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Examples of Non-Semisimple Hopf Algebra Actions on Artin-Schelter Regular Algebras

Hui-Xiang Chen[1] · Ding-Guo Wang[2] · James J. Zhang[3]

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Abstract

Let $\Bbbk$ be a base field of characteristic $p > 0$ and let U be the restricted enveloping algebra of a 2-dimensional nonabelian restricted Lie algebra. We classify all inner-faithful U-actions on noetherian Koszul Artin-Schelter regular algebras of global dimension up to three.

Keywords Hopf algebra action · Artin-Schelter regular algebra · Indecomposable module · Green ring

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1 Introduction

Invariant theory of commutative polynomial rings under finite group actions is closely connected to commutative algebra and algebraic geometry. Artin-Schelter regular algebras [1], viewed as a natural noncommutative generalization of the commutative polynomial rings, play an important role in noncommutative algebraic geometry, representation theory, and the study of noncommutative algebras [2, 3, 6]. Hopf actions (including group actions) on Artin-Schelter regular algebras have been studied extensively by many authors in recent years, see [7–9, 15, 18–20, 24–27] and so on. A very nice survey was given by Kirkman [23] a few years ago. In most papers, only semisimple Hopf algebras are considered due

Presented by: Kenneth Goodearl

✉ Ding-Guo Wang
dgwang@qfnu.edu.cn

Hui-Xiang Chen
hxchen@yzu.edu.cn

James J. Zhang
zhang@math.washington.edu

[1] School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China

[2] School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China

[3] Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA

to the fact that non-semisimple Hopf actions are much more difficult to handle. A list of significant differences between semisimple and non-semisimple actions can be found in Observation 5.1.

Recall that a Hopf H-action on an algebra A is called *inner-faithful* if there is no nonzero Hopf ideal $I \subseteq H$ such that $IA = 0$ [7, Definition 1.5]. Our goal is to construct examples of inner-faithful and homogeneous U-actions on T where U is the non-semisimple Hopf algebra given in Definition 1.1 and T is a connected graded Artin-Schelter regular algebra. The main result consists of Proposition 4.2 and Theorem 1.5 that together classify all inner-faithful U-actions on noetherian Koszul Artin-Schelter regular algebras of global dimension at most three.

Throughout let $\Bbbk$ be a base field with char $\Bbbk = p > 0$.

Definition 1.1 Let U be the $\Bbbk$-algebra generated by u and w and subject to the relations

$$u^p = 0,\ w^p = w,\ [w, u](:= wu - uw) = u. \tag{1}$$

Then U has a Hopf algebra structure with coalgebra structure and antipode determined by

$$\begin{aligned} \Delta(u) &= u \otimes 1 + 1 \otimes u, \quad \varepsilon(u) = 0, \quad S(u) = -u, \\ \Delta(w) &= w \otimes 1 + 1 \otimes w, \ \ \varepsilon(w) = 0, \ \ S(w) = -w. \end{aligned}$$

Note that $\dim_{\Bbbk} U = p^2$ and $\{u^i w^j | 0 \leqslant i, j \leqslant p-1\}$ is a $\Bbbk$-basis of U. It is easy to see that U is isomorphic, as a Hopf algebra, to the restricted enveloping algebra of the 2-dimensional nonabelian restricted Lie algebra $\mathfrak{g} := \Bbbk u \oplus \Bbbk w$ with structure determined by Eq. 1.

A very first step of understanding U-actions on Artin-Schelter regular algebras is to work out representations of U. Similar to the Taft algebras, there are exactly p^2 indecomposable U-modules up to isomorphisms, denoted by

$$\{M(l, i) | 1 \leqslant l \leqslant p, i \in \mathbb{Z}_p := \mathbb{Z}/(p)\}$$

where $\dim_{\Bbbk} M(l, i) = l$ for all l, i, see Convention 2.4 and Proposition 2.7. We also need the following tensor decomposition result.

Theorem 1.2 *Retain the notation as above. Let $r, r' \in \mathbb{Z}_p$.*

(1) *Let $1 \leqslant l \leqslant m \leqslant p$ and $l + m \leqslant p$. Then*

$$M(l, r) \otimes M(m, r') \cong \bigoplus_{i=1}^{l} M(m - l - 1 + 2i, r + r' + l - i).$$

(2) *Let $1 \leqslant l \leqslant m \leqslant p$ and $l + m > p$. Then*

$$M(l, r) \otimes M(m, r') \cong \Big(\bigoplus_{i=1}^{p-m} M(m - l - 1 + 2i, r + r' + l - i)\Big) \bigoplus \Big(\bigoplus_{i=1}^{l+m-p} M(p, r + r' + i - 1)\Big).$$

The proof of Theorem 1.2 follows from the ideas in [12]. Note that the Green ring (or the representation ring) of Hopf algebras has been studied extensively, see [10, 12, 21, 29, 30, 40–42] and more. If we can describe the Green ring of a Hopf algebra H, then it is extremely useful for understanding the representations, the fusion rules, Grothendieck group, Frobenius-Perron dimension [28, 45, 46], and many other invariants and structures of H. However, it is notoriously difficult to understand the Green ring for a general Hopf algebra (even, of small dimension, see Question 6.1). It is not a surprise that most of positive results so far concern Hopf algebras of finite or tame representation type. Using Theorem 1.2 we can present the Green ring of U – another non-semisimple Hopf algebra of finite representation type [Corollary 1.3].

For each positive integer n, define

$$f_n(y,z) = \sum_{i=0}^{[(n-1)/2]} (-1)^i \binom{n-1-i}{i} y^i z^{n-1-2i},$$

where $[(n-1)/2]$ denotes the integer part of $(n-1)/2$. Recall that $p = \operatorname{char} \Bbbk$. Let I denote the ideal of $\mathbb{Z}[y,z]$ generated by $y^p - 1$ and $(z-y-1)f_p(y,z)$.

Corollary 1.3 (Corollary 3.15) *The Green ring of U is isomorphic to the factor ring $\mathbb{Z}[y,z]/I$.*

The proof of this corollary is based on the tensor decomposition of indecomposable U-modules given in Theorem 1.2. Corollary 1.3 should be compared with [12, Theorem 3.10]. One basic message of Theorem 1.2 and Corollary 1.3 is that the representation theory of U is similar to the representation theory of the Taft algebras, though U does not have any nontrivial grouplike elements.

Another application of Theorem 1.2 is computation of the Frobenius-Perron dimension of U-modules, which was studied in [45]. We give some comments in the last section, see Remark 6.5.

Going back to our main topic, note that the explicit description of the tensor of two representations of U [Theorem 1.2] is the key to understanding U-actions on Artin-Schelter regular algebras. If T is a noetherian Artin-Schelter regular of global dimension two, then the existence of an inner-faithful U-action on T forces T to be a commutative polynomial ring. From now on Artin-Schelter is abbreviated as AS.

Proposition 1.4 *Let H be any Hopf algebra containing U as a Hopf subalgebra. If H acts inner-faithfully and homogeneously on a noetherian Koszul AS regular algebra T of global dimension two, then T is isomorphic to the commutative polynomial ring $\Bbbk[x_1, x_2]$.*

Explicit U-actions on a noetherian Koszul AS regular algebra of global dimension two are given in Proposition 4.2. The next result is a classification of all inner-faithful U-actions on T when T has global dimension 3. Historically, over an algebraically closed field of characteristic zero, AS regular algebras of global dimension three were classified by Artin, Schelter, Tate and Van den Bergh in their seminal papers [1–3]. Since our base field has positive characteristic, we have to use a different method. But the purpose of this paper is not to classify all AS regular algebras over a field of positive characteristic which is another extremely difficult project, see Remark 1.7(2).

Theorem 1.5 *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let T be a noetherian Koszul AS regular $\Bbbk$-algebra of global dimension three and let $V = T_1$. Suppose there is an inner-faithful and homogeneous U-action on T. Then one of the following occurs, up to a change of basis.*

(1) *T is the commutative polynomial ring $\Bbbk[V]$ where the left U-module V is either $M(3,i)$ or $M(2,i) \oplus M(1,j)$ for some $i, j \in \mathbb{Z}_p$.*

(2) *$p = 3$, $V = M(3, i) = \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_3$ as in Convention 2.4, and the relations of T are*

$$x_2x_1 - x_1x_2 + x_3^2 = 0, \quad x_3x_1 - x_1x_3 = 0, \quad x_3x_2 - x_2x_3 = 0.$$

In the rest of the theorem let $V = M(2, i) \oplus M(1, j) = (\Bbbk x_1 \oplus \Bbbk x_2) \oplus \Bbbk y$ for some $i, j \in \mathbb{Z}_p$.

(3) *$j = i + 1$ and the relations of T are*

$$x_1x_2 + x_1y - yx_1 = 0, \quad x_2x_1 - x_1x_2 = 0, \quad x_2^2 + x_2y - yx_2 = 0.$$

(4) *$(i - j)(2i + 1 - 2j) \neq 0$ in $\mathbb{Z}_p$, and the relations in T are*

$$yx_1 + ax_1y = 0, \quad yx_2 + ax_2y = 0, \quad x_2x_1 - x_1x_2 = 0$$

for $a \neq 0$ in $\Bbbk$.

(5) *$2i + 1 - 2j = 0$ in $\mathbb{Z}_p$, and the relations in T are*

$$yx_1 + ax_1y = 0, \quad yx_2 + ax_2y = 0, \quad x_2x_1 - x_1x_2 + \epsilon y^2 = 0$$

where $a \neq 0$, $\epsilon = 0$ or 1 and $\epsilon(a^2 - 1) = 0$.

(6) *$i = j$ in $\mathbb{Z}_p$, and the relations in T are*

$$yx_1 + ax_1y + by^2 = 0, \quad yx_2 + ax_2y = 0, \quad x_2x_1 - x_1x_2 + \epsilon x_2y = 0$$

where $a \neq 0$, $\epsilon = 0$ or 1 and $(a + 1)(b - \epsilon) = 0$.

(7) *$j = i + 2$ in $\mathbb{Z}_p$ and the relations in T are*

$$yx_2 - x_2y = 0, \quad x_1x_2 - x_2x_1 + cx_2y + by^2 = 0, \quad x_2^2 + yx_1 - x_1y + dy^2 = 0$$

where $c \neq 0$ or $d \neq 0$ only if $p = 2$ and where $b \neq 0$ only if $p = 3$.

(8) *$p = 2$, $i = j$ in $\mathbb{Z}_2$ and the relations in T are*

$$yx_2 + x_2y = 0, \quad x_1^2 + y^2 + ex_2^2 = 0, \quad x_1x_2 + x_2x_1 = 0,$$

where $e \in \Bbbk$.

(9) *$p = 2$, $i \neq j$ in $\mathbb{Z}_2$ and the relations in T are*

$$yx_2 + x_2y + by^2 = 0, \quad x_1^2 + cx_2y + y^2 + ex_2^2, \quad x_1x_2 + x_2x_1 = 0,$$

with $e \in \Bbbk$ and $(b, c) = (0, 1)$ or $(1, 0)$.

(10) *$p = 2$, $i \neq j$ in $\mathbb{Z}_2$, and the relations of T are one of the following forms:*

(10a) *$c \in \Bbbk$, and*

$$x_1^2 + c(x_2y + yx_2) + y^2 = 0, x_1x_2 + x_2x_1 = 0, x_2^2 + x_2y + yx_2 = 0.$$

(10b) *$e \in \Bbbk$, and*

$$x_1^2 + x_2y + yx_2 + ey^2 = 0, \quad x_1x_2 + x_2x_1 = 0, \quad x_2^2 + y^2 = 0.$$

Combining Theorem 1.5 with Lemma 5.2(3), we obtain

Corollary 1.6 *Let H be any Hopf algebra containing U as a Hopf subalgebra. If H acts inner-faithfully and homogeneously on a noetherian Koszul AS regular algebra T of global dimension three, then T is one of the AS regular algebras listed in Theorem 1.5.*

Remark 1.7 The following remarks aim to clarify potential confusion.

(1) The list in Theorem 1.5 is long. This is due to the fact that there are many different AS regular algebras of global dimension three. Even for the same T there could be different and non-equivalent U-actions on T.

(2) The classification of all noetherian Koszul AS regular algebras of global dimension three over a field of positive characteristic has not been done. This could be a huge project which is parallel to the work of Artin, Schelter, Tate and Van den Bergh [1–3]. The AS regular algebras listed in Theorem 1.5 form a very small portion in the class of all noetherian Koszul AS regular algebras of global dimension three.
(3) If we are given a Koszul AS regular algebra T generically, it is likely that there is no inner-faithful U-action on T. For example, let T be a skew polynomial ring $\Bbbk_{p_{ij}}[x_1, x_2, x_3]$ where $p_{ij} \neq 1$ for all $i < j$, then there is no inner-faithful U-action on T [Proposition 5.3].
(4) There are some obvious overlaps between part (1) and parts (4,5,6) in Theorem 1.5.
(5) Given a specific Hopf algebra H strictly containing U, there could be one or more AS regular algebras T listed in Theorem 1.5 on which H cannot act inner-faithfully.
(6) Given a specific Hopf algebra H containing U, there could be more than one H-action on the same T, similar to some parts of Theorem 1.5.
(7) In a weak sense, Corollary 1.6 provides a "universal" classification of H-actions on noetherian Koszul AS regular algebras of global dimension three for all H containing U. See Observation 5.5 for more comments.

There are a few reasons for us to consider to this particular Hopf algebra U.

The first one is that U is of finite representation type. This makes it possible to list all U-module $V := T_1$, which serves as an initial step in our classification. If U were of wild representation type, it is unrealistic to list all U-modules (even for a given dimension).

The second reason is that U is generated by primitive elements u and w. So u and w acts on an algebra T as derivatives. This kind of well-understood operation is helpful when we are dealing with a lot of computation.

The first two reasons make the project possible. The third reason is our motivation, namely, U is not semisimple and does not contain any nontrivial grouplike elements. The invariant theory under U-action is different from the classical invariant theory of polynomial rings under finite group actions. Observation 5.1 lists some significant differences in terms of homological properties. We would like to use the examples in this paper to further study non-semisimple Hopf actions on AS regular algebras.

Classifying U-actions on AS regular algebras is also helpful for understanding other H-actions on AS regular algebras when H is related to U, see Corollary 1.6, Remark 1.7 and Observation 5.5.

The last reason is that U appears in several different topics of recent interest. The Hopf algebra U originates from Lie theory, and is related to (small) quantum group theory,

computation of Frobenius-Perron dimension, study of the Azumaya locus of a family of PI Hopf algebras, called *iterated Hopf Ore extensions*, or *IHOEs*, as we explain next. In [5], a family of noncommutative PI Hopf algebras H in characteristic p were studied. The algebra U appears naturally as a "fiber" at every non-Azumaya point of 2-step IHOEs in [5, Proposition 8.2(3)]. The only other possible fiber of other 2-step IHOEs in [5] is the Hopf algebra $U_0 := \Bbbk[X, Y]/(X^p, Y^p)$, – the restricted enveloping algebra of the abelian Lie algebra of dimension 2 with trivial restriction – see [5, Proposition 8.2(2)]. Geometric and representation theoretic properties of the Hopf algebras in [5] are largely encoded in the properties of algebras U and U_0. Using the representations of U, we can describe all brick modules over 2-step IHOEs, see Remark 6.6.

This paper is organized as follows. Sections 2 and 3 follow from the structure of [12] and prove Theorem 1.2 and Corollary 1.3. In Section 4 we classify all U-actions on noetherian

Koszul AS regular algebras of global dimension at most three as stated in Proposition 4.2 and Theorem 1.5. We give some easy, but interesting, observations in Section 5. Section 6 contains some comments, projects, and remarks.

2 Preliminaries and Representations of *U*

Unless otherwise stated, all algebras, Hopf algebras and modules are defined over $\mathbb{k}$. All modules are left modules and all maps are $\mathbb{k}$-linear. We use $\otimes$ for $\otimes_{\mathbb{k}}$. For the theory of Hopf algebras, we refer to the standard text books [22, 32, 33, 39]. Let $\mathbb{k}^\times$ denote the multiplicative group of all nonzero elements in the field $\mathbb{k}$.

It is well-known that $\mathbb{Z}_p$ is a subfield of $\mathbb{k}$. For an integer $r \in \mathbb{Z}$, the image of r under the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}_p$ is still denoted by r. Then $\mathbb{Z}_p^\times = \{1, 2, \cdots, p-1\}$, which is a cyclic subgroup of the multiplicative group $\mathbb{k}^\times$.

Let H be a finite dimensional Hopf algebra. The *representation rings* (or the *Green rings*) $r(H)$ and $R(H)$ can be defined as follows. Recall that $r(H)$ is the abelian group generated by the isomorphism classes $[V]$ of finitely generated H-modules V modulo the relations $[M \oplus V] = [M] + [V]$. The multiplication of $r(H)$ is induced by the tensor product of H-modules, that is, $[M][V] = [M \otimes V]$. Then $r(H)$ is an associative ring. Recall that $R(H)$ is an associative $\mathbb{k}$-algebra defined by $\mathbb{k} \otimes_{\mathbb{Z}} r(H)$. Note that $r(H)$ is a free abelian group with the $\mathbb{Z}$-basis $\{[V] | V \in \text{ind}(H)\}$, where $\text{ind}(H)$ denotes the category of all finitely generated indecomposable H-modules.

For a module M over a finite dimensional algebra, let $\text{rl}(M)$ denote the Loewy length (=radical length=socle length) of M, and $\text{l}(M)$ denote the length of M. Let $P(M)$ denote the projective cover of M and $I(M)$ denote the injective hull of M.

Let U be defined as in Definition 1.1. The following facts about U are folklore. Let B denote the subalgebra of U generated by w. Then B is a p-dimensional semisimple Hopf subalgebra of U. Moreover, there is a Hopf algebra epimorphism $\pi : U \to B$ defined by $\pi(u) = 0$ and $\pi(w) = w$. It follows that $\ker \pi = (u) \supseteq J(U)$, the Jacobson radical of U. On the other hand, since $Uu = uU$ and $u^p = 0$, $J(U) \supseteq (u) = uU$, the ideal of U generated by the normal element u. Hence $\ker \pi = (u) = J(U)$. Thus, an U-module M is semisimple if and only if $u \cdot M = 0$, and moreover, M is simple if and only if $u \cdot M = 0$ and M is simple as a module over B. Note that $w^p - w = \prod_{i \in \mathbb{Z}_p}(w - i)$ over a field of characteristic p. Therefore, we have the following lemma, and its proof follows from the fact that w acts semisimply on any finite dimensional U-module.

Lemma 2.1 *For every finite dimensional U-module M and every $i \in \mathbb{Z}_p \subseteq \mathbb{k}$, let $M[i] = \{m \in M | w \cdot m = im\}$, then we have*

$$M = \bigoplus_{i \in \mathbb{Z}_p} M[i], \quad \textit{and} \quad uM[i] \subseteq M[i+1].$$

Lemma 2.2 *There are p non-isomorphic simple U-modules $\{S_i\}_{i \in \mathbb{Z}_p}$, and each S_i is 1-dimensional and determined by*

$$u \cdot x = 0, \quad \textit{and} \quad w \cdot x = ix,$$

where x is a basis element in S_i.

Note that $J(U)^m = u^m U$ for all $m \geqslant 1$. Hence $J(U)^{p-1} \neq 0$ and $J(U)^p = 0$. This means that the Loewy length of U is p. Since every simple U-module is 1-dimensional, $\mathrm{l}(M) = \dim(M)$ for all U-modules M. Let M be an U-module. Since $J(U)^s = Uu^s = u^s U$, we have $\mathrm{rad}^s(M) = u^s \cdot M$ for all $s \geqslant 1$.

Lemma 2.3 *Let $1 \leqslant l \leqslant p$ and $i \in \mathbb{Z}_p$. Then there is an algebra homomorphism $\rho_{l,i} : U \to M_l(k)$ given by*

$$\rho_{l,i}(u) = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}, \quad \rho_{l,i}(w) = \begin{pmatrix} i & & & & \\ & i+1 & & & \\ & & i+2 & & \\ & & & \ddots & \\ & & & & i+l-1 \end{pmatrix}.$$

Let $M(l,i)$ denote the corresponding left U-module.

Proof It follows from a straightforward verification. □

In some other papers when one uses a different convention, the matrix $\rho_{l,i}(u)$ in Lemma 2.3 should be replaced by its transpose. But we fix the following convention.

Convention 2.4 *By Lemma 2.3, the module $M(l,i)$ has a $\mathbb{k}$-basis $\{x_1, x_2, \cdots, x_l\}$ such that $w \cdot x_j = (i+j-1)x_j$ for all $1 \leqslant j \leqslant l$ and*

$$u \cdot x_j = \begin{cases} x_{j+1}, & 1 \leqslant j \leqslant l-1, \\ 0, & j = l. \end{cases}$$

Hence we have $x_j = u^{j-1} \cdot x_1$ for all $2 \leqslant j \leqslant l$. Such a basis is called a standard basis of $M(l,i)$.

We now list some easy facts.

Lemma 2.5 *The following hold.*

(1) *$soc(M(l,i)) = kx_l \cong S_{i+l-1}$ and $M(l,i)/rad(M(l,i)) \cong S_i$.*
(2) *$M(l,i)$ is indecomposable and uniserial.*
(3) *If $1 \leqslant l' \leqslant p$ and $i' \in \mathbb{Z}_p$, then $M(l,i) \cong M(l',i')$ if and only if $l' = l$ and $i' = i$.*

Proof It is similar to the proof of [12, Lemma 2.3]. □

The next corollary is similar to [12, Corollary 2.4].

Corollary 2.6 *The following hold.*

(1) *$M(l,i)$ is simple if and only if $l = 1$. In this case, $M(1,i) \cong S_i$.*
(2) *$M(l,i)$ is projective (respectively, injective) if and only if $l = p$.*
(3) *$M(p,i) \cong P(S_i) \cong I(S_{i-1})$.*

Proof (1) It follows from Lemma 2.5(1).

(2,3) Note that every finite dimensional Hopf algebra is self-injective as an algebra. If $l = p$, then it follows from [11, Lemma 3.5] that $M(p, i)$ is projective and injective.

Define $e_0 = 1 - w^{p-1}$ (for $i = 0$) and $e_i = \frac{1}{p-1}\sum_{j=1}^{p-1} i^{-j} w^j$ for $i \in \mathbb{Z}_p^\times$. It is easy to check that $e_0^2 = e_0$, $e_0 e_i = 0$ and $e_i e_l = \delta_{il} e_l$ for all $i, l \in \mathbb{Z}_p^\times$. That is, $\{e_0, e_1, \cdots, e_{p-1}\}$ is a set of orthogonal idempotents of U. Since $\mathbb{Z}_p^\times$ is a cyclic group of order $p-1$, $\sum_{i=1}^{p-1} i^{-j} = \delta_{j,p-1}(p-1)$ for any $1 \leqslant j \leqslant p-1$. Then one can check that $\sum_{i=1}^{p-1} e_i = w^{p-1}$, and so $\sum_{i=0}^{p-1} e_i = 1$. We also have $we_i = ie_i$ and $u^{p-1}e_i \neq 0$, where the latter follows from the fact that $\{u^i w^j \mid 0 \leqslant i, j \leqslant p-1\}$ is a $\Bbbk$-linear basis of U. Therefore,

$$Ue_i = \mathrm{span}\{e_i, ue_i, \cdots, u^{p-1}e_i\} \cong M(p, i).$$

Thus, we have the decomposition of the regular module U as follows

$$U = \bigoplus_{i=0}^{p-1} Ue_i \cong \bigoplus_{i=0}^{p-1} M(p, i).$$

Hence $M(p, i) \cong P(S_i)$, and $M(p, 0), M(p, 1), \cdots, M(p, p-1)$ are all non-isomorphic indecomposable projective U-modules. So parts (2,3) follow from Lemma 2.5. □

Since the indecomposable projective U-modules are uniserial, any indecomposable U-module is uniserial and is isomorphic to a quotient of an indecomposable projective module. Therefore, we have the following proposition (which is well-known).

Proposition 2.7 *Up to isomorphism, there are p^2 finite dimensional indecomposable U-modules as follows*

$$\{M(l, i) | 1 \leqslant l \leqslant p, i \in \mathbb{Z}_p\}.$$

It is easy to see that $\mathrm{Hom}_U(M(l, i), M(l, i)) = \Bbbk$ for all modules in Proposition 2.7.

3 Tensor Decomposition and the Green ring of *U*

Since U is cocommutative, the tensor category U-mod is symmetric. By Proposition 2.7, there are p^2 non-isomorphic indecomposable modules over U, namely,

$$\{M(l, r) | 1 \leqslant l \leqslant p, r \in \mathbb{Z}_p\}.$$

For any U-module M and $r \in \mathbb{Z}_p$, recall from Lemma 2.1 that

$$M[r] = \{m \in M | w \cdot m = rm\}.$$

Then $M[r]$ is a subspace of M. Next we list some easy facts in the following lemma.

Lemma 3.1 *Let M be an U-module. Then*

(1) *If M is indecomposable, then $dim(M[r]) \leqslant 1$ for every $r \in \mathbb{Z}_p$.*
(2) *If $M = M_1 \oplus M_2$ for some submodules M_1 and M_2, then $M[r] = M_1[r] \oplus M_2[r]$ for every $r \in \mathbb{Z}_p$.*
(3) *When M is decomposed into a direct sum of indecomposable submodules, the number of summands is at least $max\{dim(M[r]) | r \in \mathbb{Z}_p\}$.*

Lemma 3.2 *Let* $1 \leqslant l \leqslant p$ *and* $r, r' \in \mathbb{Z}_p$. *Then*

$$S_{r'} \otimes M(l, r) \cong M(l, r) \otimes S_{r'} \cong M(l, r + r')$$

as U*-modules. In particular,* $S_r \otimes S_{r'} \cong S_{r+r'}$ *and* $M(l, r) \cong S_r \otimes M(l, 0)$.

Throughout the rest of the section, let $2 \leqslant l \leqslant m \leqslant p$, and let

$$M = M(l, 0) \otimes M(m, 0).$$

Let $\{x_1, x_2, \cdots, x_l\}$ and $\{y_1, y_2, \cdots, y_m\}$ be the standard bases of $M(l, 0)$ and $M(m, 0)$, respectively, as stated in Convention 2.4. Then

$$\{x_i \otimes y_j | 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m\}$$

is a $\mathbb{k}$-basis of M. For any $2 \leqslant s \leqslant l + m$, let

$$M(s) = \text{span}\{x_i \otimes y_j | i + j = s\}.$$

Then we have $M = \bigoplus_{s=2}^{l+m} M(s)$ as $\mathbb{k}$-spaces.

Lemma 3.3 *Retain the above notation.*

(1) $u \cdot M(s) \subseteq M(s + 1)$ *for all* $2 \leqslant s \leqslant l + m$, *where* $M(l + m + 1) = 0$.
(2) $M(s) \subseteq M[s - 2]$ *for all* $2 \leqslant s \leqslant l + m$.
(3) $dim(M(s)) = \begin{cases} s - 1, & if\ 2 \leqslant s \leqslant l + 1 \\ l, & if\ l + 1 < s < m + 1 \\ l + m + 1 - s, & if\ m + 1 \leqslant s \leqslant l + m \end{cases}$

Proof It follows from a straightforward verification. □

Lemma 3.4 *The socle of* M *has the following decomposition*

$$soc(M) = \bigoplus_{2 \leqslant s \leqslant l+m} soc(M) \cap M(s).$$

Proof It follows from Lemma 3.3(1) since $\text{soc}(M) = \{z \in M | u \cdot z = 0\}$. □

Lemma 3.5 *The following statements hold.*

(1) *If* $2 \leqslant s \leqslant m$, *then* $soc(M) \cap M(s) = 0$.
(2) *If* $m + 1 \leqslant s \leqslant l + m$, *then* $dim(soc(M) \cap M(s)) = 1$.
(3) $dim(soc(M)) = l$.

Proof (1) Let $2 \leqslant s \leqslant l$ and let $z \in M(s)$. Then $z = \sum_{i=1}^{s-1} \alpha_i x_i \otimes y_{s-i}$ for some $\alpha_i \in \mathbb{k}$. A straightforward computation shows that

$$u \cdot z = \alpha_1 x_1 \otimes y_s + \sum_{2 \leqslant i \leqslant s-1} (\alpha_{i-1} + \alpha_i) x_i \otimes y_{s+1-i} + \alpha_{s-1} x_s \otimes y_1.$$

Now by an easy linear algebra argument, $u \cdot z = 0$ if and only if $z = 0$. Thus, $z \in \text{soc}(M)$ if and only if $z = 0$. This shows that $\text{soc}(M) \cap M(s) = 0$ for all $2 \leqslant s \leqslant l$.

Now let $l+1 \leqslant s \leqslant m$ and let $z \in M(s)$. In this case, $l < m$ and $z = \sum_{i=1}^{l} \alpha_i x_i \otimes y_{s-i}$ for some $\alpha_i \in \Bbbk$. Hence we have $u \cdot z = \alpha_1 x_1 \otimes y_s + \sum_{i=2}^{l} (\alpha_{i-1} + \alpha_i) x_i \otimes y_{s+1-i}$. Thus, by a similar argument as above, one can show that $z \in \mathrm{soc}(M)$ if and only if $z = 0$. Hence $\mathrm{soc}(M) \cap M(s) = 0$ for all $l + 1 \leqslant s \leqslant m$.

(2) Obviously, $u \cdot M(l+m) = 0$. Hence $M(l+m) \subseteq \mathrm{soc}(M)$, and so $M(l+m) \cap \mathrm{soc}(M) = M(l + m)$ is one dimensional. Now let $m + 1 \leqslant s < l + m$ and $z \in M(s)$. Then $z = \sum_{i=s-m}^{l} \alpha_i x_i \otimes y_{s-i}$ for some $\alpha_i \in \Bbbk$. One can check that

$$u \cdot z = \sum_{i=s+1-m}^{l} (\alpha_{i-1} + \alpha_i) x_i \otimes y_{s+1-i}.$$

Thus, $z \in \mathrm{soc}(M)$ if and only if $u \cdot z = 0$ if and only if $\alpha_{i-1} + \alpha_i = 0$ for all $s + 1 - m \leqslant i \leqslant l$. It follows that $\dim(\mathrm{soc}(M) \cap M(s)) = 1$ in this case.

(3) It follows from (1), (2) and Lemma 3.4.

□

Corollary 3.6 *Retain the above notation.*

(1) *For any $m+1 \leqslant s \leqslant l+m$, let $z_s = \sum_{i=s-m}^{l} (-1)^i x_i \otimes y_{s-i}$. Then $soc(M) \cap M(s) = \Bbbk z_s$.*

(2) *$soc(M) = span\{z_s | m + 1 \leqslant s \leqslant l + m\}$.*

(3) *$soc(M) \cong \bigoplus_{s=m+1}^{l+m} M(1, s-2) \cong \bigoplus_{s=m+1}^{l+m} S_{s-2}$.*

Proof It follows from the proof of Lemma 3.5. □

Define $\mathrm{head}(M) = M/u \cdot M$.

Corollary 3.7 *Retain the above notation. Then $head(M) \cong \bigoplus_{i=0}^{l-1} M(1, i) \cong \bigoplus_{i=0}^{l-1} S_i$.*

Proof By Lemma 3.3(1), we have $u \cdot M(s) \subseteq M(s + 1)$, and hence

$$u \cdot M = \bigoplus_{s=2}^{l+m-1} u \cdot M(s).$$

Now by Lemmas 3.3 and 3.5 and a dimension counting, we have

$$\dim(M(s+1)/u \cdot M(s)) = \begin{cases} 1, & \text{if } 2 \leqslant s \leqslant l, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $u \cdot M = (\bigoplus_{s=2}^{l} u \cdot M(s)) \bigoplus (\bigoplus_{s=l+2}^{l+m} M(s))$. Thus, as modules over $U/J(U) \cong B$, we have

$$\begin{aligned} M/u \cdot M &\cong M(2) \bigoplus (\bigoplus_{s=2}^{l} M(s+1)/u \cdot M(s)) \\ &\cong \bigoplus_{s=2}^{l+1} M(1, s-2) = \bigoplus_{i=0}^{l-1} M(1, i) \cong \bigoplus_{i=0}^{l-1} S_i. \end{aligned}$$

□

If $m = p$, then M is projective since $M(p, 0)$ is. Hence by Corollaries 2.6 and 3.7, we have

$$M \cong P(M) \cong P(M/u \cdot M) \cong \bigoplus_{i=0}^{l-1} P(S_i) \cong \bigoplus_{i=0}^{l-1} M(p, i).$$

Thus, we have the following proposition.

Proposition 3.8 *Let* $2 \leqslant l \leqslant p$ *and* $r, r' \in \mathbb{Z}_p$. *Then we have the* U*-module isomorphism*

$$M(l, r) \otimes M(p, r') \cong \bigoplus_{i=0}^{l-1} M(p, r + r' + i).$$

Proof We have already proven that $M(l, 0) \otimes M(p, 0) \cong \bigoplus_{i=0}^{l-1} M(p, i)$. Then the proposition follows from the isomorphism and Lemma 3.2. □

Lemma 3.9 *Let* $2 \leqslant l \leqslant m \leqslant p$ *and retain the above notation.*

(1) *M contains a submodule isomorphic to $M(l-1, 1) \otimes M(m-1, 0)$.*
(2) *For each $r \in \mathbb{Z}_p$, $M(l, r) \otimes M(m, 0)$ contains a submodule isomorphic to $M(l-1, r+1) \otimes M(m-1, 0)$.*
(3) *For $r \in \mathbb{Z}_p$ and $1 \leqslant s \leqslant l-1$, $M(l, r) \otimes M(m, 0)$ contains a submodule isomorphic to $M(l-s, r+s) \otimes M(m-s, 0)$.*

Proof (1) Recall that $\{x_i \otimes y_j | 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m\}$ is a basis of M. Let $N = M(l-1, 1) \otimes M(m-1, 0)$. Let $\{a_1, a_2, \cdots, a_{l-1}\}$ and $\{b_1, b_2, \cdots, b_{m-1}\}$ be the standard bases of $M(l-1, 1)$ and $M(m-1, 0)$, respectively. Then $\{a_i \otimes b_j | 1 \leqslant i \leqslant l-1, 1 \leqslant j \leqslant m-1\}$ is a basis of N. By definition, we have $w \cdot (x_i \otimes y_j) = (i + j - 2) x_i \otimes y_j$ for all $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant m$, and $w \cdot (a_i \otimes b_j) = (i + j - 1) a_i \otimes b_j$ for all (i, j).

Now define a $\Bbbk$-linear map $f : N \to M$ by

$$f(a_i \otimes b_j) = (l - i) x_i \otimes y_{j+1} + (j - m) x_{i+1} \otimes y_j,$$

where $1 \leqslant i \leqslant l-1$ and $1 \leqslant j \leqslant m-1$. It is easy to see that f is a $\Bbbk$-linear injection and that $f(w \cdot (a_i \otimes b_j)) = w \cdot f(a_i \otimes b_j)$ for all (i, j).

Then by a straightforward computation, one can check that $f(u \cdot (a_i \otimes b_j)) = u \cdot f(a_i \otimes b_j)$ for all (i, j). This finishes the proof of part (1).

(2) This follows from part (1) and Lemma 3.2.

(3) The assertion follows from induction on s and part (2). □

Theorem 3.10 *Let* $1 \leqslant l \leqslant m < p$ *and suppose that* $l + m \leqslant p$.

(1)

$$M(:= M(l,0)\otimes M(m,0)) \cong \bigoplus_{i=1}^{l} M(m-l-1+2i, l-i).$$

(2) *Let* $r, r' \in \mathbb{Z}_p$. *Then*

$$M(l,r)\otimes M(m,r') \cong \bigoplus_{i=1}^{l} M(m-l-1+2i, r+r'+l-i).$$

Proof (1) We use induction on l. If $l = 1$, it follows from Lemma 3.2 that the assertion holds. Now assume $l > 1$. Then $m \geqslant 2$. By Lemma 3.9(1), there exists a submodule N of M such that $N \cong M(l-1,1)\otimes M(m-1,0)$. By the induction hypothesis and Lemma 3.2, we have

$$N \cong S_1 \otimes M(l-1,0)\otimes M(m-1,0) \cong \bigoplus_{i=1}^{l-1} M(m-l-1+2i, l-i).$$

Therefore, one knows that

$$\mathrm{soc}(N) \cong \bigoplus_{i=1}^{l-1} M(1, l-i+(m-l-1+2i)-1) \cong \bigoplus_{i=1}^{l-1} S_{m+i-2}.$$

Now we use the $\Bbbk$-basis of M as stated before, and consider the submodule $\langle x_1\otimes y_1\rangle$ of M generated by $x_1\otimes y_1$. For the convenience, we set $x_i = 0$ for $i > l$, and $y_j = 0$ for $j > m$. Then since $1 < l+m \leqslant p$, we have

$$\begin{aligned} u^{l+m-2}\cdot(x_1\otimes y_1) &= \sum_{i=0}^{l+m-2} \tbinom{l+m-2}{i} u^i\cdot x_1 \otimes u^{l+m-2-i}\cdot y_1 \\ &= \sum_{i=0}^{l+m-2} \tbinom{l+m-2}{i} x_{i+1}\otimes y_{l+m-1-i} \\ &= \tbinom{l+m-2}{l-1} x_l \otimes y_m \neq 0. \end{aligned}$$

However, $u^{l+m-1}\cdot(x_1\otimes y_1) = 0$. It is easy to see that $w\cdot(x_1\otimes y_1) = 0$. It follows that $\langle x_1\otimes y_1\rangle$ is isomorphic to $M(l+m-1,0)$ with $\mathrm{soc}(\langle x_1\otimes y_1\rangle) = k(x_l\otimes y_m) \cong S_{l+m-2}$. Note that S_{l+m-2} is not isomorphic to any submodule of $\mathrm{soc}(N)$. Hence $N\cap\langle x_1\otimes y_1\rangle = 0$, and consequently, the sum $N+\langle x_1\otimes y_1\rangle$ of the two submodules of M is a direct sum in M. Thus, we have

$$\begin{aligned} &\dim(N+\langle x_1\otimes y_1\rangle) = \dim(N)+\dim(\langle x_1\otimes y_1\rangle) \\ &=(l-1)(m-1)+l+m-1 = lm = \dim(M). \end{aligned}$$

It follows that $M = N\oplus\langle x_1\otimes y_1\rangle \cong \bigoplus_{i=1}^{l} M(m-l-1+2i, l-i)$.

(2) It follows from Lemma 3.2 and part (1). □

Theorem 3.11 *Let* $1 \leqslant l \leqslant m < p$ *and suppose that* $l+m > p$.

(1)

$$M \cong \Big(\bigoplus_{i=1}^{p-m} M(m-l-1+2i, l-i)\Big)\bigoplus\Big(\bigoplus_{i=1}^{l+m-p} M(p, i-1)\Big).$$

(2) *Let* $r, r' \in \mathbb{Z}_p$. *Then*

$$M(l,r)\otimes M(m,r') \cong (\bigoplus_{i=1}^{p-m} M(m-l-1+2i, r+r'+l-i)) \bigoplus (\bigoplus_{i=1}^{l+m-p} M(p, r+r'+i-1)).$$

Proof (1) Since $l+m > p$ and $p > m \geqslant l \geqslant 2$, we have $1 \leqslant l+m-p = l-(p-m) \leqslant l-1$. It follows from Lemma 3.9(3) that there exists a submodule N of M such that

$$\begin{aligned} N &\cong M(l-(l+m-p), l+m-p)) \otimes M(m-(l+m-p), 0) \\ &= M(p-m, l+m) \otimes M(p-l, 0). \end{aligned}$$

Note that $(p-l)+(p-m) = 2p-(l+m) < p$ and $1 \leqslant p-m \leqslant p-l < p$. Then by Theorem 3.10(2), one gets that

$$N \cong M(p-m, l+m) \otimes M(p-l, 0) \cong \bigoplus_{i=1}^{p-m} M(m-l-1+2i, l-i)$$

and

$$\mathrm{soc}(N) \cong \bigoplus_{i=1}^{p-m} M(1, l-i+(m-l-1+2i)-1) \cong \bigoplus_{i=1}^{p-m} S_{m+i-2}.$$

Now we use the $\Bbbk$-basis of M as stated before. Let $1 \leqslant i \leqslant l+m-p$. Consider the submodule $\langle x_i \otimes y_1 \rangle$ of M generated by $x_i \otimes y_1$. At first, we have $w \cdot (x_i \otimes y_1) = (i-1)x_i \otimes y_1$. For the convenience, we set $x_j = 0$ for $j > l$, and $y_j = 0$ for $j > m$. Since $i+1+(p-1) = i+p \leqslant l+m$ and $p-m \leqslant l-i$, we have

$$\begin{aligned} u^{p-1} \cdot (x_i \otimes y_1) &= \sum_{j=0}^{p-1} \binom{p-1}{j} u^j \cdot x_i \otimes u^{p-1-j} \cdot y_1 \\ &= \sum_{j=0}^{p-1} \binom{p-1}{j} x_{i+j} \otimes y_{p-j} \\ &= \sum_{j=p-m}^{l-i} \binom{p-1}{j} x_{i+j} \otimes y_{p-j} \neq 0. \end{aligned}$$

Hence $\langle x_i \otimes y_1 \rangle = \mathrm{span}\{u^j \cdot (x_i \otimes y_1) | j = 0, 1, \cdots p-1\} \cong M(p, i-1)$, a projective (injective) module. Thus, $\mathrm{soc}(\langle x_i \otimes y_1 \rangle) \cong S_{i-2}$. Obviously,

$$S_{1-2}, S_{2-2}, \cdots, S_{(l+m-p)-2}$$

are non-isomorphic simple U-modules, and none of them is isomorphic to a submodule of N. It follows that the sum $N + \sum_{i=1}^{l+m-p} \langle x_i \otimes y_1 \rangle$ of the submodules of M is direct in M. Hence

$$\begin{aligned} &\dim(N + \sum_{i=1}^{l+m-p} \langle x_i \otimes y_1 \rangle) = \dim(N) + \sum_{i=1}^{l+m-p} \dim(\langle x_i \otimes y_1 \rangle) \\ &= (p-m)(p-l) + p(l+m-p) = lm = \dim(M). \end{aligned}$$

Thus, we have

$$\begin{aligned} M &= N \bigoplus (\bigoplus_{i=1}^{l+m-p} \langle x_i \otimes y_1 \rangle) \\ &\cong (\bigoplus_{i=1}^{p-m} M(m-l-1+2i, l-i)) \bigoplus (\bigoplus_{i=1}^{l+m-p} M(p, i-1)). \end{aligned}$$

(2) It follows from Lemma 3.2 and part (1). □

Corollary 3.12 *Let $1 \leqslant l \leqslant m < p$ and $r, r' \in \mathbb{Z}_p$.*

(1) *There is a simple summand in $M(l, r) \otimes M(m, r')$ if and only if $l = m$.*
(2) *If $l = m = 2 < p$, then $M(2, r) \otimes M(2, r') \cong S_{r+r'+1} \oplus M(3, r + r')$.*

Proof It follows from Corollary 2.6, Theorems 3.10 and 3.11. □

Proof of Theorem 1.2 (1) This is Theorem 3.10.
(2) If $m = p$, this follows from Lemma 3.2 and Proposition 3.8. If $m < p$, it is Theorem 3.11.

□

Throughout the rest of this section, let $a = [S_1]$ and $x = [M(2, 0)]$ in the Green ring $r(U)$ of U. Since U is cocommutative, $r(U)$ is a commutative ring. The following lemma is similar to [12, Lemma 3.8] (see [12, Lemma 3.8] for a proof.)

Lemma 3.13 *Retain the above notation. The following hold.*

(1) *$a^p = 1$ and $[M(l, r)] = a^r[M(l, 0)]$ for all $2 \leqslant l \leqslant p$ and $r \in \mathbb{Z}_p$.*
(2) *If $p > 2$, then $[M(l + 1, 0)] = x[M(l, 0)] - a[M(l - 1, 0)]$ for all $2 \leqslant l \leqslant p - 1$.*
(3) *$x[M(p, 0)] = (a + 1)[M(p, 0)]$.*
(4) *$r(U)$ is generated by a and x as a ring.*

Note that Lemma 3.13(1) is slightly different from [12, Lemma 3.8(1)]. The following is similar to [12, Corollary 3.9] and its proof is omitted.

Corollary 3.14 *Let $u_1, u_2, \cdots, u_p$ be a series of elements of the ring $r(U)$ defined recursively by $u_1 = 1$, $u_2 = x$ and*

$$u_l = xu_{l-1} - au_{l-2},\ \ p \geqslant l \geqslant 3.$$

Then $[M(l, 0)] = u_l$ for all $1 \leqslant l \leqslant p$ and $(x - a - 1)u_p = 0$.

Let R be the subring of $r(U)$ generated by a, and $\langle a \rangle$ the subgroup of the group of the invertible elements of $r(U)$ generated by a. Then $\langle a \rangle$ is a cyclic group of order p by Lemma 3.13(1), and $R = \mathbb{Z}\langle a \rangle$ is the group ring of $\langle a \rangle$ over $\mathbb{Z}$. Let $\mathbb{Z}[y, z]$ be the polynomial algebra over $\mathbb{Z}$ in two variables y and z. Define $f_n(y, z) \in \mathbb{Z}[y, z]$, $n \geqslant 1$, recursively, by $f_1(y, z) = 1$, $f_2(y, z) = z$ and

$$f_n(y, z) = zf_{n-1}(y, z) - yf_{n-2}(y, z),\ \ n \geqslant 3.$$

Then by [12, Lemma 3.11], for any $n \geqslant 1$, we have

$$f_n(y, z) = \sum_{i=0}^{[(n-1)/2]} (-1)^i \binom{n-1-i}{i} y^i z^{n-1-2i},$$

where $[(n-1)/2]$ denotes the integer part of $(n-1)/2$. Hence $\deg_z(f_n(y, z)) = n-1$ for all $n \geqslant 1$, where $\deg_z(f(y, z))$ denotes the degree of $f(y, z) \in \mathbb{Z}[y, z]$ in z. See [12, Section 3] for more information about $f_n(y, z)$. Let $I = (y^p - 1, (z - y - 1)f_p(y, z))$ be the ideal of $\mathbb{Z}[y, z]$ generated by $y^p - 1$ and $(z - y - 1)f_p(y, z)$.

With the above notations, we have the following corollary that is similar to [12, Theorem 3.10]. See the proof of [12, Theorem 3.10] for some details.

Corollary 3.15 *The Green ring $r(U)$ is isomorphic to the factor ring $\mathbb{Z}[y,z]/I$.*

Corollary 3.16 *The Green ring $r(U)$ is isomorphic to the Green ring $r(H_p(q))$ where $H_p(q)$ is the Taft algebra of rank p (over a possibly different base field).*

Proof This is clear by comparing Corollary 3.15 with [12, Theorem 3.10]. □

4 *U*-actions on AS regular Algebras

Recall from [1, p. 171] that a connected graded algebra T is called *Artin-Schelter regular* (or *AS regular*, for short) of dimension d if the following hold:

(a) T has global dimension $d < \infty$,
(b) $\mathrm{Ext}^i_T({}_T\Bbbk, {}_TT) = \mathrm{Ext}^i_T(\Bbbk_T, T_T) = 0$ for all $i \neq d$, where $\Bbbk = T/T_{\geq 1}$,
(c) $\mathrm{Ext}^d_T({}_T\Bbbk, {}_TT) \cong \mathrm{Ext}^d_T(\Bbbk_T, T_T) \cong \Bbbk(l)$ for some integer l,
(d) T has finite Gelfand–Kirillov dimension, see [26, Definition 1.7].

We will use the following general setting.

1. Let T be a noetherian connected graded AS regular algebra.
2. Let H be a finite-dimensional Hopf algebra acting on T inner-faithfully and homogeneously (namely, each degree i piece T_i of T is a left H-submodule of T), such that T is a left H-module algebra.

For any H-action on T, the fixed subring of the action is defined to be

$$T^H := \{a \in T \mid h \cdot a = \epsilon(h)a, \ \forall\, h \in H\}.$$

Lemma 4.1 *Let T be a graded algebra generated in degree 1 and let U act on T inner-faithfully. Then $V := T_1$ is a direct sum of indecomposable left U-modules, and at least one of which is not 1-dimensional. As a consequence,* $\dim_\Bbbk V \geq 2$.

Proof Note that U has three Hopf ideals, namely, 0, $\ker \epsilon$, and the ideal generated by u. If V is a direct sum of 1-dimensional left U-modules, then $u \cdot V = 0$. Then $u \cdot T = 0$ since T is generated by V. So the U-action is not inner-faithful, yielding a contradiction. The assertion follows. □

Note that commutative AS regular algebras are exactly commutative polynomial rings. Let T be a commutative polynomial ring $\Bbbk[x_1, \cdots, x_n]$ with $\deg x_i = 1$ for all i. Since U is cocommutative, every U-action on T is uniquely induced and uniquely determined by its action on the degree 1 piece. The following classifies completely all U-actions on noetherian AS regular algebras of global dimension 2.

Proposition 4.2 *Let U act inner-faithfully on a noetherian Koszul AS regular algebra T of global dimension 2.*

(1) *T is commutative, namely, $T = \Bbbk[V]$ where $V = T_1$. As a consequence, V is a 2-dimensional indecomposable left U-module.*

(2) *Using the notation introduced in Convention 2.4, we write V as $M(2,i) = \Bbbk x_1 \oplus \Bbbk x_2$ for some $0 \leqslant i \leqslant p-1$. Then the following hold.*

  (2a) *If $V = M(2, p-1)$, then $T^U = \Bbbk[x_1^p, x_2]$.*
  (2b) *If $V = M(2,i)$ for some $0 \leq i \leq p-2$, then $T^U = \Bbbk[x_1^p, x_2^p]$.*

Proof (1) Since the U-action on T is inner-faithful, V is not a direct sum of two 1-dimensional simples by Lemma 4.1. Hence $V = M(2,i)$ for some i, and then $T = \Bbbk\langle V\rangle/(r)$ where $r \in V \otimes V$ is the relation of T. By Lemma 3.2 and Corollary 3.6, the socle of $V \otimes V$ is two dimensional, spanned by $z_3 := -x_1 \otimes x_2 + x_2 \otimes x_1$ and $z_4 := x_2 \otimes x_2$. Since $\Bbbk r$ is a left U-module, it must be either $\Bbbk z_3$ or $\Bbbk z_4$. Since an AS regular algebra of global dimension two is a domain, r cannot be z_4. Therefore $r = z_3$, and T is commutative.

(2) By part (1), $T = \Bbbk[x_1, x_2]$. Since u and w are primitive, both of them act on T as derivatives. Then it is easy to show that $x_1^p, x_2^p \in T^U$.

Let $V = M(2,i)$. For any $0 \leq a, b \leq p-1$, it is straightforward to check that

$$u \cdot (x_1^a x_2^b) = a x_1^{a-1} x_2^{b+1}, \quad \text{and} \quad w \cdot (x_1^a x_2^b) = (ai + b(i+1)) x_1^a x_2^b. \tag{2}$$

  (2a) If $i = p-1$, then Eq. 2 implies that $x_2 \in T^U$ and that $T^U = \Bbbk[x_1^p, x_2]$.
  (2b) If $i \neq p-1$, then one sees that $T^U = \Bbbk[x_1^p, x_2^p]$ by Eq. 2. □

For a generalization of Proposition 4.2(1), see Proposition 1.4 (and Proposition 5.3).

Next we would like to determine U-actions on noetherian Koszul AS regular algebras of global dimension 3. We will use the fact that a noetherian connected graded algebra of global dimension three is a domain [38, Theorem].

Convention 4.3 *The following are assumed for the rest of this section.*

(1) *Let U act inner-faithfully and homogeneously on a noetherian Koszul AS regular algebra T of global dimension 3.*
(2) *Let $V := T_1$ be the degree 1 piece of the algebra T. It is well-known that V is a 3-dimensional left U-module.*
(3) *Let $R \subseteq V \otimes V$ be the relation space of T, namely, $T = \Bbbk\langle V\rangle/(R)$. It is well-known that R is a 3-dimensional left U-submodule of $V \otimes V$.*
(4) *If V is $M(3,i)$, then we write $V = \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_3$ using the notation introduced in Convention 2.4.*
(5) *If V is $M(2,i) \oplus S_j$ (where $S_j = M(1,j)$), then we write $M(2,i) = \Bbbk x_1 \oplus \Bbbk x_2$ using the notation introduced in Convention 2.4 and $S_j = \Bbbk y$.*
(6) *Since T is noetherian Koszul AS regular of global dimension three, the Hilbert series of T is $(1-t)^{-3}$. In particular, $\dim_{\Bbbk} T_3 = 10$.*

There is a large class of noetherian Koszul AS regular algebras of global dimension 3 and the classification of such algebras over a field of positive characteristic has not been done. We can classify all U-actions on noetherian Koszul AS regular algebras of global dimension 3 because there is no inner-faithful U-action on a generic AS regular algebra, see Proposition 5.3. The basic idea in the following is to work out the left U-module R which is a U-submodule of $V \otimes V$.

Lemma 4.4 *Suppose V is $M(3, i)$. Let $z_4 := -x_1 \otimes x_3 + x_2 \otimes x_2 - x_3 \otimes x_1$, $z_5 := x_2 \otimes x_3 - x_3 \otimes x_2$ and $z_6 := -x_3 \otimes x_3$. The following hold.*

(1) *$soc(V \otimes V) = S_{2i+2} \oplus S_{2i+3} \oplus S_{2i+4}$ where basis elements for simple modules S_{2i+2}, S_{2i+3}, S_{2i+4} respectively are z_4, z_5, z_6 respectively.*
(2) *If $p = 3$, then $V \otimes V = M(3, 2i) \oplus M(3, 2i + 1) \oplus M(3, 2i + 2)$ where the socles of $M(3, 2i)$, $M(3, 2i + 1)$, $M(3, 2i + 2)$ are generated by z_4, z_5, z_6 respectively.*
(3) *If $p \geq 5$, then $V \otimes V = M(1, 2 + 2i) \oplus M(3, 1 + 2i) \oplus M(5, 2i)$ where the socles of $M(1, 2 + 2i)$, $M(3, 1 + 2i)$, $M(5, 2i)$ respectively are generated by z_4, z_5, z_6 respectively.*

Proof (1) This follows from Lemma 3.2 and Corollary 3.6.
(2) This follows from part (1) and Proposition 3.8.
(3) This follows from part (1) and Theorem 3.10(2) when $p > 5$ and Theorem 3.11(2) when $p = 5$.

□

Lemma 4.5 *Suppose $V = M(3, i) = \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_3$. Let $W = x_3 \otimes V + V \otimes x_3$.*

(1) *$R \cap (x_3 \otimes V) = R \cap (V \otimes x_3) = 0$ and $\dim_{\Bbbk}(R \cap W) \leq 2$.*
(2) *$\dim_{\Bbbk}(R \cap W) \neq 1$.*
(3) *R does not contain three linearly independent elements of the form*

$$f_1 := x_1 \otimes x_2 + \xi_1,$$
$$f_2 := x_2 \otimes x_1 + \xi_2,$$
$$f_3 := x_2 \otimes x_2 + \xi_3,$$

where $\xi_1, \xi_2, \xi_3 \in W$.
(4) *$\dim_{\Bbbk}(R \cap W) \neq 0$.*
(5) *$\dim_{\Bbbk}(R \cap W) = 2$, x_3 is a normal element in T, and one of the following occurs:*

(5a) *$p \geq 3$, T is commutative.*
(5b) *$p = 3$ and the relations of T are*

$$x_2x_1 - x_1x_2 + x_3^2 = 0, \quad x_3x_1 - x_1x_3 = 0, \quad x_3x_2 - x_2x_3 = 0.$$

Proof Some non-essential computations are skipped. Since $V = M(3, i)$, $p \geq 3$.

(1) If $R \cap (x_3 \otimes V) \neq 0$, then there is a relation of the form $x_3 \otimes v = 0$ for some $0 \neq v \in V$. This contradicts the fact that T is a domain. Therefore $R \cap (x_3 \otimes V) = 0$. By symmetry, $R \cap (V \otimes x_3) = 0$.

The inclusion $R \cap W \to W$ induces an injective map $R \cap W \to W/(V \otimes x_3)$. Therefore $\dim_{\Bbbk} R \cap W \leq \dim_{\Bbbk} W/(V \otimes x_3) = 2$.

(2) Suppose to the contrary that $\dim_{\Bbbk} R \cap W = 1$. Let f be a basis element in $R \cap W$ and write it as

$$f = ax_3 \otimes x_1 + bx_3 \otimes x_2 + v \otimes x_3$$

for some $v \in V$. By part (1), either a or b is nonzero. Suppose first that $a \neq 0$. Then $u \cdot f = ax_3 \otimes x_2 + w \otimes x_3$ for some $w \in V$. So $\{f, u \cdot f\}$ are linearly independent elements in $R \cap W$, yielding a contradiction. Therefore $a = 0$, and in this case we may assume that $b = 1$ and

$$f = x_3 \otimes x_2 + a_1x_1 \otimes x_3 - qx_2 \otimes x_3 + c_1x_3 \otimes x_3.$$

Note that

$$(2i+3-w)\cdot f = a_1x_1\otimes x_3 - c_1x_3\otimes x_3$$

which must be zero as $\dim_{\Bbbk} R\cap W = 1$. Therefore $a_1 = c_1 = 0$ and $f = x_3\otimes x_2 - qx_2\otimes x_3$. Since $\Bbbk f$ is in the socle of $V\otimes V$, by Lemma 4.4, we have that $f_1 := z_5 = x_2\otimes x_3 - x_3\otimes x_2 \in R\cap W$.

Now let $g\in R\setminus W$ and write it as

$$g = a_{11}x_1\otimes x_1 + a_{12}x_1\otimes x_2 + a_{21}x_2\otimes x_1 + a_{22}x_2\otimes x_2 + \phi_0$$

where $\phi_0\in W$. Since $\dim_{\Bbbk} R/R\cap W = \dim_{\Bbbk} R - \dim_{\Bbbk}(R\cap W) = 2$, $u^2\cdot g\in W$. By a computation, $u^2\cdot g = 2a_{11}x_2\otimes x_2 + \phi_1$ where $\phi_1\in W$. Thus $a_{11}=0$. In this case

$$u\cdot g = (a_{12}+a_{21})x_2\otimes x_2 + a_{12}x_1\otimes x_3 + a_{21}x_3\otimes x_1 + \phi_2$$

where $\phi_2\in W$. If $3(a_{12}+a_{21})\neq 0$, then

$$\begin{aligned} u^2\cdot g &\equiv (2a_{12}+a_{21})x_2\otimes x_3 + (a_{12}+2a_{21})x_3\otimes x_2 \mod \Bbbk x_3\otimes x_3\\ &\equiv 3(a_{12}+a_{21})x_2\otimes x_3 \mod \Bbbk x_3\otimes x_3 + \Bbbk z_5\end{aligned}$$

as $u^2\cdot(a_{22}x_2\otimes x_2+\phi_0)\equiv 0$ modulo $\Bbbk x_3\otimes x_3$. Then $u^2\cdot g$ and z_5 are linearly independent elements in $R\cap W$, yielding a contradiction. Therefore $3(a_{12}+a_{21})=0$.

Since $R/R\cap W$ has dimension two, either $a_{12}\neq 0$ or $a_{21}\neq 0$. By symmetry, we assume that $a_{12}\neq 0$. So we can assume that $a_{12}=-1$. We need to consider the following two cases:

Case (2a): $a_{21}=-a_{12}=1$. Then R has two elements of the form

$$\begin{aligned} f_2 &= x_2\otimes x_1 - x_1\otimes x_2 + a_1x_1\otimes x_3 + b_1x_3\otimes x_1 + c_1x_2\otimes x_3 + d_1x_3\otimes x_3,\\ f_3 &= x_2\otimes x_2 + a_2x_1\otimes x_3 + b_2x_3\otimes x_1 + c_2x_2\otimes x_3 + d_2x_3\otimes x_3.\end{aligned}$$

It is easy to see that

$$u\cdot f_2 = x_3\otimes x_1 - x_1\otimes x_3 + a_1x_2\otimes x_3 + b_1x_3\otimes x_2 + c_1x_3\otimes x_3$$

which is in $R\cap W$ but not in $\Bbbk f_1$. This yields a contradiction.

Case (2b): $q := a_{21}\neq -a_{12}=1$. Since $3(a_{12}+a_{21})=0$, we obtain that $p=3$. We also have relations similar to f_2 and f_3 in Case (2a). Since R is a left U-module, we can choose f_2, f_3 so that $(2i+1-w)\cdot f_2 = 0 = (2i+2-w)\cdot f_3$. In this case, we have

$$\begin{aligned} f_2 &= qx_2\otimes x_1 - x_1\otimes x_2 + dx_3\otimes x_3,\\ f_3 &= x_2\otimes x_2 + ax_1\otimes x_3 + bx_3\otimes x_1.\end{aligned}$$

By easy calculation,

$$\begin{aligned} u\cdot f_2 &= (q-1)x_2\otimes x_2 - x_1\otimes x_3 + qx_3\otimes x_1,\\ u\cdot f_3 &= (1+a)x_2\otimes x_3 + (1+b)x_3\otimes x_2.\end{aligned}$$

Since $u\cdot f_2$ and $u\cdot f_3$ are relations of T, by comparing $u\cdot f_2$ with f_3 and $u\cdot f_3$ with f_1 and using the fact that $p=3$, we obtain that $a=-(q-1)^{-1}$ and $b=q(q-1)^{-1}$. This algebra can be built from $\Bbbk[x_2,x_3]$ by adding x_1 with relations f_2 and f_3. So if T is AS regular (consequently having Hilbert series $(1-t)^{-3}$), it must be an Ore extension of the form $\Bbbk[x_2,x_3][x_1;\sigma,\delta]$ for some automorphism σ and σ-derivation δ of the polynomial ring $\Bbbk[x_2,x_3]$. Now an easy ring theory argument shows that the existence of (σ,δ) forces that $q=1$, yielding a contradiction.

(3) Suppose to the contrary that R contains three linearly independent elements of the form

$$f_1 = x_1 \otimes x_2 + \xi_1,$$
$$f_2 = x_2 \otimes x_1 + \xi_2,$$
$$f_3 = x_2 \otimes x_2 + \xi_3,$$

where $\xi_1, \xi_2, \xi_3 \in W$. Then $u \cdot f_1$ and $u \cdot f_2$ must be equal to f_3. Then $R/(\Bbbk f_3)$ is a direct sum of two 1-dimensional simples. Then R is not indecomposable. Then R contains at least two of elements z_4, z_5, z_6 by Lemma 4.4(1). Clearly z_5 and z_6 are not a linear combination of these f_i. This yields a contradiction.

(4) On the contrary we suppose that $R \cap W = 0$.

First we assume that $p > 3$. If $f = x_1 \otimes x_1 + \sum_{(i,j)\neq(1,1)} a_{i,j} x_i \otimes x_j$ is in R, then $u^3 \cdot f = 3(x_2 \otimes x_3 + x_3 \otimes x_2) + c x_3 \otimes x_3$ for some $c \in \Bbbk$, which is a nonzero element in $R \cap W$, a contradiction. So R does not contain any element of the form $x_1 \otimes x_1 + \sum_{(i,j)\neq(1,1)} a_{i,j} x_i \otimes x_j$. Since $R \cap W = 0$, R must contain three elements of the form given in part (3). This contradicts part (3).

Now we assume that $p = 3$. By Lemma 4.4(1), one of the relations is $z_4 \in R$ as z_5 and z_6 are in W. By part (3), we have a relation $f \in R$ of the form

$$f = x_1 \otimes x_1 + \text{higher terms.}$$

Then $u \cdot f = x_1 \otimes x_2 + x_2 \otimes x_1 + \text{higher terms}$. Modulo z_4 and $u \cdot f$, we can assume that

$$f = x_1 \otimes x_1 + a x_1 \otimes x_2 + b x_1 \otimes x_3 + c x_3 \otimes x_1 + d x_2 \otimes x_3 + e x_3 \otimes x_2 + g x_3 \otimes x_3.$$

Then

$$(2i - w) \cdot f = -a x_1 \otimes x_2 - 2b x_1 \otimes x_3 - 2c x_3 \otimes x_1 - g x_3 \otimes x_3.$$

Since $(2i - w) \cdot f \in R$ and it is not a nonzero linear combination of z_4, f and $u \cdot f$, it must be zero. Therefore $a = b = c = g = 0$ and

$$f = x_1 \otimes x_1 + d x_2 \otimes x_3 + e x_3 \otimes x_2.$$

Consequently, we have two more relations

$$u \cdot f = x_1 \otimes x_2 + x_2 \otimes x_1 + (d + e) x_3 \otimes x_3,$$
$$-u^2 \cdot f = -x_1 \otimes x_3 + x_2 \otimes x_2 - x_3 \otimes x_1.$$

Since T is a domain one of d and e is nonzero. By symmetry, we can assume $e \neq 0$. After changing a basis element, we may assume that $e = 1$. Next we consider two cases dependent on whether $d + e$ is zero or not.

Case 1: $d + e = 0$. Since $e = 1$, $d = -1$. So three relations of T are, after we omit the $\otimes$ symbol:

$$x_3 x_2 = x_2 x_3 - x_1^2, \quad x_2 x_1 = -x_1 x_2, \quad x_3 x_1 = -x_1 x_3 + x_2^2.$$

Following ideas from Bergman's Diamond lemma [4], the next computation is referred to as *resolving the overlap ambiguity* of $x_3(x_2 x_1) = (x_3 x_2) x_1$. Using the order $x_1 < x_2 < x_3$

in the algebra T, we have

$$\begin{aligned}(x_3x_2)x_1 &= (x_2x_3 - x_1^2)x_1 = x_2(x_3x_1) - x_1^3\\ &= x_2(-x_1x_3 + x_2^2) - x_1^3 = x_1x_2x_3 + x_2^3 - x_1^3,\\ x_3(x_2x_1) &= x_3(-x_1x_2) = -(x_3x_1)x_2\\ &= -(-x_1x_3 + x_2^2)x_2 = x_1(x_3x_2) - x_2^3\\ &= x_1(x_2x_3 - x_1^2) - x_2^3 = x_1x_2x_3 - x_1^3 - x_2^3.\end{aligned}$$

Since $x_3(x_2x_1) = (x_3x_2)x_1$, we obtain that $x_1x_2x_3 + x_2^3 - x_1^3 = x_1x_2x_3 - x_1^3 - x_2^3$, or equivalently, $x_2^3 = 0$. But T is a domain, so this case cannot happen.

Case 2: $d + e = d + 1 \neq 0$. In this case, the relations of T are

$$x_3x_2 = -dx_2x_3 - x_1^2, \quad x_3^2 = -(d+e)^{-1}(x_2x_1 + x_1x_2), \quad x_3x_1 = -x_1x_3 + x_2^2.$$

By resolving the overlap ambiguity of $(x_3^2)x_3 = x_3(x_3^2)$ (details omitted, same as below), we obtain that

$$(d-1)x_2^3 = (d-1)(x_2x_1 + x_1x_2)x_3.$$

By resolving the overlap ambiguity of $(x_3)^2x_1 = x_3(x_3x_1)$, we obtain that

$$(d^2-1)x_2^2x_3 = (d + (d+1)^{-1})x_2x_1^2 - (1 + (d+1)^{-1})x_1^2x_2.$$

By resolving the overlap ambiguity of $(x_3^2)x_2 = x_3(x_3x_2)$, we obtain that

$$(1+d^2(d+1)^{-1})x_2^2x_1 = (d-1)x_1^2x_3 + (1+(d+1)^{-1})^{-1}x_1x_2^2 + (d+1)^{-1}(1-d^2)x_2x_1x_2.$$

If $d \neq 1$, by using the relations of degree 2 and three relations coming from resolving the overlap ambiguities, then T_3 is a $\Bbbk$-span of

$$\{x_1^3, x_1^2x_2, x_1x_2x_1, x_1x_2^2, x_2x_1^2, x_2x_1x_2, x_1^2x_3, x_1x_2x_3, x_2x_1x_3\}.$$

As a consequence, $\dim_{\Bbbk} T_3 = 9 < 10$, yielding a contradiction. Therefore $d = 1$. We already have $e = 1$, so the three relations of T become

$$\begin{aligned}x_1x_3 + x_3x_1 &= x_2^2,\\ x_1x_2 + x_2x_1 &= -2x_3^2 = x_3^2,\\ x_3x_2 + x_2x_3 &= -x_1^2.\end{aligned}$$

Note that, after setting $x_1 \to -x_1$ in the above algebra, the new algebra, denoted by T', has relations

$$\begin{aligned}x_1x_3 + x_3x_1 &= -x_2^2,\\ x_1x_2 + x_2x_1 &= -x_3^2,\\ x_3x_2 + x_2x_3 &= -x_1^2.\end{aligned}$$

Its Koszul dual, denoted by B, is a commutative algebra generated by y_1, y_2, y_3 subject to relations:

$$\begin{aligned}y_1y_3 - y_3y_1 &= 0, \quad y_1y_3 - y_2^2 = 0,\\ y_1y_2 - y_2y_1 &= 0, \quad y_1y_2 - y_3^2 = 0,\\ y_3y_2 - y_2y_3 &= 0, \quad y_2y_3 - y_1^2 = 0.\end{aligned}$$

It is easy to see that there is a surjective algebra map from B to $\Bbbk[t]$ by setting $y_i \to t$ for $i = 1, 2, 3$. Therefore B is not finite dimensional Frobenius. By [37, Proposition 5.10] or [31, Corollary D], T' is not AS regular. As a consequence, T is not AS regular, yielding a contradiction. Thus the assertion follows.

(5) By parts (1,2,4), $\dim_{\Bbbk} R \cap W = 2$. By the proof of part (1), the natural $\Bbbk$-linear map $\phi : R \cap W \to W/(V \otimes x_3)$ is an isomorphism. Let f_1, f_2 be two linearly independent elements in $R \cap W$. Since $\phi(f_1)$ and $\phi(f_2)$ are linearly independent, we may assume $\phi(f_1) = x_3 \otimes x_1$ and $\phi(f_2) = x_3 \otimes x_2$. In other words, we can write

$$\begin{aligned} f_1 &:= x_3 \otimes x_1 + a_1 x_1 \otimes x_3 + b_1 x_2 \otimes x_3 + c_1 x_3 \otimes x_3, \\ f_2 &:= x_3 \otimes x_2 + a_2 x_1 \otimes x_3 + b_2 x_2 \otimes x_3 + c_2 x_3 \otimes x_3. \end{aligned}$$

Therefore $x_3 T = T x_3$ and hence, x_3 is a normal element in T.

Since x_3 is normal and $\Bbbk x_3$ is a left U-module, the U-action on T induces a natural U-action on $Z := T/(x_3)$ where Z is a noetherian Koszul AS regular algebra of global dimension two. Note that the degree 1 piece of Z is $Z_1 = M(2, i)$, whence the U-action on Z is inner-faithful. By Proposition 4.2(1), Z is commutative, so $x_2 x_1 - x_1 x_2 = 0$ in Z. This implies that the third relation in T is

$$f_3 := x_2 \otimes x_1 - x_1 \otimes x_2 + a_3 x_1 \otimes x_3 + b_3 x_2 \otimes x_3 + c_3 x_3 \otimes x_3.$$

Easy computations show that

$$u \cdot f_1 = x_3 \otimes x_2 + a_1 x_2 \otimes x_3 + b_1 x_3 \otimes x_3, \tag{3}$$
$$u \cdot f_2 = x_3 \otimes x_3 + a_2 x_2 \otimes x_3 + b_2 x_3 \otimes x_3, \tag{4}$$
$$u \cdot f_3 = x_3 \otimes x_1 - x_1 \otimes x_3 + a_3 x_2 \otimes x_3 + b_3 x_3 \otimes x_3, \tag{5}$$
$$(2i + 2 - w) \cdot f_1 = -b_1 x_2 \otimes x_3 - 2c_1 x_3 \otimes x_3, \tag{6}$$
$$(2i + 3 - w) \cdot f_2 = a_2 x_1 \otimes x_3 - c_2 x_3 \otimes x_3, \tag{7}$$
$$(2i + 1 - w) \cdot f_3 = -a_3 x_1 \otimes x_3 - 2b_3 x_2 \otimes x_3 - 3c_3 x_3 \otimes x_3. \tag{8}$$

Since R is a U-submodule, the above elements are in $R \cap W$. Equations 6-8 imply that $b_1 = c_1 = a_2 = c_2 = a_3 = b_3 = 3c_3 = 0$. Equation 5 says that $f_1 = x_3 \otimes x_1 - x_1 \otimes x_3$ and that $a_1 = -1$, Eq. 3 says that $f_2 = x_3 \otimes x_2 - x_2 \otimes x_3$ and that $b_2 = -1$. Combining all these we obtain two cases:

(a) $p \geq 3$, T is commutative, or
(b) $p = 3$ and relations of T are, up to a change of basis,

$$x_2 \otimes x_1 - x_1 \otimes x_2 + x_3 \otimes x_3 = x_3 \otimes x_1 - x_1 \otimes x_3 = x_3 \otimes x_2 - x_2 \otimes x_3 = 0.$$

This finishes the proof. □

We will recycle the letter z_4 with a different meaning in the next lemma. Similar to Lemma 4.4, we have

Lemma 4.6 *Suppose* $V = M(2, i) \oplus S_j = \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk y$ *for some* $i, j \in \mathbb{Z}_p$.

(1) $V \otimes V = [M(2, i) \otimes M(2, i)] \oplus [M(2, i) \otimes S_j \oplus S_j \otimes M(2, i)] \oplus S_j \otimes S_j$.
(2) $soc(V \otimes V) = [S_{2i+1} \oplus S_{2i+2}] \oplus [S_{i+1+j} \oplus S_{j+i+1}] \oplus S_{2j}$ *with corresponding basis elements* $z_3 = -x_1 \otimes x_2 + x_2 \otimes x_1$ *and* $z_4 = x_2 \otimes x_2$ *for* $S_{2i+1} \oplus S_{2i+2}$, $x_2 \otimes y$, $y \otimes x_2$ *for* $S_{i+1+j} \oplus S_{j+i+1}$, *and* $y \otimes y$ *for* S_{2j}.

Lemma 4.7 *Suppose* $V = M(2, i) \oplus S_j$ *as in Lemma 4.6. Then* $u^2 \cdot R = 0$ *and* $\dim_{\Bbbk} u \cdot R \leq 1$.

Proof Suppose to the contrary that $u^2 \cdot R \neq 0$. Then the subspace $M(2,i) \otimes M(2,i)$ in Lemma 4.6(1) has a direct summand that is a three-dimensional indecomposable module $M(3,2i)$ (using Corollary 3.12) with socle $\Bbbk z_4 = \Bbbk x_2 \otimes x_2$. As a consequence $p \geq 3$. In this case, Lemma 4.6(1) implies that $V \otimes V = M(3,2i) \oplus N$ where $u^2 \cdot N = 0$. Pick $f \in R$ such that $u^2 \cdot f \neq 0$ and write $f = f_0 + f_1$ where $f_0 \in M(3,2i)$ and $f_1 \in N$. Then $u^2 \cdot f_0 = u^2 \cdot f \in R$. Since $u^2 \cdot f_0$ is in the socle of $M(3,2i)$, we obtain that $u^2 \cdot f$ is a nonzero element in $\Bbbk z_4$. Thus $x_2^2 = 0$ in T, which contradicts the fact that every noetherian Koszul AS regular algebra of global dimension three is a domain. Therefore $u^2 \cdot R = 0$.

Since $u^2 \cdot R = 0$, then $soc(R)$ has dimension at least 2. Therefore $u \cdot R \cong R/\ker(l_u)$ has dimension at most 1, where $l_u : R \to R$ is a left multiplication by u. □

We need the following lemma.

Lemma 4.8 *Let char* $\Bbbk = 2$.

(1) *Suppose A is generated by* x_1, x_2, y *and subject to the relations*

$$\begin{aligned} 0 &= x_1^2 + c_1 x_2 y + d_1 y x_2 + e_1 y^2, \\ 0 &= x_1 x_2 + x_2 x_1, \\ 0 &= x_2^2 + c_3 x_2 y + d_3 y x_2 + e_3 y^2. \end{aligned}$$

Then A is a noetherian Koszul AS regular algebra of global dimension 3 if and only if the parameters satisfy the following conditions

$$d_1 = c_1, \ d_3 = c_3, \ c_3^2 - e_3 \neq 0, \ c_1 e_3 - c_3 e_1 \neq 0.$$

(2) *Suppose A is generated by* x_1, x_2, y *and subject to the relations*

$$\begin{aligned} 0 &= y x_2 - q x_2 y, \\ 0 &= x_1 x_1 + a x_1 y + d y^2 + e x_2^2, \\ 0 &= x_1 x_2 + x_2 x_1 + a x_2 y. \end{aligned}$$

Then A is a noetherian Koszul AS regular algebra of global dimension 3 if and only if the parameters satisfy the following conditions.

$$a = 0, q = 1, d \neq 0.$$

In this case we can assume that $d = 1$.

(3) *Suppose A is generated by* x_1, x_2, y *and subject to the relations*

$$\begin{aligned} 0 &= y x_2 - q x_2 y + c_1 y^2, \\ 0 &= x_1 x_1 + c x_2 y + d y^2 + e x_2^2, \\ 0 &= x_1 x_2 + x_2 x_1. \end{aligned}$$

Then A is a noetherian Koszul AS regular algebra of global dimension 3 if and only if the parameters satisfy the following conditions

$$q = 1, d \neq 0, c_1 c = 0.$$

As a consequence, we may assume $d = 1$*, and* $(c_1, c) = (1, 0)$ *or* $(0, 1)$ *by a change of basis.*

To save some space, we omit the proof of Lemma 4.8 as it takes a few pages. Here are a list of ideas used in the proof.

(a) We use the fact that the Hilbert series of A must be $(1-t)^{-3}$ if A is noetherian Koszul AS regular of global dimension three. We will resolve the overlap ambiguitiy of relations (ideas from Bergman's Diamond lemma [4]) to make sure that $\dim A_3 = 10$. The condition $\dim A_3 = 10$ forces some constraints on the parameters.
(b) We study the Koszul dual of the algebra. Note that A is Koszul AS regular if and only if the Koszul dual of A is Koszul and Frobenius, see [37, Proposition 5.10] or [31, Corollary D]. Both Koszul and Frobenius properties put more constraints on the parameters.
(c) We use ideas in [6, Theorem 4.2] to show that A is noetherian Koszul AS regular.

Lemma 4.9 *Retain the hypothesis in Lemma 4.7. Recycle the letter W for $y \otimes V + V \otimes y$.*

(1) $R \cap (y \otimes V) = R \cap (V \otimes y) = 0$ *and* $\dim_{\Bbbk}(R \cap W) \leq 2$.
(2) *If* $\dim_{\Bbbk}(R \cap W) = 2$, *then y is a normal element.*
(3) *Suppose* $\dim_{\Bbbk}(R \cap W) = 1$. *Then one of the following occurs.*

(3a) *$j = i + 2$ and the relations of T are*

$$yx_2 - x_2y = 0, \quad x_1x_2 - x_2x_1 + c_2x_2y + d_2y^2 = 0, \quad x_2^2 + yx_1 - x_1y + d_3y^2 = 0$$

where $c_2 \neq 0$ or $d_3 \neq 0$ only if $p = 2$ and $d_2 \neq 0$ only if $p = 3$.

(3b) *$p = 2$, $i = j$ and the relations of T are*

$$yx_2 + x_2y = 0, \quad x_1^2 + y^2 + ex_2^2 = 0, \quad x_1x_2 + x_2x_1 = 0,$$

where $e \in \Bbbk$.

(3c) *$p = 2$, $i \neq j$ and the relations of T are*

$$yx_2 + x_2y + by^2 = 0, \quad x_1^2 + cx_2y + y^2 + ex_2^2, \quad x_1x_2 + x_2x_1 = 0,$$

with (b, c) being $(0, 1)$ or $(1, 0)$.

(4) *If R contains three linearly independent elements of the form*

$$f_1 = x_1 \otimes x_2 + \xi_1,$$
$$f_2 = x_2 \otimes x_1 + \xi_2,$$
$$f_3 = x_2 \otimes x_2 + \xi_3,$$

where $\xi_1, \xi_2, \xi_3 \in W$, then $i + 1 = j$, and, up to a change of basis, the above relations are

$$x_1x_2 + x_1y - yx_1 = 0, \quad x_2x_1 + x_1y - yx_1 = 0, \quad x_2^2 + x_2y - yx_2 = 0.$$

(5) *Suppose $R \cap W = 0$ and R is not in the situation of part (4). Then $p = 2$, $i \neq j$, and R is of the following cases:*

(5a) $c_1 \in \Bbbk$,

$$x_1^2 + c_1(x_2y + yx_2) + y^2 = 0, \quad x_1x_2 + x_2x_1 = 0, \quad x_2^2 + x_2y + yx_2 = 0.$$

(5b) $e_1 \in \Bbbk$,

$$x_1^2 + x_2y + yx_2 + e_1y^2 = 0, \quad x_1x_2 + x_2x_1 = 0, \quad x_2^2 + y^2 = 0.$$

Proof (1) It follows from the proof of Lemma 4.5(1) by replacing x_3 by y.
(2) The first paragraph of the proof of Lemma 4.5(5) works well for this situation after replacing x_3 by y.

(3) Let f be a basis element in $R \cap W$ and write it as

$$f = ay \otimes x_1 + by \otimes x_2 + v \otimes y$$

for some $v \in V$. Suppose that $a \neq 0$. Then $u \cdot f = ay \otimes x_2 + w \otimes y$ for some $w \in V$. So $\{f, u \cdot f\}$ are linearly independent elements in $R \cap W$, yielding a contradiction. Therefore $a = 0$, and in this case we may assume that $b = 1$ and

$$f = y \otimes x_2 + a_1 x_1 \otimes y - q x_2 \otimes y + c_1 y \otimes y$$

for some scalars a_1, q, c_1. Note that

$$u \cdot f = a_1 x_2 \otimes y$$

which must be zero as $\dim_{\Bbbk} R \cap W = 1$. Therefore $a_1 = 0$ and we obtain the first relation of T,

$$f_1 = y \otimes x_2 - q x_2 \otimes y + c_1 y \otimes y$$

where $c_1 \neq 0$ only if $i + 1 = j$.

Now let $g \in R \setminus W$ and write it as

$$g = a_{11} x_1 \otimes x_1 + a_{12} x_1 \otimes x_2 + a_{21} x_2 \otimes x_1 + a_{22} x_2 \otimes x_2 + \phi_0$$

where $\phi_0 \in W$. By a computation, $u^2 \cdot g = 2a_{11} x_2 \otimes x_2 + \phi_1$ where $\phi_1 \in W$. By Lemma 4.7, $u^2 \cdot g = 0$. Thus $2a_{11} = 0$. Then we have the following two cases to consider.

Case 1: $a_{11} = 0$. In this case $u \cdot g = (a_{12} + a_{21}) x_2 \otimes x_2 + \phi_2$ where $\phi_2 \in W$. Since $\dim_{\Bbbk} R/(R \cap W) = 2$, either a_{12} or a_{21} is nonzero. We may assume $a_{12} = 1$ by symmetry. If $a_{12} + a_{21} \neq 0$, we have two other relations

$$f_2 = x_1 \otimes x_2 + a_{21} x_2 \otimes x_1 + a x_1 \otimes y + b y \otimes x_1 + c x_2 \otimes y + d y \otimes y,$$
$$(1 + a_{21})^{-1} u \cdot f_2 = f_3 = x_2 \otimes x_2 + (1 + a_{21})^{-1} a x_2 \otimes y + (1 + a_{21})^{-1} b y \otimes x_2.$$

The subalgebra of T generated by x_2 and y subject to relations f_1 and f_3 is not a domain. This contradicts the fact that T is a domain. Therefore $a_{12} + a_{21} = 0$, or equivalently, $a_{12} = 1$ and $a_{21} = -1$. Now we have three relations of the form

$$\begin{aligned} f_1 &= y \otimes x_2 - q x_2 \otimes y + c_1 y \otimes y, \\ f_2 &= x_1 \otimes x_2 - x_2 \otimes x_1 + a_2 x_1 \otimes y + b_2 y \otimes x_1 + c_2 x_2 \otimes y + d_2 y \otimes y, \\ f_3 &= x_2 \otimes x_2 + a_3 x_1 \otimes y + b_3 y \otimes x_1 + c_3 x_2 \otimes y + d_3 y \otimes y. \end{aligned}$$

Since T is a domain, it may not have two relations only involving x_2 and y. Thus $a_3 x_1 \otimes y + b_3 y \otimes x_1 \neq 0$. This implies that $u \cdot f_3 = a_3 x_2 \otimes y + b_3 y \otimes x_2 \neq 0$. Therefore $u \cdot f_3$ and f_1 are linearly dependent. Replacing y by $b_3 y$, we may assume that $b_3 = 1$. Then $a_3 = -q$ and $c_1 = 0$. Similarly, we can get $a_2 = -q b_2$. After rearranging, using the fact that R is a U-module, we have $j = i + 2$ and

$$\begin{aligned} f_1 &= y \otimes x_2 - q x_2 \otimes y, \\ f_2 &= x_1 \otimes x_2 - x_2 \otimes x_1 + c_2 x_2 \otimes y + d_2 y \otimes y, \\ f_3 &= x_2 \otimes x_2 + (y \otimes x_1 - q x_1 \otimes y) + d_3 y \otimes y \end{aligned}$$

where $c_2 \neq 0$ or $d_3 \neq 0$ only if $p = 2$ and $d_2 \neq 0$ only if $p = 3$. Finally by resolving the overlap ambiguity of $(x_1 x_2) y = x_1 (x_2 y)$ with order $y < x_2 < x_1$ (details are omitted, but similar to one given in the proof of Lemma 4.5(4)), we obtain that $q = 1$. So we obtain part (3a). In this case it is easy to see that T is an Ore extension $\Bbbk[x_2, y][x_1; \delta]$.

Case 2: $a_{11} \neq 0$ (so we may assume that $a_{11} = 1$) and $p = 2$. Let $f_2 = (2i + 1 - w) \cdot g$ and $f_3 = u \cdot f_2$, we have

$$f_2 = x_1 \otimes x_1 + ax_1 \otimes y + by \otimes x_1 + cx_2 \otimes y + dy \otimes y + ex_2 \otimes x_2,$$
$$f_3 = x_1 \otimes x_2 + x_2 \otimes x_1 + ax_2 \otimes y + by \otimes x_2,$$

where $a \neq 0$ or $b \neq 0$ only if $i = j$. Replacing x_1 by $x_1 + by$, we can assume that $b = 0$. So we have three relations

$$f_1 = y \otimes x_2 - qx_2 \otimes y + c_1 y \otimes y,$$
$$f_2 = x_1 \otimes x_1 + ax_1 \otimes y + cx_2 \otimes y + dy \otimes y + ex_2 \otimes x_2,$$
$$f_3 = x_1 \otimes x_2 + x_2 \otimes x_1 + ax_2 \otimes y,$$

where $c_1 \neq 0$ only if $i \neq j$ and $a \neq 0$ only if $i = j$.

If $i = j$, then $c_1 = 0 = c$. In this case we are exactly in the situation of Lemma 4.8(2). By Lemma 4.8(2), $a = 0$, $q = 1$ and $d \neq 0$ (and we can assume $d = 1$ by changing a basis element). Therefore we obtain (3b) by setting $d = 1$.

If $i \neq j$, then $a = 0$. In this case we are exactly in the situation of Lemma 4.8(3). By Lemma 4.8(3), $q = 1$, $d \neq 0$, $c_1 c = 0$. By setting $d = 1$ and renaming c_1 to b and changing basis elements if necessary, we obtain (3c).

(4) Writing ξ_i out explicitly, we have the following three linearly independent elements in R

$$f_1 = x_1 \otimes x_2 + a_1 x_1 \otimes y + b_1 y \otimes x_1 + c_1 x_2 \otimes y + d_1 y \otimes x_2 + e_1 y \otimes y,$$
$$f_2 = x_2 \otimes x_1 + a_2 x_1 \otimes y + b_2 y \otimes x_1 + c_2 x_2 \otimes y + d_2 y \otimes x_2 + e_2 y \otimes y,$$
$$f_3 = x_2 \otimes x_2 + a_3 x_1 \otimes y + b_3 y \otimes x_1 + c_3 x_2 \otimes y + d_3 y \otimes x_2 + e_3 y \otimes y.$$

Using the fact that $R \cap W = 0$, it is easy to see that

$$u \cdot f_1 = f_3, \quad u \cdot f_2 = f_3, \quad u \cdot f_3 = 0. \tag{9}$$

Similarly, we have

$$(2i + 1 - w) \cdot f_1 = (2i + 1 - w) \cdot f_2 = (2i + 2 - w) \cdot f_3 = 0.$$

By Eq. 9, we obtain that $a_1 = a_2 = c_3$, $b_1 = b_2 = d_3$, $a_3 = b_3 = e_3 = 0$. By part (1), both c_3 and d_3 are nonzero. By using $(2i + 2 - w) \cdot f_3 = 0$, we obtain that $i + 1 = j$. Under this assumption, $(2i + 1 - w) \cdot f_1 = (2i + 1 - w) \cdot f_2 = 0$ imply that

$$c_1 = d_1 = e_1 = 0 = c_2 = d_2 = e_2.$$

Therefore we have the following relations, after setting $a = a_1$ and $b = b_1$,

$$f_1 = x_1 \otimes x_2 + ax_1 \otimes y + by \otimes x_1,$$
$$f_2 = x_2 \otimes x_1 + ax_1 \otimes y + by \otimes x_1,$$
$$f_3 = x_2 \otimes x_2 + ax_2 \otimes y + by \otimes x_2,$$

where $j = i + 1$. Using these relations, one can check that T contains a subalgebra $B := \Bbbk[x_2][y; \sigma_1, \delta]$ and $T = \sum_{n \geq 0} B x_1^n = \sum_{n \geq 0} x_1^n B$. Then T is AS regular if and only if T is an iterated Ore extension of the form $\Bbbk[x_2][y; \sigma_1, \delta_1][x_1; \sigma_2]$ for some σ_1, σ_2 and δ_1 if and only if $b = -a \neq 0$ (details are omitted). The assertion follows by setting new y to be ay.

(5) Since R is not in the situation of part (4), R contains an element $g = x_1 \otimes x_1 +$ other terms. Then $u^2 \cdot g = 2(x_2 \otimes x_2) \neq 0$. If $p > 2$, then $u^2 \cdot g \neq 0$, contradicts Lemma 4.7. Therefore $p = 2$.

For the rest of the proof, $p=2$. We have an injective U-morphism

$$R' := (R+W)/W \to (V\otimes V)/W \cong M(2,i)\otimes M(2,i),$$

so we can consider R' as a submodule of $M(2,i)\otimes M(2,i)$. By Lemma 4.6, R' contains elements $z_3 := -x_1\otimes x_2 + x_2\otimes x_1$ and $z_4 = x_2\otimes x_2$. Since R' has dimension three, it must has basis elements $x_1\otimes x_1$, z_3 and z_4. Lifting these elements from R' to R, we obtain three linearly independent elements in R (using the fact $p=2$):

$$\begin{aligned}
f_1 &= x_1\otimes x_1 + a_1x_1\otimes y + b_1y\otimes x_1 + c_1x_2\otimes y + d_1y\otimes x_2 + e_1y\otimes y,\\
f_2 &= x_1\otimes x_2 + x_2\otimes x_1 + a_2x_1\otimes y + b_2y\otimes x_1 + c_2x_2\otimes y + d_2y\otimes x_2 + e_2y\otimes y,\\
f_3 &= x_2\otimes x_2 + a_3x_1\otimes y + b_3y\otimes x_1 + c_3x_2\otimes y + d_3y\otimes x_2 + e_3y\otimes y.
\end{aligned}$$

Under this setting, $R\cap W=0$ implies that

$$u\cdot f_1 = f_2, \quad u\cdot f_2 = 0, \quad u\cdot f_3 = 0. \tag{10}$$

Further we have,

$$(2i-w)\cdot f_1 = (2i+1-w)\cdot f_2 = (2i+2-w)\cdot f_3 = 0. \tag{11}$$

Using Eqs. 10 and 11, one has

$$a_2=b_2=e_2=a_3=b_3=0, a_1=c_2, b_1=d_2,$$

and

$$(i-j)a_1=(i-j)b_1=(i-j+1)c_1=(i-j+1)d_1=(i-j+1)c_3=(i-j+1)d_3=0.$$

If $i=j$, then

$$\begin{aligned}
f_1 &= x_1\otimes x_1 + a_1x_1\otimes y + b_1y\otimes x_1 + e_1y\otimes y,\\
f_2 &= x_1\otimes x_2 + x_2\otimes x_1 + a_1x_2\otimes y + b_1y\otimes x_2,\\
f_3 &= x_2\otimes x_2 + e_3y\otimes y.
\end{aligned}$$

Since f_1 and f_3 are not of the form $v\otimes w$ for some $v,w\in V$, $e_3\neq 0$ (so we can assume that $e_3=1$) and $e_1\neq a_1b_1$. Replacing x_1 by x_1+b_1y (which will not change the U-module structure of V), we may assume that $b_1=0$; and up to a rescaling, $e_1=1$. Thus we have $i=j$ and

$$\begin{aligned}
f_1 &= x_1\otimes x_1 + ax_1\otimes y + y\otimes y,\\
f_2 &= x_1\otimes x_2 + x_2\otimes x_1 + ax_2\otimes y,\\
f_3 &= x_2\otimes x_2 + y\otimes y.
\end{aligned}$$

We claim that this is not AS regular. To see this, consider its Koszul dual B, which is generated by $y_1 := x_1^*$, $y_2 := x_2^*$, $y_3 := y^*$ subject to relations

$$y_1^2+y_2^2+y_3^2=0, \quad ay_1^2+y_1y_3=0, \quad y_3y_1=0,$$

and

$$y_1y_2+y_2y_1=0, \quad ay_1y_2+y_2y_3=0, \quad y_3y_2=0.$$

Now it is easy to check that $y_3^2(\Bbbk y_1+\Bbbk y_2+\Bbbk y_3)=0$ which implies that B is not of finite dimensional and Frobenius. By [37, Proposition 5.10] or [31, Corollary D], T is not AS regular, yielding a contradiction.

If $i\neq j$, then

$$\begin{aligned}
f_1 &= x_1\otimes x_1 + c_1x_2\otimes y + d_1y\otimes x_2 + e_1y\otimes y,\\
f_2 &= x_1\otimes x_2 + x_2\otimes x_1,\\
f_3 &= x_2\otimes x_2 + c_3x_2\otimes y + d_3y\otimes x_2 + e_3y\otimes y.
\end{aligned}$$

By Lemma 4.8(1), we have

$$d_1 = c_1,\ d_3 = c_3,\ c_3^2 - e_3 \neq 0,\ c_1e_3 - c_3e_1 \neq 0.$$

If $e_3 = 0$, then $e_1 \neq 0$ and $c_3 \neq 0$. Up to a change of basis, the relations of T are

$$\begin{aligned} x_1^2 + c_1(x_2y + yx_2) + y^2 &= 0,\\ x_1x_2 + x_2x_1 &= 0,\\ x_2^2 + x_2y + yx_2 &= 0. \end{aligned}$$

If $e_3 \neq 0$, then, up to a change of basis (by letting new y be $\alpha x_1 + \beta y$ for some scalars α, β), we may assume that $e_3 = 1$ and $c_3 = 0$, then the relations of T are

$$\begin{aligned} x_1^2 + x_2y + yx_2 + e_1y^2 &= 0,\\ x_1x_2 + x_2x_1 &= 0,\\ x_2^2 + y^2 &= 0. \end{aligned}$$

This finishes the proof. □

Lemma 4.10 *Retain the hypothesis in Lemma 4.7. If $y \in S_j \subseteq V$ is a normal element in T, then one of the following holds.*

(1) $(i-j)(2i+1-2j) \neq 0$ *in $\mathbb{Z}_p$, and the relations in T are*

$$yx_1 + ax_1y = 0, \quad yx_2 + ax_2y = 0, \quad x_2x_1 - x_1x_2 = 0$$

for $a \neq 0$.

(2) $2i + 1 - 2j = 0$ *(then $i \neq j$) in $\mathbb{Z}_p$, and the relations in T are*

$$yx_1 + ax_1y = 0, \quad yx_2 + ax_2y = 0, \quad x_2x_1 - x_1x_2 + \epsilon y^2 = 0$$

where $a \neq 0$, $\epsilon = 0$ or 1 and $\epsilon(a^2 - 1) = 0$.

(3) $i = j$ *(then $2i + 1 - 2j \neq 0$) in $\mathbb{Z}_p$, and the relations in T are*

$$yx_1 + ax_1y + by^2 = 0, \quad yx_2 + ax_2y = 0, \quad x_2x_1 - x_1x_2 + \epsilon x_2y$$

where $a \neq 0$, $\epsilon = 0$ or 1 and $(a+1)(b-\epsilon) = 0$.

Proof First of all, every algebra on the list can be written as an iterated Ore extensions of the form $\Bbbk[x_2][y;\sigma_1][x_1,\sigma_1,\delta_2]$. So these are noetherian Koszul AS regular.

Since y is normal and $\Bbbk y$ is a left U-module, the U-action on T induces naturally a U-action on $Z := T/(y)$ where Z is a noetherian Koszul AS regular algebra of global dimension two. Note that the degree 1 piece of Z is $Z_1 = M(2,i)$, whence the U-action on Z is inner-faithful. By Proposition 4.2(1), Z is commutative, so $x_2x_1 - x_1x_2 = 0$ in Z.

Let R be the relation space of T. By the previous paragraph, one can show that R has a basis elements of the form

$$\begin{aligned} r_1 &= yx_1 + a_1x_1y + b_1x_2y + c_1y^2,\\ r_2 &= yx_2 + a_2x_1y + b_2x_2y + c_2y^2,\\ r_3 &= x_2x_1 - x_1x_2 + a_3x_1y + b_3x_2y + c_3y^2. \end{aligned}$$

By the U-action on the basis elements and the fact $\Delta(u) = u \otimes 1 + 1 \otimes u$, we have

$$\begin{aligned} u \cdot r_1 &= yx_2 + a_1 x_2 y, \\ u \cdot r_2 &= a_2 x_2 y, \\ u \cdot r_3 &= a_3 x_2 y. \end{aligned}$$

Since $u \cdot R \subseteq R$, we obtain that

$$a_2 = a_3 = c_2 = 0, \quad a_1 = b_2.$$

Similarly, using the above equations and easy computations, we have

$$\begin{aligned} (i + j - w) \cdot r_1 &= -b_1 x_2 y + (i - j) c_1 y^2, \\ (i + j + 1 - w) \cdot r_2 &= 0, \\ (2i + 1 - w) \cdot r_3 &= (i - j) b_3 x_2 y + (2i + 1 - 2j) c_3 y^2. \end{aligned}$$

Therefore

$$b_1 = 0$$

and

$$(i - j) c_1 = 0, \tag{12}$$
$$(i - j) b_3 = 0, \tag{13}$$
$$(2i + 1 - 2j) c_3 = 0. \tag{14}$$

Now we have three relations of the form

$$\begin{aligned} r_1 &= yx_1 + a_1 x_1 y + c_1 y^2, \\ r_2 &= yx_2 + a_1 x_2 y, \\ r_3 &= x_2 x_1 - x_1 x_2 + b_3 x_2 y + c_3 y^2. \end{aligned}$$

with coefficients satisfying (12)-(14).

Similar to the process of resolving the overlap ambiguity, we calculate

$$\begin{aligned} y(x_2 x_1) &= y[x_1 x_2 - b_3 x_2 y - c_3 y^2] \\ &= (yx_1) x_2 - b_3 (yx_2) y - c_3 y^3 \\ &= (-a_1 x_1 y - c_1 y^2) x_2 - b_3 (-a_1 x_2 y) y - c_3 y^3 \\ &= a_1^2 x_1 x_2 y - a_1^2 c_1 x_2 y^2 + a_1 b_3 x_2 y^2 - c_3 y^3 \\ (yx_2) x_1 &= -a_1 (x_2 y) x_1 = -a_1 x_2 (yx_1) \\ &= -a_1 x_2 [-a_1 x_1 y - c_1 y^2] \\ &= a_1^2 (x_2 x_1) y + a_1 c_1 x_2 y^2 \\ &= a_1^2 [x_1 x_2 - b_3 x_2 y - c_3 y^2] y + a_1 c_1 x_2 y^2 \\ &= a_1^2 x_1 x_2 y - a_1^2 b_3 x_2 y^2 + a_1 c_1 x_2 y^2 - a_1^2 c_3 y^3. \end{aligned}$$

Since $\{x_1^i x_2^j y^k \mid i, j, k \geq\}$ is a $\Bbbk$-linear basis, we have

$$c_3 (a_1^2 - 1) = 0 \tag{15}$$
$$-a_1^2 c_1 + a_1 b_3 = -a_1^2 b_3 + a_1 c_1. \tag{16}$$

Since R does not contain an element of the form $v \otimes w$ for some $v, w \in V$, $a_1 \neq 0$. Now the system of Eqs. 12-16 has the following solutions:

(a) $(i-j)(2i+1-2j) \neq 0$ in $\mathbb{Z}_p$, $r_1 = yx_1 + ax_1y$, $r_2 = yx_2 + ax_2y$, $r_3 = x_2x_1 - x_1x_2$. This is case (1).
(b) $2i+1-2j = 0$ (then $i \neq j$) in $\mathbb{Z}_p$, $r_1 = yx_1 + ax_1y$, $r_2 = yx_2 + ax_2y$, $r_3 = x_2x_1 - x_1x_2 + cy^2$ where $c(a^2-1) = 0$. Replacing y by $\sqrt{c}y$, we obtain the case (2).
(c) $i = j$ (then $2i+1-2j \neq 0$) in $\mathbb{Z}_p$, $r_1 = yx_1 + ax_1y + cy^2$, $r_2 = yx_2 + ax_2y$, $r_3 = x_2x_1 - x_1x_2 + bx_2y$ where $(a+1)(b-c) = 0$. This is case (3) after rescaling.

This finishes the proof. □

Proof of Theorem 1.5 First of all, it is routine to check that all algebras in Theorem 1.5 are noetherian connected graded Koszul AS regular of global dimension three, as we did in the proofs of Lemmas 4.5, 4.8, 4.9 and 4.10.

If T is commutative, we only need to specify the U-action on $V := T_1$. There are two cases: either $V = M(3,i)$ or $V = M(2,i) \oplus S_j$. This is part (1).

Next we suppose that T is not commutative. If $V = M(3,i)$, then, by Lemma 4.5, only part (2) can occur.

For the rest of the proof, we assume that T is not commutative and $V = M(2,i) \oplus S_j$. We use $\dim_{\Bbbk} R \cap W$ to classify the pairs (U, T).

If $\dim_{\Bbbk} R \cap W = 2$, by Lemma 4.9(2), y is a normal element. By Lemma 4.10, we obtain parts (4,5,6). If $\dim_{\Bbbk} R \cap W = 1$, by Lemma 4.9(3) we obtain parts (7,8,9). If $\dim_{\Bbbk} R \cap W = 0$, by Lemma 4.9(4,5), we obtain parts (3) and (10). This finishes the proof. □

5 Easy Observations

From the limited information in the global dimension 2 case, we see differences between the semisimple and non-semisimple actions.

Observation 5.1 The following remarks demonstrate differences between semisimple and non-semisimple Hopf actions on AS regular algebras. Suppose we are in the setting of Proposition 4.2.

(1) Let p be an odd prime and let $i = \frac{p-1}{2}$. Let $V = M(2,i)$. Then the relation of $T = \Bbbk[V]$ is of the form $r = x_1 \otimes x_2 - x_2 \otimes x_1 \in V \otimes V$. By definition, $w \cdot x_j = (i+j-1)x_j$ for $j = 1, 2$. Hence

$$\begin{aligned} w \cdot r &= (w \otimes 1 + 1 \otimes w)(x_1 \otimes x_2 - x_2 \otimes x_1) \\ &= (2i+1)(x_1 \otimes x_2 - x_2 \otimes x_1) \\ &= p(x_1 \otimes x_2 - x_2 \otimes x_1) = pr = 0. \end{aligned}$$

It is clear that $u \cdot r = 0$. Therefore the U-action on $\Bbbk r$ is trivial. If we use the statement in [7, Theorem 2.1] as our definition of trivial homological determinant, then the U-action on T has trivial homological determinant. By Proposition 4.2(2), T^U is AS regular. Therefore [7, Theorem 0.6] fails without H being semisimple.

(2) Suppose the U-action on T has trivial homological determinant as in part (1). Since T is a free module over T^U, $\mathrm{End}_{T^U}(T)$ is a matrix algebra over T^U. So $T\#U$ is not isomorphic to $\mathrm{End}_{T^U}(T)$, consequently, [8, Theorem 0.3] fails without H being semisimple.

(3) Suppose the U-action on T has trivial homological determinant as in part (1). Note that $\mathrm{CMreg}(T) = 0$ and

$$\mathrm{CMreg}(T^U) = \mathrm{CMreg}(\Bbbk[x_1^p, x_2^p]) = 2p - 2 \neq 0,$$

where the definition of Castelnuovo-Mumford regularity, denoted CMreg, can be found in [26, Definition 2.9]. This example shows that [26, Lemma 2.15(2)] fails without H being semisimple.

(4) In Proposition 4.2(2), T^U is AS regular. We are wondering if there is a version of the Shephard-Todd-Chevalley Theorem for non-semisimple Hopf actions on AS regular algebras, even if U does not have any nontrivial grouplike element. In the case of Proposition 4.2(2a), we have that $\dim U = p^2$ and that the product of degrees of generators is p. Hence the conclusion of both [26, Proposition 1.8(4)] and [18, Conjecture 0.3] fails. Since U is not semisimple, we are wondering if U is still qualified to be called a *reflection Hopf algebra*, see [25, Definition 3.2].

(5) When H is semisimple (with mild hypotheses), by [16, Corollary 3.10], the rank of T as a left T^H-module is equal to $\dim_{\Bbbk} H$. In Proposition 4.2(2a), the rank of T as a left T^U-module is p, while $\dim_{\Bbbk} U = p^2$. Therefore [16, Corollary 3.10] fails without H being semisimple.

(6) When H is semisimple, acting on an AS regular algebra T inner-faithfully, T is usually a left free H-module. In the commutative case see, for example, [36, Theorem 6.19(2,3)]. In the noncommutative case, this was verified for many examples, see [18, 19]. However, in the non-semisimple case, T is never a free H-module as ${}_H T_0 \cong {}_H\Bbbk$ cannot be projective.

(7) Consider the case when $p = 2$ and $i = 1$ in Proposition 4.2(2a). Then it is clear that $T_0 = \Bbbk \cong S_0$, $T_1 = \Bbbk x_1 + \Bbbk x_2 \cong M(2, 1)$, and $T_2 = (\Bbbk x_1 + \Bbbk x_2)x_2 + \Bbbk x_1^2 \cong M(2, 1) \oplus S_0$. For $i \geq 3$, $T_i \cong \begin{cases} M(2,1)^m & i = 2m, \\ M(2,1)^m \oplus S_0 & i = 2m+1. \end{cases}$ Therefore neither S_1 nor $M(2, 0)$ appears as a direct summand of T. Further $\mathrm{ann}_U(T) = \Bbbk wu \neq 0$. So the U-action on T is not faithful, though it is inner-faithful. However, in the semisimple case, an inner-faithful H-action on an AS regular algebra is expected to be faithful.

Next we verify the claim made in Remark 1.7(3).

Lemma 5.2 *Let H be a Hopf algebra containing K as a Hopf subalgebra. Let H act on an algebra T inner-faithfully.*

(1) *Suppose every nonzero Hopf ideal of K contains a nonzero skew primitive element. Then the induced K-action on T is inner-faithful.*
(2) *If K is pointed, then the induced K-action on T is inner-faithful.*
(3) *If $K = U$, then the induced U-action on T is inner-faithful.*

Proof (1) If the K-action on T is not inner-faithful, then $x \cdot T = 0$ for some nonzero skew primitive in K. It is clear that x is also a skew primitive element in H. Let I be the ideal of H generated by x. Since x is a primitive element, I is a Hopf ideal of H. It is clear that $I \cdot T = 0$ as $x \cdot T = 0$. Therefore the H-action is not inner-faithful. The assertion follows.

(2) By [33, Corollary 5.4.7], every nonzero Hopf ideal of the pointed Hopf algebra K contains a nonzero skew primitive. The assertion follows from part (1).

(3) This is a special case of (2) as U is pointed (in fact, connected).

□

Proof of Proposition 1.4 Let H act inner-faithfully on a noetherian Koszul AS regular algebra T of global dimension two. By Lemma 5.2, the induced U-action on T is inner-faithful. By Proposition 4.2(1), T is $\Bbbk[x_1, x_2]$. □

Let m be an integer ≥ 2. Let $\{p_{ij} \mid 1 \leq i < j \leq m\}$ be a set of nonzero scalars. Define $p_{ij} = p_{ji}^{-1}$ if $i > j$. Recall that the skew polynomial ring is defined to be

$$\Bbbk_{p_{ij}}[x_1, \cdots, x_m] = \frac{\Bbbk\langle x_1, \cdots, x_m\rangle}{(x_j x_i = p_{ij} x_i x_j, \ \forall\, i < j)}.$$

Proposition 5.3 *Suppose $p_{ij} \neq 1$ for all $i < j$. Let $T = \Bbbk_{p_{ij}}[x_1, \cdots, x_m]$ and let H be a Hopf algebra containing $\Bbbk[u]/(u^p)$, with u being primitive, as a Hopf subalgebra. Then there is no inner-faithful homogeneous H-action on T.*

Proof By Lemma 5.2(2), we may assume $H = \Bbbk[u]/(u^p)$ with u being primitive. Suppose to the contrary that there is an inner-faithful homogeneous H-action on T. By [24, Lemma 5.9(d)] and cocommutativity of H, H acts on the Koszul dual (equal to the Ext-algebra) of T, denoted by

$$B := \mathrm{Ext}^*_T(\Bbbk, \Bbbk) = \frac{\Bbbk\langle y_1, \cdots, y_m\rangle}{(y_j y_i = -p_{ij}^{-1} y_i y_j, \ \forall\, i < j, \, y_i^2, \ \forall\, i)}$$

where $y_i = x_i^*$ for each i.

Since u is primitive, u acts on B as a derivation. For every i, write $u(y_i) = \sum_j a_{ji} y_j$. Then

$$\begin{aligned} 0 = u(y_i^2) &= (\sum_j a_{ji} y_j) y_i + y_i (\sum_j a_{ji} y_j) \\ &= \sum_j a_{ji} (y_j y_i + y_i y_j) = \sum_{j \neq i} a_{ji} (y_j y_i + y_i y_j) \\ &= \sum_{j \neq i} a_{ji} (1 - p_{ij}^{-1}) y_i y_j \end{aligned}$$

which implies that $a_{ji} = 0$ for all $j \neq i$. Equivalently, $u(y_i) = a_{ii} y_i$ for each i. Since $u^p = 0$, we obtain that $a_{ii}^p = 0$, or equivalently, $a_{ii} = 0$. Thus $u \cdot B_1 = 0$. Recall that $B_1 = (T_1)^*$. Hence $u \cdot T_1 = 0$. Consequently, $u \cdot T = 0$ and the H-action is not inner-faithful, yielding a contradiction. □

As an immediately consequence of Lemma 5.2(2) and Proposition 5.3, we have the following very simple universal non-existence result.

Corollary 5.4 *Let H be a nontrivial finite dimensional connected local Hopf algebra. Then there is no inner-faithful homogeneous H-action on $T = \Bbbk_{p_{ij}}[x_1, \cdots, x_m]$ where $p_{ij} \neq 1$ for all $i < j$.*

Proof Note that every nontrivial finite dimensional connected local Hopf algebra contains $\Bbbk[u]/(u^p)$. Therefore the assertion follows from Lemma 5.2(2) and Proposition 5.3. □

It is easy to check that $\Bbbk[u]/(u^p)$ acts inner-faithfully on both the polynomial ring $\Bbbk[x_1, x_2]$ and the Jordan plane $\Bbbk\langle x_1, x_2\rangle/(x_1x_2 - x_2x_1 - x_1^2)$.

Note that most of p^3-dimensional connected Hopf algebras in the classification [34, 35, 43, 44] contain a Hopf subalgebra of the form $\Bbbk[u]/(u^p)$. So Proposition 5.3 applies to these Hopf algebras. The next observation is a continuation of Remark 1.7(7).

Observation 5.5 Let H be a Hopf algebra containing U as a Hopf subalgebra. If H acts on a noetherian Koszul AS regular algebra T of global dimension three, then it induces naturally a U-action on T. This induced U-action must be inner-faithful by Lemma 5.2(3). As a consequence, T must be one of the algebras listed in Theorem 1.5. This basically gives a proof of Corollary 1.6.

(1) Theorem 1.5 is helpful for understanding explicit H-actions on T when H contains U. Even if H is of wild representation type, we can start from the list of T in Theorem 1.5 to work out all possible H-actions on T. This strategy is different from the one in the proof of Theorem 1.5.
(2) Suggested by Zhuang's result [48, Theorem 1.1], every pointed Hopf algebra over a field of positive characteristic is expected to contain one of the special Hopf algebras such as U, U_0, or Taft algebras and so on (this list should be short). By understanding actions on noetherian Koszul AS regular algebras of global dimension three under Hopf algebras from this list, we should get a pretty good picture of Hopf actions on Koszul AS regular algebras of global dimension three for all pointed Hopf algebra over a field of positive characteristic.
(3) The classification of p^3-dimensional connected or pointed Hopf algebras is undergoing in [34, 35, 43, 44]. It is known from their work that U appears as a Hopf subalgebra in many of their examples. Therefore Theorem 1.5 is really helpful for understanding the actions on noetherian Koszul AS regular algebras of global dimension three under the Hopf algebras listed in [34, 35, 43, 44].

6 Comments, Projects, and Remarks

In this final section we randomly collect some general comments, related projects, remarks and questions related to the projects in the previous sections.

As noted in [12, Remark 3.13(2)], the representations of the Drinfeld double of a Hopf algebra H are generally much more complicated than the representations of H. So we start with the following question.

Question 6.1 *What is the Green ring of the Drinfeld double of U?*

The work [10, 11] is closely related to this question.

The next few projects concern the Hopf actions of AS regular algebras of global dimension three or higher. A test case is the next.

Project 6.2 *Classify all U-actions on noetherian Koszul AS regular algebras of global dimension 4.*

The tensor decomposition in Theorem 1.2 is again the key to this project.

Project 6.3 *For all U-actions given in Theorem 1.5, work out the fixed subrings T^U and understand the connection between the U-actions and the properties of T^U.*

This would be a very interesting project as little is known about non-semisimple Hopf actions on AS regular algebras.

As noted before Taft algebras are similar to U in several aspects. So the following project seems doable.

Project 6.4 *Let H be a Taft algebra $H_n(q)$ of dimension n^2 over a field of arbitrary characteristic* [12, p. 767]. *Classify all inner-faithful H-actions on noetherian Koszul AS regular algebras of global dimension three.*

Unrelated to Hopf actions, the results in Section 3 are useful for the computation of Frobenius-Perron dimensions of representations of U.

The Frobenius-Perron dimension of an object in a semisimple finite tensor (or fusion) category was introduced by Etingof-Nikshych-Ostrik in 2005 [17]. Since then it has become an extremely useful invariant in the study of fusion categories and representations of semisimple (weak and/or quasi-)Hopf algebras. Recently, a new definition of Frobenius-Perron dimension was introduced in [13, 14] where the original definition was extended from an object in a semisimple finite tensor category to an endofunctor of any $\mathbb{k}$-linear category. In particular, it is defined for objects in non-semisimple $\mathbb{k}$-linear monoidal categories. This new Frobenius-Perron dimension has been computed in various cases, see [13, 14, 45–47].

Remark 6.5 (1) Xu computed Frobenius-Perron dimensions of representations of Taft algebras in [45], whose method can be used to give Frobenius-Perron dimension of representations of U for some small prime p. For example,

(a) let $p = 2$ and X be a finite dimensional U-module, then $\operatorname{fpdim}(X)$ equals to the $\mathbb{k}$-dimension of X;

(b) let $p = 3$ and $X = \sum_{i=0}^{2} \sum_{l=1}^{3} M(l, i)^{\oplus a_{li}}$, then

$$\operatorname{fpdim}(X) = \frac{1}{2}\left[(\alpha + \gamma) + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}\right]$$

where

$$\alpha = a_{10} + a_{11} + a_{12} + 2(a_{20} + a_{21} + a_{22}) + 3(a_{30} + a_{31} + a_{32}),$$
$$\beta = a_{12} + a_{21} + a_{22} + a_{30} + a_{31} + a_{32},$$
$$\gamma = a_{10} + a_{22} + a_{31}.$$

This formula is similar to [45, Proposition 7.1]. Since our Convention 2.4 is slightly different from the one used in [45], the formula does not match up with [45, Proposition 7.1] exactly.

(2) It would be interesting to work out a formula of $\operatorname{fpdim}(X)$ when $p = 5$.

A general algebra B is usually of wild representation type, then it is impossible to understand all indecomposable left B-modules. Sometime it is possible to work out all brick modules which are a special class of indecomposable module. A left B-module is called a *brick* if $\operatorname{Hom}_B(M, M) = \mathbb{k}$. Brick modules are fundamental objects in the study of

Frobenius-Perron dimension of endofunctors [13, 46]. Even when B is of wild representation type, it would be extremely helpful to understand all brick modules. The next remark is a consequence of Proposition 2.7.

Remark 6.6 Let H be a 2-step iterated Hopf Ore extension given in [5]. In parts (2,3), Let Z be the center of H and let M be a brick left H-module. Let $\mathfrak{m} = \mathrm{ann}_H(M)$.

(1) Suppose H is commutative. Then each brick left H-module is 1-dimensional and there is a one-to-one correspondence between brick H-modules and a closed point in Spec H.

(2) Suppose H is noncommutative and $d_0 = 0$ as in [5, Proposition 8.2]. Then

$$M \cong \begin{cases} \text{the unique 1-dimensional simple associated to } \mathfrak{m} & \text{if } \mathfrak{m} \text{ is not Azumaya,} \\ \text{the unique } p\text{-dimensional simple associated to } \mathfrak{m} & \text{if } \mathfrak{m} \text{ is Azumaya.} \end{cases}$$

(3) Suppose H is noncommutative and $d_0 \neq 0$ as in [5, Proposition 8.2]. Then

$$M \cong \begin{cases} \text{one of } p^2 \text{ indecomopsable modules associated to } \mathfrak{m} & \text{if } \mathfrak{m} \text{ is not Azumaya,} \\ \text{the unique } p\text{-dimensional simple associated to } \mathfrak{m} & \text{if } \mathfrak{m} \text{ is Azumaya.} \end{cases}$$

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