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OPTIMAL STOPPING WITH EXPECTATION CONSTRAINTS

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We analyze an optimal stopping problem with a series of inequality-type and equality-type expectation constraints in a general non-Markovian framework. We show that the optimal stopping problem with expectation constraints (OSEC) in an arbitrary probability setting is equivalent to the constrained problem in weak formulation (an optimization over joint laws of stopping rules with Brownian motion and state dynamics on an enlarged canonical space) and thus the OSEC value is independent of a specific probabilistic setup. Using a martingale-problem formulation, we make an equivalent characterization of the probability classes in weak formulation, which implies that the OSEC value function is upper semi-analytic. Then we exploit a measurable selection argument to establish a dynamic programming principle in weak formulation for the OSEC value function, in which the conditional expected costs act as additional states for constraint levels at the intermediate horizon.

1. Introduction. In this article, we study a continuous-time optimal stopping problem with a series of inequality-type and equality-type expectation constraints in a general non-Markovian framework.

Given a historical path $\mathbf{x}|_{[0,t]}$, let the state of the game $\mathcal{X}^{t,\mathbf{x}}_{\cdot}$ evolve according to some SDE on a probability space $(\mathcal{Q},\mathcal{F},\mathfrak{p})$ whose drift and diffusion coefficients depend on the past trajectories of the solution. The player decides an exercise time τ to maximize her expected reward while being subject to a series of constraints: for $i\in\mathbb{N}$, the expectation of some accumulative cost $\int_t^\tau g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr$ should not overpass certain level y_i and the expectation of some other accumulative cost $\int_t^\tau h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr$ should exactly hit certain level z_i. This optimal stopping problem with expectation constraints (OSEC for short), or optimization problem over constrained stopping times, has many applications in various economic, engineering and financial areas such as travel problem with fuel constraint, evaluation of American-type derivatives, quickest detection problem, etc.

Let $V(t,\mathbf{x},y,z)$ denote the OSEC value with $(y,z):=\big(\{y_i\},\{z_i\}\big)$. We aim to study the measurability of this value function and establish an associated dynamic programming principle (DPP) without imposing any continuity condition on reward and cost functions in time and state variables. Inspired by [30] and [31], we embed the constrained stopping rule τ together with the Brownian and state information into an enlarged canonical space $\overline{\Omega}$ and regard their joint distribution as a new type of controls. Then the optimization of the expected reward over constrained stopping times transforms into a maximal expectation of reward functional over a class $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ of probability measures on $\overline{\Omega}$ under which three canonical coordinates $(\overline{W},\overline{X},\overline{T})$ serve as Brownian motion, state process and constrained stopping rules respectively.

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One of our achievements is to show that the two optimization problems are equivalent: the value $V(t,\mathbf{x},y,z)$ of OSEC in strong formulation (i.e., on $\mathcal{Q}$) is equal to the value $\overline{V}(t,\mathbf{x},y,z)$ of OSEC in weak formulation (i.e., over $\overline{\Omega}$). This result indicates that the OSEC value is actually a robust value, independent of a specific probability model.

A dynamic programming principle of a stochastic optimization problem allows one to maximize/minimize the problem stage by stage in a backward recursive way. It requires the problem value function to be measurable so that one can do optimization at an intermediate horizon first. To show the measurability of the OSEC value functions, we construct a Polish space of stopping times (which is of independent interest) and exploit the martingale-problem formulation of [63] to describe the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ as a series of probabilistic tests on stochastic behaviors of the canonical coordinates of $\overline{\Omega}$. Under such countable characterization, the set-valued mapping $(t,\mathbf{x},y,z)\mapsto\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ has Borel-measurable graph and the OSEC value function $V=\overline{V}$ is thus upper semi-analytic in $(t,\mathbf{x},y,z)$.

In the next step we establish a DPP for $\overline{V}$ in weak formulation, which takes conditional expectations of the remaining costs as additional states for constraint levels at the intermediate horizon. For the subsolution side of this DPP, we use the regular conditional probability distribution to indicate that the probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$, $\forall\,(t,\mathbf{x},y,z)$ are stable under conditioning. For the supersolution side of the DPP, we employ a measurable selection theorem in the analytic-set theory to paste a class of locally $\varepsilon-$optimal probability measures. By the martingale-problem formulation again, the canonical coordinates $(\overline{W},\overline{X})$ are still Brownian motion and the state process under the pasted probability measure. Finally we make a delicate analysis to show that the third canonical coordinate $\overline{T}$ serves as a constrained stopping time under the pasted probability measure. To wit, the probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$'s are also stable under pasting (or concatenation).

Relevant Literature.

Since Arrow et al. [2] and Snell [58], the theory of (unconstrained) optimal stopping has been plentifully developed over decades. Expositions of this theory are presented in monographs [26, 57, 29, 37]. For the recent development of the optimal stopping under model uncertainty/non-linear expectations and the closely related controller-stopper-games, see [38, 39, 24, 27, 40, 55, 7, 8, 5, 23, 4, 28, 9, 47, 10, 11].

Kennedy [41] initiated the study of optimal stopping problem with expectation constraint. The author used a *Lagrange multiplier* method to reformulate a discrete-time optimal stopping problem with first-moment constraint as a minimax problem and showed that the optimal value of the dual problem is equal to that of the primal problem. Since then, the Lagrangian technique has been prevailing in research of OSEC (see e.g. [52, 42, 3, 66, 43, 65]) and has been applied to various economic/financial problems such as Markov decision processes with constrained stopping times [34, 33], mean-variance optimal control/stopping problem [48, 49], quickest detection problem [50], etc.

Recently, Ankirchner et al. [1] and Miller [44] took different approaches to optimal stopping problems for diffusion processes with expectation constraints by transforming them to stochastic optimization problems with martingale controls. The former characterizes the value function in terms of a Hamilton-Jacobi-Bellman equation and obtains a verification theorem, while the latter embeds the optimal stopping problem with first-moment constraint into a time-inconsistent (unconstrained) stopping problem. However, the authors only postulate dynamic programming principles for their corresponding problems. In contrast, we rigorously prove in this article a dynamic programming principle for the optimal stopping problem with expectation constraints.

In their study of a continuous-time stochastic optimization problem of controlled Markov processes, El Karoui, Huu Nguyen and Jeanblanc-Picqué [30] regarded joint laws of state and control processes as *control rules* on the product space of canonical state space and control space. Then they used a measurable selection theorem in the analytic-set theory to establish a DPP without assuming any regularity on the reward functional. Nutz & van Handel [46] and Neufeld & Nutz [45] came up with a similar idea to address a superheging problem under volatility uncertainty. They modeled the "uncertainty" by path-dependent classes of controlled-diffusion laws and explored the analytic measurability of these classes. Using the measurable selection techniques, the authors obtained DPP result in a form of time-consistency of a sub-linear expectation and they thus established a duality formula for the robust superhedging of measurable claims. The approach of [46, 45] was later extended to derive DPPs of various non-Markovian control problems, see [53] for a dual formulation of robust semi-static trading and its application to martingale optimal transportation and see [54] for stochastic control of a class of nonlinear kernels and its relation to second-order backward stochastic differential equations. Since the class of controlled-diffusion laws is naturally different from the class of stopping-time laws, the results of these works are not applicable to our optimal stopping problem with expectation constraints.

In [32, 31], El Karoui and Tan utilized the measurable selection argument to attain the DPP for a general stochastic control/stopping problem by embedding stopping times with controlled diffusions into an enlarged canonical space in the spirit of [30]. However, the probability class they considered in weak formulation is not suitable for optimal stopping with expectation constraints, see our Remark 3.4 for details. In this paper, we make a more accurate description of probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ in which the third canonical coordinate serves as some constrained stopping time. We construct a Polish space of stopping times and use it to show the Borel measurability of the graph $\big[\big[\overline{\mathcal{P}}\big]\big]$. We also verify the stability of probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ under conditioning and concatenation so that we can exploit measurable selection theorem to establish a DPP for the OSEC value function.

A closely related topic to our research is optimal stopping with constraint on the distribution of stopping times. Bayraktar and Miller [6] studied the problem of optimally stopping a Brownian motion with the restriction that the distribution of the stopping time must equal a given measure with finitely many atoms, and obtained a dynamic programming result which relates each of the sequential optimal control problems. Källblad [35] used measure-valued martingales to transform the distribution-constrained optimal stopping problem to a stochastic control problem and derived a DPP by measurable selection arguments. From the perspective of optimal transport, Beiglböck et al. [13] gave a geometric interpretation of optimal stopping times of a Brownian motion with distribution constraint.

As to the stochastic control problems with expectation constraints, Pfeiffer et al. [51] obtained a duality result by a Lagrange relaxation approach and Yu et al. [25] used the measurable selection argument to derive a DPP result. Moreover, for stochastic control problems with state constraints, stochastic target problems with controlled losses and related geometric DPP, see [18, 19, 21, 59, 60, 61, 22, 16, 20, 17].

The rest of the paper is organized as follows: Section 2 introduces the optimal stopping problem with expectation constraints in a generic probabilistic setting. Section 3 shows that the constrained optimal stopping problem can be equivalently embedded into an enlarged canonical space: i.e., the OSEC in strong formulation has the same value as the OSEC in weak formulation. In Section 4, we construct a Polish space of stopping times and use the martingale-problem formulation to make a countable characterization of the probability class in weak formulation, from which we deduce that the OSEC value function is upper semi-analytic. Then in Section 5, we utilize a measurable selection argument to establish a dynamic programming principle in weak formulation for the OSEC value function. We defer the proofs of our results to Section 6 and put some technical lemmata in the appendix.

We close this section by a description of our notation and a review of the martingale-problem formulation.

1.1. *Notation and Preliminaries.* Throughout this paper, let us denote $a^+ := a \vee 0$ and $a^- := (-a) \vee 0$ for any $a \in \mathbb{R}$. We set $\mathbb{Q}_+ := \mathbb{Q} \cap [0,\infty)$, $\mathbb{Q}_+^{2,<} := \big\{(s,r) \in \mathbb{Q}_+ \times \mathbb{Q}_+ : s < r\big\}$ and set $\Re := (-\infty,\infty]^{\mathbb{N}}$ as the product of countably many copies of $(-\infty,\infty]$. On $\mathbb{T} := [0,\infty]$ we define a metric $\rho_+(t_1,t_2) := \big|\arctan(t_1) - \arctan(t_2)\big|$, $\forall\, t_1,t_2 \in \mathbb{T}$ and consider the induced topology by ρ_+.

For a general topological space $\big(\mathbb{X}, \mathfrak{T}(\mathbb{X})\big)$, we denote its Borel sigma-field by $\mathscr{B}(\mathbb{X})$ and let $\mathfrak{P}(\mathbb{X})$ be the set of all probability measures on $\big(\mathbb{X}, \mathscr{B}(\mathbb{X})\big)$. Recall that a topological space $\mathbb{X}$ is called a *Borel space* if it is homeomorphic to a Borel subset of a complete separable metric space.

Let $n \in \mathbb{N}$. For any $x \in \mathbb{R}^n$ and $\delta \in (0,\infty)$, let $O_\delta(x)$ denote the open ball centered at x with radius δ and let $\overline{O}_\delta(x)$ be its closure. For any $x, \widetilde{x} \in \mathbb{R}^n$ we denote the usual inner product by $x \cdot \widetilde{x} := \sum_{i=1}^n x_i \widetilde{x}_i$, and for any $n \times n-$real matrices $A, \widetilde{A}$ we denote the Frobenius inner product by $A : \widetilde{A} := trace\big(A\widetilde{A}^T\big)$, where $\widetilde{A}^T$ is the transpose of $\widetilde{A}$. Let $\big\{\mathcal{E}_i^n\big\}_{i \in \mathbb{N}}$ be a countable subbase of the Euclidean topology $\mathfrak{T}(\mathbb{R}^n)$ on $\mathbb{R}^n$. Then $\mathscr{O}(\mathbb{R}^n) := \Big\{ \underset{i=1}{\overset{n}{\cap}} \mathcal{E}_{k_i}^n : \{k_i\}_{i=1}^n \subset \mathbb{N}\Big\} \cup \{\emptyset, \mathbb{R}^n\}$ forms a countable base of $\mathfrak{T}(\mathbb{R}^n)$ and thus $\mathscr{B}(\mathbb{R}^n) = \sigma\big(\mathscr{O}(\mathbb{R}^n)\big)$. We also set $\widehat{\mathscr{O}}(\mathbb{R}^n) := \underset{k \in \mathbb{N}}{\cup} \big(\mathbb{Q}_+ \times \mathscr{O}(\mathbb{R}^n)\big)^k$. For any $\varphi \in C^2(\mathbb{R}^n)$, let $D\varphi$ be its gradient, $D^2\varphi$ be its Hessian matrix and denote $D^0\varphi := \varphi$. For $i = 1, \cdots, n$, define $\varphi_i(x) := x_i$, $\forall\, x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We let $\mathfrak{C}(\mathbb{R}^n)$ be the collection of these coordinate functions and their products, i.e., $\mathfrak{C}(\mathbb{R}^n) := \{\varphi_i\}_{i=1}^n \cup \{\varphi_i \varphi_j\}_{i,j=1}^n$.

Let $(\Omega, \mathcal{F}, P)$ be a generic probability space. For subsets A_1, A_2 of Ω, we denote $A_1 \Delta A_2 := (A_1 \cap A_2^c) \cup (A_2 \cap A_1^c)$. For a random variable ξ on Ω with values in a measurable space $(\mathcal{Q}, \mathcal{G})$, we say ξ is $\mathcal{F}/\mathcal{G}-$measurable if its induced sigma-field $\xi^{-1}(\mathcal{G}) := \{\xi^{-1}(\mathcal{A}) : \forall\, \mathcal{A} \in \mathcal{G}\}$ is included in $\mathcal{F}$. For a sub-sigma-field $\mathfrak{F}$ of $\mathcal{F}$, define $\mathscr{N}_P(\mathfrak{F}) := \big\{\mathcal{N} \subset \Omega : \mathcal{N} \subset A \text{ for some } A \in \mathfrak{F} \text{ with } P(A) = 0\big\}$, which collects all $P-$null sets with respect to $\mathfrak{F}$. For two sub-sigma-fields $\mathfrak{F}_1, \mathfrak{F}_2$ of $\mathcal{F}$, we denote $\mathfrak{F}_1 \vee \mathfrak{F}_2 := \sigma(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. Let $t \in [0,\infty)$. For a filtration $\mathbf{F} = \{\mathcal{F}_s\}_{s \in [t,\infty)}$ of $\mathcal{F}$, we set $\mathcal{F}_\infty := \sigma\Big(\underset{s \in [t,\infty)}{\cup} \mathcal{F}_s\Big)$ and refer to filtration $\mathbf{F}^P = \big\{\mathcal{F}_s^P := \sigma\big(\mathcal{F}_s \cup \mathscr{N}_P(\mathcal{F}_\infty)\big)\big\}_{s \in [t,\infty)}$ as the $P-$augmentation of $\mathbf{F}$. For a process $X = \{X_s\}_{s \in [t,\infty)}$ on Ω with values in a topological space, denote its raw filtration by $\mathbf{F}^X = \big\{\mathcal{F}_s^X := \sigma(X_r; r \in [t,s])\big\}_{s \in [t,\infty)}$ and denote the $P-$augmentation of $\mathbf{F}^X$ by $\mathbf{F}^{X,P} = \big\{\mathcal{F}_s^{X,P} := \sigma\big(\mathcal{F}_s^X \cup \mathscr{N}_P(\mathcal{F}_\infty^X)\big)\big\}_{s \in [t,\infty)}$. We call X a continuous process if its paths are all continuous. When the time variable s of X has complicated form, we may write $X(s,\omega)$ as $X_s(\omega)$ for readability. By default, a Brownian motion $\{B_s\}_{s \in [t,\infty)}$ on $(\Omega, \mathcal{F}, P)$ is with respect to its raw filtration $\mathbf{F}^B$ unless stated otherwise.

Fix $d, l \in \mathbb{N}$. Let $\Omega_0 = \big\{\omega \in C([0,\infty); \mathbb{R}^d) : \omega(0) = 0\big\}$ be the space of all $\mathbb{R}^d-$valued continuous paths starting from $\mathbf{0}$, which is a Polish space under the topology of locally uniform convergence. Let P_0 be the Wiener measure on $\big(\Omega_0, \mathscr{B}(\Omega_0)\big)$, under which the canonical process $W = \{W_s\}_{s \in [0,\infty)}$ of Ω_0 is a $d-$dimensional standard Brownian motion. For any $t \in [0,\infty)$, $W_s^t := W_s - W_t$, $s \in [t,\infty)$ is also a Brownian motion on $\big(\Omega_0, \mathscr{B}(\Omega_0), P_0\big)$. Let $\Omega_X = C([0,\infty); \mathbb{R}^l)$ be the space of all $\mathbb{R}^l-$valued continuous paths endowed with the topology of locally uniform convergence. The function $\mathfrak{l}_1(t, \omega_0) := \omega_0(t \wedge \cdot)$ is continuous in $(t, \omega_0) \in [0,\infty) \times \Omega_0$ while the function

$$\mathfrak{l}_2(t, \omega_X) := \omega_X(t \wedge \cdot) \tag{1.1}$$

is continuous in $(t, \omega_X) \in [0,\infty) \times \Omega_X$.

Let $b\colon (0,\infty)\times\Omega_X \mapsto \mathbb{R}^l$ and $\sigma\colon (0,\infty)\times\Omega_X \mapsto \mathbb{R}^{l\times d}$ be two Borel-measurable functions such that for any $t\in(0,\infty)$ and any $\omega_X,\omega'_X\in\Omega_X$

$$\big|b(t,\omega_X)-b(t,\omega'_X)\big|+\big|\sigma(t,\omega_X)-\sigma(t,\omega'_X)\big|\le\kappa(t)\big\|\omega_X-\omega'_X\big\|_t \tag{1.2}$$

$$\text{and}\quad \int_0^t\big(|b(r,\mathbf{0})|^2+|\sigma(r,\mathbf{0})|^2\big)dr<\infty, \tag{1.3}$$

where $\kappa\colon(0,\infty)\mapsto(0,\infty)$ is some non-decreasing function and $\big\|\omega_X-\omega'_X\big\|_t:=\sup\limits_{s\in[0,t]}\big|\omega_X(s)-\omega'_X(s)\big|$. Under conditions (1.2) and (1.3), SDEs with coefficients (b,σ) are well-posed (see e.g. Theorem V.12.1 of [56]):

PROPOSITION 1.1. *Let $(\Omega,\mathcal{F},P)$ be a probability space. Given $t\in[0,\infty)$, let $\{B_s\}_{s\in[t,\infty)}$ be a $d-$dimensional Brownian motion with respect to a right-continuous complete filtration $\mathbf{F}=\{\mathcal{F}_s\}_{s\in[t,\infty)}$ on $(\Omega,\mathcal{F},P)$. For any $\mathbf{x}\in\Omega_X$, the SDE*

$$X_s=\mathbf{x}(t)+\int_t^s b(r,X_{r\wedge\cdot})dr+\int_t^s\sigma(r,X_{r\wedge\cdot})dB_r,\quad\forall\, s\in[t,\infty) \tag{1.4}$$

with initial condition $X\big|_{[0,t]}=\mathbf{x}|_{[0,t]}$ admits a unique strong solution $X^{t,\mathbf{x}}=\{X^{t,\mathbf{x}}_s\}_{s\in[0,\infty)}$ on $(\Omega,\mathcal{F},\{\mathcal{F}_s\}_{s\in[t,\infty)},P)$ $\big($i.e., $X^{t,\mathbf{x}}$ is an $\{\mathcal{F}_{s\vee t}\}_{s\in[0,\infty)}-$ adapted continuous process satisfying (1.4) and $P\big\{X^{t,\mathbf{x}}_s=\widetilde{X}^{t,\mathbf{x}}_s,\forall\, s\in[0,\infty)\big\}=1$ if $\big\{\widetilde{X}^{t,\mathbf{x}}_s\big\}_{s\in[0,\infty)}$ is another $\{\mathcal{F}_{s\vee t}\}_{s\in[0,\infty)}-$adapted continuous process satisfying (1.4)$\big)$.

Let ${}^oX^{t,\mathbf{x}}=\{{}^oX^{t,\mathbf{x}}_s\}_{s\in[0,\infty)}$ be the unique strong solution of (1.4) on $(\Omega,\mathcal{F},P)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0\big)$ with $(B,\mathbf{F})=\big(W^t,\mathbf{F}^{W^t,P_0}\big)$ and denote by $\mathscr{H}_o$ the collection of all $(-\infty,\infty]-$valued Borel-measurable functions ϕ on $(0,\infty)\times\Omega_X$ such that

$$E_{P_0}\Big[\int_t^\infty\phi^-(r,{}^oX^{t,\mathbf{x}}_{r\wedge\cdot})dr\Big]<\infty,\quad\forall\,(t,\mathbf{x})\in[0,\infty)\times\Omega_X.$$

Moreover, we take the conventions $\inf\emptyset:=\infty$, $\sup\emptyset:=-\infty$ and $(+\infty)+(-\infty)=-\infty$. In particular, on a measure space $(\Omega,\mathcal{F},\mathfrak{m})$, one can define the integral $\int_\Omega\xi\,d\mathfrak{m}:=\int_\Omega\xi^+\,d\mathfrak{m}-\int_\Omega\xi^-\,d\mathfrak{m}$ for any $[-\infty,\infty]-$valued $\mathcal{F}-$measurable random variable ξ on Ω.

1.2. *Review of Martingale-Problem Formulation of SDEs.* In this subsection, we consider a general measurable space $(\Omega,\mathcal{F})$. Let $\{B_s\}_{s\in[0,\infty)}$ be an $\mathbb{R}^d-$valued continuous process on Ω with $B_0=\mathbf{0}$ and let $X=\{X_s\}_{s\in[0,\infty)}$ be an $\mathbb{R}^l-$valued continuous process on Ω such that (B_s,X_s) is $\mathcal{F}-$measurable for each $s\in[0,\infty)$.

Let $t\in[0,\infty)$. We set $B^t_s:=B_s-B_t$, $\forall\, s\in[t,\infty)$ and define filtration $\mathbf{F}^t=\{\mathcal{F}^t_s\}_{s\in[t,\infty)}$ by $\mathcal{F}^t_s:=\mathcal{F}^{B^t}_s\vee\mathcal{F}^X_s=\sigma\big(B^t_r;r\in[t,s]\big)\vee\sigma(X_r;r\in[0,s])$, $\forall\, s\in[t,\infty)$. For any $\varphi\in C^2(\mathbb{R}^{d+l})$, define

$$M^t_s(\varphi):=\varphi\big(B^t_s,X_s\big)-\int_t^s\overline{b}\big(r,X_{r\wedge\cdot}\big)\cdot D\varphi\big(B^t_r,X_r\big)dr-\frac12\int_t^s\overline{\sigma}\,\overline{\sigma}^T\big(r,X_{r\wedge\cdot}\big):D^2\varphi(B^t_r,X_r)dr,$$

$\forall\, s\in[t,\infty)$, where $\overline{b}(r,\omega_X):=\begin{pmatrix}0\\ b(r,\omega_X)\end{pmatrix}\in\mathbb{R}^{d+l}$, $\overline{\sigma}(r,\omega_X):=\begin{pmatrix}I_{d\times d}\\ \sigma(r,\omega_X)\end{pmatrix}\in\mathbb{R}^{(d+l)\times d}$, $\forall\,(r,\omega_X)\in(0,\infty)\times\Omega_X$. Clearly, $\big\{M^t_s(\varphi)\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^t-$adapted continuous process. For any $n\in\mathbb{N}$ and $\mathfrak{a}\in\mathbb{R}^{d+l}$, set $\tau^t_n(\mathfrak{a}):=\inf\big\{s\in[t,\infty):|(B^t_s,X_s)-\mathfrak{a}|\ge n\big\}\wedge(t+n)$, which is an $\mathbf{F}^t-$stopping time. In particular, we denote $\tau^t_n(\mathbf{0})$ by τ^t_n.

In virtue of [63], we have the following martingale-problem formulation of SDEs with coefficients (b,σ) on Ω.

PROPOSITION 1.2. *Let $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$ and let P be a probability measure on $(\Omega,\mathcal{F})$ such that $P\big\{X_s=\mathbf{x}(s),\,\forall\,s\in[0,t]\big\}=1$. Then $\big\{M^t_{s\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^t-$adapted continuous process under P for any $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$ and the following statements are equivalent on $(\Omega,\mathcal{F},P)$:*
(i) The process B^t is a Brownian motion and $P\{X_s=X^{t,\mathbf{x}}_s,\,\forall\,s\in[0,\infty)\}=1$, where $\big\{X^{t,\mathbf{x}}_s\big\}_{s\in[0,\infty)}$ is the unique $\big\{\mathcal{F}^{B^t,P}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process solving SDE (1.4).
(ii) $\big\{M^t_{s\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^t-$martingale for any $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$.
(iii) $\big\{M^t_{s\wedge\tau^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^t-$martingale for any $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$.

Under either of these situations, one clearly has $P\big\{X^{t,\mathbf{x}}_s={}^oX^{t,\mathbf{x}}_s(B),\,\forall\,s\in[0,\infty)\big\}=1$ and $E_P\big[\int_t^\infty\phi^-(r,X^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]=E_{P_0}\big[\int_t^\infty\phi^-(r,{}^oX^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]<\infty$ for any $\phi\in\mathscr{H}_o$.

2. Optimal Stopping with Expectation Constraints. Let $(\mathcal{Q},\mathcal{F},\mathfrak{p})$ be a probability space equipped with a $d-$dimensional standard Brownian motion $\{\mathcal{B}_s\}_{s\in[0,\infty)}$.

Let $t\in[0,\infty)$. We set $\mathcal{B}^t_s:=\mathcal{B}_s-\mathcal{B}_t$, $\forall\,s\in[t,\infty)$, which is also a Brownian motion on $(\mathcal{Q},\mathcal{F},\mathfrak{p})$. For any $\mathbf{x}\in\Omega_X$, Proposition 1.1 shows that the SDE

$$\mathcal{X}_s=\mathbf{x}(t)+\int_t^s b(r,\mathcal{X}_{r\wedge\cdot})dr+\int_t^s\sigma(r,\mathcal{X}_{r\wedge\cdot})d\mathcal{B}_r,\quad\forall\,s\in[t,\infty)\tag{2.1}$$

with initial condition $\mathcal{X}\big|_{[0,t]}=\mathbf{x}|_{[0,t]}$ admits a unique strong solution $\mathcal{X}^{t,\mathbf{x}}=\big\{\mathcal{X}^{t,\mathbf{x}}_s\big\}_{s\in[0,\infty)}$ on $\big(\mathcal{Q},\mathcal{F},\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}},\mathfrak{p}\big)$ $\big($i.e., $\mathcal{X}^{t,\mathbf{x}}$ is the unique $\big\{\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process solving SDE (2.1)$\big)$. Let $\mathcal{S}_t$ collect all $[t,\infty]-$valued $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping times.

Let $f\in\mathscr{H}_o$, $\{g_i,h_i\}_{i\in\mathbb{N}}\subset\mathscr{H}_o$ and let $\pi:[0,\infty)\times\Omega_X\mapsto(-\infty,\infty]$ be a Borel-measurable function bounded from below by some $c_\pi\in(-\infty,0)$.

Given a historical path $\mathbf{x}|_{[0,t]}$, the state of the game then evolves along process $\{\mathcal{X}^{t,\mathbf{x}}_s\}_{s\in[t,\infty)}$. The player of the game need to select an exercise time $\tau\in\mathcal{S}_t$ to cease the game, at which she will receive an accumulative reward $\int_t^\tau f\big(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot}\big)\,dr$ plus a terminal reward $\pi\big(\tau,\mathcal{X}^{t,\mathbf{x}}_{\tau\wedge\cdot}\big)$ (both random rewards can take negative values). The player intends to maximize the expectation of her total wealth, but her choice τ is subject to a series of expectation constraints

$$E_{\mathfrak{p}}\Big[\int_t^\tau g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\Big]\le y_i,\quad E_{\mathfrak{p}}\Big[\int_t^\tau h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\Big]=z_i,\quad\forall\,i\in\mathbb{N}\tag{2.2}$$

for some $(y,z)=\big(\{y_i\}_{i\in\mathbb{N}},\{z_i\}_{i\in\mathbb{N}}\big)\in\Re\times\Re$. One can view $\int_t^\tau g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr$ or $\int_t^\tau h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr$ as certain accumulative costs. So the value of this optimal stopping problem with expectation constraints (OSEC for short) is

$$V(t,\mathbf{x},y,z):=\sup_{\tau\in\mathcal{S}_{t,\mathbf{x}}(y,z)}E_{\mathfrak{p}}\Big[\int_t^\tau f\big(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot}\big)dr+\mathbf{1}_{\{\tau<\infty\}}\pi\big(\tau,\mathcal{X}^{t,\mathbf{x}}_{\tau\wedge\cdot}\big)\Big],\tag{2.3}$$

where $\mathcal{S}_{t,\mathbf{x}}(y,z):=\big\{\tau\in\mathcal{S}_t:E_{\mathfrak{p}}\big[\int_t^\tau g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]\le y_i,\,E_{\mathfrak{p}}\big[\int_t^\tau h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]=z_i,\,\forall\,i\in\mathbb{N}\big\}$.

REMARK 2.1. Let $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$.
1) (finitely many constraints) For $i\in\mathbb{N}$, the constraint $E_{\mathfrak{p}}\big[\int_t^\tau g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]\le y_i$ holds for any $\tau\in\mathcal{S}_t$ if $y_i=\infty$, and the constraint $E_{\mathfrak{p}}\big[\int_t^\tau h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]=z_i$ holds for any $\tau\in\mathcal{S}_t$ if $\big(h_i(\cdot,\cdot),z_i\big)=(0,0)$.
1a) If we take $\big(y_i,h_i(\cdot,\cdot),z_i\big)=(\infty,0,0)$, $\forall\,i\in\mathbb{N}$, there is no expectation constraint at all.

1b) If one takes $y_i=\infty$, $\forall\, i\ge 2$ and $\big(h_i(\cdot,\cdot),z_i\big)=(0,0)$, $\forall\, i\in\mathbb{N}$, (2.2) reduces to a single constraint $E_{\mathfrak{p}}\big[\int_t^\tau g_1(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]\le y_1$. In addition, if $y_1\ge 0$, then $t\in\mathcal{S}_{t,\mathbf{x}}(y,\mathbf{0})$.
1c) If one takes $y_i=\infty$, $\forall\, i\in\mathbb{N}$ and $\big(h_i(\cdot,\cdot),z_i\big)=(0,0)$, $\forall\, i\ge 2$, (2.2) degenerates to $E_{\mathfrak{p}}\big[\int_t^\tau h_1(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]=z_1$.
1d) If we take $\big(y_i,h_i(\cdot,\cdot),z_i\big)=(\infty,0,0)$, $\forall\, i\ge 2$, (2.2) becomes a couple of constraints $E_{\mathfrak{p}}\big[\int_t^\tau g_1(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]\le y_1$ and $E_{\mathfrak{p}}\big[\int_t^\tau h_1(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]=z_1$.
1e) If we take $g_2=-g_1$, $y_2\ge -y_1$; $y_i=\infty$, $\forall\, i\ge 3$ and $\big(h_i(\cdot,\cdot),z_i\big)=(0,0)$, $\forall\, i\in\mathbb{N}$, (2.2) becomes a range constraint $-y_2\le E_{\mathfrak{p}}\big[\int_t^\tau g_1(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]\le y_1$.
2) (moment constraints) Let $i\in\mathbb{N}$, $a\in(0,\infty)$ and $q\in[1,\infty)$. If $g_i(s,\mathbf{x})=aqs^{q-1}$, $\forall\,(s,\mathbf{x})\in(0,\infty)\times\Omega_X$ $\big($resp. $h_i(s,\mathbf{x})=aqs^{q-1}$, $\forall\,(s,\mathbf{x})\in(0,\infty)\times\Omega_X\big)$, then the expectation constraint $E_{\mathfrak{p}}\big[\int_t^\tau g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]\le y_i$ $\big($resp. $E_{\mathfrak{p}}\big[\int_t^\tau h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]=z_i\big)$ specifies as a moment constraint $E_{\mathfrak{p}}\big[a(\tau^q-t^q)\big]\le y_i$ $\big($resp. $E_{\mathfrak{p}}\big[a(\tau^q-t^q)\big]=z_i\big)$.

We would like to study the measurability of value function V and derive a dynamic programming principle for V without imposing any continuity condition on functions f,π,g_i's and h_i's in time and state variables. Inspired by [30], we will use mapping $\omega\mapsto\big(\mathcal{B}_\cdot(\omega),\mathcal{X}^{t,\mathbf{x}}_\cdot(\omega),\tau(\omega)\big)$ to transfer the OSEC onto an enlarged canonical space and regard joint laws of $(\mathcal{B}_\cdot,\mathcal{X}^{t,\mathbf{x}}_\cdot,\tau)$ as a new type of controls.

3. Weak Formulation.

In this section, we study the optimal stopping problem with expectation constraints in a weak formulation or over an enlarged canonical space

$$\overline{\Omega}:=\Omega_0\times\Omega_X\times\mathbb{T}.$$

Clearly, $\overline{\Omega}$ is a Borel space under the product topology. Let $\mathfrak{P}\big(\overline{\Omega}\big)$ be the space of all probability measures on $\big(\overline{\Omega},\mathscr{B}(\overline{\Omega})\big)$ equipped with the topology of weak convergence, which is also a Borel space (see e.g. Corollary 7.25.1 of [14]). For any $\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big)$, set $\mathscr{B}_{\overline{P}}(\overline{\Omega}):=\sigma\big(\mathscr{B}(\overline{\Omega})\cup\mathscr{N}_{\overline{P}}(\mathscr{B}(\overline{\Omega}))\big)$. We define the canonical coordinates on $\overline{\Omega}$ by

$$\overline{W}_s(\overline{\omega}):=\omega_0(s),\;\; \overline{X}_s(\overline{\omega}):=\omega_X(s),\;\; s\in[0,\infty) \quad\text{and}\quad \overline{T}(\overline{\omega}):=\mathfrak{t},\;\; \forall\,\overline{\omega}=\big(\omega_0,\omega_X,\mathfrak{t}\big)\in\overline{\Omega},$$

in which one can regard $\overline{W}$ as a canonical coordinate for Brownian motion, $\overline{X}$ as a canonical coordinate for the state process, and $\overline{T}$ as a canonical coordinate for stopping rules. Given $t\in[0,\infty)$, we also set $\overline{W}^t_s:=\overline{W}_s-\overline{W}_t$, $\forall\, s\in[t,\infty)$.

The weak formulation of the OSEC relies on the following probability classes of $\mathfrak{P}\big(\overline{\Omega}\big)$.

DEFINITION 3.1. For any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, let $\overline{\mathcal{P}}_{t,\mathbf{x}}$ be the collection of all probability measures $\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big)$ satisfying:
(D1) The process $\overline{W}^t$ is a $d-$dimensional Brownian motion on $\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}\big)$.
(D2) $\overline{P}\big\{\overline{X}_s=\overline{\mathscr{X}}^{t,\mathbf{x}}_s,\;\forall\, s\in[0,\infty)\big\}=1$, where $\big\{\overline{\mathscr{X}}^{t,\mathbf{x}}_s\big\}_{s\in[0,\infty)}$ is an $\big\{\mathcal{F}^{\overline{W}^t,\overline{P}}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process that uniquely solves the following SDE on $\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}\big)$:

$$\overline{\mathscr{X}}_s=\mathbf{x}(t)+\int_t^s b\big(r,\overline{\mathscr{X}}_{r\wedge\cdot}\big)dr+\int_t^s\sigma\big(r,\overline{\mathscr{X}}_{r\wedge\cdot}\big)d\overline{W}_r,\qquad\forall\, s\in[t,\infty) \tag{3.1}$$

with initial condition $\overline{\mathscr{X}}\big|_{[0,t]}=\mathbf{x}\big|_{[0,t]}$.
(D3) There exists a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}$ on Ω_0 such that $\overline{P}\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}=1$.

Let $t\in[0,\infty)$. For any $s\in[t,\infty)$, define $\overline{\mathcal{F}}^t_s:=\mathcal{F}^{\overline{W}^t}_s\vee\mathcal{F}^{\overline{X}}_s=\sigma\big(\overline{W}^t_r;r\in[t,s]\big)\vee\sigma\big(\overline{X}_r;r\in[0,s]\big)$, which is countably generated by

$$\big\{\overline{X}_r^{-1}(\mathcal{O}):r\in\mathbb{Q}\cap[0,t],\mathcal{O}\in\mathscr{O}(\mathbb{R}^l)\big\}\cup\big\{(\overline{W}^t_r,\overline{X}_r)^{-1}(\mathcal{O}'):r\in\mathbb{Q}\cap(t,s],\mathcal{O}'\in\mathscr{O}(\mathbb{R}^{d+l})\big\}.$$

We denote the filtration $\big\{\overline{\mathcal{F}}^t_s\big\}_{s\in[t,\infty)}$ by $\overline{\mathbf{F}}^t$. For any $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$,

$$\overline{M}^t_s(\varphi):=\varphi\big(\overline{W}^t_s,\overline{X}_s\big)-\int_t^s\overline{b}\big(r,\overline{X}_{r\wedge\cdot}\big)\cdot D\varphi\big(\overline{W}^t_r,\overline{X}_r\big)dr-\frac{1}{2}\int_t^s\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{X}_{r\wedge\cdot}\big):D^2\varphi\big(\overline{W}^t_r,\overline{X}_r\big)dr,$$

$\forall\, s\in[t,\infty)$ is an $\overline{\mathbf{F}}^t-$adapted continuous process and

$$\overline{\tau}^t_n(\mathfrak{a}):=\inf\big\{s\in[t,\infty):\big|(\overline{W}^t_s,\overline{X}_s)-\mathfrak{a}\big|\ge n\big\}\wedge(t+n)$$

is an $\overline{\mathbf{F}}^t-$stopping time. We will simply denote $\overline{\tau}^t_n(\mathbf{0})$ by $\overline{\tau}^t_n$.

Let us also define a shifted canonical process on $\overline{\Omega}$ by $\overline{\mathscr{W}}^t_{\mathfrak{s}}(\overline{\omega}):=\overline{W}_{t+\mathfrak{s}}(\overline{\omega})-\overline{W}_t(\overline{\omega})=\overline{W}^t_{t+\mathfrak{s}}(\overline{\omega})$, $\forall\,(\mathfrak{s},\overline{\omega})\in[0,\infty)\times\overline{\Omega}$. $\big($Note: the subscript $\mathfrak{s}\in[0,\infty)$ of $\overline{\mathscr{W}}^t$ is the relative time after t while the subscript $s\in[t,\infty)$ of $\overline{W}^t$ is the real time.$\big)$

According to the martingale-problem formulation of SDEs (Proposition 1.2), we have an alternative description of the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$:

REMARK 3.1. Let $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$. In Definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$, (D1) and (D2) is equivalent to
(D1$'$) $\overline{P}\{\overline{X}_s=\mathbf{x}(s),\,\forall\,s\in[0,t]\}=1$ and $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\overline{\mathbf{F}}^t,\overline{P}\big)-$martingale for any $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$,

while (D3) is equivalent to
(D3$'$) There exists a $[0,\infty]-$valued $\mathbf{F}^{W,P_0}-$stopping time $\ddot{\tau}$ on Ω_0 such that $\overline{P}\big\{\overline{T}=t+\ddot{\tau}\big(\overline{\mathscr{W}}^t\big)\big\}=1$.

REMARK 3.2. Let $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$ and let $\overline{P}\in\mathfrak{P}(\overline{\Omega})$ satisfy (D1) and (D2) of Definition 3.1.
(1) For any $\phi\in\mathscr{H}_o$, Proposition 1.2 shows that

$$E_{\overline{P}}\Big[\int_t^\infty\phi^-\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]=E_{\overline{P}}\Big[\int_t^\infty\phi^-\big(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot}\big)dr\Big]<\infty.$$

(2) Let $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$. As $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\overline{\mathbf{F}}^t,\overline{P}\big)-$martingale, the optional sampling theorem implies that for any two $[t,\infty]-$valued $\overline{\mathbf{F}}^t-$stopping times $\overline{\zeta}_1,\overline{\zeta}_2$ with $\overline{\zeta}_1\le\overline{\zeta}_2$, $\overline{P}-$a.s.,

$$\begin{aligned}(3.2)\qquad & E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\zeta}_2\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi)-\overline{M}^t_{\overline{\zeta}_1\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]\\ &=E_{\overline{P}}\Big[E_{\overline{P}}\Big[\overline{M}^t_{\overline{\zeta}_2\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi)-\overline{M}^t_{\overline{\zeta}_1\wedge\overline{\tau}^t_n(\mathfrak{a})}(\varphi)\Big|\overline{\mathcal{F}}^t_{\overline{\zeta}_1}\Big]\mathbf{1}_{\overline{A}}\Big]=0,\quad\forall\,\overline{A}\in\overline{\mathcal{F}}^t_{\overline{\zeta}_1}.\end{aligned}$$

Let $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, $(y,z)=\big(\{y_i\}_{i\in\mathbb{N}},\{z_i\}_{i\in\mathbb{N}}\big)\in\Re\times\Re$ and set

$$\overline{R}(t):=\int_{\overline{T}\wedge t}^{\overline{T}}f\big(r,\overline{X}_{r\wedge\cdot}\big)dr+\mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big).$$

Given a historical state path $\mathbf{x}|_{[0,t]}$, the value of the optimal stopping problem with expectation constraints

$$E_{\overline{P}}\bigg[\int_t^{\overline{T}} g_i(r,\overline{X}_{r\wedge\cdot})dr\bigg]\le y_i,\quad E_{\overline{P}}\bigg[\int_t^{\overline{T}} h_i(r,\overline{X}_{r\wedge\cdot})dr\bigg]=z_i,\quad \forall\, i\in\mathbb{N} \tag{3.3}$$

in weak formulation is

$$\overline{V}(t,\mathbf{x},y,z):=\sup_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)} E_{\overline{P}}\big[\overline{R}(t)\big]=\sup_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)} E_{\overline{P}}\Big[\int_t^{\overline{T}} f(r,\overline{X}_{r\wedge\cdot})dr+\mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big],$$

where $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z):=\Big\{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}: E_{\overline{P}}\big[\int_t^{\overline{T}} g_i(r,\overline{X}_{r\wedge\cdot})dr\big]\le y_i,\ E_{\overline{P}}\big[\int_t^{\overline{T}} h_i(r,\overline{X}_{r\wedge\cdot})dr\big]=z_i,\ \forall\, i\in \mathbb{N}\Big\}$. We will simply call $\overline{V}(t,\mathbf{x},y,z)$ the *weak* value of the optimal stopping problem with expectation constraints. In case $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)=\emptyset$, $\overline{V}(t,\mathbf{x},y,z)=-\infty$ by the convention $\sup\emptyset:=-\infty$.

We can consider another weak value function of the OSEC: Let $\mathbf{w}\in\Omega_0$ and define $\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}:=\big\{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}: \overline{P}\big\{\overline{W}_s=\mathbf{w}(s),\ \forall\, s\in[0,t]\big\}=1\big\}$ as the subclass of $\overline{\mathcal{P}}_{t,\mathbf{x}}$ given the historical Brownian path $\mathbf{w}|_{[0,t]}$. The weak value of the optimal stopping problem with expectation constraints (3.3) given $(\mathbf{w},\mathbf{x})\big|_{[0,t]}$ is $\overline{V}(t,\mathbf{w},\mathbf{x},y,z):=\sup\limits_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}(y,z)} E_{\overline{P}}\big[\overline{R}(t)\big]$, where $\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}(y,z):=\big\{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z): \overline{P}\big\{\overline{W}_s=\mathbf{w}(s),\ \forall\, s\in[0,t]\big\}=1\big\}$.

One of our main results (Theorem 3.1 below) exposes that the value function $V(t,\mathbf{x},y,z)$ in (2.3) coincides with the weak value function $\overline{V}(t,\mathbf{x},y,z)$, and is even equal to $\overline{V}(t,\mathbf{w},\mathbf{x},y,z)$.

THEOREM 3.1. *Let* $(t,\mathbf{w},\mathbf{x},y,z)\in[0,\infty)\times\Omega_0\times\Omega_X\times\Re\times\Re$. *Then* $V(t,\mathbf{x},y,z)=\overline{V}(t,\mathbf{x},y,z)=\overline{V}(t,\mathbf{w},\mathbf{x},y,z)$, *and* $\mathcal{S}_{t,\mathbf{x}}(y,z)\neq\emptyset\Leftrightarrow\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)\neq\emptyset\Leftrightarrow\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}(y,z)\neq\emptyset$.

Theorem 3.1 demonstrates that the value of the OSEC is independent of a specific probabilistic setup and is also indifferent to the Brownian history. This result even allows us to deal with the robust case:

REMARK 3.3. Let $\big\{(\mathcal{Q}_\alpha,\mathcal{F}_\alpha,\mathfrak{p}_\alpha)\big\}_{\alpha\in\mathfrak{A}}$ be a family of probability spaces, where $\mathfrak{A}$ is a countable or uncountable index set $\big($e.g. one can consider a non-dominated class $\{\mathfrak{p}_\alpha\}_{\alpha\in\mathfrak{A}}$ of probability measures on a measurable space $(\mathcal{Q},\mathcal{F})\big)$.

Given $\alpha\in\mathfrak{A}$, let $\mathcal{B}^\alpha=\{\mathcal{B}^\alpha_s\}_{s\in[0,\infty)}$ be a $d-$dimensional standard Brownian motion on $(\mathcal{Q}_\alpha,\mathcal{F}_\alpha,\mathfrak{p}_\alpha)$. For any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, set $\mathcal{B}^{\alpha,t}_s:=\mathcal{B}^\alpha_s-\mathcal{B}^\alpha_t$, $s\in[t,\infty)$ and let $\mathcal{X}^{\alpha,t,\mathbf{x}}=\big\{\mathcal{X}^{\alpha,t,\mathbf{x}}_s\big\}_{s\in[0,\infty)}$ be the unique $\big\{\mathcal{F}^{\mathcal{B}^{\alpha,t},\mathfrak{p}_\alpha}_{s\vee t}\big\}_{s\in[0,\infty)}-$adapted continuous process solving the SDE

$$\mathcal{X}_s=\mathbf{x}(t)+\int_t^s b(r,\mathcal{X}_{r\wedge\cdot})dr+\int_t^s\sigma(r,\mathcal{X}_{r\wedge\cdot})d\mathcal{B}^\alpha_r,\quad \forall\, s\in[t,\infty)$$

with initial condition $\mathcal{X}\big|_{[0,t]}=\mathbf{x}\big|_{[0,t]}$ on $\big(\mathcal{Q}_\alpha,\mathcal{F}_\alpha,\mathbf{F}^{\mathcal{B}^{\alpha,t},\mathfrak{p}_\alpha},\mathfrak{p}_\alpha\big)$.

Then we know from Theorem 3.1 that for any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$ and $(y,z)=\big(\{y_i\}_{i\in\mathbb{N}},\{z_i\}_{i\in\mathbb{N}}\big)\in\Re\times\Re$

$$\overline{V}(t,\mathbf{x},y,z)=\sup_{\alpha\in\mathfrak{A}}\ \sup_{\tau_\alpha\in\mathcal{S}^\alpha_{t,\mathbf{x}}(y,z)} E_{\mathfrak{p}_\alpha}\Big[\int_t^{\tau_\alpha} f\big(r,\mathcal{X}^{\alpha,t,\mathbf{x}}_{r\wedge\cdot}\big)dr+\mathbf{1}_{\{\tau_\alpha<\infty\}}\pi\big(\tau_\alpha,\mathcal{X}^{\alpha,t,\mathbf{x}}_{\tau_\alpha\wedge\cdot}\big)\Big],$$

where $\mathcal{S}^\alpha_{t,\mathbf{x}}(y,z)$ collects all $[t,\infty]-$valued $\mathbf{F}^{B^{\alpha,t},\mathfrak{p}_\alpha}-$stopping times τ_α satisfying

$$E_{\mathfrak{p}_\alpha}\Big[\int_t^{\tau_\alpha} g_i\big(r,\mathcal{X}^{\alpha,t,\mathbf{x}}_{r\wedge\cdot}\big)dr\Big]\le y_i\quad\text{and}\quad E_{\mathfrak{p}_\alpha}\Big[\int_t^{\tau_\alpha} h_i\big(r,\mathcal{X}^{\alpha,t,\mathbf{x}}_{r\wedge\cdot}\big)dr\Big]=z_i,\quad \forall\, i\in\mathbb{N}.$$

To wit, the weak value $\overline{V}(t,\mathbf{x},y,z)$ is also equal to the robust value of the OSEC under model uncertainty.

The equivalence between strong and weak formulation of an (unconstrained) optimal stopping problem was discussed in [31]. However, their argument may not be applicable to optimal stopping with expectation constraints:

REMARK 3.4. When $\big(y_i,h_i(\cdot,\cdot),z_i\big)=(\infty,0,0)$, $\forall\, i\in\mathbb{N}$, the unconstrained version of Theorem 3.1 states that for any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, $V(t,\mathbf{x}):=\sup\limits_{\tau\in\mathcal{S}_t}E_{\mathfrak{p}}\big[\int_t^\tau f\big(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot}\big)dr+\mathbf{1}_{\{\tau<\infty\}}\pi\big(\tau,\mathcal{X}^{t,\mathbf{x}}_{\tau\wedge\cdot}\big)\big]$ is equal to $\overline{V}(t,\mathbf{x}):=\sup\limits_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}}E_{\overline{P}}\big[\overline{R}(t)\big]$. On the other hand, [31] showed that for any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, $V(t,\mathbf{x})$ equals $\overline{\overline{V}}(t,\mathbf{x}):=\sup\limits_{\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}}E_{\overline{P}}\big[\overline{R}(t)\big]$, where $\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}$ collects all $\overline{P}\in\mathfrak{P}(\overline{\Omega})$ satisfying (D1), (D2) of Definition 3.1 and "$\overline{P}\big\{\overline{T}\ge t\big\}=1$" (We summarize [31]'s result in our terms for an easy comparison with our work). As $\overline{\mathcal{P}}_{t,\mathbf{x}}\subset\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}$, the equality $V(t,\mathbf{x})=\sup\limits_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}}E_{\overline{P}}\big[\overline{R}(t)\big]=\sup\limits_{\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}}E_{\overline{P}}\big[\overline{R}(t)\big]$ indicates that the probability classes $\overline{\mathcal{P}}_{t,\mathbf{x}}$'s are more accurate than $\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}$'s to describe the (unconstrained) optimal stopping problem in weak formulation.

The condition (D3) of Definition 3.1 is necessary for the expectation-constraint case. Without it, the weak value $\overline{\overline{V}}(t,\mathbf{x},y,z):=\sup\limits_{\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}(y,z)}E_{\overline{P}}\big[\overline{R}(t)\big]$ $\Big($with $\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}(y,z):=\Big\{\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}:E_{\overline{P}}\big[\int_t^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\big]\le y_i,\ E_{\overline{P}}\big[\int_t^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot})dr\big]=z_i,\ \forall\, i\in\mathbb{N}\Big\}\Big)$ may not be equal to $V(t,\mathbf{x},y,z)$ for the following reason:

In Proposition 4.3 of [31], the key to show $\overline{\overline{V}}(t,\mathbf{x})\le V(t,\mathbf{x})$, or $E_{\overline{P}}\big[\overline{R}(t)\big]\le V(t,\mathbf{x})$ for a given $\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}$, relies on transforming the hitting times of process $\big\{E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\in[t,s]\}}\big|\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty\big]\big\}_{s\in[t,\infty)}$ to a member of $\mathcal{S}_t$. More precisely, the so-called *Property* (K) assures an $\mathbf{F}^{W^t,P_0}-$adapted càdlàg process $\widehat{\vartheta}_\cdot$ such that $\widehat{\vartheta}_s(\overline{W})=E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\in[t,s]\}}\big|\mathcal{F}^{\overline{W}^t}_s\big]=E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\in[t,s]\}}\big|\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty\big]$, $\overline{P}-$a.s. for any $s\in[t,\infty)$. It follows that $E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\in[t,s]\}}\mathbf{1}_{\{\overline{\mathscr{X}}^{t,\mathbf{x}}\in A\}}\big|\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty\big]=\mathbf{1}_{\{\overline{\mathscr{X}}^{t,\mathbf{x}}_\cdot\in A\}}\widehat{\vartheta}_s(\overline{W})=\int_t^s\mathbf{1}_{\{\overline{\mathscr{X}}^{t,\mathbf{x}}\in A\}}\widehat{\vartheta}(dr,\overline{W})$, $\overline{P}-$a.s. for any $(s,A)\in[t,\infty)\times\mathscr{B}(\Omega_X)$, where $\overline{\mathscr{X}}^{t,\mathbf{x}}=\big\{\overline{\mathscr{X}}^{t,\mathbf{x}}_s\big\}_{s\in[0,\infty)}$ is the unique solution of SDE (3.1). Let Φ be a nonnegative Borel-measurable function on $[0,\infty)\times\Omega_X$. Then a standard approximation argument and the "change-of-variable" formula yield that $E_{\overline{P}}\big[\Phi(\overline{T},\overline{X})\big|\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty\big]=\int_t^\infty\Phi(r,\overline{X})\widehat{\vartheta}(dr,\overline{W})=\int_0^1\Phi(\varrho(\overline{W},\lambda),\overline{X})d\lambda$, $\overline{P}-$a.s., where $\varrho(\omega_0,\lambda):=\inf\big\{s\in[t,\infty):\widehat{\vartheta}_s(\omega_0)>\lambda\big\}$, $\forall\,(\omega_0,\lambda)\in\Omega_0\times(0,1)$. Since the joint $\overline{P}-$distribution of $\big(\overline{W},\overline{\mathscr{X}}^{t,\mathbf{x}}\big)$ is equal to the joint $\mathfrak{p}-$distribution of $(\mathcal{B},\mathcal{X}^{t,\mathbf{x}})$,

$$E_{\overline{P}}\big[\Phi(\overline{T},\overline{X})\big]=\int_0^1E_{\overline{P}}\big[\Phi(\varrho(\overline{W},\lambda),\overline{\mathscr{X}}^{t,\mathbf{x}})\big]d\lambda=\int_0^1E_{\mathfrak{p}}\big[\Phi(\varrho(\mathcal{B},\lambda),\mathcal{X}^{t,\mathbf{x}})\big]d\lambda. \tag{3.4}$$

As $\tau_\lambda:=\varrho(\mathcal{B},\lambda)\in\mathcal{S}_t$ for each $\lambda\in(0,1)$, taking Φ to be the total reward function implies that

$$\begin{aligned}E_{\overline{P}}\big[\overline{R}(t)\big]&=\int_0^1E_{\mathfrak{p}}\Big[\int_t^{\tau_\lambda}f\big(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot}\big)dr+\mathbf{1}_{\{\tau_\lambda<\infty\}}\pi\big(\tau_\lambda,\mathcal{X}^{t,\mathbf{x}}_{\tau_\lambda\wedge\cdot}\big)\Big]d\lambda\\&\le\int_0^1V(t,\mathbf{x})d\lambda=V(t,\mathbf{x}).\end{aligned} \tag{3.5}$$

However, this argument is not applicable to the expectation-constraint case: Given a $\overline{P}\in\overline{\overline{\mathcal{P}}}_{t,\mathbf{x}}(y,z)$, since τ_λ may not belong to $\mathcal{S}_{t,\mathbf{x}}(y,z)$ for a.e. $\lambda\in(0,1)$, one can not get $E_{\overline{P}}\big[\overline{R}(t)\big]\le V(t,\mathbf{x},y,z)$ like (3.5). Actually, for each $\lambda\in(0,1)$, τ_λ is only of $\mathcal{S}_{t,\mathbf{x}}(y_\lambda,z_\lambda)$ with $(y_\lambda,z_\lambda)=\big(\{y^i_\lambda\}_{i\in\mathbb{N}},\{z^i_\lambda\}_{i\in\mathbb{N}}\big)$, $(y^i_\lambda,z^i_\lambda):=\Big(E_{\mathfrak{p}}\big[\int_t^{\tau_\lambda}g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big],E_{\mathfrak{p}}\big[\int_t^{\tau_\lambda}h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]\Big)$. For $i\in\mathbb{N}$, choosing accumulative cost functions for Φ in (3.4) renders that

$$\int_0^1 E_{\mathfrak{p}}\Big[\int_t^{\tau_\lambda}g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\Big]d\lambda=E_{\overline{P}}\Big[\int_t^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\Big]\le y_i$$

and similarly $\int_0^1 E_{\mathfrak{p}}\big[\int_t^{\tau_\lambda}h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]d\lambda=z_i$, so $V\big(t,\mathbf{x},\{\int_0^1 y_\lambda d\lambda\}_{i\in\mathbb{N}},\{\int_0^1 z_\lambda d\lambda\}_{i\in\mathbb{N}}\big)\le V(t,\mathbf{x},y,z)$. Then the attempt to show $E_{\overline{P}}\big[\overline{R}(t)\big]\le V(t,\mathbf{x},y,z)$ reduces to deriving a Jensen-type inequality:

$$\int_0^1 V(t,\mathbf{x},y_\lambda,z_\lambda)d\lambda\le V\Big(t,\mathbf{x},\Big\{\int_0^1 y_\lambda d\lambda\Big\}_{i\in\mathbb{N}},\Big\{\int_0^1 z_\lambda d\lambda\Big\}_{i\in\mathbb{N}}\Big).$$

But this does not hold since the value function V is not concave in level z of equality-type expectation constraints.

4. The Measurability of OSEC Values. In this section, using the martingale-problem formulation of SDEs, we characterize the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$ by countably many stochastic behaviors of the canonical coordinates $(\overline{W},\overline{X},\overline{T})$ of $\overline{\Omega}$. This will enable us to analyze the measurability of value functions of the optimal stopping problem with expectation constraints.

Let $\mathfrak{S}$ be the equivalence classes of all $[0,\infty]-$valued $\mathbf{F}^{W,P_0}-$stopping times on Ω_0 in the sense that $\tau_1,\tau_2\in\mathfrak{S}$ are equivalent if $P_0\{\tau_1=\tau_2\}=1$. We endow $\mathfrak{S}$ with the metric

$$\rho_{\mathfrak{S}}(\tau_1,\tau_2):=E_{P_0}\big[\rho_+(\tau_1,\tau_2)\big],\quad \forall\,\tau_1,\tau_2\in\mathfrak{S}.$$

LEMMA 4.1. *$\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$ is a complete separable metric space, i.e., a Polish space.*

For any $\tau\in\mathfrak{S}$, we define its joint distribution with W under P_0 by $\Gamma(\tau):=P_0\circ(W,\tau)^{-1}\in\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$.

LEMMA 4.2. *The mapping $\Gamma:\mathfrak{S}\mapsto\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$ is a continuous injection from $\mathfrak{S}$ into $\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$.*

We can use Remark 3.1 and Lemma 4.2 to decompose the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$ as the intersection of countable many action sets of processes $(\overline{W},\overline{X},\overline{T})$:

PROPOSITION 4.1. *For any $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$ is the intersection of the following three subsets of $\mathfrak{P}\big(\overline{\Omega}\big)$:*

i) $\overline{\mathcal{P}}^1_{t,\mathbf{x}}:=\big\{\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big):\overline{P}\{\overline{X}_s=\mathbf{x}(s),\ \forall\,s\in[0,t]\}=1\big\}$.

ii) $\overline{\mathcal{P}}^2_t:=\Big\{\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big):\,E_{\overline{P}}\Big[\Big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\Big)\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^t_{t+s_i},\overline{X}_{t+s_i})\in\mathcal{O}_i\}}\Big]=0,\ \forall\,(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N},\ \forall\,(\mathfrak{s},\mathfrak{r})\in\mathbb{Q}^{2,<}_+,\ \forall\,\{(s_i,\mathcal{O}_i)\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})\Big\}$.

iii) $\overline{\mathcal{P}}^3_t:=\big\{\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big):\overline{P}\circ(\overline{\mathscr{W}}^t,\overline{T}-t)^{-1}\in\Gamma(\mathfrak{S})\big\}$.

Based on the countable decomposition of the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}$ by Proposition 4.1, the next proposition shows that the graph of probability classes $\big\{\overline{\mathcal{P}}_{t,\mathbf{x}}\big\}_{(t,\mathbf{x})\in[0,\infty)\times\Omega_X}$ is a Borel subset of $[0,\infty)\times\Omega_X\times\mathfrak{P}(\overline{\Omega})$, which is crucial for the measurability of the value functions $V=\overline{V}$.

LEMMA 4.3. *The mapping* $\overline{\Gamma}(t,\overline{P}):=\overline{P}\circ\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)^{-1}\in\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$, $\forall\,(t,\overline{P})\in[0,\infty)\times\mathfrak{P}(\overline{\Omega})$ *is continuous.*

PROPOSITION 4.2. *The graph* $\langle\langle\overline{\mathcal{P}}\rangle\rangle:=\big\{\big(t,\mathbf{x},\overline{P}\big)\in[0,\infty)\times\Omega_X\times\mathfrak{P}(\overline{\Omega}):\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}\big\}$ *is a Borel subset of* $[0,\infty)\times\Omega_X\times\mathfrak{P}(\overline{\Omega})$.

Set $\overline{D}:=\big\{(t,\mathbf{x},y,z)\in[0,\infty)\times\Omega_X\times\Re\times\Re:\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)\neq\emptyset\big\}$ and $\overline{\mathcal{D}}:=\big\{(t,\mathbf{w},\mathbf{x},y,z)\in[0,\infty)\times\Omega_0\times\Omega_X\times\Re\times\Re:\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}(y,z)\neq\emptyset\big\}$.

COROLLARY 4.1. *The graph* $[[\,\overline{\mathcal{P}}\,]]:=\big\{\big(t,\mathbf{x},y,z,\overline{P}\big)\in\overline{D}\times\mathfrak{P}(\overline{\Omega}):\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)\big\}$ *is a Borel subset of* $\overline{D}\times\mathfrak{P}(\overline{\Omega})$ *and the graph* $\{\{\overline{\mathcal{P}}\}\}:=\big\{\big(t,\mathbf{w},\mathbf{x},y,z,\overline{P}\big)\in\overline{\mathcal{D}}\times\mathfrak{P}(\overline{\Omega}):\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}(y,z)\big\}$ *is a Borel subset of* $\overline{\mathcal{D}}\times\mathfrak{P}(\overline{\Omega})$.

By Corollary 4.1, the value function $\overline{V}$ is upper semi-analytic and is thus universally measurable.

THEOREM 4.1. *The value function* $\overline{V}(t,\mathbf{x},y,z)$ *is upper semi-analytic on* $\overline{D}$ *and the value function* $\overline{V}(t,\mathbf{w},\mathbf{x},y,z)$ *is upper semi-analytic on* $\overline{\mathcal{D}}$.

5. Dynamic Programming Principle for $\overline{V}$. In this section, we explore a dynamic programming principle (DPP) for the value function $\overline{V}$ in weak formulation, which takes the conditional expected integrals of constraint functions as additional states.

Given $t\in[0,\infty)$, let $\overline{\gamma}$ be a $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping time and let $\overline{P}\in\mathfrak{P}(\overline{\Omega})$. According to Lemma 1.3.3 and Theorem 1.1.8 of [63], $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ is countably generated and there is thus a family $\big\{\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\}_{\overline{\omega}\in\overline{\Omega}}$ of probability measures in $\mathfrak{P}(\overline{\Omega})$, called the *regular conditional probability distribution* (r.c.p.d.) of $\overline{P}$ with respect to $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, such that

(5.1) for any $\overline{A}\in\mathscr{B}(\overline{\Omega})$, the mapping $\overline{\omega}\mapsto\overline{P}^t_{\overline{\gamma},\overline{\omega}}(\overline{A})$ is $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable;

(5.2) for any $(-\infty,\infty]-$valued, $\mathscr{B}_{\overline{P}}(\overline{\Omega})-$measurable random variable $\overline{\xi}$ that is bounded from below under $\overline{P}$, it holds for all $\overline{\omega}\in\overline{\Omega}$ except on a $\overline{\mathcal{N}}_{\overline{\xi}}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ that $\overline{\xi}$ is $\mathscr{B}_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}(\overline{\Omega})-$measurable and $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\,\overline{\xi}\,\big]=E_{\overline{P}}\big[\,\overline{\xi}\,\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})$;

(5.3) for some $\overline{\mathcal{N}}_0\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}(\overline{A})=\mathbf{1}_{\{\overline{\omega}\in\overline{A}\}}$, $\forall\,(\overline{\omega},\overline{A})\in\overline{\mathcal{N}}^c_0\times\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$.

Let $\overline{\omega}\in\overline{\Omega}$ and set $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}:=\big\{\overline{\omega}'\in\overline{\Omega}:\overline{W}^t_r(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}),\ \forall\,r\in[t,\overline{\gamma}(\overline{\omega})]\big\}$. We know from Galmarino's test that

$$\overline{\gamma}(\overline{\omega}')=\overline{\gamma}(\overline{\omega}),\quad\forall\,\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}},\tag{5.4}$$

and $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}$ is thus $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable. Since $\overline{\omega}\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}$ for any $\overline{\omega}\in\overline{\Omega}$, (5.3) shows that

$$\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\big)=\mathbf{1}_{\big\{\overline{\omega}\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\big\}}=1,\quad\forall\,\overline{\omega}\in\overline{\mathcal{N}}^c_0.\tag{5.5}$$

For any $i\in\mathbb{N}$, define $\overline{Y}^i_{\overline{P}}(\overline{\gamma}):=E_{\overline{P}}\Big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}} g_i(r,\overline{X}_{r\wedge\cdot})dr\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big]$ and $\overline{Z}^i_{\overline{P}}(\overline{\gamma}):=E_{\overline{P}}\Big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}} h_i (r,\overline{X}_{r\wedge\cdot})dr\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big]$. So $\big(\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big):=\Big(\big\{\overline{Y}^i_{\overline{P}}(\overline{\gamma})\big\}_{i\in\mathbb{N}},\big\{\overline{Z}^i_{\overline{P}}(\overline{\gamma})\big\}_{i\in\mathbb{N}}\Big)$ is an $\Re\times\Re-$valued $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable random variable.

In terms of the r.c.p.d. $\big\{\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\}_{\overline{\omega}\in\overline{\Omega}}$, the probability class $\big\{\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z):(t,\mathbf{x},y,z)\in\overline{D}\big\}$ is stable under conditioning as follows. It will play an important role in deriving the sub-solution side of the DPP for $\overline{V}$.

PROPOSITION 5.1. *Given $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$, let $\overline{\gamma}$ be a $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping time and let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}$. There exists a $\overline{P}-$null set $\overline{\mathcal{N}}$ such that*

$$\overline{P}^t_{\overline{\gamma},\overline{\omega}}\in\overline{\mathcal{P}}_{\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\Big(\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big),\quad\forall\,\overline{\omega}\in\big\{\overline{T}\ge\overline{\gamma}\big\}\cap\overline{\mathcal{N}}^c. \tag{5.6}$$

Now, we are ready to present a dynamic programming principle in weak formulation for the value function $\overline{V}$, in which $\big(\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)$ act as additional states for constraint levels at the intermediate horizon $\overline{\gamma}$.

THEOREM 5.1. *Given $(t,\mathbf{x},y,z)\in\overline{D}$, let $\big\{\overline{\gamma}_{\overline{P}}\big\}_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}$ be a family of $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping times. Then*

$$\begin{aligned}\overline{V}(t,\mathbf{x},y,z)=\sup_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}E_{\overline{P}}\bigg[&\mathbf{1}_{\{\overline{T}<\overline{\gamma}_{\overline{P}}\}}\Big(\int_t^{\overline{T}}f(r,\overline{X}_{r\wedge\cdot})dr+\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big)\\&+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}_{\overline{P}}\}}\Big(\int_t^{\overline{\gamma}_{\overline{P}}}f(r,\overline{X}_{r\wedge\cdot})dr+\overline{V}\Big(\overline{\gamma}_{\overline{P}},\overline{X}_{\overline{\gamma}_{\overline{P}}\wedge\cdot},\overline{Y}_{\overline{P}}\big(\overline{\gamma}_{\overline{P}}\big),\overline{Z}_{\overline{P}}\big(\overline{\gamma}_{\overline{P}}\big)\Big)\Big)\bigg].\end{aligned}\tag{5.7}$$

6. Proofs.

Proof of Proposition 1.2: 1) Set $\mathcal{N}:=\big\{\omega\in\Omega: X_s(\omega)\neq\mathbf{x}(s)\text{ for some }s\in[0,t]\big\}\in\mathscr{N}_P(\mathcal{F}^X_t)$ and let $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$. We denote $c^\varphi_n(\mathfrak{a}):=\sup_{|(w,x)|\le n+\mathfrak{a}}\big(\sum_{i=0}^2|D^i\varphi(w,x)|\big)+\big|\varphi(0,\mathbf{x}(t))\big|<\infty$ and $c^n_{t,\mathbf{x}}(\mathfrak{a}):=\big[d/2+\kappa(t+n)(\|\mathbf{x}\|_t+n+\mathfrak{a})+\kappa^2(t+n)(\|\mathbf{x}\|_t+n+\mathfrak{a})^2\big]n+\int_t^{t+n}\big(|b(r,\mathbf{0})|+|\sigma(r,\mathbf{0})|^2\big)dr<\infty$. Given $\omega\in\mathcal{N}^c$, since $\big\|X_{r\wedge\cdot}(\omega)\big\|_r=\sup_{r'\in[0,r]}\big|X_{r'}(\omega)\big|\le\|\mathbf{x}\|_t\vee(n+\mathfrak{a})$, $\forall\,r\in\big[t,(\tau^t_n(\mathfrak{a}))(\omega)\big]$, we can deduce from (1.2), (1.3) and Cauchy-Schwarz inequality that

$$\begin{aligned}\sup_{s\in[t,(\tau^t_n(\mathfrak{a}))(\omega)]}\big|\big(M^t_s(\varphi)\big)(\omega)\big|\le&\sup_{s\in[t,(\tau^t_n(\mathfrak{a}))(\omega)]}\big|\varphi\big(B^t_s(\omega),X_s(\omega)\big)\big|\\&+c^\varphi_n(\mathfrak{a})\int_t^{(\tau^t_n(\mathfrak{a}))(\omega)}\Big(\big|b(r,X_{r\wedge\cdot}(\omega))\big|+\frac{1}{2}\big(d+\big|\sigma(r,X_{r\wedge\cdot}(\omega))\big|^2\big)\Big)dr\\\le&\,c^\varphi_n(\mathfrak{a})+c^\varphi_n(\mathfrak{a})\int_t^{(\tau^t_n(\mathfrak{a}))(\omega)}\Big(\kappa(r)\big\|X_{r\wedge\cdot}(\omega)\big\|_r+|b(r,\mathbf{0})|+d/2\\&+\kappa^2(r)\big\|X_{r\wedge\cdot}(\omega)\big\|_r^2+|\sigma(r,\mathbf{0})|^2\Big)dr\le c^\varphi_n(\mathfrak{a})(1+c^n_{t,\mathbf{x}}(\mathfrak{a})).\end{aligned}\tag{6.1}$$

So $\big\{M^t_{s\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^t-$adapted continuous process under P.

2) We next show that (i) implies (ii): Suppose that (i) holds and let $(\varphi,n,\mathfrak{a})\in C^2(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{R}^{d+l}$. We simply denote $\Xi^{t,\mathbf{x}}_s:=(B^t_s,X^{t,\mathbf{x}}_s)$, $\forall\,s\in[t,\infty)$ and set $\tau^{t,\mathbf{x}}_n(\mathfrak{a}):=\inf\big\{s\in[t,\infty):$

$|\Xi^{t,\mathbf{x}}_s-\mathfrak{a}|\ge n\big\}\wedge(t+n)$, which is an $\mathbf{F}^{B^t,P}-$stopping time. Applying Itô's formula yields that $P-$a.s.

$$M^{t,\mathbf{x}}_s(\varphi):=\varphi(\Xi^{t,\mathbf{x}}_s)-\int_t^s \overline{b}\big(r,X^{t,\mathbf{x}}_{r\wedge\cdot}\big)\cdot D\varphi(\Xi^{t,\mathbf{x}}_r)dr-\frac12\int_t^s \overline{\sigma}\,\overline{\sigma}^T\big(r,X^{t,\mathbf{x}}_{r\wedge\cdot}\big):D^2\varphi(\Xi^{t,\mathbf{x}}_r)dr$$
$$=\varphi\big(0,\mathbf{x}(t)\big)+\int_t^s D\varphi(\Xi^{t,\mathbf{x}}_r)\cdot\overline{\sigma}\big(r,X^{t,\mathbf{x}}_{r\wedge\cdot}\big)dB_r,\quad s\in[t,\infty).$$

For any $\omega\in\Omega$, an analogy to (6.1) shows that $\sup\limits_{s\in[t,(\tau^{t,\mathbf{x}}_n(\mathfrak{a}))(\omega)]}\big|\big(M^{t,\mathbf{x}}_s(\varphi)\big)(\omega)\big|\le c^\varphi_n(\mathfrak{a})(1+c^n_{t,\mathbf{x}}(\mathfrak{a}))$ and $\int_t^{(\tau^{t,\mathbf{x}}_n(\mathfrak{a}))(\omega)}\big|D\varphi(\Xi^{t,\mathbf{x}}_r(\omega))\cdot\overline{\sigma}(r,X^{t,\mathbf{x}}_{r\wedge\cdot}(\omega))\big|^2dr\le(c^\varphi_n(\mathfrak{a}))^2\big[d+2\kappa^2(t+n)(\|\mathbf{x}\|_t+n+\mathfrak{a})^2\big]n+2(c^\varphi_n(\mathfrak{a}))^2\int_t^{t+n}|\sigma(r,\mathbf{0})|^2dr<\infty$. So

$$\big\{M^{t,\mathbf{x}}_{s\wedge\tau^{t,\mathbf{x}}_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}\text{ is a bounded }\mathbf{F}^{B^t,P}-\text{martingale.}\tag{6.2}$$

Set $\mathcal{N}_{t,\mathbf{x}}:=\{\omega\in\Omega:\ X_s(\omega)\ne X^{t,\mathbf{x}}_s(\omega)\text{ for some }s\in[0,\infty)\}\in\mathscr{N}_P\big(\mathcal{F}^{B^t,P}_\infty\vee\mathcal{F}^X_\infty\big)$. For any $(s,\omega)\in[0,\infty)\times\mathcal{N}^c_{t,\mathbf{x}}$,

$$X^{t,\mathbf{x}}_s(\omega)=X_s(\omega),\ \big(M^{t,\mathbf{x}}_{s\vee t}(\varphi)\big)(\omega)=\big(M^t_{s\vee t}(\varphi)\big)(\omega)\text{ and }(\tau^{t,\mathbf{x}}_n(\mathfrak{a}))(\omega)=(\tau^t_n(\mathfrak{a}))(\omega).\tag{6.3}$$

Fix $t_1,t_2\in[t,\infty)$ with $t_1<t_2$. Let $\big\{(s_i,\mathcal{E}_i)\big\}_{i=1}^m\subset[t,t_1]\times\mathscr{B}(\mathbb{R}^d)$ and $\big\{(r_j,A_j)\big\}_{j=1}^k\subset[0,t_1]\times\mathscr{B}(\mathbb{R}^l)$. We can derive from (6.2) and (6.3) that $E_P\Big[\mathbf{1}_{\mathcal{N}^c_{t,\mathbf{x}}}\big(M^t_{t_2\wedge\tau^t_n(\mathfrak{a})}(\varphi)-M^t_{t_1\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big)\prod_{i=1}^m\mathbf{1}_{(B^t_{s_i})^{-1}(\mathcal{E}_i)}\prod_{j=1}^k\mathbf{1}_{X^{-1}_{r_j}(A_j)}\Big]=E_P\Big[\mathbf{1}_{\mathcal{N}^c_{t,\mathbf{x}}}\big(M^{t,\mathbf{x}}_{t_2\wedge\tau^{t,\mathbf{x}}_n(\mathfrak{a})}(\varphi)-M^{t,\mathbf{x}}_{t_1\wedge\tau^{t,\mathbf{x}}_n(\mathfrak{a})}(\varphi)\big)\prod_{i=1}^m\mathbf{1}_{(B^t_{s_i})^{-1}(\mathcal{E}_i)}\prod_{j=1}^k\mathbf{1}_{(X^{t,\mathbf{x}}_{r_j})^{-1}(A_j)}\Big]=0$. So the Lambda-system

$$\Lambda:=\big\{A\in\mathcal{F}:E_P\big[\big(M^t_{t_2\wedge\tau^t_n(\mathfrak{a})}(\varphi)-M^t_{t_1\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big)\mathbf{1}_A\big]=0\big\}$$

contains the Pi-system $\Big\{\Big(\bigcap\limits_{i=1}^m(B^t_{s_i})^{-1}(\mathcal{E}_i)\Big)\cap\Big(\bigcap\limits_{j=1}^k X^{-1}_{r_j}(A_j)\Big):\ \big\{(s_i,\mathcal{E}_i)\big\}_{i=1}^m\subset[t,t_1]\times\mathscr{B}(\mathbb{R}^d),\ \big\{(r_j,A_j)\big\}_{j=1}^k\subset[0,t_1]\times\mathscr{B}(\mathbb{R}^l)\Big\}$, which generates $\mathcal{F}^t_{t_1}$. Dynkin's Pi-Lambda Theorem (see e.g Theorem 3.2 of [15]) renders $\mathcal{F}^t_{t_1}\subset\Lambda$, i.e., $E_P\big[\big(M^t_{t_2\wedge\tau^t_n(\mathfrak{a})}(\varphi)-M^t_{t_1\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big)\mathbf{1}_A\big]=0,\ \forall\,A\in\mathcal{F}^t_{t_1}$. Hence, $\big\{M^t_{s\wedge\tau^t_n(\mathfrak{a})}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^t-$martingale.

3) As $\mathfrak{C}(\mathbb{R}^{d+l})\subset C^2(\mathbb{R}^{d+l})$, (ii)$\Rightarrow$(iii) is straightforward. It remains to show that (iii) gives rise to (i).

3a) Let $\mathbf{F}^{t,P}=\big\{\mathcal{F}^{t,P}_s\big\}_{s\in[t,\infty)}$ be the $P-$augmentation of $\mathbf{F}^t$ $\Big($i.e., $\mathcal{F}^{t,P}_s:=\sigma(\mathcal{F}^t_s\cup\mathscr{N}_P(\mathcal{F}^t_\infty))$ with $\mathcal{F}^t_\infty:=\sigma\Big(\bigcup\limits_{s\in[t,\infty)}\mathcal{F}^t_s\Big)\Big)$ We define $\mathcal{F}^{t,P}_{s+}:=\bigcap\limits_{\varepsilon>0}\mathcal{F}^{t,P}_{s+\varepsilon},\ \forall\,s\in[t,\infty)$ and set $\mathbf{G}^{t,P}=\big\{\mathcal{G}^{t,P}_s:=\mathcal{F}^{t,P}_{s+}\big\}_{s\in[t,\infty)}$.

Let $i,j\in\{1,\cdots,d\}$. We set $\phi_i(w,x):=w_i$ and $\phi_{ij}(w,x):=w_iw_j$ for any $w=(w_1,\cdots,w_d)\in\mathbb{R}^d$ and $x\in\mathbb{R}^l$. Clearly, $\phi_i,\phi_{ij}\in\mathfrak{C}(\mathbb{R}^{d+l})$. One can calculate that $M^t_s(\phi_i)=B^{t,i}_s$, $M^t_s(\phi_{ij})=B^{t,i}_sB^{t,j}_s-\delta_{ij}(s-t)$, $\forall\,s\in[t,\infty)$, where $B^t_s=\big(B^{t,1}_s,\cdots,B^{t,d}_s\big)$ and δ_{ij} is the $(i,j)-$element of the identity matrix $I_{d\times d}$.

Let $n\in\mathbb{N}$. By (iii), $\big\{M^t_{s\wedge\tau^t_n}(\phi_i)\big\}_{s\in[t,\infty)}$ and $\big\{M^t_{s\wedge\tau^t_n}(\phi_{ij})\big\}_{s\in[t,\infty)}$ are bounded $\mathbf{F}^t-$martingales and are thus bounded $\mathbf{F}^{t,P}-$ martingales. The *optional sampling* theorem (e.g. Theorem 1.3.22 of [36]) implies that they are further $\mathbf{G}^{t,P}-$martingales. Since $\lim\limits_{n\to\infty}\uparrow\tau^t_n=\infty$, we see that $\big\{M^t_s(\phi_i)=B^{t,i}_s\big\}_{s\in[t,\infty)}$ and $\big\{M^t_s(\phi_{ij})=B^{t,i}_sB^{t,j}_s-\delta_{ij}(s-t)\big\}_{s\in[t,\infty)}$ are $\mathbf{G}^{t,P}-$local martingales. Lévy's characterization theorem then yields that B^t is a Brownian motion with respect to filtration $\mathbf{G}^{t,P}$ and is thus a Brownian motion with respect to filtration $\mathbf{F}^{B^t}$.

3b) We simply denote $\Xi_s := (B^t_s, X_s)$, $\beta_s := \overline{b}\big(s, X_{s\wedge\cdot}\big)$ and $\alpha_s := \overline{\sigma}\,\overline{\sigma}^T\big(s, X_{s\wedge\cdot}\big)$, $\forall\, s \in [t,\infty)$. Let $i,j \in \{1,\cdots,d+l\}$. We set $\psi_i(\boxminus) := \boxminus_i$ and $\psi_{ij}(\boxminus) := \boxminus_i \boxminus_j$ for any $\boxminus = \big(\boxminus_1,\cdots,\boxminus_{d+l}\big) \in \mathbb{R}^{d+l}$. Similar to $M^t_\cdot(\phi_i)$ and $M^t_\cdot(\phi_{ij})$, the processes $M^t_s(\psi_i) = \Xi^{(i)}_s - \int_t^s \beta^{(i)}_r dr$ and $M^t_s(\psi_{ij}) = \Xi^{(i)}_s \Xi^{(j)}_s - \int_t^s \beta^{(i)}_r \Xi^{(j)}_r dr - \int_t^s \beta^{(j)}_r \Xi^{(i)}_r dr - \int_t^s (\alpha_r)_{ij} dr$, $s \in [t,\infty)$ are $\mathbf{G}^{t,P}-$local martingales. Using the *integration by parts* formula, we obtain that $P-$a.s.

$$
\begin{aligned}
\Xi^{(i)}_s \Xi^{(j)}_s - M^t_s(\psi_i) M^t_s(\psi_j) &= M^t_s(\psi_i) \int_t^s \beta^{(j)}_r dr + M^t_s(\psi_j) \int_t^s \beta^{(i)}_r dr + \int_t^s \beta^{(i)}_r dr \cdot \int_t^s \beta^{(j)}_r dr \\
&= \int_t^s M^t_r(\psi_i) \beta^{(j)}_r dr + \int_t^s \Big(\int_t^r \beta^{(j)}_{r'} dr' \Big) dM^t_r(\psi_i) + \int_t^s M^t_r(\psi_j) \beta^{(i)}_r dr \\
&\quad + \int_t^s \Big(\int_t^r \beta^{(i)}_{r'} dr' \Big) dM^t_r(\psi_j) + \int_t^s \Big(\int_t^r \beta^{(i)}_{r'} dr' \Big) \beta^{(j)}_r dr + \int_t^s \Big(\int_t^r \beta^{(j)}_{r'} dr' \Big) \beta^{(i)}_r dr \\
&= \int_t^s \Big[\Xi^{(i)}_r \beta^{(j)}_r + \Xi^{(j)}_r \beta^{(i)}_r \Big] dr + \int_t^s \Big(\int_t^r \beta^{(i)}_{r'} dr' \Big) dM^t_r(\psi_i) + \int_t^s \Big(\int_t^r \beta^{(j)}_{r'} dr' \Big) dM^t_r(\psi_j),
\end{aligned}
$$

$\forall\, s \in [t,\infty)$. So $M^t_s(\psi_i) M^t_s(\psi_j) - \int_t^s (\alpha_r)_{ij} dr = M^t_s(\psi_{ij}) - \int_t^s \big(\int_t^r \beta^{(i)}_{r'} dr' \big) dM^t_r(\psi_i) - \int_t^s \big(\int_t^r \beta^{(j)}_{r'} dr' \big) dM^t_r(\psi_j)$, $s \in [t,\infty)$ is also an $\mathbf{G}^{t,P}-$local martingale, which implies that the quadratic variation of the $\mathbf{G}^{t,P}-$local martingale $M^t_s := \big(M^t_s(\psi_1),\cdots,M^t_s(\psi_{d+l}) \big) = \Xi_s - \int_t^s \beta_r dr$, $s \in [t,\infty)$ is $\big\langle M^t, M^t \big\rangle_s = \int_t^s \alpha_r dr$, $s \in [t,\infty)$.

Let $n \in \mathbb{N}$, $a \in \mathbb{R}^l$ and set $\mathcal{H}^a_s := \begin{pmatrix} -\sigma^T\big(s, X_{s\wedge\cdot}\big) a \\ a \end{pmatrix}$, $\forall\, s \in (t,\infty)$. The stochastic exponential of the $\mathbf{G}^{t,P}-$ martingale $\big\{ \int_t^{\tau^t_n \wedge s} \mathcal{H}^a_r \cdot dM^t_r \big\}_{s \in [t,\infty)}$ is

$$
\begin{aligned}
\exp\Big\{ \int_t^{\tau^t_n \wedge s} \mathcal{H}^a_r \cdot dM^t_r - \frac{1}{2} \int_t^{\tau^t_n \wedge s} (\mathcal{H}^a_r)^T \alpha_r \mathcal{H}^a_r dr \Big\} &= \exp\Big\{ \int_t^{\tau^t_n \wedge s} \mathcal{H}^a_r \cdot d\Xi_r - \int_t^{\tau^t_n \wedge s} \mathcal{H}^a_r \cdot \beta_r dr \Big\} \\
&= \exp\Big\{ a \cdot \Big(\int_t^{\tau^t_n \wedge s} dX_r - \int_t^{\tau^t_n \wedge s} \sigma\big(r, X_{r\wedge\cdot}\big) dB_r - \int_t^{\tau^t_n \wedge s} b\big(r, X_{r\wedge\cdot}\big) dr \Big) \Big\}, \quad s \in [t,\infty).
\end{aligned}
$$

Letting a vary over $\mathbb{R}^l$ yields that $P-$a.s., $X_{\tau^t_n \wedge s} = \mathbf{x}(t) + \int_t^{\tau^t_n \wedge s} b\big(r, X_{r\wedge\cdot}\big) dr + \int_t^{\tau^t_n \wedge s} \sigma\big(r, X_{r\wedge\cdot}\big) dB_r$, $\forall\, s \in [t,\infty)$. Sending $n \to \infty$ then renders that $P-$a.s., $X_s = \mathbf{x}(t) + \int_t^s b\big(r, X_{r\wedge\cdot}\big) dr + \int_t^s \sigma\big(r, X_{r\wedge\cdot}\big) dB_r$, $\forall\, s \in [t,\infty)$. Viewing SDE (1.4) on $\big(\Omega, \mathcal{F}, \mathbf{G}^{t,P}, P\big)$, we know from Proposition 1.1 that there is a unique $\big\{ \mathcal{G}^{t,P}_{s\vee t} \big\}_{s \in [0,\infty)}-$adapted continuous process satisfying (1.4). Hence, $P\{X_s = X^{t,\mathbf{x}}_s,\ \forall\, s \in [0,\infty)\} = 1$. □

Proof of Theorem 3.1: Fix $(t,\mathbf{w},\mathbf{x}) \in [0,\infty) \times \Omega_0 \times \Omega_X$ and $(y,z) = \big(\{y_i\}_{i\in\mathbb{N}}, \{z_i\}_{i\in\mathbb{N}}\big) \in \Re \times \Re$.

1) We first show that $V(t,\mathbf{x},y,z) \le \overline{V}(t,\mathbf{w},\mathbf{x},y,z)$: If $\mathcal{S}_{t,\mathbf{x}}(y,z) = \emptyset$, then $V(t,\mathbf{x},y,z) = -\infty \le \overline{V}(t,\mathbf{w},\mathbf{x},y,z)$. So we assume $\mathcal{S}_{t,\mathbf{x}}(y,z) \neq \emptyset$ and let $\tau \in \mathcal{S}_{t,\mathbf{x}}(y,z)$. Define a process $\mathcal{B}^{t,\mathbf{w}}_s(\omega) := \mathbf{w}(s\wedge t) + \mathcal{B}^t_{s\vee t}(\omega)$, $\forall\, (s,\omega) \in [0,\infty) \times \mathcal{Q}$ and define a mapping $\Psi \colon \mathcal{Q} \mapsto \overline{\Omega}$ by

$$
\Psi(\omega) := \big(\mathcal{B}^{t,\mathbf{w}}(\omega), \mathcal{X}^{t,\mathbf{x}}(\omega), \tau(\omega) \big) \in \overline{\Omega}, \quad \forall\, \omega \in \mathcal{Q}.
$$

It holds for any $(s,\omega) \in [t,\infty) \times \mathcal{Q}$ that $\overline{W}^t_s(\Psi(\omega)) = \overline{W}_s(\Psi(\omega)) - \overline{W}_t(\Psi(\omega)) = \mathcal{B}^{t,\mathbf{w}}_s(\omega) - \mathcal{B}^{t,\mathbf{w}}_t(\omega) = \mathcal{B}^t_s(\omega)$. Since $\mathcal{X}^{t,\mathbf{x}} = \big\{ \mathcal{X}^{t,\mathbf{x}}_s \big\}_{s \in [0,\infty)}$ is an $\big\{ \mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_{s\vee t} \big\}_{s \in [0,\infty)}-$adapted continuous process and since τ is an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping time, we can deduce that the mapping Ψ is $\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_\infty/\mathscr{B}(\overline{\Omega})-$measurable and is $\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_s/\overline{\mathcal{F}}^t_s-$measurable for any $s\in[t,\infty)$. Let $\overline{P}_\Psi\in\mathfrak{P}(\overline{\Omega})$ be the probability measure induced by Ψ, i.e., $\overline{P}_\Psi(\overline{A}):=\mathfrak{p}\big(\Psi^{-1}(\overline{A})\big)$, $\forall\,\overline{A}\in\mathscr{B}(\overline{\Omega})$.

Fix $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. We define an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$adapted continuous process $\mathcal{M}^{t,\mathbf{x}}_s(\varphi):=\varphi\big(\mathcal{B}^t_s,\mathcal{X}^{t,\mathbf{x}}_s\big)-\int_t^s\overline{b}\big(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot}\big)\cdot D\varphi\big(\mathcal{B}^t_r,\mathcal{X}^{t,\mathbf{x}}_r\big)dr-\frac{1}{2}\int_t^s\overline{\sigma}\,\overline{\sigma}^T\big(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot}\big):D^2\varphi\big(\mathcal{B}^t_r,\mathcal{X}^{t,\mathbf{x}}_r\big)dr$, $\forall\,s\in[t,\infty)$ and define an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping time $\mathsf{T}^{\,t,\mathbf{x}}_n:=\inf\big\{s\in[t,\infty):\big|(\mathcal{B}^t_s,\mathcal{X}^{t,\mathbf{x}}_s)\big|\ge n\big\}\wedge(t+n)$. Applying Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X)=(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B},\mathcal{X}^{t,\mathbf{x}})$ yields that $\big\{\mathcal{M}^{t,\mathbf{x}}_{s\wedge\mathsf{T}^{t,\mathbf{x}}_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}},\mathfrak{p}\big)-$martingale.

Since $\overline{P}_\Psi\big\{\overline{X}_s=\mathbf{x}(s),\,\forall\,s\in[0,t]\big\}=\mathfrak{p}\big\{\overline{X}_s(\Psi)=\mathbf{x}(s),\,\forall\,s\in[0,t]\big\}=\mathfrak{p}\big\{\mathcal{X}^{t,\mathbf{x}}_s=\mathbf{x}(s),\,\forall\,s\in[0,t]\big\}=1$, using Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}_\Psi,\overline{W},\overline{X}\big)$ shows that $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\overline{\mathbf{F}}^t-$adapted continuous process under $\overline{P}_\Psi$. Given $\omega\in\mathcal{Q}$, since $\overline{W}^t_s\big(\Psi(\omega)\big)=\mathcal{B}^t_s(\omega)$, $\forall\,s\in[t,\infty)$, we see that $\big(\overline{M}^t_s(\varphi)\big)\big(\Psi(\omega)\big)=\big(\mathcal{M}^{t,\mathbf{x}}_s(\varphi)\big)(\omega)$, $\forall\,s\in[t,\infty)$ and $\overline{\tau}^t_n\big(\Psi(\omega)\big)=\mathsf{T}^{t,\mathbf{x}}_n(\omega)$. Then

$$\begin{aligned}\big(\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)\big)\big(\Psi(\omega)\big)&=\big(\overline{M}^t(\varphi)\big)\big(s\wedge\overline{\tau}^t_n(\Psi(\omega)),\Psi(\omega)\big)=\big(\overline{M}^t(\varphi)\big)\big(s\wedge\mathsf{T}^{t,\mathbf{x}}_n(\omega),\Psi(\omega)\big)\\&=\big(\mathcal{M}^{t,\mathbf{x}}(\varphi)\big)\big(s\wedge\mathsf{T}^{t,\mathbf{x}}_n(\omega),\omega\big)=\big(\mathcal{M}^{t,\mathbf{x}}_{s\wedge\mathsf{T}^{t,\mathbf{x}}_n}(\varphi)\big)(\omega),\ \ \forall\,(s,\omega)\in[t,\infty)\times\mathcal{Q}.\end{aligned}\tag{6.4}$$

Let $t_1,t_2\in[t,\infty)$ with $t_1<t_2$ and let $\overline{A}\in\overline{\mathcal{F}}^t_{t_1}$. As $\Psi^{-1}(\overline{A})\in\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_{t_1}$, the $\big(\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}},\mathfrak{p}\big)-$martingality of $\big\{\mathcal{M}^{t,\mathbf{x}}_{s\wedge\mathsf{T}^{t,\mathbf{x}}_n}(\varphi)\big\}_{s\in[t,\infty)}$ and (6.4) imply that

$$\begin{aligned}E_{\overline{P}_\Psi}\Big[\big(\overline{M}^t_{t_2\wedge\overline{\tau}^t_n}(\varphi)-\overline{M}^t_{t_1\wedge\overline{\tau}^t_n}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]&=E_{\mathfrak{p}}\Big[\Big(\big(\overline{M}^t_{t_2\wedge\overline{\tau}^t_n}(\varphi)\big)(\Psi)-\big(\overline{M}^t_{t_1\wedge\overline{\tau}^t_n}(\varphi)\big)(\Psi)\Big)\mathbf{1}_{\Psi^{-1}(\overline{A})}\Big]\\&=E_{\mathfrak{p}}\Big[\big(\mathcal{M}^{t,\mathbf{x}}_{t_2\wedge\mathsf{T}^t_n}(\varphi)-\mathcal{M}^{t,\mathbf{x}}_{t_1\wedge\mathsf{T}^t_n}(\varphi)\big)\mathbf{1}_{\Psi^{-1}(\overline{A})}\Big]=0.\end{aligned}$$

So $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\big(\overline{\mathbf{F}}^t,\overline{P}_\Psi\big)-$martingale. By Remark 3.1, $\overline{P}_\Psi$ satisfies (D1) and (D2) of Definition 3.1.

Since $W^t_s(\mathcal{B}^{t,\mathbf{w}}(\omega))=\mathcal{B}^{t,\mathbf{w}}_s(\omega)-\mathcal{B}^{t,\mathbf{w}}_t(\omega)=\mathcal{B}^t_s(\omega)$ for any $(s,\omega)\in[t,\infty)\times\Omega$, taking $(\Omega,\mathcal{F},P,B,\Phi)=(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B},\mathcal{B}^{t,\mathbf{w}})$ in Lemma A.2 (2) shows that $\mathfrak{p}\big\{\tau=\widehat{\tau}(\mathcal{B}^{t,\mathbf{w}})\big\}=1$ for some $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}$ on Ω_0, it follows that $\overline{P}_\Psi\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}=\mathfrak{p}\big\{\overline{T}(\Psi)=\widehat{\tau}(\overline{W}(\Psi))\big\}=\mathfrak{p}\big\{\tau=\widehat{\tau}(\mathcal{B}^{t,\mathbf{w}})\big\}=1$. As $\overline{W}_s\big(\Psi(\omega)\big)=\mathcal{B}^{t,\mathbf{w}}_s(\omega)=\mathbf{w}(s)$, $\forall\,(s,\omega)\in[0,t]\times\mathcal{Q}$, it is clear that $\overline{P}_\Psi\big\{\overline{W}_s=\mathbf{w}(s),\,\forall\,s\in[0,t]\big\}=\mathfrak{p}\big\{\overline{W}_s(\Psi)=\mathbf{w}(s),\,\forall\,s\in[0,t]\big\}=1$. Thus $\overline{P}_\Psi\in\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}$. For any $i\in\mathbb{N}$, $E_{\overline{P}_\Psi}\Big[\int_t^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]=E_{\mathfrak{p}}\Big[\int_t^{\overline{T}(\Psi)}g_i\big(r,\overline{X}_{r\wedge\cdot}(\Psi)\big)dr\Big]=E_{\mathfrak{p}}\big[\int_t^\tau g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]\le y_i$ and $E_{\overline{P}_\Psi}\Big[\int_t^{\overline{T}}h_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]=E_{\mathfrak{p}}\big[\int_t^\tau h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]=z_i$, which means that $\overline{P}_\Psi\in\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}(y,z)$. Similarly, we can deduce that

$$\begin{aligned}&E_{\mathfrak{p}}\Big[\int_t^\tau f(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr+\mathbf{1}_{\{\tau<\infty\}}\pi\big(\tau,\mathcal{X}^{t,\mathbf{x}}_{\tau\wedge\cdot}\big)\Big]\\&\qquad=E_{\overline{P}_\Psi}\Big[\int_t^{\overline{T}}f\big(r,\overline{X}_{r\wedge\cdot}\big)dr+\mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big]\le\overline{V}(t,\mathbf{w},\mathbf{x},y,z).\end{aligned}$$

Taking supremum over $\tau\in\mathcal{S}_{t,\mathbf{x}}(y,z)$ yields that $V(t,\mathbf{x},y,z)\le\overline{V}(t,\mathbf{w},\mathbf{x},y,z)$.

2) As $\overline{\mathcal{P}}_{t,\mathbf{w},\mathbf{x}}(y,z)\subset\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$, we automatically have $\overline{V}(t,\mathbf{w},\mathbf{x},y,z)\le\underline{\overline{V}}(t,\mathbf{x},y,z)$. It remains to demonstrate that $\overline{V}(t,\mathbf{x},y,z)\le V(t,\mathbf{x},y,z)$. If $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)=\emptyset$, then $\overline{V}(t,\mathbf{x},y,z)=-\infty\le V(t,\mathbf{x},y,z)$.

Assume $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)\neq\emptyset$ and let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$. Given $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$, $\overline{M}^{t,\mathbf{x}}_s(\varphi):=\varphi(\overline{W}^t_s,\overline{\mathscr{X}}^{t,\mathbf{x}}_s)-\int_t^s\overline{b}(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot})\cdot D\varphi(\overline{W}^t_r,\overline{\mathscr{X}}^{t,\mathbf{x}}_r)dr-\frac{1}{2}\int_t^s\overline{\sigma}\,\overline{\sigma}^T(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot}):D^2\varphi(\overline{W}^t_r,\overline{\mathscr{X}}^{t,\mathbf{x}}_r)dr$, $s\in[t,\infty)$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$adapted continuous process and $\overline{\tau}^{t,\mathbf{x}}_n:=\inf\{s\in[t,\infty):\big|(\overline{W}^t_s,\overline{\mathscr{X}}^{t,\mathbf{x}}_s)\big|\geq n\}\wedge(t+n)$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$stopping time. Since $\overline{W}^t$ is a Brownian motion under $\overline{P}$ by (D1) of Definition 3.1, applying Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X)=(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{W},\overline{\mathscr{X}}^{t,\mathbf{x}})$ shows that

$$\big\{\overline{M}^{t,\mathbf{x}}_{s\wedge\overline{\tau}^{t,\mathbf{x}}_n}(\varphi)\big\}\text{ is a bounded }\mathbf{F}^{\overline{W}^t,\overline{P}}-\text{martingale.}\tag{6.5}$$

Let $(\mathfrak{x}_o,\mathfrak{t}_o)$ be an arbitrary pair in $\Omega_X\times[t,\infty]$ and define a mapping $\Psi_o:\mathcal{Q}\mapsto\overline{\Omega}$ by $\Psi_o(\omega):=(\mathcal{B}(\omega),\mathfrak{x}_o,\mathfrak{t}_o)\in\overline{\Omega}$, $\forall\,\omega\in\mathcal{Q}$. (Actually, we are indifferent to the second and third components of $\Psi_o(\omega)$.) Since $\overline{W}^t_s(\Psi_o(\omega))=\overline{W}_s(\Psi_o(\omega))-\overline{W}_t(\Psi_o(\omega))=\mathcal{B}^t_s(\omega)$ for any $(s,\omega)\in[t,\infty)\times\mathcal{Q}$ and since $\overline{W}^t$ is a Brownian motion under $\overline{P}$ by (D1) of Definition 3.1, applying Lemma A.1 with $t_0=t$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B})$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{W})$ and $\Phi=\Psi_o$ yields that

$$\Psi_o^{-1}\big(\mathcal{F}^{\overline{W}^t}_s\big)=\mathcal{F}^{\mathcal{B}^t}_s,\ \Psi_o^{-1}\big(\mathcal{F}^{\overline{W}^t,\overline{P}}_s\big)\subset\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_s,\ \forall\,s\in[t,\infty]\quad\text{and}\tag{6.6}$$

$$\big(\mathfrak{p}\circ\Psi_o^{-1}\big)(\overline{A})=\overline{P}(\overline{A}),\quad\forall\,\overline{A}\in\mathcal{F}^{\overline{W}^t,\overline{P}}_\infty.\tag{6.7}$$

Then $\mathscr{X}^{t,\mathbf{x}}_s(\omega):=\overline{\mathscr{X}}^{t,\mathbf{x}}_s(\Psi_o(\varpi))$, $s\in[0,\infty)$ defines an $\{\mathcal{F}^{\mathcal{B}^t,\mathfrak{p}}_{s\vee t}\}_{s\in[0,\infty)}-$adapted continuous process.

Let $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. We define an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$adapted continuous process $\mathscr{M}^{t,\mathbf{x}}_s(\varphi):=\varphi(\mathcal{B}^t_s,\mathscr{X}^{t,\mathbf{x}}_s)-\int_t^s\overline{b}(r,\mathscr{X}^{t,\mathbf{x}}_{r\wedge\cdot})\cdot D\varphi(\mathcal{B}^t_r,\mathscr{X}^{t,\mathbf{x}}_r)dr-\frac{1}{2}\int_t^s\overline{\sigma}\,\overline{\sigma}^T(r,\mathscr{X}^{t,\mathbf{x}}_{r\wedge\cdot}):D^2\varphi(\mathcal{B}^t_r,\mathscr{X}^{t,\mathbf{x}}_r)dr$, $\forall\,s\in[t,\infty)$ and define an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping time $\zeta^{t,\mathbf{x}}_n:=\inf\{s\in[t,\infty):\big|(\mathcal{B}^t_s,\mathscr{X}^{t,\mathbf{x}}_s)\big|\geq n\}\wedge(t+n)$.

Applying Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X)=(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B},\mathscr{X}^{t,\mathbf{x}})$ and using an analogy to (6.4) renders that $\big\{\mathscr{M}^{t,\mathbf{x}}_{s\wedge\zeta^{t,\mathbf{x}}_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$adapted continuous process under $\mathfrak{p}$ satisfying

$$\big(\overline{M}^{t,\mathbf{x}}_{s\wedge\overline{\tau}^{t,\mathbf{x}}_n}(\varphi)\big)\big(\Psi_o(\omega)\big)=\big(\mathscr{M}^{t,\mathbf{x}}_{s\wedge\zeta^{t,\mathbf{x}}_n}(\varphi)\big)(\omega),\quad\forall\,(s,\omega)\in[t,\infty)\times\mathcal{Q}.\tag{6.8}$$

Let $t_1,t_2\in[t,\infty)$ with $t_1<t_2$ and let $A\in\mathcal{F}^{\mathcal{B}^t}_{t_1}$. Since $\Psi_o^{-1}(\overline{A})=A$ for some $\overline{A}\in\mathcal{F}^{\overline{W}^t}_{t_1}$ by (6.6), we can derive from (6.5), (6.7) and (6.8) that $0=E_{\overline{P}}\Big[\big(\overline{M}^{t,\mathbf{x}}_{t_2\wedge\overline{\tau}^{t,\mathbf{x}}_n}(\varphi)-\overline{M}^{t,\mathbf{x}}_{t_1\wedge\overline{\tau}^{t,\mathbf{x}}_n}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=E_{\mathfrak{p}}\Big[\Big(\overline{M}^{t,\mathbf{x}}_{t_2\wedge\overline{\tau}^{t,\mathbf{x}}_n}(\varphi)-\overline{M}^{t,\mathbf{x}}_{t_1\wedge\overline{\tau}^{t,\mathbf{x}}_n}(\varphi)\Big)(\Psi_o)\mathbf{1}_{\Psi_o^{-1}(\overline{A})}\Big]=E_{\mathfrak{p}}\Big[\big(\mathscr{M}^{t,\mathbf{x}}_{t_2\wedge\zeta^{t,\mathbf{x}}_n}(\varphi)-\mathscr{M}^{t,\mathbf{x}}_{t_1\wedge\zeta^{t,\mathbf{x}}_n}(\varphi)\big)\mathbf{1}_A\Big]$, which implies that $\big\{\mathscr{M}^{t,\mathbf{x}}_{s\wedge\zeta^{t,\mathbf{x}}_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$martingale. Then an application of Proposition 1.2 with $(\Omega,\mathcal{F},P,B,X)=(\mathcal{Q},\mathcal{F},\mathfrak{p},\mathcal{B},\mathscr{X}^{t,\mathbf{x}})$ shows that

$$\mathfrak{p}\{\mathscr{X}^{t,\mathbf{x}}_s=\mathcal{X}^{t,\mathbf{x}}_s,\ \forall\,s\in[0,\infty)\}=1.\tag{6.9}$$

By (D3) of Definition 3.1, there exists a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\gamma}$ on Ω_0 such that $\overline{P}\{\overline{T}=\widehat{\gamma}(\overline{W})\}=1$. Lemma A.2 (1) renders that $\gamma:=\widehat{\gamma}(\mathcal{B})$ is an $\mathbf{F}^{\mathcal{B}^t,\mathfrak{p}}-$stopping time on $\mathcal{Q}$ while $\widehat{\gamma}(\overline{W})$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$stopping time on $\overline{\Omega}$. For any $i\in\mathbb{N}$, we can deduce from (D2) of Definition 3.1, (6.7) and (6.9) that

$$\begin{aligned}y_i&\geq E_{\overline{P}}\bigg[\int_t^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\bigg]=E_{\overline{P}}\bigg[\int_t^{\widehat{\gamma}(\overline{W})}g_i(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot})dr\bigg]\\&=E_{\mathfrak{p}}\bigg[\int_t^{\widehat{\gamma}(\overline{W}(\Psi_o))}g_i(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot}(\Psi_o))dr\bigg]=E_{\mathfrak{p}}\bigg[\int_t^{\gamma}g_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\bigg],\end{aligned}\tag{6.10}$$

and similarly that $E_{\mathfrak{p}}\big[\int_t^{\gamma} h_i(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big]=E_{\overline{P}}\big[\int_t^{\overline{T}} h_i(r,\overline{X}_{r\wedge\cdot})dr\big]=z_i$. So $\gamma\in\mathcal{S}_{t,\mathbf{x}}(y,z)$. Analogous to (6.10),

$$E_{\overline{P}}\bigg[\int_t^{\overline{T}} f\big(r,\overline{X}_{r\wedge\cdot}\big)dr+\mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\bigg]=E_{\mathfrak{p}}\bigg[\int_t^{\gamma} f\big(r,\mathcal{X}^{t,\mathbf{x}}_{r\wedge\cdot}\big)dr+\mathbf{1}_{\{\gamma<\infty\}}\pi\big(\gamma,\mathcal{X}^{t,\mathbf{x}}_{\gamma\wedge\cdot}\big)\bigg]$$
$$\le V(t,\mathbf{x},y,z).$$

Taking supremum over $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ yields that $\overline{V}(t,\mathbf{x},y,z)\le V(t,\mathbf{x},y,z)$. □

Proof of Lemma 4.1: 1) We first show that the metric space $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$ is complete. Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$ such that $\sup\limits_{k\in\mathbb{N}}\rho_{\mathfrak{S}}(\tau_n,\tau_{n+k})<2^{-n}$ for any $n\in\mathbb{N}$.

For any $n\in\mathbb{N}$, the monotone convergence theorem implies that

$$E_{P_0}\Big[\sup_{k\in\mathbb{N}}\big|\arctan(\tau_n)-\arctan(\tau_{n+k})\big|\Big]\le E_{P_0}\Big[\sum_{k\in\mathbb{N}}\big|\arctan(\tau_{n+k-1})-\arctan(\tau_{n+k})\big|\Big]$$
$$=\sum_{k\in\mathbb{N}}\rho_{\mathfrak{S}}\big(\tau_{n+k-1},\tau_{n+k}\big)\le\sum_{k\in\mathbb{N}}2^{1-n-k}=2^{1-n}.$$

So $\lim\limits_{n\to\infty}E_{P_0}\Big[\sup\limits_{k\in\mathbb{N}}\big|\arctan(\tau_n)-\arctan(\tau_{n+k})\big|\Big]=0$. Then one can extract a subsequence $\big\{\tau_{n_j}\big\}_{j\in\mathbb{N}}$ of $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{j\to\infty}\Big(\sup\limits_{k\in\mathbb{N}}\big|\arctan(\tau_{n_j}(\omega_0))-\arctan(\tau_{k+n_j}(\omega_0))\big|\Big)=0$ for all $\omega_0\in\Omega_0$ except on a P_0-null set $\mathcal{N}$. Given $\omega_0\in\mathcal{N}^c$, we see that $\lim\limits_{j\to\infty}\Big(\sup\limits_{\ell\in\mathbb{N}}\big|\arctan(\tau_{n_j}(\omega_0))-\arctan(\tau_{n_{j+\ell}}(\omega_0))\big|\Big)=0$, i.e., $\big\{\arctan\big(\tau_{n_j}(\omega_0)\big)\big\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $[0,\pi/2]$. Let $\xi_*(\omega_0)$ be the limit of $\big\{\arctan\big(\tau_{n_j}(\omega_0)\big)\big\}_{j\in\mathbb{N}}$ in $[0,\pi/2]$.

As $\mathbf{F}^{W,P_0}$ is a right-continuous complete filtration, Lemma 1.2.11 of [36] implies that $\tau_*:=\varliminf\limits_{j\to\infty}\tau_{n_j}$ is an $\mathbf{F}^{W,P_0}-$stopping time on Ω_0 satisfying

$$\arctan\big(\tau_*(\omega_0)\big)=\arctan\Big(\sup_{j\in\mathbb{N}}\inf_{\ell\ge j}\tau_{n_\ell}(\omega_0)\Big)=\sup_{j\in\mathbb{N}}\arctan\Big(\inf_{\ell\ge j}\tau_{n_\ell}(\omega_0)\Big)$$
$$=\sup_{j\in\mathbb{N}}\inf_{\ell\ge j}\arctan\Big(\tau_{n_\ell}(\omega_0)\Big)=\varliminf_{j\to\infty}\arctan\big(\tau_{n_j}(\omega_0)\big)=\lim_{j\to\infty}\arctan\big(\tau_{n_j}(\omega_0)\big)=\xi_*(\omega_0),$$

$\forall\,\omega_0\in\mathcal{N}^c$. Applying the bounded convergence theorem renders that $\lim\limits_{j\to\infty}\rho_{\mathfrak{S}}(\tau_{n_j},\tau_*)=\lim\limits_{j\to\infty}E_{P_0}\big[\big|\arctan(\tau_{n_j})-\arctan(\tau_*)\big|\big]=0$.

We next let $\{\tau_n\}_{n\in\mathbb{N}}$ be a general Cauchy sequence in $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$. For any $j\in\mathbb{N}$, there exists $n_j\in\mathbb{N}$ such that $\sup\limits_{k\in\mathbb{N}}\rho_{\mathfrak{S}}(\tau_{n_j},\tau_{k+n_j})<2^{-j}$. In particular, the subsequence $\big\{\widetilde{\tau}_j:=\tau_{n_j}\big\}_{j\in\mathbb{N}}$ of $\{\tau_n\}_{n\in\mathbb{N}}$ satisfies that $\sup\limits_{\ell\in\mathbb{N}}\rho_{\mathfrak{S}}\big(\widetilde{\tau}_j,\widetilde{\tau}_{j+\ell}\big)<2^{-j}$ for any $j\in\mathbb{N}$ and thus has a limit $\widetilde{\tau}_*$ in $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$ by the above argument. Let $\varepsilon\in(0,1)$. There exists a $\mathfrak{k}\in\mathbb{N}$ with $\mathfrak{k}\ge1-\log_2\varepsilon$ such that $\rho_{\mathfrak{S}}\big(\widetilde{\tau}_{\mathfrak{k}},\widetilde{\tau}_*\big)\le\varepsilon/2$. Then it holds for any $j\ge n_{\mathfrak{k}}$ that $\rho_{\mathfrak{S}}\big(\tau_j,\widetilde{\tau}_*\big)\le\rho_{\mathfrak{S}}\big(\tau_j,\widetilde{\tau}_{\mathfrak{k}}\big)+\rho_{\mathfrak{S}}\big(\widetilde{\tau}_{\mathfrak{k}},\widetilde{\tau}_*\big)\le\sup\limits_{\ell\in\mathbb{N}}\rho_{\mathfrak{S}}\big(\tau_{n_{\mathfrak{k}}},\tau_{n_{\mathfrak{k}}+\ell}\big)+\varepsilon/2<2^{-\mathfrak{k}}+\varepsilon/2\le\varepsilon$. So $\lim\limits_{j\to\infty}\rho_{\mathfrak{S}}\big(\tau_j,\widetilde{\tau}_*\big)=0$, which shows the completeness of $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$.

2) We need some technical preparation for constructing a countable dense subset of $\mathfrak{S}$.

Fix $s\in[0,\infty)$. Given $\delta\in\mathbb{Q}_+$, set $O^s_\delta(\omega_0):=\Big\{\omega_0'\in\Omega_0\colon \sup\limits_{r\in[0,s]}\big|\omega_0'(r)-\omega_0(r)\big|<\delta\Big\}$. Since Ω_0 is a continuous-path space, we can deduce that

$$\begin{aligned}O^s_\delta(\omega_0)&=\underset{n\in\mathbb{N}}{\cup}\ \underset{r\in(0,s)\cap\mathbb{Q}}{\cap}\big\{\omega_0'\in\Omega_0: |\omega_0'(r)-\omega_0(r)|\le\delta-\delta/n\big\}\\&=\underset{n\in\mathbb{N}}{\cup}\ \underset{r\in(0,s)\cap\mathbb{Q}}{\cap}\big\{\omega_0'\in\Omega_0: W_r(\omega_0')\in\overline{O}_{\delta-\delta/n}\big(\omega_0(r)\big)\big\}\in\mathcal{F}^W_s.\end{aligned}$$

Let $\mathfrak{T}_s(\Omega_0)$ collect the empty set $\emptyset$ and all subsets $\mathcal{O}$ of Ω_0 such that for any $\omega_0\in\mathcal{O}$ there exists some $\delta\in(0,1)$ satisfying $O^s_\delta(\omega_0)\subset\mathcal{O}$. Obviously, $\mathfrak{T}_s(\Omega_0)$ forms a topology on Ω_0.

We claim that for any $A\in\mathcal{F}^W_s$ and $\varepsilon\in(0,1)$,

(6.11) there are $\mathcal{O}_1,\mathcal{O}_2\in\mathfrak{T}_s(\Omega_0)$ such that $\mathcal{O}_1^c\subset A\subset\mathcal{O}_2$ and $P_0(A\cap\mathcal{O}_1)\vee P_0(A^c\cap\mathcal{O}_2)<\varepsilon$.

To see this, we define $\Lambda_s:=\{A\in\mathscr{B}(\Omega_0)$: for any $\varepsilon\in(0,1)$ there exist $\mathcal{O}_1,\mathcal{O}_2$ in $\mathfrak{T}_s(\Omega_0)$ such that $\mathcal{O}_1^c\subset A\subset\mathcal{O}_2$ and $P_0(A\cap\mathcal{O}_1)\vee P_0(A^c\cap\mathcal{O}_2)<\varepsilon\}$. Clearly, $\emptyset,\Omega_0\in\Lambda_s$ as they both belong to $\mathfrak{T}_s(\Omega_0)$. It is also easy to see that $A^c\in\Lambda_s$ if $A\in\Lambda_s$.

Let $\{A_n\}_{n\in\mathbb{N}}\subset\Lambda_s$ and $\varepsilon\in(0,1)$. For any $n\in\mathbb{N}$, there exist $\mathcal{O}^1_n,\mathcal{O}^2_n$ in $\mathfrak{T}_s(\Omega_0)$ such that $(\mathcal{O}^1_n)^c\subset A_n\subset\mathcal{O}^2_n$ and $P_0(A_n\cap\mathcal{O}^1_n)\vee P_0(A_n^c\cap\mathcal{O}^2_n)<\varepsilon 2^{-1-n}$. The set $\mathcal{O}_2:=\underset{n\in\mathbb{N}}{\cup}\mathcal{O}^2_n\in\mathfrak{T}_s(\Omega_0)$ contains $A:=\underset{n\in\mathbb{N}}{\cup}A_n$ and satisfies $P_0(A^c\cap\mathcal{O}_2)\le\sum\limits_{n\in\mathbb{N}}P_0(A^c\cap\mathcal{O}^2_n)\le\sum\limits_{n\in\mathbb{N}}P_0(A_n^c\cap\mathcal{O}^2_n)<\varepsilon/2$. Similarly, it holds for $\mathcal{E}:=\underset{n\in\mathbb{N}}{\cap}\mathcal{O}^1_n$ that $P_0(A\cap\mathcal{E})\le\sum\limits_{n\in\mathbb{N}}P_0(A_n\cap\mathcal{O}^1_n)<\varepsilon/2$. We can find an $N\in\mathbb{N}$ such that $P_0\Big(\overset{N}{\underset{n=1}{\cap}}\mathcal{O}^1_n\Big)<P_0(\mathcal{E})+\varepsilon/2$. Then $\mathcal{O}_1:=\overset{N}{\underset{n=1}{\cap}}\mathcal{O}^1_n$ is a set of $\mathfrak{T}_s(\Omega_0)$ satisfying that $\mathcal{O}_1^c=\overset{N}{\underset{n=1}{\cup}}(\mathcal{O}^1_n)^c\subset\underset{n\in\mathbb{N}}{\cup}A_n=A$ and $P_0(A\cap\mathcal{O}_1)=P_0(A\cap\mathcal{E})+P_0\big(A\cap(\mathcal{O}_1\backslash\mathcal{E})\big)\le P_0(A\cap\mathcal{E})+P_0(\mathcal{O}_1\backslash\mathcal{E})<\varepsilon$, which shows $\underset{n\in\mathbb{N}}{\cup}A_n=A\in\Lambda_s$. Hence Λ_s is a sigma-field of Ω_0.

Let $r\in[0,s]$ and let O be a nonempty open subset of $\mathbb{R}^d$. Given $\omega_0\in W_r^{-1}(O)$, there exists $\delta\in(0,1)$ such that $O_\delta\big(W_r(\omega_0)\big)\subset O$. As $W_r\big(O^s_\delta(\omega_0)\big)\subset O_\delta\big(W_r(\omega_0)\big)$, we obtain that $O^s_\delta(\omega_0)\subset W_r^{-1}(O)$ and thus $W_r^{-1}(O)\in\mathfrak{T}_s(\Omega_0)$. Let $\varepsilon\in(0,1)$ and define closed sets $D_n:=\{x\in\mathbb{R}^d: \mathrm{dist}(x,O^c)\ge 1/n\}$, $\forall\, n\in\mathbb{N}$. Since $\underset{n\in\mathbb{N}}{\cap}W_r^{-1}(O\backslash D_n)=W_r^{-1}\Big(\underset{n\in\mathbb{N}}{\cap}(O\backslash D_n)\Big)=\emptyset$, there exists N such that $P_0\big(W_r^{-1}(O\backslash D_N)\big)<\varepsilon$. Similar to the inclusion $W_r^{-1}(O)\in\mathfrak{T}_s(\Omega_0)$, one has $\mathcal{O}_1:=W_r^{-1}(D_N^c)\in\mathfrak{T}_s(\Omega_0)$. Since $\mathcal{O}_1^c=W_r^{-1}(D_N)\subset W_r^{-1}(O)$ and $P_0(W_r^{-1}(O)\cap\mathcal{O}_1)=P_0\Big(W_r^{-1}(O\cap D_N^c)\Big)<\varepsilon$, we see that $W_r^{-1}(O)\in\Lambda_s$. It follows that $\mathcal{F}^W_s=\sigma\big(W_r^{-1}(O); r\in[0,s]$, open subset O of $\mathbb{R}^d\big)\subset\Lambda_s$. So (6.11) holds.

Let $\big\{\omega^i_0\big\}_{i\in\mathbb{N}}$ be a countable dense subset of Ω_0 and let $s\in[0,\infty)$. We set $\Theta_s:=\big\{O^s_\delta(\omega^i_0)\colon \delta\in\mathbb{Q}_+,\, i\in\mathbb{N}\big\}\subset\mathcal{F}^W_s$. Let $A\in\mathcal{F}^W_s$ and $\varepsilon\in(0,1)$. By (6.11), there exists $\mathcal{O}_2\in\mathfrak{T}_s(\Omega_0)$ such that $A\subset\mathcal{O}_2$ and $P_0(\mathcal{O}_2)-P_0(A)<\varepsilon$. As usual, $\mathcal{O}_2$ is the union of some sequence $\big\{O_i\big\}_{i\in\mathbb{N}}$ in Θ_s. So A satisfies that

$$A\subset\underset{i\in\mathbb{N}}{\cup}O_i\quad\text{and}\quad P_0(A)>P_0\Big(\underset{i\in\mathbb{N}}{\cup}O_i\Big)-\varepsilon. \tag{6.12}$$

3) Now we are ready to demonstrate the separability of $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$.

Given $q\in\mathbb{Q}_+$, let us simply denote by $\{O^q_j\}_{j\in\mathbb{N}}$ the countable sub-collection $\Theta_q=\big\{O^q_\delta(\omega^i_0)\colon \delta\in\mathbb{Q}_+,\, i\in\mathbb{N}\big\}$ of $\mathcal{F}^W_q$ and define $\Upsilon^q_{k,\alpha}:=\Big\{q\mathbf{1}_{\underset{j\in I}{\cup}O^q_j}+\alpha\mathbf{1}_{\underset{j\in I}{\cap}(O^q_j)^c}: I\subset\{1,\cdots,k\}\Big\}\subset\mathfrak{S}$, $\forall\, k\in\mathbb{N}$, $\forall\,\alpha\in\mathbb{N}\cap[q,\infty)$. For any $k,n\in\mathbb{N}$, we set $\widehat{\Upsilon}_{k,n}:=\underset{\alpha\in\mathbb{N}}{\cup}\Big\{\overset{2^n\alpha}{\underset{i=1}{\wedge}}\tau_i: \tau_i\in\Upsilon^{i2^{-n}}_{k,\alpha},\, i=$

$1,\cdots,2^n\alpha\Big\}$, which is a countable subset of $\mathfrak{S}$. Then $\widehat{\Upsilon}:=\underset{k,n\in\mathbb{N}}{\cup}\widehat{\Upsilon}_{k,n}$ is also a countable subset of $\mathfrak{S}$. To show $\widehat{\Upsilon}$ is dense in $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$, we let $\tau\in\mathfrak{S}$, $\varepsilon\in(0,1)$ and try to pick $\gamma\in\widehat{\Upsilon}$ such that

$$\rho_{\mathfrak{S}}(\tau,\gamma)<\varepsilon. \tag{6.13}$$

Since $\lim\limits_{\alpha\to\infty}\rho_{\mathfrak{S}}(\tau,\tau\wedge\alpha)=\lim\limits_{\alpha\to\infty}E_{P_0}\big[\big|\arctan(\tau)-\arctan(\tau\wedge\alpha)\big|\big]=0$, one can find $\widehat{\alpha}\in\mathbb{N}$ such that $\rho_{\mathfrak{S}}(\tau,\tau\wedge\widehat{\alpha})<\varepsilon/4$.

Let $n\in\mathbb{N}$ and set $s^n_0:=0$. Given $i\in\{0,1,\cdots,2^n\widehat{\alpha}\}$, we set $s^n_i:=i2^{-n}$ and $A^n_i:=\{s^n_{i-1}\le\tau<s^n_i\}\in\mathcal{F}^{W,P_0}_{s^n_i}$. By e.g. Problem 2.7.3 of [36], there exists $\mathcal{A}^n_i\in\mathcal{F}^W_{s^n_i}$ such that $\mathcal{N}^n_i:=A^n_i\Delta\mathcal{A}^n_i\in\mathscr{N}_{P_0}(\mathcal{F}^W_\infty)$. Define $\widetilde{\mathcal{A}}^n_i:=\mathcal{A}^n_i\Big\backslash\Big(\underset{j<i}{\cup}\mathcal{A}^n_j\Big)\in\mathcal{F}^W_{s^n_i}$ and $\widetilde{\mathcal{A}}_n:=\overset{2^n\widehat{\alpha}}{\underset{i=1}{\cup}}\widetilde{\mathcal{A}}^n_i=\overset{2^n\widehat{\alpha}}{\underset{i=1}{\cup}}\mathcal{A}^n_i\in\mathcal{F}^W_{\widehat{\alpha}}$. The $\mathbf{F}^{W,P_0}-$stopping time $\tau_n:=\sum\limits_{i=1}^{2^n\widehat{\alpha}}s^n_i\mathbf{1}_{A^n_i}+\widehat{\alpha}\mathbf{1}_{\{\tau\ge\widehat{\alpha}\}}$ coincides with the $\mathbf{F}^W-$stopping time $\widetilde{\tau}_n:=\sum\limits_{i=1}^{2^n\widehat{\alpha}}s^n_i\mathbf{1}_{\widetilde{\mathcal{A}}^n_i}+\widehat{\alpha}\mathbf{1}_{\widetilde{\mathcal{A}}^c_n}$ over $\Omega_n:=\Big(\overset{2^n\widehat{\alpha}}{\underset{i=1}{\cup}}\big(A^n_i\cap\widetilde{\mathcal{A}}^n_i\big)\Big)\cup\Big(\{\tau\ge\widehat{\alpha}\}\cap\widetilde{\mathcal{A}}^c_n\Big)$. We can deduce that

$$\begin{aligned} A^n_i\backslash\widetilde{\mathcal{A}}^n_i&=A^n_i\cap\Big[\big(\mathcal{A}^n_i\big)^c\cup\Big(\underset{j<i}{\cup}\mathcal{A}^n_j\Big)\Big]=\big(A^n_i\cap(\mathcal{A}^n_i)^c\big)\cup\Big(\underset{j<i}{\cup}\big(\mathcal{A}^n_j\cap A^n_i\big)\Big)\\ &\subset\big(A^n_i\Delta\mathcal{A}^n_i\big)\cup\Big(\underset{j<i}{\cup}\big(\mathcal{A}^n_j\cap(A^n_j)^c\big)\Big)\subset\underset{j\le i}{\cup}\mathcal{N}^n_j\in\mathscr{N}_{P_0}(\mathcal{F}^W_\infty), \end{aligned} \tag{6.14}$$

for $i=1,\cdots,2^n\widehat{\alpha}$ and that $\{\tau\ge\widehat{\alpha}\}\cap\widetilde{\mathcal{A}}_n=\overset{2^n\widehat{\alpha}}{\underset{i=1}{\cup}}\big(\{\tau\ge\widehat{\alpha}\}\cap\mathcal{A}^n_i\big)\subset\overset{2^n\widehat{\alpha}}{\underset{i=1}{\cup}}\big((A^n_i)^c\cap\mathcal{A}^n_i\big)\subset\overset{2^n\widehat{\alpha}}{\underset{i=1}{\cup}}\mathcal{N}^n_i\in\mathscr{N}_{P_0}(\mathcal{F}^W_\infty)$. Putting them together shows that $\Omega^c_n=\Big(\overset{2^n\widehat{\alpha}}{\underset{i=1}{\cup}}\big(A^n_i\backslash\widetilde{\mathcal{A}}^n_i\big)\Big)\cup\Big(\{\tau\ge\widehat{\alpha}\}\cap\widetilde{\mathcal{A}}_n\Big)$ belongs to $\mathscr{N}_{P_0}(\mathcal{F}^W_\infty)$. To wit,

$$\tau_n=\widetilde{\tau}_n,\quad P_0-a.s. \tag{6.15}$$

Since $\lim\limits_{n\to\infty}\downarrow\tau_n=\tau\wedge\widehat{\alpha}$, one has $\lim\limits_{n\to\infty}\rho_{\mathfrak{S}}(\tau\wedge\widehat{\alpha},\tau_n)=\lim\limits_{n\to\infty}E_{P_0}\big[\big|\arctan(\tau\wedge\widehat{\alpha})-\arctan(\tau_n)\big|\big]=0$. So there exists $\mathfrak{n}\in\mathbb{N}$ such that $\rho_{\mathfrak{S}}\big(\tau\wedge\widehat{\alpha},\tau_{\mathfrak{n}}\big)<\varepsilon/4$.

Given $i\in\{1,\cdots,2^{\mathfrak{n}}\widehat{\alpha}\}$, we know from (6.12) that for some sequence $\{O^i_j\}_{j\in\mathbb{N}}$ in $\Theta_{s^{\mathfrak{n}}_i}=\big\{O^{s^{\mathfrak{n}}_i}_j\big\}_{j\in\mathbb{N}}$

$$\widetilde{\mathcal{A}}^{\mathfrak{n}}_i\subset\underset{j\in\mathbb{N}}{\cup}O^i_j\quad\text{and}\quad P_0\big(\widetilde{\mathcal{A}}^{\mathfrak{n}}_i\big)>P_0\Big(\underset{j\in\mathbb{N}}{\cup}O^i_j\Big)-\frac{\varepsilon}{2^{2+2\mathfrak{n}}\pi\widehat{\alpha}^2}. \tag{6.16}$$

And we can find $\ell_i\in\mathbb{N}$ such that $\mathcal{O}_i:=\overset{\ell_i}{\underset{j=1}{\cup}}O^i_j\in\mathcal{F}^W_{s^{\mathfrak{n}}_i}$ satisfies

$$P_0(\mathcal{O}_i)>P_0\Big(\underset{j\in\mathbb{N}}{\cup}O^i_j\Big)-\frac{\varepsilon}{2^{2+2\mathfrak{n}}\pi\widehat{\alpha}^2}. \tag{6.17}$$

Clearly, $\gamma_i:=s^{\mathfrak{n}}_i\mathbf{1}_{\mathcal{O}_i}+\widehat{\alpha}\mathbf{1}_{\mathcal{O}^c_i}\in\Upsilon^{s^{\mathfrak{n}}_i}_{k_i,\widehat{\alpha}}$ for some $k_i\in\mathbb{N}$.

Let $i\in\{1,\cdots,2^{\mathfrak{n}}\widehat{\alpha}\}$ and set $\widetilde{\mathcal{O}}_i:=\mathcal{O}_i\backslash\underset{\mathfrak{i}<i}{\cup}\mathcal{O}_{\mathfrak{i}}\in\mathcal{F}^W_{s^{\mathfrak{n}}_i}$. Analogous to (6.14), $\widetilde{A}^{\mathfrak{n}}_i\backslash\widetilde{\mathcal{O}}_i=\widetilde{A}^{\mathfrak{n}}_i\cap\Big[\mathcal{O}^c_i\cup\big(\underset{\mathfrak{i}<i}{\cup}\mathcal{O}_{\mathfrak{i}}\big)\Big]\subset\Big(\big(\underset{j\in\mathbb{N}}{\cup}O^i_j\big)\cap\mathcal{O}^c_i\Big)\cup\Big(\underset{\mathfrak{i}<i}{\cup}\big(\mathcal{O}_{\mathfrak{i}}\cap(\widetilde{A}^{\mathfrak{n}}_{\mathfrak{i}})^c\big)\Big)$. So (6.16) and (6.17) yield that

$$P_0(\widetilde{A}^{\mathfrak{n}}_i\backslash\widetilde{\mathcal{O}}_i)\le P_0\Big(\big(\underset{j\in\mathbb{N}}{\cup}O^i_j\big)\cap\mathcal{O}^c_i\Big)+\sum_{\mathfrak{i}<i}P_0\Big(\big(\underset{j\in\mathbb{N}}{\cup}O^{\mathfrak{i}}_j\big)\cap(\widetilde{A}^{\mathfrak{n}}_{\mathfrak{i}})^c\Big) \tag{6.18}$$

$$<\sum_{\mathfrak{i}\le i}\frac{\varepsilon}{2^{2+2\mathfrak{n}}\pi\widehat{\alpha}^2}=\frac{i\varepsilon}{2^{2+2\mathfrak{n}}\pi\widehat{\alpha}^2}\le\frac{\varepsilon}{2^{2+\mathfrak{n}}\pi\widehat{\alpha}}.$$

Define $\widetilde{\mathcal{O}}:=\underset{i=1}{\overset{2^{\mathfrak{n}}\widehat{\alpha}}{\cup}}\widetilde{\mathcal{O}}_i=\underset{i=1}{\overset{2^{\mathfrak{n}}\widehat{\alpha}}{\cup}}\mathcal{O}_i\in\mathcal{F}^W_{\widehat{\alpha}}$ and $\mathfrak{k}:=\max\{k_i: i=1,\cdots,2^{\mathfrak{n}}\widehat{\alpha}\}$. Then $\gamma:=\underset{i=1}{\overset{2^{\mathfrak{n}}\widehat{\alpha}}{\wedge}}\gamma_i$ is a stopping time of $\widehat{\Upsilon}_{\mathfrak{k},\mathfrak{n}}$ (and is thus of $\widehat{\Upsilon}$). In particular, $\gamma=\sum_{i=1}^{2^{\mathfrak{n}}\widehat{\alpha}}s^{\mathfrak{n}}_i\mathbf{1}_{\widetilde{\mathcal{O}}_i}+\widehat{\alpha}\mathbf{1}_{\widetilde{\mathcal{O}}^c}$ is equal to $\widetilde{\tau}_{\mathfrak{n}}=\sum_{i=1}^{2^{\mathfrak{n}}\widehat{\alpha}}s^{\mathfrak{n}}_i\mathbf{1}_{\widetilde{\mathcal{A}}^{\mathfrak{n}}_i}+\widehat{\alpha}\mathbf{1}_{\widetilde{\mathcal{A}}^c_{\mathfrak{n}}}$ over $\widehat{\mathcal{A}}:=\Big(\underset{i=1}{\overset{2^{\mathfrak{n}}\widehat{\alpha}}{\cup}}\big(\widetilde{\mathcal{O}}_i\cap\widetilde{\mathcal{A}}^{\mathfrak{n}}_i\big)\Big)\cup\big(\widetilde{\mathcal{O}}^c\cap\widetilde{\mathcal{A}}^c_{\mathfrak{n}}\big)\in\mathcal{F}^W_{\widehat{\alpha}}$. Since (6.16) implies that

$$P_0\big(\widetilde{\mathcal{O}}\cap\widetilde{\mathcal{A}}^c_{\mathfrak{n}}\big)\le\sum_{i=1}^{2^{\mathfrak{n}}\widehat{\alpha}}P_0\big(\widetilde{\mathcal{O}}_i\cap\widetilde{\mathcal{A}}^c_{\mathfrak{n}}\big)\le\sum_{i=1}^{2^{\mathfrak{n}}\widehat{\alpha}}P_0\big(\widetilde{\mathcal{O}}_i\cap(\widetilde{\mathcal{A}}^{\mathfrak{n}}_i)^c\big)\le\sum_{i=1}^{2^{\mathfrak{n}}\widehat{\alpha}}P_0\big(\mathcal{O}_i\backslash\widetilde{\mathcal{A}}^{\mathfrak{n}}_i\big)$$
$$\le\sum_{i=1}^{2^{\mathfrak{n}}\widehat{\alpha}}P_0\Big\{\Big(\underset{j\in\mathbb{N}}{\cup}O^i_j\Big)\Big\backslash\widetilde{\mathcal{A}}^{\mathfrak{n}}_i\Big\}<\sum_{i=1}^{2^{\mathfrak{n}}\widehat{\alpha}}\frac{\varepsilon}{2^{2+2\mathfrak{n}}\pi\widehat{\alpha}^2}=\frac{\varepsilon}{2^{2+\mathfrak{n}}\pi\widehat{\alpha}},$$

(6.18) renders that $P_0\big(\widehat{\mathcal{A}}^c\big)=P_0\Big\{\Big(\underset{i=1}{\overset{2^{\mathfrak{n}}\widehat{\alpha}}{\cup}}\big(\widetilde{\mathcal{A}}^{\mathfrak{n}}_i\backslash\widetilde{\mathcal{O}}_i\big)\Big)\cup\big(\widetilde{\mathcal{O}}\cap\widetilde{\mathcal{A}}^c_{\mathfrak{n}}\big)\Big\}=\sum_{i=1}^{2^{\mathfrak{n}}\widehat{\alpha}}P_0\big(\widetilde{\mathcal{A}}^{\mathfrak{n}}_i\backslash\widetilde{\mathcal{O}}_i\big)+P_0\big(\widetilde{\mathcal{O}}\cap\widetilde{\mathcal{A}}^c_{\mathfrak{n}}\big)<(2^{\mathfrak{n}}\widehat{\alpha}+1)\frac{\varepsilon}{2^{2+\mathfrak{n}}\pi\widehat{\alpha}}\le\frac{\varepsilon}{2\pi}$ and thus $\rho_{\mathfrak{S}}\big(\widetilde{\tau}_{\mathfrak{n}},\gamma\big)=E_{P_0}\big[\mathbf{1}_{\widehat{\mathcal{A}}^c}\big|\arctan(\widetilde{\tau}_{\mathfrak{n}})-\arctan(\gamma)\big|\big]\le\pi P_0\big(\widehat{\mathcal{A}}^c\big)<\varepsilon/2$. By (6.15), it follows that $\rho_{\mathfrak{S}}(\tau,\gamma)\le\rho_{\mathfrak{S}}(\tau,\tau\wedge\widehat{\alpha})+\rho_{\mathfrak{S}}\big(\tau\wedge\widehat{\alpha},\tau_{\mathfrak{n}}\big)+\rho_{\mathfrak{S}}\big(\widetilde{\tau}_{\mathfrak{n}},\gamma\big)<\varepsilon$, proving (6.13). Therefore, $\big(\mathfrak{S},\rho_{\mathfrak{S}}\big)$ is a complete separable space. □

Proof of Lemma 4.2: 1) We first show that Γ is injective: Set $\mathbb{Q}_\pi:=\big(\mathbb{Q}\cap[0,\pi/2)\big)\cup\{\pi/2\}$ and let $\tau_1,\tau_2\in\mathfrak{S}$ such that $\Gamma(\tau_1)=\Gamma(\tau_2)$.

Given $q\in\mathbb{Q}_\pi$ and $n\in\mathbb{N}$, we define $\mathcal{E}^q_n:=(q-1/n,q+1/n)\cap[0,\pi/2]$ and $A^i_{n,q}:=\big\{\arctan(\tau_i)\in\mathcal{E}^q_n\big\}\in\mathcal{F}^{W,P_0}_\infty$ for $i=1,2$. Then $A_{n,q}:=A^1_{n,q}\cap(A^2_{n,q})^c$ satisfies

$$\begin{aligned}(6.19)\quad P_0(A_{n,q})&=P_0\big\{\omega_0\in(A^2_{n,q})^c:\tau_1(\omega_0)\in\tan(\mathcal{E}^q_n)\big\}=\big(\Gamma(\tau_1)\big)\big((A^2_{n,q})^c\times\tan(\mathcal{E}^q_n)\big)\\&=\big(\Gamma(\tau_2)\big)\big((A^2_{n,q})^c\times\tan(\mathcal{E}^q_n)\big)=P_0\big\{\omega_0\in(A^2_{n,q})^c:\tau_2(\omega_0)\in\tan(\mathcal{E}^q_n)\big\}=P_0(\emptyset)=0.\end{aligned}$$

Clearly, $\underset{n\in\mathbb{N}}{\cup}\;\underset{q\in\mathbb{Q}_\pi}{\cup}A_{n,q}\subset\{\tau_1\ne\tau_2\}$. To see the reverse inclusion, we let $\omega_0\in\Big(\underset{n\in\mathbb{N}}{\cup}\;\underset{q\in\mathbb{Q}_\pi}{\cup}A_{n,q}\Big)^c$ and let $n\in\mathbb{N}$. There exists $\mathfrak{q}=\mathfrak{q}(n)\in\mathbb{Q}_\pi$ such that $\arctan\big(\tau_1(\omega_0)\big)\in\mathcal{E}^{\mathfrak{q}}_n$, or $\omega_0\in\big\{\arctan(\tau_1)\in\mathcal{E}^{\mathfrak{q}}_n\big\}=A^1_{n,\mathfrak{q}}$. As $\omega_0\in A^c_{n,\mathfrak{q}}$, we see that $\omega_0\in A^2_{n,\mathfrak{q}}$, i.e., $\arctan(\tau_2(\omega_0))$ also belongs to $\mathcal{E}^{\mathfrak{q}}_n$. It follows that $\rho_+\big(\tau_1(\omega_0),\tau_2(\omega_0)\big)=\big|\arctan(\tau_1(\omega_0))-\arctan(\tau_2(\omega_0))\big|<2/n$. Letting $n\to\infty$ yields that $\tau_1(\omega_0)=\tau_2(\omega_0)$. So $\underset{n\in\mathbb{N}}{\cup}\;\underset{q\in\mathbb{Q}_\pi}{\cup}A_{n,q}=\{\tau_1\ne\tau_2\}$. It follows from (6.19) that $P_0\{\tau_1\ne\tau_2\}=0$, which means that $\tau_1=\tau_2$ in $\mathfrak{S}$. Hence, the mapping $\Gamma:\mathfrak{S}\mapsto\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$ is injective.

2) We next discuss the continuity of Γ: Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{S}$ that converges to a $\tau\in\mathfrak{S}$ under $\rho_{\mathfrak{S}}$. We need to show that $P^n:=\Gamma(\tau_n)$ converges to $P:=\Gamma(\tau)$ under the weak topology of $\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$, i.e.

$$(6.20)\quad \lim_{n\to\infty}\int_{(\omega_0,\mathfrak{t})\in\Omega_0\times\mathbb{T}}\phi(\omega_0,\mathfrak{t})P^n\big(d(\omega_0,\mathfrak{t})\big)=\int_{(\omega_0,\mathfrak{t})\in\Omega_0\times\mathbb{T}}\phi(\omega_0,\mathfrak{t})P\big(d(\omega_0,\mathfrak{t})\big)$$

for any bounded continuous function $\phi:\Omega_0\times\mathbb{T}\mapsto\mathbb{R}$.

Let ϕ be a bounded continuous function on $\Omega_0\times\mathbb{T}$. For (6.20), it suffices to show that for any subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ of $\{\tau_n\}_{n\in\mathbb{N}}$, we can find a subsequence $\big\{\tau_{n'_k}\big\}_{k\in\mathbb{N}}$ of $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ satisfying (6.20).

Let $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ be an arbitrary subsequence of $\{\tau_n\}_{n\in\mathbb{N}}$. As $0=\lim\limits_{k\to\infty}\rho_{\mathfrak{S}}\big(\tau_{n_k},\tau\big)=\lim\limits_{k\to\infty}E_{P_0}\big[\rho_+(\tau_{n_k},\tau)\big]$, one can extract a subsequence $\big\{n'_k\big\}_{k\in\mathbb{N}}$ from $\big\{n_k\big\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty}\rho_+\big(\tau_{n'_k}(\omega_0),\tau(\omega_0)\big)=0$ for all $\omega_0\in\Omega_0$ except on a P_0-null set $\mathcal{N}$. Given $\omega_0\in\mathcal{N}^c$, since $\lim\limits_{k\to\infty}\rho_+\big(\tau_{n'_k}(\omega_0),\tau(\omega_0)\big)=0$, the continuity of ϕ renders that $\lim\limits_{k\to\infty}\phi\big(\omega_0,\tau_{n'_k}(\omega_0)\big)=\phi\big(\omega_0,\tau(\omega_0)\big)$. Applying the bounded convergence theorem yields that

$$\lim_{k\to\infty}\int_{(\omega_0,\mathfrak{t})\in\Omega_0\times\mathbb{T}}\phi(\omega_0,\mathfrak{t})P^{n'_k}\big(d(\omega_0,\mathfrak{t})\big)=\lim_{k\to\infty}\int_{\Omega_0}\phi\big(\omega_0,\tau_{n'_k}(\omega_0)\big)P_0(d\omega_0)$$
$$=\int_{\Omega_0}\phi\big(\omega_0,\tau(\omega_0)\big)P_0(d\omega_0)=\int_{(\omega_0,\mathfrak{t})\in\Omega_0\times\mathbb{T}}\phi(\omega_0,\mathfrak{t})P\big(d(\omega_0,\mathfrak{t})\big).\qquad\square$$

Proof of Proposition 4.1: Fix $(t,\mathbf{x})\in[0,\infty)\times\Omega_X$.

1) Let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}$. It is clear that $\overline{P}\in\overline{\mathcal{P}}^1_{t,\mathbf{x}}$. Let $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. By (D1$'$) of Remark 3.1, $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $(\overline{\mathbf{F}}^t,\overline{P})-$martingale. For any $(\mathfrak{s},\mathfrak{r})\in\mathbb{Q}^{2,<}_+$ and $\{(s_i,\mathcal{O}_i)\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})$, as $\big\{(\overline{W}^t_{t+s_i},\overline{X}_{t+s_i})\in\mathcal{O}_i\big\}\in\overline{\mathcal{F}}^t_{t+\mathfrak{s}}$ for $i=1,\cdots,k$, one directly has $E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod\limits_{i=1}^k\mathbf{1}_{\{(\overline{W}^t_{t+s_i},\overline{X}_{t+s_i})\in\mathcal{O}_i\}}\Big]=0$. So $\overline{P}\in\overline{\mathcal{P}}^2_t$.

By (D3$'$) of Remark 3.1, there exists a $[0,\infty]-$valued $\mathbf{F}^{W,P_0}-$stopping time $\ddot{\tau}$ on Ω_0 such that $\overline{P}\big\{\overline{T}=t+\ddot{\tau}\big(\overline{\mathscr{W}}^t\big)\big\}=1$. Since $\overline{\mathscr{W}}^t_{\mathfrak{s}}=\overline{W}_{t+\mathfrak{s}}-\overline{W}_t$, $\mathfrak{s}\in[0,\infty)$ is a Brownian motion under $\overline{P}$ by (D1) of Definition 3.1, applying Lemma A.1 with $t_0=0$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{\mathscr{W}}^t\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W\big)$ and $\Phi=\overline{\mathscr{W}}^t$ shows that

$$\overline{P}\circ\big(\overline{\mathscr{W}}^t\big)^{-1}(A_0)=P_0(A_0),\qquad\forall\,A_0\in\mathcal{F}^{W,P_0}_\infty.\tag{6.21}$$

For any $\mathcal{A}_0\in\mathscr{B}(\Omega_0)=\mathcal{F}^W_\infty$ and $\mathcal{E}\in\mathscr{B}(\mathbb{T})$, since $\ddot{\tau}^{-1}(\mathcal{E})\in\mathcal{F}^{W,P_0}_\infty$, we can derive that

$$\overline{P}\circ\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)^{-1}(\mathcal{A}_0\times\mathcal{E})=\overline{P}\big\{\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)\in\mathcal{A}_0\times\mathcal{E}\big\}=\overline{P}\big\{\big(\overline{\mathscr{W}}^t,\ddot{\tau}(\overline{\mathscr{W}}^t)\big)\in\mathcal{A}_0\times\mathcal{E}\big\}$$
$$=\overline{P}\circ\big(\overline{\mathscr{W}}^t\big)^{-1}\big\{(W,\ddot{\tau})\in\mathcal{A}_0\times\mathcal{E}\big\}=\overline{P}\circ\big(\overline{\mathscr{W}}^t\big)^{-1}\big(\mathcal{A}_0\cap\ddot{\tau}^{-1}(\mathcal{E})\big)=P_0\big(\mathcal{A}_0\cap\ddot{\tau}^{-1}(\mathcal{E})\big)$$
$$=P_0\big\{(W,\ddot{\tau})\in\mathcal{A}_0\times\mathcal{E}\big\}=P_0\circ(W,\ddot{\tau})^{-1}(\mathcal{A}_0\times\mathcal{E}).$$

Then Dynkin's Pi-Lambda Theorem implies that $\overline{P}\circ\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)^{-1}=P_0\circ(W,\ddot{\tau})^{-1}$ on $\mathscr{B}\big(\Omega_0\times\mathbb{T}\big)$. i.e., $\overline{P}\circ\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)^{-1}=\Gamma(\ddot{\tau})\in\Gamma(\mathfrak{S})$. So $\overline{P}$ also belongs to $\overline{\mathcal{P}}^3_t$, which shows $\overline{\mathcal{P}}_{t,\mathbf{x}}\subset\overline{\mathcal{P}}^1_{t,\mathbf{x}}\cap\overline{\mathcal{P}}^2_t\cap\overline{\mathcal{P}}^3_t$.

2a) Let $\overline{P}\in\overline{\mathcal{P}}^1_{t,\mathbf{x}}\cap\overline{\mathcal{P}}^2_t$. To see that $\overline{P}$ satisfies (D1$'$) of Remark 3.1, we take $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. As $\overline{P}\big\{\overline{X}_s=\mathbf{x}(s),\,\forall\,s\in[0,t]\big\}=1$, applying Proposition 1.2 with $\big(\Omega,\mathcal{F},P,B,X\big)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{W},\overline{X}\big)$ implies that $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is a bounded $\overline{\mathbf{F}}^t-$adapted continuous process under $\overline{P}$.

Let $(\mathfrak{s},\mathfrak{r})\in\mathbb{Q}^{2,<}_+$, $\big\{(t_i,\mathcal{O}_i)\big\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,t]\big)\times\mathscr{O}(\mathbb{R}^l)$ and $\big\{(s_j,\mathcal{O}'_j)\big\}_{j=1}^m\subset\big(\mathbb{Q}\cap(0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})$. If $\mathbf{x}(t_{\mathfrak{i}})\notin\mathcal{O}_{\mathfrak{i}}$ for some $\mathfrak{i}\in\{1,\cdots,k\}$, then $\overline{P}\{\overline{X}_{t_{\mathfrak{i}}}\in\mathcal{O}_{\mathfrak{i}}\}=0$ and thus

$$E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod_{i=1}^k\mathbf{1}_{\{\overline{X}_{t_i}\in\mathcal{O}_i\}}\prod_{j=1}^m\mathbf{1}_{\{(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j})\in\mathcal{O}'_j\}}\Big]=0.$$

On the other hand, if $\mathbf{x}(t_i)\in\mathcal{O}_i$ for each $i\in\{1,\cdots,k\}$, then

$$E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod_{i=1}^k\mathbf{1}_{\{\overline{X}_{t_i}\in\mathcal{O}_i\}}\prod_{j=1}^m\mathbf{1}_{\{(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j})\in\mathcal{O}'_j\}}\Big]$$

$$=E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod_{j=1}^m\mathbf{1}_{\{(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j})\in\mathcal{O}'_j\}}\Big]=0.$$

So the Lambda-system $\overline{\Lambda}^{t,n}_{\mathfrak{s},\mathfrak{r}}:=\Big\{\overline{A}\in\mathscr{B}(\overline{\Omega}): E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=0\Big\}$ includes the Pi-system $\Big\{\Big(\bigcap_{i=1}^k\overline{X}^{-1}_{t_i}(\mathcal{O}_i)\Big)\cap\Big(\bigcap_{j=1}^m(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j})^{-1}(\mathcal{O}'_j)\Big): \{(t_i,\mathcal{O}_i)\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,t]\big)\times\mathscr{O}(\mathbb{R}^l),\ \{(s_j,\mathcal{O}'_j)\}_{j=1}^m\subset\big(\mathbb{Q}\cap(0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})\Big\}$, which generates $\overline{\mathcal{F}}^t_{t+\mathfrak{s}}$. Dynkin's Pi-Lambda Theorem renders that $\overline{\mathcal{F}}^t_{t+\mathfrak{s}}\subset\overline{\Lambda}^{t,n}_{\mathfrak{s},\mathfrak{r}}$, i.e.,

$$E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=0,\quad\forall\,\overline{A}\in\overline{\mathcal{F}}^t_{t+\mathfrak{s}}.\tag{6.22}$$

Let $t\le s<r<\infty$ and $\overline{A}\in\overline{\mathcal{F}}^t_s$. Taking $(\mathfrak{s},\mathfrak{r})=\Big(\frac{\lceil(s-t)2^k\rceil}{2^k},\frac{1+\lceil(r-t)2^k\rceil}{2^k}\Big)$, $k\in\mathbb{N}$ in (6.22) and sending $k\to\infty$, we can deduce from the continuity of bounded process $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ that $E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge r}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge s}(\varphi)\big)\mathbf{1}_{\overline{A}}\Big]=0$. So $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)\big\}_{s\in[t,\infty)}$ is an $\big(\overline{\mathbf{F}}^t,\overline{P}\big)-$martingale. By Remark 3.1, $\overline{P}$ satisfies (D1) and (D2) of Definition 3.1.

2b) Let $\overline{P}\in\overline{\mathcal{P}}^1_{t,\mathbf{x}}\cap\overline{\mathcal{P}}^2_t\cap\overline{\mathcal{P}}^3_t$. There exists a $[0,\infty]-$valued $\mathbf{F}^{W,P_0}-$stopping time $\ddot{\tau}$ on Ω_0 such that $\overline{P}\circ\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)^{-1}=\Gamma(\ddot{\tau})=P_0\circ(W,\ddot{\tau})^{-1}$. We still have (6.21) since $\overline{\mathscr{W}}^t$ is a Brownian motion under $\overline{P}$ by (D1) of Definition 3.1. Given $D\in\mathscr{B}(\Omega_0\times\mathbb{T})$, taking $A_0=\big(W,\ddot{\tau}\big)^{-1}(D)\in\mathcal{F}^{W,P_0}_\infty$ in (6.21) yields that

$$\begin{aligned}\overline{P}\big\{\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)\in D\big\}&=\overline{P}\circ\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)^{-1}(D)=P_0\circ(W,\ddot{\tau})^{-1}(D)\\&=\overline{P}\circ\big(\overline{\mathscr{W}}^t\big)^{-1}\big(\big(W,\ddot{\tau}\big)^{-1}(D)\big)=\overline{P}\big\{\big(\overline{\mathscr{W}}^t,\ddot{\tau}(\overline{\mathscr{W}}^t)\big)\in D\big\}.\end{aligned}$$

So the joint distribution of $(\overline{\mathscr{W}}^t,\overline{T}-t)$ is the same as that of $\big(\overline{\mathscr{W}}^t,\ddot{\tau}(\overline{\mathscr{W}}^t)\big)$ under $\overline{P}$. In particular, the $\overline{P}-$law of $\overline{T}$ is equal to the $\overline{P}-$law of $t+\ddot{\tau}(\overline{\mathscr{W}}^t)$ and therefore $\overline{P}$ satisfies (D3') of Remark 3.1 or equivalently (D3) of Definition 3.1. □

Proof of Lemma 4.3: Let $\{t_n\}_{n\in\mathbb{N}}\subset[0,\infty)$ converge to $t\in[0,\infty)$ and let $\{\overline{P}_n\}_{n\in\mathbb{N}}\subset\mathfrak{P}\big(\overline{\Omega}\big)$ converge to $\overline{P}\in\mathfrak{P}\big(\overline{\Omega}\big)$ under the weak topology of $\mathfrak{P}\big(\overline{\Omega}\big)$ $\big($i.e., $\lim_{n\to\infty}\int_{\overline{\omega}\in\overline{\Omega}}\phi(\overline{\omega})\overline{P}_n(d\overline{\omega})=\int_{\overline{\omega}\in\overline{\Omega}}\phi(\overline{\omega})\overline{P}(d\overline{\omega})$ for any bounded continuous function $\phi:\overline{\Omega}\mapsto\mathbb{R}\big)$. To see that $\big\{\overline{\Gamma}(t_n,\overline{P}_n)=\overline{P}_n\circ\big(\overline{\mathscr{W}}^{t_n},\overline{T}-t_n\big)^{-1}\big\}_{n\in\mathbb{N}}$ converges to $\overline{\Gamma}(t,\overline{P})=\overline{P}\circ\big(\overline{\mathscr{W}}^t,\overline{T}-t\big)^{-1}$ under the weak topology of $\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$, we let $\psi:\Omega_0\times\mathbb{T}\mapsto\mathbb{R}$ be a bounded continuous function and show that

$$\begin{aligned}&\lim_{n\to\infty}\int_{(\omega_0,\lambda)\in\Omega_0\times\mathbb{T}}\psi(\omega_0,\lambda)\big(\overline{P}_n\circ(\overline{\mathscr{W}}^{t_n},\overline{T}-t_n)^{-1}\big)\big(d(\omega_0,\lambda)\big)\\&\quad=\int_{(\omega_0,\lambda)\in\Omega_0\times\mathbb{T}}\psi(\omega_0,\lambda)\big(\overline{P}\circ(\overline{\mathscr{W}}^t,\overline{T}-t)^{-1}\big)\big(d(\omega_0,\lambda)\big).\end{aligned}$$

Set $\|\psi\|_\infty := \sup\limits_{(\omega_0,\lambda)\in\Omega_0\times\mathbb{T}} \big|\psi(\omega_0,\lambda)\big|$ and let $\varepsilon\in(0,1)$. Since the weakly convergent sequence $\{\overline{P}_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathfrak{P}\big(\overline{\Omega}\big)$, Prohorov's Theorem yields that $\{\overline{P}_n\}_{n\in\mathbb{N}}$ is tight, i.e., $\sup\limits_{n\in\mathbb{N}} \overline{P}_n\big(\overline{\mathcal{K}}^c_\varepsilon\big) \le \dfrac{\varepsilon}{4\|\psi\|_\infty}$ for some compact subset $\overline{\mathcal{K}}_\varepsilon$ of $\overline{\Omega}$.

The topology of locally uniform convergence on Ω_0 implies that $(s,\overline{\omega})\mapsto \overline{\mathscr{W}}^s(\overline{\omega})$ is a continuous mapping from $[0,\infty)\times\overline{\Omega}$ to Ω_0 and $\overline{\Phi}(s,\overline{\omega}) := \big(\overline{\mathscr{W}}^s(\overline{\omega}), \overline{T}(\overline{\omega})-s\big)$ is thus a continuous mapping from $[0,\infty)\times\overline{\Omega}$ to $\Omega_0\times\mathbb{T}$. There exists $\delta\in(0,1)$ such that $\big|\psi\circ\overline{\Phi}(s,\overline{\omega}) - \psi\circ\overline{\Phi}(s',\overline{\omega}')\big| < \varepsilon/4$ for any $(s,\overline{\omega}),(s',\overline{\omega}')\in[0,t+1]\times\overline{\mathcal{K}}_\varepsilon$ with $|s-s'|\vee\rho_{\overline{\Omega}}(\overline{\omega},\overline{\omega}')<\delta$. And one can find $N\in\mathbb{N}$ such that $\big|\int_{\overline{\omega}\in\overline{\Omega}}\psi\circ\overline{\Phi}(t,\overline{\omega})\overline{P}_n(d\overline{\omega}) - \int_{\overline{\omega}\in\overline{\Omega}}\psi\circ\overline{\Phi}(t,\overline{\omega})\overline{P}(d\overline{\omega})\big| < \frac{\varepsilon}{4}$ and $|t_n-t|<\delta$ for any $n\ge N$.

For any $n\ge N$, we can deduce that

$$
\begin{aligned}
&\Big|\int_{(\omega_0,\lambda)\in\Omega_0\times\mathbb{T}}\psi(\omega_0,\lambda)\big(\overline{P}_n\circ(\overline{\mathscr{W}}^{t_n},\overline{T}-t_n)^{-1}\big)\big(d(\omega_0,\lambda)\big)\\
&\qquad\qquad -\int_{(\omega_0,\lambda)\in\Omega_0\times\mathbb{T}}\psi(\omega_0,\lambda)\big(\overline{P}\circ(\overline{\mathscr{W}}^{t},\overline{T}-t)^{-1}\big)\big(d(\omega_0,\lambda)\big)\Big|\\
&\le\Big|\int_{\overline{\omega}\in\overline{\Omega}}\Big(\psi\big(\overline{\Phi}(t_n,\overline{\omega})\big)-\psi\big(\overline{\Phi}(t,\overline{\omega})\big)\Big)\overline{P}_n(d\overline{\omega})\Big|\\
&\qquad\qquad +\Big|\int_{\overline{\omega}\in\overline{\Omega}}\psi\big(\overline{\Phi}(t,\overline{\omega})\big)\overline{P}_n(d\overline{\omega})-\int_{\overline{\omega}\in\overline{\Omega}}\psi\big(\overline{\Phi}(t,\overline{\omega})\big)\overline{P}(d\overline{\omega})\Big|\\
&<\int_{\overline{\omega}\in\overline{\mathcal{K}}_\varepsilon}\Big|\psi\big(\overline{\Phi}(t_n,\overline{\omega})\big)-\psi\big(\overline{\Phi}(t,\overline{\omega})\big)\Big|\overline{P}_n(d\overline{\omega})+\int_{\overline{\omega}\in\overline{\mathcal{K}}^c_\varepsilon}\Big|\psi\big(\overline{\Phi}(t_n,\overline{\omega})\big)\Big|\overline{P}_n(d\overline{\omega})\\
&\qquad\qquad +\int_{\overline{\omega}\in\overline{\mathcal{K}}^c_\varepsilon}\Big|\psi\big(\overline{\Phi}(t,\overline{\omega})\big)\Big|\overline{P}_n(d\overline{\omega})+\varepsilon/4\\
&\le\frac{\varepsilon}{4}\overline{P}_n(\overline{\mathcal{K}}_\varepsilon)+2\|\psi\|_\infty\overline{P}_n\big(\overline{\mathcal{K}}^c_\varepsilon\big)+\varepsilon/4\le\varepsilon. \qquad\square
\end{aligned}
$$

Proof of Proposition 4.2: According to Proposition 4.1, $\langle\langle\overline{\mathcal{P}}\rangle\rangle$ is the intersection of $\langle\langle\overline{\mathcal{P}}\rangle\rangle_1 := \big\{\big(t,\mathbf{x},\overline{P}\big)\in[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big): \overline{P}\in\overline{\mathcal{P}}^1_{t,\mathbf{x}}\big\}$ and $\langle\langle\overline{\mathcal{P}}\rangle\rangle_i := \big\{\big(t,\mathbf{x},\overline{P}\big)\in[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big): \overline{P}\in\overline{\mathcal{P}}^i_t\big\}$ for $i=2,3$.

1) Since the function $\mathfrak{l}_2(t,\omega_X):=\omega_X(t\wedge\cdot)$ is continuous in $(t,\omega_X)\in[0,\infty)\times\Omega_X$, the mapping $\psi_X(t,\mathbf{x},\overline{\omega}):=\mathbf{1}_{\{\mathfrak{l}_2(t,\overline{X}(\overline{\omega}))-\mathfrak{l}_2(t,\mathbf{x})=0\}}$, $(t,\mathbf{x},\overline{\omega})\in[0,\infty)\times\Omega_X\times\overline{\Omega}$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}(\overline{\Omega})-$measurable. Lemma A.3 implies that $\Psi_X(t,\mathbf{x},\overline{P}) := \int_{\overline{\omega}\in\overline{\Omega}}\psi_X(t,\mathbf{x},\overline{\omega})\overline{P}(d\overline{\omega}) = \overline{P}\big\{\overline{X}_s=\mathbf{x}(s),\,\forall\,s\in[0,t]\big\}$, $(t,\mathbf{x},\overline{P})\in[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big)$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}\big(\mathfrak{P}\big(\overline{\Omega}\big)\big)-$measurable. So $\langle\langle\overline{\mathcal{P}}\rangle\rangle_1 = \big\{(t,\mathbf{x},\overline{P})\in[0,\infty)\times\Omega_X\times\mathfrak{P}\big(\overline{\Omega}\big): \Psi_X(t,\mathbf{x},\overline{P})=1\big\}\in\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}\big(\mathfrak{P}\big(\overline{\Omega}\big)\big)$.

2) Since $W(s,\omega_0):=\omega_0(s)$ is continuous in $(s,\omega_0)\in[0,\infty)\times\Omega_0$ and $W^X(s,\omega_X):=\omega_X(s)$ is continuous in $(s,\omega_X)\in[0,\infty)\times\Omega_X$, the function $\Xi(t,\mathfrak{s},\omega_0,\omega_X) := \big(W(t+\mathfrak{s},\omega_0)-W(t,\omega_0), W^X(t+\mathfrak{s},\omega_X)\big)$ is continuous in $(t,\mathfrak{s},\omega_0,\omega_X)\in[0,\infty)\times[0,\infty)\times\Omega_0\times\Omega_X$.

Let $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$. The measurability of functions $b,\sigma,\mathfrak{l}_2,\Xi$ imply that the mapping

$$
\begin{aligned}
\mathcal{H}_\varphi(t,\mathfrak{s},r,\omega_0,\omega_X) := \mathbf{1}_{\{t\le r\le t+\mathfrak{s}\}}\Big\{&\overline{b}\big(r,\mathfrak{l}_2(r,\omega_X)\big)\cdot D\varphi\big(\Xi(t,r-t,\omega_0,\omega_X)\big)\\
&+\frac{1}{2}\overline{\sigma}\overline{\sigma}^T\big(r,\mathfrak{l}_2(r,\omega_X)\big):D^2\varphi\big(\Xi(t,r-t,\omega_0,\omega_X)\big)\Big\},
\end{aligned}
$$

$(t,\mathfrak{s},r,\omega_0,\omega_X)\in[0,\infty)\times[0,\infty)\times(0,\infty)\times\Omega_0\times\Omega_X$ is Borel-measurable. For each $(t,\mathfrak{s},\omega_0,\omega_X)\in[0,\infty)\times[0,\infty)\times\Omega_0\times\Omega_X$, an analogy to (6.1) renders that $\int_0^\infty\big|\mathcal{H}_\varphi(t,\mathfrak{s},r,\omega_0,\omega_X)\big|dr<\infty$. So $\mathcal{I}_\varphi(t,\mathfrak{s},\omega_0,\omega_X):=\int_0^\infty\mathcal{H}_\varphi(t,\mathfrak{s},r,\omega_0,\omega_X)dr$ is a real-valued, Borel-measurable mapping on $[0,\infty)\times[0,\infty)\times\Omega_0\times\Omega_X$. Define $\mathscr{T}_n(t,\omega_0,\omega_X):=\inf\big\{\mathfrak{s}\in[0,\infty):|\Xi(t,\mathfrak{s},\omega_0,\omega_X)|\ge n\big\}$, $(t,\omega_0,\omega_X)\in[0,\infty)\times\Omega_0\times\Omega_X$, which is also Borel-measurable since for any $a\in[0,\infty)$,

$$
\begin{aligned}
&\big\{(t,\omega_0,\omega_X)\in[0,\infty)\times\Omega_0\times\Omega_X:\mathscr{T}_n(t,\omega_0,\omega_X)>a\big\}\\
&\quad=\Big\{(t,\omega_0,\omega_X)\in[0,\infty)\times\Omega_0\times\Omega_X:\sup_{a'\in[0,a]}|\Xi(t,a',\omega_0,\omega_X)|<n\Big\}\\
&\quad=\Big(\underset{k\in\mathbb{N}}{\cup}\ \underset{q\in\mathbb{Q}\cap[0,a]}{\cap}\big\{(t,\omega_0,\omega_X)\in[0,\infty)\times\Omega_0\times\Omega_X:|\Xi(t,q,\omega_0,\omega_X)|\le n-1/k\big\}\Big)\\
&\quad\in\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X).
\end{aligned}
$$

For any $\mathfrak{s}\in[0,\infty)$, since the path-valued random variables $(\overline{W},\overline{X})$ on $\overline{\Omega}$ are $\mathscr{B}(\Omega_0)\otimes\mathscr{B}(\Omega_X)-$measurable, we can derive from the Borel measurability of $\mathcal{I}_\varphi$ and $\mathscr{T}_n$ that the mapping

$$
\begin{aligned}
(6.23)\qquad \overline{M}^{\varphi,n}_{\mathfrak{s}}(t,\overline{\omega})&:=(\varphi\circ\Xi-\mathcal{I}_\varphi)\big(t,\mathscr{T}_n\big(t,\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega})\big)\wedge n\wedge\mathfrak{s},\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega})\big)\\
&=\big(\overline{M}^t(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega})\wedge(t+\mathfrak{s}),\overline{\omega}\big),\quad(t,\overline{\omega})\in[0,\infty)\times\overline{\Omega}
\end{aligned}
$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\overline{\Omega})-$measurable, where we used the fact $\overline{\tau}^t_n(\overline{\omega})=t+\mathscr{T}_n\big(t,\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega})\big)\wedge n$.

Let $\theta:=\big(\varphi,n,(\mathfrak{s},\mathfrak{r}),\{(s_i,\mathcal{O}_i)\}_{i=1}^k\big)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})$. Since $\bar{\mathfrak{f}}_\theta(t,\overline{\omega}):=\big(\overline{M}^{\varphi,n}_{\mathfrak{r}}(t,\overline{\omega})-\overline{M}^{\varphi,n}_{\mathfrak{s}}(t,\overline{\omega})\big)\times\prod_{i=1}^k\mathbf{1}_{\{\Xi(t,s_i\wedge\mathfrak{s},\overline{W}(\overline{\omega}),\overline{X}(\overline{\omega}))\in\mathcal{O}_i\}}$, $(t,\overline{\omega})\in[0,\infty)\times\overline{\Omega}$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\overline{\Omega})-$measurable by (6.23), applying Lemma A.3 yields that the mapping $(t,\overline{P})\mapsto\int_{\overline{\omega}\in\overline{\Omega}}\bar{\mathfrak{f}}_\theta(t,\overline{\omega})\overline{P}(d\overline{\omega})$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable and the set $\Big\{(t,\mathbf{x},\overline{P})\in[0,\infty)\times\Omega_X\times\mathfrak{P}(\overline{\Omega}):E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^t_{t+s_i\wedge\mathfrak{s}},\overline{X}_{t+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}\Big]=0\Big\}$ is thus Borel-measurable. Letting θ run through the countable collection $\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})$ shows $\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle_2\in\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)$.

3) We know from Lemma 4.1 and Lemma 4.2 that the mapping $\Gamma:\mathfrak{S}\ni\tau\mapsto P_0\circ(W,\tau)^{-1}\in\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$ is a continuous injection from the Polish space $\mathfrak{S}$ to $\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$ and the image $\Gamma(\mathfrak{S})$ is thus a Lusin subset of $\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$. According to Theorem A.6 of [64], $\Gamma(\mathfrak{S})$ is even a Borel subset of the Borel space $\mathfrak{P}\big(\Omega_0\times\mathbb{T}\big)$. Then Lemma 4.3 implies $\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle_3=\big\{(t,\mathbf{x},\overline{P})\in[0,\infty)\times\Omega_X\times\mathfrak{P}(\overline{\Omega}):\overline{\Gamma}(t,\overline{P})\in\Gamma(\mathfrak{S})\big\}\in\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)$. Totally, $\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle=\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle_1\cap\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle_2\cap\langle\!\langle\overline{\mathcal{P}}\rangle\!\rangle_3$ is a Borel subset of $[0,\infty)\times\Omega_X\times\mathfrak{P}(\overline{\Omega})$. □

Proof of Corollary 4.1: 1) Let $i\in\mathbb{N}$. By the measurability of functions g_i and $\mathfrak{l}_2$ (defined in (1.1)), the mapping $\mathfrak{g}_i(t,s,r,\omega_X):=\mathbf{1}_{\{t\wedge s\le r\le s\}}g_i\big(r,\mathfrak{l}_2(r,\omega_X)\big)$ is Borel-measurable in $(t,s,r,\omega_X)\in[0,\infty)\times[0,\infty)\times(0,\infty)\times\Omega_X$. It follows that

$$
(6.24)\qquad \overline{\mathcal{I}}_{g_i}(t,\overline{\omega}):=\int_0^\infty\mathfrak{g}_i\big(t,\overline{T}(\overline{\omega}),r,\overline{X}(\overline{\omega})\big)dr=\int_{\overline{T}(\overline{\omega})\wedge t}^{\overline{T}(\overline{\omega})}g_i\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega})\big)dr\,,
$$

$(t,\overline{\omega})\in[0,\infty)\times\overline{\Omega}$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\overline{\Omega})-$measurable. Lemma A.3 implies that

$$
\overline{\Phi}_{g_i}(t,\overline{P}):=\int_{\overline{\omega}\in\overline{\Omega}}\overline{\mathcal{I}}_{g_i}(t,\overline{\omega})\overline{P}(d\overline{\omega})=E_{\overline{P}}\Big[\int_{\overline{T}\wedge t}^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big],\ \ (t,\overline{P})\in[0,\infty)\times\mathfrak{P}\big(\overline{\Omega}\big)
$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable. Similarly, $\overline{\Phi}_{h_i}(t,\overline{P}):=E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}h_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\big]$, $(t,\overline{P})\in[0,\infty)\times\mathfrak{P}(\overline{\Omega})$ is $\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable. Then the set

$$\mathscr{D}:=\big\{(t,\mathbf{x},y,z,\overline{P})\in[0,\infty)\times\Omega_X\times\Re\times\Re\times\mathfrak{P}(\overline{\Omega}):\overline{\Phi}_{g_i}(t,\overline{P})\le y_i,\ \overline{\Phi}_{h_i}(t,\overline{P})=z_i,\ \forall\,i\in\mathbb{N}\big\}$$

is Borel-measurable. Since $\langle\langle\overline{\mathcal{P}}\rangle\rangle\in\mathscr{B}[0,\infty)\otimes\mathscr{B}(\Omega_X)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)$ by Proposition 4.1, using the projection $\overline{\Pi}_1(t,\mathbf{x},y,z,\overline{P}):=\big(t,\mathbf{x},\overline{P}\big)$ yields that $[[\overline{\mathcal{P}}]]=\big\{(t,\mathbf{x},y,z,\overline{P})\in[0,\infty)\times\Omega_X\times\Re\times\Re\times\mathfrak{P}(\overline{\Omega}):\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}};\ E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\big]\le y_i,\ E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}h_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\big]=z_i,\ \forall\,i\in\mathbb{N}\big\}=\overline{\Pi}_1^{-1}\big(\langle\langle\overline{\mathcal{P}}\rangle\rangle\big)\cap\mathscr{D}$ is a Borel subset of $\overline{D}\times\mathfrak{P}(\overline{\Omega})$.

2) Similarly to $\Psi_X(t,\mathbf{x},\overline{P})$, defined in the proof of Proposition 4.2, the mapping

$$\Psi_W(t,\mathbf{w},\overline{P}):=\overline{P}\big\{\overline{W}_s=\mathbf{w}(s),\ \forall\,s\in[0,t]\big\}$$

is Borel-measurable in $(t,\mathbf{w},\overline{P})\in[0,\infty)\times\Omega_0\times\mathfrak{P}(\overline{\Omega})$. By the projections $\overline{\Pi}_2(t,\mathbf{w},\mathbf{x},y,z,\overline{P}):=\big(t,\mathbf{x},\overline{P}\big)$, $\overline{\Pi}_3(t,\mathbf{w},\mathbf{x},y,z,\overline{P}):=\big(t,\mathbf{x},y,z,\overline{P}\big)$ and $\overline{\Pi}_4(t,\mathbf{w},\mathbf{x},y,z,\overline{P}):=(t,\mathbf{w},\overline{P})$, we can deduce that $\{\{\overline{\mathcal{P}}\}\}=\overline{\Pi}_2^{-1}\big(\langle\langle\overline{\mathcal{P}}\rangle\rangle\big)\cap\overline{\Pi}_3^{-1}(\mathscr{D})\cap\overline{\Pi}_4^{-1}\big(\Psi_W^{-1}(1)\big)$ is a Borel subset of $\overline{\mathcal{D}}\times\mathfrak{P}(\overline{\Omega})$. $\square$

Proof of Theorem 4.1: Analogous to (6.24), $\overline{\mathcal{I}}_f(t,\overline{\omega}):=\int_{\overline{T}(\overline{\omega})\wedge t}^{\overline{T}(\overline{\omega})}f\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega})\big)dr$ is Borel-measurable in $(t,\overline{\omega})\in[0,\infty)\times\overline{\Omega}$. Since the measurability of functions π and $\mathfrak{l}_2$ $\big($defined in (1.1)$\big)$ implies that the mapping $(s,\overline{\omega})\mapsto\pi\big(s,\mathfrak{l}_2(s,\overline{X}(\overline{\omega}))\big)=\pi\big(s,\overline{X}_{s\wedge\cdot}(\overline{\omega})\big)$ is $\mathscr{B}(0,\infty)\otimes\mathscr{B}(\overline{\Omega})-$measurable, the random variable $\overline{\phi}_\pi(\overline{\omega}):=\mathbf{1}_{\{\overline{T}(\overline{\omega})<\infty\}}\pi\big(\overline{T}(\overline{\omega}),\overline{X}\big(\overline{T}(\overline{\omega})\wedge\cdot,\overline{\omega}\big)\big)$, $\overline{\omega}\in\overline{\Omega}$ is $\mathscr{B}(\overline{\Omega})-$measurable. Lemma A.3 shows that

$$\overline{\mathscr{V}}(t,\overline{P}):=\int_{\overline{\omega}\in\overline{\Omega}}\big(\overline{\mathcal{I}}_f(t,\overline{\omega})+\overline{\phi}_\pi(\overline{\omega})\big)\overline{P}(d\overline{\omega})=E_{\overline{P}}\big[\overline{R}(t)\big],\ (t,\overline{P})\in[0,\infty)\times\mathfrak{P}(\overline{\Omega})$$

is $\mathscr{B}[0,\infty)\otimes\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable. Then Corollary 4.1 and Proposition 7.47 of [14] yield that $\overline{V}(t,\mathbf{x},y,z)=\sup\limits_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}\overline{\mathscr{V}}(t,\overline{P})=\sup\limits_{(t,\mathbf{x},y,z,\overline{P})\in[[\overline{\mathcal{P}}]]}\overline{\mathscr{V}}(t,\overline{P})$ is upper semi-analytic on $\overline{D}$ and $\overline{V}(t,\mathbf{w},\mathbf{x},y,z)=\sup\limits_{(t,\mathbf{w},\mathbf{x},y,z,\overline{P})\in\{\{\overline{\mathcal{P}}\}\}}\overline{\mathscr{V}}(t,\overline{P})$ is upper semi-analytic on $\overline{\mathcal{D}}$. $\square$

Proof of Proposition 5.1: Let us set $t_{\overline{\omega}}:=\overline{\gamma}(\overline{\omega})\ge t$ for any $\overline{\omega}\in\overline{\Omega}$.
1) We first demonstrate that for $\overline{P}-$a.s. $\overline{\omega}\in\overline{\Omega}$, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ belongs to $\overline{\mathcal{P}}^1_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\cap\overline{\mathcal{P}}^2_{t_{\overline{\omega}}}$ and thus satisfies (D1) and (D2) in Definition 3.1 of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$ according to Part (2a) of the proof of Proposition 4.1.
1a) By (D2) in Definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$, $\overline{\mathcal{N}}_X:=\big\{\overline{\omega}\in\overline{\Omega}:\overline{X}_s(\overline{\omega})\ne\overline{\mathscr{X}}^{t,\mathbf{x}}_s(\overline{\omega})$ for some $s\in[0,\infty)\big\}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. As $\big\{\overline{\mathscr{X}}^{t,\mathbf{x}}_s\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^{\overline{W}^t,\overline{P}}-$adapted continuous process, one can construct an $\mathbb{R}^l-$valued $\mathbf{F}^{\overline{W}^t}-$predictable process $\big\{\overline{K}^t_s\big\}_{s\in[t,\infty)}$ such that $\overline{\mathcal{N}}_K:=\big\{\overline{\omega}\in\overline{\Omega}:\overline{K}^t_s(\overline{\omega})\ne\overline{\mathscr{X}}^{t,\mathbf{x}}_s(\overline{\omega})$ for some $s\in[t,\infty)\big\}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$ $\big($see e.g. Lemma 2.4 of [62]$\big)$. Since $\overline{\mathbf{K}}^t_{\overline{\gamma},\overline{\omega}}:=\bigcap\limits_{r\in\mathbb{Q}\cap(t,\infty)}\big\{\overline{\omega}'\in\overline{\Omega}:\overline{K}^t_{\overline{\gamma}\wedge r}(\overline{\omega}')=\overline{K}^t_{\overline{\gamma}\wedge r}(\overline{\omega})\big\}$ is an $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}-$measurable set including $\overline{\omega}$, (5.3) shows that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\overline{\mathbf{K}}^t_{\overline{\gamma},\overline{\omega}}\big)=1$, $\forall\,\overline{\omega}\in\overline{\mathcal{N}}^c_0$.

Given $\overline{\omega}\in\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c$, we can deduce from (5.4) that for any $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c$

$$\overline{\omega}\in\overline{\mathbf{K}}^t_{\overline{\gamma},\overline{\omega}}\iff\overline{X}_s(\overline{\omega}')=\mathbf{x}(s),\ \forall\,s\in[0,t]\text{ and }\overline{K}^t_{\overline{\gamma}(\overline{\omega})\wedge r}(\overline{\omega}')=\overline{K}^t_{\overline{\gamma}(\overline{\omega})\wedge r}(\overline{\omega}),\ \forall\,r\in\mathbb{Q}\cap(t,\infty)$$

$$\iff \overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\ \forall\, s\in[0,t] \text{ and } \overline{X}_{\overline{\gamma}(\overline{\omega})\wedge r}(\overline{\omega}')=\overline{X}_{\overline{\gamma}(\overline{\omega})\wedge r}(\overline{\omega}),\ \forall\, r\in\mathbb{Q}\cap(t,\infty)$$

$$\iff \overline{X}_r(\overline{\omega}')=\overline{X}_{\overline{\gamma}\wedge r}(\overline{\omega}),\ \forall\, r\in[0,\overline{\gamma}(\overline{\omega})]. \tag{6.25}$$

And (5.2) shows that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K)=E_{\overline{P}}\big[\mathbf{1}_{\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K}\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})=0$ for all $\overline{\omega}\in\overline{\Omega}$ except on a $\widehat{\mathcal{N}}_{X,K}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$.

Set $\overline{\mathcal{N}}_1:=\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\cup\widehat{\mathcal{N}}_{X,K}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. Given $\overline{\omega}\in\big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_1\big)^c$, taking $\overline{P}(\cdot)$ in (6.25) and using (5.5) yield that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\{\overline{\omega}'\in\overline{\Omega}\colon \overline{X}_r(\overline{\omega}')=\overline{X}_{\overline{\gamma}\wedge r}(\overline{\omega}),\ \forall\, r\in[0,t_{\overline{\omega}}]\big\}=1$, i.e., $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\in\overline{\mathcal{P}}^1_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$.

1b) For any $\varphi\in\mathfrak{C}(\mathbb{R}^{d+l})$ and $q\in\mathbb{Q}^d$, define a function $\varphi_q(w,x):=\varphi(w-q,x)$, $(w,x)\in\mathbb{R}^{d+l}$. We set $\mathscr{C}:=\{\varphi_q:\varphi\in\mathfrak{C}(\mathbb{R}^{d+l}),q\in\mathbb{Q}^l\}$, which is a countable sub-collection of $C^2(\mathbb{R}^{d+l})$. For any $n\in\mathbb{N}$, define an $\overline{\mathbf{F}}^t-$stopping time by $\overline{\zeta}_n(\overline{\omega}):=\inf\big\{r\in[\overline{\gamma}(\overline{\omega}),\infty)\colon |\overline{W}^t_r(\overline{\omega})-\overline{W}^t_{\overline{\gamma}}(\overline{\omega})|^2+|\overline{X}_r(\overline{\omega})|^2\geq n^2\big\}\wedge\big(\overline{\gamma}(\overline{\omega})+n\big)$, $\overline{\omega}\in\overline{\Omega}$.

Let $\theta:=\big(\phi,n,j,(\mathfrak{s},\mathfrak{r}),\{(s_i,\mathcal{O}_i)\}_{i=1}^k\big)\in\mathscr{C}\times\mathbb{N}\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})$. Since $\big\{\overline{M}^t_{s\wedge\overline{\tau}^t_j}(\phi)\big\}_{s\in[t,\infty)}$ is a bounded $(\overline{\mathbf{F}}^t,\overline{P})-$martingale by applying Proposition 1.2 with $\big(\Omega,\mathcal{F},P,B,X\big)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P},\overline{W},\overline{X}\big)$, the optional sampling theorem implies that

$$E_{\overline{P}}\Big[\overline{M}^t_{(\overline{\gamma}+\mathfrak{r})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)\Big|\overline{\mathcal{F}}^t_{\overline{\gamma}+\mathfrak{s}}\Big]=\overline{M}^t_{(\overline{\gamma}+\mathfrak{s})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi),\quad \overline{P}-\text{a.s.}$$

Set $\overline{\xi}_\theta:=\overline{M}^t_{(\overline{\gamma}+\mathfrak{r})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)-\overline{M}^t_{(\overline{\gamma}+\mathfrak{s})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)=\mathbf{1}_{\{\overline{\tau}^t_j>\overline{\gamma}\}}\big(\overline{M}^t_{(\overline{\gamma}+\mathfrak{r})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)-\overline{M}^t_{(\overline{\gamma}+\mathfrak{s})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j}(\phi)\big)$ and $\overline{\eta}_\theta:=\prod\limits_{i=1}^k\mathbf{1}_{\{(\overline{W}^t_{\overline{\gamma}+s_i\wedge\mathfrak{s}}-\overline{W}^t_{\overline{\gamma}},\overline{X}_{\overline{\gamma}+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}\in\overline{\mathcal{F}}^t_{\overline{\gamma}+\mathfrak{s}}$. As $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\subset\overline{\mathcal{F}}^t_{\overline{\gamma}}\subset\overline{\mathcal{F}}^t_{\overline{\gamma}+\mathfrak{s}}$, the tower property renders that $E_{\overline{P}}\big[\overline{\xi}_\theta\overline{\eta}_\theta\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big]=E_{\overline{P}}\Big[\overline{\eta}_\theta E_{\overline{P}}\big[\overline{\xi}_\theta\big|\overline{\mathcal{F}}^t_{\overline{\gamma}+\mathfrak{s}}\big]\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big]=0$, $\overline{P}-$a.s. By (5.2) again, there exists an $\overline{\mathcal{N}}_\theta\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ such that

$$E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{\xi}_\theta\overline{\eta}_\theta\big]=E_{\overline{P}}\big[\overline{\xi}_\theta\overline{\eta}_\theta\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})=0,\quad \forall\,\overline{\omega}\in\overline{\mathcal{N}}^c_\theta. \tag{6.26}$$

Define $\overline{\mathcal{N}}_2:=\bigcup\big\{\overline{\mathcal{N}}_\theta:\theta\in\mathscr{C}\times\mathbb{N}\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})\big\}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$ and fix $\overline{\omega}\in\big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_1\cup\overline{\mathcal{N}}_2\big)^c$. We let $\big(\varphi,n,(\mathfrak{s},\mathfrak{r}),\{(s_i,\mathcal{O}_i)\}_{i=1}^k\big)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}\times\mathbb{Q}^{2,<}_+\times\widehat{\mathscr{O}}(\mathbb{R}^{d+l})$ and let $j\in\mathbb{N}$. There exists a sequence $\{q_m=q_m(\overline{\omega})\}_{m\in\mathbb{N}}$ of $\mathbb{Q}^d$ that converges to $\overline{W}^t_{\overline{\gamma}}(\overline{\omega})$.

Let $m\in\mathbb{N}$. We set $\theta_m:=\big(\varphi_{q_m},n,j,(\mathfrak{s},\mathfrak{r}),\{(s_i,\mathcal{O}_i)\}_{i=1}^k\big)$ and define

$$\begin{aligned}\delta^{j,m}_{\overline{\omega}}&:=\sup_{|(w,x)|\leq j}\Big(\sum_{i=0}^2\big|D^i\varphi_{q_m}(w,x)-D^i\varphi\big(w-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),x\big)\big|\Big)\\&=\sup_{|(w,x)|\leq j}\Big(\sum_{i=0}^2\big|D^i\varphi(w-q_m,x)-D^i\varphi\big(w-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),x\big)\big|\Big).\end{aligned}$$

Given $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\mathcal{N}}^c_X\cap\big\{\overline{\tau}^t_j>\overline{\gamma}\big\}$, (5.4) implies that $\overline{\tau}^t_j(\overline{\omega}')>\overline{\gamma}(\overline{\omega}')=t_{\overline{\omega}}$ and $\overline{\zeta}_n(\overline{\omega}')=\inf\big\{r\in[t_{\overline{\omega}},\infty)\colon|\overline{W}_r(\overline{\omega}')-\overline{W}_{t_{\overline{\omega}}}(\overline{\omega}')|^2+|\overline{X}_r(\overline{\omega}')|^2\geq n^2\big\}\wedge\big(t_{\overline{\omega}}+n\big)=\overline{\tau}^{t_{\overline{\omega}}}_n(\overline{\omega}')$. As $\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_{t_{\overline{\omega}}}(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega})$, $\forall\, r\in[t_{\overline{\omega}},\infty)$, it holds for any $t_{\overline{\omega}}\leq s_1\leq s_2<\infty$ that

$$\begin{aligned}\big(\overline{M}^{t_{\overline{\omega}}}_{s_2}(\varphi)-\overline{M}^{t_{\overline{\omega}}}_{s_1}(\varphi)\big)(\overline{\omega}')&=\varphi\big(\overline{W}^t_{s_2}(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{X}_{s_2}(\overline{\omega}')\big)-\varphi\big(\overline{W}^t_{s_1}(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{X}_{s_1}(\overline{\omega}')\big)\\&\quad-\int_{s_1}^{s_2}\overline{b}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}')\big)\cdot D\varphi\big(\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{X}_r(\overline{\omega}')\big)dr\end{aligned}$$

$$-\frac{1}{2}\int_{s_1}^{s_2}\overline{\sigma}\,\overline{\sigma}^T\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}')\big):D^2\varphi(\overline{W}^t_r(\overline{\omega}')-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\overline{X}_r(\overline{\omega}'))dr.$$

Since $\big|\big(\overline{W}^t_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)\big|\le j$ for any $r\in[t_{\overline{\omega}},\overline{\tau}^t_j(\overline{\omega}')]$, an analogy to (6.1) shows that for any $t_{\overline{\omega}}\le s_1\le s_2\le\overline{\tau}^t_j(\overline{\omega}')$

$$\Big|\big(\overline{M}^{t_{\overline{\omega}}}_{s_2}(\varphi)-\overline{M}^{t_{\overline{\omega}}}_{s_1}(\varphi)-\overline{M}^t_{s_2}(\varphi_{q_m})+\overline{M}^t_{s_1}(\varphi_{q_m})\big)(\overline{\omega}')\Big|$$
$$\le 2\delta^{j,m}_{\overline{\omega}}+\delta^{j,m}_{\overline{\omega}}\int_t^{\overline{\tau}^t_j(\overline{\omega}')}\Big(\big|\overline{b}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}')\big)\big|+\frac{1}{2}\big|\overline{\sigma}\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega}')\big)\big|^2\Big)dr\le\delta^{j,m}_{\overline{\omega}}(2+c^j_{t,\mathbf{x}}),$$

where $c^j_{t,\mathbf{x}}:=\big[d/2+\kappa(t+j)(\|\mathbf{x}\|_t+j)+\kappa^2(t+j)(\|\mathbf{x}\|_t+j)^2\big]j+\int_t^{t+j}\big(|b(r,\mathbf{0})|+|\sigma(r,\mathbf{0})|^2\big)dr<\infty$. Taking $s_1=\big((\overline{\gamma}+\mathfrak{s})\wedge\overline{\zeta}_n\wedge\overline{\tau}^t_j\big)(\overline{\omega}')=(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n(\overline{\omega}')\wedge\overline{\tau}^t_j(\overline{\omega}')$ and $s_2=(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n(\overline{\omega}')\wedge\overline{\tau}^t_j(\overline{\omega}')$ yields $\Big|\big(\overline{M}^{t_{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)-\overline{M}^{t_{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)\big)(\overline{\omega}')-\overline{\xi}_{\theta_m}(\overline{\omega}')\Big|\le\delta^{j,m}_{\overline{\omega}}(2+c^j_{t,\mathbf{x}})$. As $\overline{\eta}_{\theta_m}(\overline{\omega}')=\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}}(\overline{\omega}'),\overline{X}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}}(\overline{\omega}'))\in\mathcal{O}_i\}}$ by (5.4), we see from (5.5) that

$$E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\Big[\mathbf{1}_{\{\overline{\tau}^t_j>\overline{\gamma}\}}\Big|\Big(\overline{M}^{t_{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)-\overline{M}^{t_{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)\Big)$$
$$\times\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}},\overline{X}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}-\overline{\xi}_{\theta_m}\overline{\eta}_{\theta_m}\Big|\Big]\le\delta^{j,m}_{\overline{\omega}}(2+c^j_{t,\mathbf{x}}).$$

The uniform continuity of $D^i\varphi$'s over compact sets implies $\lim_{m\to\infty}\downarrow\delta^{j,m}_{\overline{\omega}}=0$, and one can then deduce from (6.26) that

$$E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\Big[\mathbf{1}_{\{\overline{\tau}^t_j>\overline{\gamma}\}}\Big(\overline{M}^{t_{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)-\overline{M}^{t_{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n\wedge\overline{\tau}^t_j}(\varphi)\Big)\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}},\overline{X}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}\Big]$$
$$=\lim_{m\to\infty}E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{\xi}_{\theta_m}\overline{\eta}_{\theta_m}\big]=0. \tag{6.27}$$

Since $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\{\overline{\omega}'\in\overline{\Omega}:\overline{X}_r(\overline{\omega}')=\overline{X}_{\overline{\gamma}\wedge r}(\overline{\omega}),\ \forall\,r\in[0,t_{\overline{\omega}}]\big\}=1$ by Part (1a), applying Proposition 1.2 with $\big(\Omega,\mathcal{F},P,B,X\big)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}^t_{\overline{\gamma},\overline{\omega}},\overline{W},\overline{X}\big)$ and $(t,\mathbf{x})=\big(t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})\big)$ renders that $\big\{\overline{M}^{t_{\overline{\omega}}}_{s\wedge\overline{\tau}^{t_{\overline{\omega}}}_n}(\varphi)\big\}_{s\in[t_{\overline{\omega}},\infty)}$ is a bounded process under $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$. As $\lim_{j\to\infty}\uparrow\overline{\tau}^t_j(\overline{\omega}')=\infty$ for any $\overline{\omega}'\in\overline{\Omega}$, letting $j\to\infty$ in (6.27) and using the bounded convergence theorem, we obtain that $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\Big[\Big(\overline{M}^{t_{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{r})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n}(\varphi)-\overline{M}^{t_{\overline{\omega}}}_{(t_{\overline{\omega}}+\mathfrak{s})\wedge\overline{\tau}^{t_{\overline{\omega}}}_n}(\varphi)\Big)\prod_{i=1}^k\mathbf{1}_{\{(\overline{W}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}},\overline{X}_{t_{\overline{\omega}}+s_i\wedge\mathfrak{s}})\in\mathcal{O}_i\}}\Big]=0$. Hence, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\in\overline{\mathcal{P}}^1_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\cap\overline{\mathcal{P}}^2_{t_{\overline{\omega}}}$ for any $\overline{\omega}\in\big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_1\cup\overline{\mathcal{N}}_2\big)^c$.

2) We next show that for $\overline{P}-$a.s. $\overline{\omega}\in\overline{\Omega}$, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ satisfies (D3) in Definition 3.1 of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$.

By (D3) in Definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$, there is a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}$ such that $\overline{P}\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}=1$. Since Lemma A.2 (1) implies that $\widehat{\tau}(\overline{W})$ is a $[t,\infty]-$valued $\mathbf{F}^{\overline{W}^t,\overline{P}}-$stopping time on $\overline{\Omega}$, applying Lemma A.4 with $(\mathfrak{P}_t,\tau)=\big(\{\overline{P}\},\widehat{\tau}\big)$ assures that there exists $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ satisfying

$$\big\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\big\}\Delta\overline{\mathcal{A}}_*\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big). \tag{6.28}$$

Let $n,i\in\mathbb{N}$. Set $s^n_i:=t+i2^{-n}$ and $A^n_i:=\big\{s^n_{i-1}\le\widehat{\tau}<s^n_i\big\}\in\mathcal{F}^{W^t,P_0}_{s^n_i}$ with $s^n_0:=t$. Using Lemma A.5, we can find $\overline{\mathcal{N}}^n_i\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$ such that for any $(s,\overline{\omega})\in[t,s^n_i]\times\overline{\Omega}$, there exists

$A^{s,\overline{\omega}}_{n,i}\in\mathcal{F}^{W^s}_{s^n_i}$ satisfying $\mathbf{1}_{\{\overline{W}(\overline{\omega}')\in A^n_i\}}=\mathbf{1}_{\{\overline{W}(\overline{\omega}')\in A^{s,\overline{\omega}}_{n,i}\}}$, $\forall\,\overline{\omega}'\in\overline{\mathbf{W}}^t_{s,\overline{\omega}}\cap\big(\overline{\mathcal{N}}^n_i\big)^c$. For each $\overline{\omega}\in\{\overline{\gamma}\le s^n_i\}$, taking $s=t_{\overline{\omega}}$ yields some $A^{\overline{\omega}}_{n,i}=A^{t_{\overline{\omega}},\overline{\omega}}_{n,i}\in\mathcal{F}^{W^{t_{\overline{\omega}}}}_{s^n_i}$ such that

$$\mathbf{1}_{\{\overline{W}(\overline{\omega}')\in A^n_i\}}=\mathbf{1}_{\{\overline{W}(\overline{\omega}')\in A^{\overline{\omega}}_{n,i}\}},\quad \forall\,\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\overline{\mathcal{N}}^n_i\big)^c. \tag{6.29}$$

Set $\overline{\mathcal{N}}_{\widehat{\tau}}:=\underset{n,i\in\mathbb{N}}{\cup}\overline{\mathcal{N}}^n_i\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. By (5.2), it holds for any $\overline{\omega}\in\overline{\Omega}$ except on an $\overline{\mathcal{N}}_3\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ that

$$\begin{aligned}&\overline{P}^t_{\overline{\gamma},\overline{\omega}}\Big(\big(\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}\Delta\overline{\mathcal{A}}_*\big)\cup\overline{\mathcal{N}}_{\widehat{\tau}}\cup\big\{\overline{T}\ne\widehat{\tau}(\overline{W})\big\}\Big)\\ &\quad=E_{\overline{P}}\Big[\mathbf{1}_{(\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}\Delta\overline{\mathcal{A}}_*)\cup\overline{\mathcal{N}}_{\widehat{\tau}}\cup\{\overline{T}\ne\widehat{\tau}(\overline{W})\}}\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big](\overline{\omega})=0.\end{aligned} \tag{6.30}$$

Fix $\overline{\omega}\in\overline{\mathcal{N}}^c_0\cap\overline{\mathcal{N}}^c_3\cap\overline{\mathcal{A}}_*$. For any $n\in\mathbb{N}$, set $\mathfrak{i}_n(\overline{\omega}):=\lfloor 2^n(t_{\overline{\omega}}-t)\rfloor+1>2^n(t_{\overline{\omega}}-t)$ and defines a $(t_{\overline{\omega}},\infty]-$valued $\mathbf{F}^{W^{t_{\overline{\omega}}}}-$stopping time:

$$\widehat{\tau}^{\overline{\omega}}_n(\omega_0):=\sum_{i=\mathfrak{i}_n(\overline{\omega})}^\infty\mathbf{1}_{\{\omega_0\in A^{\overline{\omega}}_{n,i}\}}s^n_i+\infty\mathbf{1}_{\big\{\omega_0\in\underset{i=\mathfrak{i}_n(\overline{\omega})}{\overset{\infty}{\cap}}(A^{\overline{\omega}}_{n,i})^c\big\}},\quad\forall\,\omega_0\in\Omega_0.$$

As $\mathbf{F}^{W^{t_{\overline{\omega}}},P_0}$ is a right-continuous complete filtration, Lemma I.2.11 of [36] implies that $\widehat{\tau}^{\overline{\omega}}(\omega_0):=\underset{n\to\infty}{\underline{\lim}}\,\widehat{\tau}^{\overline{\omega}}_n(\omega_0)$, $\forall\,\omega_0\in\Omega_0$ is a $[t_{\overline{\omega}},\infty]-$valued $\mathbf{F}^{W^{t_{\overline{\omega}}},P_0}-$stopping time.

Let $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\mathcal{N}}^c_{\widehat{\tau}}\cap\big\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\big\}$ and $n\in\mathbb{N}$. Since (5.4) shows that $\widehat{\tau}\big(\overline{W}(\overline{\omega}')\big)\ge\overline{\gamma}(\overline{\omega}')=t_{\overline{\omega}}$, (6.29) renders that

$$\begin{aligned}&\sum_{i\in\mathbb{N}}\mathbf{1}_{\{s^n_{i-1}\le\widehat{\tau}(\overline{W}(\overline{\omega}'))<s^n_i\}}s^n_i+\infty\mathbf{1}_{\{\widehat{\tau}(\overline{W}(\overline{\omega}'))=\infty\}}\\ &\quad=\sum_{i=\mathfrak{i}_n(\overline{\omega})}^\infty\mathbf{1}_{\{\overline{W}(\overline{\omega}')\in A^n_i\}}s^n_i+\infty\mathbf{1}_{\big\{\overline{W}(\overline{\omega}')\in\underset{i=\mathfrak{i}_n(\overline{\omega})}{\overset{\infty}{\cap}}(A^n_i)^c\big\}}\\ &\quad=\sum_{i=\mathfrak{i}_n(\overline{\omega})}^\infty\mathbf{1}_{\{\overline{W}(\overline{\omega}')\in A^{\overline{\omega}}_{n,i}\}}s^n_i+\infty\mathbf{1}_{\big\{\overline{W}(\overline{\omega}')\in\underset{i=\mathfrak{i}_n(\overline{\omega})}{\overset{\infty}{\cap}}(A^{\overline{\omega}}_{n,i})^c\big\}}=\widehat{\tau}^{\overline{\omega}}_n\big(\overline{W}(\overline{\omega}')\big).\end{aligned}$$

Sending $n\to\infty$ reaches that $\widehat{\tau}\big(\overline{W}(\overline{\omega}')\big)=\underset{n\to\infty}{\lim}\downarrow\widehat{\tau}^{\overline{\omega}}_n(\overline{W}(\overline{\omega}'))=\widehat{\tau}^{\overline{\omega}}(\overline{W}(\overline{\omega}'))$. So $\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\mathcal{N}}^c_{\widehat{\tau}}\cap\big\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\big\}\cap\big\{\overline{T}=\widehat{\tau}^{\overline{\omega}}(\overline{W})\big\}=\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\overline{\mathcal{N}}^c_{\widehat{\tau}}\cap\big\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\big\}\cap\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}$. Since $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\{\overline{T}\ne\widehat{\tau}(\overline{W})\big\}=0$, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}\Delta\overline{\mathcal{A}}_*\big)=0$ by (6.30) and since $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, we can deduce from (5.5) and (5.3) that $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\big\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\big\}\cap\big\{\overline{T}=\widehat{\tau}^{\overline{\omega}}(\overline{W})\big\}\big)=\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big(\big\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\big\}\cap\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}\big)=\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\big\}=\overline{P}^t_{\overline{\gamma},\overline{\omega}}(\overline{\mathcal{A}}_*)=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\}}=1$. Hence, for any $\overline{\omega}\in\overline{\mathcal{N}}^c_0\cap\overline{\mathcal{N}}^c_3\cap\overline{\mathcal{A}}_*$, $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\{\overline{T}=\widehat{\tau}^{\overline{\omega}}(\overline{W})\big\}=1$, i.e., $\overline{P}^t_{\overline{\gamma},\overline{\omega}}$ satisfies (D3) in Definition 3.1 of $\overline{\mathcal{P}}_{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}$.

3) Let $i\in\mathbb{N}$. According to (5.2), it holds for all $\overline{\omega}\in\overline{\Omega}$ except on $\overline{\mathcal{N}}^i_{g,h}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big)$ that $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\big]=\big(\overline{Y}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})$ and $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot})dr\big]=\big(\overline{Z}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})$. Given $\overline{\omega}\in\Big(\overline{\mathcal{N}}_0\cup\overline{\mathcal{N}}_3\cup\overline{\mathcal{N}}^i_{g,h}\Big)^c\cap\overline{\mathcal{A}}_*$, (5.4), (5.5) and $\overline{P}^t_{\overline{\gamma},\overline{\omega}}\big\{\overline{T}=\widehat{\tau}^{\overline{\omega}}(\overline{W})\ge t_{\overline{\omega}}\big\}=1$ from Part (2) imply that $\big(\overline{Y}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\big]=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\mathbf{1}_{\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}}\int_{\overline{T}\wedge t_{\overline{\omega}}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\big]$ $=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{t_{\overline{\omega}}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\big]$ and similarly that $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\int_{t_{\overline{\omega}}}^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot})dr\big]=\big(\overline{Z}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})$.

Hence,

$$\overline{P}^t_{\overline{\gamma},\overline{\omega}} \in \overline{\mathcal{P}}_{\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\Big(\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big), \quad \forall\, \overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*, \tag{6.31}$$

where $\overline{\mathcal{N}}_* := \overline{\mathcal{N}}_0 \cup \overline{\mathcal{N}}_1 \cup \overline{\mathcal{N}}_2 \cup \overline{\mathcal{N}}_3 \cup \Big(\underset{i\in\mathbb{N}}{\cup}\overline{\mathcal{N}}^i_{g,h}\Big) \in \mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. In particular, (5.6) holds for $\overline{P}-$null set $\overline{\mathcal{N}} := \overline{\mathcal{N}}_* \cup \big\{\overline{T}\neq\widehat{\tau}(\overline{W})\big\} \cup \big(\{\widehat{\tau}(\overline{W})\geq\overline{\gamma}\}\Delta\overline{\mathcal{A}}_*\big) \in \mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$. □

Proof of Theorem 5.1: We first show the measurability of the random variable

$$\mathbf{1}_{\{\overline{T}\geq\overline{\gamma}_{\overline{P}}\}}\overline{V}\big(\overline{\gamma}_{\overline{P}},\overline{X}_{\overline{\gamma}_{\overline{P}}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}_{\overline{P}}),\overline{Z}_{\overline{P}}(\overline{\gamma}_{\overline{P}})\big)$$

for each $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ so that the right hand side of (5.7) is well-defined.

Let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ and simply denote $\overline{\gamma}_{\overline{P}}$ by $\overline{\gamma}$. Like in Part (1a) of the proof of Proposition 5.1, we still set $\overline{\mathcal{N}}_X := \big\{\overline{\omega}\in\overline{\Omega}\colon \overline{X}_s(\overline{\omega})\neq\overline{\mathscr{X}}^{t,\mathbf{x}}_s(\overline{\omega}) \text{ for some } s\in[0,\infty)\big\} \in \mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$ and let $\big\{\overline{K}^t_s\big\}_{s\in[t,\infty)}$ be the $\mathbf{F}^{\overline{W}^t}-$predictable process such that $\overline{\mathcal{N}}_K := \big\{\overline{\omega}\in\overline{\Omega}\colon \overline{K}^t_s(\overline{\omega})\neq\overline{\mathscr{X}}^{t,\mathbf{x}}_s(\overline{\omega}) \text{ for some } s\in[t,\infty)\big\} \in \mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. Since $\overline{X}|_{[0,t)} = \mathbf{x}|_{[0,t)}$ and $\overline{X}|_{[t,\infty)} = \overline{K}^t$ on $\big(\overline{\mathcal{N}}_X\cup\overline{\mathcal{N}}_K\big)^c$, one can deduce that the path-valued random variable $\overline{X}_{\overline{\gamma}\wedge\cdot}\colon\overline{\Omega}\mapsto\Omega_X$ is $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)\big)/\mathscr{B}(\Omega_X)-$measurable.

Set $\breve{\Omega} := [0,\infty)\times\Omega_X\times\Re\times\Re\supset\overline{D}$. Let $\widehat{\tau}$ be the $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time with $\overline{P}\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}=1$ and let $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, $\overline{\mathcal{N}}_*\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$ be as in (6.28) and (6.31). For any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$, we know from (6.31) that $\Big(\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big)\in\overline{D}$. By the measurability of $\overline{X}_{\overline{\gamma}\wedge\cdot}$,

$$\begin{aligned}\breve{\Psi}(\overline{\omega}) :=\, & \mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}^c_*\cup\overline{\mathcal{N}}_*\}}(t,\mathbf{x},y,z)\\ & +\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}\Big(\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big)\in\overline{D}, \quad \forall\,\overline{\omega}\in\overline{\Omega}\end{aligned} \tag{6.32}$$

is a $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)\big)/\mathscr{B}(\overline{D})-$measurable random variable, which induces a probability measure $\breve{P} := \overline{P}\circ\breve{\Psi}^{-1}$ on $\big(\breve{\Omega},\mathscr{B}(\breve{\Omega})\big)$. Then $\breve{\Psi}$ is further $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)\big)/\sigma\big(\mathscr{B}(\overline{D})\cup\mathscr{N}_{\breve{P}}\big(\mathscr{B}(\overline{D})\big)\big)-$measurable.

As the universally measurable function $(t',\mathbf{x}',y',z')\mapsto\overline{V}(t',\mathbf{x}',y',z')$ is $\sigma\big(\mathscr{B}(\overline{D})\cup\mathscr{N}_{\breve{P}}\big(\mathscr{B}(\overline{D})\big)\big)/\mathscr{B}[-\infty,\infty]-$measurable by Theorem 4.1,

$$\begin{aligned}\breve{V}(\overline{\omega}) :=\, & \mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}\overline{V}\big(\breve{\Psi}(\overline{\omega})\big)\\ =\, & \mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}\overline{V}\Big(\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big), \quad \forall\,\overline{\omega}\in\overline{\Omega}\end{aligned} \tag{6.33}$$

is $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)\big)/\mathscr{B}[-\infty,\infty]-$measurable. We see from (6.28) that $(\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*)\Delta\{\overline{T}\geq\overline{\gamma}\}\subset\big(\overline{\mathcal{A}}_*\Delta\{\overline{T}\geq\overline{\gamma}\}\big)\cup\overline{\mathcal{N}}_*\subset\big(\overline{\mathcal{A}}_*\Delta\{\widehat{\tau}(\overline{W})\geq\overline{\gamma}\}\big)\cup\{\overline{T}\neq\widehat{\tau}(\overline{W})\}\cup\overline{\mathcal{N}}_*\in\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$, where $\widehat{\tau}$ is the $\mathbf{F}^{W^t,P_0}-$stopping time with $\overline{P}\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}=1$. It follows that

$$\mathbf{1}_{\{\overline{T}(\overline{\omega})\geq\overline{\gamma}(\overline{\omega})\}}\overline{V}\Big(\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big), \quad \overline{\omega}\in\overline{\Omega}$$

is $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)\big)/\mathscr{B}[-\infty,\infty]-$ measurable and the right hand side of (5.7) is thus well-defined.

For any $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping time $\overline{\zeta}$, we denote

$$\overline{R}(\overline{\zeta}) := \int_{\overline{T}\wedge\overline{\zeta}}^{\overline{T}} f(r,\overline{X}_{r\wedge\cdot})dr + \mathbf{1}_{\{\overline{T}<\infty\}}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big).$$

(I) (sub-solution side) Fix $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ and simply denote $\overline{\gamma}_{\overline{P}}$ by $\overline{\gamma}$.

Let $\widehat{\tau}$ be the $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time with $\overline{P}\{\overline{T}=\widehat{\tau}(\overline{W})\}=1$ and let $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, $\overline{\mathcal{N}}_*\in\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)$ be as in (6.28) and (6.31). By (5.2), there is a $\overline{\mathcal{N}}_{f,\pi}\in\mathscr{N}_{\overline{P}}(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}})$ such that $E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{R}(\overline{\gamma})\big]=E_{\overline{P}}\big[\overline{R}(\overline{\gamma})\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})$ for any $\overline{\omega}\in\overline{\mathcal{N}}^c_{f,\pi}$. For any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}_*\cup\overline{\mathcal{N}}_{f,\pi}\big)^c$, as $\overline{\mathcal{N}}_0\subset\overline{\mathcal{N}}_*$, (5.4), (5.5) and (6.31) imply that

$$E_{\overline{P}}\big[\overline{R}(\overline{\gamma})\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big](\overline{\omega})=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{R}(\overline{\gamma})\big]=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\mathbf{1}_{\overline{\mathbf{W}}_{\overline{\gamma},\overline{\omega}}}\overline{R}(\overline{\gamma}(\overline{\omega}))\big]=E_{\overline{P}^t_{\overline{\gamma},\overline{\omega}}}\big[\overline{R}(\overline{\gamma}(\overline{\omega}))\big]$$
$$\le\overline{V}\big(\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\big).$$

Since $\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}=\mathbf{1}_{\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}}=\mathbf{1}_{\overline{\mathcal{A}}_*}$, $\overline{P}-$a.s. by (6.28) and since $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, the tower property renders that

$$E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big]=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*}\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big]$$
$$\ge E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*}E_{\overline{P}}\big[\overline{R}(\overline{\gamma})\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big]\Big]=E_{\overline{P}}\Big[E_{\overline{P}}\big[\mathbf{1}_{\overline{\mathcal{A}}_*}\overline{R}(\overline{\gamma})\big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\big]\Big]=E_{\overline{P}}\big[\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\overline{R}(\overline{\gamma})\big].$$

It follows that $E_{\overline{P}}\big[\overline{R}(t)\big]\le E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f(r,\overline{X}_{r\wedge\cdot})dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)+\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)\Big]$. Letting $\overline{P}$ vary over $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ yields that

$$\overline{V}(t,\mathbf{x},y,z)=\sup_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}E_{\overline{P}}\big[\overline{R}(t)\big]\le\sup_{\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)}E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}<\overline{\gamma}_{\overline{P}}\}}\Big(\int_t^{\overline{T}}f(r,\overline{X}_{r\wedge\cdot})dr+\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big)$$
$$+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}_{\overline{P}}\}}\Big(\int_t^{\overline{\gamma}_{\overline{P}}}f(r,\overline{X}_{r\wedge\cdot})dr+\overline{V}\big(\overline{\gamma}_{\overline{P}},\overline{X}_{\overline{\gamma}_{\overline{P}}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}_{\overline{P}}),\overline{Z}_{\overline{P}}(\overline{\gamma}_{\overline{P}})\big)\Big)\Big].$$

(II) (super-solution side) Let $\overline{P}\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ and simply denote $\overline{\gamma}_{\overline{P}}$ by $\overline{\gamma}$. We shall show that

$$\overline{V}(t,\mathbf{x},y,z)\ge E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f(r,\overline{X}_{r\wedge\cdot})dr$$
$$+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\Big].\tag{6.34}$$

As $\mathcal{F}^{\overline{W}^t}_t=\{\emptyset,\overline{\Omega}\}$, the $[t,\infty)-$valued $\mathbf{F}^{\overline{W}^t}-$stopping time $\overline{\gamma}$ satisfies either $\{\overline{\gamma}=t\}=\overline{\Omega}$ or $\{\overline{\gamma}>t\}=\overline{\Omega}$.

Suppose first that $\{\overline{\gamma}=t\}=\overline{\Omega}$: for any $i\in\mathbb{N}$, $\overline{Y}^i_{\overline{P}}(t)=E_{\overline{P}}\big[\int_{\overline{T}\wedge t}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\big|\mathcal{F}^{\overline{W}^t}_t\big]=E_{\overline{P}}\big[\int_t^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\big]\le y_i$ and $\overline{Z}^i_{\overline{P}}(t)=E_{\overline{P}}\big[\int_t^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot})dr\big]=z_i$. Then

$$E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f(r,\overline{X}_{r\wedge\cdot})dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\Big]$$
$$=E_{\overline{P}}\big[\overline{V}\big(t,\overline{X}_{t\wedge\cdot},\overline{Y}_{\overline{P}}(t),\overline{Z}_{\overline{P}}(t)\big)\big]\le\overline{V}\big(t,\mathbf{x},y,z\big).$$

Let us assume $\{\overline{\gamma}>t\}=\overline{\Omega}$ in the rest of this proof and set $\overline{\mathcal{N}}_X:=\big\{\overline{\omega}\in\overline{\Omega}:\overline{X}_s(\overline{\omega})\ne\overline{\mathscr{X}}^{t,\mathbf{x}}_s(\overline{\omega})\text{ for some }s\in[0,\infty)\big\}\in\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)$.

II.a) Define a random variable $\overline{W}^{t,\overline{\gamma}}:\overline{\Omega}\mapsto\Omega_0$ by $\overline{W}^{t,\overline{\gamma}}_r(\overline{\omega}):=\overline{W}^t\big((r\vee t)\wedge\overline{\gamma}(\overline{\omega}),\overline{\omega}\big)$, $\forall\,(r,\overline{\omega})\in[0,\infty)\times\overline{\Omega}$, which is clearly $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}/\mathscr{B}(\Omega_0)-$measurable.

Set $\ddot{\Omega}:=[0,\infty)\times\Omega_0\times\Omega_X\times\Re\times\Re\supset\overline{\mathcal{D}}$ and pick up an arbitrary element $\mathbf{w}$ from Ω_0. We let $\widehat{\tau}$ be the $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time with $\overline{P}\{\overline{T}=\widehat{\tau}(\overline{W})\}=1$ and let $\overline{\mathcal{A}}_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$, $\overline{\mathcal{N}}_*\in\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)$ be as in (6.28) and (6.31). Since $(t,\mathbf{x},y,z)\in\overline{D}$, Theorem 3.1 and (6.31) show that $(t,\mathbf{w},\mathbf{x},y,z)\in\overline{\mathcal{D}}$ and that $\Big(\overline{\gamma}(\overline{\omega}),\overline{W}^{t,\overline{\gamma}}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big)\in\overline{\mathcal{D}}$ for any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$. Similarly to (6.32),

$$
\begin{aligned}
&\ddot{\Psi}(\overline{\omega}):=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}^c_*\cup\overline{\mathcal{N}}_*\}}(t,\mathbf{w},\mathbf{x},y,z)\\
&\quad+\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}\Big(\overline{\gamma}(\overline{\omega}),\overline{W}^{t,\overline{\gamma}}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big)\in\overline{\mathcal{D}},
\end{aligned}
\tag{6.35}
$$

$\forall\,\overline{\omega}\in\overline{\Omega}$ is $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)\big)/\mathscr{B}(\overline{\mathcal{D}})-$measurable, which induces a probability measure $\ddot{P}:=\overline{P}\circ\ddot{\Psi}^{-1}$ on $\big(\ddot{\Omega},\mathscr{B}(\ddot{\Omega})\big)$. Then $\ddot{\Psi}$ is further $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)\big)/\sigma\big(\mathscr{B}(\overline{\mathcal{D}})\cup\mathscr{N}_{\ddot{P}}(\mathscr{B}(\overline{\mathcal{D}}))\big)-$measurable.

II.b) Fix $\varepsilon\in(0,1)$ through Part (II.e).

According to Jankov-von Neumann Theorem (Proposition 7.50 of [14]), Corollary 4.1 and Theorem 4.1, there exists an analytically measurable function $\overline{\mathbf{Q}}_\varepsilon:\overline{\mathcal{D}}\mapsto\mathfrak{P}(\overline{\Omega})$ such that for any $(\mathfrak{t},\mathfrak{w},\mathfrak{x},\mathfrak{y},\mathfrak{z})\in\overline{\mathcal{D}}$, $\overline{\mathbf{Q}}_\varepsilon(\mathfrak{t},\mathfrak{w},\mathfrak{x},\mathfrak{y},\mathfrak{z})$ belongs to $\overline{\mathcal{P}}_{\mathfrak{t},\mathfrak{w},\mathfrak{x}}(\mathfrak{y},\mathfrak{z})$ and satisfies

$$
E_{\overline{\mathbf{Q}}_\varepsilon(\mathfrak{t},\mathfrak{w},\mathfrak{x},\mathfrak{y},\mathfrak{z})}\big[\overline{R}(\mathfrak{t})\big]\ge\begin{cases}\overline{V}(\mathfrak{t},\mathfrak{w},\mathfrak{x},\mathfrak{y},\mathfrak{z})-\varepsilon, & \text{if } \overline{V}(\mathfrak{t},\mathfrak{w},\mathfrak{x},\mathfrak{y},\mathfrak{z})<\infty;\\ 1/\varepsilon, & \text{if } \overline{V}(\mathfrak{t},\mathfrak{w},\mathfrak{x},\mathfrak{y},\mathfrak{z})=\infty.\end{cases}
\tag{6.36}
$$

As $\overline{\mathbf{Q}}_\varepsilon$ is universally measurable, it is also $\sigma\big(\mathscr{B}(\overline{\mathcal{D}})\cup\mathscr{N}_{\ddot{P}}(\mathscr{B}(\overline{\mathcal{D}}))\big)/\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable,

$$
\overline{Q}^{\overline{\omega}}_\varepsilon:=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}^c_*\cup\overline{\mathcal{N}}_*\}}\overline{P}+\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\}}\overline{\mathbf{Q}}_\varepsilon\big(\ddot{\Psi}(\overline{\omega})\big),\quad\forall\,\overline{\omega}\in\overline{\Omega}
$$

is thus $\sigma\big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)\big)/\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable.

Given a $[0,\infty]-$valued $\mathscr{B}(\overline{\Omega})-$measurable random variable $\overline{\phi}$, Proposition 7.25 of [14] implies that the mapping $\mathfrak{P}(\overline{\Omega})\ni\overline{Q}\mapsto E_{\overline{Q}}\big[\overline{\phi}\big]$ is $\mathscr{B}\big(\mathfrak{P}(\overline{\Omega})\big)-$measurable. The measurability of $\{\overline{Q}^{\overline{\omega}}_\varepsilon\}_{\overline{\omega}\in\overline{\Omega}}$ renders that

$$
\text{the random variable } \overline{\Omega}\ni\overline{\omega}\mapsto E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{\phi}\big] \text{ is } \sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}(\overline{\mathcal{F}}^t_\infty)\Big)-\text{measurable.}
\tag{6.37}
$$

Let $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$ and denote $t_{\overline{\omega}}:=\overline{\gamma}(\overline{\omega})$. We know from (6.35) that

$$
\overline{Q}^{\overline{\omega}}_\varepsilon=\overline{\mathbf{Q}}_\varepsilon\big(\ddot{\Psi}(\overline{\omega})\big)\in\overline{\mathcal{P}}_{\overline{\gamma}(\overline{\omega}),\overline{W}^{t,\overline{\gamma}}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}\Big(\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\Big).
\tag{6.38}
$$

By (D3) in Definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$, there is a $[t_{\overline{\omega}},\infty]-$valued $\mathbf{F}^{W^{t_{\overline{\omega}}},P_0}-$stopping time $\widehat{\tau}_{\overline{\omega}}$ with

$$
\overline{Q}^{\overline{\omega}}_\varepsilon\big(\{\overline{T}=\widehat{\tau}_{\overline{\omega}}(\overline{W})\}\big)=1.
\tag{6.39}
$$

Set $\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}}:=\big\{\overline{\omega}'\in\overline{\Omega}:(\overline{W}_s,\overline{X}_s)(\overline{\omega}')=(\overline{W}^{t,\overline{\gamma}}_s,\overline{X}_s)(\overline{\omega}),\,\forall\,s\in[0,\overline{\gamma}(\overline{\omega})]\big\}$ and $\overline{\Xi}^t_{\overline{\gamma},\overline{\omega}}:=\big\{\overline{\omega}'\in\overline{\Omega}:\overline{W}^t_s(\overline{\omega}')=\overline{W}^t_s(\overline{\omega}),\,\forall\,s\in\big[t,\overline{\gamma}(\overline{\omega})\big];\,\overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\,\forall\,s\in[0,\overline{\gamma}(\overline{\omega})]\big\}$. Since $\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}}\subset\big\{\overline{\omega}'\in\overline{\Omega}:\overline{W}_s(\overline{\omega}')=0,\,\forall\,s\in[0,t];\overline{W}_s(\overline{\omega}')=\overline{W}^t_s(\overline{\omega}),\,\forall\,s\in(t,\overline{\gamma}(\overline{\omega})];\overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\,\forall\,s\in[0,\overline{\gamma}(\overline{\omega})]\big\}\subset\overline{\Xi}^t_{\overline{\gamma},\overline{\omega}}\subset\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}$, we see from (6.38) that

$$
\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}}\big)=1,\quad\text{and thus}\quad\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\big)=\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\Xi}^t_{\overline{\gamma},\overline{\omega}}\big)=1.
\tag{6.40}
$$

Let $\overline{A}\in\mathscr{B}(\overline{\Omega})$. We claim that

$$
\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\mathcal{A}}\cap\overline{A}\big)=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}\}}\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}\big),\quad\forall\,\overline{\mathcal{A}}\in\overline{\mathcal{F}}^t_{\overline{\gamma}},\ \forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*.
\tag{6.41}
$$

To see this, we take $\overline{A}\in\overline{\mathcal{F}}^t_{\overline{\gamma}}$. Let $\overline{\omega}_1\in\overline{A}\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$ and set $s_1:=\overline{\gamma}(\overline{\omega}_1)$. Since $\overline{A}\cap\{\overline{\gamma}\le s_1\}$ is an $\overline{\mathcal{F}}^t_{s_1}-$measurable set including $\overline{\omega}_1$, one can deduce that

$$\overline{\Xi}^t_{\overline{\gamma},\overline{\omega}_1}=\big\{\overline{\omega}'\in\overline{\Omega}\colon \overline{W}^t_r(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}_1),\ \forall\, r\in[t,s_1];\ \overline{X}_r(\overline{\omega}')=\overline{X}_r(\overline{\omega}_1),\ \forall\, r\in[0,s_1]\big\}$$

$$\text{is also contained in } \overline{A}\cap\{\overline{\gamma}\le s_1\}. \tag{6.42}$$

By (6.40), $\overline{Q}^{\overline{\omega}_1}_\varepsilon\big(\overline{A}\big)=1$ and thus $\overline{Q}^{\overline{\omega}_1}_\varepsilon\big(\overline{A}\cap\overline{A}\big)=\overline{Q}^{\overline{\omega}_1}_\varepsilon\big(\overline{A}\big)=\mathbf{1}_{\{\overline{\omega}_1\in\overline{A}\}}\overline{Q}^{\overline{\omega}_1}_\varepsilon\big(\overline{A}\big)$. We next let $\overline{\omega}_2\in\overline{A}^c\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$ and set $s_2:=\overline{\gamma}(\overline{\omega}_2)$. As $\overline{A}^c\cap\{\overline{\gamma}\le s_2\}$ is an $\overline{\mathcal{F}}^t_{s_2}-$measurable set including $\overline{\omega}_2$, $\overline{\Xi}^t_{\overline{\gamma},\overline{\omega}_2}=\big\{\overline{\omega}'\in\overline{\Omega}\colon \overline{W}^t_r(\overline{\omega}')=\overline{W}^t_r(\overline{\omega}_2),\ \forall\, r\in[t,s_2];\ \overline{X}_r(\overline{\omega}')=\overline{X}_r(\overline{\omega}_2),\ \forall\, r\in[0,s_2]\big\}$ is also included in $\overline{A}^c\cap\{\overline{\gamma}\le s_2\}$. We correspondingly have $\overline{Q}^{\overline{\omega}_2}_\varepsilon(\overline{A}^c)=1$ and thus $\overline{Q}^{\overline{\omega}_2}_\varepsilon\big(\overline{A}\cap\overline{A}\big)=0=\mathbf{1}_{\{\overline{\omega}_2\in\overline{A}\}}\overline{Q}^{\overline{\omega}_2}_\varepsilon\big(\overline{A}\big)$.

Consider a pasted probability measure $\overline{P}_\varepsilon\in\mathfrak{P}\big(\overline{\Omega}\big)$:

$$\overline{P}_\varepsilon(\overline{A}):=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{A}\big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_\varepsilon(\overline{A})\overline{P}(d\overline{\omega}),\quad \forall\,\overline{A}\in\mathscr{B}(\overline{\Omega}). \tag{6.43}$$

In particular, taking $\overline{A}=\overline{\Omega}$ in (6.41) renders that

$$\overline{P}_\varepsilon(\overline{A})=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{A}\big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\omega}\in\overline{A}\}}\overline{P}(d\overline{\omega})=\overline{P}(\overline{A}),\quad \forall\,\overline{A}\in\overline{\mathcal{F}}^t_{\overline{\gamma}}. \tag{6.44}$$

In the next three parts, we demonstrate that $\overline{P}_\varepsilon$ also belongs to $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$, i.e., the probability class $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ is stable under the pasting (6.43).

II.c) We first show that $\overline{P}_\varepsilon$ is of $\overline{\mathcal{P}}^1_{t,\mathbf{x}}\cap\overline{\mathcal{P}}^2_t$ and thus satisfies (D1) and (D2) in Definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

II.c.1) Set $\overline{\Omega}_X:=\big\{\overline{X}_s=\mathbf{x}(s),\ \forall\, s\in[0,t]\big\}$. By the proof of Proposition 5.1, $\overline{\Omega}^c_X\subset\overline{\mathcal{N}}_X=\big\{\overline{\omega}\in\overline{\Omega}\colon \overline{X}_s(\overline{\omega})\neq\overline{\mathscr{X}}^{t,\mathbf{x}}_s(\overline{\omega})$ for some $s\in[0,\infty)\big\}\subset\overline{\mathcal{N}}_1\subset\overline{\mathcal{N}}_*$. Given $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\subset\overline{\Omega}_X$, one has $\overline{X}_s(\overline{\omega})=\mathbf{x}(s),\ \forall\, s\in[0,t]$ and thus $\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}}\subset\big\{\overline{\omega}'\in\overline{\Omega}\colon \overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\ \forall\, s\in[0,t]\big\}=\overline{\Omega}_X$. As $\overline{P}\big(\overline{\Omega}_X\big)\ge\overline{P}(\overline{\mathcal{N}}^c_*)=1$, (6.40) implies that $\overline{P}_\varepsilon\big(\overline{\Omega}_X\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\overline{\Omega}_X\big)+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*}1\cdot\overline{P}(d\overline{\omega})=\overline{P}\big(\overline{\mathcal{A}}^c_*\big)+\overline{P}(\overline{\mathcal{A}}_*)=1$, i.e., $\overline{P}_\varepsilon\in\overline{\mathcal{P}}^1_{t,\mathbf{x}}$.

II.c.2) We need some technical preparation for checking $\overline{P}_\varepsilon\in\overline{\mathcal{P}}^2_t$: Let $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$ and set $\mathfrak{a}_{\overline{\omega}}:=\big(-\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\mathbf{0}\big)\in\mathbb{R}^{d+l}$. We define an $\overline{\mathbf{F}}^{t_{\overline{\omega}}}-$stopping time $\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}'):=\inf\big\{s\in[t_{\overline{\omega}},\infty)\colon \big|(\overline{W}^{t_{\overline{\omega}}}_s,\overline{X}_s)(\overline{\omega}')-\mathfrak{a}_{\overline{\omega}}\big|\ge n\big\}$, $\overline{\omega}'\in\overline{\Omega}$.

Given $\varphi\in\mathfrak{C}(\mathbb{R}^{d+l})$, define a $C^2(\mathbb{R}^{d+l})$ function $\varphi_{\overline{\omega}}(w,x):=\varphi\big(w+\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),x\big)$, $(w,x)\in\mathbb{R}^{d+l}$. For $i=0,1,2$ and $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}$, since $D^i\varphi\big(\overline{W}^t_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)=D^i\varphi\big(\overline{W}^t_r(\overline{\omega}')-\overline{W}^t(\overline{\gamma}(\overline{\omega}),\overline{\omega}')+\overline{W}^t(\overline{\gamma}(\overline{\omega}),\overline{\omega}),\overline{X}_r(\overline{\omega}')\big)=D^i\varphi_{\overline{\omega}}\big(\overline{W}^{t_{\overline{\omega}}}_r(\overline{\omega}'),\overline{X}_r(\overline{\omega}')\big)$, $\forall\, r\in[t_{\overline{\omega}},\infty)$, one has

$$\big(\overline{M}^t_{r_2}(\varphi)-\overline{M}^t_{r_1}(\varphi)\big)(\overline{\omega}')=\big(\overline{M}^{t_{\overline{\omega}}}_{r_2}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}}}_{r_1}(\varphi_{\overline{\omega}})\big)(\overline{\omega}'),\ \forall\, t_{\overline{\omega}}\le r_1\le r_2<\infty. \tag{6.45}$$

Let $\Big\{\overline{\mathfrak{X}}^{\overline{\omega}}_s=\overline{\mathscr{X}}^{t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega})}_s\Big\}_{s\in[0,\infty)}$ be the $\Big\{\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}},\overline{Q}^{\overline{\omega}}_\varepsilon}_{s\vee t_{\overline{\omega}}}\Big\}_{s\in[0,\infty)}-$adapted continuous process that uniquely solves the following SDE on $\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{Q}^{\overline{\omega}}_\varepsilon\big)$:

$$\overline{\mathscr{X}}_s=\overline{X}_{\overline{\gamma}}(\overline{\omega})+\int_{t_{\overline{\omega}}}^s b(r,\overline{\mathscr{X}}_{r\wedge\cdot})dr+\int_{t_{\overline{\omega}}}^s\sigma(r,\overline{\mathscr{X}}_{r\wedge\cdot})d\overline{W}_r,\quad s\in[t_{\overline{\omega}},\infty)$$

with initial condition $\overline{\mathscr{X}}_s=\overline{X}_s(\overline{\omega}),\ \forall\, s\in[0,t_{\overline{\omega}}]$. By (6.38), $\overline{\mathcal{N}}^{\overline{\omega}}_X:=\big\{\overline{\omega}'\in\overline{\Omega}:\overline{X}_s(\overline{\omega}')\neq \overline{\mathfrak{X}}^{\overline{\omega}}_s(\overline{\omega}')$ for some $s\in[0,\infty)\big\}\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big(\overline{\mathcal{F}}^{t_{\overline{\omega}}}_\infty\big)$. And there exists an $\mathbb{R}^l-$valued $\mathbf{F}^{\overline{W}^{t_{\overline{\omega}}}}-$ predictable process $\big\{K^{\overline{\omega}}_s\big\}_{s\in[t_{\overline{\omega}},\infty)}$ such that $\overline{\mathcal{N}}^{\overline{\omega}}_K:=\big\{\overline{\omega}'\in\overline{\Omega}:K^{\overline{\omega}}_s(\overline{\omega}')\neq\overline{\mathfrak{X}}^{\overline{\omega}}_s(\overline{\omega}')$ for some $s\in[t_{\overline{\omega}},\infty)\big\}\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big(\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_\infty\big)$.

Let $(\varphi,n)\in\mathfrak{C}(\mathbb{R}^{d+l})\times\mathbb{N}$, $(\mathfrak{s},\mathfrak{r})\in\mathbb{Q}^{2,<}_+$ and $\{(s_i,\mathcal{O}_i)\}_{i=1}^k\subset\big(\mathbb{Q}\cap[0,\mathfrak{s}]\big)\times\mathscr{O}(\mathbb{R}^{d+l})$. Denote $\overline{M}^{t,n}_s(\varphi):=\overline{M}^t_{s\wedge\overline{\tau}^t_n}(\varphi)$, $s\in[t,\infty)$ and set $\widehat{A}:=\underset{i=1}{\overset{k}{\cap}}(\overline{W}^t_{t+s_i},\overline{X}_{t+s_i})^{-1}(\mathcal{O}_i)\in\overline{\mathcal{F}}^t_{t+\mathfrak{s}}$.

(i) To verify $E_{\overline{P}_\varepsilon}\big[\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\big]=0$, we first show that

$$E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\{\overline{\gamma}>t+\mathfrak{s}\}}\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\Big]=0. \tag{6.46}$$

Since $\{\overline{\gamma}>t+\mathfrak{s}\}\cap\widehat{A}=\{\overline{\gamma}>t+\mathfrak{s}\}\cap\Big(\underset{i=1}{\overset{k}{\cap}}(\overline{W}^t_{\overline{\gamma}\wedge(t+s_i)},\overline{X}_{\overline{\gamma}\wedge(t+s_i)})^{-1}(\mathcal{O}_i)\Big)\in\overline{\mathcal{F}}^t_{\overline{\gamma}\wedge(t+\mathfrak{s})}$ and $\overline{M}^{t,n}_{\overline{\gamma}\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,n}_{\overline{\gamma}\wedge(t+\mathfrak{s})}(\varphi)\in\overline{\mathcal{F}}^t_{\overline{\gamma}\wedge(t+\mathfrak{r})}\subset\overline{\mathcal{F}}^t_{\overline{\gamma}}$, using (6.44) and applying (3.2) with $\big(\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2\big)=\big(\mathbf{0},\overline{\gamma}\wedge(t+\mathfrak{s}),\overline{\gamma}\wedge(t+\mathfrak{r})\big)$ yield that

$$\begin{aligned}&E_{\overline{P}_\varepsilon}\Big[\big(\overline{M}^{t,n}_{\overline{\gamma}\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>t+\mathfrak{s}\}\cap\widehat{A}}\Big]\\&\quad=E_{\overline{P}}\Big[\big(\overline{M}^t_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge\overline{\gamma}\wedge(t+\mathfrak{s})}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>t+\mathfrak{s}\}\cap\widehat{A}}\Big]=0.\end{aligned}\tag{6.47}$$

And (6.41) implies that

$$\begin{aligned}E_{\overline{P}_\varepsilon}&\Big[\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{\overline{\gamma}\wedge(t+\mathfrak{r})}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>t+\mathfrak{s}\}\cap\widehat{A}}\Big]=E_{\overline{P}_\varepsilon}\Big[\big(\overline{M}^{t,n}_{\overline{\gamma}\vee(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,n}_{\overline{\gamma}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>t+\mathfrak{s}\}\cap\widehat{A}}\Big]\\&=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*}\big(\overline{M}^{t,n}_{\overline{\gamma}\vee(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,n}_{\overline{\gamma}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>t+\mathfrak{s}\}\cap\widehat{A}}\Big]\\&\quad+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\gamma}(\overline{\omega})>t+\mathfrak{s}\}}\mathbf{1}_{\{\overline{\omega}\in\widehat{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{M}^{t,n}_{\overline{\gamma}\vee(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,n}_{\overline{\gamma}}(\varphi)\big]\overline{P}(d\overline{\omega}).\end{aligned}\tag{6.48}$$

Taking $\big(\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2\big)=\big(\mathbf{0},\overline{\gamma},\overline{\gamma}\vee(t+\mathfrak{r})\big)$ in (3.2) renders that

$$\begin{aligned}&E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*}\big(\overline{M}^{t,n}_{\overline{\gamma}\vee(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,n}_{\overline{\gamma}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>t+\mathfrak{s}\}\cap\widehat{A}}\Big]\\&\quad=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*}\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee(t+\mathfrak{r}))}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge\overline{\gamma}}(\varphi)\big)\mathbf{1}_{\{\overline{\gamma}>t+\mathfrak{s}\}\cap\widehat{A}}\Big]=0.\end{aligned}\tag{6.49}$$

Fix $\overline{\omega}\in\{\overline{\tau}^t_n>\overline{\gamma}\}\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$ and set $\mathfrak{r}_{\overline{\omega}}:=t_{\overline{\omega}}\vee(t+\mathfrak{r})$. As $t_{\overline{\omega}}=\overline{\gamma}(\overline{\omega})<\overline{\tau}^t_n(\overline{\omega})\le t+n$, $n_{\overline{\omega}}:=\lceil t+n-t_{\overline{\omega}}\rceil\in\mathbb{N}$. Using (3.2) with $(t,\mathbf{x},\overline{P},\varphi,n,\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2)=\big(t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\overline{Q}^{\overline{\omega}}_\varepsilon,\varphi_{\overline{\omega}},n_{\overline{\omega}},\mathfrak{a}_{\overline{\omega}},t_{\overline{\omega}},\mathfrak{r}_{\overline{\omega}}\wedge(t+n)\big)$ yields that

$$\begin{aligned}0&=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t_{\overline{\omega}}+n_{\overline{\omega}})\wedge\mathfrak{r}_{\overline{\omega}}\wedge(t+n)}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t_{\overline{\omega}}+n_{\overline{\omega}})\wedge t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\Big]\\&=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge\mathfrak{r}_{\overline{\omega}}\wedge(t+n)}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\Big].\end{aligned}\tag{6.50}$$

Because $\overline{\omega}\in\big\{\overline{\omega}'\in\overline{\Omega}:\overline{\tau}^t_n(\overline{\omega}')>\overline{\gamma}(\overline{\omega})\big\}\in\overline{\mathcal{F}}^t_{\overline{\gamma}(\overline{\omega})}$, an analogy to (6.42) shows that $\Xi^t_{\overline{\gamma},\overline{\omega}}\subset\big\{\overline{\omega}'\in\overline{\Omega}:\overline{\tau}^t_n(\overline{\omega}')>\overline{\gamma}(\overline{\omega})\big\}$. Let $\overline{\omega}'\in\overline{\Xi}^t_{\overline{\gamma},\overline{\omega}}$. Since $\inf\big\{s\in[t,\infty):\big|(\overline{W}^t_s,\overline{X}_s)(\overline{\omega}')\big|\ge n\big\}\ge\overline{\tau}^t_n(\overline{\omega}')>\overline{\gamma}(\overline{\omega})$, one has $\big|(\overline{W}^t_s,\overline{X}_s)(\overline{\omega}')\big|<n,\ \forall\, s\in[t,t_{\overline{\omega}}]$ and thus

$$\begin{aligned}&\inf\big\{s\in[t,\infty):|(\overline{W}^t_s,\overline{X}_s)(\overline{\omega}')|\ge n\big\}\\&\quad=\inf\big\{s\in[t_{\overline{\omega}},\infty):\big|\overline{W}^t_s(\overline{\omega}')-\overline{W}^t(\overline{\gamma}(\overline{\omega}),\overline{\omega}'),\overline{X}_s(\overline{\omega}')\big)+\big(\overline{W}^t_{\overline{\gamma}}(\overline{\omega}),\mathbf{0}\big)\big|\ge n\big\}=\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}').\end{aligned}$$

It follows that $\overline{\tau}^t_n(\overline{\omega}')=\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge(t+n)$. Taking $(r_1,r_2)=\big(t_{\overline{\omega}},\overline{\tau}^t_n(\overline{\omega}')\wedge\mathfrak{r}_{\overline{\omega}}\big)$ in (6.45), we can deduce from (5.4) that

$$\begin{aligned}&\big(\overline{M}^t(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge(\overline{\gamma}(\overline{\omega}')\vee(t+\mathfrak{r})),\overline{\omega}'\big)-\big(\overline{M}^t(\varphi)\big)\big(\overline{\gamma}(\overline{\omega}'),\overline{\omega}'\big)\\&\quad=\big(\overline{M}^t(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge(\overline{\gamma}(\overline{\omega})\vee(t+\mathfrak{r})),\overline{\omega}'\big)-\big(\overline{M}^t(\varphi)\big)\big(\overline{\gamma}(\overline{\omega}),\overline{\omega}'\big)\\&\quad=\big(\overline{M}^{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge\mathfrak{r}_{\overline{\omega}},\overline{\omega}'\big)-\big(\overline{M}^{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)(t_{\overline{\omega}},\overline{\omega}')\\&\quad=\big(\overline{M}^{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)\big(\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge(t+n)\wedge\mathfrak{r}_{\overline{\omega}},\overline{\omega}'\big)-\big(\overline{M}^{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)(t_{\overline{\omega}},\overline{\omega}').\end{aligned}$$

As $\{\overline{\tau}^t_n>\overline{\gamma}\}\in\overline{\mathcal{F}}^t_{\overline{\tau}^t_n\wedge\overline{\gamma}}\subset\overline{\mathcal{F}}^t_{\overline{\gamma}}$, (6.41), (6.40), and (6.50) then imply that

$$\begin{aligned}&E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{M}^{t,n}_{\overline{\gamma}\vee(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,n}_{\overline{\gamma}}(\varphi)\big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\{\overline{\tau}^t_n>\overline{\gamma}\}}\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee(t+\mathfrak{r}))}(\varphi)-\overline{M}^t_{\overline{\gamma}}(\varphi)\big)\big]\\&\quad=\mathbf{1}_{\{\overline{\tau}^t_n(\overline{\omega})>\overline{\gamma}(\overline{\omega})\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{M}^t_{\overline{\tau}^t_n\wedge(\overline{\gamma}\vee(t+\mathfrak{r}))}(\varphi)-\overline{M}^t_{\overline{\gamma}}(\varphi)\big]\\&\quad=\mathbf{1}_{\{\overline{\tau}^t_n(\overline{\omega})>\overline{\gamma}(\overline{\omega})\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t+n)\wedge\mathfrak{r}_{\overline{\omega}}}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}}}_{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\Big]=0,\quad\forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*.\end{aligned}$$

So $\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\gamma}(\overline{\omega})>t+\mathfrak{s}\}}\mathbf{1}_{\{\overline{\omega}\in\widehat{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{M}^{t,n}_{\overline{\gamma}\vee(t+\mathfrak{r})}(\varphi)-\overline{M}^{t,n}_{\overline{\gamma}}(\varphi)\big]\overline{P}(d\overline{\omega})=0$, which together with (6.47)–(6.49) leads to (6.46).

(ii) If $\mathfrak{s}=0$, as $\{\overline{\gamma}>t\}=\overline{\Omega}$, (6.46) directly gives $E_{\overline{P}_\varepsilon}\big[\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_t(\varphi)\big)\mathbf{1}_{\widehat{A}}\big]=0$.

Next, let $\mathfrak{s}>0$. In this case, we can assume with loss of generality that $0=s_1<\cdots<s_k=\mathfrak{s}$ with $k\geq 2$. As $\overline{\mathcal{A}}^c_*\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\subset\overline{\mathcal{F}}^t_{\overline{\gamma}}$, one has $\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\leq t+\mathfrak{s}\}\in\overline{\mathcal{F}}^t_{t+\mathfrak{s}}$. Applying (3.2) with $(\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2)=(\mathbf{0},t+\mathfrak{s},t+\mathfrak{r})$ yields that

$$\begin{aligned}&E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\leq t+\mathfrak{s}\}}\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\Big]\\&\quad=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\leq t+\mathfrak{s}\}}\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\mathbf{1}_{\widehat{A}}\Big]=0.\end{aligned}\tag{6.51}$$

Fix $i\in\{1,\cdots,k-1\}$ and fix $\overline{\omega}\in\big\{\overline{\tau}^t_n>\overline{\gamma}\big\}\cap\big\{t+s_i<\overline{\gamma}\leq t+s_{i+1}\big\}\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$. Since $\overline{\mathbf{W}}^t_{\gamma,\overline{\omega}}\subset\{\overline{\gamma}>t+s_i\}$ by (5.4), $\overline{A}_i:=\underset{j=1}{\overset{i}{\cap}}\big(\overline{W}^t_{\overline{\gamma}\wedge(t+s_j)},\overline{X}_{\overline{\gamma}\wedge(t+s_j)}\big)^{-1}(\mathcal{O}_j)\in\overline{\mathcal{F}}^t_{\overline{\gamma}}$ satisfies

$$\overline{\mathbf{W}}^t_{\gamma,\overline{\omega}}\cap\Big(\underset{j=1}{\overset{i}{\cap}}\big(\overline{W}^t_{t+s_j},\overline{X}_{t+s_j}\big)^{-1}(\mathcal{O}_j)\Big)=\overline{\mathbf{W}}^t_{\gamma,\overline{\omega}}\cap\overline{A}_i.\tag{6.52}$$

Also, (5.4) shows that $\overline{\mathbf{W}}^t_{\gamma,\overline{\omega}}\subset\{\overline{\gamma}\leq t+\mathfrak{s}\}$ and thus $\overline{\mathbf{W}}^t_{\gamma,\overline{\omega}}\cap\{\overline{\tau}^t_n\leq\overline{\gamma}\}\subset\{\overline{\tau}^t_n\leq t+\mathfrak{s}\}$. By (6.40),

$$\begin{aligned}&E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\{\overline{\tau}^t_n\leq\overline{\gamma}\}}\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\big]\\&\quad\leq E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\{\overline{\tau}^t_n\leq t+\mathfrak{s}\}}\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\mathbf{1}_{\widehat{A}}\big]=0.\end{aligned}\tag{6.53}$$

Define $A^{\overline{\omega}}_i:=\underset{j=i+1}{\overset{k}{\cap}}\big(W^{t_{\overline{\omega}}}_{t+s_j},K^{\overline{\omega}}_{t+s_j}\big)^{-1}(\mathcal{O}_{j,\overline{\omega}})\in\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_{t+\mathfrak{s}}$ with $\mathcal{O}_{j,\overline{\omega}}:=\big\{\mathfrak{x}+\mathfrak{a}_{\overline{\omega}}:\mathfrak{x}\in\mathcal{O}_j\big\}\in\mathscr{B}(\mathbb{R}^{d+l})$. Since $t_{\overline{\omega}}=\overline{\gamma}(\overline{\omega})<\overline{\tau}^t_n(\overline{\omega})\leq t+n$ and since $t_{\overline{\omega}}\leq t+s_{i+1}\leq t+\mathfrak{s}$, we set $n_{\overline{\omega}}$ as in Step (i) and using (3.2) with $(t,\mathbf{x},\overline{P},\varphi,n,\mathfrak{a},\overline{\zeta}_1,\overline{\zeta}_2)=\big(t_{\overline{\omega}},\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\overline{Q}^{\overline{\omega}}_\varepsilon,\varphi_{\overline{\omega}},n_{\overline{\omega}},\mathfrak{a}_{\overline{\omega}},t+n\wedge\mathfrak{s},t+n\wedge\mathfrak{r}\big)$ renders that

$$\begin{aligned}0&=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\Big(\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t_{\overline{\omega}}+n_{\overline{\omega}})\wedge(t+n\wedge\mathfrak{r})}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t_{\overline{\omega}}+n_{\overline{\omega}})\wedge(t+n\wedge\mathfrak{s})}(\varphi_{\overline{\omega}})\Big)\mathbf{1}_{A^{\overline{\omega}}_i}\Big]\\&=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\Big(\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t+n\wedge\mathfrak{r})}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t+n\wedge\mathfrak{s})}(\varphi_{\overline{\omega}})\Big)\mathbf{1}_{A^{\overline{\omega}}_i}\Big].\end{aligned}\tag{6.54}$$

Given $j\in\{i+1,\cdots,k\}$ and $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\mathcal{N}^{\overline{\omega}}_X\cup\mathcal{N}^{\overline{\omega}}_K\big)^c$, we can derive that $(\overline{W}^t,\overline{X})(t+s_j,\overline{\omega}')\in\mathcal{O}_j$ if and only if $\big(\overline{W}^{t_{\overline{\omega}}},K^{\overline{\omega}}\big)(t+s_j,\overline{\omega}')=\big(\overline{W}^t_{t+s_j}(\overline{\omega}')-\overline{W}^t\big(\overline{\gamma}(\overline{\omega}),\overline{\omega}'\big),\overline{\mathfrak{X}}^{\overline{\omega}}_{t+s_j}(\overline{\omega}')\big)=(\overline{W}^t,\overline{X})(t+s_j,\overline{\omega}')+\mathfrak{a}_{\overline{\omega}}\in\mathcal{O}_{j,\overline{\omega}}$. So (6.52) implies that

$$\widehat{A}\cap\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\mathcal{N}^{\overline{\omega}}_X\cup\mathcal{N}^{\overline{\omega}}_K\big)^c=\overline{A}_i\cap A^{\overline{\omega}}_i\cap\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\big(\mathcal{N}^{\overline{\omega}}_X\cup\mathcal{N}^{\overline{\omega}}_K\big)^c. \tag{6.55}$$

Let $\overline{\omega}'\in\overline{\Xi}^t_{\overline{\gamma},\overline{\omega}}$. Like in Step (i), we still have $\overline{\tau}^t_n(\overline{\omega}')=\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge(t+n)$ since $\overline{\gamma}(\overline{\omega})<\overline{\tau}^t_n(\overline{\omega})$. Taking $(r_1,r_2)=\big(\overline{\tau}^t_n(\overline{\omega}')\wedge(t+\mathfrak{s}),\overline{\tau}^t_n(\overline{\omega}')\wedge(t+\mathfrak{r})\big)$ in (6.45) shows that $\big(\overline{M}^t(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge(t+\mathfrak{r}),\overline{\omega}'\big)-\big(\overline{M}^t(\varphi)\big)\big(\overline{\tau}^t_n(\overline{\omega}')\wedge(t+\mathfrak{s}),\overline{\omega}'\big)=\big(\overline{M}^{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)\big(\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge(t+n\wedge\mathfrak{r}),\overline{\omega}'\big)-\big(\overline{M}^{t_{\overline{\omega}}}(\varphi_{\overline{\omega}})\big)\big(\overline{\zeta}^n_{\overline{\omega}}(\overline{\omega}')\wedge(t+n\wedge\mathfrak{s}),\overline{\omega}'\big)$. Then we can deduce from (6.53), (6.55), (6.40), (6.41) and (6.54) that for any $\overline{\omega}\in\{t+s_i<\overline{\gamma}\le t+s_{i+1}\}\cap\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$

$$\begin{aligned}
E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\Big]&=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\mathbf{1}_{\{\overline{\tau}^t_n>\overline{\gamma}\}}\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\Big]\\
&=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\mathbf{1}_{\overline{A}_i\cap A^{\overline{\omega}}_i}\mathbf{1}_{\{\overline{\tau}^t_n>\overline{\gamma}\}}\big(\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{r})}(\varphi)-\overline{M}^t_{\overline{\tau}^t_n\wedge(t+\mathfrak{s})}(\varphi)\big)\Big]\\
&=\mathbf{1}_{\{\overline{\omega}\in\overline{A}_i\}}\mathbf{1}_{\{\overline{\tau}^t_n(\overline{\omega})>\overline{\gamma}(\overline{\omega})\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\Big(\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t+n\wedge\mathfrak{r})}(\varphi_{\overline{\omega}})-\overline{M}^{t_{\overline{\omega}}}_{\overline{\zeta}^n_{\overline{\omega}}\wedge(t+n\wedge\mathfrak{s})}(\varphi_{\overline{\omega}})\Big)\mathbf{1}_{A^{\overline{\omega}}_i}\Big]=0,
\end{aligned}$$

and thus $\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{t+s_i<\overline{\gamma}(\overline{\omega})\le t+s_{i+1}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\big]\overline{P}(d\overline{\omega})=0$. Taking summation from $i=1$ through $i=k-1$, we obtain from (6.51) that

$$\begin{aligned}
E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\{\overline{\gamma}\le t+\mathfrak{s}\}}\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\Big]&=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*\cap\{\overline{\gamma}\le t+\mathfrak{s}\}\cap\widehat{A}}\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\Big]\\
&\quad+\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\mathbf{1}_{\{\overline{\gamma}(\overline{\omega})\le t+\mathfrak{s}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\Big]\overline{P}(d\overline{\omega})=0.
\end{aligned}$$

Adding it to (6.46) yield $E_{\overline{P}_\varepsilon}\big[\big(\overline{M}^{t,n}_{t+\mathfrak{r}}(\varphi)-\overline{M}^{t,n}_{t+\mathfrak{s}}(\varphi)\big)\mathbf{1}_{\widehat{A}}\big]=0$.

Hence, $\overline{P}_\varepsilon\in\overline{\mathcal{P}}^2_t$. According to Part (2a) of the proof of Proposition 4.1, $\overline{P}_\varepsilon$ satisfies (D1) and (D2) in Definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

II.d) In this part, we show that $\overline{P}_\varepsilon\big\{\overline{T}=\widehat{\tau}_\varepsilon(\overline{W})\big\}=1$ for some $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}_\varepsilon$, i.e., $\overline{P}_\varepsilon$ satisfies (D3) in Definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

II.d.1) For any $s\in[t,\infty)$, there is a $[0,1]-$valued $\mathcal{F}^{W^t}_s-$measurable random variable ϑ^ε_s on Ω_0 such that

$$\vartheta^\varepsilon_s\big(\overline{W}(\overline{\omega})\big)=E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\Big|\mathcal{F}^{\overline{W}^t}_s\Big](\overline{\omega}),\quad\forall\,\overline{\omega}\in\overline{\Omega}. \tag{6.56}$$

Since $\overline{W}^t$ is a Brownian motion under $\overline{P}_\varepsilon$ by Part (II.c), applying Lemma A.1 with $t_0=t$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}_\varepsilon,\overline{W}\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W\big)$ and $\Phi=\overline{W}$ yields that $\big\{\vartheta^\varepsilon_s(\overline{W})\big\}_{s\in[t,\infty)}$ is an $\mathbf{F}^{\overline{W}^t}-$adapted process and that $E_{P_0}[\vartheta^\varepsilon_s]=E_{\overline{P}_\varepsilon}\big[\vartheta^\varepsilon_s(\overline{W})\big]=E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big]$ is right-continuous in $s\in[t,\infty)$. As $\mathbf{F}^{W^t,P_0}$ is a right-continuous complete filtration, the process $\{\vartheta^\varepsilon_s\}_{s\in[t,\infty)}$ admits a $[0,1]-$valued $\mathbf{F}^{W^t,P_0}-$adapted càdlàg modification $\big\{\widehat{\vartheta}^\varepsilon_s\big\}_{s\in[t,\infty)}$. Define a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time by

$$\mathsf{T}_\varepsilon(\omega_0):=\inf\big\{s\in[t,\infty):\widehat{\vartheta}^\varepsilon_s(\omega_0)=1\big\}. \tag{6.57}$$

As $\overline{W}^t$ is also a Brownian motion under $\overline{P}_\varepsilon$ by Part (II.c), using Lemma A.1 with $t_0=t$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}_\varepsilon,\overline{W}\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W\big)$ and $\Phi=\overline{W}$ implies that $\widehat{\tau}(\overline{W})$ and $\mathsf{T}_\varepsilon(\overline{W})$ are $[t,\infty]-$valued $\mathbf{F}^{\overline{W}^t,\overline{P}_\varepsilon}-$stopping times. Then

$$\overline{\tau}_\varepsilon:=\widehat{\tau}(\overline{W})\mathbf{1}_{\{\widehat{\tau}(\overline{W})<\overline{\gamma}\}}+\big(\mathsf{T}_\varepsilon(\overline{W})\vee\overline{\gamma}\big)\mathbf{1}_{\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}}$$

is also a $[t,\infty]-$valued $\mathbf{F}^{\overline{W}^t,\overline{P}_\varepsilon}-$stopping time. According to Lemma A.4, there exists two $[t,\infty]-$valued $\mathscr{B}(\overline{\Omega})-$measurable random variables $\overline{\xi}$ and $\overline{\xi}_\varepsilon$ such that

$$\text{both } \big\{\widehat{\tau}(\overline{W})\neq\overline{\xi}\big\} \text{ and } \big\{\mathsf{T}_\varepsilon(\overline{W})\neq\overline{\xi}_\varepsilon\big\} \text{ belong to } \mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)\cap\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big). \tag{6.58}$$

We can also update (6.28) to:

$$\big\{\widehat{\tau}(\overline{W})\geq\overline{\gamma}\big\}\Delta\overline{\mathcal{A}}_*\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)\cap\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big). \tag{6.59}$$

Since $\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\mathcal{A}}^c_*\cap\{\overline{T}=\overline{\xi}\}\big)=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}^c_*\}}\overline{Q}^{\overline{\omega}}_\varepsilon\{\overline{T}=\overline{\xi}\}=0,\ \forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*$ by (6.41), one has $\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\cap\{\overline{T}=\overline{\xi}\}\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\{\overline{T}=\overline{\xi}\}\big)$. It follows from (6.59), (6.58) and $\overline{P}\big\{\overline{T}=\widehat{\tau}(\overline{W})\big\}=1$ that

$$\begin{aligned}
&\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\cap\{\overline{T}=\overline{\tau}_\varepsilon\}\big)=\overline{P}_\varepsilon\big(\{\widehat{\tau}(\overline{W})<\overline{\gamma}\}\cap\{\overline{T}=\overline{\tau}_\varepsilon\}\big)=\overline{P}_\varepsilon\big(\{\widehat{\tau}(\overline{W})<\overline{\gamma}\}\cap\{\overline{T}=\widehat{\tau}(\overline{W})\}\big)\\
&\qquad=\overline{P}_\varepsilon\big(\overline{\mathcal{A}}^c_*\cap\{\overline{T}=\overline{\xi}\}\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\{\overline{T}=\overline{\xi}\}\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\cap\{\overline{T}=\widehat{\tau}(\overline{W})\}\big)=\overline{P}\big(\overline{\mathcal{A}}^c_*\big).
\end{aligned}\tag{6.60}$$

II.d.2) We next show that $\overline{P}_\varepsilon\big(\overline{\mathcal{A}}_*\cap\{\overline{T}=\overline{\tau}_\varepsilon\}\big)=\overline{P}(\overline{\mathcal{A}}_*)$ and thus $\overline{P}_\varepsilon\{\overline{T}=\overline{\tau}_\varepsilon\}=1$.

As $\big\{\mathsf{T}_\varepsilon(\overline{W})\neq\overline{\xi}_\varepsilon\big\}\in\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$, there exists $\overline{A}^\varepsilon_\xi\in\mathcal{F}^{\overline{W}^t}_\infty\subset\mathscr{B}(\overline{\Omega})$ such that $\big\{\mathsf{T}_\varepsilon(\overline{W})\neq\overline{\xi}_\varepsilon\big\}\subset\overline{A}^\varepsilon_\xi$ and $\overline{P}_\varepsilon\big(\overline{A}^\varepsilon_\xi\big)=0$. By (6.37), the random variable $\overline{\omega}\mapsto\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}^\varepsilon_\xi\big)$ is $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)\Big)-$measurable. Since $0\leq\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}^\varepsilon_\xi\big)\overline{P}(d\overline{\omega})\leq\overline{P}_\varepsilon\big(\overline{A}^\varepsilon_\xi\big)=0$, we can find $\overline{\mathcal{N}}^\varepsilon_\xi\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$ such that

$$\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{A}^\varepsilon_\xi\big)=0 \text{ and thus } \overline{Q}^{\overline{\omega}}_\varepsilon\big\{\mathsf{T}_\varepsilon(\overline{W})\neq\overline{\xi}_\varepsilon\big\}=0,\quad\forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}^\varepsilon_\xi\big)^c. \tag{6.61}$$

Let $s\in\mathbb{Q}\cap[t,\infty)$ and pick a countable Pi-system $\big\{\overline{\mathcal{O}}_j\big\}_{j\in\mathbb{N}}$ that generates $\mathcal{F}^{\overline{W}^t}_s$. We also let $j\in\mathbb{N}$ and $\overline{A}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$. As $\overline{\mathcal{A}}_*\cap\overline{A}\cap\{\overline{\gamma}\leq s\}\in\mathcal{F}^{\overline{W}^t}_s$, it holds $\overline{P}_\varepsilon-$a.s. that $\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})=\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{\mathcal{O}}_j\cap\{\overline{\gamma}\leq s\}}E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big|\mathcal{F}^{\overline{W}^t}_s\Big]=E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\leq s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big|\mathcal{F}^{\overline{W}^t}_s\Big]$. Then (6.41) and (6.56) imply that

$$\begin{aligned}
&\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]\overline{P}(d\overline{\omega})=E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]\\
&\quad=E_{\overline{P}_\varepsilon}\Big[E_{\overline{P}_\varepsilon}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\leq s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big|\mathcal{F}^{\overline{W}^t}_s\Big]\Big]=E_{\overline{P}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{A}}_*\cap\overline{A}\cap\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\leq s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big]\\
&\quad=\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\leq s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big]\overline{P}(d\overline{\omega}).
\end{aligned}$$

So $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$ is contained in the Lambda-system

$$\begin{aligned}
\overline{\Lambda}^\varepsilon_{s,j}:=\Big\{\overline{A}\in\mathscr{B}_{\overline{P}}(\overline{\Omega}):&\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]\overline{P}(d\overline{\omega})\\
&=\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\leq s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big]\overline{P}(d\overline{\omega})\Big\}.
\end{aligned}$$

As $\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)$ is closed under intersection, Dynkin's Pi-Lambda Theorem shows that $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)\Big)\subset\overline{\Lambda}^\varepsilon_{s,j}$, i.e., for any $\overline{A}\in\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\mathscr{B}(\overline{\Omega})\big)\Big)$

$$\begin{aligned}
&\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]\overline{P}(d\overline{\omega})\\
&\qquad=\int_{\overline{\omega}\in\overline{\Omega}}\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{A}\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\leq s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big]\overline{P}(d\overline{\omega}).
\end{aligned}\tag{6.62}$$

Since $\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})$ and $\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}$ are $\mathscr{B}(\overline{\Omega})-$measurable, we see from (6.37) that the random variables $\overline{\Omega}\ni\overline{\omega}\mapsto E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]$ and $\overline{\Omega}\ni\overline{\omega}\mapsto E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big]$ are $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)\Big)-$measurable. Letting $\overline{A}$ vary over $\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)\Big)$ in (6.62) yields that $\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]=\mathbf{1}_{\{\overline{\omega}\in\overline{\mathcal{A}}_*\}}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\{\overline{T}\in[\overline{\gamma},s]\}}\big]$ for all $\overline{\omega}\in\overline{\Omega}$ except on some $\overline{\mathcal{N}}^\varepsilon_{s,j}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. It then follows from (6.39) that

$$E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\{\widehat{\tau}_{\overline{\omega}}(\overline{W})\in[\overline{\gamma},s]\}}\big],\quad\forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\cap\big(\overline{\mathcal{N}}^\varepsilon_{s,j}\big)^c. \tag{6.63}$$

By Lemma A.1, $\overline{\mathcal{N}}^\vartheta_s:=\overline{W}^{-1}\big(\{\widehat{\vartheta}^\varepsilon_s\ne\vartheta^\varepsilon_s\}\big)$ belongs to $\mathscr{N}_{\overline{P}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. An analogy to (6.61) gives $\overline{\mathcal{N}}^\varepsilon_{s,0}\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$ such that $\overline{Q}^{\overline{\omega}}_\varepsilon\big(\overline{\mathcal{N}}^\vartheta_s\big)=0$ for any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}^\varepsilon_{s,0}\big)^c$.

Set $\overline{\mathcal{N}}^\varepsilon_*:=\overline{\mathcal{N}}_*\cup\Big(\underset{s\in\mathbb{Q}\cap[t,\infty)}{\cup}\ \underset{j=0}{\overset{\infty}{\cup}}\overline{\mathcal{N}}^\varepsilon_{s,j}\Big)\cup\overline{\mathcal{N}}^\varepsilon_\xi\in\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)$. We fix $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}^\varepsilon_*\big)^c$ and let $s\in\mathbb{Q}\cap[t,\infty)$.

• When $s<t_{\overline{\omega}}$: Since (6.63), (5.4) and (6.40) show that $E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}}\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{t_{\overline{\omega}}\le s\}}\mathbf{1}_{\{\widehat{\tau}_{\overline{\omega}}(\overline{W})\in[t_{\overline{\omega}},s]\}}\big]=0$ for any $j\in\mathbb{N}$, Dynkin's Pi-Lambda Theorem implies that $E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{E}}}\vartheta^\varepsilon_s(\overline{W})\big]=0$ for any $\overline{\mathcal{E}}\in\mathcal{F}^{\overline{W}^t}_s$. Letting $\mathcal{E}$ vary over $\mathcal{F}^{\overline{W}^t}_s$ reaches that $\vartheta^\varepsilon_s\big(\overline{W}(\overline{\omega}')\big)=0$ for all $\overline{\omega}'\in\overline{\Omega}$ except on some $\overline{\mathfrak{N}}^{\overline{\omega}}_{s,1}\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$.

• When $s\ge t_{\overline{\omega}}$: Applying Lemma A.1 with $t_0=t_{\overline{\omega}}$, $(\Omega_1,\mathcal{F}_1,P_1,B^1)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{Q}^{\overline{\omega}}_\varepsilon,\overline{W}\big)$, $(\Omega_2,\mathcal{F}_2,P_2,B^2)=\big(\Omega_0,\mathscr{B}(\Omega_0),P_0,W\big)$ and $\Phi=\overline{W}$ yields that $\widehat{\tau}_{\overline{\omega}}(\overline{W})$ is a $[t_{\overline{\omega}},\infty]-$valued $\mathbf{F}^{\overline{W}^{t_{\overline{\omega}}},\overline{Q}^{\overline{\omega}}_\varepsilon}-$stopping time and thus $\big\{\widehat{\tau}_{\overline{\omega}}(\overline{W})\in[\overline{\gamma}\wedge s,s]\big\}\in\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}},\overline{Q}^{\overline{\omega}}_\varepsilon}_s$. By Problem 2.7.3 of [36], there is $\overline{\mathcal{A}}^{\overline{\omega}}_s\in\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_s$ such that $\overline{\mathfrak{N}}^{\overline{\omega}}_{s,2}:=\overline{\mathcal{A}}^{\overline{\omega}}_s\,\Delta\big\{\widehat{\tau}_{\overline{\omega}}(\overline{W})\in[\overline{\gamma}\wedge s,s]\big\}\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big(\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_\infty\big)$. Then we see from (6.63) that $E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\vartheta^\varepsilon_s(\overline{W})\big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\{\widehat{\tau}_{\overline{\omega}}(\overline{W})\in[\overline{\gamma}\wedge s,s]\}}\big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{O}}_j}\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\overline{\mathcal{A}}^{\overline{\omega}}_s}\big]$ for any $j\in\mathbb{N}$, and we know from Dynkin's Pi-Lambda Theorem that $E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{E}}}\vartheta^\varepsilon_s(\overline{W})\big]=E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\mathbf{1}_{\overline{\mathcal{E}}}\mathbf{1}_{\{\overline{\gamma}\le s\}}\mathbf{1}_{\overline{\mathcal{A}}^{\overline{\omega}}_s}\big]$ for any $\overline{\mathcal{E}}\in\mathcal{F}^{\overline{W}^t}_s$. As $\mathcal{F}^{\overline{W}^{t_{\overline{\omega}}}}_s=\sigma\big(\overline{W}^{t_{\overline{\omega}}}_r;r\in[t_{\overline{\omega}},s]\big)=\sigma\big(\overline{W}^t_r-\overline{W}^t_{t_{\overline{\omega}}};r\in[t_{\overline{\omega}},s]\big)\subset\sigma\big(\overline{W}^t_r;r\in[t,s]\big)=\mathcal{F}^{\overline{W}^t}_s$, letting $\mathcal{E}$ run over $\mathcal{F}^{\overline{W}^t}_s$ renders that $\vartheta^\varepsilon_s\big(\overline{W}(\overline{\omega}')\big)=\mathbf{1}_{\{\overline{\gamma}(\overline{\omega}')\le s\}}\mathbf{1}_{\{\overline{\omega}'\in\overline{\mathcal{A}}^{\overline{\omega}}_s\}}$ for all $\overline{\omega}'\in\overline{\Omega}$ except on some $\overline{\mathfrak{N}}^{\overline{\omega}}_{s,3}\in\mathscr{N}_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$.

Let $\overline{\omega}'\in\overline{\mathbf{W}}^t_{\overline{\gamma},\overline{\omega}}\cap\Big(\underset{s\in\mathbb{Q}\cap[t,\infty)}{\cup}\overline{\mathcal{N}}^\vartheta_s\Big)^c\cap\Big(\underset{s\in\mathbb{Q}\cap[t,t_{\overline{\omega}})}{\cup}\overline{\mathfrak{N}}^{\overline{\omega}}_{s,1}\Big)^c\cap\Big(\underset{s\in\mathbb{Q}\cap[t_{\overline{\omega}},\infty)}{\cup}\overline{\mathfrak{N}}^{\overline{\omega}}_{s,2}\cup\overline{\mathfrak{N}}^{\overline{\omega}}_{s,3}\Big)^c\cap\big\{\mathsf{T}_\varepsilon(\overline{W})\ne\overline{\xi}_\varepsilon\big\}$. The above analysis and (5.4) show that

$$\begin{aligned}\widehat{\vartheta}^\varepsilon_s\big(\overline{W}(\overline{\omega}')\big)=\vartheta^\varepsilon_s\big(\overline{W}(\overline{\omega}')\big)&=\mathbf{1}_{\{s\ge t_{\overline{\omega}}\}}\mathbf{1}_{\{\overline{\omega}'\in\overline{\mathcal{A}}^{\overline{\omega}}_s\}}=\mathbf{1}_{\{s\ge t_{\overline{\omega}}\}}\mathbf{1}_{\big\{\widehat{\tau}_{\overline{\omega}}(\overline{W}(\overline{\omega}'))\in[\overline{\gamma}(\overline{\omega}')\wedge s,s]\big\}}\\&=\mathbf{1}_{\{s\ge t_{\overline{\omega}}\}}\mathbf{1}_{\big\{\widehat{\tau}_{\overline{\omega}}(\overline{W}(\overline{\omega}'))\in[t_{\overline{\omega}},s]\big\}},\quad\forall\,s\in\mathbb{Q}\cap[t,\infty).\end{aligned}$$

So the right-continuity of process $\widehat{\vartheta}^\varepsilon_\cdot$ gives that $\widehat{\vartheta}^\varepsilon_s\big(\overline{W}(\overline{\omega}')\big)=\mathbf{1}_{\{s\ge t_{\overline{\omega}}\}}\mathbf{1}_{\big\{\widehat{\tau}_{\overline{\omega}}(\overline{W}(\overline{\omega}'))\in[t_{\overline{\omega}},s]\big\}}$, $\forall\,s\in[t,\infty)$, and we can deduce from (6.57) that

$$\overline{\xi}_\varepsilon(\overline{\omega}')=\mathsf{T}_\varepsilon\big(\overline{W}(\overline{\omega}')\big)=\inf\big\{s\in[t,\infty):\widehat{\vartheta}^\varepsilon_s\big(\overline{W}(\overline{\omega}')\big)=1\big\}=\widehat{\tau}_{\overline{\omega}}\big(\overline{W}(\overline{\omega}')\big)\ge t_{\overline{\omega}}=\overline{\gamma}(\overline{\omega}').$$

In particular, one has $\overline{\xi}_\varepsilon(\overline{\omega}')\vee\overline{\gamma}(\overline{\omega}')=\widehat{\tau}_{\overline{\omega}}\big(\overline{W}(\overline{\omega}')\big)$, which together with (6.40), (6.61) and (6.39) implies that $1=\overline{Q}^{\overline{\omega}}_{\overline{\omega}}\big\{\overline{\xi}_\varepsilon\vee\overline{\gamma}=\widehat{\tau}_{\overline{\omega}}(\overline{W})\big\}=\overline{Q}^{\overline{\omega}}_{\overline{\omega}}\big\{\overline{\xi}_\varepsilon\vee\overline{\gamma}=\overline{T}\big\}$, $\forall\,\overline{\omega}\in\overline{\mathcal{A}}_*\cap\big(\overline{\mathcal{N}}^\varepsilon_*\big)^c$. Then (6.59), (6.58) and (6.41) render that

$$\overline{P}_\varepsilon\big(\overline{\mathcal{A}}_*\cap\{\overline{T}=\overline{\tau}_\varepsilon\}\big)=\overline{P}_\varepsilon\big(\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}\cap\{\overline{T}=\overline{\tau}_\varepsilon\}\big)=\overline{P}_\varepsilon\big(\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}\cap\{\overline{T}=\mathsf{T}_\varepsilon(\overline{W})\vee\overline{\gamma}\}\big)$$

$$=\overline{P}_\varepsilon\big(\overline{\mathcal{A}}_*\cap\{\overline{T}=\overline{\xi}_\varepsilon\vee\overline{\gamma}\}\big)=\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}\overline{Q}^{\overline{\omega}}_\varepsilon\{\overline{T}=\overline{\xi}_\varepsilon\vee\overline{\gamma}\}\overline{P}(d\overline{\omega})=\overline{P}(\overline{\mathcal{A}}_*).$$

Adding it to (6.60) yields $\overline{P}_\varepsilon\{\overline{T}=\overline{\tau}_\varepsilon\}=1$. Moreover, applying Lemma A.2 (2) with $(\Omega,\mathcal{F},P,B)=\big(\overline{\Omega},\mathscr{B}(\overline{\Omega}),\overline{P}_\varepsilon,\overline{W}\big)$ and $\Phi=\overline{W}$, we can find a $[t,\infty]-$valued $\mathbf{F}^{W^t,P_0}-$stopping time $\widehat{\tau}_\varepsilon$ on Ω_0 such that $\overline{\tau}_\varepsilon=\widehat{\tau}_\varepsilon(\overline{W})$, $\overline{P}_\varepsilon-$a.s. Hence, $\overline{P}_\varepsilon$ satisfies (D3) in Definition 3.1 of $\overline{\mathcal{P}}_{t,\mathbf{x}}$.

II.e) Fix $i\in\mathbb{N}$. Since $\big\{\int_t^s g_i(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big\}_{s\in[t,\infty)}$ and $\big\{\int_t^s h_i(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot})dr\big\}_{s\in[t,\infty)}$ are two $\mathbf{F}^{\overline{W}^t,\overline{P}}-$adapted continuous processes, Lemma 2.4 of [62] assures two $\mathbf{F}^{\overline{W}^t}-$predictable processes $\{\overline{\Phi}^i_s\}_{s\in[t,\infty)}$ and $\{\overline{\Psi}^i_s\}_{s\in[t,\infty)}$ such that $\overline{\mathcal{N}}^{i,1}_{g,h}:=\Big\{\overline{\omega}\in\overline{\Omega}:\overline{\Phi}^i_s(\overline{\omega})\neq\int_t^s g_i\big(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot}(\overline{\omega})\big)dr$ or $\overline{\Psi}^i_s(\overline{\omega})\neq\int_t^s h_i\big(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot}(\overline{\omega})\big)dr$ for some $s\in[t,\infty)\Big\}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. By Remark 3.2 (1), $E_{\overline{P}}\big[\int_t^\infty g_i^-(r,\overline{X}_{r\wedge\cdot})\vee h_i^-(r,\overline{X}_{r\wedge\cdot})dr\big]<\infty$. So it holds for any $\overline{\omega}\in\overline{\Omega}$ except on some $\overline{\mathcal{N}}^{i,2}_{g,h}\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{X}}_\infty\big)$ that $\int_t^\infty g_i^-\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega})\big)\vee h_i^-\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega})\big)dr<\infty$.

For any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\cap\overline{\mathcal{N}}^c_X\cap\big(\overline{\mathcal{N}}^{i,1}_{g,h}\cup\overline{\mathcal{N}}^{i,2}_{g,h}\big)^c$, since $\overline{\Omega}^t_{\overline{\gamma},\overline{\omega}}\subset\big\{\overline{\omega}'\in\overline{\Omega}:\overline{X}_s(\overline{\omega}')=\overline{X}_s(\overline{\omega}),\ \forall\, s\in\big[0,\overline{\gamma}(\overline{\omega})\big]\big\}$, (6.40) and (6.38) show that

$$E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\int_t^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]=\int_t^{\overline{\gamma}(\overline{\omega})}g_i\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega})\big)dr+E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\int_{\overline{\gamma}(\overline{\omega})}^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]$$
$$\le\int_t^{\overline{\gamma}(\overline{\omega})}g_i\big(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot}(\overline{\omega})\big)dr+\big(\overline{Y}^i_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})$$
$$=\overline{\Phi}^i_{\overline{\gamma}}(\overline{\omega})+E_{\overline{P}}\Big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big](\overline{\omega})\tag{6.64}$$

and similarly that $E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\int_t^{\overline{T}}h_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]=\overline{\Psi}^i_{\overline{\gamma}}(\overline{\omega})+E_{\overline{P}}\Big[\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}h_i(r,\overline{X}_{r\wedge\cdot})dr\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big](\overline{\omega})$. Since $\overline{\mathcal{A}}_*,\overline{\Phi}^i_{\overline{\gamma}}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ and since $\mathbf{1}_{\overline{\mathcal{A}}_*}=\mathbf{1}_{\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}}=\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}$, $\overline{P}-$a.s. by (6.28), we can deduce from the tower property that

$$\int_{\overline{\omega}\in\overline{\mathcal{A}}_*}E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\Big[\int_t^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]\overline{P}(d\overline{\omega})\le E_{\overline{P}}\bigg[E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*}\Big(\overline{\Phi}^i_{\overline{\gamma}}+\int_{\overline{T}\wedge\overline{\gamma}}^{\overline{T}}g_i(r,\overline{X}_{r\wedge\cdot})dr\Big)\Big|\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\Big]\bigg]$$
$$=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*}\Big(\int_t^{\overline{\gamma}}g_i\big(r,\overline{\mathscr{X}}^{t,\mathbf{x}}_{r\wedge\cdot}\big)dr+\int_{\overline{\gamma}}^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot})dr\Big)\Big]=E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*}\int_t^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]$$

and thus $E_{\overline{P}_\varepsilon}\big[\int_t^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\big]\le E_{\overline{P}}\big[\int_t^{\overline{T}}g_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\big]\le y_i$. Analogously, we have $E_{\overline{P}_\varepsilon}\big[\int_t^{\overline{T}}h_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\big]=E_{\overline{P}}\big[\int_t^{\overline{T}}h_i\big(r,\overline{X}_{r\wedge\cdot}\big)dr\big]=z_i$. Hence, $\overline{P}_\varepsilon$ belongs to $\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$.

II.f) Let $\breve{V}$ be the function defined in (6.33) and set $D^V_\infty:=\big\{\overline{\omega}\in\overline{\Omega}:\breve{V}(\overline{\omega})=\infty\big\}=\big\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*:\overline{V}\big(\breve{\Psi}(\overline{\omega})\big)=\infty\big\}\in\sigma\Big(\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}\cup\mathscr{N}_{\overline{P}}\big(\overline{\mathcal{F}}^t_\infty\big)\Big)$. By Theorem 3.1, D^V_∞ is also equal to $\big\{\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*:\overline{V}\big(\ddot{\Psi}(\overline{\omega})\big)=\infty\big\}$. As $E_{\overline{P}}\big[\int_t^\infty f^-(r,\overline{X}_{r\wedge\cdot})dr\big]<\infty$, there exists a $\overline{\mathcal{N}}_f\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{X}}_\infty\big)$ such that $\int_t^\infty f^-\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega})\big)dr<\infty$ for any $\overline{\omega}\in\overline{\mathcal{N}}^c_f$.

Let $\varepsilon\in(0,1)$. For any $\overline{\omega}\in\overline{\mathcal{A}}_*\cap\overline{\mathcal{N}}^c_*\cap\overline{\mathcal{N}}^c_f$, an analogy to (6.64), (6.36) and Theorem 3.1 imply that

$$E_{\overline{Q}^{\overline{\omega}}_\varepsilon}\big[\overline{R}(t)\big]=\int_t^{\overline{\gamma}(\overline{\omega})}f\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega})\big)dr+E_{\mathbf{Q}_\varepsilon(\ddot{\Psi}(\overline{\omega}))}\big[\overline{R}(\overline{\gamma}(\overline{\omega}))\big]\ge\int_t^{\overline{\gamma}(\overline{\omega})}f\big(r,\overline{X}_{r\wedge\cdot}(\overline{\omega})\big)dr$$

$$+\mathbf{1}_{\{\overline{\omega}\in(D^V_\infty)^c\}}\Big(\overline{V}\big(\overline{\gamma}(\overline{\omega}),\overline{X}_{\overline{\gamma}\wedge\cdot}(\overline{\omega}),\big(\overline{Y}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega}),\big(\overline{Z}_{\overline{P}}(\overline{\gamma})\big)(\overline{\omega})\big)-\varepsilon\Big)+\frac{1}{\varepsilon}\mathbf{1}_{\{\overline{\omega}\in D^V_\infty\}}.$$

Since $\overline{P}_\varepsilon\in\overline{\mathcal{P}}_{t,\mathbf{x}}(y,z)$ and since $\mathbf{1}_{\overline{\mathcal{A}}_*}=\mathbf{1}_{\{\widehat{\tau}(\overline{W})\ge\overline{\gamma}\}}=\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}$, $\overline{P}-$a.s. by (6.28),

$$\overline{V}(t,\mathbf{x},y,z)\ge E_{\overline{P}_\varepsilon}\big[\overline{R}(t)\big]\ge E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}^c_*}\overline{R}(t)+\mathbf{1}_{\overline{\mathcal{A}}_*}\int_t^{\overline{\gamma}}f\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]$$

$$+E_{\overline{P}}\Big[\mathbf{1}_{\overline{\mathcal{A}}_*}\Big(\mathbf{1}_{(D^V_\infty)^c}\big[\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)-\varepsilon\big]+\frac{1}{\varepsilon}\mathbf{1}_{D^V_\infty}\Big)\Big]$$

$$\ge E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f\big(r,\overline{X}_{r\wedge\cdot}\big)dr+\mathbf{1}_{(D^V_\infty)^c}\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\Big]$$

$$-\varepsilon+\frac{1}{\varepsilon}\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big). \tag{6.65}$$

To verify (6.34), we set $\overline{\mathcal{I}}^t_{\overline{P}}:=\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f(r,\overline{X}_{r\wedge\cdot})dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)+\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)$.

- If $\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big)=0$, then $\overline{V}(t,\mathbf{x},y,z)\ge E_{\overline{P}}\Big[\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)+\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f\big(r,\overline{X}_{r\wedge\cdot}\big)dr+\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\Big]-\varepsilon$ holds for any $\varepsilon\in(0,1)$. Letting $\varepsilon\to0$ gives (6.34).
- If $\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big)>0$ and $E_{\overline{P}}\big[\big(\overline{\mathcal{I}}^t_{\overline{P}}\big)^-\big]=\infty$, then $E_{\overline{P}}\big[\overline{\mathcal{I}}^t_{\overline{P}}\big]=-\infty\le\overline{V}(t,\mathbf{x},y,z)$, so (6.34) holds automatically.
- If $\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big)>0$ and $E_{\overline{P}}\big[\big(\overline{\mathcal{I}}^t_{\overline{P}}\big)^-\big]<\infty$, since Remark 3.2 (1) shows that

$$E_{\overline{P}}\Big[-\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}}\overline{R}(t)-\mathbf{1}_{\{\overline{T}\ge\overline{\gamma}\}}\Big(\int_t^{\overline{\gamma}}f\big(r,\overline{X}_{r\wedge\cdot}\big)dr+\mathbf{1}_{(D^V_\infty)^c}\overline{V}\big(\overline{\gamma},\overline{X}_{\overline{\gamma}\wedge\cdot},\overline{Y}_{\overline{P}}(\overline{\gamma}),\overline{Z}_{\overline{P}}(\overline{\gamma})\big)\Big)\Big]$$

$$=E_{\overline{P}}\Big[-\mathbf{1}_{(D^V_\infty)^c}\overline{\mathcal{I}}^t_{\overline{P}}-\mathbf{1}_{D^V_\infty}\int_t^{\overline{\gamma}\wedge\overline{T}}f\big(r,\overline{X}_{r\wedge\cdot}\big)dr-\mathbf{1}_{\{\overline{T}<\overline{\gamma}\}\cap D^V_\infty}\pi\big(\overline{T},\overline{X}_{\overline{T}\wedge\cdot}\big)\Big]$$

$$\le E_{\overline{P}}\Big[\big(\overline{\mathcal{I}}^t_{\overline{P}}\big)^-+\int_t^\infty f^-\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]-c_\pi<\infty,$$

we can deduce from (6.65) that for any $\varepsilon\in(0,1)$

$$\overline{V}(t,\mathbf{x},y,z)\ge-E_{\overline{P}}\Big[\big(\overline{\mathcal{I}}^t_{\overline{P}}\big)^-+\int_t^\infty f^-\big(r,\overline{X}_{r\wedge\cdot}\big)dr\Big]+c_\pi-\varepsilon+\frac{1}{\varepsilon}\overline{P}\big(\{\overline{T}\ge\overline{\gamma}\}\cap D^V_\infty\big).$$

Sending $\varepsilon\to0$ yields $\overline{V}(t,\mathbf{x},y,z)=\infty$, so (6.34) still holds. This completes the proof of Theorem 5.1. □

APPENDIX

In this appendix, we list some technical lemmata needed to verify our main results, we refer interested readers to our ArXiv version [12] for detailed proofs of these lemmata.

LEMMA A.1. *Let $t_0\in[0,\infty)$. For $i=1,2$, let $(\Omega_i,\mathcal{F}_i,P_i)$ be a probability space and let $B^i=\{B^i_s\}_{s\in[0,\infty)}$ be an $\mathbb{R}^d-$valued continuous process on Ω with $B^i_0=\mathbf{0}$ such that $\mathfrak{B}^i_s:=B^i_s-B^i_{t_0}$, $s\in[t_0,\infty)$ is a Brownian motion on $(\Omega_i,\mathcal{F}_i,P_i)$. Let $\Phi:\Omega_1\mapsto\Omega_2$ be a mapping such that $\mathfrak{B}^2_s(\Phi(\omega))=\mathfrak{B}^1_s(\omega)$ for any $(s,\omega)\in[t_0,\infty)\times\Omega_1$, then (i) $\Phi^{-1}\big(\mathcal{F}^{\mathfrak{B}^2}_s\big)=\mathcal{F}^{\mathfrak{B}^1}_s$, $\forall\, s\in[t_0,\infty]$; (ii) $\Phi^{-1}\big(\mathscr{N}_{P_2}(\mathcal{F}^{\mathfrak{B}^2}_\infty)\big)\subset\mathscr{N}_{P_1}(\mathcal{F}^{\mathfrak{B}^1}_\infty)$; (iii) $\Phi^{-1}\big(\mathcal{F}^{\mathfrak{B}^2,P_2}_s\big)\subset\mathcal{F}^{\mathfrak{B}^1,P_1}_s$, $\forall\, s\in[t_0,\infty]$ and (iv) $P_1\circ\Phi^{-1}(A)=P_2(A)$ for any $A\in\mathcal{F}^{\mathfrak{B}^2,P_2}_\infty$.*

LEMMA A.2. *Let* $(\Omega,\mathcal{F},P)$ *be a probability space and let* $t\in[0,\infty)$. *Let* $B=\{B_s\}_{s\in[0,\infty)}$ *be an* $\mathbb{R}^d$*−valued continuous process on* Ω *with* $B_0=\mathbf{0}$ *such that* $B^t_s:=B_s-B_t$, $s\in[t,\infty)$ *is a Brownian motion on* $(\Omega,\mathcal{F},P)$.
(1) For any $[t,\infty]$*−valued* $\mathbf{F}^{W^t,P_0}$*−stopping time* $\widehat{\tau}$ *on* Ω_0, $\widehat{\tau}(B)$ *is an* $\mathbf{F}^{B^t,P}$*−stopping time on* Ω.
(2) Let $\Phi:\Omega\mapsto\Omega_0$ *be a mapping such that* $W^t_s(\Phi(\omega))=B^t_s(\omega)$ *for any* $(s,\omega)\in[t,\infty)\times\Omega$. *For any* $[t,\infty]$*−valued* $\mathbf{F}^{B^t,P}$*−stopping time* τ *on* Ω, *there exists a* $[t,\infty]$*−valued* $\mathbf{F}^{W^t,P_0}$*−stopping time* $\widehat{\tau}$ *on* Ω_0 *such that* $\tau=\widehat{\tau}(\Phi)$, P*−a.s.*

LEMMA A.3. *Let* $\mathfrak{X}$ *be a topological space and let* $\mathfrak{Y}$ *be a Borel space. If* $\mathfrak{f}:\mathfrak{X}\times\mathfrak{Y}\mapsto(-\infty,\infty]$ *is a* $\mathscr{B}(\mathfrak{X})\otimes\mathscr{B}(\mathfrak{Y})$*−measurable function bounded from below, then* $\phi_{\mathfrak{f}}(x,P):=\int_{y\in\mathfrak{Y}}\mathfrak{f}(x,y)P(dy)$, $(x,P)\in\mathfrak{X}\times\mathfrak{P}(\mathfrak{Y})$ *is* $\mathscr{B}(\mathfrak{X})\otimes\mathscr{B}\big(\mathfrak{P}(\mathfrak{Y})\big)$*−measurable.*

LEMMA A.4. *Given* $t\in[0,\infty)$, *let* τ *be a* $[t,\infty]$*−valued* $\mathbf{F}^{W^t,P_0}$*−stopping time and let* $\mathfrak{P}_t$ *be a subset of* $\mathfrak{P}(\overline{\Omega})$ *such that* $\overline{W}^t$ *is a Brownian motion under each* $\overline{P}\in\mathfrak{P}_t$. *There is a* $[t,\infty]$*−valued* $\mathscr{B}(\overline{\Omega})$*−measurable random variable* $\overline{\xi}$ *such that* $\{\tau(\overline{W})\neq\overline{\xi}\}\in\underset{\overline{P}\in\mathfrak{P}_t}{\cap}\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$. *If* $\overline{\gamma}$ *is a* $[t,\infty)$*−valued* $\mathbf{F}^{\overline{W}^t}$*−stopping time, one can find* $\overline{A}\in\mathcal{F}^{\overline{W}^t}_{\overline{\gamma}}$ *such that* $\{\tau(\overline{W})\geq\overline{\gamma}\}\Delta\overline{A}\in\underset{\overline{P}\in\mathfrak{P}_t}{\cap}\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$.

LEMMA A.5. *Let* $t\in[0,\infty)$ *and* $\overline{P}\in\mathfrak{P}(\overline{\Omega})$. *For any* $(s,\overline{\omega})\in[t,\infty)\times\overline{\Omega}$, *set* $\overline{\mathbf{W}}^t_{s,\overline{\omega}}:=\big\{\overline{\omega}'\in\overline{\Omega}:\overline{W}^t_a(\overline{\omega}')=\overline{W}^t_a(\overline{\omega}),\ \forall\,a\in[t,s]\big\}$. *Then for any* $r\in[t,\infty)$, $\mathcal{F}^{W^t,P_0}_r\subset\mathscr{S}_r:=\Big\{A\subset\Omega_0:\exists\,\overline{\mathcal{N}}_r\in\mathscr{N}_{\overline{P}}\big(\mathcal{F}^{\overline{W}^t}_\infty\big)$ *such that for any* $(s,\overline{\omega})\in[t,r]\times\overline{\Omega}$, *there exists* $A^{s,\overline{\omega}}\in\mathcal{F}^{W^s}_r$ *satisfying* $\mathbf{1}_{\{\overline{W}(\overline{\omega}')\in A\}}=\mathbf{1}_{\{\overline{W}(\overline{\omega}')\in A^{s,\overline{\omega}}\}},\ \forall\,\overline{\omega}'\in\overline{\mathbf{W}}^t_{s,\overline{\omega}}\cap\overline{\mathcal{N}}^c_r\Big\}$.

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