^\vartheta_n:=\inf\big\{s\in[t,\infty):\big|(\mathcal{B}^t_s,\mathscr{X}^\vartheta_s)\big|\ge n\big\}\wedge(t+n)$. Since $\overline{\nu}_s(\Psi_o)=\widehat{\nu}_s\big(\overline{W}(\Psi_o)\big)=\widehat{\nu}_s(\mathcal{B})=\nu_s,\ \forall\,s\in[t,\infty)$, applying Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X,\mu)=\big(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B},\mathscr{X}^\vartheta,\nu\big)$ and using an analogy to (6.3) yield that $\big\{\mathscr{M}^\vartheta_{s\wedge\zeta^\vartheta_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$adapted continuous process under $\mathfrak{p}$ satisfying

$$
\big(\overline{\mathcal{M}}^\vartheta_{s\wedge\overline{\tau}^\vartheta_n}(\varphi)\big)\big(\Psi_o(\omega)\big)=\big(\mathscr{M}^\vartheta_{s\wedge\zeta^\vartheta_n}(\varphi)\big)(\omega),\quad\ \forall\,(s,\omega)\in[t,\infty)\times\mathcal{Q}.\tag{6.7}
$$

Let $t_1,t_2\in[t,\infty)$ with $t_1<t_2$ and let $A\in\mathcal{F}^{\mathcal{B}^t}_{t_1}$. Since $\Psi_o^{-1}(\overline{A})=A$ for some $\overline{A}\in\mathcal{F}^{\overline{W}^t}_{t_1}$ by (6.6), we can derive from (6.5), (6.6) and (6.7) that $0=E_{\overline{P}}\Big[\big(\overline{\mathcal{M}}^{\vartheta}_{t_2\wedge\overline{\tau}^{\vartheta}_n}(\varphi)-\overline{\mathcal{M}}^{\vartheta}_{t_1\wedge\overline{\tau}^{\vartheta}_n}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=E_{\mathfrak{p}}\Big[\Big(\big(\overline{\mathcal{M}}^{\vartheta}_{t_2\wedge\overline{\tau}^{\vartheta}_n}(\varphi)\big)(\Psi_o)-\big(\overline{\mathcal{M}}^{\vartheta}_{t_1\wedge\overline{\tau}^{\vartheta}_n}(\varphi)\big)(\Psi_o)\Big)\mathbf{1}_{\Psi_o^{-1}(\overline{A})}\Big]$ $=E_{\mathfrak{p}}\Big[\big(\mathscr{M}^{\vartheta}_{t_2\wedge\zeta^{\vartheta}_n}(\varphi)-\mathscr{M}^{\vartheta}_{t_1\wedge\zeta^{\vartheta}_n}(\varphi)\big)\mathbf{1}_A\Big]$, which implies that $\big\{\mathscr{M}^{\vartheta}_{s\wedge\zeta^{\vartheta}_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$martingale. Then an application of Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X,\mu)=\big(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B},\mathscr{X}^{\vartheta},\nu\big)$ shows that

$$\mathfrak{p}\{\mathscr{X}^{\vartheta}_s=\mathcal{X}^{t,\mathbf{x},\nu}_s,\ \ \forall\, s\in[0,\infty)\}=1. \tag{6.8}$$

By (D4) of Definition 3.1, there exists a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\gamma}$ on Ω_0 such that $\overline{P}\{\overline{T}=\widehat{\gamma}(\overline{W})\}=1$. Lemma A.2 (1) renders that $\gamma:=\widehat{\gamma}(\mathcal{B})$ is an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping time on $\mathcal{Q}$ while $\widehat{\gamma}(\overline{W})$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$stopping time on $\overline{\Omega}$. For any $i\in\mathbb{N}$, we can deduce from (D3), (6.6) and (6.8) that

$$\begin{aligned} y_i \ge\ & E_{\overline{P}}\bigg[\int_t^{\overline{T}} g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\bigg]=E_{\overline{P}}\bigg[\int_t^{\widehat{\gamma}(\overline{W})} g_i\big(r,\overline{\mathscr{X}}^{\vartheta}_{r\wedge\cdot},\widehat{\nu}_r(\overline{W})\big)dr\bigg]=E_{\mathfrak{p}}\bigg[\int_t^{\widehat{\gamma}(\overline{W}(\Psi_o))} g_i\big(r,\overline{\mathscr{X}}^{\vartheta}_{r\wedge\cdot}(\Psi_o),\widehat{\nu}_r(\overline{W}(\Psi_o))\big)dr\bigg]\\ =\ & E_{\mathfrak{p}}\bigg[\int_t^{\widehat{\gamma}(\mathcal{B})} g_i(r,\mathscr{X}^{\vartheta}_{r\wedge\cdot},\widehat{\nu}_r(\mathcal{B}))dr\bigg]=E_{\mathfrak{p}}\bigg[\int_t^{\gamma} g_i(r,\mathcal{X}^{t,\mathbf{x},\nu}_{r\wedge\cdot},\nu_r)dr\bigg], \end{aligned} \tag{6.9}$$

and similarly $E_{\mathfrak{p}}\big[\int_t^{\gamma} h_i(r,\mathcal{X}^{t,\mathbf{x},\nu}_{r\wedge\cdot},\nu_r)dr\big]=E_{\overline{P}}\big[\int_t^{\overline{T}} h_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]=z_i$. So $(\nu,\gamma)\in\mathcal{C}_{t,\mathbf{x}}(y,z)$. Analogous to (6.9),

$$E_{\overline{P}}\bigg[\int_t^{\overline{T}} f\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr+\mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\bigg]=E_{\mathfrak{p}}\bigg[\int_t^{\gamma} f\big(r,\mathcal{X}^{t,\mathbf{x},\nu}_{r\wedge\cdot},\nu_r\big)dr+\mathbf{1}_{\{\gamma<\infty\}}\pi\big(\gamma,\mathcal{X}^{t,\mathbf{x},\nu}_{\gamma\wedge\cdot}\big)\bigg]\le V(t,\mathbf{x},y,z).$$

Taking supremum over $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ yields that $\overline{V}(t,\mathbf{x},y,z)\le V(t,\mathbf{x},y,z)$. □

Proof of Lemma 4.2: 1) *We first show that Γ is injective*:

Set $\mathbb{Q}_\pi:=\big(\mathbb{Q}\cap[0,\pi/2)\big)\cup\{\pi/2\}$ and let $(\mu^1,\tau_1),(\mu^2,\tau_2)\in\mathfrak{U}\times\mathfrak{S}$ such that $\Gamma(\mu^1,\tau_1)=\Gamma(\mu^2,\tau_2)$. *We make a countable decomposition of the set* $\big\{\omega_0\in\Omega_0:\mu^1_\cdot(\omega_0)\neq\mu^2_\cdot(\omega_0)\ \textit{or}\ \tau_1(\omega_0)\neq\tau_2(\omega_0)\big\}$:

Let $q\in\mathbb{Q}_\pi$ and $n\in\mathbb{N}$. We set $\mathcal{E}^q_n:=(q-1/n,q+1/n)\cap[0,\pi/2]$. Also let $k,j\in\mathbb{N}$ and $i=1,2$. We know from (1.1) and Lemma A.3 that $A^i_{n,k,j,q}:=(\mu^i)^{-1}\big(\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big)\big)\cap\big\{\arctan(\tau_i)\in\mathcal{E}^q_n\big\}$ belongs to $\mathcal{F}^{W,P_0}_\infty$. Then $A_{n,k,j,q}:=A^1_{n,k,j,q}\cap(A^2_{n,k,j,q})^c$ satisfies that

$$\begin{aligned} P_0(A_{n,k,j,q})\ =\ & P_0\big\{\omega_0\in(A^2_{n,k,j,q})^c:\mu^1_\cdot(\omega_0)\in\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big),\tau_1(\omega_0)\in\tan(\mathcal{E}^q_n)\big\}\\ =\ & P_0\big\{\omega_0\in\Omega_0:W_\cdot(\omega_0)\in(A^2_{n,k,j,q})^c,\mu^1_\cdot(\omega_0)\in\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big),\tau_1(\omega_0)\in\tan(\mathcal{E}^q_n)\big\}\\ =\ & \big(\Gamma(\mu^1,\tau_1)\big)\big((A^2_{n,k,j,q})^c\times\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big)\times\tan(\mathcal{E}^q_n)\big)=\big(\Gamma(\mu^2,\tau_2)\big)\big((A^2_{n,k,j,q})^c\times\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big)\times\tan(\mathcal{E}^q_n)\big)\\ =\ & P_0\big\{\omega_0\in(A^2_{n,k,j,q})^c:\mu^2_\cdot(\omega_0)\in\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big),\tau_2(\omega_0)\in\tan(\mathcal{E}^q_n)\big\}=P_0\big((A^2_{n,k,j,q})^c\cap A^2_{n,k,j,q}\big)=0. \end{aligned} \tag{6.10}$$

We claim that

$$\mathcal{A}:=\big\{\omega_0\in\Omega_0:\mu^1_\cdot(\omega_0)\neq\mu^2_\cdot(\omega_0)\big\}\cup\big\{\omega_0\in\Omega_0:\tau_1(\omega_0)\neq\tau_2(\omega_0)\big\}\ \text{is equal to}\ \underset{n,k,j\in\mathbb{N}}{\cup}\ \underset{q\in\mathbb{Q}_\pi}{\cup} A_{n,k,j,q}. \tag{6.11}$$

Clearly, $\underset{n,k,j\in\mathbb{N}}{\cup}\ \underset{q\in\mathbb{Q}_\pi}{\cup} A_{n,k,j,q}\subset\mathcal{A}$. Assume that $\mathcal{A}\cap\Big(\underset{n,k,j\in\mathbb{N}}{\cup}\ \underset{q\in\mathbb{Q}_\pi}{\cup} A_{n,k,j,q}\Big)^c$ is not empty and has an element ω_0.

Given $n,j\in\mathbb{N}$, since the proof of Lemma 1.1 selected $\{\mathfrak{m}_k\}_{k\in\mathbb{N}}$ as a countable dense subset of the topological space $\big(\mathfrak{P}([0,\infty)\times\mathbb{U}),\mathfrak{T}_\sharp\big(\mathfrak{P}([0,\infty)\times\mathbb{U})\big)\big)$, there exist $\mathfrak{k}=\mathfrak{k}(n,j)\in\mathbb{N}$ and $\mathfrak{q}=\mathfrak{q}(n)\in\mathbb{Q}_\pi$ such that $\mathfrak{i}_{\mathbb{J}}\big(\mu^1_\cdot(\omega_0)\big)\in O_{\frac{1}{n}}(\mathfrak{m}_{\mathfrak{k}},\phi_j)$ and $\arctan\big(\tau_1(\omega_0)\big)\in\mathcal{E}^{\mathfrak{q}}_n$. This shows $\omega_0\in(\mu^1)^{-1}\big(\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_{\mathfrak{k}},\phi_j)\big)\big)\cap\big\{\arctan(\tau_1)\in\mathcal{E}^{\mathfrak{q}}_n\big\}=A^1_{n,\mathfrak{k},j,\mathfrak{q}}$. Since $\omega_0\in A^c_{n,\mathfrak{k},j,\mathfrak{q}}$, we see that $\omega_0\in A^2_{n,\mathfrak{k},j,\mathfrak{q}}$, i.e., $\big(\mu^2_\cdot(\omega_0),\arctan(\tau_2(\omega_0))\big)$ also belongs to $\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_{\mathfrak{k}},\phi_j)\big)\times\mathcal{E}^{\mathfrak{q}}_n$. It follows that $\big|\int_0^\infty e^{-s}\big[\phi_j\big(s,\mu^1_s(\omega_0)\big)-\phi_j\big(s,\mu^2_s(\omega_0)\big)\big]ds\big|<2/n$ and $\rho_+\big(\tau_1(\omega_0),\tau_2(\omega_0)\big)=\big|\arctan(\tau_1(\omega_0))-\arctan(\tau_2(\omega_0))\big|<2/n$. Letting $n\to\infty$ yields that $\int_0^\infty e^{-s}\phi_j\big(s,\mu^1_s(\omega_0)\big)ds=\int_0^\infty e^{-s}\phi_j\big(s,\mu^2_s(\omega_0)\big)ds$ and $\tau_1(\omega_0)=\tau_2(\omega_0)$.

As $\{\phi_j\}_{j\in\mathbb{N}}$ is dense in $\widehat{C}_b\big([0,\infty)\times\mathbb{U}\big)$ by Proposition 7.20 of [7], the dominated convergence theorem implies that $\int_0^\infty e^{-s}\,\phi\big(s,\mu^1_s(\omega_0)\big)ds=\int_0^\infty e^{-s}\,\phi\big(s,\mu^2_s(\omega_0)\big)ds$ holds for any $\phi\in\widehat{C}_b\big([0,\infty)\times\mathbb{U}\big)$. By a standard approximation, this equality also holds for any bounded Borel-measurable functions ϕ on $[0,\infty)\times\mathbb{U}$. For any $s\in[0,\infty)$, taking $\phi(r,u)=\mathbf{1}_{\{r\in[0,s]\}}\mathscr{I}(u)$ gives that $\int_0^s e^{-r}\mathscr{I}\big(\mu^1_r(\omega_0)\big)dr=\int_0^s e^{-r}\mathscr{I}\big(\mu^2_r(\omega_0)\big)dr$. Then we obtain that $\mu^1_s(\omega_0)=\mu^2_s(\omega_0)$ for a.e. $s\in(0,\infty)$ or $\mu^1_\cdot(\omega_0)=\mu^2_\cdot(\omega_0)$ in $\mathbb{J}$. A contradiction appears. So the claim (6.11) holds.

By (6.10), one has $P_0\{\omega_0\in\Omega_0: \mu^1_\cdot(\omega_0)\neq\mu^2_\cdot(\omega_0)\}=P_0\{\omega_0\in\Omega_0: \tau_1(\omega_0)\neq\tau_2(\omega_0)\}=0$. The former together with Fubini Theorem renders that $(ds\times dP_0)\{(s,\omega_0)\in[0,\infty)\times\Omega_0: \mu^1_s(\omega_0)\neq\mu^2_s(\omega_0)\}=0$ or $\mu^1=\mu^2$ in $\mathfrak{U}$, while the latter directly means $\tau_1=\tau_2$ in $\mathfrak{S}$. Hence, the mapping $\Gamma: \mathfrak{U}\times\mathfrak{S}\mapsto\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$ is injective.

2) *We next discuss the continuity of* Γ:

Let $\{\mu^n\}_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{U}$ converging to a $\mu\in\mathfrak{U}$ under $\rho_{\mathfrak{U}}$ and let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{S}$ converging to a $\tau\in\mathfrak{S}$ under $\rho_{\mathfrak{S}}$ $\big($i.e., $\lim_{n\to\infty}\rho_{\mathfrak{U}}(\mu^n,\mu)=\lim_{n\to\infty}\rho_{\mathfrak{S}}(\tau_n,\tau)=0\big)$. We need to demonstrate that $P^n:=\Gamma(\mu^n,\tau_n)$ converges to $P:=\Gamma(\mu,\tau)$ under the weak topology of $\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$, i.e.,

$$\lim_{n\to\infty}\int_{(\omega_0,\mathfrak{u},\mathfrak{t})\in\Omega_0\times\mathbb{J}\times\mathbb{T}}\psi(\omega_0,\mathfrak{u},\mathfrak{t})P^n\big(d(\omega_0,\mathfrak{u},\mathfrak{t})\big)=\int_{(\omega_0,\mathfrak{u},\mathfrak{t})\in\Omega_0\times\mathbb{J}\times\mathbb{T}}\psi(\omega_0,\mathfrak{u},\mathfrak{t})P\big(d(\omega_0,\mathfrak{u},\mathfrak{t})\big) \tag{6.12}$$

for any bounded continuous function $\psi: \Omega_0\times\mathbb{J}\times\mathbb{T}\mapsto\mathbb{R}$.

Let ψ be a real-valued bounded continuous function on $\Omega_0\times\mathbb{J}\times\mathbb{T}$. For (6.12), it suffices to show that for any subsequence $\big\{(\mu^{n_k},\tau_{n_k})\big\}_{k\in\mathbb{N}}$ of $\big\{(\mu^n,\tau_n)\big\}_{n\in\mathbb{N}}$, we can find a subsequence $\big\{(\mu^{n'_k},\tau_{n'_k})\big\}_{k\in\mathbb{N}}$ of $\big\{(\mu^{n_k},\tau_{n_k})\big\}_{k\in\mathbb{N}}$ that satisfies (6.12).

Let $\big\{(\mu^{n_k},\tau_{n_k})\big\}_{k\in\mathbb{N}}$ be an arbitrary subsequence of $\big\{(\mu^n,\tau_n)\big\}_{n\in\mathbb{N}}$. Since $0=\lim_{k\to\infty}\rho_{\mathfrak{U}}(\mu^{n_k},\mu)=\lim_{k\to\infty}E_{P_0}\big[\int_0^\infty e^{-s}\big(1\wedge\rho_{\mathbb{U}}(\mu^{n_k}_s,\mu_s)\big)ds\big]$, there exists a subsequence $\Big\{\mu^{\widetilde{n}_k}\Big\}_{k\in\mathbb{N}}$ of $\big\{\mu^{n_k}\big\}_{k\in\mathbb{N}}$ such that for all $\omega_0\in\Omega_0$ except on a P_0-null set $\mathcal{N}_1$, $\lim_{k\to\infty}\rho_{\mathbb{U}}\Big(\mu^{\widetilde{n}_k}_s(\omega_0),\mu_s(\omega_0)\Big)=0$ for a.e. $s\in(0,\infty)$. For any $\phi\in C_b\big([0,\infty)\times\mathbb{U}\big)$, the dominated convergence theorem implies that $\lim_{k\to\infty}\int_0^\infty\phi(s,u)\mathfrak{i}_{\mathbb{J}}\big(\mu^{\widetilde{n}_k}_\cdot(\omega_0)\big)(ds,du)=\lim_{k\to\infty}\int_0^\infty e^{-s}\phi\Big(s,\mu^{\widetilde{n}_k}_s(\omega_0)\Big)ds=\int_0^\infty e^{-s}\phi\big(s,\mu_s(\omega_0)\big)ds=\int_0^\infty\phi(s,u)\mathfrak{i}_{\mathbb{J}}\big(\mu_\cdot(\omega_0)\big)(ds,du)$. Namely, $\Big\{\mathfrak{i}_{\mathbb{J}}\big(\mu^{\widetilde{n}_k}_\cdot(\omega_0)\big)\Big\}_{k\in\mathbb{N}}$ converges to $\mathfrak{i}_{\mathbb{J}}\big(\mu_\cdot(\omega_0)\big)$ under the weak topology $\mathfrak{T}_\sharp\big(\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)\big)$ of $\mathfrak{P}\big([0,\infty)\times\mathbb{U}\big)$, or equivalently, $\big\{\mu^{\widetilde{n}_k}_\cdot(\omega_0)\big\}_{k\in\mathbb{N}}$ converges to $\mu_\cdot(\omega_0)$ under the induced topology $\mathfrak{T}_\sharp(\mathbb{J})$ of $\mathbb{J}$.

As $0=\lim_{k\to\infty}\rho_{\mathfrak{S}}\big(\tau_{\widetilde{n}_k},\tau\big)=\lim_{k\to\infty}E_{P_0}\big[\rho_+(\tau_{\widetilde{n}_k},\tau)\big]$, one can extract a subsequence $\big\{n'_k\big\}_{k\in\mathbb{N}}$ from $\big\{\widetilde{n}_k\big\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\rho_+\big(\tau_{n'_k}(\omega_0),\tau(\omega_0)\big)=0$ for all $\omega_0\in\Omega_0$ except on a P_0-null set $\mathcal{N}_2$. Given $\omega_0\in(\mathcal{N}_1\cup\mathcal{N}_2)^c$, since $\Big\{\mu^{n'_k}_\cdot(\omega_0)\Big\}_{k\in\mathbb{N}}$ also converges to $\mu_\cdot(\omega_0)$ under $\mathfrak{T}_\sharp(\mathbb{J})$, the continuity of ψ and the bounded convergence theorem yield that

$$\begin{aligned}\lim_{k\to\infty}\int_{(\omega_0,\mathfrak{u},\mathfrak{t})\in\Omega_0\times\mathbb{J}\times\mathbb{T}}&\psi(\omega_0,\mathfrak{u},\mathfrak{t})P^{n'_k}\big(d(\omega_0,\mathfrak{u},\mathfrak{t})\big)=\lim_{k\to\infty}\int_{\Omega_0}\psi\Big(\omega_0,\mu^{n'_k}_\cdot(\omega_0),\tau_{n'_k}(\omega_0)\Big)P_0(d\omega_0)\\&=\int_{\Omega_0}\psi\big(\omega_0,\mu_\cdot(\omega_0),\tau(\omega_0)\big)P_0(d\omega_0)=\int_{(\omega_0,\mathfrak{u},\mathfrak{t})\in\Omega_0\times\mathbb{J}\times\mathbb{T}}\psi(\omega_0,\mathfrak{u},\mathfrak{t})P\big(d(\omega_0,\mathfrak{u},\mathfrak{t})\big).\end{aligned}$$ □

Proof of Proposition 4.1: Fix $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$.

1) Let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}$, which is clearly of $\overline{\mathcal{P}}^1_{t,\mathbf{x}}$. *We show that* $\overline{P}$ *also belongs to* $\overline{\mathcal{P}}^2_t$ *and* $\overline{\mathcal{P}}^3_t$.

By (D1′) and (D4′) of Remark 3.1, there exist a $\mathbb{U}-$valued, $\mathbf{F}^W-$predictable process $\ddot{\mu}=\{\ddot{\mu}_{\mathfrak{s}}\}_{\mathfrak{s}\in[0,\infty)}$ and a $[0,\infty]-$valued $\mathbf{F}^{W,P_0}-$stopping time $\ddot{\tau}$ on Ω_0 such that $\overline{P}\big\{\overline{\mathscr{U}}^t_{\mathfrak{s}}=\ddot{\mu}_{\mathfrak{s}}(\overline{\mathscr{W}}^t)$ for a.e. $\mathfrak{s}\in(0,\infty)\big\}=\overline{P}\big\{\overline{T}=t+\ddot{\tau}\big(\overline{\mathscr{W}}^t\big)\big\}=1$. Fubini Theorem shows that $\ddot{\mathscr{N}}:=\big\{\omega_0\in\Omega_0: \ddot{\mu}_\cdot(\omega_0)\notin\mathbb{J}\big\}$ is an $\mathcal{F}^{W,P_0}_\infty-$measurable set with zero P_0-measure or $\ddot{\mathscr{N}}\in\mathscr{N}_{P_0}\big(\mathcal{F}^W_\infty\big)$. Then $\breve{\mu}_{\mathfrak{s}}:=\ddot{\mu}_{\mathfrak{s}}\mathbf{1}_{\ddot{\mathscr{N}}^c}+u_0\mathbf{1}_{\ddot{\mathscr{N}}}$, $\mathfrak{s}\in[0,\infty)$ is an $\mathbf{F}^{W,P_0}-$predictable process with all paths in $\mathbb{J}$.

1a) *We first show that* $\overline{Q}_{t,\overline{P}}=\Gamma(\breve{\mu},\ddot{\tau})$ *and thus* $\overline{P}\in\overline{\mathcal{P}}^2_t$:

As $\overline{\mathscr{W}}^t_{\mathfrak{s}}=\overline{W}_{t+\mathfrak{s}}-\overline{W}_t$, $\mathfrak{s}\in[0,\infty)$ is a Brownian motion under $\overline{P}$ by (D2), using Lemma A.1 with $t_0=0$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{\mathscr{W}}^t\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W\big)$ and $\Phi=\overline{\mathscr{W}}^t$ yields that

$$\big(\overline{\mathscr{W}}^t\big)^{-1}(A_0)\in\mathcal{F}^{\overline{\mathscr{W}}^t,\overline{P}}_\infty=\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty \quad\text{and}\quad \overline{P}\circ\big(\overline{\mathscr{W}}^t\big)^{-1}(A_0)=P_0(A_0), \quad \forall\, A_0\in\mathcal{F}^{W,P_0}_\infty. \tag{6.13}$$

For any $A_0\in\mathscr{B}(\Omega_0)=\mathcal{F}^W_\infty$, $\mathcal{A}\in\mathscr{B}(\mathbb{J})$ and $\mathcal{E}\in\mathscr{B}(\mathbb{T})$, since $\breve{\mu}^{-1}(\mathcal{A})\in\mathcal{F}^{W,P_0}_\infty$ by Lemma A.3 and since $\ddot{\tau}^{-1}(\mathcal{E})\in\mathcal{F}^{W,P_0}_\infty$, we can derive that

$$\begin{aligned}\overline{Q}_{t,\overline{P}}(A_0\times\mathcal{A}\times\mathcal{E})&=\overline{P}\big\{\big(\overline{\mathscr{W}}^t,\overline{\mathscr{U}}^t,\overline{T}-t\big)\in A_0\times\mathcal{A}\times\mathcal{E}\big\}=\overline{P}\big\{\big(\overline{\mathscr{W}}^t,\ddot{\mu}(\overline{\mathscr{W}}^t),\ddot{\tau}(\overline{\mathscr{W}}^t)\big)\in A_0\times\mathcal{A}\times\mathcal{E}\big\}\\&=\overline{P}\big\{\big(\overline{\mathscr{W}}^t,\breve{\mu}(\overline{\mathscr{W}}^t),\ddot{\tau}(\overline{\mathscr{W}}^t)\big)\in A_0\times\mathcal{A}\times\mathcal{E}\big\}=\overline{P}\circ(\overline{\mathscr{W}}^t)^{-1}\big\{(W,\breve{\mu},\ddot{\tau})\in A_0\times\mathcal{A}\times\mathcal{E}\big\}=\overline{P}\circ(\overline{\mathscr{W}}^t)^{-1}\big(A_0\cap\breve{\mu}^{-1}(\mathcal{A})\cap\ddot{\tau}^{-1}(\mathcal{E})\big)\\&=P_0\big(A_0\cap\breve{\mu}^{-1}(\mathcal{A})\cap\ddot{\tau}^{-1}(\mathcal{E})\big)=P_0\big\{(W,\breve{\mu},\ddot{\tau})\in A_0\times\mathcal{A}\times\mathcal{E}\big\}=\big(\Gamma(\breve{\mu},\ddot{\tau})\big)(A_0\times\mathcal{A}\times\mathcal{E}).\end{aligned}$$

Then Dynkin's Pi-Lambda Theorem implies that $\overline{Q}_{t,\overline{P}}=\Gamma(\breve{\mu},\ddot{\tau})$ on $\mathscr{B}(\Omega_0\times\mathbb{J}\times\mathbb{T})$. So $\overline{P}$ belongs to $\overline{\mathcal{P}}^2_t$.

1b) Let $\widehat{\mu}=\{\widehat{\mu}_s\}_{s\in[t,\infty)}$ be the $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process in (D1) such that the complement of $\overline{\Omega}_\mu:=\big\{\overline{\omega}\in\overline{\Omega}:\overline{U}_s(\overline{\omega})=\overline{\mu}_s(\overline{\omega})$ for a.e. $s\in(t,\infty)\big\}$ is of $\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$, where $\overline{\mu}_s(\overline{\omega}):=\widehat{\mu}_s\big(\overline{W}(\overline{\omega})\big)$, $\forall\,(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$. Given $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$, (D2$'$) of Remark 3.1 shows that $\big\{\overline{M}^{t,\overline{\mu}}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $(\overline{\mathbf{F}}^t,\overline{P})-$martingale. For any $(\mathfrak{s},\mathfrak{r})\in\mathbb{Q}^{2,<}_+$ and $\{(s_i,\mathcal{O}_i)\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})$, since $\overline{M}^t_s(\varphi)=\overline{M}^{t,\overline{\mu}}_s(\varphi)$, $\forall\,s\in[t,\infty)$ on $\overline{\Omega}_\mu$ and since $\big\{(\overline{W}^t_{t+s_i},\overline{X}_{t+s_i})\in\mathcal{O}_i\big\}\in\overline{\mathcal{F}}^t_{t+\mathfrak{s}}$ for $i=1,\cdots,k$, one has $E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod\limits_{i=1}^k\mathbf{1}_{\{(\overline{W}^t_{t+s_i},\overline{X}_{t+s_i})\in\mathcal{O}_i\}}\Big]=E_{\overline{P}}\Big[\big(\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod\limits_{i=1}^k\mathbf{1}_{\{(\overline{W}^t_{t+s_i},\overline{X}_{t+s_i})\in\mathcal{O}_i\}}\Big]=0$. So $\overline{P}$ is also of $\overline{\mathcal{P}}^3_t$, which shows $\overline{\mathcal{P}}_{t,\mathbf{x}}\subset\overline{\mathcal{P}}^1_{t,\mathbf{x}}\cap\overline{\mathcal{P}}^2_t\cap\overline{\mathcal{P}}^3_t$.

2) Next, let $\overline{P}\in\overline{\mathcal{P}}^1_{t,\mathbf{x}}\cap\overline{\mathcal{P}}^2_t\cap\overline{\mathcal{P}}^3_t$. *We need to demonstrate that $\overline{P}$ also belongs to $\overline{\mathcal{P}}_{t,\mathbf{x}}$.*

2a) Fix $i,j\in\{1,\cdots,d\}$. We set $\phi_i(w,x):=w_i$ and $\phi_{ij}(w,x):=w_iw_j$ for any $w=(w_1,\cdots,w_d)\in\mathbb{R}^d$ and $x\in\mathbb{R}^l$. Clearly, $\phi_i,\phi_{ij}\in\mathfrak{C}(\mathbb{R}^{d+l})$.

We verify that $\overline{M}^t_s(\phi_i)=\overline{W}^{t,i}_s$, $\overline{M}^t_s(\phi_{ij})=\overline{W}^{t,i}_s\overline{W}^{t,j}_s-\delta_{ij}(s-t)$, $\forall\,s\in[t,\infty)$ are $\mathbf{F}^{\overline{W}^t}-$local martingales, where $\overline{W}^t_s=\big(\overline{W}^{t,1}_s,\cdots,\overline{W}^{t,d}_s\big)$. Then $\overline{P}$ satisfies (D2) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$ by Lévy's characterization theorem.

Let $n\in\mathbb{N}$ and $(\mathfrak{s},\mathfrak{r})\in\mathbb{Q}^{2,<}_+$. As $\overline{P}\in\overline{\mathcal{P}}^3_t$, it holds for any $\big\{(s_i,\mathcal{O}_i)\big\}_{i=1}^k\subset\big(\mathbb{Q}\cap(0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^d)$ that $E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\phi_i)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\phi_i)\big)\prod\limits_{i=1}^k\mathbf{1}_{\{\overline{W}^t_{t+s_i}\in\mathcal{O}_i\}}\Big]=0$ and $E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\phi_{ij})-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\phi_{ij})\big)\prod\limits_{i=1}^k\mathbf{1}_{\{\overline{W}^t_{t+s_i}\in\mathcal{O}_i\}}\Big]=0$. So the Lambda-system $\overline{\Lambda}^{t,n}_{\mathfrak{s},\mathfrak{r}}:=\Big\{\overline{A}\in\mathscr{B}(\overline{\Omega}):E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\phi_i)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\phi_i)\big)\mathbf{1}_{\overline{A}}\Big]=0$ and $E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\phi_{ij})-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\phi_{ij})\big)\mathbf{1}_{\overline{A}}\Big]=0\Big\}$ includes the Pi-system $\Big\{\Big(\bigcap\limits_{i=1}^k(\overline{W}^t_{t+s_i})^{-1}(\mathcal{O}_i)\Big):\big\{(s_i,\mathcal{O}_i)\big\}_{i=1}^k\subset\big(\mathbb{Q}\cap(0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^d)\Big\}$, which generates $\mathcal{F}^{\overline{W}^t}_{t+\mathfrak{s}}$. Dynkin's Pi-Lambda Theorem renders that $\mathcal{F}^{\overline{W}^t}_{t+\mathfrak{s}}\subset\overline{\Lambda}^{t,n}_{\mathfrak{s},\mathfrak{r}}$, i.e.,

$$E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\phi_i)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\phi_i)\big)\mathbf{1}_{\overline{A}}\Big]=0 \text{ and } E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\phi_{ij})-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\phi_{ij})\big)\mathbf{1}_{\overline{A}}\Big]=0,\ \ \forall\,\overline{A}\in\mathcal{F}^{\overline{W}^t}_{t+\mathfrak{s}}. \tag{6.14}$$

Let $t\le s<r<\infty$ and $\overline{A}\in\mathcal{F}^{\overline{W}^t}_s$. Taking $(\mathfrak{s},\mathfrak{r})=\Big(\frac{\lceil(s-t)2^m\rceil}{2^m},\frac{1+\lceil(r-t)2^m\rceil}{2^m}\Big)$, $m\in\mathbb{N}$ in (6.14) and sending $m\to\infty$, we can deduce from the continuity of bounded processes $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\phi_i)\big\}_{s\in[t,\infty)}$ and $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\phi_{ij})\big\}_{s\in[t,\infty)}$ that $E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge r}(\phi_i)-\overline{M}^t_{\overline{\tau}^t_n\wedge s}(\phi_i)\big)\mathbf{1}_{\overline{A}}\Big]=0$ and $E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge r}(\phi_{ij})-\overline{M}^t_{\overline{\tau}^t_n\wedge s}(\phi_{ij})\big)\mathbf{1}_{\overline{A}}\Big]=0$. So $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\phi_i)\big\}_{s\in[t,\infty)}$ and $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\phi_{ij})\big\}_{s\in[t,\infty)}$ are two $\big(\mathbf{F}^{\overline{W}^t},\overline{P}\big)-$martingales. As $\lim\limits_{n\to\infty}\uparrow\overline{\tau}^t_n=\infty$, we see that $\big\{\overline{M}^t_s(\phi_i)=\overline{W}^{t,i}_s\big\}_{s\in[t,\infty)}$ and $\big\{\overline{M}^t_s(\phi_{ij})=\overline{W}^{t,i}_s\overline{W}^{t,j}_s-\delta_{ij}(s-t)\big\}_{s\in[t,\infty)}$ are $\mathbf{F}^{\overline{W}^t}-$local martingales. Lévy's characterization theorem implies that $\overline{W}^t$ is a Brownian motion on $\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}\big)$. So $\overline{P}$ satisfies (D2) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$. We still have (6.13) since $\overline{\mathscr{W}}^t$ is also a Brownian motion under $\overline{P}$.

As $\overline{P}\in\overline{\mathcal{P}}^2_t$, there exist a $\mathbb{U}-$valued, $\mathbf{F}^{W,P_0}-$predictable process $\ddot{\nu}=\big\{\ddot{\nu}_s\big\}_{s\in[0,\infty)}$ with all paths in $\mathbb{J}$ and a $[0,\infty]-$valued $\mathbf{F}^{W,P_0}-$stopping time $\ddot{\gamma}$ on Ω_0 such that $\overline{Q}_{t,\overline{P}}=\Gamma\big(\ddot{\nu},\ddot{\gamma}\big)=P_0\circ\big(W,\ddot{\nu},\ddot{\gamma}\big)^{-1}$. Given $D\in\mathscr{B}(\Omega_0\times\mathbb{J}\times\mathbb{T})$, taking $A_0=\big(W,\ddot{\nu},\ddot{\gamma}\big)^{-1}(D)\in\mathcal{F}^{W,P_0}_\infty$ in (6.13) yields that

$$\overline{P}\big\{(\overline{\mathscr{W}}^t,\overline{\mathscr{U}}^t,\overline{T}-t)\in D\big\}=\overline{Q}_{t,\overline{P}}(D)=P_0\circ\big(W,\ddot{\nu},\ddot{\gamma}\big)^{-1}(D)=\overline{P}\circ(\overline{\mathscr{W}}^t)^{-1}\big((W,\ddot{\nu},\ddot{\gamma})^{-1}(D)\big)=\overline{P}\big\{\big(\overline{\mathscr{W}}^t,\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t),\ddot{\gamma}(\overline{\mathscr{W}}^t)\big)\in D\big\},$$

which shows the joint $\overline{P}-$distribution of $(\overline{\mathscr{W}}^t,\overline{\mathscr{U}}^t,\overline{T})$ is the same as the joint $\overline{P}-$distribution of $\big(\overline{\mathscr{W}}^t,\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t),t+\ddot{\gamma}(\overline{\mathscr{W}}^t)\big)$. Similar to Part (2b) in the proof of [4, Proposition 4.1], we can use the equality $\overline{P}\circ(\overline{\mathscr{W}}^t,\overline{T})^{-1}=\overline{P}\circ\big(\overline{\mathscr{W}}^t,t+\ddot{\gamma}(\overline{\mathscr{W}}^t)\big)^{-1}$ to derive that $\overline{P}\{\overline{T}=t+\ddot{\gamma}(\overline{\mathscr{W}}^t)\}=1$. Namely, $\overline{P}$ satisfies (D4$'$) or equivalently (D4) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

2b) *We next show $\overline{P}\big\{\overline{\mathscr{U}}^t_{\mathfrak{s}}=\ddot{\nu}_{\mathfrak{s}}(\overline{\mathscr{W}}^t)$ for a.e. $\mathfrak{s}\in(0,\infty)\big\}=1$ and thus $\overline{P}$ satisfies (D1$'$) or equivalently (D1) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.*

Fix $\mathcal{A}\in\mathscr{B}(\mathbb{J})$ and define $\Lambda:=\big\{\overline{A}\in\mathscr{B}_{\overline{P}}(\overline{\Omega}):\overline{P}\big(\overline{A}\cap\{\overline{\mathscr{U}}^t\in\mathcal{A}\}\big)=\overline{P}\big(\overline{A}\cap\{\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t)\in\mathcal{A}\}\big)\big\}$. As $\overline{P}\{\overline{\mathscr{U}}^t\in\mathcal{A}\}=\overline{P}\{(\overline{\mathscr{W}}^t,\overline{\mathscr{U}}^t)\in\Omega_0\times\mathcal{A}\}=\overline{P}\{(\overline{\mathscr{W}}^t,\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t))\in\Omega_0\times\mathcal{A}\}=\overline{P}\{\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t)\in\mathcal{A}\}$, we see that $\overline{\Omega}\in\Lambda$ and Λ is thus a Lambda-system.

For any $(\mathfrak{s},\mathcal{E})\in[0,\infty)\times\mathscr{B}(\mathbb{R}^d)$, since $W_{\mathfrak{s}}:\Omega_0\mapsto\mathbb{R}^d$ is a continuous function, one has $W^{-1}_{\mathfrak{s}}(\mathcal{E})\in\mathscr{B}(\Omega_0)$. Then it holds for any $\{(\mathfrak{s}_i,\mathcal{E}_i)\}_{i=1}^N\subset[0,\infty)\times\mathscr{B}(\mathbb{R}^d)$ that

$$\overline{P}\Big(\Big(\bigcap_{i=1}^N(\mathscr{W}^t_{\mathfrak{s}_i})^{-1}(\mathcal{E}_i)\Big)\cap\{\overline{\mathscr{U}}^t\in\mathcal{A}\}\Big)=\overline{P}\Big(\Big\{\overline{\mathscr{W}}^t\in\bigcap_{i=1}^N W_{\mathfrak{s}_i}^{-1}(\mathcal{E}_i)\Big\}\cap\{\overline{\mathscr{U}}^t\in\mathcal{A}\}\Big)=\overline{P}\Big\{\big(\overline{\mathscr{W}}^t,\overline{\mathscr{U}}^t\big)\in\bigcap_{i=1}^N W_{\mathfrak{s}_i}^{-1}(\mathcal{E}_i)\times\mathcal{A}\Big\}$$
$$=\overline{P}\Big\{\big(\overline{\mathscr{W}}^t,\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t)\big)\in\bigcap_{i=1}^N W_{\mathfrak{s}_i}^{-1}(\mathcal{E}_i)\times\mathcal{A}\Big\}=\overline{P}\Big(\Big(\bigcap_{i=1}^N(\overline{\mathscr{W}}^t_{\mathfrak{s}_i})^{-1}(\mathcal{E}_i)\Big)\cap\{\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t)\in\mathcal{A}\}\Big).$$

So Λ contains the Pi-system $\Big\{\bigcap_{i=1}^N(\overline{\mathscr{W}}^t_{\mathfrak{s}_i})^{-1}(\mathcal{E}_i):(\mathfrak{s}_i,\mathcal{E}_i)\in[0,\infty)\times\mathscr{B}(\mathbb{R}^d),i=1,\cdots,N\Big\}$, which generates $\mathcal{F}_\infty^{\overline{\mathscr{W}}^t}=\mathcal{F}_\infty^{\overline{W}^t}$. Dynkin's Pi-Lambda Theorem implies that $\mathcal{F}_\infty^{\overline{W}^t,\overline{P}}\subset\Lambda$, i.e.

$$\overline{P}\big(\overline{A}\cap\{\overline{\mathscr{U}}^t\in\mathcal{A}\}\big)=\overline{P}\big(\overline{A}\cap\{\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t)\in\mathcal{A}\}\big),\quad\forall\,\overline{A}\in\mathcal{F}_\infty^{\overline{W}^t,\overline{P}},\ \forall\,\mathcal{A}\in\mathscr{B}(\mathbb{J}).\tag{6.15}$$

Let $n,k,j\in\mathbb{N}$. By Lemma A.3, $\mathcal{O}_{n,k,j}:=\big\{\omega_0\in\Omega_0:\ddot{\nu}_\cdot(\omega_0)\in\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big\}\in\mathcal{F}_\infty^{W,P_0}$. Since (6.13) shows that $\big(\overline{\mathscr{W}}^t\big)^{-1}(\mathcal{O}_{n,k,j})\in\mathcal{F}_\infty^{\overline{W}^t,\overline{P}}$, applying (6.15) with $\overline{A}=\big(\overline{\mathscr{W}}^t\big)^{-1}(\mathcal{O}^c_{n,k,j})$ and $\mathcal{A}=\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)$ yields that $\overline{P}\Big((\overline{\mathscr{W}}^t)^{-1}(\mathcal{O}^c_{n,k,j})\cap\{\overline{\mathscr{U}}^t\in\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big\}\Big)=\overline{P}\Big((\overline{\mathscr{W}}^t)^{-1}(\mathcal{O}^c_{n,k,j})\cap\{\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t)\in\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big\}\Big)=\overline{P}\big((\overline{\mathscr{W}}^t)^{-1}(\mathcal{O}^c_{n,k,j})\cap(\overline{\mathscr{W}}^t)^{-1}(\mathcal{O}_{n,k,j})\big)=0$.

Similar to (6.11), we can deduce that $\big\{\overline{\omega}\in\overline{\Omega}:\overline{\mathscr{U}}^t(\overline{\omega})\neq\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t(\overline{\omega}))\big\}=\bigcup_{n,k,j\in\mathbb{N}}\Big((\overline{\mathscr{W}}^t)^{-1}(\mathcal{O}^c_{n,k,j})\cap\{\overline{\mathscr{U}}^t\in\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big\}\Big)$. It follows that $\overline{P}\big\{\overline{\omega}\in\overline{\Omega}:\overline{\mathscr{U}}^t(\overline{\omega})\neq\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t(\overline{\omega}))\big\}=0$ and thus

$$\overline{P}\big\{\overline{\omega}\in\overline{\Omega}:\overline{\mathscr{U}}^t_{\mathfrak{s}}(\overline{\omega})=\ddot{\nu}_{\mathfrak{s}}(\overline{\mathscr{W}}^t(\overline{\omega}))\text{ for a.e. }\mathfrak{s}\in(0,\infty)\big\}=\overline{P}\big\{\overline{\omega}\in\overline{\Omega}:\overline{\mathscr{U}}^t(\overline{\omega})=\ddot{\nu}_\cdot(\overline{\mathscr{W}}^t(\overline{\omega}))\big\}=1.\tag{6.16}$$

Like Lemma 2.4 of [44], we can construct a $[0,1]-$valued $\mathbf{F}^W-$predictable process $\{\breve{\nu}_{\mathfrak{s}}\}_{\mathfrak{s}\in[0,\infty)}$ on Ω_0 such that $\breve{\nu}_{\mathfrak{s}}(\omega_0)=\mathscr{I}\big(\ddot{\nu}_{\mathfrak{s}}(\omega_0)\big)$ for $d\mathfrak{s}\times dP_0-$a.s. $(\mathfrak{s},\omega_0)\in[0,\infty)\times\Omega_0$. By Fubini Theorem, it holds for all $\omega_0\in\Omega_0$ except on a $\mathcal{N}_\nu\in\mathscr{N}_{P_0}\big(\mathcal{F}_\infty^W\big)$ that $\breve{\nu}_{\mathfrak{s}}(\omega_0)=\mathscr{I}\big(\ddot{\nu}_{\mathfrak{s}}(\omega_0)\big)$ for a.e. $\mathfrak{s}\in[0,\infty)$. Define $\nu^o_{\mathfrak{s}}(\omega_0):=\mathscr{I}^{-1}\big(\breve{\nu}_{\mathfrak{s}}(\omega_0)\big)\mathbf{1}_{\{\breve{\nu}_{\mathfrak{s}}(\omega_0)\in\mathfrak{E}\}}+u_0\mathbf{1}_{\{\breve{\nu}_{\mathfrak{s}}(\omega_0)\notin\mathfrak{E}\}}$, $(\mathfrak{s},\omega_0)\in[0,\infty)\times\Omega_0$, which is a $\mathbb{U}-$valued $\mathbf{F}^{W^t}-$predictable process. Given $\overline{\omega}\in\big(\overline{\mathscr{W}}^t\big)^{-1}(\mathcal{N}^c_\nu)$, it holds for a.e. $\mathfrak{s}\in(0,\infty)$ that $\breve{\nu}_{\mathfrak{s}}\big(\overline{\mathscr{W}}^t(\overline{\omega})\big)=\mathscr{I}\big(\ddot{\nu}_{\mathfrak{s}}\big(\overline{\mathscr{W}}^t(\overline{\omega})\big)\big)$ and thus $\nu^o_{\mathfrak{s}}\big(\overline{\mathscr{W}}^t(\overline{\omega})\big)=\ddot{\nu}_{\mathfrak{s}}\big(\overline{\mathscr{W}}^t(\overline{\omega})\big)$. As $\big(\overline{\mathscr{W}}^t\big)^{-1}(\mathcal{N}_\nu)\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}_\infty^{\overline{W}^t}\big)$, (6.16) leads to that $\overline{P}\big\{\overline{\omega}\in\overline{\Omega}:\overline{\mathscr{U}}^t_{\mathfrak{s}}(\overline{\omega})=\nu^o_{\mathfrak{s}}(\overline{\mathscr{W}}^t(\overline{\omega}))\text{ for a.e. }\mathfrak{s}\in(0,\infty)\big\}=1$. To wit, $\overline{P}$ satisfies (D1$'$) or equivalently (D1) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

2c) Let $\widehat{\nu}=\{\widehat{\nu}_s\}_{s\in[t,\infty)}$ be the $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process in (D1) such that the complement of $\overline{\Omega}_\nu:=\big\{\overline{\omega}\in\overline{\Omega}:\overline{U}_s(\overline{\omega})=\overline{\nu}_s(\overline{\omega})\text{ for a.e. }s\in(t,\infty)\big\}$ is of $\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$, where $\overline{\nu}_s(\overline{\omega}):=\widehat{\nu}_s\big(\overline{W}(\overline{\omega})\big)$, $\forall\,(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$.

Let $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. *We show that $\overline{M}^{t,\overline{\nu}}_{s\wedge\overline{\tau}^t_n}(\varphi)$, $s\in[t,\infty)$ is an $\big(\overline{\mathbf{F}}^t,\overline{P}\big)-$martingale and thus $\overline{P}$ satisfies (D3) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$ according to the martingale-problem formulation.*

As $\overline{P}\big\{\overline{X}_s=\mathbf{x}(s),\,\forall\,s\in[0,t]\big\}=1$, applying Proposition 1.2 with $\big(\Omega,\mathcal{F},P,B,X,\mu\big)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{W},\overline{X},\overline{\nu}\big)$ implies that $\big\{\overline{M}^{t,\overline{\nu}}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\overline{\mathbf{F}}^t-$adapted continuous process under $\overline{P}$.

Let $(\mathfrak{s},\mathfrak{r})\in\mathbb{Q}^{2,<}_+$, $\big\{(t_i,\mathcal{O}_i)\big\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,t]\big)\times\mathscr{O}(\mathbb{R}^l)$ and $\big\{(s_j,\mathcal{O}'_j)\big\}_{j=1}^m\subset\big(\mathbb{Q}\cap(0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})$. If $\mathbf{x}(t_{\mathfrak{i}})\notin\mathcal{O}_{\mathfrak{i}}$ for some $\mathfrak{i}\in\{1,\cdots,k\}$, then $\overline{P}\{\overline{X}_{t_{\mathfrak{i}}}\in\mathcal{O}_{\mathfrak{i}}\}=0$ and thus $E_{\overline{P}}\Big[\big(\overline{M}^{t,\overline{\nu}}_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,\overline{\nu}}_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod_{i=1}^k\mathbf{1}_{\{\overline{X}_{t_i}\in\mathcal{O}_i\}}\prod_{j=1}^m\mathbf{1}_{\{(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j})\in\mathcal{O}'_j\}}\Big]=0$. On the other hand, if $\mathbf{x}(t_i)\in\mathcal{O}_i$ for each $i\in\{1,\cdots,k\}$, since $\overline{M}^t_s(\varphi)=\overline{M}^{t,\overline{\nu}}_s(\varphi)$, $\forall\,s\in[t,\infty)$ on $\overline{\Omega}_\nu$, then

$$E_{\overline{P}}\Big[\big(\overline{M}^{t,\overline{\nu}}_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,\overline{\nu}}_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod_{i=1}^k\mathbf{1}_{\{\overline{X}_{t_i}\in\mathcal{O}_i\}}\prod_{j=1}^m\mathbf{1}_{\{(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j})\in\mathcal{O}'_j\}}\Big]$$
$$=E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod_{j=1}^m\mathbf{1}_{\{(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j})\in\mathcal{O}'_j\}}\Big]=0.$$

Since $\overline{\mathcal{F}}^t_{t+\mathfrak{s}}$ is generated by the Pi-system $\Big\{\Big(\bigcap_{i=1}^k\overline{X}_{t_i}^{-1}(\mathcal{O}_i)\Big)\cap\Big(\bigcap_{j=1}^m(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j})^{-1}(\mathcal{O}'_j)\Big):\big\{(t_i,\mathcal{O}_i)\big\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,t]\big)\times\mathscr{O}(\mathbb{R}^l),\ \big\{(s_j,\mathcal{O}'_j)\big\}_{j=1}^m\subset\big(\mathbb{Q}\cap(0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})\Big\}$, Dynkin's Pi-Lambda Theorem renders that (c.f. (6.14))

$$E_{\overline{P}}\Big[\big(\overline{M}^{t,\overline{\nu}}_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,\overline{\nu}}_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=0,\quad\forall\,\overline{A}\in\overline{\mathcal{F}}^t_{t+\mathfrak{s}}.\tag{6.17}$$

Let $t \le s < r < \infty$ and $\overline{A} \in \overline{\mathcal{F}}^t_s$. Taking $(\mathfrak{s},\mathfrak{r}) = \Big(\frac{\lceil (s-t)2^k \rceil}{2^k}, \frac{1+\lceil (r-t)2^k \rceil}{2^k}\Big)$, $k \in \mathbb{N}$ in (6.17) and letting $k \to \infty$, we can deduce from the continuity of bounded process $\big\{\overline{M}^{t,\overline{\nu}}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ that

$$E_{\overline{P}}\Big[\big(\overline{M}^{t,\overline{\nu}}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\nu}}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big] = 0, \quad \text{i.e., } \big\{\overline{M}^{t,\overline{\nu}}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)} \text{ is an } \big(\overline{\mathbf{F}}^t, \overline{P}\big)\text{−martingale.} \tag{6.18}$$

By Remark 3.1 (ii), $\overline{P}$ satisfies (D3) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$. □

Proof of Proposition 4.2: According to Proposition 4.1, $\langle\langle\overline{\mathcal{P}}\rangle\rangle$ is the intersection of $\langle\langle\overline{\mathcal{P}}\rangle\rangle_1 := \big\{\big(t,\mathbf{x},\overline{P}\big) \in [0,\infty) \times \Omega_X \times \mathfrak{P}\big(\overline{\Omega}\big) : \overline{P} \in \overline{\mathcal{P}}^1_{t,\mathbf{x}}\big\}$ and $\langle\langle\overline{\mathcal{P}}\rangle\rangle_i := \big\{\big(t,\mathbf{x},\overline{P}\big) \in [0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big) : \overline{P} \in \overline{\mathcal{P}}^i_t\big\}$ for $i=2,3$. Similar to the proof of [4, Proposition 4.2], one can easily show that $\langle\langle\overline{\mathcal{P}}\rangle\rangle_1$ is a Borel subset of $[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big)$.

1) Since Ω_0 is a Polish space and since $\mathbb{J}$ is a Borel space, we know from Proposition 7.13 and Corollary 7.25.1 of [7] that $\Omega_0\times\mathbb{J}\times\mathbb{T}$ with the product topology is a Borel space and $\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$ is also a Borel space. As Lemma 4.1 and Lemma 4.2 show that $\Gamma : \mathfrak{U}\times\mathfrak{S} \ni (\mu,\tau) \mapsto P_0\circ(W,\mu,\tau)^{-1} \in \mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$ is a continuous injection from the Polish space $\mathfrak{U}\times\mathfrak{S}$ to $\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$, the image $\Gamma(\mathfrak{U}\times\mathfrak{S})$ is a Lusin subset of $\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$. Theorem A.6 of [46] implies that $\Gamma(\mathfrak{U}\times\mathfrak{S})$ is even a Borel subset of the Borel space $\mathfrak{P}\big(\Omega_0\times\mathbb{J}\times\mathbb{T}\big)$. Then Lemma 4.3 yields $\langle\langle\overline{\mathcal{P}}\rangle\rangle_2 = \big\{\big(t,\mathbf{x},\overline{P}\big) \in [0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big) : \overline{Q}_{t,\overline{P}} \in \Gamma(\mathfrak{U}\times\mathfrak{S})\big\} \in \mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}\big(\mathfrak{P}\big(\overline{\Omega}\big)\big)$.

2) *We next show that $\langle\langle\overline{\mathcal{P}}\rangle\rangle_3$ is a countable union of Borel-measurable subsets of $[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big)$, so $\langle\langle\overline{\mathcal{P}}\rangle\rangle_3$ is also Borel-measurable*:

Since $W(s,\omega_0) := \omega_0(s)$ is continuous in $(s,\omega_0) \in [0,\infty)\times\Omega_0$ and $W^X(s,\omega_X) := \omega_X(s)$ is continuous in $(s,\omega_X) \in [0,\infty)\times\Omega_X$, the function $\Xi(t,\mathfrak{s},\omega_0,\omega_X) := \big(W(t+\mathfrak{s},\omega_0) - W(t,\omega_0), W^X(t+\mathfrak{s},\omega_X)\big)$ is continuous in $(t,\mathfrak{s},\omega_0,\omega_X) \in [0,\infty)\times[0,\infty)\times\Omega_0\times\Omega_X$. For any $n\in\mathbb{N}$, the mapping $\mathscr{T}_n(t,\omega_0,\omega_X) := \inf\big\{\mathfrak{s}\in[0,\infty) : |\Xi(t,\mathfrak{s},\omega_0,\omega_X)| \ge n\big\}$, $(t,\omega_0,\omega_X) \in [0,\infty)\times\Omega_0\times\Omega_X$ is Borel-measurable since for any $a\in[0,\infty)$,

$$\begin{aligned}\big\{(t,\omega_0,\omega_X)\in[0,\infty)\times\Omega_0\times\Omega_X : \mathscr{T}_n(t,\omega_0,\omega_X) > a\big\} &= \Big\{(t,\omega_0,\omega_X)\in[0,\infty)\times\Omega_0\times\Omega_X : \sup_{\mathfrak{s}'\in[0,\mathfrak{s}]} |\Xi(t,\mathfrak{s}',\omega_0,\omega_X)| < n\Big\} \\ &= \Big(\underset{k\in\mathbb{N}}{\cup}\ \underset{q\in\mathbb{Q}\cap[0,\mathfrak{s}]}{\cap}\big\{(t,\omega_0,\omega_X)\in[0,\infty)\times\Omega_0\times\Omega_X : |\Xi(t,q,\omega_0,\omega_X)| \le n-1/k\big\}\Big) \in \mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X).\end{aligned}$$

Let $\varphi\in\mathscr{P}(\mathbb{R}^{d+l})$. Since the function $H_\varphi(r,\mathfrak{x},\boxminus,u) := \overline{b}(r,\mathfrak{x},u)\cdot D\varphi(\boxminus) + \frac{1}{2}\overline{\sigma}\,\overline{\sigma}^T(r,\mathfrak{x},u) : D^2\varphi(\boxminus)$, $\forall\,(r,\mathfrak{x},\boxminus,u) \in (0,\infty)\times\Omega_X\times\mathbb{R}^{d+l}\times\mathbb{U}$ is Borel-measurable, Lemma 1.3 (2) shows that the mapping

$$\mathcal{I}_\varphi(t,\mathfrak{s},\omega_0,\omega_X,\mathfrak{u}) := \int_t^{t+\mathfrak{s}} H_\varphi\big(r,\mathfrak{l}_2(r,\omega_X),\Xi(t,(r-t)^+,\omega_0,\omega_X),\mathfrak{u}(r)\big)dr,\ \ \forall\,(t,\mathfrak{s},\omega_0,\omega_X,\mathfrak{u})\in[0,\infty)\times[0,\infty)\times\Omega_0\times\Omega_X\times\mathbb{J}$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})$−measurable.

Given $n\in\mathbb{N}$ and $\mathfrak{s}\in[0,\infty)$, since the random variables $(\overline{W},\overline{X},\overline{U})$ on $\overline{\Omega}$ are $\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\mathbb{J})$−measurable, we can derive from the Borel measurability of $\mathcal{I}_\varphi$ and $\mathscr{T}_n$ that the mapping

$$\begin{aligned}\overline{M}^\varphi_{n,\mathfrak{s}}(t,\overline{\omega}) &:= (\varphi\circ\Xi)\big(t,\mathscr{T}_n\big(t,\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega})\big)\wedge n\wedge\mathfrak{s},\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega})\big) - \mathcal{I}_\varphi\big(t,\mathscr{T}_n\big(t,\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega})\big)\wedge n\wedge\mathfrak{s},\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega}),\overline{U}(\overline{\omega})\big) \\ &= \big(\overline{M}^t(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega})\wedge(t+\mathfrak{s}),\overline{\omega}\big), \quad \forall\,(t,\overline{\omega})\in[0,\infty)\times\overline{\Omega}\end{aligned} \tag{6.19}$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\overline{\Omega})$−measurable, where we used the fact $\overline{\tau}^t_n(\overline{\omega}) = t + \mathscr{T}_n\big(t,\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega})\big)\wedge n$.

Let $\theta := \big(\varphi,n,(\mathfrak{s},\mathfrak{r}),\{(s_i,\mathcal{O}_i)\}_{i=1}^k\big) \in \mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})$. Since $\mathfrak{f}_\theta(t,\overline{\omega}) := \big(\overline{M}^\varphi_{n,\mathfrak{r}}(t,\overline{\omega}) - \overline{M}^\varphi_{n,\mathfrak{s}}(t,\overline{\omega})\big)\times\prod_{i=1}^k \mathbf{1}_{\{\Xi(t,s_i\wedge\mathfrak{s},\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega}))\in\mathcal{O}_i\}}$, $(t,\overline{\omega})\in[0,\infty)\times\overline{\Omega}$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\overline{\Omega})$−measurable by (6.19), an application of Lemma A.3 of [4] yields that the mapping $(t,\overline{P}) \mapsto \int_{\overline{\omega}\in\overline{\Omega}} \mathfrak{f}_\theta(t,\overline{\omega})\overline{P}(d\overline{\omega})$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\mathfrak{P}\big(\overline{\Omega}\big)\big)$−measurable and the set $\Big\{(t,\mathbf{x},\overline{P}) \in [0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big) : E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi) - \overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod_{i=1}^k \mathbf{1}_{\{(\overline{W}^t_{t+s_i\wedge\mathfrak{s}},\overline{X}_{t+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}\Big] = 0\Big\}$ is thus Borel-measurable. Letting θ run through the countable collection $\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})$ shows $\langle\langle\overline{\mathcal{P}}\rangle\rangle_3 \in \mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}\big(\mathfrak{P}\big(\overline{\Omega}\big)\big)$.

Totally, $\langle\langle\overline{\mathcal{P}}\rangle\rangle = \langle\langle\overline{\mathcal{P}}\rangle\rangle_1 \cap \langle\langle\overline{\mathcal{P}}\rangle\rangle_2 \cap \langle\langle\overline{\mathcal{P}}\rangle\rangle_3$ is a Borel subset of $[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big)$. □

Proof of Proposition 5.1: We set $t_{\overline{\omega}} := \overline{\gamma}(\overline{\omega}) \ge t$ for any $\overline{\omega}\in\overline{\Omega}$.

1) *We first use (R2) of r.c.p.d. Definition to show that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ satisfies (D1) of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$ for $\overline{P}$−a.s. $\overline{\omega}\in\overline{\Omega}$:*

By (D1) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$, there is a $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process $\widehat{\mu}=\{\widehat{\mu}_s\}_{s\in[t,\infty)}$ on Ω_0 such that the $\overline{P}-$measure of $\overline{\Omega}_\mu:=\big\{\overline{U}_r=\widehat{\mu}_r(\overline{W})$ for a.e. $r\in(t,\infty)\big\}$ is equal to 1. And (R2) assures a $\overline{\mathcal{N}}_\mu\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ such that

$$\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\overline{\Omega}_\mu\big)=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\mathbf{1}_{\overline{\Omega}_\mu}\big]=E_{\overline{P}}\big[\mathbf{1}_{\overline{\Omega}_\mu}\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})=1,\quad \forall\,\overline{\omega}\in\overline{\mathcal{N}}^c_\mu. \tag{6.20}$$

According to Lemma A.5, for any $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$ we can find a $\mathbb{U}-$valued, $\mathbf{F}^{W^s}-$predictable process $\widehat{\mu}^{s,\overline{\omega}}=\big\{\widehat{\mu}^{s,\overline{\omega}}_r\big\}_{r\in[s,\infty)}$ on Ω_0 such that $\widehat{\mu}^{s,\overline{\omega}}_r\big(\overline{W}(\overline{\omega}')\big)=\widehat{\mu}_r\big(\overline{W}(\overline{\omega}')\big)$, $\forall\,(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$, where $\overline{\mathbf{W}}^t_{s,\overline{\omega}}:=\big\{\overline{\omega}'\in\overline{\Omega}\colon \overline{W}^t_r(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}),\ \forall\,r\in[t,s]\big\}$.

Let $\overline{\omega}\in\big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_\mu\big)^c$. We set $\widehat{\mu}^{\overline{\omega}}_r(\omega_0):=\widehat{\mu}^{t_{\overline{\omega}},\overline{\omega}}_r(\omega_0)$, $\forall\,(r,\omega_0)\in[t_{\overline{\omega}},\infty)\times\Omega_0$, which is a $\mathbb{U}-$valued, $\mathbf{F}^{W^{t_{\overline{\omega}}}}-$predictable process satisfying

$$\widehat{\mu}^{\overline{\omega}}_r\big(\overline{W}(\overline{\omega}')\big)=\widehat{\mu}_r\big(\overline{W}(\overline{\omega}')\big),\quad \forall\,(r,\overline{\omega}')\in[t_{\overline{\omega}},\infty)\times\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}. \tag{6.21}$$

It follows that $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big\{\overline{\omega}'\in\overline{\Omega}\colon \overline{U}_r(\overline{\omega}')=\widehat{\mu}^{\overline{\omega}}_r\big(\overline{W}(\overline{\omega}')\big)$ for a.e. $r\in(t_{\overline{\omega}},\infty)\big\}\supset\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}_\mu$, Then (5.2) and (6.20) render that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\{\overline{\omega}'\in\overline{\Omega}\colon\overline{U}_r(\overline{\omega}')=\widehat{\mu}^{\overline{\omega}}_r\big(\overline{W}(\overline{\omega}')\big)$ for a.e. $r\in(t_{\overline{\omega}},\infty)\big\}=1$. So $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ satisfies (D1) of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$ with $\widehat{\mu}=\widehat{\mu}^{\overline{\omega}}$ for any $\overline{\omega}\in\big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_\mu\big)^c$.

2) *Via a delicate analysis of $\overline{P}-$null sets, we exploit the martingale-problem formulation of controlled SDEs* (3.1) *with control $\widehat{\mu}^{\overline{\omega}}$ to show that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ satisfies (D2) and (D3) of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$ for $\overline{P}-$a.s. $\overline{\omega}\in\overline{\Omega}$.*

2a) Set $\overline{\mu}_r:=\widehat{\mu}_r(\overline{W})$, $r\in[t,\infty)$. From (D3) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$ we have $\overline{\mathcal{N}}_X:=\big\{\overline{\omega}\in\overline{\Omega}\colon\overline{X}_s(\overline{\omega})\neq\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s(\overline{\omega})$ for some $s\in[0,\infty)\big\}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. As $\big\{\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$adapted continuous process, an analogy to Lemma 2.4 of [44] allows us to construct an $\mathbb{R}^l-$valued $\mathbf{F}^{\overline{W}^t}-$predictable process $\big\{\overline{K}^t_s\big\}_{s\in[t,\infty)}$ such that $\overline{\mathcal{N}}_K:=\big\{\overline{\omega}\in\overline{\Omega}:\overline{K}^t_s(\overline{\omega})\neq\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s(\overline{\omega})$ for some $s\in[t,\infty)\big\}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. By (R2), it holds for all $\overline{\omega}\in\overline{\Omega}$ except on a $\widehat{\mathcal{N}}_{X,K}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ that

$$\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)=E_{\overline{P}}\big[\mathbf{1}_{\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K}\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})=0. \tag{6.22}$$

Since $\overline{\mathbf{K}}^t_{\overline{\gamma},\overline{\omega}}:=\underset{r\in\mathbb{Q}\cap(t,\infty)}{\cap}\big\{\overline{\omega}'\in\overline{\Omega}:\overline{K}^t_{\overline{\gamma}\wedge r}(\overline{\omega}')=\overline{K}^t_{\overline{\gamma}\wedge r}(\overline{\omega})\big\}$ is an $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable set including $\overline{\omega}$, (R3) shows that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\overline{\mathbf{K}}^t_{\overline{\gamma},\overline{\omega}}\big)=1$, $\forall\,\overline{\omega}\in\overline{\mathcal{N}}^c_0$. For any $\overline{\omega}\in\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c$, we can deduce from (5.1) that

$$\begin{aligned}
\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c\cap\overline{\mathbf{K}}^t_{\overline{\gamma},\overline{\omega}}&=\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c\cap\big\{\overline{\omega}'\in\overline{\Omega}\colon\overline{X}_s(\overline{\omega}')=\mathbf{x}(s),\ \forall\,s\in[0,t];\ \overline{K}^t_{\overline{\gamma}(\overline{\omega})\wedge r}(\overline{\omega}')=\overline{K}^t_{\overline{\gamma}(\overline{\omega})\wedge r}(\overline{\omega}),\ \forall\,r\in\mathbb{Q}\cap(t,\infty)\big\}\\
&=\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c\cap\big\{\overline{\omega}'\in\overline{\Omega}\colon\overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\ \forall\,s\in[0,t];\ \overline{X}_{\overline{\gamma}(\overline{\omega})\wedge r}(\overline{\omega}')=\overline{X}_{\overline{\gamma}(\overline{\omega})\wedge r}(\overline{\omega}),\ \forall\,r\in\mathbb{Q}\cap(t,\infty)\big\}\\
&=\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c\cap\big\{\overline{\omega}'\in\overline{\Omega}\colon\overline{X}_r(\overline{\omega}')=\overline{X}_{\overline{\gamma}\wedge r}(\overline{\omega}),\ \forall\,r\in[0,\overline{\gamma}(\overline{\omega})]\big\}.
\end{aligned} \tag{6.23}$$

Set $\overline{\mathcal{N}}_1:=\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\cup\widehat{\mathcal{N}}_{X,K}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. Given $\overline{\omega}\in\big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_1\big)^c$, taking $\overline{P}(\cdot)$ in (6.23) and using (5.2) yield that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\{\overline{\omega}'\in\overline{\Omega}\colon\overline{X}_r(\overline{\omega}')=\overline{X}_{\overline{\gamma}\wedge r}(\overline{\omega}),\ \forall\,r\in[0,t_{\overline{\omega}}]\big\}=1$.

2b) For any $\varphi\in\mathfrak{C}(\mathbb{R}^{d+l})$ and $q\in\mathbb{Q}^d$, define a function $\varphi_q(w,x):=\varphi(w-q,x)$, $(w,x)\in\mathbb{R}^{d+l}$. We set $\mathscr{C}:=\{\varphi_q:\varphi\in\mathfrak{C}(\mathbb{R}^{d+l}),q\in\mathbb{Q}^l\}$, which is a countable sub-collection of $C^2(\mathbb{R}^{d+l})$. For any $n\in\mathbb{N}$, define an $\overline{\mathbf{F}}^t-$stopping time by $\overline{\zeta}_n(\overline{\omega}):=\inf\big\{r\in[\overline{\gamma}(\overline{\omega}),\infty):|\overline{W}^t_r(\overline{\omega})-\overline{W}^t_{\overline{\gamma}}(\overline{\omega})|^2+|\overline{X}_r(\overline{\omega})|^2\geq n^2\big\}\wedge\big(\overline{\gamma}(\overline{\omega})+n\big)$, $\overline{\omega}\in\overline{\Omega}$.

Let $\theta:=\big(\phi,n,j,(\mathfrak{s},\mathfrak{r}),\{(s_i,\mathcal{O}_i)\}_{i=1}^k\big)\in\mathscr{C}\times\mathbb{N}\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})$. Since $\big\{\overline{M}^{t,\overline{\mu}}_{s\wedge\overline{\tau}^t_j}(\phi)\big\}_{s\in[t,\infty)}$ is a bounded $(\overline{\mathbf{F}}^t,\overline{P})-$martingale by applying Proposition 1.2 with $\big(\Omega,\mathcal{F},P,B,X,\mu\big)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{W},\overline{X},\overline{\mu}\big)$, the optional sampling theorem implies that $E_{\overline{P}}\Big[\overline{M}^{t,\overline{\mu}}_{(\overline{\gamma}+\mathfrak{r})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)\Big|\overline{\mathcal{F}}^t_{\overline{\gamma}+\mathfrak{s}}\Big]=\overline{M}^{t,\overline{\mu}}_{(\overline{\gamma}+\mathfrak{s})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)$, $\overline{P}-$a.s. Set $\overline{\xi}_\theta:=\overline{M}^{t,\overline{\mu}}_{(\overline{\gamma}+\mathfrak{r})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)-\overline{M}^{t,\overline{\mu}}_{(\overline{\gamma}+\mathfrak{s})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)=\mathbf{1}_{\{\overline{\tau}^t_j>\overline{\gamma}\}}\Big(\overline{M}^{t,\overline{\mu}}_{(\overline{\gamma}+\mathfrak{r})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)-\overline{M}^{t,\overline{\mu}}_{(\overline{\gamma}+\mathfrak{s})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)\Big)$ and set $\overline{\eta}_\theta:=\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^t_{\overline{\gamma}+s_i\wedge\mathfrak{s}}-\overline{W}^t_{\overline{\gamma}},\overline{X}_{\overline{\gamma}+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}\in\overline{\mathcal{F}}^t_{\overline{\gamma}+\mathfrak{s}}$. As $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\subset\overline{\mathcal{F}}^t_{\overline{\gamma}}\subset\overline{\mathcal{F}}^t_{\overline{\gamma}+\mathfrak{s}}$, the tower property renders that $E_{\overline{P}}\big[\overline{\xi}_\theta\overline{\eta}_\theta\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big]=E_{\overline{P}}\Big[\overline{\eta}_\theta E_{\overline{P}}\big[\overline{\xi}_\theta\big|\overline{\mathcal{F}}^t_{\overline{\gamma}+\mathfrak{s}}\big]\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big]=0$, $\overline{P}-$a.s. By (R2) again, there exists an $\overline{\mathcal{N}}_\theta\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ such that

$$E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{\xi}_\theta\overline{\eta}_\theta\big]=E_{\overline{P}}\big[\overline{\xi}_\theta\overline{\eta}_\theta\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})=0,\quad\forall\,\overline{\omega}\in\overline{\mathcal{N}}^c_\theta. \tag{6.24}$$

Define $\overline{\mathcal{N}}_2:=\bigcup\big\{\overline{\mathcal{N}}_\theta:\theta\in\mathscr{C}\times\mathbb{N}\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})\big\}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. We fix $\overline{\omega}\in\big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_1\cup\overline{\mathcal{N}}_2\cup\overline{\mathcal{N}}_\mu\big)^c$ and set $\overline{\mu}^{\overline{\omega}}_r(\overline{\omega}'):=\widehat{\mu}^{\overline{\omega}}_r(\overline{\omega}')$, $(r,\overline{\omega}')\in[t_{\overline{\omega}},\infty)\times\overline{\Omega}$. Let $\big(\varphi,n,(\mathfrak{s},\mathfrak{r}),\{(s_i,\mathcal{O}_i)\}_{i=1}^k\big)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})$ and let $j\in\mathbb{N}$.

There exists a sequence $\{q_m=q_m(\overline{\omega})\}_{m\in\mathbb{N}}$ of $\mathbb{Q}^d$ that converges to $\overline{W}^t_{\overline{\gamma}}(\overline{\omega})$. Let $m\in\mathbb{N}$. We set $\theta_m:=\big(\varphi_{q_m},n,j,(\mathfrak{s},\mathfrak{r}),\{(s_i,\mathcal{O}_i)\}_{i=1}^k\big)$ and define $\delta^{j,m}_{\overline{\omega}}:=\underset{|(w,x)|\le j}{\sup}\Big(\sum_{i=0}^2\big|D^i\varphi_{q_m}(w,x)-D^i\varphi\big(w-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),x\big)\big|\Big)=\underset{|(w,x)|\le j}{\sup}\Big(\sum_{i=0}^2\big|D^i\varphi(w-q_m,x)-D^i\varphi\big(w-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),x\big)\big|\Big)$.

Given $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\mathcal{N}}^c_X\cap\{\overline{\tau}^t_j>\overline{\gamma}\}$, (5.1) implies that $\overline{\tau}^t_j(\overline{\omega}')>\overline{\gamma}(\overline{\omega}')=t_{\overline{\omega}}$ and $\overline{\zeta}_n(\overline{\omega}')=\inf\big\{r\in[t_{\overline{\omega}},\infty):|\overline{W}_r(\overline{\omega}')-\overline{W}_{t_{\overline{\omega}}}(\overline{\omega}')|^2+|\overline{X}_r(\overline{\omega}')|^2\ge n^2\big\}\wedge(t_{\overline{\omega}}+n)=\overline{\tau}^{t_{\overline{\omega}}}_n(\overline{\omega}')$. Since $\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_{t_{\overline{\omega}}}(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega})$, $\forall\, r\in[t_{\overline{\omega}},\infty)$, (6.21) shows that for any $t_{\overline{\omega}}\le s_1\le s_2<\infty$

$$\begin{aligned}\big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{s_2}(\varphi)-\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{s_1}(\varphi)\big)(\overline{\omega}')&=\varphi\big(\overline{W}^{t_{\overline{\omega}}}_{s_2}(\overline{\omega}'),\overline{X}_{s_2}(\overline{\omega}')\big)-\varphi\big(\overline{W}^{t_{\overline{\omega}}}_{s_1}(\overline{\omega}'),\overline{X}_{s_1}(\overline{\omega}')\big)-\int_{s_1}^{s_2}\overline{b}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\widehat{\mu}^{\overline{\omega}}_r(\overline{W}(\overline{\omega}'))\big)\cdot D\varphi\big(\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)dr\\&\quad-\frac12\int_{s_1}^{s_2}\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\widehat{\mu}^{\overline{\omega}}_r(\overline{W}(\overline{\omega}'))\big):D^2\varphi(\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}'))dr\\&=\varphi\big(\overline{W}^t_{s_2}(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{X}_{s_2}(\overline{\omega}')\big)-\varphi\big(\overline{W}^t_{s_1}(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{X}_{s_1}(\overline{\omega}')\big)-\int_{s_1}^{s_2}\overline{b}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\widehat{\mu}_r(\overline{W}(\overline{\omega}'))\big)\cdot D\varphi\big(\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{X}_r(\overline{\omega}')\big)dr\\&\quad-\frac12\int_{s_1}^{s_2}\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\widehat{\mu}_r(\overline{W}(\overline{\omega}'))\big):D^2\varphi(\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{X}_r(\overline{\omega}'))dr.\end{aligned}$$

As $\big|\big(\overline{W}^t_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)\big|\le j$ for any $r\in[t_{\overline{\omega}},\overline{\tau}^t_j(\overline{\omega}')]$, we can deduce from (1.2), (1.3) and Cauchy-Schwarz inequality that for any $t_{\overline{\omega}}\le s_1\le s_2\le\overline{\tau}^t_j(\overline{\omega}')$

$$\begin{aligned}&\Big|\big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{s_2}(\varphi)-\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{s_1}(\varphi)-\overline{M}^{t,\overline{\mu}}_{s_2}(\varphi_{q_m})+\overline{M}^{t,\overline{\mu}}_{s_1}(\varphi_{q_m})\big)(\overline{\omega}')\Big|\\&\le2\delta^{j,m}_{\overline{\omega}}+\delta^{j,m}_{\overline{\omega}}\int_t^{\overline{\tau}^t_j(\overline{\omega}')}\Big(\big|\overline{b}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\widehat{\mu}_r(\overline{W}(\overline{\omega}'))\big)\big|+\frac12\big|\overline{\sigma}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\widehat{\mu}_r(\overline{W}(\overline{\omega}'))\big)\big|^2\Big)dr\\&\le2\delta^{j,m}_{\overline{\omega}}+\delta^{j,m}_{\overline{\omega}}\int_t^{\overline{\tau}^t_j(\overline{\omega}')}\Big(\kappa(r)\big\|\overline{X}_{r\wedge\cdot}(\overline{\omega}')\big\|_r+\big|b\big(r,\mathbf{0},\widehat{\mu}_r(\overline{W}(\overline{\omega}'))\big)\big|+\frac d2+\kappa^2(r)\big\|\overline{X}_{r\wedge\cdot}(\overline{\omega}')\big\|_r^2+\big|\sigma\big(r,\mathbf{0},\widehat{\mu}_r(\overline{W}(\overline{\omega}'))\big)\big|^2\Big)dr\le\delta^{j,m}_{\overline{\omega}}(2+c^j_{t,\mathbf{x}}),\end{aligned}$$

where $c^j_{t,\mathbf{x}}:=\big[d/2+\kappa(t+j)(\|\mathbf{x}\|_t+j)+\kappa^2(t+j)(\|\mathbf{x}\|_t+j)^2\big]j+\int_t^{t+j}\underset{u\in\mathbb{U}}{\sup}\big(|b(r,\mathbf{0},u)|+|\sigma(r,\mathbf{0},u)|^2\big)dr<\infty$. Taking $s_1=\big((\overline{\gamma}+\mathfrak{s})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j\big)(\overline{\omega}')=(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n(\overline{\omega}')\wedge\overline{\tau}^t_j(\overline{\omega}')$ and $s_2=(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n(\overline{\omega}')\wedge\overline{\tau}^t_j(\overline{\omega}')$ yields that $\Big|\big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)-\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)\big)(\overline{\omega}')-\overline{\xi}_{\theta_m}(\overline{\omega}')\Big|\le\delta^{j,m}_{\overline{\omega}}(2+c^j_{t,\mathbf{x}})$. As $\overline{\eta}_{\theta_m}(\overline{\omega}')=\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}}(\overline{\omega}'),\overline{X}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}}(\overline{\omega}'))\in\mathcal{O}_i\}}$ by (5.1), we see from (5.2) and (6.22) that

$$E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\bigg[\mathbf{1}_{\{\overline{\tau}^t_j>\overline{\gamma}\}}\bigg|\Big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)-\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)\Big)\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}},\overline{X}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}-\overline{\xi}_{\theta_m}\overline{\eta}_{\theta_m}\bigg|\bigg]\le\delta^{j,m}_{\overline{\omega}}(2+c^j_{t,\mathbf{x}}).$$

The uniform continuity of $D^i\varphi$'s over compact sets implies $\lim_{m\to\infty}\downarrow\delta^{j,m}_{\overline{\omega}}=0$, and one can then obtain from (6.24) that

$$E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\bigg[\mathbf{1}_{\{\overline{\tau}^t_j>\overline{\gamma}\}}\Big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)-\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)\Big)\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}},\overline{X}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}\bigg]=\lim_{m\to\infty}E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{\xi}_{\theta_m}\overline{\eta}_{\theta_m}\big]=0.\quad(6.25)$$

Since $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\{\overline{\omega}'\in\overline{\Omega}:\overline{X}_r(\overline{\omega}')=\overline{X}_{\overline{\gamma}\wedge r}(\overline{\omega}),\ \forall\, r\in[0,t_{\overline{\omega}}]\}=1$ by Part (2a), applying Proposition 1.2 with $\big(\Omega,\mathcal{F},P,B,X\big)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}^t_{\overline{\gamma},\overline{\omega}},\overline{W},\overline{X}\big)$ and $(t,\mathbf{x},\mu)=\big(t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\overline{\mu}^{\overline{\omega}}\big)$ renders that $\big\{\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{s\wedge\overline{\tau}^{t_{\overline{\omega}}}_n}(\varphi)\big\}_{s\in[t_{\overline{\omega}},\infty)}$ is a bounded process under $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$. As $\lim_{j\to\infty}\uparrow\overline{\tau}^t_j(\overline{\omega}')=\infty$ for any $\overline{\omega}'\in\overline{\Omega}$, letting $j\to\infty$ in (6.25) and using the bounded convergence theorem reach that $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\Big[\Big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n}(\varphi)-\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n}(\varphi)\Big)\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}},\overline{X}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}\Big]=0$. Following similar arguments to those that lead to (6.18), we can derive that $\big\{\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{s\wedge\overline{\tau}^{t_{\overline{\omega}}}_n}(\varphi)\big\}_{s\in[t_{\overline{\omega}},\infty)}$ is a $\big(\overline{\mathbf{F}}^{t_{\overline{\omega}}},\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big)$−martingale. Then Remark 3.1 (ii) shows that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ satisfies (D2)+(D3) of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$ for any $\overline{\omega}\in\big(\mathcal{N}_0\cup\mathcal{N}_1\cup\mathcal{N}_2\cup\mathcal{N}_\mu\big)^c$.

3) *It remains to show that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ satisfies (D4) of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$ and thus $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\in\overline{\mathcal{P}}_{\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\Big(\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big)$ for $\overline{P}-$a.s. $\overline{\omega}\in\overline{\Omega}$.*

By (D4) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$, there is a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}$ such that $\overline{P}\{\overline{T}=\widehat{\tau}(\overline{W})\}=1$. Analogous to Part (2) in the proof of [4, Proposition 5.1], we can find $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ satisfying

$$\big\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\big\}\Delta\overline{\mathcal{A}}_*\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big), \tag{6.26}$$

and there exists $\overline{\mathcal{N}}_3\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ such that for any $\overline{\omega}\in\overline{\mathcal{N}}^c_0\cap\overline{\mathcal{N}}^c_3\cap\overline{\mathcal{A}}_*$, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\{\overline{T}=\widehat{\tau}^{\overline{\omega}}(\overline{W})\}=1$ for some $[t_{\overline{\omega}},\infty]-$valued $\mathbf{F}^{W^{t_{\overline{\omega}}},P_0}-$stopping time $\widehat{\tau}^{\overline{\omega}}$, namely, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ satisfies (D4) of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$.

Let $i\in\mathbb{N}$. According to (R2), it holds for all $\overline{\omega}\in\overline{\Omega}$ except on a $\overline{\mathcal{N}}^i_{g,h}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ that $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]=\big(\overline{Y}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})$ and $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]=\big(\overline{Z}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})$. Given $\overline{\omega}\in\Big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_3\cup\overline{\mathcal{N}}^i_{g,h}\Big)^c\cap\overline{\mathcal{A}}_*$, (5.1), (5.2) and $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\{\overline{T}=\widehat{\tau}^{\overline{\omega}}(\overline{W})\ge t_{\overline{\omega}}\}=1$ imply $\big(\overline{Y}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\mathbf{1}_{\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}}\int_{\overline{T}\wedge t_{\overline{\omega}}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]$ $=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{t_{\overline{\omega}}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]$ and similarly $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{t_{\overline{\omega}}}^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]=\big(\overline{Z}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})$. Therefore,

$$\overline{P}^t_{\overline{\gamma},\overline{\omega}}\in\overline{\mathcal{P}}_{\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\Big(\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big),\quad\forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*, \tag{6.27}$$

where $\overline{\mathcal{N}}_*:=\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_1\cup\overline{\mathcal{N}}_2\cup\overline{\mathcal{N}}_3\cup\overline{\mathcal{N}}_\mu\cup\Big(\underset{i\in\mathbb{N}}{\cup}\overline{\mathcal{N}}^i_{g,h}\Big)\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. In particular, (5.3) holds for $\overline{P}-$null set $\overline{\mathcal{N}}:=\overline{\mathcal{N}}_*\cup\big\{\overline{T}\neq\widehat{\tau}(\overline{W})\big\}\cup\big(\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}\Delta\overline{\mathcal{A}}_*\big)\in\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$. □

Proof of Theorem 5.1: For any $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping time $\overline{\zeta}$, we denote $\overline{R}(\overline{\zeta}):=\int_{\overline{T}\wedge\overline{\zeta}}^{\overline{T}}f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)$.

(I) (sub-solution side) Fix $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ and simply denote $\overline{\gamma}_{\overline{P}}$ by $\overline{\gamma}$.

Let $\widehat{\tau}$ be the $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time with $\overline{P}\{\overline{T}=\widehat{\tau}(\overline{W})\}=1$ and let $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, $\overline{\mathcal{N}}_*\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$ be as in (6.26) and (6.27). By (R2), there is a $\overline{\mathcal{N}}_{f,\pi}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ such that $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{R}(\overline{\gamma})\big]=E_{\overline{P}}\big[\overline{R}(\overline{\gamma})\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})$ for any $\overline{\omega}\in\overline{\mathcal{N}}^c_{f,\pi}$. For any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}_*\cup\overline{\mathcal{N}}_{f,\pi}\big)^c$, as $\overline{\mathcal{N}}_0\subset\overline{\mathcal{N}}_*$, (5.1), (5.2) and (6.27) imply that

$$E_{\overline{P}}\big[\overline{R}(\overline{\gamma})\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{R}(\overline{\gamma})\big]=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\mathbf{1}_{\overline{\mathbf{W}}^t_{\overline{\tau},\overline{\omega}}}\overline{R}(\overline{\gamma}(\overline{\omega}))\big]=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{R}(\overline{\gamma}(\overline{\omega}))\big]\le\overline{V}\big(\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\big).$$

Since $\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}=\mathbf{1}_{\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}}=\mathbf{1}_{\overline{\mathcal{A}}_*}$, $\overline{P}-$a.s. by (6.26) and since $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, the tower property renders that

$$\begin{aligned}E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big]&=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*}\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big]\ge E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*}E_{\overline{P}}\big[\overline{R}(\overline{\gamma})\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big]\Big]\\&=E_{\overline{P}}\Big[E_{\overline{P}}\big[\mathbf{1}_{\overline{\mathcal{A}}_*}\overline{R}(\overline{\gamma})\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big]\Big]=E_{\overline{P}}\big[\mathbf{1}_{\overline{\mathcal{A}}_*}\overline{R}(\overline{\gamma})\big]=E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\overline{R}(\overline{\gamma})\big].\end{aligned}$$

It follows that $E_{\overline{P}}\big[\overline{R}(t)\big]\le E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\Big]$. Letting $\overline{P}$ vary over $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ yields that $\overline{V}(t,\mathbf{x},y,z)=\underset{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}{\sup}E_{\overline{P}}\big[\overline{R}(t)\big]\le\underset{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}{\sup}E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}<\overline{\gamma}_{\overline{P}}\}}\Big(\int_t^{\overline{T}}f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}_{\overline{P}}\}}\Big(\int_t^{\overline{\gamma}_{\overline{P}}}f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\overline{V}\big(\overline{\gamma}_{\overline{P}},\overline{X}_{\overline{\gamma}_{\overline{P}}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}_{\overline{P}}),\overline{Z}_{\overline{P}}(\overline{\gamma}_{\overline{P}})\big)\Big)\Big]$.

(II) (super-solution side) Let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ and simply denote $\overline{\gamma}_{\overline{P}}$ by $\overline{\gamma}$. *We shall show that*

$$E_{\overline{P}}\bigg[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\bigg]\le\overline{V}(t,\mathbf{x},y,z). \tag{6.28}$$

As $\mathcal{F}^{\overline{W}^t}_t=\{\emptyset,\overline{\Omega}\}$, the $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping time $\overline{\gamma}$ satisfies either $\{\overline{\gamma}=t\}=\overline{\Omega}$ or $\{\overline{\gamma}>t\}=\overline{\Omega}$.

Suppose first that $\{\overline{\gamma}=t\}=\overline{\Omega}$: for any $i\in\mathbb{N}$, $\overline{Y}^i_{\overline{P}}(t)=E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big|\mathcal{F}^{\overline{W}^t}_t\big]=E_{\overline{P}}\big[\int_t^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]\le y_i$ and $\overline{Z}^i_{\overline{P}}(t)=E_{\overline{P}}\big[\int_t^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\big]=z_i$. Then

$$E_{\overline{P}}\bigg[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\bigg]=E_{\overline{P}}\big[\overline{V}\big(t,\overline{X}_{t\wedge\cdot},\overline{Y}_{\overline{P}}(t),\overline{Z}_{\overline{P}}(t)\big)\big]\le\overline{V}\big(t,\mathbf{x},y,z\big).$$

In the rest of this proof, we assume $\{\overline{\gamma}>t\}=\overline{\Omega}$ and show that the inequality (6.28) *also holds in this situation. As the argument is quite lengthy, we split it into several parts and make brief description at the beginning of each step.*

By (D1) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$, there exists a $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process $\widehat{\mu}=\{\widehat{\mu}_s\}_{s\in[t,\infty)}$ on Ω_0 such that the $\overline{P}-$measure of $\overline{\Omega}_\mu:=\{\overline{U}_r=\widehat{\mu}_r(\overline{W})$ for a.e. $r\in(t,\infty)\}$ is equal to 1. Since $\overline{\mu}_s:=\widehat{\mu}_s(\overline{W}),\ \forall\,s\in[t,\infty)$ is a $\mathbb{U}-$valued $\mathbf{F}^{\overline{W}^t}-$predictable process on $\overline{\Omega}$, the $[0,1)-$valued $\mathbf{F}^{\overline{W}^t}-$adapted continuous process $\overline{J}^{\,t}_s:=\int_t^s e^{-r}\mathscr{I}(\overline{\mu}_r)dr,\ \forall\,s\in[t,\infty)$ satisfies that for any $\overline{\omega}\in\overline{\Omega}_\mu$

$$\overline{\Upsilon}^t_s(\overline{\omega})=\int_t^s e^{-r}\mathscr{I}\big(\overline{U}_r(\overline{\omega})\big)dr=\int_t^s e^{-r}\mathscr{I}\big(\overline{\mu}_r(\overline{\omega})\big)dr=\overline{J}^{\,t}_s(\overline{\omega}),\quad \forall\,s\in[t,\infty). \tag{6.29}$$

Set $\overline{\mathcal{N}}_X:=\big\{\overline{\omega}\in\overline{\Omega}:\ \overline{X}_s(\overline{\omega})\neq\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s(\overline{\omega})$ for some $s\in[0,\infty)\big\}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. We also let $\widehat{\tau}$ be the $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time with $\overline{P}\{\overline{T}=\widehat{\tau}(\overline{W})\}=1$ and let $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}},\ \overline{\mathcal{N}}_*\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$ be as in (6.26) and (6.27).

II.a) *Let us define a truncation $\big(\overline{W}^{t,\overline{\gamma}},\overline{U}^{t,\overline{\gamma}}\big)$ of $(\overline{W},\overline{U})$ over the stochastic interval $[\![t,\overline{\gamma}]\!]$ by*

$$\big(\overline{W}^{t,\overline{\gamma}}_r,\overline{U}^{t,\overline{\gamma}}_r\big)(\overline{\omega}):=\Big(\overline{W}^t\big((r\vee t)\wedge\overline{\gamma}(\overline{\omega}),\overline{\omega}\big),\mathbf{1}_{\{r\in[0,t)\cup(\overline{\gamma}(\overline{\omega}),\infty)\}}u_0+\mathbf{1}_{\{r\in[t,\overline{\gamma}(\overline{\omega})]\}}\overline{U}_r(\overline{\omega})\Big)\in\Omega_0\times\mathbb{J},\quad \forall\,(r,\overline{\omega})\in[0,\infty)\times\overline{\Omega}.$$

We will embed $\big(\overline{W}^{t,\overline{\gamma}},\overline{U}^{t,\overline{\gamma}}\big)$ with $\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)$ into another enlarged canonical space $\ddot{\Omega}:=[0,\infty)\times\Omega_0\times\mathbb{J}\times\Omega_X\times\Re\times\Re$ via a measurable mapping $\ddot{\Psi}$.

II.a.1) Clearly, $\overline{W}^{t,\overline{\gamma}}$ is $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}/\mathscr{B}(\Omega_0)-$measurable. To show the measurability of $\overline{U}^{t,\overline{\gamma}}$, we let $\varphi\in L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)$. The $\mathbf{F}^{\overline{W}^t}-$predictability of $\{\overline{\mu}_s\}_{s\in[t,\infty)}$ implies that $\big\{\varphi\big(s,\overline{\mu}_s\big)\big\}_{s\in[t,\infty)}$ is also an $\mathbf{F}^{\overline{W}^t}-$predictable process and $\int_t^{\overline{\gamma}}\varphi(s,\overline{\mu}_s)ds$ is thus $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable. Then $\overline{\xi}_\mu:=\int_0^t\varphi(s,u_0)ds+\int_t^{\overline{\gamma}}\varphi(s,\overline{\mu}_s)ds+\int_{\overline{\gamma}}^\infty\varphi(s,u_0)ds$ is an $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable random variable such that $\overline{\xi}_\mu(\overline{\omega})=\int_0^\infty\varphi\big(s,\overline{U}^{t,\overline{\gamma}}_s(\overline{\omega})\big)ds=I_\varphi\big(\overline{U}^{t,\overline{\gamma}}(\overline{\omega})\big)$ for any $\overline{\omega}\in\overline{\Omega}_\mu$. Since $\overline{\Omega}_\mu=\big\{\overline{\omega}\in\overline{\Omega}:\ \overline{\Upsilon}^t_s(\overline{\omega})=\int_t^s e^{-r}\mathscr{I}(\overline{U}_r(\overline{\omega}))dr=\int_t^s e^{-r}\mathscr{I}\big(\overline{\mu}_r(\overline{\omega})\big)dr,\ \forall\,s\in(t,\infty)\big\}=\underset{s\in\mathbb{Q}\in(t,\infty)}{\cap}\big\{\overline{\omega}\in\overline{\Omega}:\ \overline{\Upsilon}^t_s(\overline{\omega})=\int_t^s e^{-r}\mathscr{I}\big(\overline{\mu}_r(\overline{\omega})\big)dr\big\}\in\mathcal{F}^{\overline{W}^t,\overline{\Upsilon}^t,\overline{P}}_\infty$, it holds for any $\mathcal{E}\in\mathscr{B}(\mathbb{R})$ that $\big(\overline{U}^{t,\overline{\gamma}}\big)^{-1}\big((I_\varphi)^{-1}(\mathcal{E})\big)=\big\{\overline{\omega}\in\overline{\Omega}:\ I_\varphi\big(\overline{U}^{t,\overline{\gamma}}(\overline{\omega})\big)\in\mathcal{E}\big\}=\big\{\overline{\omega}\in\overline{\Omega}_\mu:\ \overline{\xi}_\mu(\overline{\omega})\in\mathcal{E}\big\}\cup\big\{\overline{\omega}\in\overline{\Omega}^c_\mu:\ I_\varphi\big(\overline{U}^{t,\overline{\gamma}}(\overline{\omega})\big)\in\mathcal{E}\big\}\in\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t,\overline{\Upsilon}^t}_\infty\big)\Big)$. By Lemma 1.3 (1), the sigma-field $\Big\{A\subset\mathbb{J}:\ \big(\overline{U}^{t,\overline{\gamma}}\big)^{-1}(A)\in\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t,\overline{\Upsilon}^t}_\infty\big)\Big)\Big\}$ includes all generating sets of $\mathscr{B}(\mathbb{J})$ and thus contains $\mathscr{B}(\mathbb{J})$. Hence, $\overline{U}^{t,\overline{\gamma}}$ is $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t,\overline{\Upsilon}^t}_\infty\big)\Big)-$measurable.

Since $\big\{\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$adapted continuous process, we can emulate Lemma 2.4 of [44] to construct an $\mathbb{R}^l-$valued $\mathbf{F}^{\overline{W}^t}-$predictable process $\big\{\overline{K}^t_s\big\}_{s\in[t,\infty)}$ such that $\overline{\mathcal{N}}_K:=\big\{\overline{\omega}\in\overline{\Omega}:\ \overline{K}^t_s(\overline{\omega})\neq\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s(\overline{\omega})$ for some $s\in[t,\infty)\big\}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. Since $\overline{X}|_{[0,t)}=\mathbf{x}|_{[0,t)}$ and $\overline{X}|_{[t,\infty)}=\overline{K}^t$ on $\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c$, one can deduce that the random variable $\overline{X}_{\overline{\gamma}\wedge\cdot}:\overline{\Omega}\mapsto\Omega_X$ is $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)\big)/\mathscr{B}(\Omega_X)-$measurable.

II.a.2) Set $\overline{\mathcal{G}}^t_s:=\mathcal{F}^{\overline{W}^t}_s\vee\mathcal{F}^{\overline{\Upsilon}^t}_s\vee\mathcal{F}^{\overline{X}}_s=\sigma\big(\overline{W}^t_r;r\in[t,s]\big)\vee\sigma\big(\overline{\Upsilon}^t_r;r\in[t,s]\big)\vee\sigma\big(\overline{X}_r;r\in[0,s]\big),\ \forall\,s\in[t,\infty)$. We arbitrarily pick $(\mathbf{w},\mathbf{u})$ from $\Omega_0\times\mathbb{J}$.

Since $(t,\mathbf{x},y,z)\in D_{\overline{\mathcal{P}}}$, Theorem 3.1 and (6.27) show that $(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)\in\mathcal{D}_{\overline{\mathcal{P}}}$ and that $\Big(\overline{\gamma}(\overline{\omega}),\overline{W}^{t,\overline{\gamma}}(\overline{\omega}),\overline{U}^{t,\overline{\gamma}}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big)\in\mathcal{D}_{\overline{\mathcal{P}}}$ for any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$. By the measurability of random variables $\overline{W}^{t,\overline{\gamma}}$, $\overline{U}^{t,\overline{\gamma}}$ and $\overline{X}_{\overline{\gamma}\wedge\cdot}$ in Step (II.a.1),

$$\ddot{\Psi}(\overline{\omega}):=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}^c_*\cup\overline{\mathcal{N}}_*\}}(t,\mathbf{w},\mathbf{u},\mathbf{x},y,z)+\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}\Big(\overline{\gamma}(\overline{\omega}),\overline{W}^{t,\overline{\gamma}}(\overline{\omega}),\overline{U}^{t,\overline{\gamma}}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big)\in\mathcal{D}_{\overline{\mathcal{P}}}, \tag{6.30}$$

$\forall\,\overline{\omega}\in\overline{\Omega}$ is $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_\infty\big)\Big)\Big/\mathscr{B}(\mathcal{D}_{\overline{\mathcal{P}}})-$measurable, which induces a probability measure $\ddot{P}:=\overline{P}\circ\ddot{\Psi}^{-1}$ on $\big(\ddot{\Omega},\mathscr{B}(\ddot{\Omega})\big)$. Then $\ddot{\Psi}$ is further $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_\infty\big)\Big)\Big/\sigma\big(\mathscr{B}(\mathcal{D}_{\overline{\mathcal{P}}})\cup\mathscr{N}_{\ddot{P}}(\mathscr{B}(\mathcal{D}_{\overline{\mathcal{P}}}))\big)-$measurable.

II.b) Fix $\varepsilon\in(0,1)$ through Part (II.f). For any $\overline{\omega}\in\overline{\Omega}$, we may denote $t_{\overline{\omega}}:=\overline{\gamma}(\overline{\omega})$.

According to Jankov-von Neumann Theorem (Proposition 7.50 of [7]), Corollary 4.1 and Theorem 4.1, there exists an analytically measurable function $\overline{\mathbf{Q}}_\varepsilon:\mathcal{D}_{\overline{\mathcal{P}}}\mapsto\mathfrak{P}(\overline{\Omega})$ such that for any $(\mathfrak{t},\mathfrak{w},\mathfrak{u},\mathfrak{x},\mathfrak{y},\mathfrak{z})\in\mathcal{D}_{\overline{\mathcal{P}}}$, $\overline{\mathbf{Q}}_\varepsilon(\mathfrak{t},\mathfrak{w},\mathfrak{u},\mathfrak{x},\mathfrak{y},\mathfrak{z})$ belongs to $\overline{\mathcal{P}}_{\mathfrak{t},\mathfrak{w},\mathfrak{u},\mathfrak{x}}(\mathfrak{y},\mathfrak{z})$ and satisfies

$$E_{\overline{\mathbf{Q}}_\varepsilon(\mathfrak{t},\mathfrak{w},\mathfrak{u},\mathfrak{x},\mathfrak{y},\mathfrak{z})}\big[\,\overline{R}(\mathfrak{t})\big] \ge \begin{cases} \overline{V}(\mathfrak{t},\mathfrak{w},\mathfrak{u},\mathfrak{x},\mathfrak{y},\mathfrak{z})-\varepsilon, & \text{if } \overline{V}(\mathfrak{t},\mathfrak{w},\mathfrak{u},\mathfrak{x},\mathfrak{y},\mathfrak{z})<\infty; \\ 1/\varepsilon, & \text{if } \overline{V}(\mathfrak{t},\mathfrak{w},\mathfrak{u},\mathfrak{x},\mathfrak{y},\mathfrak{z})=\infty. \end{cases} \tag{6.31}$$

II.b.1) *We can use the composite mapping $\overline{\mathbf{Q}}_\varepsilon\big(\ddot{\Psi}(\overline{\omega})\big)$ to construct a pasted probability measure $\overline{P}_\varepsilon$ as follows*:

Since $\overline{\mathbf{Q}}_\varepsilon$ is universally measurable, it is also $\sigma\big(\mathscr{B}(\mathcal{D}_{\overline{\mathcal{P}}}) \cup \mathscr{N}_{\ddot{P}}(\mathscr{B}(\mathcal{D}_{\overline{\mathcal{P}}}))\big)/\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable and $\overline{Q}^{\overline{\omega}}_\varepsilon := \mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}^c_*\cup\overline{\mathcal{N}}_*\}}\overline{P}+\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}\overline{\mathbf{Q}}_\varepsilon\big(\ddot{\Psi}(\overline{\omega})\big)$, $\forall\,\overline{\omega}\in\overline{\Omega}$ is thus $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}} \cup \mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_\infty\big)\Big)\Big/\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable.

Given a $[0,\infty]-$valued $\mathscr{B}(\overline{\Omega})-$measurable random variable $\overline{\phi}$, Proposition 7.25 of [7] implies that the mapping $\mathfrak{P}(\overline{\Omega}) \ni \overline{Q}\mapsto E_{\overline{Q}}\big[\,\overline{\phi}\,\big]$ is $\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable. The measurability of $\big\{\overline{Q}^{\overline{\omega}}_\varepsilon\big\}_{\overline{\omega}\in\overline{\Omega}}$ renders that

$$\text{the random variable } \overline{\Omega} \ni \overline{\omega}\mapsto E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\,\overline{\phi}\,\big] \text{ is } \sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}} \cup \mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_\infty\big)\Big)-\text{measurable.} \tag{6.32}$$

Hence, we can define a pasted probability measure $\overline{P}_\varepsilon\in\mathfrak{P}(\overline{\Omega})$:

$$\overline{P}_\varepsilon(\overline{A}):=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{A}\big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_\varepsilon(\overline{A})\overline{P}(d\overline{\omega}). \quad \forall\,\overline{A}\in\mathscr{B}(\overline{\Omega}). \tag{6.33}$$

II.b.2) *To show $\overline{P}_\varepsilon=\overline{P}$ over $\overline{\mathcal{G}}^t_{\overline{\gamma}}$, we need to discuss some properties of $\overline{Q}^{\overline{\omega}}_\varepsilon$.*

Let $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$. By (6.30), one has

$$\overline{Q}^{\overline{\omega}}_\varepsilon=\overline{\mathbf{Q}}_\varepsilon\big(\ddot{\Psi}(\overline{\omega})\big)\in\overline{\mathcal{P}}_{\overline{\gamma}(\overline{\omega}),\overline{W}^{t,\overline{\gamma}}(\overline{\omega}),\overline{U}^{t,\overline{\gamma}}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\Big(\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big). \tag{6.34}$$

Set $\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}} := \big\{\overline{\omega}' \in \overline{\Omega}:\, (\overline{W}_s,\overline{X}_s)(\overline{\omega}') = (\overline{W}^{t,\overline{\gamma}}_s,\overline{X}_s)(\overline{\omega}),\,\forall\,s\in[0,\overline{\gamma}(\overline{\omega})];\; \overline{U}_s(\overline{\omega}')=\overline{U}^{t,\overline{\gamma}}_s(\overline{\omega}) \text{ for a.e. } s\in(0,\overline{\gamma}(\overline{\omega}))\big\}$ and $\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}} := \big\{\overline{\omega}' \in \overline{\Omega}:\, (\overline{W}^t_s,\overline{\Upsilon}^t_s)(\overline{\omega}') = (\overline{W}^t_s,\overline{\Upsilon}^t_s)(\overline{\omega}),\,\forall\,s\in\big[t,\overline{\gamma}(\overline{\omega})\big];\,\overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\,\forall\,s\in[0,\overline{\gamma}(\overline{\omega})]\big\}$. Since $\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}} \subset \big\{\overline{\omega}' \in \overline{\Omega}: \overline{W}_s(\overline{\omega}')=0,\,\forall\,s\in[0,t];\overline{W}_s(\overline{\omega}')=\overline{W}^t_s(\overline{\omega}),\,\forall\,s\in(t,\overline{\gamma}(\overline{\omega})];\overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\,\forall\,s\in[0,\overline{\gamma}(\overline{\omega})];\,\overline{U}_s(\overline{\omega}')=\overline{U}_s(\overline{\omega}) \text{ for a.e. } s\in(t,\overline{\gamma}(\overline{\omega}))\big\}\subset\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\subset\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}$, we see from (6.34) that

$$\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}}\big)=1, \quad \text{and thus} \quad \overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\big)=\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\big)=1. \tag{6.35}$$

By (6.34), there is a $\mathbb{U}-$valued, $\mathbf{F}^{W^{t_{\overline{\omega}}}}-$predictable process $\widehat{\mu}^{\overline{\omega}}=\big\{\widehat{\mu}^{\overline{\omega}}_s\big\}_{s\in[t_{\overline{\omega}},\infty)}$ on Ω_0 such that the $\overline{Q}^{\overline{\omega}}_\varepsilon-$measure of $\overline{\Omega}^{\overline{\omega}}_\mu := \big\{\overline{\omega}' \in \overline{\Omega}:\, \overline{U}_s(\overline{\omega}') = \overline{\mu}^{\overline{\omega}}_s(\overline{\omega}') \text{ for a.e. } s\in(t_{\overline{\omega}},\infty)\big\}$ is equal to 1 with $\overline{\mu}^{\overline{\omega}}_s := \widehat{\mu}^{\overline{\omega}}_s(\overline{W})$, $\forall\,s\in[t_{\overline{\omega}},\infty)$ and that $\overline{\mathcal{N}}^{\overline{\omega}}_X := \big\{\overline{\omega}' \in \overline{\Omega}:\, \overline{X}_s(\overline{\omega}') \neq \overline{\mathscr{X}}^{\overline{\omega}}_s(\overline{\omega}')\ \text{ for some } s\in[0,\infty)\big\} \in \mathscr{N}_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big(\overline{\mathcal{F}}^{t_{\overline{\omega}}}_\infty\big)$, where $\Big\{\overline{\mathscr{X}}^{\overline{\omega}}_s = \overline{\mathscr{X}}^{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\overline{\mu}^{\overline{\omega}}}_s\Big\}_{s\in[0,\infty)}$ is an $\Big\{\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}},\overline{Q}^{\overline{\omega}}_\varepsilon}_{s\vee t_{\overline{\omega}}}\Big\}_{s\in[0,\infty)}-$adapted continuous process that uniquely solves the following SDE with the open-loop control $\overline{\mu}^{\overline{\omega}}$ on $\big(\overline{\Omega},\mathscr{B}\big(\overline{\Omega}\big),\overline{Q}^{\overline{\omega}}_\varepsilon\big)$:

$$\overline{\mathscr{X}}_s=\overline{X}\big(t_{\overline{\omega}},\overline{\omega}\big)+\int_{t_{\overline{\omega}}}^s b\big(r,\overline{\mathscr{X}}_{r\wedge\cdot},\overline{\mu}^{\overline{\omega}}_r\big)dr+\int_{t_{\overline{\omega}}}^s\sigma\big(r,\overline{\mathscr{X}}_{r\wedge\cdot},\overline{\mu}^{\overline{\omega}}_r\big)d\overline{W}_r, \quad \forall\,s\in[t_{\overline{\omega}},\infty)$$

with initial condition $\overline{\mathscr{X}}_s=\overline{X}_s(\overline{\omega})$, $\forall\,s\in[0,t_{\overline{\omega}}]$.

Since $\big\{\overline{\mu}^{\overline{\omega}}_s=\widehat{\mu}^{\overline{\omega}}_s(\overline{W})\big\}_{s\in[t_{\overline{\omega}},\infty)}$ is a $\mathbb{U}-$valued, $\mathbf{F}^{\overline{W}^{t_{\overline{\omega}}}}-$predictable process on $\overline{\Omega}$, the $[0,1)-$valued $\mathbf{F}^{\overline{W}^{t_{\overline{\omega}}}}-$adapted continuous process $\overline{J}^{\overline{\omega}}_s:=\int_{t_{\overline{\omega}}}^s e^{-r}\mathscr{I}\big(\overline{\mu}^{\overline{\omega}}_r\big)dr$, $\forall\,s\in[t_{\overline{\omega}},\infty)$ satisfies that for any $\overline{\omega}'\in\overline{\Omega}^{\overline{\omega}}_\mu$

$$\overline{\Upsilon}^{\overline{\omega}}_s(\overline{\omega}')=\int_{t_{\overline{\omega}}}^s e^{-r}\mathscr{I}\big(\overline{U}_r(\overline{\omega}')\big)dr=\int_{t_{\overline{\omega}}}^s e^{-r}\mathscr{I}\big(\overline{\mu}^{\overline{\omega}}_r(\overline{\omega}')\big)dr=\overline{J}^{\overline{\omega}}_s(\overline{\omega}'), \quad \forall\,s\in[t_{\overline{\omega}},\infty). \tag{6.36}$$

Like Lemma 2.4 of [44], one can construct an $\mathbb{R}^l-$valued $\mathbf{F}^{\overline{W}^{t_{\overline{\omega}}}}-$predictable process $\big\{\overline{K}^{\overline{\omega}}_s\big\}_{s\in[t_{\overline{\omega}},\infty)}$ such that $\overline{\mathcal{N}}^{\overline{\omega}}_K := \big\{\overline{\omega}'\in\overline{\Omega}: \overline{K}^{\overline{\omega}}_s(\overline{\omega}')\neq\overline{\mathscr{X}}^{\overline{\omega}}_s(\overline{\omega}') \text{ for some } s\in[t_{\overline{\omega}},\infty)\big\}\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big(\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_\infty\big)$.

II.b.3) Let $\overline{A}\in\mathscr{B}(\overline{\Omega})$. *We claim that*

$$\overline{Q}^{\overline{\omega}}_\varepsilon(\overline{\mathcal{A}}\cap\overline{A})=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}\}}\overline{Q}^{\overline{\omega}}_\varepsilon(\overline{A}), \quad \forall\,\overline{\mathcal{A}}\in\overline{\mathcal{G}}^t_{\overline{\gamma}},\ \forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*. \tag{6.37}$$

To see this, we take $\overline{\mathcal{A}}\in\overline{\mathcal{G}}^t_{\overline{\gamma}}$. Let $\overline{\omega}_1\in\overline{\mathcal{A}}\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$ and set $s_1:=\overline{\gamma}(\overline{\omega}_1)$. Since $\overline{\mathcal{A}}\cap\{\overline{\gamma}\le s_1\}$ is an $\overline{\mathcal{G}}^t_{s_1}-$measurable set including $\overline{\omega}_1$, one can deduce that

$$\begin{aligned}\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}_1} = \big\{\overline{\omega}'\in\overline{\Omega}\colon (\overline{W}^t_r,\overline{\Upsilon}^t_r)(\overline{\omega}')=(\overline{W}^t_r,\overline{\Upsilon}^t_r)(\overline{\omega}_1),\ \forall\, r\in[t,s_1];\\ \overline{X}_r(\overline{\omega}')=\overline{X}_r(\overline{\omega}_1),\ \forall\, r\in[0,s_1]\big\} \text{ is also contained in } \overline{\mathcal{A}}\cap\{\overline{\gamma}\le s_1\}.\end{aligned} \tag{6.38}$$

By (6.35), $\overline{Q}^{\overline{\omega}_1}_{\varepsilon}\big(\overline{\mathcal{A}}\big)=1$ and thus $\overline{Q}^{\overline{\omega}_1}_{\varepsilon}\big(\overline{\mathcal{A}}\cap\overline{A}\big)=\overline{Q}^{\overline{\omega}_1}_{\varepsilon}\big(\overline{A}\big)=\mathbf{1}_{\{\overline{\omega}_1\in\overline{\mathcal{A}}\}}\overline{Q}^{\overline{\omega}_1}_{\varepsilon}(\overline{A})$. We next let $\overline{\omega}_2\in\overline{\mathcal{A}}^c\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$ and set $s_2:=\overline{\gamma}(\overline{\omega}_2)$. As $\overline{\mathcal{A}}^c\cap\{\overline{\gamma}\le s_2\}$ is an $\overline{\mathcal{G}}^t_{s_2}-$measurable set including $\overline{\omega}_2$, $\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}_2}=\big\{\overline{\omega}'\in\overline{\Omega}:(\overline{W}^t_r,\overline{\Upsilon}^t_r)(\overline{\omega}')=(\overline{W}^t_r,\overline{\Upsilon}^t_r)(\overline{\omega}_2),\ \forall\, r\in[t,s_2];\ \overline{X}_r(\overline{\omega}')=\overline{X}_r(\overline{\omega}_2),\ \forall\, r\in[0,s_2]\big\}$ is also included in $\overline{\mathcal{A}}^c\cap\{\overline{\gamma}\le s_2\}$. We correspondingly have $\overline{Q}^{\overline{\omega}_2}_{\varepsilon}(\overline{\mathcal{A}}^c)=1$ and thus $\overline{Q}^{\overline{\omega}_2}_{\varepsilon}\big(\overline{\mathcal{A}}\cap\overline{A}\big)=0=\mathbf{1}_{\{\overline{\omega}_2\in\overline{\mathcal{A}}\}}\overline{Q}^{\overline{\omega}_2}_{\varepsilon}(\overline{A})$, proving (6.37).

In particular, taking $\overline{A}=\overline{\Omega}$ in (6.37) renders that

$$\overline{P}_{\varepsilon}(\overline{\mathcal{A}})=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{\mathcal{A}}\big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}\}}\overline{P}(d\overline{\omega})=\overline{P}(\overline{\mathcal{A}}),\quad \forall\,\overline{\mathcal{A}}\in\overline{\mathcal{G}}^t_{\overline{\gamma}}. \tag{6.39}$$

In the next four Parts (II.c)$-$(II.f)*, we demonstrate that $\overline{P}_{\varepsilon}$ also belongs to $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$, i.e., the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ is stable under the pasting* (6.33)*.*

II.c) *In this part, we demonstrate that $\overline{W}^t$ is a Brownian motion with respect to the filtration $\overline{\mathbf{G}}^t:=\{\overline{\mathcal{G}}^t_s\}_{s\in[t,\infty)}$ under $\overline{P}_{\varepsilon}$.* More precisely, let $t\le s<r<\infty$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, we need to verify that

$$\overline{P}_{\varepsilon}\big\{\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\big\}=\overline{P}_{\varepsilon}\big\{\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\big\}\overline{P}_{\varepsilon}\big(\overline{A}\big)=\phi(r-s,\mathcal{E})\overline{P}_{\varepsilon}\big(\overline{A}\big),\quad \forall\,\overline{A}\in\overline{\mathcal{G}}^t_s,$$

where $\phi(a,\mathcal{E}):=\big(2\pi a\big)^{-d/2}\int_{z\in\mathcal{E}}e^{-\frac{z^2}{2a}}dz,\ \forall\, a\in(0,\infty)$.

II.c.1) *We first show $\overline{P}_{\varepsilon}\big(\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\big)=\phi(r-s,\mathcal{E})$ based on the fact that $\overline{W}^{t_{\overline{\omega}}}$ is a Brownian motion under $\overline{Q}^{\overline{\omega}}_{\varepsilon}$ and $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ by* (6.34) *and* (6.27) *respectively.*

Let $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$. Since $\{\overline{\gamma}\ge r\}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}\wedge r}\subset\overline{\mathcal{G}}^t_{\overline{\gamma}}$ and since $\{\overline{\gamma}\ge r\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})=\{\overline{\gamma}\ge r\}\cap(\overline{W}^t_{\overline{\gamma}\wedge r}-\overline{W}^t_{\overline{\gamma}\wedge s})^{-1}(\mathcal{E})\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}\wedge r}\subset\overline{\mathcal{G}}^t_{\overline{\gamma}}$, (6.37) implies that

$$\mathbf{1}_{\{\overline{\gamma}(\overline{\omega})\ge r\}}\overline{Q}^{\overline{\omega}}_{\varepsilon}\Big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\Big)=\overline{Q}^{\overline{\omega}}_{\varepsilon}\Big(\{\overline{\gamma}\ge r\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\Big)=\mathbf{1}_{\{\overline{\gamma}(\overline{\omega})\ge r\}\cap\big\{\overline{\omega}\in(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big\}}. \tag{6.40}$$

- If $\overline{\gamma}(\overline{\omega})\le s$, since $(\overline{W}^t_r-\overline{W}^t_s)(\overline{\omega}')=\overline{W}_r(\overline{\omega}')-\overline{W}_s(\overline{\omega}')=\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}')-\overline{W}^{t_{\overline{\omega}}}_s(\overline{\omega}')$, $\forall\,\overline{\omega}'\in\overline{\Omega}$ and since $\overline{W}^{t_{\overline{\omega}}}$ is a Brownian motion with respect to the filtration $\mathbf{F}^{\overline{W}^{t_{\overline{\omega}}}}$ under $\overline{Q}^{\overline{\omega}}_{\varepsilon}$ by (6.34),

$$\overline{Q}^{\overline{\omega}}_{\varepsilon}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)=\overline{Q}^{\overline{\omega}}_{\varepsilon}\Big\{\Big(\overline{W}^{t_{\overline{\omega}}}_r-\overline{W}^{t_{\overline{\omega}}}_s\Big)^{-1}(\mathcal{E})\Big\}=\phi(r-s,\mathcal{E}). \tag{6.41}$$

- We next suppose that $s<\overline{\gamma}(\overline{\omega})<r$ and set $\mathcal{E}_{\overline{\omega}}:=\big\{\mathfrak{x}-\overline{W}^t_{\overline{\gamma}}(\overline{\omega})+\overline{W}^t_s(\overline{\omega}):\mathfrak{x}\in\mathcal{E}\big\}\in\mathscr{B}(\mathbb{R}^d)$. For any $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}$, $(\overline{W}^t_r-\overline{W}^t_s)(\overline{\omega}')\in\mathcal{E}$ if and only if $\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}')=\overline{W}_r(\overline{\omega}')-\overline{W}(\overline{\gamma}(\overline{\omega}),\overline{\omega}')=\overline{W}^t_r(\overline{\omega}')-\overline{W}^t\big(\overline{\gamma}(\overline{\omega}),\overline{\omega}'\big)=\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_s(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega})+\overline{W}^t_s(\overline{\omega})\in\mathcal{E}_{\overline{\omega}}$, which shows

$$(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}=\Big(\overline{W}^{t_{\overline{\omega}}}_r\Big)^{-1}(\mathcal{E}_{\overline{\omega}})\cap\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\,. \tag{6.42}$$

So (6.35) gives that

$$\overline{Q}^{\overline{\omega}}_{\varepsilon}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)=\overline{Q}^{\overline{\omega}}_{\varepsilon}\Big\{\Big(\overline{W}^{t_{\overline{\omega}}}_r\Big)^{-1}(\mathcal{E}_{\overline{\omega}})\Big\}=\phi\big(r-t_{\overline{\omega}},\mathcal{E}_{\overline{\omega}}\big). \tag{6.43}$$

By (R2), there exists $\overline{\mathcal{N}}_{s,r,\mathcal{E}}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ such that $E_{\overline{P}}\Big[\mathbf{1}_{(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})}\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big](\overline{\omega})=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\Big[\mathbf{1}_{(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})}\Big]$ for any $\overline{\omega}\in\overline{\mathcal{N}}^c_{s,r,\mathcal{E}}$. Given $\overline{\omega}\in\{s<\overline{\gamma}<r\}\cap\overline{\mathcal{N}}^c_{s,r,\mathcal{E}}\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$, since $\overline{W}^{t_{\overline{\omega}}}$ is a Brownian motion with respect to the filtration $\mathbf{F}^{\overline{W}^{t_{\overline{\omega}}}}$ under $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ by (6.27), we can deduce from (5.2), (6.42) and (6.43) that

$$\begin{aligned}E_{\overline{P}}\Big[\mathbf{1}_{(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})}\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big](\overline{\omega}) &= \overline{P}^t_{\overline{\gamma},\overline{\omega}}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)=\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\big)=\overline{P}^t_{\overline{\gamma},\overline{\omega}}\Big\{\Big(\overline{W}^{t_{\overline{\omega}}}_r\Big)^{-1}(\mathcal{E}_{\overline{\omega}})\cap\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\Big\}\\ &= \overline{P}^t_{\overline{\gamma},\overline{\omega}}\Big\{\Big(\overline{W}^{t_{\overline{\omega}}}_r\Big)^{-1}(\mathcal{E}_{\overline{\omega}})\Big\}=\phi\big(r-t_{\overline{\omega}},\mathcal{E}_{\overline{\omega}}\big)=\overline{Q}^{\overline{\omega}}_{\varepsilon}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big).\end{aligned} \tag{6.44}$$

Since $\{s<\overline{\gamma}<r\}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}\wedge r}\subset\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ and since $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, it follows that $\overline{P}\big(\{s<\overline{\gamma}<r\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{\mathcal{A}}_*\big)=E_{\overline{P}}\Big[\mathbf{1}_{\{s<\overline{\gamma}<r\}\cap\overline{\mathcal{A}}_*}E_{\overline{P}}\big[\mathbf{1}_{(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})}\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big]\Big]=\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{s<\overline{\gamma}(\overline{\omega})<r\}\cap\{\overline{\omega}\in\overline{\mathcal{A}}_*\}}\overline{Q}^{\overline{\omega}}_{\varepsilon}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)\overline{P}(d\overline{\omega})$. Then we see from (6.41) and (6.40) that

$$\begin{aligned}\overline{P}_{\varepsilon}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)&=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_{\varepsilon}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)\overline{P}(d\overline{\omega})\\&=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)+\phi(r-s,\mathcal{E})\overline{P}\big(\{\overline{\gamma}\le s\}\cap\overline{\mathcal{A}}_*\big)+\overline{P}\big(\{s<\overline{\gamma}<r\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{\mathcal{A}}_*\big)\\&\quad+\overline{P}\big(\{\overline{\gamma}\ge r\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{\mathcal{A}}_*\big)\\&=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le s\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)+\phi(r-s,\mathcal{E})\overline{P}\big(\{\overline{\gamma}\le s\}\cap\overline{\mathcal{A}}_*\big)+\overline{P}\big(\{\overline{\gamma}>s\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big).\end{aligned}$$

Since $\overline{W}^t$ is a Brownian motion under $\overline{P}$ and since $\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le s\}\in\mathcal{F}^{\overline{W}^t}_s$,

$$\overline{P}_{\varepsilon}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)=\phi(r-s,\mathcal{E})\Big(\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le s\}\big)+\overline{P}\big(\{\overline{\gamma}\le s\}\cap\overline{\mathcal{A}}_*\big)+\overline{P}\{\overline{\gamma}>s\}\Big)=\phi(r-s,\mathcal{E}).$$

II.c.2) *We next show that*

$$\overline{P}_{\varepsilon}\Big(\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\Big)=\phi(r-s,\mathcal{E})\overline{P}_{\varepsilon}(\overline{A}),\quad\forall\,\overline{A}\in\overline{\mathcal{G}}^t_s.\tag{6.45}$$

(i) Let $\overline{A}\in\overline{\mathcal{G}}^t_s$. Since $\overline{A}\cap\{\overline{\gamma}>s\}\in\overline{\mathcal{G}}^t_{\overline{\gamma}}$, one can derive from (6.37), (6.40), (6.44), the tower property and (6.39) that

$$\begin{aligned}\overline{P}_{\varepsilon}\Big(\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\cap\{\overline{\gamma}>s\}\Big)&=\overline{P}\Big(\overline{\mathcal{A}}^c_*\cap\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\cap\{\overline{\gamma}>s\}\Big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\omega}\in\overline{A}\cap\{\overline{\gamma}>s\}\}}\overline{Q}^{\overline{\omega}}_{\varepsilon}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)\overline{P}(d\overline{\omega})\\&=\overline{P}\Big(\overline{\mathcal{A}}^c_*\cap\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\cap\{\overline{\gamma}>s\}\Big)+\overline{P}\Big(\overline{\mathcal{A}}_*\cap\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\cap\{\overline{\gamma}\ge r\}\Big)\\&\quad+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\omega}\in\overline{A}\cap\{s<\overline{\gamma}<r\}\}}E_{\overline{P}}\Big[\mathbf{1}_{(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})}\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big](\overline{\omega})\overline{P}(d\overline{\omega})\\&=\overline{P}\Big(\overline{\mathcal{A}}^c_*\cap\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\cap\{\overline{\gamma}>s\}\Big)+\overline{P}\Big(\overline{\mathcal{A}}_*\cap\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\cap\{\overline{\gamma}\ge r\}\Big)\\&\quad+E_{\overline{P}}\Big[E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{\mathcal{A}}_*\cap\overline{A}\cap\{s<\overline{\gamma}<r\}\}}\mathbf{1}_{(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})}\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big]\Big]\\&=\overline{P}\Big(\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\cap\{\overline{\gamma}>s\}\Big)=\overline{P}\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\big)\times\overline{P}\big(\overline{A}\cap\{\overline{\gamma}>s\}\big)=\phi(r-s,\mathcal{E})\overline{P}_{\varepsilon}\big(\overline{A}\cap\{\overline{\gamma}>s\}\big),\end{aligned}\tag{6.46}$$

where we used the independence of $\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})$ from $\mathcal{F}^{\overline{W}^t}_s$ under $\overline{P}$ in the fifth equality above. This equality directly verifies (6.45) for the case "$s=t$" as we assume $\{\overline{\gamma}>t\}=\overline{\Omega}$ (see the fifth line below (6.28)).

(ii) *We then demonstrate* (6.45) *for the complicated case "$s>t$"*:

Let $0\le t_1\le\cdots\le t_n\le t$, $\{\mathcal{E}^o_i\}_{i=1}^n\subset\mathscr{B}(\mathbb{R}^l)$ and set $\overline{A}_X:=\underset{i=1}{\overset{n}{\cap}}\overline{X}^{-1}_{t_i}(\mathcal{E}^o_i)\in\mathcal{F}^{\overline{X}}_t\subset\overline{\mathcal{G}}^t_{\overline{\gamma}}$. We also let $t=s_1<s_2<\cdots<s_{m-1}<s_m=s$ with $k\ge2$, let $\{\mathcal{E}_j\}_{j=1}^m\subset\mathscr{B}(\mathbb{R}^{d+1+l})$ and set $\overline{A}_m:=\underset{j=1}{\overset{m}{\cap}}\big(\overline{W}^t_{s_j},\overline{\Upsilon}^t_{s_j},\overline{X}_{s_j}\big)^{-1}(\mathcal{E}_j)\in\overline{\mathcal{G}}^t_s$. Taking $\overline{A}=\overline{A}_X\cap\overline{A}_m$ in (6.46) renders that

$$\overline{P}_{\varepsilon}\Big(\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}_X\cap\overline{A}_m\cap\{\overline{\gamma}>s\}\Big)=\phi(r-s,\mathcal{E})\overline{P}_{\varepsilon}\big(\overline{A}_X\cap\overline{A}_m\cap\{\overline{\gamma}>s\}\big).\tag{6.47}$$

By (6.29), the set $\overline{\mathcal{A}}:=\Big(\underset{i=1}{\overset{n}{\cap}}\{\mathbf{x}(t_i)\in\mathcal{E}^o_i\}\Big)\cap\Big(\underset{j=1}{\overset{m}{\cap}}\big(\overline{W}^t_{s_j},\overline{J}^t_{s_j},\overline{K}^t_{s_j}\big)^{-1}(\mathcal{E}_j)\Big)\in\mathcal{F}^{\overline{W}^t}_s$ satisfies $\overline{\Omega}_\mu\cap\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c\cap\overline{A}_X\cap\overline{A}_m=\overline{\Omega}_\mu\cap\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c\cap\overline{\mathcal{A}}$. Since $\overline{W}^t$ is a Brownian motion under $\overline{P}$ and since $\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le s\}\in\mathcal{F}^{\overline{W}^t}_s$,

$$\begin{aligned}\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}_X\cap\overline{A}_m\cap\{\overline{\gamma}\le s\}\big)&=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{\mathcal{A}}\cap\{\overline{\gamma}\le s\}\big)\\&=\overline{P}\big(\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\big)\times\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{\mathcal{A}}\cap\{\overline{\gamma}\le s\}\big)=\phi(r-s,\mathcal{E})\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{A}_X\cap\overline{A}_m\cap\{\overline{\gamma}\le s\}\big).\end{aligned}\tag{6.48}$$

It remains to show that $\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_{\varepsilon}\big(\{\overline{\gamma}\le s\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{A}_X\cap\overline{A}_m\big)\overline{P}(d\overline{\omega})=\phi(r-s,\mathcal{E})\times\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_{\varepsilon}\big(\{\overline{\gamma}\le s\}\cap\overline{A}_X\cap\overline{A}_m\big)\overline{P}(d\overline{\omega})$.

Fix $k=1,\cdots,m-1$. We set $\overline{A}^{\overline{\gamma}}_k:=\underset{j=1}{\overset{k}{\cap}}\big(\overline{W}^t_{\overline{\gamma}\wedge s_j},\overline{\Upsilon}^t_{\overline{\gamma}\wedge s_j},\overline{X}_{\overline{\gamma}\wedge s_j}\big)^{-1}(\mathcal{E}_j)\in\overline{\mathcal{G}}^t_{\overline{\gamma}\wedge s_{m-1}}\subset\overline{\mathcal{G}}^t_{\overline{\gamma}}$ and let $\overline{\omega}\in\{s_k<\overline{\gamma}\le s_{k+1}\}\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$. By (5.1),

$$\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\Big(\underset{j=1}{\overset{k}{\cap}}\big(\overline{W}^t_{s_j},\overline{\Upsilon}^t_{s_j},\overline{X}_{s_j}\big)^{-1}(\mathcal{E}_j)\Big)=\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\Big(\underset{j=1}{\overset{k}{\cap}}\big(\overline{W}^t_{\overline{\gamma}\wedge s_j},\overline{\Upsilon}^t_{\overline{\gamma}\wedge s_j},\overline{X}_{\overline{\gamma}\wedge s_j}\big)^{-1}(\mathcal{E}_j)\Big)=\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{A}^{\overline{\gamma}}_k. \tag{6.49}$$

We set $\mathfrak{a}_{\overline{\omega}}:=\big(\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{\Upsilon}^t_{\overline{\gamma}}(\overline{\omega}),\mathbf{0}\big)\in\mathbb{R}^{d+1+l}$ and define $\overline{\mathcal{A}}^{\overline{\omega}}_k:=\underset{j=k+1}{\overset{m}{\cap}}\Big(\overline{W}^{t_{\overline{\omega}}}_{s_j},\overline{J}^{\overline{\omega}}_{s_j},\overline{K}^{\overline{\omega}}_{s_j}\Big)^{-1}\big(\mathcal{E}_{j,\overline{\omega}}\big)\in\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_s$, where $\mathcal{E}_{j,\overline{\omega}}:=\big\{\mathfrak{x}-\mathfrak{a}_{\overline{\omega}}:\mathfrak{x}\in\mathcal{E}_j\big\}\in\mathscr{B}(\mathbb{R}^{d+1+l})$. For $j=k+1,\cdots,m$ and $\overline{\omega}'\in\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_\mu\cap\big(\overline{\mathcal{N}}^{\overline{\omega}}_X\cup\overline{\mathcal{N}}^{\overline{\omega}}_K\big)^c$, one can deduce from (6.36) that $\big(\overline{W}^t_{s_j},\overline{\Upsilon}^t_{s_j},\overline{X}_{s_j}\big)(\overline{\omega}')\in\mathcal{E}_j$ if and only if $\big(\overline{W}^{t_{\overline{\omega}}}_{s_j},\overline{J}^{\overline{\omega}}_{s_j},\overline{K}^{\overline{\omega}}_{s_j}\big)(\overline{\omega}')=\big(\overline{W}^{t_{\overline{\omega}}}_{s_j},\overline{\Upsilon}^{\overline{\omega}}_{s_j},\overline{X}_{s_j}\big)(\overline{\omega}')=\big(\overline{W}^t_{s_j}(\overline{\omega}')-\overline{W}^t_{t_{\overline{\omega}}}(\overline{\omega}'),\overline{\Upsilon}^t_{s_j}(\overline{\omega}')-\overline{\Upsilon}^t_{t_{\overline{\omega}}}(\overline{\omega}'),\overline{X}_{s_j}(\overline{\omega}')\big)=\big(\overline{W}^t_{s_j},\overline{\Upsilon}^t_{s_j},\overline{X}_{s_j}\big)(\overline{\omega}')-\mathfrak{a}_{\overline{\omega}}\in\mathcal{E}_{j,\overline{\omega}}$. So $\Big(\underset{j=k+1}{\overset{m}{\cap}}\big(\overline{W}^t_{s_j},\overline{\Upsilon}^t_{s_j},\overline{X}_{s_j}\big)^{-1}(\mathcal{E}_j)\Big)\cap\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_\mu\cap\big(\overline{\mathcal{N}}^{\overline{\omega}}_X\cup\overline{\mathcal{N}}^{\overline{\omega}}_K\big)^c=\overline{\mathcal{A}}^{\overline{\omega}}_k\cap\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_\mu\cap\big(\overline{\mathcal{N}}^{\overline{\omega}}_X\cup\overline{\mathcal{N}}^{\overline{\omega}}_K\big)^c$, which together with (6.49) shows that $\overline{A}_m\cap\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_\mu\cap\big(\overline{\mathcal{N}}^{\overline{\omega}}_X\cup\overline{\mathcal{N}}^{\overline{\omega}}_K\big)^c=\overline{A}^{\overline{\gamma}}_k\cap\overline{\mathcal{A}}^{\overline{\omega}}_k\cap\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_\mu\cap\big(\overline{\mathcal{N}}^{\overline{\omega}}_X\cup\overline{\mathcal{N}}^{\overline{\omega}}_K\big)^c$.

As $\overline{W}^{t_{\overline{\omega}}}$ is a Brownian motion with respect to the filtration $\mathbf{F}^{\overline{W}^{t_{\overline{\omega}}}}$ under $\overline{Q}^{\overline{\omega}}_\varepsilon$ by (6.34), we can deduce from (6.35) and (6.37) that

$$\begin{aligned}\overline{Q}^{\overline{\omega}}_\varepsilon\Big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{A}_X\cap\overline{A}_m\Big)&=\overline{Q}^{\overline{\omega}}_\varepsilon\Big\{\overline{A}_X\cap\overline{A}^{\overline{\gamma}}_k\cap\overline{\mathcal{A}}^{\overline{\omega}}_k\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\Big\}=\mathbf{1}_{\{\overline{\omega}\in\overline{A}_X\cap\overline{A}^{\overline{\gamma}}_k\}}\overline{Q}^{\overline{\omega}}_\varepsilon\Big\{\overline{\mathcal{A}}^{\overline{\omega}}_k\cap\Big(\overline{W}^{t_{\overline{\omega}}}_r-\overline{W}^{t_{\overline{\omega}}}_s\Big)^{-1}(\mathcal{E})\Big\}\\&=\mathbf{1}_{\{\overline{\omega}\in\overline{A}_X\cap\overline{A}^{\overline{\gamma}}_k\}}\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\mathcal{A}}^{\overline{\omega}}_k\big)\times\overline{Q}^{\overline{\omega}}_\varepsilon\Big\{\Big(\overline{W}^{t_{\overline{\omega}}}_r-\overline{W}^{t_{\overline{\omega}}}_s\Big)^{-1}(\mathcal{E})\Big\}=\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}_X\cap\overline{A}^{\overline{\gamma}}_k\cap\overline{\mathcal{A}}^{\overline{\omega}}_k\big)\phi(r-s,\mathcal{E})=\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}_X\cap\overline{A}_m\big)\phi(r-s,\mathcal{E}),\end{aligned}$$

and thus $\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\omega}\in\{s_k<\overline{\gamma}\le s_{k+1}\}\}}\overline{Q}^{\overline{\omega}}_\varepsilon\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{A}_X\cap\overline{A}_m\big)\overline{P}(d\overline{\omega})=\phi(r-s,\mathcal{E})\times\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\omega}\in\{s_k<\overline{\gamma}\le s_{k+1}\}\}}\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}_X\cap\overline{A}_m\big)\overline{P}(d\overline{\omega})$. Since $\{\overline{\gamma}>0\}=\overline{\Omega}$ and since $\{\overline{\gamma}\le s\}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}\wedge s}\subset\overline{\mathcal{G}}^t_{\overline{\gamma}}$, taking summation from $k=1$ through $k=m-1$ and using (6.37) yield that

$$\begin{aligned}\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_\varepsilon\big(\{\overline{\gamma}\le s\}\cap(\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{A}_X\cap\overline{A}_m\big)\overline{P}(d\overline{\omega})&=\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\omega}\in\{\overline{\gamma}\le s\}\}}\overline{Q}^{\overline{\omega}}_\varepsilon\big((\overline{W}^t_r-\overline{W}^t_s)^{-1}(\mathcal{E})\cap\overline{A}_X\cap\overline{A}_m\big)\overline{P}(d\overline{\omega})\\&=\phi(r-s,\mathcal{E})\times\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\omega}\in\{\overline{\gamma}\le s\}\}}\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}_X\cap\overline{A}_m\big)\overline{P}(d\overline{\omega})=\phi(r-s,\mathcal{E})\times\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_\varepsilon\big(\{\overline{\gamma}\le s\}\cap\overline{A}_X\cap\overline{A}_m\big)\overline{P}(d\overline{\omega}).\end{aligned}$$

Adding it to (6.47) and (6.48) reaches $\overline{P}_\varepsilon\big(\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}_X\cap\overline{A}_m\big)=\phi(r-s,\mathcal{E})\overline{P}_\varepsilon\big(\overline{A}_X\cap\overline{A}_m\big)$. So the Lambda-system $\overline{\Lambda}^t_{s,r}:=\Big\{\overline{A}\in\mathscr{B}(\overline{\Omega}):\overline{P}_\varepsilon\big(\big(\overline{W}^t_r-\overline{W}^t_s\big)^{-1}(\mathcal{E})\cap\overline{A}\big)=\phi(r-s,\mathcal{E})\overline{P}_\varepsilon\big(\overline{A}\big)\Big\}$ contains the Pi-system $\Big\{\Big(\underset{i=1}{\overset{n}{\cap}}\overline{X}^{-1}_{t_i}(\mathcal{E}^o_i)\Big)\cap\Big(\underset{j=1}{\overset{m}{\cap}}\big(\overline{W}^t_{s_j},\overline{\Upsilon}^t_{s_j},\overline{X}_{s_j}\big)^{-1}(\mathcal{E}_j)\Big):0\le t_1\le\cdots\le t_n\le t=s_1<s_2<\cdots<s_{m-1}<s_m=s,\ \{\mathcal{E}^o_i\}^n_{i=1}\subset\mathscr{B}(\mathbb{R}^l),\ \{\mathcal{E}_j\}^m_{j=1}\subset\mathscr{B}(\mathbb{R}^{d+1+l})\Big\}$, which generates $\overline{\mathcal{G}}^t_s$. In light of Dynkin's Pi-Lambda Theorem, we obtain $\overline{\mathcal{G}}^t_s\subset\overline{\Lambda}^t_{s,r}$, proving (6.45).

Hence, $\overline{W}^t$ is a Brownian motion with respect to the filtration $\overline{\mathbf{G}}^t$ under $\overline{P}_\varepsilon$. Namely, $\overline{P}_\varepsilon$ satisfies (D2) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

II.d) *In this part, we demonstrate that $\overline{P}_\varepsilon$ satisfies (D1) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.*

For any $s\in[t,\infty)$, there is a $[0,1]-$valued $\mathcal{F}^{W^t}_s-$measurable random variable Υ^ε_s on Ω_0 such that

$$\Upsilon^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=E_{\overline{P}_\varepsilon}\Big[\overline{\Upsilon}^t_s\big|\mathcal{F}^{\overline{W}^t}_s\Big](\overline{\omega}),\quad\forall\,\overline{\omega}\in\overline{\Omega}.$$

Since $\overline{W}^t$ is a Brownian motion with respect to the filtration $\overline{\mathbf{G}}^t$ under $\overline{P}_\varepsilon$ by Part (II.c), applying Lemma A.1 with $t_0=t$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}_\varepsilon,\overline{W}\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W\big)$ and $\Phi=\overline{W}$ yields that $\big\{\Upsilon^\varepsilon_s(\overline{W})\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^{\overline{W}^t}-$adapted process and that $E_{P_0}\big[\Upsilon^\varepsilon_s\big]=E_{\overline{P}_\varepsilon}\big[\Upsilon^\varepsilon_s(\overline{W})\big]=E_{\overline{P}_\varepsilon}\big[\overline{\Upsilon}^t_s\big]$ is right-continuous in $s\in[t,\infty)$. As $\mathbf{F}^{W^t,P_0}$ is a right-continuous complete filtration, the process $\{\Upsilon^\varepsilon_s\}_{s\in[t,\infty)}$ admits a càdlàg modification $\big\{\widehat{\Upsilon}^\varepsilon_s\big\}_{s\in[t,\infty)}$, which is a $[0,1]-$valued $\mathbf{F}^{W^t,P_0}-$adapted process.

II.d.1) *Define a process $\widehat{\mathscr{U}}^\varepsilon_s(\omega_0):=e^s\varliminf_{\delta\to0+}\frac{1}{\delta}\Big(\widehat{\Upsilon}^\varepsilon_s(\omega_0)-\widehat{\Upsilon}^\varepsilon_{(s-\delta)\vee t}(\omega_0)\Big)$, $\forall\,(s,\omega_0)\in[t,\infty)\times\Omega_0$. We use it to construct in (6.54) a $\mathbb{U}-$valued $\mathbf{F}^{W^t}-$predictable process $\mu^\varepsilon_\cdot$ satisfying $\overline{P}_\varepsilon\big\{\overline{U}_s=\mu^\varepsilon_s(\overline{W})$ for a.e. $s\in(t,\infty)\big\}=1$.*

As $\overline{W}^t$ is a Brownian motion both under $\overline{P}$ and under $\overline{P}_\varepsilon$, taking $t_0=t$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}$ or $\overline{P}_\varepsilon,\overline{W}\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W\big)$ and $\Phi=\overline{W}$ in Lemma A.1 renders that

$$\text{the process } \big\{\widehat{\Upsilon}^\varepsilon_s(\overline{W})\big\}_{s\in[t,\infty)} \text{ is both } \mathbf{F}^{\overline{W}^t,\overline{P}}\text{−adapted and } \mathbf{F}^{\overline{W}^t,\overline{P}_\varepsilon}\text{−adapted}$$
$$\text{as well as that } \overline{W}^{-1}(\mathcal{N})\in\mathscr{N}_{\overline{P}}(\mathcal{F}^{\overline{W}^t}_\infty)\cap\mathscr{N}_{\overline{P}_\varepsilon}(\mathcal{F}^{\overline{W}^t}_\infty),\quad \forall\,\mathcal{N}\in\mathscr{N}_{P_0}\big(\mathcal{F}^{W^t}_\infty\big). \tag{6.50}$$

Given $s\in[t,\infty)$, since $\big\{\widehat{\Upsilon}^\varepsilon_s\neq\Upsilon^\varepsilon_s\big\}\in\mathscr{N}_{P_0}\big(\mathcal{F}^{W^t}_\infty\big)$, (6.50) shows that $\widehat{\mathcal{N}}^\varepsilon_s:=\big\{\widehat{\Upsilon}^\varepsilon_s(\overline{W})\neq\Upsilon^\varepsilon_s(\overline{W})\big\}=\overline{W}^{-1}\Big(\big\{\widehat{\Upsilon}^\varepsilon_s\neq\Upsilon^\varepsilon_s\big\}\Big)\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)\cap\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. Applying Lemma A.6 with $\big(\overline{P},\overline{\mathcal{W}},\mathfrak{F}.,\overline{\xi}\big)=\big(\overline{P}_\varepsilon,\{\overline{W}^t_s\}_{s\in[t,\infty)},\overline{\mathbf{G}}^t,\overline{\Upsilon}^t_s\big)$, we obtain

$$\widehat{\Upsilon}^\varepsilon_s(\overline{W})=\Upsilon^\varepsilon_s(\overline{W})=E_{\overline{P}_\varepsilon}\Big[\overline{\Upsilon}^t_s\Big|\mathcal{F}^{\overline{W}^t}_s\Big]=E_{\overline{P}_\varepsilon}\Big[\overline{\Upsilon}^t_s\Big|\mathcal{F}^{\overline{W}^t}_\infty\Big],\quad \overline{P}_\varepsilon\text{−a.s.} \tag{6.51}$$

Let $\overline{\zeta}$ be a $[t,\infty)$−valued $\mathbf{F}^{\overline{W}^t}$−stopping time, let $\overline{A}\in\mathcal{F}^{\overline{W}^t}_{\overline{\zeta}}$ and let $n\in\mathbb{N}$. We set $s^n_i:=t+i2^{-n}$, $\forall\,i\in\mathbb{N}\cup\{0\}$ and define $\overline{\zeta}_n:=\sum_{i\in\mathbb{N}}s^n_i\mathbf{1}_{\{\overline{\zeta}\in[s^n_{i-1},s^n_i)\}}$. For any $i\in\mathbb{N}$, one can deduce from (6.51) and the monotone convergence theorem that $\widehat{\Upsilon}^\varepsilon_{\overline{\zeta}_n}(\overline{W})=\sum_{i\in\mathbb{N}}\mathbf{1}_{\{\overline{\zeta}\in[s^n_{i-1},s^n_i)\}}\widehat{\Upsilon}^\varepsilon_{s^n_i}(\overline{W})=\sum_{i\in\mathbb{N}}\mathbf{1}_{\{\overline{\zeta}\in[s^n_{i-1},s^n_i)\}}E_{\overline{P}_\varepsilon}\Big[\overline{\Upsilon}^t_{s^n_i}\big|\mathcal{F}^{\overline{W}^t}_\infty\Big]=E_{\overline{P}_\varepsilon}\Big[\sum_{i\in\mathbb{N}}\mathbf{1}_{\{\overline{\zeta}\in[s^n_{i-1},s^n_i)\}}\overline{\Upsilon}^t_{s^n_i}\big|\mathcal{F}^{\overline{W}^t}_\infty\Big]=E_{\overline{P}_\varepsilon}\Big[\overline{\Upsilon}^t_{\overline{\zeta}_n}\big|\mathcal{F}^{\overline{W}^t}_\infty\Big]$, $\overline{P}_\varepsilon$−a.s. As $\overline{\zeta}=\lim\limits_{n\to\infty}\downarrow\overline{\zeta}_n$, the right-continuity of process $\widehat{\Upsilon}^\varepsilon$, $\overline{\Upsilon}^t$ and the bounded convergence theorem imply that

$$\widehat{\Upsilon}^\varepsilon_{\overline{\zeta}}(\overline{W})=\lim_{n\to\infty}\widehat{\Upsilon}^\varepsilon_{\overline{\zeta}_n}(\overline{W})=\lim_{n\to\infty}E_{\overline{P}_\varepsilon}\Big[\overline{\Upsilon}^t_{\overline{\zeta}_n}\big|\mathcal{F}^{\overline{W}^t}_\infty\Big]=E_{\overline{P}_\varepsilon}\Big[\overline{\Upsilon}^t_{\overline{\zeta}}\big|\mathcal{F}^{\overline{W}^t}_\infty\Big],\quad \overline{P}_\varepsilon\text{−a.s.} \tag{6.52}$$

As the right-continuous $\mathbf{F}^{W^t,P_0}$−adapted process $\big\{\widehat{\Upsilon}^\varepsilon_s\big\}_{s\in[t,\infty)}$ is $\mathbf{F}^{W^t,P_0}$−optional, the $[-\infty,\infty]$−valued process $\{\widehat{\mathscr{U}}^\varepsilon_s\}_{s\in[t,\infty)}$ is also $\mathbf{F}^{W^t,P_0}$−optional. Similar to Lemma 2.4 of [44], we can construct a $[-\infty,\infty]$−valued $\mathbf{F}^{W^t}$−predictable process $\mathscr{U}^\varepsilon=\{\mathscr{U}^\varepsilon_s\}_{s\in[t,\infty)}$ with $\mathscr{U}^\varepsilon_s(\omega_0)=\widehat{\mathscr{U}}^\varepsilon_s(\omega_0)$ for $ds\times dP_0$−a.s. $(s,\omega_0)\in[t,\infty)\times\Omega_0$. Using Fubini Theorem, one can find a $\mathcal{N}^\varepsilon_{\mathscr{U}}\in\mathscr{N}_{P_0}\big(\mathcal{F}^{W^t}_\infty\big)$ such that for any $\omega_0\in\big(\mathcal{N}^\varepsilon_{\mathscr{U}}\big)^c$,

$$\mathscr{U}^\varepsilon_s(\omega_0)=\widehat{\mathscr{U}}^\varepsilon_s(\omega_0)\quad\text{for a.e. } s\in[t,\infty). \tag{6.53}$$

By (6.50), $\overline{\mathcal{N}}^\varepsilon_{\mathscr{U}}:=\overline{W}^{-1}\big(\mathcal{N}^\varepsilon_{\mathscr{U}}\big)\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)\cap\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. We define a $\mathbb{U}$−valued $\mathbf{F}^{W^t}$−predictable process by

$$\mu^\varepsilon_s(\omega_0):=\mathscr{I}^{-1}\big(\mathscr{U}^\varepsilon_s(\omega_0)\big)\mathbf{1}_{\{\mathscr{U}^\varepsilon_s(\omega_0)\in\mathfrak{E}\}}+u_0\mathbf{1}_{\{\mathscr{U}^\varepsilon_s(\omega_0)\notin\mathfrak{E}\}}\in\mathbb{U},\quad \forall\,(s,\omega_0)\in[t,\infty)\times\Omega_0. \tag{6.54}$$

II.d.2) *We next show that* $\overline{P}_\varepsilon\big\{\overline{\omega}\in\overline{\Omega}\colon \mu^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\overline{U}_s(\overline{\omega})$ *for a.e.* $s\in(t,\overline{\gamma}(\overline{\omega}))\big\}=1$.

Let $s\in[t,\infty)$ and let $\overline{A}\in\mathcal{F}^{\overline{W}^t}_\infty$. Since $\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\in\overline{\mathcal{G}}^t_{\overline{\gamma}}$ and $\overline{J}^t_{\overline{\gamma}\wedge s}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}\wedge s}\subset\overline{\mathcal{G}}^t_{\overline{\gamma}}$, (6.37) and (6.29) show that $E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\overline{A}}\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big]=E_{\overline{P}}\Big[\mathbf{1}_{\overline{A}^c_*\cap\overline{A}}\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\Big]+\int_{\overline{\omega}\in\overline{A}_*}\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}(\overline{\omega})\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}\big)\overline{P}(d\overline{\omega})=E_{\overline{P}}\big[\mathbf{1}_{\overline{A}^c_*\cap\overline{A}}\overline{J}^t_{\overline{\gamma}\wedge s}\big]+\int_{\overline{\omega}\in\overline{A}_*}\overline{J}^t_{\overline{\gamma}\wedge s}(\overline{\omega})\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}\big)\overline{P}(d\overline{\omega})=E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\overline{A}}\overline{J}^t_{\overline{\gamma}\wedge s}\big]$. Letting $\overline{A}$ varies over $\mathcal{F}^{\overline{W}^t}_\infty$ and taking $\overline{\zeta}=\overline{\gamma}\wedge s$ in (6.52), we obtain $\overline{J}^t_{\overline{\gamma}\wedge s}=E_{\overline{P}_\varepsilon}\big[\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big|\mathcal{F}^{\overline{W}^t}_\infty\big]=\widehat{\Upsilon}^\varepsilon_{\overline{\gamma}\wedge s}(\overline{W})$, $\overline{P}_\varepsilon$−a.s. By the right-continuity of processes $\overline{J}^t$ and $\widehat{\Upsilon}^\varepsilon$, it holds for all $\overline{\omega}\in\overline{\Omega}$ except on a $\overline{\mathcal{N}}^\varepsilon_1\in\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$ that $\widehat{\Upsilon}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\overline{J}^t_s(\overline{\omega})$, $\forall\,s\in\big[t,\overline{\gamma}(\overline{\omega})\big]$.

Let $\overline{\omega}\in\big(\overline{\mathcal{N}}^\varepsilon_{\mathscr{U}}\cup\overline{\mathcal{N}}^\varepsilon_1\big)^c$. Since the Lebesgue differentiation theorem yields that

$$\lim_{\delta\to0+}\frac{1}{\delta}\Big(\widehat{\Upsilon}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)-\widehat{\Upsilon}^\varepsilon_{(s-\delta)\vee t}\big(\overline{W}(\overline{\omega})\big)\Big)=\lim_{\delta\to0+}\frac{1}{\delta}\Big(\overline{J}^t_s(\overline{\omega})-\overline{J}^t_{(s-\delta)\vee t}(\overline{\omega})\Big)=\lim_{\delta\to0+}\frac{1}{\delta}\int_{(s-\delta)\vee t}^s e^{-r}\mathscr{I}\big(\overline{\mu}_r(\overline{\omega})\big)dr=e^{-s}\mathscr{I}\big(\overline{\mu}_s(\overline{\omega})\big)$$

for a.e. $s\in\big(t,\overline{\gamma}(\overline{\omega})\big)$, we see from (6.53) that $\mathscr{U}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\widehat{\mathscr{U}}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=e^s\lim\limits_{\delta\to0+}\frac{1}{\delta}\Big(\widehat{\Upsilon}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)-\widehat{\Upsilon}^\varepsilon_{(s-\delta)\vee t}\big(\overline{W}(\overline{\omega})\big)\Big)=\mathscr{I}\big(\overline{\mu}_s(\overline{\omega})\big)$ for a.e. $s\in\big(t,\overline{\gamma}(\overline{\omega})\big)$. Thus,

$$\widehat{A}^\varepsilon_1:=\big\{\overline{\omega}\in\overline{\Omega}\colon \mu^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\overline{\mu}_s(\overline{\omega})\text{ for a.e. } s\in(t,\overline{\gamma}(\overline{\omega}))\big\}\supset\big(\overline{\mathcal{N}}^\varepsilon_{\mathscr{U}}\cup\overline{\mathcal{N}}^\varepsilon_1\big)^c. \tag{6.55}$$

As $\big\{\int_t^s e^{-r}\mathscr{I}\big(\mu^\varepsilon_r\big)dr\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^{W^t}$−adapted process, Lemma A.1 (1) shows that $\big\{\int_t^s e^{-r}\mathscr{I}\big(\mu^\varepsilon_r(\overline{W})\big)dr\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^{\overline{W}^t}$−adapted continuous process, which together with the $\mathbf{F}^{\overline{W}^t}$−adaptedness of continuous process $\overline{J}^t$ implies

$$\widehat{A}^\varepsilon_1=\Big\{\int_t^{\overline{\gamma}\wedge s}e^{-r}\mathscr{I}\big(\mu^\varepsilon_r(\overline{W})\big)dr=\overline{J}^t_{\overline{\gamma}\wedge s},\ \forall\,s\in[t,\infty)\Big\}=\bigcap_{s\in\mathbb{Q}\cap[t,\infty)}\Big\{\int_t^{\overline{\gamma}\wedge s}e^{-r}\mathscr{I}\big(\mu^\varepsilon_r(\overline{W})\big)dr=\overline{J}^t_{\overline{\gamma}\wedge s}\Big\}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\subset\overline{\mathcal{G}}^t_{\overline{\gamma}}. \tag{6.56}$$

Then we can derive from (6.39) and (6.55) that $1=\overline{P}_\varepsilon\big((\overline{\mathcal{N}}^{\,\varepsilon}_{\mathscr{U}}\cup\overline{\mathcal{N}}^{\,\varepsilon}_1)^c\big)\le\overline{P}_\varepsilon\big(\widehat{A}^\varepsilon_1\big)=\overline{P}\big(\widehat{A}^\varepsilon_1\big)$. Similar to (6.56),

$$\overline{\mathcal{A}}^\varepsilon_1:=\big\{\overline{\omega}\in\overline{\Omega}\colon \mu^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\overline{U}_s(\overline{\omega}) \text{ for a.e. } s\in(t,\overline{\gamma}(\overline{\omega}))\big\}=\underset{s\in\mathbb{Q}\cap[t,\infty)}{\cap}\Big\{\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}=\int_t^{\overline{\gamma}\wedge s}e^{-r}\mathscr{I}\big(\mu^\varepsilon_r(\overline{W})\big)dr\Big\}\in\overline{\mathcal{G}}^t_{\overline{\gamma}}.$$

Applying (6.39) again renders that

$$\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^\varepsilon_1\big)=\overline{P}\big(\overline{\mathcal{A}}^\varepsilon_1\big)=\overline{P}\big(\overline{\Omega}_\mu\cap\overline{\mathcal{A}}^\varepsilon_1\big)=\overline{P}\big(\overline{\Omega}_\mu\cap\widehat{A}^\varepsilon_1\big)=\overline{P}\big(\widehat{A}^\varepsilon_1\big)=1. \tag{6.57}$$

II.d.3) *The demonstration of* $\overline{P}_\varepsilon\big\{\overline{\omega}\in\overline{\Omega}\colon \mu^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\overline{U}_s(\overline{\omega}) \text{ for a.e. } s\in\big(\overline{\gamma}(\overline{\omega}),\infty\big)\big\}=1$ *is technically involved, we will carefully verify it.*

(i) *Set* $\overline{\mathcal{A}}^\varepsilon_2:=\big\{\overline{\omega}\in\overline{\Omega}\colon \mu^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\overline{U}_s(\overline{\omega}) \text{ for a.e. } s\in\big(\overline{\gamma}(\overline{\omega}),\infty\big)\big\}$. *We first show that* $\overline{P}\big(\overline{\mathcal{A}}^c_*\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{\mathcal{A}}^\varepsilon_2\big)$.

Let $s\in[t,\infty)$ and let $\overline{A}\in\mathcal{F}^{\overline{W}^t}_\infty$. Since $E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\overline{A}}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\big]=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\overline{A}}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\Big]=E_{\overline{P}}\big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\overline{A}}\big(\overline{J}^{\,t}_s-\overline{J}^{\,t}_{\overline{\gamma}\wedge s}\big)\big]=E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\overline{A}}\big(\overline{J}^{\,t}_s-\overline{J}^{\,t}_{\overline{\gamma}\wedge s}\big)\big]$ by (6.29), letting $\overline{A}$ varies over $\mathcal{F}^{\overline{W}^t}_\infty$ and using (6.52) again yield that

$$\mathbf{1}_{\overline{\mathcal{A}}^c_*}\big(\overline{J}^{\,t}_s-\overline{J}^{\,t}_{\overline{\gamma}\wedge s}\big)=E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{A}}^c_*}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\big|\mathcal{F}^{\overline{W}^t}_\infty\big]=\mathbf{1}_{\overline{\mathcal{A}}^c_*}E_{\overline{P}_\varepsilon}\big[\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\big|\mathcal{F}^{\overline{W}^t}_\infty\big]=\mathbf{1}_{\overline{\mathcal{A}}^c_*}\big(\widehat{\Upsilon}^\varepsilon_s(\overline{W})-\widehat{\Upsilon}^\varepsilon_{\overline{\gamma}\wedge s}(\overline{W})\big),\quad \overline{P}_\varepsilon\text{-a.s.}$$

The right-continuity of processes $\overline{J}^{\,t}$ and $\widehat{\Upsilon}^\varepsilon$ assures an $\overline{\mathcal{N}}^{\,\varepsilon}_2\in\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$ such that for any $\overline{\omega}\in\big(\overline{\mathcal{A}}_*\cup\overline{\mathcal{N}}^{\,\varepsilon}_2\big)^c$ and any $s\in[\overline{\gamma}(\overline{\omega}),\infty)$, one has $\widehat{\Upsilon}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)-\widehat{\Upsilon}^\varepsilon\big(\overline{\gamma}(\overline{\omega}),\overline{W}(\overline{\omega})\big)=\overline{J}^{\,t}_s(\overline{\omega})-\overline{J}^{\,t}\big(\overline{\gamma}(\overline{\omega}),\overline{\omega}\big)$.

Let $\overline{\omega}\in\big(\overline{\mathcal{A}}_*\cup\overline{\mathcal{N}}^{\,\varepsilon}_{\mathscr{U}}\cup\overline{\mathcal{N}}^{\,\varepsilon}_2\big)^c$. Since the Lebesgue differentiation theorem implies that $\lim\limits_{\delta\to0+}\frac{1}{\delta}\Big(\widehat{\Upsilon}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)-\widehat{\Upsilon}^\varepsilon\big((s-\delta)\vee\overline{\gamma}(\overline{\omega}),\overline{W}(\overline{\omega})\big)\Big)=\lim\limits_{\delta\to0+}\frac{1}{\delta}\Big(\overline{J}^{\,t}_s(\overline{\omega})-\overline{J}^{\,t}\big((s-\delta)\vee\overline{\gamma}(\overline{\omega}),\overline{\omega}\big)\Big)=\lim\limits_{\delta\to0+}\frac{1}{\delta}\int_{(s-\delta)\vee\overline{\gamma}(\overline{\omega})}^s e^{-r}\mathscr{I}\big(\overline{\mu}_r(\overline{\omega})\big)dr=e^{-s}\mathscr{I}\big(\overline{\mu}_s(\overline{\omega})\big)$ for a.e. $s\in\big(\overline{\gamma}(\overline{\omega}),\infty\big)$, (6.53) shows that $\mathscr{U}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\widehat{\mathscr{U}^\varepsilon_s}\big(\overline{W}(\overline{\omega})\big)=e^s\lim\limits_{\delta\to0+}\frac{1}{\delta}\Big(\widehat{\Upsilon}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)-\widehat{\Upsilon}^\varepsilon\big((s-\delta)\vee\overline{\gamma}(\overline{\omega}),\overline{W}(\overline{\omega})\big)\Big)=\mathscr{I}\big(\overline{\mu}_s(\overline{\omega})\big)$ for a.e. $s\in\big(\overline{\gamma}(\overline{\omega}),\infty\big)$. So $\widehat{A}^\varepsilon_2:=\big\{\overline{\omega}\in\overline{\Omega}\colon \mu^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=\overline{\mu}_s(\overline{\omega}) \text{ for a.e. } s\in\big(\overline{\gamma}(\overline{\omega}),\infty\big)\big\}\supset\overline{\mathcal{A}}^c_*\cap\big(\overline{\mathcal{N}}^{\,\varepsilon}_{\mathscr{U}}\cup\overline{\mathcal{N}}^{\,\varepsilon}_2\big)^c$. As $\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\big)=\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\cap(\overline{\mathcal{N}}^{\,\varepsilon}_{\mathscr{U}}\cup\overline{\mathcal{N}}^{\,\varepsilon}_2)^c\big)\le\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\cap\widehat{A}^\varepsilon_2\big)\le\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\big)$, we obtain that

$$\overline{P}\big(\overline{\mathcal{A}}^c_*\big)=\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\big)=\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\cap\widehat{A}^\varepsilon_2\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\widehat{A}^\varepsilon_2\big)=\overline{P}\big(\overline{\Omega}_\mu\cap\overline{\mathcal{A}}^c_*\cap\widehat{A}^\varepsilon_2\big)=\overline{P}\big(\overline{\Omega}_\mu\cap\overline{\mathcal{A}}^c_*\cap\overline{\mathcal{A}}^\varepsilon_2\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{\mathcal{A}}^\varepsilon_2\big). \tag{6.58}$$

(ii) *We next show that* $\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\mathcal{A}}^\varepsilon_2\big)=1$ *for* $\overline{P}$*-a.s.* $\overline{\omega}\in\overline{\mathcal{A}}_*$, *then* $\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^\varepsilon_2\big)=1$ *easily follows.*

We denote $\mathbb{Q}_t:=\{t+q\colon q\in\mathbb{Q}_+\}$ and set $\overline{\mathcal{N}}^{\,\varepsilon}_\Upsilon:=\underset{s\in\mathbb{Q}_t}{\cup}\big\{\widehat{\Upsilon}^\varepsilon_s(\overline{W})\ne\Upsilon^\varepsilon_s(\overline{W})\big\}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)\cap\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$.

Fix $s\in\mathbb{Q}_t$. Given $k\in\mathbb{N}$, we set $s^k_i:=t+i2^{-k}(s-t)$ for $i=0,1,\cdots,2^k$. Since $\big\{\Upsilon^\varepsilon_r(\overline{W})\big\}_{r\in[t,\infty)}$ is a $[0,1]$-valued $\mathbf{F}^{\overline{W}^t}$-adapted process, the random variable $\overline{\xi}^{\,\varepsilon,k}_s:=\mathbf{1}_{\{\overline{\gamma}\ge s\}}\Upsilon^\varepsilon_s(\overline{W})+\sum_{i=1}^{2^k}\mathbf{1}_{\{s^k_{i-1}\le\overline{\gamma}<s^k_i\}}\Upsilon^\varepsilon_{s^k_i}(\overline{W})\in[0,1]$ is $\mathcal{F}^{\overline{W}^t}_s$-measurable. Then $\overline{\xi}^{\,\varepsilon}_s:=\varliminf\limits_{k\to\infty}\overline{\xi}^{\,\varepsilon,k}_s$ is also a $[0,1]$-valued $\mathcal{F}^{\overline{W}^t}_s$-measurable random variable.

For any $\overline{\omega}\in\big(\overline{\mathcal{N}}^{\,\varepsilon}_\Upsilon\big)^c$, the right-continuity of process $\widehat{\Upsilon}^\varepsilon$ shows that $\widehat{\Upsilon}^\varepsilon\big(\overline{\gamma}(\overline{\omega})\wedge s,\overline{W}(\overline{\omega})\big)=\lim\limits_{k\to\infty}\Big(\mathbf{1}_{\{\overline{\gamma}(\overline{\omega})\ge s\}}\widehat{\Upsilon}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)+\sum_{i=1}^{2^k}\mathbf{1}_{\{s^k_{i-1}\le\overline{\gamma}(\overline{\omega})<s^k_i\}}\widehat{\Upsilon}^\varepsilon_{s^k_i}\big(\overline{W}(\overline{\omega})\big)\Big)=\lim\limits_{k\to\infty}\overline{\xi}^{\,\varepsilon,k}_s(\overline{\omega})=\overline{\xi}^{\,\varepsilon}_s(\overline{\omega})$ and thus

$$\widehat{\Upsilon}^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)-\widehat{\Upsilon}^\varepsilon\big(\overline{\gamma}(\overline{\omega})\wedge s,\overline{W}(\overline{\omega})\big)=\Upsilon^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)-\overline{\xi}^{\,\varepsilon}_s(\overline{\omega}). \tag{6.59}$$

As $\overline{\mathcal{N}}^{\,\varepsilon}_{\mathscr{U}}\cup\overline{\mathcal{N}}^{\,\varepsilon}_\Upsilon\in\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$, there exists $\overline{A}^\varepsilon_U\in\mathcal{F}^{\overline{W}^t}_\infty$ such that $\overline{\mathcal{N}}^{\,\varepsilon}_{\mathscr{U}}\cup\overline{\mathcal{N}}^{\,\varepsilon}_\Upsilon\subset\overline{A}^\varepsilon_U$ and $\overline{P}_\varepsilon\big(\overline{A}^\varepsilon_U\big)=0$, which implies $0\le\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}^\varepsilon_U\big)\overline{P}(d\overline{\omega})\le\overline{P}_\varepsilon\big(\overline{A}^\varepsilon_U\big)=0$. Since the random variable $\overline{\Omega}\ni\overline{\omega}\mapsto\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}^\varepsilon_U\big)$ is $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_\infty\big)\Big)$-measurable by (6.32), there exists $\overline{\mathfrak{N}}^\varepsilon_U\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_\infty\big)$ such that for any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}_*\cup\overline{\mathfrak{N}}^\varepsilon_U\big)^c$

$$\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}^\varepsilon_U\big)=0\quad\text{and thus}\quad\overline{\mathcal{N}}^{\,\varepsilon}_{\mathscr{U}}\cup\overline{\mathcal{N}}^{\,\varepsilon}_\Upsilon\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big). \tag{6.60}$$

Let $\big\{\overline{O}_j\big\}_{j\in\mathbb{N}}$ be a countable Pi-system that generates $\mathcal{F}^{\overline{W}^t}_s$. Let $j\in\mathbb{N}$ and $\overline{A}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$. One can deduce from (6.37), (6.59) and (6.52) that

$$\begin{aligned}
&\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\mathbf{1}_{\overline{O}_j}\Big(\Upsilon^\varepsilon_s(\overline{W})-\overline{\xi}^{\,\varepsilon}_s\Big)\Big]\overline{P}(d\overline{\omega})=E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{O}_j}\Big(\Upsilon^\varepsilon_s(\overline{W})-\overline{\xi}^{\,\varepsilon}_s\Big)\Big]=E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{O}_j}\Big(\widehat{\Upsilon}^\varepsilon_s(\overline{W})-\widehat{\Upsilon}^\varepsilon_{\overline{\gamma}\wedge s}(\overline{W})\Big)\Big]\\
&\quad=E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{O}_j}E_{\overline{P}_\varepsilon}\Big[\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\Big|\mathcal{F}^{\overline{W}^t}_\infty\Big]\Big]=E_{\overline{P}_\varepsilon}\Big[E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\Big|\mathcal{F}^{\overline{W}^t}_\infty\Big]\Big]\\
&\quad=E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\Big]=\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\big]\overline{P}(d\overline{\omega}).
\end{aligned}$$

So the Lambda-system $\overline{\Lambda}^{\varepsilon}_{s,j}:=\Big\{\overline{A}\in\mathscr{B}_{\overline{P}}(\overline{\Omega}):\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\big[\mathbf{1}_{\overline{O}_j}\big(\Upsilon^{\varepsilon}_s(\overline{W})-\overline{\xi}^{\varepsilon}_s\big)\big]\overline{P}(d\overline{\omega})=\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\big[\mathbf{1}_{\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\big]\overline{P}(d\overline{\omega})\Big\}$ contains $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ and $\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$. As $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$ is closed under intersection, we know from Dynkin's Pi-Lambda Theorem that $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)\big)\subset\overline{\Lambda}^{\varepsilon}_{s,j}$ or

$$\begin{aligned}&\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\overline{O}_j}\Big(\Upsilon^{\varepsilon}_s(\overline{W})-\overline{\xi}^{\varepsilon}_s\Big)\Big]\overline{P}(d\overline{\omega})\\&\qquad=\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\big[\mathbf{1}_{\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\big]\overline{P}(d\overline{\omega}),\quad\forall\,\overline{A}\in\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)\Big).\end{aligned}\tag{6.61}$$

Taking $\phi=\mathbf{1}_{\overline{O}_j}\big(\Upsilon^{\varepsilon}_s(\overline{W})-\overline{\xi}^{\varepsilon}_s\big)$ and $\phi=\mathbf{1}_{\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)$ respectively in (6.32) shows that $\overline{\Omega}\ni\overline{\omega}\mapsto E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\big[\mathbf{1}_{\overline{O}_j}\big(\Upsilon^{\varepsilon}_s(\overline{W})-\overline{\xi}^{\varepsilon}_s\big)\big]$ and $\overline{\Omega}\ni\overline{\omega}\mapsto E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\big[\mathbf{1}_{\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\big]$ are two $[-1,1]-$valued $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_{\infty}\big)\Big)-$measurable random variables. Letting $\overline{A}$ run through $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_{\infty}\big)\Big)$ in (6.61), we can find an $\overline{\mathfrak{N}}^{\varepsilon}_{s,j}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_{\infty}\big)$ such that for any $\overline{\omega}\in\big(\overline{\mathfrak{N}}^{\varepsilon}_{s,j}\big)^c$

$$\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\overline{O}_j}\Big(\Upsilon^{\varepsilon}_s(\overline{W})-\overline{\xi}^{\varepsilon}_s\Big)\Big]=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\Big].\tag{6.62}$$

Set $\overline{\mathfrak{N}}^{\varepsilon}_s:=\underset{j\in\mathbb{N}}{\cup}\overline{\mathfrak{N}}^{\varepsilon}_{s,j}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_{\infty}\big)$ and let $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}_*\cup\overline{\mathfrak{N}}^{\varepsilon}_U\cup\overline{\mathfrak{N}}^{\varepsilon}_s\big)^c$. Since it holds for any $\overline{\omega}'\in\mathbf{W}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_{\mu}$ that $\overline{\Upsilon}^t_s(\overline{\omega}')-\overline{\Upsilon}^t\big(\overline{\gamma}(\overline{\omega}')\wedge s,\overline{\omega}'\big)=\overline{\Upsilon}^t_s(\overline{\omega}')-\overline{\Upsilon}^t\big(\overline{\gamma}(\overline{\omega})\wedge s,\overline{\omega}'\big)=\int_{\overline{\gamma}(\overline{\omega})\wedge s}^{s}e^{-r}\mathscr{I}\big(\overline{U}_r(\overline{\omega}')\big)dr=\int_{\overline{\gamma}(\overline{\omega})}^{\overline{\gamma}(\overline{\omega})\vee s}e^{-r}\mathscr{I}\big(\overline{\mu}^{\overline{\omega}}_r(\overline{\omega}')\big)dr=\overline{\Upsilon}^{\overline{\omega}}_{t_{\overline{\omega}}\vee s}(\overline{\omega}')$ by (5.1), we can deduce from (6.59), (6.60), (6.62), (6.35) and (6.36) that for any $j\in\mathbb{N}$

$$E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\overline{O}_j}\Big(\widehat{\Upsilon}^{\varepsilon}_s(\overline{W})-\widehat{\Upsilon}^{\varepsilon}_{\overline{\gamma}\wedge s}(\overline{W})\Big)\Big]=E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\overline{O}_j}\Big(\Upsilon^{\varepsilon}_s(\overline{W})-\overline{\xi}^{\varepsilon}_s\Big)\Big]=E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\overline{O}_j}\big(\overline{\Upsilon}^t_s-\overline{\Upsilon}^t_{\overline{\gamma}\wedge s}\big)\Big]=E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\overline{O}_j}\overline{\Upsilon}^{\overline{\omega}}_{t_{\overline{\omega}}\vee s}\Big]=E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\overline{O}_j}\overline{J}^{\overline{\omega}}_{t_{\overline{\omega}}\vee s}\Big].$$

Then Dynkin's Pi-Lambda Theorem implies that the Lambda-system $\Big\{\mathcal{E}\in\mathscr{B}_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}(\overline{\Omega}):E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\mathcal{E}}\Big(\widehat{\Upsilon}^{\varepsilon}_s(\overline{W})-\widehat{\Upsilon}^{\varepsilon}_{\overline{\gamma}\wedge s}(\overline{W})\Big)\Big]=E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\mathcal{E}}\overline{J}^{\overline{\omega}}_{t_{\overline{\omega}}\vee s}\Big]\Big\}$ includes $\mathcal{F}^{\overline{W}^t}_s$ and thus contains $\mathcal{F}^{\overline{W}^t,\overline{Q}^{\overline{\omega}}_{\varepsilon}}_s=\sigma\Big(\mathcal{F}^{\overline{W}^t}_s\cup\mathscr{N}_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\big(\mathcal{F}^{\overline{W}^t}_{\infty}\big)\Big)$:

$$E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\mathcal{E}}\Big(\widehat{\Upsilon}^{\varepsilon}_s(\overline{W})-\widehat{\Upsilon}^{\varepsilon}_{\overline{\gamma}\wedge s}(\overline{W})\Big)\Big]=E_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\Big[\mathbf{1}_{\mathcal{E}}\overline{J}^{\overline{\omega}}_{t_{\overline{\omega}}\vee s}\Big],\quad\forall\,\mathcal{E}\in\mathcal{F}^{\overline{W}^t,\overline{Q}^{\overline{\omega}}_{\varepsilon}}_s.\tag{6.63}$$

If $\overline{\gamma}(\overline{\omega})\ge s$, then $\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_{t_{\overline{\omega}}\vee s}=\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_{t_{\overline{\omega}}}=\{\emptyset,\overline{\Omega}\}\subset\mathcal{F}^{\overline{W}^t}_s$; if $\overline{\gamma}(\overline{\omega})<s$, then $\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_{t_{\overline{\omega}}\vee s}=\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_s=\sigma\Big(\big(\overline{W}^{t_{\overline{\omega}}}_r\big)^{-1}(\mathcal{E}):r\in\big[\overline{\gamma}(\overline{\omega}),s\big],\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)\Big)=\sigma\Big(\big(\overline{W}^t_r-\overline{W}^t_{t_{\overline{\omega}}}\big)^{-1}(\mathcal{E}):r\in\big[\overline{\gamma}(\overline{\omega}),s\big],\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)\Big)\subset\mathcal{F}^{\overline{W}^t}_s$. In both cases, we see that $\overline{J}^{\overline{\omega}}_{t_{\overline{\omega}}\vee s}\in\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_{t_{\overline{\omega}}\vee s}\subset\mathcal{F}^{\overline{W}^t}_s$. Since $\widehat{\Upsilon}^{\varepsilon}_s(\overline{W})-\widehat{\Upsilon}^{\varepsilon}_{\overline{\gamma}\wedge s}(\overline{W})$ is $\mathcal{F}^{\overline{W}^t,\overline{Q}^{\overline{\omega}}_{\varepsilon}}_s-$measurable by (6.59) and (6.60), letting $\mathcal{E}$ run through $\mathcal{F}^{\overline{W}^t,\overline{Q}^{\overline{\omega}}_{\varepsilon}}_s$ in (6.63), we can find some $\overline{\mathfrak{N}}^{\varepsilon}_{s,\overline{\omega}}\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\big(\mathcal{F}^{\overline{W}^t}_{\infty}\big)$ such that

$$\widehat{\Upsilon}^{\varepsilon}_s\big(\overline{W}(\overline{\omega}')\big)-\widehat{\Upsilon}^{\varepsilon}\big(\overline{\gamma}(\overline{\omega}')\wedge s,\overline{W}(\overline{\omega}')\big)=\overline{J}^{\overline{\omega}}_{t_{\overline{\omega}}\vee s}(\overline{\omega}'),\quad\forall\,\overline{\omega}'\in\big(\overline{\mathfrak{N}}^{\varepsilon}_{s,\overline{\omega}}\big)^c.\tag{6.64}$$

Now, set $\overline{\mathfrak{N}}^{\varepsilon}_{\sharp}:=\underset{s\in\mathbb{Q}_t}{\cup}\overline{\mathfrak{N}}^{\varepsilon}_s=\underset{s\in\mathbb{Q}_t}{\cup}\underset{j\in\mathbb{N}}{\cup}\overline{\mathfrak{N}}^{\varepsilon}_{s,j}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_{\infty}\big)$ and fix $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}_*\cup\overline{\mathfrak{N}}^{\varepsilon}_U\cup\overline{\mathfrak{N}}^{\varepsilon}_{\sharp}\big)^c$. We also set $\overline{\mathfrak{N}}^{\varepsilon}_{\sharp,\overline{\omega}}:=\underset{s\in\mathbb{Q}_t}{\cup}\overline{\mathfrak{N}}^{\varepsilon}_{s,\overline{\omega}}\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_{\varepsilon}}\big(\mathcal{F}^{\overline{W}^t}_{\infty}\big)$ and let $\overline{\omega}'\in\mathbf{W}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}^{\varepsilon}_{\mathscr{U}}\cup\overline{\mathfrak{N}}^{\varepsilon}_{\sharp,\overline{\omega}}\big)^c$. The right-continuity of process $\widehat{\Upsilon}^{\varepsilon}$, (5.1) and (6.64) render that $\widehat{\Upsilon}^{\varepsilon}_s\big(\overline{W}(\overline{\omega}')\big)-\widehat{\Upsilon}^{\varepsilon}\big(t_{\overline{\omega}},\overline{W}(\overline{\omega}')\big)=\int_{t_{\overline{\omega}}}^{s}e^{-r}\mathscr{I}\big(\overline{\mu}^{\overline{\omega}}_r(\overline{\omega}')\big)dr$, $\forall\,s\in[t_{\overline{\omega}},\infty)$. By the Lebesgue differentiation theorem, we have $\lim\limits_{\delta\to0+}\frac{1}{\delta}\Big(\widehat{\Upsilon}^{\varepsilon}_s\big(\overline{W}(\overline{\omega}')\big)-\widehat{\Upsilon}^{\varepsilon}\big((s-\delta)\vee t_{\overline{\omega}},\overline{W}(\overline{\omega}')\big)\Big)=\lim\limits_{\delta\to0+}\frac{1}{\delta}\int_{(s-\delta)\vee t_{\overline{\omega}}}^{s}e^{-r}\mathscr{I}\big(\overline{\mu}^{\overline{\omega}}_r(\overline{\omega}')\big)dr=e^{-s}\mathscr{I}\big(\overline{\mu}^{\overline{\omega}}_s(\overline{\omega}')\big)$ for a.e. $s\in(t_{\overline{\omega}},\infty)$ and thus $\mathscr{U}^{\varepsilon}_s\big(\overline{W}(\overline{\omega}')\big)=\widehat{\mathscr{U}^{\varepsilon}_s}\big(\overline{W}(\overline{\omega}')\big)=e^s\lim\limits_{\delta\to0+}\frac{1}{\delta}\Big(\widehat{\Upsilon}^{\varepsilon}_s\big(\overline{W}(\overline{\omega}')\big)-\widehat{\Upsilon}^{\varepsilon}\big((s-\delta)\vee t_{\overline{\omega}},\overline{W}(\overline{\omega}')\big)\Big)=\mathscr{I}\big(\overline{\mu}^{\overline{\omega}}_s(\overline{\omega}')\big)$ for a.e. $s\in(t_{\overline{\omega}},\infty)$. It follows that $\widehat{A}^{\varepsilon}_{2,\overline{\omega}}:=\big\{\overline{\omega}'\in\overline{\Omega}:\,\mu^{\varepsilon}_s\big(\overline{W}(\overline{\omega}')\big)=\overline{\mu}^{\overline{\omega}}_s(\overline{\omega}')$ for a.e. $s\in(t_{\overline{\omega}},\infty)\big\}\supset\mathbf{W}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}^{\varepsilon}_{\mathscr{U}}\cup\overline{\mathfrak{N}}^{\varepsilon}_{\sharp,\overline{\omega}}\big)^c$. Then (6.35) and (6.60) show that $1=\overline{Q}^{\overline{\omega}}_{\varepsilon}\Big\{\mathbf{W}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}^{\varepsilon}_{\mathscr{U}}\cup\overline{\mathfrak{N}}^{\varepsilon}_{\sharp,\overline{\omega}}\big)^c\Big\}\le\overline{Q}^{\overline{\omega}}_{\varepsilon}\big(\widehat{A}^{\varepsilon}_{2,\overline{\omega}}\big)$. Since $\mathbf{W}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_{\mu}\cap\overline{\mathcal{A}}^{\varepsilon}_2=\mathbf{W}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_{\mu}\cap\big\{\overline{\omega}'\in\overline{\Omega}:\,\mu^{\varepsilon}_s\big(\overline{W}(\overline{\omega}')\big)=\overline{U}_s(\overline{\omega}')$ for a.e. $s\in\big(t_{\overline{\omega}},\infty\big)\big\}=\mathbf{W}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\Omega}^{\overline{\omega}}_{\mu}\cap\widehat{A}^{\varepsilon}_{2,\overline{\omega}}$, we further see that

$$\overline{Q}^{\overline{\omega}}_{\varepsilon}\big(\overline{\mathcal{A}}^{\varepsilon}_2\big)=\overline{Q}^{\overline{\omega}}_{\varepsilon}\big(\widehat{A}^{\varepsilon}_{2,\overline{\omega}}\big)=1,\quad\forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}_*\cup\overline{\mathfrak{N}}^{\varepsilon}_U\cup\overline{\mathfrak{N}}^{\varepsilon}_{\sharp}\big)^c,\tag{6.65}$$

which together with (6.57) and (6.58) yields that

$$\overline{P}_{\varepsilon}\big\{\overline{U}_s=\mu^{\varepsilon}_s(\overline{W}),\ \text{for a.e. }s\in(t,\infty)\big\}=\overline{P}_{\varepsilon}\big(\overline{\mathcal{A}}^{\varepsilon}_1\cap\overline{\mathcal{A}}^{\varepsilon}_2\big)=\overline{P}_{\varepsilon}\big(\overline{\mathcal{A}}^{\varepsilon}_2\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{\mathcal{A}}^{\varepsilon}_2\big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_{\varepsilon}\big(\overline{\mathcal{A}}^{\varepsilon}_2\big)\overline{P}(d\overline{\omega})=\overline{P}\big(\overline{\mathcal{A}}^c_*\big)+\overline{P}\big(\overline{\mathcal{A}}_*\big)=1.$$

Hence, $\overline{P}_\varepsilon$ satisfies (D1) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$. We set $\overline{\mu}^\varepsilon_s:=\mu^\varepsilon_s(\overline{W})$, $\forall\, s\in[t,\infty)$.

II.e) *In this part, we use the martingale-problem formulation to demonstrate that* $\big\{\overline{M}^{t,\overline{\mu}^\varepsilon}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ *is a bounded* $\big(\overline{\mathbf{F}}^t,\overline{P}_\varepsilon\big)$*−martingale for any* $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$*. Consequently,* $\overline{P}_\varepsilon$ *satisfies (D2′) of* $\overline{\mathcal{P}}_{t,\mathbf{x}}$ *and thus satisfies (D3) of* $\overline{\mathcal{P}}_{t,\mathbf{x}}$*.*

II.e.1) Fix $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$ and set $\overline{\Omega}_X:=\big\{\overline{X}_s=\mathbf{x}(s),\ \forall\, s\in[0,t]\big\}$. We know from the proof of Proposition 5.1 that $\overline{\Omega}^c_X\subset\overline{\mathcal{N}}_X=\big\{\overline{\omega}\in\overline{\Omega}:\ \overline{X}_s(\overline{\omega})\neq\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_s(\overline{\omega})\text{ for some } s\in[0,\infty)\big\}\subset\overline{\mathcal{N}}_1\subset\overline{\mathcal{N}}_*$. Given $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\subset\overline{\Omega}_X$, one has $\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}}\subset\big\{\overline{\omega}'\in\overline{\Omega}:\ \overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\ \forall\, s\in[0,t]\big\}=\overline{\Omega}_X$ and (6.35) implies that $\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\Omega}_X\big)=1$. As $\overline{P}\big(\overline{\Omega}_X\big)=1$ by (D2′) of Remark 3.1, one can deduce that $\overline{P}_\varepsilon\big(\overline{\Omega}_X\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{\Omega}_X\big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*}1\cdot\overline{P}(d\overline{\omega})=\overline{P}\big(\overline{\mathcal{A}}^c_*\big)+\overline{P}(\overline{\mathcal{A}}_*)=1$. Applying Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X,\mu)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}_\varepsilon,\overline{W},\overline{X},\overline{\mu}^\varepsilon\big)$ renders that $\big\{\overline{M}^{t,\overline{\mu}^\varepsilon}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\overline{\mathbf{F}}^t$−adapted process under $\overline{P}_\varepsilon$.

To show the $\overline{P}_\varepsilon$−martingality of $\big\{\overline{M}^{t,\overline{\mu}^\varepsilon}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$, we let $t\le s<r<\infty$, $\{(s_i,\mathcal{E}_i)\}_{i=1}^k\subset[t,s]\times\mathscr{B}(\mathbb{R}^{d+l})$ and set $\overline{A}:=\bigcap\limits_{i=1}^k(\overline{W}^t_{s_i},\overline{X}_{s_i})^{-1}(\mathcal{E}_i)\in\overline{\mathcal{F}}^t_s$. We need to verify that

$$E_{\overline{P}_\varepsilon}\big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big]=0. \tag{6.66}$$

If $t+n\le s$, one directly has $E_{\overline{P}_\varepsilon}\big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big]=E_{\overline{P}_\varepsilon}\big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n}(\varphi)\big)\mathbf{1}_{\overline{A}}\big]=0$ since $\overline{\tau}^t_n\le t+n\le s$.

II.e.2) Assume $t+n>s$. *To obtain* (6.66) *for the this case, we first show that*

$$E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\{\overline{\gamma}>s\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=0. \tag{6.67}$$

(i) For any $\overline{\omega}\in\widehat{A}^\varepsilon_1$ and any $t\le r_1\le r_2\le\overline{\gamma}(\overline{\omega})$, we have $\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{r_2}(\varphi)\big)(\overline{\omega})-\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{r_1}(\varphi)\big)(\overline{\omega})=\big(\overline{M}^{t,\overline{\mu}}_{r_2}(\varphi)\big)(\overline{\omega})-\big(\overline{M}^{t,\overline{\mu}}_{r_1}(\varphi)\big)(\overline{\omega})$. So $\mathbf{1}_{\widehat{A}^\varepsilon_1}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge s}(\varphi)\big)=\mathbf{1}_{\widehat{A}^\varepsilon_1}\big(\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge s}(\varphi)\big)$ and (6.57) renders that $E_{\overline{P}_\varepsilon}\Big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]=E_{\overline{P}_\varepsilon}\Big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge s}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]=E_{\overline{P}_\varepsilon}\Big[\big(\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge s}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]$. Since $\{\overline{\gamma}>s\}\cap\overline{A}=\{\overline{\gamma}>s\}\cap\Big(\bigcap\limits_{i=1}^k(\overline{W}^t_{\overline{\gamma}\wedge s_i},\overline{X}_{\overline{\gamma}\wedge s_i})^{-1}(\mathcal{E}_i)\Big)\in\overline{\mathcal{F}}^t_{\overline{\gamma}\wedge s}$ and $\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge s}(\varphi)\in\overline{\mathcal{F}}^t_{\overline{\gamma}\wedge r}$, using (6.39) and applying (3.2) with $(\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2)=\big(\mathbf{0},\overline{\gamma}\wedge s,\overline{\gamma}\wedge r\big)$ yield that

$$E_{\overline{P}_\varepsilon}\Big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]=E_{\overline{P}}\Big[\big(\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge s}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]=0. \tag{6.68}$$

It also holds for any $\overline{\omega}\in\widehat{A}^\varepsilon_2$ and any $\overline{\gamma}(\overline{\omega})\le r_1\le r_2<\infty$ that

$$\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{r_2}(\varphi)\big)(\overline{\omega})-\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{r_1}(\varphi)\big)(\overline{\omega})=\big(\overline{M}^{t,\overline{\mu}}_{r_2}(\varphi)\big)(\overline{\omega})-\big(\overline{M}^{t,\overline{\mu}}_{r_1}(\varphi)\big)(\overline{\omega}). \tag{6.69}$$

As $\overline{P}\Big(\overline{\mathcal{A}}^c_*\cap\big(\widehat{A}^\varepsilon_2\big)^c\Big)=0$ by (6.58), taking $\big(\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2\big)=\big(\mathbf{0},\overline{\gamma},\overline{\gamma}\vee r\big)$ in (3.2), we obtain

$$\begin{aligned}
E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]&=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\widehat{A}^\varepsilon_2}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]\\
=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\widehat{A}^\varepsilon_2}\big(\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]&=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*}\big(\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge\overline{\gamma}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]=0.
\end{aligned}$$

It follows from (6.37) that

$$E_{\overline{P}_\varepsilon}\Big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge r}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>s\}\cap\overline{A}}\Big]=\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\gamma}(\overline{\omega})>s\}}\mathbf{1}_{\{\overline{\omega}\in\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}}(\varphi)\big]\overline{P}(d\overline{\omega}). \tag{6.70}$$

(ii) For $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}_*\cup\overline{\mathfrak{N}}^\varepsilon_U\cup\overline{\mathfrak{N}}^\varepsilon_\sharp\big)^c$, we define a $C^2(\mathbb{R}^{d+l})$ function $\varphi_{\overline{\omega}}(w,x):=\varphi\big(w+\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),x\big)$, $(w,x)\in\mathbb{R}^{d+l}$ and define an $\overline{\mathbf{F}}^{t_{\overline{\omega}}}$−stopping time $\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}'):=\inf\big\{s\in[t_{\overline{\omega}},\infty):\ \big|(\overline{W}^{t_{\overline{\omega}}}_s,\overline{X}_s)(\overline{\omega}')-\mathfrak{a}_{\overline{\omega}}\big|\ge n\big\}$, $\overline{\omega}'\in\overline{\Omega}$ with $\mathfrak{a}_{\overline{\omega}}:=\big(-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\mathbf{0}\big)\in\mathbb{R}^{d+l}$. For $i=0,1,2$ and $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\widehat{A}^\varepsilon_{2,\overline{\omega}}$, since $D^i\varphi\big(\overline{W}^t_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)=D^i\varphi\big(\overline{W}^t_r(\overline{\omega}')-\overline{W}^t(\overline{\gamma}(\overline{\omega}),\overline{\omega}')+\overline{W}^t(\overline{\gamma}(\overline{\omega}),\overline{\omega}),\overline{X}_r(\overline{\omega}')\big)=$

$D^i\varphi_{\overline{\omega}}\big(\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)$, $\forall\, r\in[t_{\overline{\omega}},\infty)$, we obtain

$$
\begin{aligned}
&\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{r_2}(\varphi)\big)(\overline{\omega}')-\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{r_1}(\varphi)\big)(\overline{\omega}')\\
&=-\int_{r_1}^{r_2}\overline{b}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\overline{\mu}^\varepsilon_r(\overline{\omega}')\big)\cdot D\varphi\big(\overline{W}^t_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)dr-\frac{1}{2}\int_{r_1}^{r_2}\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\overline{\mu}^\varepsilon_r(\overline{\omega}')\big):D^2\varphi\big(\overline{W}^t_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)dr\\
&=-\int_{r_1}^{r_2}\overline{b}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\overline{\mu}^{\overline{\omega}}_r(\overline{\omega}')\big)\cdot D\varphi_{\overline{\omega}}\big(\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)dr-\frac{1}{2}\int_{r_1}^{r_2}\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}'),\overline{\mu}^{\overline{\omega}}_r(\overline{\omega}')\big):D^2\varphi_{\overline{\omega}}\big(\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)dr\\
&=\Big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{r_2}(\varphi_{\overline{\omega}})\Big)(\overline{\omega}')-\Big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{r_1}(\varphi_{\overline{\omega}})\Big)(\overline{\omega}'),\quad \forall\, t_{\overline{\omega}}\le r_1\le r_2<\infty. \qquad (6.71)
\end{aligned}
$$

(iii) Fix $\overline{\omega}\in\{\overline{\tau}^t_n>\overline{\gamma}\}\cap\overline{\mathcal{A}}_*\cap\big(\mathcal{N}_*\cup\overline{\mathfrak{N}}^\varepsilon_U\cup\overline{\mathfrak{N}}^\varepsilon_\sharp\big)^c$ and set $\mathfrak{r}_{\overline{\omega}}:=t_{\overline{\omega}}\vee r$. Since $t_{\overline{\omega}}=\overline{\gamma}(\overline{\omega})<\overline{\tau}^t_n(\overline{\omega})\le t+n$, applying (3.2) with $(t,\mathbf{x},\overline{P},\overline{\mu},\varphi,\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2)=\big(t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\overline{Q}^{\overline{\omega}}_\varepsilon,\overline{\mu}^{\overline{\omega}},\varphi_{\overline{\omega}},\mathfrak{a}_{\overline{\omega}},t_{\overline{\omega}},\mathfrak{r}_{\overline{\omega}}\wedge(t+n)\big)$ yields that

$$
0=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\overline{M}^{t_{\overline{\omega}},\,\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t_{\overline{\omega}}+n)\wedge\mathfrak{r}_{\overline{\omega}}\wedge(t+n)}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}},\,\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t_{\overline{\omega}}+n)\wedge t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\Big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\overline{M}^{t_{\overline{\omega}},\,\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge\mathfrak{r}_{\overline{\omega}}\wedge(t+n)}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\Big]. \qquad (6.72)
$$

Because $\overline{\omega}\in\{\overline{\omega}'\in\overline{\Omega}:\overline{\tau}^t_n(\overline{\omega}')>\overline{\gamma}(\overline{\omega})\}\in\overline{\mathcal{F}}^t_{\overline{\gamma}(\overline{\omega})}\subset\overline{\mathcal{G}}^t_{\overline{\gamma}(\overline{\omega})}$, an analogy to (6.38) shows that $\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\subset\{\overline{\omega}'\in\overline{\Omega}:\overline{\tau}^t_n(\overline{\omega}')>\overline{\gamma}(\overline{\omega})\}$. Let $\overline{\omega}'\in\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}}\cap\widehat{A}^\varepsilon_{2,\overline{\omega}}$. Since $\inf\{s\in[t,\infty):\big|(\overline{W}^t_s,\overline{X}_s)(\overline{\omega}')\big|\ge n\}\ge\overline{\tau}^t_n(\overline{\omega}')>\overline{\gamma}(\overline{\omega})$, one has $\big|(\overline{W}^t_s,\overline{X}_s)(\overline{\omega}')\big|<n$, $\forall\, s\in[t,t_{\overline{\omega}}]$ and thus

$$
\inf\{s\in[t,\infty):|(\overline{W}^t_s,\overline{X}_s)(\overline{\omega}')|\ge n\}=\inf\big\{s\in[t_{\overline{\omega}},\infty):\big|\big(\overline{W}^t_s(\overline{\omega}')-\overline{W}^t(\overline{\gamma}(\overline{\omega}),\overline{\omega}'),\overline{X}_s(\overline{\omega}')\big)+\big(\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\mathbf{0}\big)\big|\ge n\big\}=\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}').
$$

It follows that $\overline{\tau}^t_n(\overline{\omega}')=\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge(t+n)$. Taking $(r_1,r_2)=\big(t_{\overline{\omega}},\overline{\tau}^t_n(\overline{\omega}')\wedge\mathfrak{r}_{\overline{\omega}}\big)$ in (6.71), we can deduce from (5.1) that

$$
\begin{aligned}
&\big(\overline{M}^{t,\overline{\mu}^\varepsilon}(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge(\overline{\gamma}(\overline{\omega}')\vee r),\overline{\omega}'\big)-\big(\overline{M}^{t,\overline{\mu}^\varepsilon}(\varphi)\big)\big(\overline{\gamma}(\overline{\omega}'),\overline{\omega}'\big)=\big(\overline{M}^{t,\overline{\mu}^\varepsilon}(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge(\overline{\gamma}(\overline{\omega})\vee r),\overline{\omega}'\big)-\big(\overline{M}^{t,\overline{\mu}^\varepsilon}(\varphi)\big)\big(\overline{\gamma}(\overline{\omega}),\overline{\omega}'\big)\\
&\quad=\big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)\big(\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge(t+n)\wedge\mathfrak{r}_{\overline{\omega}},\overline{\omega}'\big)-\big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)(t_{\overline{\omega}},\overline{\omega}').
\end{aligned}
$$

As $\{\overline{\tau}^t_n>\overline{\gamma}\}\in\overline{\mathcal{F}}^t_{\overline{\tau}^t_n\wedge\overline{\gamma}}\subset\overline{\mathcal{G}}^t_{\overline{\gamma}}$, (6.37), (6.35), (6.65) and (6.72) then imply that

$$
\begin{aligned}
&E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}}(\varphi)\big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\{\overline{\tau}^t_n>\overline{\gamma}\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\gamma}}(\varphi)\big)\big]=\mathbf{1}_{\{\overline{\tau}^t_n(\overline{\omega})>\overline{\gamma}(\overline{\omega})\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\gamma}}(\varphi)\big]\\
&\quad=\mathbf{1}_{\{\overline{\tau}^t_n(\overline{\omega})>\overline{\gamma}(\overline{\omega})\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t+n)\wedge\mathfrak{r}_{\overline{\omega}}}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\Big]=0,\quad\forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\mathcal{N}_*\cup\overline{\mathfrak{N}}^\varepsilon_U\cup\overline{\mathfrak{N}}^\varepsilon_\sharp\big)^c.
\end{aligned}
$$

Thus $\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\gamma}(\overline{\omega})>s\}}\mathbf{1}_{\{\overline{\omega}\in\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee r)}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge\overline{\gamma}}(\varphi)\big]\overline{P}(d\overline{\omega})=0$, which together with (6.70) and (6.68) leads to (6.67) for the case "$t+n>s$".

II.e.3) *If $t+n>s=t$, as $\{\overline{\gamma}>t\}=\overline{\Omega}$, (6.67) directly becomes (6.66). We further assume $t+n>s>t$ and continue to verify (6.66). In this case, we can also assume without loss of generality that $t=s_1<\cdots<s_k=s$ with $k\ge 2$.*

Since (6.69) renders that $\mathbf{1}_{\widehat{A}^\varepsilon_2}\mathbf{1}_{\{\overline{\gamma}\le s\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)=\mathbf{1}_{\widehat{A}^\varepsilon_2}\mathbf{1}_{\{\overline{\gamma}\le s\}}\big(\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)$ and since $\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le s\}\in\mathcal{F}^{\overline{W}^t}_s\subset\overline{\mathcal{F}}^t_s$, using $\overline{P}\Big(\overline{\mathcal{A}}^c_*\cap\big(\widehat{A}^\varepsilon_2\big)^c\Big)=0$ and taking $(\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2)=(\mathbf{0},s,r)$ in (3.2) yield that

$$
E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le s\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le s\}}\big(\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=0. \qquad (6.73)
$$

(i) Fix $i\in\{1,\cdots,k-1\}$ and fix $\overline{\omega}\in\{\overline{\tau}^t_n>\overline{\gamma}\}\cap\{s_i<\overline{\gamma}\le s_{i+1}\}\cap\overline{\mathcal{A}}_*\cap\big(\mathcal{N}_*\cup\overline{\mathfrak{N}}^\varepsilon_U\cup\overline{\mathfrak{N}}^\varepsilon_\sharp\big)^c$. Since $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\subset\{\overline{\gamma}>s_i\}$ by (5.1), $\overline{A}_i:=\underset{j=1}{\overset{i}{\cap}}\big(\overline{W}^t_{\overline{\gamma}\wedge s_j},\overline{X}_{\overline{\gamma}\wedge s_j}\big)^{-1}(\mathcal{E}_j)\in\overline{\mathcal{F}}^t_{\overline{\gamma}}$ satisfies that

$$
\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\Big(\underset{j=1}{\overset{i}{\cap}}\big(\overline{W}^t_{s_j},\overline{X}_{s_j}\big)^{-1}(\mathcal{E}_j)\Big)=\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{A}_i. \qquad (6.74)
$$

Also, (5.1) shows that $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\subset\{\overline{\gamma}\le s\}$ and thus $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\{\overline{\tau}^t_n\le\overline{\gamma}\}\subset\{\overline{\tau}^t_n\le s\}$. We see from (6.35) that

$$
E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\{\overline{\tau}^t_n\le\overline{\gamma}\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\{\overline{\tau}^t_n\le\overline{\gamma}\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big]=0. \qquad (6.75)
$$

Define $\overline{A}^{\overline{\omega}}_i := \underset{j=i+1}{\overset{k}{\cap}} \big(\overline{W}^{t_{\overline{\omega}}}_{s_j}, \overline{K}^{\overline{\omega}}_{s_j}\big)^{-1}(\mathcal{E}_{j,\overline{\omega}}) \in \mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_s$ with $\mathcal{E}_{j,\overline{\omega}} := \big\{\mathfrak{x}+\mathfrak{a}_{\overline{\omega}} : \mathfrak{x} \in \mathcal{E}_j\big\} \in \mathscr{B}(\mathbb{R}^{d+l})$. Using (3.2) with $(t,\mathbf{x},\overline{P},\overline{\mu},\varphi,\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2) = \big(t_{\overline{\omega}}, \overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}), \overline{Q}^{\overline{\omega}}_\varepsilon, \overline{\mu}^{\overline{\omega}}, \varphi_{\overline{\omega}}, \mathfrak{a}_{\overline{\omega}}, s, (t+n)\wedge r\big)$ renders that

$$
\begin{aligned}
0 &= E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\Big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t_{\overline{\omega}}+n)\wedge(t+n)\wedge r)}(\varphi_{\overline{\omega}}) - \overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t_{\overline{\omega}}+n)\wedge s}(\varphi_{\overline{\omega}})\Big)\mathbf{1}_{\overline{A}^{\overline{\omega}}_i}\Big] \\
&= E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\Big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t+n)\wedge r)}(\varphi_{\overline{\omega}}) - \overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge s}(\varphi_{\overline{\omega}})\Big)\mathbf{1}_{\overline{A}^{\overline{\omega}}_i}\Big].
\end{aligned} \tag{6.76}
$$

(ii) Let $\overline{\omega}' \in \overline{\Theta}^t_{\overline{\gamma},\overline{\omega}} \cap \widehat{A}^\varepsilon_{2,\overline{\omega}}$. Like in Step (i), we still have $\overline{\Theta}^t_{\overline{\gamma},\overline{\omega}} \subset \big\{\overline{\omega}' \in \overline{\Omega} : \overline{\tau}^t_n(\overline{\omega}') > \overline{\gamma}(\overline{\omega})\big\}$ and $\overline{\tau}^t_n(\overline{\omega}') = \overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}') \wedge (t+n)$. As $\overline{\tau}^t_n(\overline{\omega}') \wedge s \ge \overline{\gamma}(\overline{\omega})$, taking $(r_1,r_2) = \big(\overline{\tau}^t_n(\overline{\omega}')\wedge s, \overline{\tau}^t_n(\overline{\omega}')\wedge r\big)$ in (6.71) shows that $\big(\overline{M}^{t,\overline{\mu}^\varepsilon}(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge r, \overline{\omega}'\big) - \big(\overline{M}^{t,\overline{\mu}^\varepsilon}(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge s, \overline{\omega}'\big) = \big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)\big(\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge(t+n)\wedge r, \overline{\omega}'\big) - \big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)\big(\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge s, \overline{\omega}'\big)$. It follows from (6.35), (6.65) and (6.76) that

$$
E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}^{\overline{\omega}}_i}\big] = E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\Big(\overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t+n)\wedge r}(\varphi_{\overline{\omega}}) - \overline{M}^{t_{\overline{\omega}},\overline{\mu}^{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge s}(\varphi_{\overline{\omega}})\Big)\mathbf{1}_{\overline{A}^{\overline{\omega}}_i}\Big] = 0. \tag{6.77}
$$

For any $j \in \{i+1,\cdots,k\}$ and $\overline{\omega}' \in \overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}} \cap \big(\mathcal{N}^{\overline{\omega}}_X \cup \mathcal{N}^{\overline{\omega}}_K\big)^c$, (5.1) implies that $\big(\overline{W}^t_{s_j}, \overline{X}_{s_j}\big)(\overline{\omega}') \in \mathcal{E}_j$ if and only if $\big(\overline{W}^{t_{\overline{\omega}}}_{s_j}, \overline{K}^{\overline{\omega}}_{s_j}\big)(\overline{\omega}') = \big(\overline{W}^t_{s_j}(\overline{\omega}') - \overline{W}^t\big(\overline{\gamma}(\overline{\omega}), \overline{\omega}'\big), \overline{\mathscr{X}}^{\overline{\omega}}_{s_j}(\overline{\omega}')\big) = \big(\overline{W}^t_{s_j}, \overline{X}_{s_j}\big)(\overline{\omega}') + \mathfrak{a}_{\overline{\omega}} \in \mathcal{E}_{j,\overline{\omega}}$. By (6.74), one has $\overline{A} \cap \overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}} \cap \big(\mathcal{N}^{\overline{\omega}}_X \cup \mathcal{N}^{\overline{\omega}}_K\big)^c = \overline{A}_i \cap \overline{A}^{\overline{\omega}}_i \cap \overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}} \cap \big(\mathcal{N}^{\overline{\omega}}_X \cup \mathcal{N}^{\overline{\omega}}_K\big)^c$. Then we can deduce from (6.75), (6.35), (6.37) and (6.77) that

$$
\begin{aligned}
&E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big] = E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\{\overline{\tau}^t_n > \overline{\gamma}\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big] = E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\{\overline{\tau}^t_n > \overline{\gamma}\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}_i\cap\overline{A}^{\overline{\omega}}_i}\big] \\
&= \mathbf{1}_{\{\overline{\omega}\in\overline{A}_i\}}\mathbf{1}_{\{\overline{\tau}^t_n(\overline{\omega}) > \overline{\gamma}(\overline{\omega})\}} E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}^{\overline{\omega}}_i}\Big] = 0, \quad \forall\, \overline{\omega} \in \{s_i < \overline{\gamma} \le s_{i+1}\} \cap \overline{\mathcal{A}}_* \cap \big(\overline{\mathcal{N}}_* \cup \overline{\mathfrak{N}}^\varepsilon_U \cup \overline{\mathfrak{N}}^\varepsilon_\sharp\big)^c,
\end{aligned}
$$

and thus $\int_{\overline{\omega}\in\overline{\mathcal{A}}_*} \mathbf{1}_{\{s_i < \overline{\gamma}(\overline{\omega}) \le s_{i+1}\}} E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big]\overline{P}(d\overline{\omega}) = 0$. Taking summation from $i=1$ through $i=k-1$, we obtain from (6.73) that $E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\{\overline{\gamma}\le s\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big] = E_{\overline{P}}\big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le s\}}\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big] + \int_{\overline{\omega}\in\overline{\mathcal{A}}_*} \mathbf{1}_{\{\overline{\gamma}(\overline{\omega})\le s\}} E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big]\overline{P}(d\overline{\omega}) = 0$. Adding it to (6.67) yields (6.66) for the case "$t+n > s > t$".

II.e.4) We know from (6.66) that the Lambda-system $\Big\{\overline{A} \in \mathscr{B}(\overline{\Omega}) : E_{\overline{P}_\varepsilon}\big[\big(\overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge r}(\varphi) - \overline{M}^{t,\overline{\mu}^\varepsilon}_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\big] = 0\Big\}$ contains the Pi-system $\Big\{\underset{i=1}{\overset{k}{\cap}} \big(\overline{W}^t_{s_i}, \overline{X}_{s_i}\big)^{-1}(\mathcal{E}_i) : \{(s_i,\mathcal{E}_i)\}_{i=1}^k \subset [t,s] \times \mathscr{B}(\mathbb{R}^{d+l})\Big\}$ and thus includes $\overline{\mathcal{F}}^t_s$ thanks to Dynkin's Pi-Lambda Theorem. Hence, $\big\{\overline{M}^{t,\overline{\mu}^\varepsilon}_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\overline{\mathbf{F}}^t, \overline{P}_\varepsilon\big)-$martingale. According to Remark 3.1 (ii), $\overline{P}_\varepsilon$ satisfies (D3) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

Similar to Part (II.d) in the proof of [4, Theorem 5.1], we can construct a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}_\varepsilon$ with $\overline{P}_\varepsilon\big\{\overline{T} = \widehat{\tau}_\varepsilon(\overline{W})\big\} = 1$, So $\overline{P}_\varepsilon$ also satisfies (D4) of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

II.f) *In this part, we show that $\overline{P}_\varepsilon$ belongs to the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$.*

Fix $i \in \mathbb{N}$. Since $\big\{\int_t^s g_i\big(r, \overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_{r\wedge\cdot}, \overline{\mu}_r\big)dr\big\}_{s\in[t,\infty)}$ and $\big\{\int_t^s h_i\big(r, \overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_{r\wedge\cdot}, \overline{\mu}_r\big)dr\big\}_{s\in[t,\infty)}$ are two $\mathbf{F}^{\overline{W}^t,\overline{P}}-$adapted continuous processes, Lemma 2.4 of [44] assures two $\mathbf{F}^{\overline{W}^t}-$predictable processes $\big\{\overline{\Phi}^i_s\big\}_{s\in[t,\infty)}$ and $\big\{\overline{\Psi}^i_s\big\}_{s\in[t,\infty)}$ such that $\overline{\mathcal{N}}^{i,1}_{g,h} := \Big\{\overline{\omega} \in \overline{\Omega} : \overline{\Phi}^i_s(\overline{\omega}) \ne \int_t^s g_i\big(r, \overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_{r\wedge\cdot}(\overline{\omega}), \overline{\mu}_r(\overline{\omega})\big)dr$ or $\overline{\Psi}^i_s(\overline{\omega}) \ne \int_t^s h_i\big(r, \overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_{r\wedge\cdot}(\overline{\omega}), \overline{\mu}_r(\overline{\omega})\big)dr$ for some $s \in [t,\infty)\Big\} \in \mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. By Remark 3.2 (1), $E_{\overline{P}}\big[\int_t^\infty g_i^-\big(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r\big) \vee h_i^-\big(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r\big)dr\big] < \infty$. So it holds for any $\overline{\omega} \in \overline{\Omega}$ except on some $\overline{\mathcal{N}}^{i,2}_{g,h} \in \mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$ that $\int_t^\infty g_i^-\big(r, \overline{X}_{r\wedge\cdot}(\overline{\omega}), \overline{U}_r(\overline{\omega})\big) \vee h_i^-\big(r, \overline{X}_{r\wedge\cdot}(\overline{\omega}), \overline{U}_r(\overline{\omega})\big)dr < \infty$.

For any $\overline{\omega} \in \overline{\mathcal{A}}_* \cap \overline{\mathcal{N}}^c_* \cap \overline{\mathcal{N}}^c_X \cap \overline{\Omega}_\mu \cap \big(\overline{\mathcal{N}}^{i,1}_{g,h} \cup \overline{\mathcal{N}}^{i,2}_{g,h}\big)^c$, as $\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}} \subset \big\{\overline{\omega}' \in \overline{\Omega} : \overline{X}_s(\overline{\omega}') = \overline{X}_s(\overline{\omega}),\ \forall\, s \in [0,\overline{\gamma}(\overline{\omega})]; \overline{U}_s(\overline{\omega}') = \overline{U}_s(\overline{\omega})$ for a.e. $s \in (t,\overline{\gamma}(\overline{\omega}))\big\}$, (6.35) and (6.34) show that

$$
\begin{aligned}
E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\int_t^{\overline{T}} g_i\big(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r\big)dr\Big] &= \int_t^{\overline{\gamma}(\overline{\omega})} g_i\big(r, \overline{X}_{r\wedge\cdot}(\overline{\omega}), \overline{U}_r(\overline{\omega})\big)dr + E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\int_{\overline{\gamma}(\overline{\omega})}^{\overline{T}} g_i\big(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r\big)dr\Big] \\
&\le \int_t^{\overline{\gamma}(\overline{\omega})} g_i\big(r, \overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_{r\wedge\cdot}(\overline{\omega}), \overline{\mu}_r(\overline{\omega})\big)dr + \big(\overline{Y}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}) = \overline{\Phi}^i_{\overline{\gamma}}(\overline{\omega}) + E_{\overline{P}}\Big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}} g_i\big(r, \overline{X}_{r\wedge\cdot}, \overline{U}_r\big)dr\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big](\overline{\omega})
\end{aligned} \tag{6.78}
$$

and similarly that $E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\int_t^{\overline{T}} h_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\Big]=\overline{\Psi}^i_{\overline{\gamma}}(\overline{\omega})+E_{\overline{P}}\Big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}} h_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big](\overline{\omega})$. Since $\overline{\mathcal{A}}_*,\overline{\Phi}^i_{\overline{\gamma}}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ and since $\mathbf{1}_{\overline{\mathcal{A}}_*}=\mathbf{1}_{\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}}=\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}$, $\overline{P}-$a.s. by (6.26), we can deduce from the tower property that

$$\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\int_t^{\overline{T}} g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\Big]\overline{P}(d\overline{\omega})\le E_{\overline{P}}\bigg[E_{\overline{P}}\bigg[\mathbf{1}_{\overline{\mathcal{A}}_*}\Big(\overline{\Phi}^i_{\overline{\gamma}}+\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}} g_i(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr\Big)\bigg|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\bigg]\bigg]$$

$$=E_{\overline{P}}\bigg[\mathbf{1}_{\overline{\mathcal{A}}_*}\Big(\int_t^{\overline{\gamma}} g_i\big(r,\overline{\mathscr{X}}^{t,\mathbf{x},\overline{\mu}}_{r\wedge\cdot},\overline{\mu}_r\big)dr+\int_{\overline{\gamma}}^{\overline{T}} g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\Big)\bigg]=E_{\overline{P}}\bigg[\mathbf{1}_{\overline{\mathcal{A}}_*}\int_t^{\overline{T}} g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\bigg]$$

and thus $E_{\overline{P}_\varepsilon}\big[\int_t^{\overline{T}} g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]\le E_{\overline{P}}\big[\int_t^{\overline{T}} g_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]\le y_i$. Analogously, we have $E_{\overline{P}_\varepsilon}\big[\int_t^{\overline{T}} h_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]$ $=E_{\overline{P}}\big[\int_t^{\overline{T}} h_i\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]=z_i$. Hence, $\overline{P}_\varepsilon$ is of $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$.

II.g) *With all technical preparation above, we eventually verify the inequality* (6.28) *for the situation* $\big\{\overline{\gamma}>t\big\}=\overline{\Omega}$.

Since the mapping $\ddot{\Psi}$ is $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{G}}^t_\infty\big)\Big)\Big/\sigma\big(\mathscr{B}(\mathcal{D}_{\overline{\mathcal{P}}})\cup\mathscr{N}_{\ddot{P}}(\mathscr{B}(\mathcal{D}_{\overline{\mathcal{P}}}))\big)-$measurable by Step (II.a.2), Theorem 4.1 renders that $D^V_\infty:=\big\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*:\overline{V}\big(\ddot{\Psi}(\overline{\omega})\big)=\infty\big\}$ is $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)\Big)-$measurable. As $E_{\overline{P}}\big[\int_t^\infty f^-\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]<\infty$, there is a $\overline{\mathcal{N}}_f\in\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$ such that $\int_t^\infty f^-\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}),\overline{U}_r(\overline{\omega})\big)dr<\infty$ for any $\overline{\omega}\in\overline{\mathcal{N}}^c_f$.

Let $\varepsilon\in(0,1)$. For any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\cap\overline{\mathcal{N}}^c_f$, an analogy to (6.78), (6.31) and Theorem 3.1 imply that

$$\begin{aligned}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{R}(t)\big]&=\int_t^{\overline{\gamma}(\overline{\omega})} f\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}),\overline{U}_r(\overline{\omega})\big)dr+E_{\overline{\mathbf{Q}}_\varepsilon(\ddot{\Psi}(\overline{\omega}))}\big[\overline{R}(\overline{\gamma}(\overline{\omega}))\big]\\ &\ge\int_t^{\overline{\gamma}(\overline{\omega})} f\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}),\overline{U}_r(\overline{\omega})\big)dr+\mathbf{1}_{\{\overline{\omega}\in(D^V_\infty)^c\}}\Big(\overline{V}\big(\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\big)-\varepsilon\Big)+\frac{1}{\varepsilon}\mathbf{1}_{\{\overline{\omega}\in D^V_\infty\}}.\end{aligned}$$

Since $\overline{P}_\varepsilon\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ and since $\mathbf{1}_{\overline{\mathcal{A}}_*}=\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}$, $\overline{P}-$a.s.

$$\begin{aligned}\overline{V}(t,\mathbf{x},y,z)&\ge E_{\overline{P}_\varepsilon}\big[\overline{R}(t)\big]\ge E_{\overline{P}}\bigg[\mathbf{1}_{\overline{\mathcal{A}}^c_*}\overline{R}(t)+\mathbf{1}_{\overline{\mathcal{A}}_*}\Big(\int_t^{\overline{\gamma}} f\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr+\mathbf{1}_{(D^V_\infty)^c}\big[\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)-\varepsilon\big]+\frac{1}{\varepsilon}\mathbf{1}_{D^V_\infty}\Big)\bigg]\\ &=E_{\overline{P}}\bigg[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}} f\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr+\mathbf{1}_{(D^V_\infty)^c}\big[\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)-\varepsilon\big]+\frac{1}{\varepsilon}\mathbf{1}_{D^V_\infty}\Big)\bigg]\\ &\ge E_{\overline{P}}\bigg[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}} f\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr+\mathbf{1}_{(D^V_\infty)^c}\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\bigg]-\varepsilon+\frac{1}{\varepsilon}\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big).\end{aligned}\tag{6.79}$$

To verify (6.28), we set $\overline{\mathcal{I}}^t_{\overline{P}}:=\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}} f(r,\overline{X}_{r\wedge\cdot},\overline{U}_r)dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)$.

- If $\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big)=0$, then $\overline{V}(t,\mathbf{x},y,z)\ge E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}} f\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\Big]$ $-\varepsilon$ holds for any $\varepsilon\in(0,1)$. Letting $\varepsilon\to0$ gives (6.28).
- If $\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big)>0$ and $E_{\overline{P}}\big[\big(\overline{\mathcal{I}}^t_{\overline{P}}\big)^-\big]=\infty$, then $E_{\overline{P}}\big[\overline{\mathcal{I}}^t_{\overline{P}}\big]=-\infty\le\overline{V}(t,\mathbf{x},y,z)$, so (6.28) holds automatically.
- If $\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big)>0$ and $E_{\overline{P}}\big[\big(\overline{\mathcal{I}}^t_{\overline{P}}\big)^-\big]<\infty$, since Remark 3.2 (1) shows that $E_{\overline{P}}\Big[-\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)-\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}} f\big(r,\overline{X}_{r\wedge\cdot},$ $\overline{U}_r\big)dr+\mathbf{1}_{(D^V_\infty)^c}\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\Big]=E_{\overline{P}}\Big[-\mathbf{1}_{(D^V_\infty)^c}\overline{\mathcal{I}}^t_{\overline{P}}-\mathbf{1}_{D^V_\infty}\int_t^{\overline{\gamma}\wedge\overline{T}} f\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr-\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}\cap D^V_\infty}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big]\le$ $E_{\overline{P}}\big[\big(\overline{\mathcal{I}}^t_{\overline{P}}\big)^-+\int_t^\infty f^-\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\big]-c_\pi<\infty$, we can deduce from (6.79) that

$$\overline{V}(t,\mathbf{x},y,z)\ge-E_{\overline{P}}\Big[\big(\overline{\mathcal{I}}^t_{\overline{P}}\big)^-+\int_t^\infty f^-\big(r,\overline{X}_{r\wedge\cdot},\overline{U}_r\big)dr\Big]+c_\pi-\varepsilon+\frac{1}{\varepsilon}\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big),\qquad\forall\,\varepsilon\in(0,1).$$

Sending $\varepsilon\to0$ yields $\overline{V}(t,\mathbf{x},y,z)=\infty$, so (6.28) still holds. This completes the proof of Theorem 5.1. □

A Appendix

Lemma A.1. *Let $t_0\in[0,\infty)$. For $i=1,2$, let $(\Omega_i,\mathcal{F}_i,P_i)$ be a probability space and let $B^i=\{B^i_s\}_{s\in[0,\infty)}$ be an $\mathbb{R}^d-$valued continuous process on Ω with $B^i_0=\mathbf{0}$ such that $\mathfrak{B}^i_s:=B^i_s-B^i_{t_0}$, $s\in[t_0,\infty)$ is a Brownian motion on $(\Omega_i,\mathcal{F}_i,P_i)$. Let $\Phi:\Omega_1\mapsto\Omega_2$ be a mapping such that $\mathfrak{B}^2_s(\Phi(\omega))=\mathfrak{B}^1_s(\omega)$ for any $(s,\omega)\in[t_0,\infty)\times\Omega_1$, then (i) $\Phi^{-1}\big(\mathcal{F}^{\mathfrak{B}^2}_s\big)=\mathcal{F}^{\mathfrak{B}^1}_s$, $\forall\,s\in[t_0,\infty]$; (ii) $\Phi^{-1}\big(\mathscr{N}_{P_2}(\mathcal{F}^{\mathfrak{B}^2}_\infty)\big)\subset\mathscr{N}_{P_1}(\mathcal{F}^{\mathfrak{B}^1}_\infty)$; (iii) $\Phi^{-1}\big(\mathcal{F}^{\mathfrak{B}^2,P_2}_s\big)\subset\mathcal{F}^{\mathfrak{B}^1,P_1}_s$, $\forall\,s\in[t_0,\infty]$ and (iv) $P_1\circ\Phi^{-1}(A)=P_2(A)$ for any $A\in\mathcal{F}^{\mathfrak{B}^2,P_2}_\infty$.*

The proof of Lemma A.1 is basic. We refer interested readers to the ArXiv version of [4] for it. We also recall the following result from [4].

Lemma A.2. *Let $(\Omega,\mathcal{F},P)$ be a probability space and let $t\in[0,\infty)$. Let $B=\{B_s\}_{s\in[0,\infty)}$ be an $\mathbb{R}^d-$valued continuous process on Ω with $B_0=\mathbf{0}$ such that $B^t_s:=B_s-B_t$, $s\in[t,\infty)$ is a Brownian motion on $(\Omega,\mathcal{F},P)$.*
(1) For any $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}$ on Ω_0, $\widehat{\tau}(B)$ is an $\mathbf{F}^{B^t,P}-$stopping time on Ω.
(2) Let $\Phi:\Omega\mapsto\Omega_0$ be a mapping such that $W^t_s(\Phi(\omega))=B^t_s(\omega)$ for any $(s,\omega)\in[t,\infty)\times\Omega$. For any $[t,\infty]-$valued $\mathbf{F}^{B^t,P}-$stopping time τ on Ω, there exists a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}$ on Ω_0 such that $\tau=\widehat{\tau}(\Phi)$, $P-$a.s.

Lemma A.3. *Let $\mu=\{\mu_s\}_{s\in[0,\infty)}$ be a $\mathbb{U}-$valued, $\mathbf{F}^{W,P_0}-$predictable process on Ω_0 with all paths in $\mathbb{J}$. Then $\mu^{-1}(A):=\{\omega_0\in\Omega_0:\mu_\cdot(\omega_0)\in A\}\in\mathcal{F}^{W,P_0}_\infty$ for any $A\in\mathscr{B}(\mathbb{J})$.*

Proof: Let $\varphi\in L^0\big((0,\infty)\times\mathbb{U};\mathbb{R}\big)$. The $\mathbf{F}^{W,P_0}-$predictability of μ implies that $\nu_s(\omega_0):=\varphi\big(s,\mu_s(\omega_0)\big)$, $(s,\omega_0)\in(0,\infty)\times\Omega_0$ is also an $\mathbf{F}^{W,P_0}-$predictable process and $\int_0^\infty\nu_s ds=\int_0^\infty\varphi(s,\mu_s)ds$ is thus $\mathcal{F}^{W,P_0}_\infty-$measurable. Then it holds for any $\mathcal{E}\in\mathscr{B}(\mathbb{R})$ that $\mu^{-1}\big((I_\varphi)^{-1}(\mathcal{E})\big)=\big\{\omega_0\in\Omega_0: I_\varphi\big(\mu_\cdot(\omega_0)\big)\in\mathcal{E}\big\}=\big\{\omega_0\in\Omega_0:\int_0^\infty\varphi\big(s,\mu_s(\omega_0)\big)ds\in\mathcal{E}\big\}\in\mathcal{F}^{W,P_0}_\infty$, which together with Lemma 1.3 (1) shows that the sigma-field $\big\{A\subset\mathbb{J}:\mu^{-1}(A)\in\mathcal{F}^{W,P_0}_\infty\big\}$ includes all generating sets of $\mathscr{B}(\mathbb{J})$ and thus contains $\mathscr{B}(\mathbb{J})$. □

Lemma A.4. *For any $t\in[0,\infty)$ and $(\omega_0,\mathfrak{u})\in\Omega_0\times\mathbb{J}$, define $\mathscr{W}^t_\mathfrak{s}(\omega_0):=\omega_0(t+\mathfrak{s})-\omega_0(t)$ and $\mathscr{U}^t_\mathfrak{s}(\mathfrak{u}):=\mathfrak{u}(t+\mathfrak{s})$, $\forall\,\mathfrak{s}\in[0,\infty)$. Then $(t,\omega_0)\mapsto\mathscr{W}^t(\omega_0)$ is a continuous mapping from $[0,\infty)\times\Omega_0$ to Ω_0 and $(t,\mathfrak{u})\mapsto\mathscr{U}^t(\mathfrak{u})$ is a continuous mapping from $[0,\infty)\times\mathbb{J}$ to $\mathbb{J}$.*

Proof: 1) Let $(t,\omega_0)\in[0,\infty)\times\Omega_0$ and let $\varepsilon\in(0,1)$. Set $N:=\lceil 2-\log_2\varepsilon\rceil$ and $T:=\lceil t+1\rceil$. Since $\omega_0(s)$ is uniformly continuous in $s\in[0,N+T]$, there exists $\delta=\delta(t,\omega_0,\varepsilon)\in\Big(0,\dfrac{\varepsilon}{2^{T+3}}\Big)$ such that $|\omega_0(s_1)-\omega_0(s_2)|\le\dfrac{\varepsilon}{4N}$ for any $s_1,s_2\in[0,N+T]$ with $|s_2-s_1|<\delta$.

For any $(t',\omega'_0)\in[0,\infty)\times\Omega_0$ with $|t'-t|\vee\rho_{\Omega_0}(\omega_0,\omega'_0)<\delta$, we can deduce that

$$
\begin{aligned}
\rho_{\Omega_0}(\mathscr{W}^t(\omega_0),\mathscr{W}^{t'}(\omega'_0)) &= \sum_{n\in\mathbb{N}}\Big(2^{-n}\wedge\sup_{\mathfrak{s}\in[0,n]}\big|\omega_0(t+\mathfrak{s})-\omega_0(t)-\omega'_0(t'+\mathfrak{s})+\omega'_0(t')\big|\Big)\\
&\le\sum_{n=1}^N\Big(2^{-n}\wedge 2\sup_{\mathfrak{s}\in[0,n]}\big|\omega_0(t+\mathfrak{s})-\omega'_0(t'+\mathfrak{s})\big|\Big)+\sum_{n=N+1}^\infty 2^{-n}\\
&\le 2\sum_{n=1}^N\Big(2^{-n}\wedge\sup_{\mathfrak{s}\in[0,n]}\big|\omega_0(t+\mathfrak{s})-\omega_0(t'+\mathfrak{s})\big|\Big)+2\sum_{n=1}^N\Big(2^{-n}\wedge\sup_{\mathfrak{s}\in[0,n]}\big|\omega_0(t'+\mathfrak{s})-\omega'_0(t'+\mathfrak{s})\big|\Big)+2^{-N}\\
&\le 2N\sup_{\substack{s_1,s_2\in[0,N+T]\\|s_2-s_1|<\delta}}\big|\omega_0(s_2)-\omega_0(s_1)\big|+2\sum_{n=1}^N\Big(2^{-n}\wedge\sup_{s\in[0,n+T]}\big|\omega_0(s)-\omega'_0(s)\big|\Big)+\varepsilon/4\\
&\le 3\varepsilon/4+2\sum_{n=1}^N 2^T\Big(2^{-n-T}\wedge\sup_{s\in[0,n+T]}\big|\omega_0(s)-\omega'_0(s)\big|\Big)\le 3\varepsilon/4+2^{1+T}\rho_{\Omega_0}\big(\omega_0,\omega'_0\big)<\varepsilon.
\end{aligned}
$$

So $(t,\omega_0)\mapsto\mathscr{W}^t(\omega_0)$ is a continuous mapping from $[0,\infty)\times\Omega_0$ to Ω_0.
2) We next discuss the continuity of mapping $(t,\mathfrak{u})\mapsto\mathscr{U}^t(\mathfrak{u})$ from $[0,\infty)\times\mathbb{J}$ to $\mathbb{J}$.

Denote by $\mathfrak{T}[0,\infty)$ the Euclidean topology on $[0,\infty)$. As $\mathfrak{T}_\sharp(\mathbb{J})$ is generated by the subbase $\big\{\mathfrak{i}_\mathbb{J}^{-1}\big(O_{\frac1n}(\mathfrak{m}_k,\phi_j)\big)\big\}_{n,k,j\in\mathbb{N}}$, it suffices to show that $A_{n,k,j}:=\big\{(t,\mathfrak{u})\in[0,\infty)\times\mathbb{J}:\mathscr{U}^t(\mathfrak{u})\in\mathfrak{i}_\mathbb{J}^{-1}\big(O_{\frac1n}(\mathfrak{m}_k,\phi_j)\big)\big\}$ belongs to the product topology $\mathfrak{T}[0,\infty)\otimes\mathfrak{T}_\sharp(\mathbb{J})$ for any $n,k,j\in\mathbb{N}$.

Fix $n,k,j\in\mathbb{N}$ and set $\|\phi_j\|_\infty:=\sup_{(s,u)\in[0,\infty)\times\mathbb{U}}\big|\phi_j(s,u)\big|$. We pick $(t,\mathfrak{u})\in A_{n,k,j}$ and set $\mathfrak{c}:=\dfrac{1}{5e^{t+1}(\|\phi_j\|_\infty+1)}\Big(1/n-\big|\int_0^\infty e^{-\mathfrak{s}}\phi_j(\mathfrak{s},\mathfrak{u}(t+\mathfrak{s}))d\mathfrak{s}-\int_0^\infty\int_\mathbb{U}\phi_j(t,u)\mathfrak{m}_k(dt,du)\big|\Big)>0$. By the uniform continuity of ϕ_j, there exists $\lambda\in(0,\mathfrak{c})$ such that $\big|\phi_j(s_1,u_1)-\phi_j(s_2,u_2)\big|<\mathfrak{c}$ for any $(s_1,u_1),(s_2,u_2)\in[0,\infty)\times\mathbb{U}$ with $|s_1-s_2|\vee\rho_\mathbb{U}(u_1,u_2)<\lambda$. We define another function ϕ^λ_j of $\widehat{C}_b\big([0,\infty)\times\mathbb{U}\big)$ by $\phi^\lambda_j(s,u):=\big((1+(s-t)/\lambda)^+\wedge 1\big)\phi_j\big((s-t)^+,u\big)$, $\forall\,(s,u)\in[0,\infty)\times\mathbb{U}$. Clearly, $\mathcal{O}_\lambda(\mathfrak{u}):=\mathfrak{i}_\mathbb{J}^{-1}\big(O_\lambda(\mathfrak{i}_\mathbb{J}(\mathfrak{u}),\phi^\lambda_j)\big)$ is a member of $\mathfrak{T}_\sharp(\mathbb{J})$.

Let $(t',\mathfrak{u}')\in\big((t-\lambda)^+,t+\lambda\big)\times\mathcal{O}_\lambda(\mathfrak{u})$. Since $\lambda>\big|\int_0^\infty e^{-s}\big[\phi^\lambda_j(s,\mathfrak{u}(s))-\phi^\lambda_j(s,\mathfrak{u}'(s))\big]ds\big|=\Big|\int_{(t-\lambda)^+}^t e^{-s}\big(1+(s-t)/\lambda\big)\big[\phi_j(0,\mathfrak{u}(s))-\phi_j(0,\mathfrak{u}'(s))\big]ds+\int_t^\infty e^{-s}\big[\phi_j(s-t,\mathfrak{u}(s))-\phi_j(s-t,\mathfrak{u}'(s))\big]ds\Big|$, one has

$$\mathcal{I}_1:=\Big|\int_t^\infty e^{-s+t}\big[\phi_j(s-t,\mathfrak{u}(s))-\phi_j(s-t,\mathfrak{u}'(s))\big]ds\Big|\le e^t\Big|\int_{(t-\lambda)^+}^t e^{-s}\big(1+(s-t)/\lambda\big)\big(\phi_j\big(0,\mathfrak{u}(s)\big)-\phi_j\big(0,\mathfrak{u}'(s)\big)\big)ds\Big|+e^t\lambda$$
$$\le\ e^t\|\phi_j\|_\infty e^{-(t-\lambda)^+}\lambda+e^t\lambda\le e\lambda\|\phi_j\|_\infty+e^t\lambda.$$

We can also estimate:

- $\mathcal{I}_2:=\big|\int_t^\infty e^{-s+t}\big[\phi_j(s-t,\mathfrak{u}'(s))-\phi_j((s-t')^+,\mathfrak{u}'(s))\big]ds\big|\le e^t\int_t^\infty e^{-s}\big|\phi_j(s-t,\mathfrak{u}'(s))-\phi_j((s-t')^+,\mathfrak{u}'(s))\big|ds\le\mathfrak{c}$.
- $\mathcal{I}_3:=\big|\int_t^\infty e^{-s}(e^t-e^{t'})\phi_j((s-t')^+,\mathfrak{u}'(s))ds\big|\le\|\phi_j\|_\infty\big|e^{t'}-e^t\big|e^{-t}\le\|\phi_j\|_\infty e^{t\vee t'-t}|t-t'|\le e\lambda\|\phi_j\|_\infty$.
- $\mathcal{I}_4:=\big|\int_t^\infty e^{-s+t'}\phi_j((s-t')^+,\mathfrak{u}'(s))ds-\int_{t'}^\infty e^{-s+t'}\phi_j(s-t',\mathfrak{u}'(s))ds\big|\le\int_{t\wedge t'}^{t\vee t'}e^{-s+t'}\big|\phi_j((s-t')^+,\mathfrak{u}'(s))\big|ds\le\|\phi_j\|_\infty e^{t'-t\wedge t'}|t-t'|\le e\lambda\|\phi_j\|_\infty$.

Putting them together leads to that

$$\Big|\int_0^\infty e^{-\mathfrak{s}}\phi_j(\mathfrak{s},\mathscr{U}^{t'}_{\mathfrak{s}}(\mathfrak{u}'))d\mathfrak{s}-\int_0^\infty\int_{\mathbb{U}}\phi_j(t,u)\mathfrak{m}_k(dt,du)\Big|$$
$$\le\Big|\int_0^\infty e^{-\mathfrak{s}}\phi_j(\mathfrak{s},\mathfrak{u}'(t'+\mathfrak{s}))d\mathfrak{s}-\int_0^\infty e^{-\mathfrak{s}}\phi_j(\mathfrak{s},\mathfrak{u}(t+\mathfrak{s}))d\mathfrak{s}\Big|+\Big|\int_0^\infty e^{-\mathfrak{s}}\phi_j(\mathfrak{s},\mathfrak{u}(t+\mathfrak{s}))d\mathfrak{s}-\int_0^\infty\int_{\mathbb{U}}\phi_j(t,u)\mathfrak{m}_k(dt,du)\Big|$$
$$=\Big|\int_{t'}^\infty e^{-s+t'}\phi_j(s-t',\mathfrak{u}'(s))ds-\int_t^\infty e^{-s+t}\phi_j(s-t,\mathfrak{u}(s))ds\Big|+(1/n-4e^{t+1}(\|\phi_j\|_\infty+1)\mathfrak{c})$$
$$\le\sum_{i=1}^4\mathcal{I}_i+\big[1/n-5e^{t+1}(\|\phi_j\|_\infty+1)\mathfrak{c}\big]<1/n.$$

This shows $\mathscr{U}^{t'}(\mathfrak{u}')\in\mathfrak{i}_{\mathbb{J}}^{-1}\big(O_{\frac{1}{n}}(\mathfrak{m}_k,\phi_j)\big)$ and thus $(t,\mathfrak{u})\in\big((t-\lambda)^+,t+\lambda\big)\times\mathcal{O}_\lambda(\mathfrak{u})\subset A_{n,k,j}$. Then $A_{n,k,j}$ is an open set of $\mathfrak{T}[0,\infty)\otimes\mathfrak{T}_\sharp(\mathbb{J})$, proving the lemma. □

Lemma A.5. *Given $t\in[0,\infty)$, let $\mu=\{\mu_r\}_{r\in[t,\infty)}$ be a $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process on Ω_0. For any $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$, there exists a $\mathbb{U}-$valued, $\mathbf{F}^{W^s}-$predictable process $\big\{\mu^{s,\overline{\omega}}_r\big\}_{r\in[s,\infty)}$ on Ω_0 such that $\mu^{s,\overline{\omega}}_r\big(\overline{W}(\overline{\omega}')\big)=\mu_r\big(\overline{W}(\overline{\omega}')\big)$, $\forall\,(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$, where $\overline{\mathbf{W}}^t_{s,\overline{\omega}}:=\big\{\overline{\omega}'\in\overline{\Omega}:\overline{W}^t_r(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}),\ \ \forall\,r\in[t,s]\big\}$.*

Proof: 1) Define $\Lambda:=\Big\{D\subset[t,\infty)\times\Omega_0:$ for any $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$ there exists $D^{s,\overline{\omega}}\in\mathscr{P}^{W^s}$ such that $\mathbf{1}_{\{(r,\overline{W}(\overline{\omega}'))\in D\}}=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^{s,\overline{\omega}}\big\}},\ \forall\,(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}\Big\}$. Clearly, $\emptyset\in\Lambda$ with $D^{s,\overline{\omega}}=\emptyset$ for any $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$. Given $D\in\Lambda$, by taking the complement of each $D^{s,\overline{\omega}}$, we see that $D^c\in\Lambda$. Let $\{D_n\}_{n\in\mathbb{N}}\subset\Lambda$. For any $n\in\mathbb{N}$ and $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$, there exists $D^{s,\overline{\omega}}_n\in\mathscr{P}^{W^s}$ satisfying $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D_n\big\}}=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^{s,\overline{\omega}}_n\big\}}$ for any $(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$. Then for any $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$, the $\mathbf{F}^{W^s}-$predictable set $\underset{n\in\mathbb{N}}{\cap}D^{s,\overline{\omega}}_n$ satisfies that

$$\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in\underset{n\in\mathbb{N}}{\cap}D_n\big\}}=\prod_{n\in\mathbb{N}}\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D_n\big\}}=\prod_{n\in\mathbb{N}}\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^{s,\overline{\omega}}_n\big\}}=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in\underset{n\in\mathbb{N}}{\cap}D^{s,\overline{\omega}}_n\big\}},$$

$\forall\,(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$. Namely, $\underset{n\in\mathbb{N}}{\cap}D_n\in\Lambda$. Hence, Λ is a sigma−field of $[t,\infty)\times\Omega_0$.

2) We next demonstrate that Λ contains all generating sets of the $\mathbf{F}^{W^t}-$predictable sigma−field $\mathscr{P}^{W^t}$ of $[t,\infty)\times\Omega_0$:

$$\{t\}\times A\text{ for }A\in\mathcal{F}^{W^t}_t\quad\text{ and }\quad(q,\infty)\times A\text{ for }q\in[t,\infty)\cap\mathbb{Q},\ A\in\mathcal{F}^{W^t}_{q-}.$$

(2a) For any $s\in[t,\infty)$ and $(r,\overline{\omega}')\in[s,\infty)\times\overline{\Omega}$, one has $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in\{t\}\times\Omega_0\big\}}=\mathbf{1}_{\{r=t\}}$ and $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in(t,\infty)\times\Omega_0\big\}}=\mathbf{1}_{\{r=(t,\infty)\}}$. They show that $\{t\}\times\Omega_0\in\Lambda$ with $D^{s,\overline{\omega}}=\mathbf{1}_{\{s=t\}}(\{s\}\times\Omega_0)+\mathbf{1}_{\{s>t\}}\emptyset,\ \forall\,(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$ and that $(t,\infty)\times\Omega_0\in\Lambda$ with $D^{s,\overline{\omega}}=\mathbf{1}_{\{s=t\}}((s,\infty)\times\Omega_0)+\mathbf{1}_{\{s>t\}}([s,\infty)\times\Omega_0),\ \forall\,(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$. Since $\mathcal{F}^{W^t}_t=\mathcal{F}^{W^t}_{t-}=\{\emptyset,\Omega_0\}$ and since $\emptyset\in\Lambda$, we see that $\{t\}\times A\in\Lambda$ for any $A\in\mathcal{F}^{W^t}_t$ and that $(t,\infty)\times A\in\Lambda$ for any $A\in\mathcal{F}^{W^t}_{t-}$.
(2b) Fix $q\in(t,\infty)\cap\mathbb{Q}$ and set $\widehat{\Lambda}_q:=\big\{A\subset\Omega_0:(q,\infty)\times A\in\Lambda\big\}$. Clearly, $\emptyset\in\widehat{\Lambda}_q$. Since it holds for any $s\in[t,\infty)$ and $(r,\overline{\omega}')\in[s,\infty)\times\overline{\Omega}$ that $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in(q,\infty)\times\Omega_0\big\}}=\mathbf{1}_{\{r\in(q,\infty)\}}$, we obtain that $(q,\infty)\times\Omega_0\in\Lambda$ with $D^{s,\overline{\omega}}=\mathbf{1}_{\{s\le q\}}((q,\infty)\times$

$\Omega_0)+\mathbf{1}_{\{s>q\}}([s,\infty)\times\Omega_0)$, $\forall\,(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$. So Ω_0 belongs to $\widehat{\Lambda}_q$. Given $A\in\widehat{\Lambda}_q$, since $(q,\infty)\times A\in\Lambda$ and $(q,\infty)\times\Omega_0\in\Lambda$, we can deduce that $(q,\infty)\times A^c=\big((q,\infty)\times\Omega_0\big)\cap\big((q,\infty)\times A\big)^c\in\Lambda$ and thus $A^c\in\widehat{\Lambda}_q$. If $\{A_n\}_{n\in\mathbb{N}}\in\widehat{\Lambda}_q$, then $(q,\infty)\times\Big(\underset{n\in\mathbb{N}}{\cup}A_n\Big)=\underset{n\in\mathbb{N}}{\cup}\big((q,\infty)\times A_n\big)\in\Lambda$, i.e., $\underset{n\in\mathbb{N}}{\cup}A_n\in\widehat{\Lambda}_q$. Thus $\widehat{\Lambda}_q$ is a sigma-field of Ω_0.

Let $q'\in[t,q)$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$. We show by three cases that $(q,\infty)\times(W^t_{q'})^{-1}(\mathcal{E})\in\Lambda$:

(i) If $(s,\overline{\omega})\in[t,q')\times\overline{\Omega}$, then $\mathcal{E}_{s,\overline{\omega}}:=\big\{\mathfrak{x}-\overline{W}^t_s(\overline{\omega})\,:\,\forall\,\mathfrak{x}\in\mathcal{E}\big\}\in\mathscr{B}(\mathbb{R}^d)$ satisfies that $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in(q,\infty)\times(W^t_{q'})^{-1}(\mathcal{E})\big\}}=\mathbf{1}_{\{r\in(q,\infty)\}}\mathbf{1}_{\{\overline{W}^t_{q'}(\overline{\omega}')-\overline{W}^t_s(\overline{\omega}')\in\mathcal{E}_{s,\overline{\omega}}\}}=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in(q,\infty)\times(W^s_{q'})^{-1}(\mathcal{E}_{s,\overline{\omega}})\big\}}$, $\forall\,(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$.

(ii) If $(s,\overline{\omega})\in[q',\infty)\times(\overline{W}^t_{q'})^{-1}(\mathcal{E})$, it holds for any $(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$ that $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in(q,\infty)\times(W^t_{q'})^{-1}(\mathcal{E})\big\}}=\mathbf{1}_{\{r\in(q,\infty)\}}\mathbf{1}_{\{\overline{W}^t_{q'}(\overline{\omega}')\in\mathcal{E}\}}=\mathbf{1}_{\{r\in(q,\infty)\}}\mathbf{1}_{\{\overline{W}^t_{q'}(\overline{\omega})\in\mathcal{E}\}}=\mathbf{1}_{\{r\in(q,\infty)\}}=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in(q,\infty)\times\Omega_0\big\}}$.

(iii) If $(s,\overline{\omega})\in[q',\infty)\times(\overline{W}^t_{q'})^{-1}(\mathcal{E}^c)$, it holds for any $(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$ that $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in(q,\infty)\times(W^t_{q'})^{-1}(\mathcal{E})\big\}}=\mathbf{1}_{\{r\in(q,\infty)\}}\mathbf{1}_{\{\overline{W}^t_{q'}(\overline{\omega})\in\mathcal{E}\}}=0=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in\emptyset\big\}}$.

So $(q,\infty)\times(W^t_{q'})^{-1}(\mathcal{E})\in\Lambda$ with $D^{s,\overline{\omega}}=\mathbf{1}_{\{(s,\overline{\omega})\in[t,q')\times\overline{\Omega}\}}\big((q,\infty)\times(W^s_{q'})^{-1}(\mathcal{E}_{s,\overline{\omega}})\big)+\mathbf{1}_{\big\{(s,\overline{\omega})\in[q',\infty)\times(\overline{W}^t_{q'})^{-1}(\mathcal{E}^c)\big\}}\emptyset+\mathbf{1}_{\big\{(s,\overline{\omega})\in[q',q]\times(\overline{W}^t_{q'})^{-1}(\mathcal{E})\big\}}\big((q,\infty)\times\Omega_0\big)+\mathbf{1}_{\big\{(s,\overline{\omega})\in(q,\infty)\times(\overline{W}^t_{q'})^{-1}(\mathcal{E})\big\}}\big([s,\infty)\times\Omega_0\big)$ for any $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$. It follows that $\mathcal{F}^{W^t}_{q-}=\sigma\big(W^t_{q'};q'\in[t,q)\big)\subset\widehat{\Lambda}_q$. Then Λ contains all generating sets of $\mathscr{P}^{W^t}$ and thus includes $\mathscr{P}^{W^t}$.

3) Let $\{\mu_s\}_{s\in[t,\infty)}$ be a general $\mathbb{U}-$valued, $\mathbf{F}^{W^t}-$predictable process on Ω_0.

Let $n\in\mathbb{N}$, we set $a^n_i:=i2^{-n}$, $\forall\,i\in\{0,1,\cdots,1+2^n\}$ and $D^n_i:=\big\{(r,\omega_0)\in[t,\infty)\times\Omega_0:\mathscr{I}\big(\mu_r(\omega_0)\big)\in[a^n_i,a^n_{i+1})\big\}\in\mathscr{P}^{W^t}\subset\Lambda$, $\forall\,i\in\{0,1,\cdots,2^n\}$. So for $i=0,1,\cdots,2^n$ and $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$, there exists $D^{s,\overline{\omega},n}_i\in\mathscr{P}^{W^s}$ satisfying $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^n_i\big\}}=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^{s,\overline{\omega},n}_i\big\}}$, $\forall\,(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$.

Fix $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$. For any $n\in\mathbb{N}$, define an $\mathbf{F}^{W^s}-$predictable process $\nu^{s,\overline{\omega},n}$ by $\nu^{s,\overline{\omega},n}_r(\omega_0):=\sum\limits_{i=1}^{2^n}\mathbf{1}_{\big\{(r,\omega_0)\in\widetilde{D}^{s,\overline{\omega},n}_i\big\}}a^n_i$, $\forall\,(r,\omega_0)\in[s,\infty)\times\Omega_0$, where $\widetilde{D}^{s,\overline{\omega},n}_1=D^{s,\overline{\omega},n}_1$ and $\widetilde{D}^{s,\overline{\omega},n}_i:=D^{s,\overline{\omega},n}_i\Big\backslash\Big(\underset{j=1}{\overset{i-1}{\cup}}D^{s,\overline{\omega},n}_j\Big)\in\mathscr{P}^{W^s}$ for $i=2,\cdots,2^n$. Then $\underline{\nu}^{s,\overline{\omega}}_r(\omega_0):=\varliminf\limits_{n\to\infty}\nu^{s,\overline{\omega},n}_r(\omega_0)$, $\forall\,(r,\omega_0)\in[s,\infty)\times\Omega_0$ is a $[0,1]-$valued, $\mathbf{F}^{W^s}-$predictable process and

$$\mu^{s,\overline{\omega}}_r(\omega_0):=\mathscr{I}^{-1}\big(\underline{\nu}^{s,\overline{\omega}}_r(\omega_0)\big)\mathbf{1}_{\big\{\underline{\nu}^{s,\overline{\omega}}_r(\omega_0)\in\mathfrak{E}\big\}}+u_0\mathbf{1}_{\big\{\underline{\nu}^{s,\overline{\omega}}_r(\omega_0)\notin\mathfrak{E}\big\}},\quad\forall\,(r,\omega_0)\in[s,\infty)\times\Omega_0$$

defines a $\mathbb{U}-$valued, $\mathbf{F}^{W^s}-$predictable process.

Let $(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$ and let $n\in\mathbb{N}$. For any $i=2,\cdots,2^n$, since $0\le\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^{s,\overline{\omega},n}_i\cap(\overset{i-1}{\underset{j=1}{\cup}}D^{s,\overline{\omega},n}_j)\big\}}\le\sum_{j=1}^{i-1}\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^{s,\overline{\omega},n}_i\cap D^{s,\overline{\omega},n}_j\big\}}=\sum_{j=1}^{i-1}\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^n_i\cap D^n_j\big\}}=0$, we obtain $\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in\widetilde{D}^{s,\overline{\omega},n}_i\big\}}=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^{s,\overline{\omega},n}_i\big\}}=\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^n_i\big\}}$. It follows that $\nu^{s,\overline{\omega},n}_r(\overline{W}(\overline{\omega}'))=\sum_{i=1}^{2^n}\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in\widetilde{D}^{s,\overline{\omega},n}_i\big\}}a^n_i=\sum_{i=1}^{2^n}\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^n_i\big\}}a^n_i$, $\forall\,n\in\mathbb{N}$ and thus $\mathscr{I}\big(\mu_r(\overline{W}(\overline{\omega}'))\big)=\lim\limits_{n\to\infty}\uparrow\sum_{i=1}^{2^n}\mathbf{1}_{\big\{(r,\overline{W}(\overline{\omega}'))\in D^n_i\big\}}a^n_i=\lim\limits_{n\to\infty}\uparrow\nu^{s,\overline{\omega},n}_r\big(\overline{W}(\overline{\omega}')\big)=\underline{\nu}^{s,\overline{\omega}}_r\big(\overline{W}(\overline{\omega}')\big)$. Then $\mu^{s,\overline{\omega}}_r\big(\overline{W}(\overline{\omega}')\big)=\mu_r\big(\overline{W}(\overline{\omega}')\big)$, $\forall\,(r,\overline{\omega}')\in[s,\infty)\times\overline{\mathbf{W}}^t_{s,\overline{\omega}}$. □

Lemma A.6. *Given $t\in[0,\infty)$ and $\overline{P}\in\mathfrak{P}(\overline{\Omega})$, let $\overline{\mathcal{W}}=\{\overline{\mathcal{W}}_s\}_{s\in[t,\infty)}$ be a $d-$dimensional Brownian motion with respect to some filtration $\mathfrak{F}_\cdot=\{\mathfrak{F}_s\}_{s\in[t,\infty)}$ on $\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}\big)$. For any $s\in[t,\infty)$, if $\overline{\xi}$ is a real-valued, $\mathfrak{F}_s-$measurable random variable that is $\overline{P}-$integrable, then $E_{\overline{P}}\big[\,\overline{\xi}\big|\mathcal{F}^{\overline{\mathcal{W}}}_\infty\big]=E_{\overline{P}}\big[\,\overline{\xi}\big|\mathcal{F}^{\overline{\mathcal{W}}}_s\big]$, $\overline{P}-$a.s.*

Proof: Fix $s\in[t,\infty)$. Let $t\le t_1<\cdots<t_n\le s=s_0<s_1<\cdots<s_k<\infty$ and let $\{\mathcal{E}_i\}_{i=1}^n\cup\{\mathcal{E}'_j\}_{j=1}^k\subset\mathscr{B}(\mathbb{R}^d)$.

Set $\psi_k(x):=1$, $\forall\,x\in\mathbb{R}^d$. For $j=k,\cdots,1$ recursively, the Markov property of the Brownian motion $\overline{\mathcal{W}}$ shows that there exists another Borel-measurable function $\psi_{j-1}:\mathbb{R}^d\mapsto\mathbb{R}$ satisfying $E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{W}}^{-1}_{s_j}(\mathcal{E}'_j)}\psi_j\big(\overline{\mathcal{W}}_{s_j}\big)\big|\mathfrak{F}_{s_{j-1}}\Big]=\psi_{j-1}\big(\overline{\mathcal{W}}_{s_{j-1}}\big)$,

$\overline{P}-$a.s. Then we can deduce that

$$\begin{aligned}
&E_{\overline{P}}\bigg[\prod_{j=1}^{k}\mathbf{1}_{\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)}\Big|\mathfrak{F}_s\bigg]=E_{\overline{P}}\bigg[E_{\overline{P}}\bigg[\prod_{j=1}^{k}\mathbf{1}_{\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)}\Big|\mathfrak{F}_{s_{k-1}}\bigg]\Big|\mathfrak{F}_s\bigg]=E_{\overline{P}}\bigg[\prod_{j=1}^{k-1}\mathbf{1}_{\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)}E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{W}}_{s_k}^{-1}(\mathcal{E}'_k)}\Big|\mathfrak{F}_{s_{k-1}}\Big]\Big|\mathfrak{F}_s\bigg]\\
&\quad=E_{\overline{P}}\bigg[\prod_{j=1}^{k-1}\mathbf{1}_{\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)}\psi_{k-1}\big(\overline{\mathcal{W}}_{s_{k-1}}\big)\Big|\mathfrak{F}_s\bigg]=E_{\overline{P}}\bigg[\prod_{j=1}^{k-2}\mathbf{1}_{\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)}E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{W}}_{s_{k-1}}^{-1}(\mathcal{E}'_{k-1})}\psi_{k-1}\big(\overline{\mathcal{W}}_{s_{k-1}}\big)\Big|\mathfrak{F}_{s_{k-2}}\Big]\Big|\mathfrak{F}_s\bigg]\\
&\quad=E_{\overline{P}}\bigg[\prod_{j=1}^{k-2}\mathbf{1}_{\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)}\psi_{k-2}\big(\overline{\mathcal{W}}_{s_{k-2}}\big)\Big|\mathfrak{F}_s\bigg]=\cdots=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{W}}_{s_1}^{-1}(\mathcal{E}'_1)}\psi_1\big(\overline{\mathcal{W}}_{s_1}\big)\big|\mathfrak{F}_s\Big]=\psi_0\big(\overline{\mathcal{W}}_s\big),\quad \overline{P}-\text{a.s.}
\end{aligned}$$

It follows that $E_{\overline{P}}\Big[\mathbf{1}_{\underset{i=1}{\overset{n}{\cap}}\overline{\mathcal{W}}_{t_i}^{-1}(\mathcal{E}_i)}\mathbf{1}_{\underset{j=1}{\overset{k}{\cap}}\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)}\Big|\mathfrak{F}_s\Big]=\prod_{i=1}^{n}\mathbf{1}_{\overline{\mathcal{W}}_{t_i}^{-1}(\mathcal{E}_i)}E_{\overline{P}}\Big[\prod_{j=1}^{k}\mathbf{1}_{\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)}\Big|\mathfrak{F}_s\Big]=\prod_{i=1}^{n}\mathbf{1}_{\overline{\mathcal{W}}_{t_i}^{-1}(\mathcal{E}_i)}\psi_0\big(\overline{\mathcal{W}}_s\big)$, $\overline{P}-$a.s. By Dynkin's Pi-Lambda Theorem, the Lambda-system $\big\{A\in\mathcal{F}_\infty^{\overline{\mathcal{W}}}: E_{\overline{P}}\big[\mathbf{1}_A\big|\mathfrak{F}_s\big]=\overline{\beta},\ \ \overline{P}-\text{a.s. for some real}-\text{valued } \mathcal{F}_s^{\overline{\mathcal{W}}}-\text{measurable random variable } \overline{\beta} \text{ on } \overline{\Omega}\big\}$ contains the Pi-system $\Big\{\Big(\underset{i=1}{\overset{n}{\cap}}\overline{\mathcal{W}}_{t_i}^{-1}(\mathcal{E}_i)\Big)\bigcap\Big(\underset{j=1}{\overset{k}{\cap}}\overline{\mathcal{W}}_{s_j}^{-1}(\mathcal{E}'_j)\Big): t\le t_1<\cdots< t_n\le s=s_0<s_1<\cdots<s_k<\infty,\ \{\mathcal{E}_i\}_{i=1}^n\cup\{\mathcal{E}'_j\}_{j=1}^k\subset\mathscr{B}(\mathbb{R}^d)\Big\}$ and thus includes $\mathcal{F}_\infty^{\overline{\mathcal{W}}}$.

Let $\overline{\xi}$ be a real-valued, $\mathfrak{F}_s-$measurable random variable that is $\overline{P}-$integrable and let $A\in\mathcal{F}_\infty^{\overline{\mathcal{W}}}$. There exists a real$-$valued $\mathcal{F}_s^{\overline{\mathcal{W}}}-$measurable random variable $\overline{\beta}$ such that $E_{\overline{P}}\big[\mathbf{1}_A\big|\mathfrak{F}_s\big]=\overline{\beta}$, $\overline{P}-$a.s. Since it holds for any $\mathcal{A}_s\in\mathcal{F}_s^{\overline{\mathcal{W}}}\subset\mathfrak{F}_s$ that $E_{\overline{P}}[\mathbf{1}_{\mathcal{A}_s}\mathbf{1}_A]=E_{\overline{P}}\Big[\mathbf{1}_{\mathcal{A}_s}E_{\overline{P}}\big[\mathbf{1}_A\big|\mathfrak{F}_s\big]\Big]=E_{\overline{P}}\big[\mathbf{1}_{\mathcal{A}_s}\overline{\beta}\big]$, we see that $E_{\overline{P}}\big[\mathbf{1}_A\big|\mathcal{F}_s^{\overline{\mathcal{W}}}\big]=\overline{\beta}=E_{\overline{P}}\big[\mathbf{1}_A\big|\mathfrak{F}_s\big]$, $\overline{P}-$a.s. Then the tower property implies that

$$\begin{aligned}
E_{\overline{P}}\big[\mathbf{1}_A\overline{\xi}\big]&=E_{\overline{P}}\Big[\overline{\xi}E_{\overline{P}}\big[\mathbf{1}_A\big|\mathfrak{F}_s\big]\Big]=E_{\overline{P}}\Big[\overline{\xi}E_{\overline{P}}\big[\mathbf{1}_A\big|\mathcal{F}_s^{\overline{\mathcal{W}}}\big]\Big]=E_{\overline{P}}\Big[E_{\overline{P}}\big[\overline{\xi}E_{\overline{P}}[\mathbf{1}_A|\mathcal{F}_s^{\overline{\mathcal{W}}}]\big|\mathcal{F}_s^{\overline{\mathcal{W}}}\big]\Big]\\
&=E_{\overline{P}}\Big[E_{\overline{P}}\big[\mathbf{1}_A\big|\mathcal{F}_s^{\overline{\mathcal{W}}}\big]E_{\overline{P}}\big[\overline{\xi}\big|\mathcal{F}_s^{\overline{\mathcal{W}}}\big]\Big]=E_{\overline{P}}\Big[E_{\overline{P}}\Big[\mathbf{1}_AE_{\overline{P}}\big[\overline{\xi}|\mathcal{F}_s^{\overline{\mathcal{W}}}\big]\big|\mathcal{F}_s^{\overline{\mathcal{W}}}\Big]\Big]=E_{\overline{P}}\Big[\mathbf{1}_AE_{\overline{P}}\big[\overline{\xi}\big|\mathcal{F}_s^{\overline{\mathcal{W}}}\big]\Big].
\end{aligned}$$

As A runs through $\mathcal{F}_\infty^{\overline{\mathcal{W}}}$, we obtain that $E_{\overline{P}}\big[\overline{\xi}\big|\mathcal{F}_\infty^{\overline{\mathcal{W}}}\big]=E_{\overline{P}}\big[\overline{\xi}\big|\mathcal{F}_s^{\overline{\mathcal{W}}}\big]$, $\overline{P}-$a.s. □

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