ON THE GENERIC BEHAVIOR OF THE SPECTRAL NORM

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Abstract. The main result of the paper is that for any closed symplectic manifold the spectral norm of the iterates of a Hamiltonian diffeomorphism is locally uniformly bounded away from zero C^∞-generically.

Contents

1. Introduction

We show that for a Hamiltonian diffeomorphism φ of a closed symplectic manifold M the spectral norm over $\mathbb{Q}$ of the iterates φ^k is locally uniformly bounded away from zero C^∞-generically in φ, without any additional assumptions on M.

The question of the behavior of the sequence $\gamma\big(\varphi^k\big)$ of spectral norms goes back to the work of Polterovich, [Po02]. Recently, there has been renewed interest in the problem whether and when this sequence is bounded away from zero. There are several reasons for this question, amounting roughly speaking to the fact that one can obtain pretty strong results on the symplectic dynamics of φ when the sequence is *not* bounded away from zero:

$$\underline{\gamma}(\varphi) := \liminf_{k\to\infty} \gamma\big(\varphi^k\big) = 0. \tag{1.1}$$

Among these are, for instance, Lagrangian Poincaré recurrence, [GG18, JS], and the variant of the strong closing lemma from [ÇS]. Simultaneously, fairly explicit criteria

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for this sequence to be bounded away from zero have been established, based on the crossing energy theorem from [GG14, GG18]; see, e.g., [ÇGG22b] and Theorem 3.1. Let us now provide some more context for the question.

First, note that the condition (1.1) can be interpreted as that φ is γ-rigid or, in other words, a γ-approximate identity.

This notion is a particular case of a much more general concept. Namely, consider a class of diffeomorphisms φ or even homeomorphisms of a manifold M, which we assume here to be closed. For instance, this can be the class of all diffeomorphisms or of Hamiltonian diffeomorphisms when M is symplectic, etc. Assume furthermore that this class is equipped with some norm $\|\cdot\|$, e.g., the C^0- or C^1-norm or the γ- or Hofer-norm in the Hamiltonian case. A map φ is said to be $\|\cdot\|$-*rigid* if $\varphi^{k_i} \to id$ with respect to $\|\cdot\|$, i.e., $\left\|\varphi^{k_i}\right\| \to 0$, for some sequence $k_i \to \infty$. The term "rigid" is somewhat overused in dynamics and also frequently confused with structural stability, and in [GG19] we proposed to call such a map φ a $\|\cdot\|$-*approximate identity*, or a $\|\cdot\|$-*a.i.* for the sake of brevity. We refer the reader to, e.g., [Br, GG19, ÇS] for a further discussion of approximate identities, aka rigid maps, in different contexts and further references. Here we only mention that C^r-a.i. is obviously C^s-a.i. for any $s \leq r$ and, when M is aspherical or $M = \mathbb{CP}^n$, a C^0-a.i. is also a γ-a.i.; see [BHS, Sh22b].

Zeroing in on γ-a.i.'s we note that there are rather few examples of such maps. The most dynamically interesting examples are Hamiltonian pseudo-rotations. This class of maps has been extensively studied in a variety of settings by dynamical systems methods and more recently from the perspective of symplectic topology and Floer theory; see, e.g., [AK, A-Z, Br, FK, GG18, JS, LRS] and references therein.

While the official definitions of Hamiltonian pseudo-rotations vary, these are, roughly speaking, Hamiltonian diffeomorphisms with a finite and minimal possible number of periodic points (in the sense of Arnold's conjecture); see [GG18, Sh20, Sh21]. For instance, when $M = \mathbb{CP}^n$ this number is $n + 1$. Most likely, for many symplectic manifolds this condition can be relaxed. Namely, in all examples of Hamiltonian diffeomorphisms φ with finitely many periodic points, all periodic points are fixed points and their number is minimal possible. Thus φ is a pseudo-rotation. For a certain class of manifolds M, including $\mathbb{CP}^n$, this has been established rigorously under a minor non-degeneracy assumption; see [Sh22a] and also [ÇGG22a]. Moreover, in all examples to date of Hamiltonian diffeomorphisms φ with finitely many periodic points, φ is non-degenerate.

In general, the relation between pseudo-rotations and γ-a.i.'s is not obvious. All known Hamiltonian pseudo-rotations are γ-a.i.'s and for $M = \mathbb{CP}^n$ this is proved in [GG18] by using the results from [GG09a]. The converse is not true: for instance any element φ of a Hamiltonian torus action is a γ-a.i., although φ need not have isolated fixed points. (It is conceivable that for a strongly non-degenerate γ-a.i., the periodic points are necessarily the fixed points: in the obvious notation, $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$. However, a map φ with the latter property need not be a γ-a.i. For instance, $\gamma\big(\varphi^k\big)$ can grow linearly for such a map; see Remark 4.10.)

Most closed symplectic manifolds (M,ω) admit no pseudo-rotations, i.e., every Hamiltonian diffeomorphism of M has infinitely many periodic points. This statement (for a specific manifold M) is usually referred to as the Conley conjecture. To date, the Conley conjecture has been shown to hold unless there exists $A \in \pi_2(M)$ such that $\langle [\omega], A\rangle > 0$ and $\langle c_1(TM), A\rangle > 0$; see [Çi18, GG15, GG17] and references therein. In particular, the Conley conjecture holds when M is symplectically aspherical or negative monotone. Furthermore, for a broad class of closed symplectic manifolds, φ has infinitely many periodic points C^∞-generically; see [GG09b, Su21] and Section 4.2.

Although the classes of Hamiltonian pseudo-rotations and γ-a.i.'s are certainly different, there is a clear parallel between these two classes and their existence conditions on M.

Conjecture. *Let M be closed symplectic manifold.*

(i) *The manifold M admits no γ-a.i.'s unless there exists $A \in \pi_2(M)$ such that $\langle [\omega], A\rangle > 0$ and $\langle c_1(TM), A\rangle > 0$.*

(ii) *A Hamiltonian diffeomorphism $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ is a γ-a.i. if and only if all iterates φ^k are Morse–Bott non-degenerate and $\dim \mathrm{H}_*\big(\mathrm{Fix}(\varphi^k); \mathbb{F}\big) = n+1$ for all $k \in \mathbb{N}$ and any ground field $\mathbb{F}$.*

This conjecture is supported by some evidence. For instance, M does not admit periodic Hamiltonian diffeomorphisms φ (i.e., $\varphi^N = id$ for some $N > 1$) when M satisfies the conditions of (i); see [AS, Po02]. In addition, $\mathrm{Fix}(\varphi^k)$ is Morse–Bott non-degenerate whenever φ is periodic. This is a consequence of the equivariant Darboux lemma; see, e.g., [GS, Thm. 22.2]. Moreover, aspherical or negative monotone symplectic manifolds do not admit C^1-a.i.'s; see [Po02] and [Su23]. Further results and references along these lines can be found in [AS]. In [ÇGG22b] both assertions are proved in dimension two for strongly non-degenerate Hamiltonian diffeomorphisms; see Corollary 3.4. Moreover, in the setting of (i) the sequence of the spectral norms $\gamma\big(\varphi^p\big)$ over $\mathbb{Z}/p\mathbb{Z}$, where p ranges through all primes, is separated away from zero, [Sh23]. As we have already mentioned the "if" part of (ii) is established in [GG18] without any non-degeneracy assumption when $|\operatorname{Per}(\varphi)| = n+1$. With this in mind, Part (ii) of the conjecture asserts, in particular, that every pseudo-rotation of $\mathbb{CP}^n$ is strongly non-degenerate.

Remark 1.1. While Part (ii) of the conjecture might extend to some other ambient symplectic manifolds M, some restriction on M is necessary. For instance, the torus $\mathbb{T}^{2n}$ equipped with an irrational symplectic structure admits a Hamiltonian diffeomorphism φ such that the conditions of (ii) are satisfied but $\gamma\big(\varphi^k\big) \to \infty$; see [Ze] and also [Çi23] for further constructions of this type with complicated dynamics.

In a similar vein, the main result of this paper can be thought of as the γ-a.i. analogue of the aforementioned theorem on the C^∞-generic Conley conjecture, although at this moment the proof of the latter requires some additional conditional conditions on the underlying manifold; see Section 4.2.

Remark 1.2. Overall, rather little is known about the behavior of the γ-norm under iterations. For a certain class of manifolds, including $\mathbb{CP}^n$, the spectral norm is *a priori* bounded from above, [EP, KS]. However, such manifolds appear to be rare; see Remark 4.10. Also, the sequence $\gamma\big(\varphi^k\big)$ is bounded from above when $\operatorname{supp}\varphi$ is displaceable in M, but not much beyond these facts and the results of this paper is known about the behavior of this sequence. For instance, when M is a surface of positive genus, it is not known if $\gamma\big(\varphi^k\big)$ necessarily grows linearly or can be bounded from above when φ is strongly non-degenerate or, as the opposite extreme, autonomous and $\operatorname{supp}\varphi$ is not displaceable.

Remark 1.3. It is worth keeping in mind that in contrast with some other dynamics concepts, in most if not all settings a.i.'s are sensitive to reparametrization. To be more specific, let an a.i. φ be the time-one map of the flow of a vector field X and let ψ be the time-one map of fX for some function $f > 0$. Then, in general, ψ need not be an a.i. For instance, assume that X is a solid rotation vector field on $M = S^2$ and $f \neq const$. Then one can show that ψ is not a C^0-a.i., and hence not a C^r-a.i. for any $r \geq 0$. Apparently, the same is true for the γ-norm, but this fact is yet to be proved rigorously; cf. item (ii) of the Conjecture.

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2. Preliminaries and notation

In this section we very briefly set our notation and conventions which are quite standard and spelled out in more detail in, e.g., [ÇS]. The reader may find it convenient to jump to Section 3 and consult this section only as needed.

Throughout the paper, all manifolds, functions and maps are assumed to be C^∞-smooth unless specifically stated otherwise.

Let (M^{2n}, ω) be a closed symplectic manifold. A *Hamiltonian diffeomorphism* $\varphi = \varphi_H = \varphi_H^1$ is the time-one map of the time-dependent flow $\varphi^t = \varphi_H^t$ of a 1-periodic in time Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. We set $H_t = H(t, \cdot)$. The Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. We say that φ is *non-degenerate* if all fixed points of φ are non-degenerate, and *strongly non-degenerate* if all periodic points of φ are non-degenerate. We will denote by $Ham(M, \omega)$ the group of Hamiltonian diffeomorphisms of (M, ω).

Recall that the *spectral norm*, also known as the γ*-norm*, of φ is defined as

$$\gamma(\varphi) = \inf_H \big\{ \operatorname{c}(H) + \operatorname{c}\left(H^{\mathrm{inv}}\right) \mid \varphi = \varphi_H \big\},$$

where $H^{\mathrm{inv}}(x) = -H_t\big(\varphi_H^t(x)\big)$ is the Hamiltonian generating the flow $\big(\varphi_H^t\big)^{-1}$ and $\operatorname{c} = \operatorname{c}_{[M]}$ is the spectral invariant associated with the fundamental class $[M] \in$

$\mathrm{H}_{2n}(M)$. (Here we can take as H^{inv} any Hamiltonian generating this flow with the same time/space average as H.) The infimum is taken over all 1-periodic in time Hamiltonians H generating φ, i.e., $\varphi = \varphi_H$. The *Hofer norm* of φ is defined as

$$\|\varphi\|_H = \inf_H \int_{S^1} \big(\max_M H_t - \min_M H_t \big)\, dt,$$

where the infimum is again taken over all 1-periodic in time Hamiltonians H generating φ. Then

$$\gamma(\varphi) \leq \|\varphi\|_H.$$

We refer the reader to, e.g., [Oh05a, Oh05b, Sc, Vi] and also, e.g., [ÇS, EP, GG09a, KS, Po01, Us08, Us11], for the original treatment and a detailed discussion of spectral invariants and these norms, and for further references.

Here we are interested in the behavior of $\gamma\big(\varphi^k\big)$, $k \in \mathbb{N}$, and in particular in the question when this sequence is bounded away from zero. As in the introduction, set

$$\underline{\gamma}(\varphi) = \liminf_{k\to\infty} \gamma\big(\varphi^k\big) \in [0,\, \infty].$$

These definitions implicitly depend on the construction of the filtered Floer homology $\mathrm{HF}^a(H)$ for the action window $(-\infty,\, a)$. In this paper we do not in general assume that the class $[\omega]$ is rational or that φ is non-degenerate. Hence, we feel, a word is due on the specifics of the definitions.

Assume first that H is non-degenerate. Then we utilize Pardon's VFC package, [Pa], to define the filtered Floer homology $\mathrm{HF}^a(H)$ over $\mathbb{Q}$ and spectral invariants; see, e.g., [ÇS, Us08]. To be more specific, $\mathrm{HF}^a(H)$ is the homology of the subcomplex $\mathrm{CF}^a(H)$ of the Floer complex $\mathrm{CF}(H)$ generated by Floer chains with action below a. Virtually any choice of the *Novikov field* can be used here. We take the standard Novikov field

$$\Lambda = \Big\{ \sum_{A\in\Gamma} b_A A \,\Big|\, b_A \in \mathbb{Q} \text{ and } \#\{b_A \neq 0,\, \omega(A) > c\} < \infty\ \forall\, c \in \mathbb{R} \Big\},$$

where $\Gamma = \pi_2(M)/\big(\ker[\omega] \cap \ker c_1(TM)\big)$. Alternatively, we could have used the universal Novikov field. Then, for any $\alpha \in \mathrm{H}_*(M) \otimes \Lambda$, the spectral invariant $\mathrm{c}_\alpha(H)$ is defined as

$$\mathrm{c}_\alpha(H) = \inf\{a \in \mathbb{R} \mid \alpha \in \operatorname{im} \iota_a\}, \tag{2.1}$$

where

$$\iota_a\colon\ \mathrm{HF}^a(H) \to \mathrm{HF}(H) \cong \mathrm{H}_*(M) \otimes \Lambda \tag{2.2}$$

is the natural inclusion-induced map and the identification on the right is the PSS-isomorphism. We note that all spectral invariants necessarily belong to the action spectrum $\mathcal{S}(H)$ of H when H is non-degenerate, [Us08].

When H is not necessarily non-degenerate, we set

$$\mathrm{c}_\alpha(H) := \inf_{\tilde H \geq H} \mathrm{c}_\alpha(\tilde H) = \sup_{\tilde H \leq H} \mathrm{c}_\alpha(\tilde H) = \lim_{\tilde H \to H} \mathrm{c}_\alpha(\tilde H),$$

where $\tilde H$ is non-degenerate and the convergence $\tilde H \to H$ is taken to be C^0. The second and third equalities and the existence of the limit follow from that c_α is monotone and $\mathrm{c}_\alpha(\tilde{H} + const) = \mathrm{c}_\alpha(\tilde{H}) + const$. Alternatively, we could have set

$$\mathrm{HF}^a(H) = \varinjlim_{\tilde{H} \geq H} \mathrm{HF}^a(\tilde{H}),$$

and then used (2.1) and (2.2) to get the same result.

Defined in this way, spectral invariants c_α can be easily shown to have all the standard properties: $\mathrm{c}_\alpha(H)$ is monotone and Lipschitz continuous in H with Lipschitz constant one; $\mathrm{c}_\alpha(H + const) = \mathrm{c}_\alpha(H) + const$; etc. (We refer the reader to, e.g., [ÇS] for more details.) The exception is that $\mathrm{c}_\alpha(H)$ has been proven to be spectral, i.e., an element of $\mathcal{S}(H)$, only when $[\omega]$ is rational or H is non-degenerate; see [EP, Oh05b, Us08].

3. Main results

The key to bounding $\underline{\gamma}$ from below is the following fact connecting the behavior of $\gamma\big(\varphi^k\big)$ with the dynamics of φ and, in particular, its hyperbolic points.

Theorem 3.1. *Let $\varphi\colon M \to M$ be a Hamiltonian diffeomorphism of a closed symplectic manifold M with more than $\dim \mathrm{H}_*(M)$ hyperbolic periodic points. Then $\underline{\gamma}(\varphi) > 0$. Moreover, $\underline{\gamma}$ is locally uniformly bounded away from zero near φ, i.e., there exists $\delta > 0$, possibly depending on φ, and a sufficiently C^∞-small neighborhood $\mathcal{U}$ of φ such that*

$$\underline{\gamma}(\psi) > \delta \text{ for all } \psi \in \mathcal{U}.$$

Without the moreover part, this theorem was originally proved in [ÇGG22b]. We give a complete proof in Section 4. Let us emphasize that in Theorem 3.1 we impose no non-degeneracy requirements on φ, and also that the property of φ to have more than $\dim \mathrm{H}_*(M)$ hyperbolic periodic points, or more than any fixed number of hyperbolic periodic points, is open in C^1-topology.

Example 3.2. Assume that M is a closed surface and $\mathrm{h_{top}}(\varphi) > 0$. Then φ has infinitely many hyperbolic periodic points, [Ka]. Hence, $\underline{\gamma}(\varphi) > 0$. Moreover, $\underline{\gamma}(\psi) > \delta$ for some $\delta > 0$ and all ψ which are C^∞-close to φ. Also note in connection with Theorem 3.3 and Corollary 3.4 below that $\mathrm{h_{top}} > 0$ is a C^∞-generic condition in dimension two, [LCS].

The requirement of the theorem that the number of hyperbolic points is greater $\dim \mathrm{H}_*(M)$ can be further relaxed by looking only at the odd/even-degree homology of M, depending on whether $n = \dim M/2$ is odd or even; see Remark 4.2.

The main result of the paper is the following theorem relying on Theorem 3.1 and the strong closing lemma from [ÇS].

Theorem 3.3. *Let M be a closed symplectic manifold. The function $\underline{\gamma}$ is locally uniformly bounded away from zero on a C^∞-open and dense set of Hamiltonian diffeomorphisms $\varphi\colon M \to M$, i.e., for every φ in this set there exists $\delta > 0$, possibly depending on φ but not on ψ, such that*

$$\underline{\gamma}(\psi) > \delta$$

whenever ψ is sufficiently C^∞-close to φ.

We note that we do not assert here that in general the set of Hamiltonian diffeomorphisms φ with $\underline{\gamma}(\varphi) > 0$ is itself C^∞-open, but rather that this set contains a set which is C^∞-open and dense. Nor do we impose any restrictions on the (symplectic) topology of M or require any of the iterates φ^k to be non-degenerate. The proof of Theorem 3.3 given in Section 4.1 is based on a variant of the Birkhoff–Lewis–Moser theorem. The key new ingredient of the proof is the strong closing lemma from [ÇS]. It is also worth pointing out that if we replaced the statement that the set is C^∞-dense by that it is C^1-dense, the theorem would turn into an easy consequence of already known facts; see Remark 4.5.

In several situations, Theorem 3.3 can be made slightly more precise. For instance, we have the following result, also originally proved in [ÇGG22b] without the moreover part.

Corollary 3.4. *Assume that M is a surface and φ is strongly non-degenerate. Then $\underline{\gamma}(\varphi) > 0$ when M has positive genus. When M is the two-sphere, $\underline{\gamma}(\varphi) = 0$ if and only if φ is a pseudo-rotation. Moreover, $\underline{\gamma}$ is locally uniformly bounded from 0 on the set of all strongly non-degenerate Hamiltonian diffeomorphisms φ when M has positive genus and on the set of such φ with at least three fixed points when $M = S^2$.*

Proof. When M has positive genus, a Conley conjecture type argument guarantees that φ has infinitely many hyperbolic periodic points; see [FH, GG15, SZ] or [LCS]. Thus, in this case, the statement follows directly from Theorem 3.1.

Concentrating on $M = S^2$, first note that for all, not necessarily non-degenerate, pseudo-rotations of $\mathbb{CP}^n$, the sequence $\gamma\big(\varphi^k\big)$ contains a subsequence converging to zero, and hence $\underline{\gamma}(\varphi) = 0$; see [GG18]. In the opposite direction, when $M = S^2$, the existence of one positive hyperbolic periodic point is enough to ensure that $\underline{\gamma}(\varphi) > 0$ and, moreover, $\underline{\gamma}$ is locally uniformly bounded away from zero; see Remark 4.2. Hence, more generally, without any non-degeneracy assumption, if $\underline{\gamma}(\varphi) = 0$, then all periodic points of φ are elliptic. For strongly non-degenerate Hamiltonian diffeomorphisms φ, this forces φ to be a pseudo-rotation. ☐

Since the Hofer norm is bounded from below by the spectral norm, we have the following.

Corollary 3.5. *In all results from this section, we can replace the spectral norm by the Hofer norm.*

We refer the reader to the next section for further refinements of Theorems 3.1 and 3.3.

Remark 3.6. Throughout the paper all homology groups are taken over $\mathbb{Q}$. This choice of the background coefficient field is necessitated by the use of Floer theory for an arbitrary closed symplectic manifold M. When M is weakly monotone, $\mathbb{Q}$ can be replaced by any coefficient field.

4. Proofs and refinements

In section 4.1, we prove Theorems 3.1 and 3.3. In Section 4.2, we refine the latter result under certain additional assumptions on M and further comment on the class of γ-a.i.'s.

4.1. **Proofs of Theorems 3.1 and 3.3.**

Proof of Theorem 3.1. By the conditions of the theorem, for some $N \in \mathbb{N}$, the Hamiltonian diffeomorphism φ has more than $\dim \mathrm{H}_*(M)$ hyperbolic N-periodic points. We denote the set of these points by $\mathcal{K}$. Thus $|\mathcal{K}| > \dim \mathrm{H}_*(M)$ and clearly $\mathcal{K}$ is a locally maximal hyperbolic set. Furthermore, every point in $\mathcal{K}$ is also ℓN-periodic for all $\ell \in \mathbb{N}$. For $\epsilon > 0$, denote by $b_\epsilon(\varphi)$ the number of bars in the barcode of φ of length greater than ϵ including infinite bars; see, e.g., [ÇGG21]. Then, we claim that for a sufficiently small $\epsilon > 0$ and any $\ell \in \mathbb{N}$,

$$b_\epsilon\big(\varphi^{\ell N}\big) \geq \dim \mathrm{H}_*(M) + \lceil (|\mathcal{K}| - \dim \mathrm{H}_*(M))/2 \rceil > \dim \mathrm{H}_*(M). \tag{4.1}$$

In particular, $\varphi^{\ell N}$ has at least one finite bar of length greater than $\epsilon > 0$.

This inequality is essentially a consequence of [ÇGG21, Prop. 3.8 and 6.2]. To prove (4.1), first note that the number of infinite bars in the barcode of any Hamiltonian diffeomorphism is equal to $\dim \mathrm{H}_*(M)$. Secondly, it follows from [ÇGG21, Prop. 6.2] and the proof of [ÇGG21, Prop. 3.8] that every periodic point in $\mathcal{K}$ appears as an "end point" of a bar of length greater than $\epsilon > 0$. Combining these two facts, we conclude that $\varphi^{\ell N}$ has at least $\lceil (|\mathcal{K}| - \dim \mathrm{H}_*(M))/2 \rceil$ finite bars of length greater than $\epsilon > 0$, and (4.1) follows.

Furthermore, since the crossing energy lower bound in [ÇGG21, Thm. 6.1] is stable under C^∞-small perturbations of the Hamiltonian, for every positive $\eta < \epsilon$ the same is true for any Hamiltonian diffeomorphism Ψ which is C^∞-close to φ^N. Namely,

$$b_\eta\big(\Psi^\ell\big) > \dim \mathrm{H}_*(M),$$

and hence the barcode of Ψ^ℓ has a finite bar of length greater than η.

Also, recall that as is proved in [KS, Thm. A], for any φ,

$$\beta_{\max}(\varphi) \leq \gamma(\varphi),$$

where the left-hand side is the *boundary depth*, i.e., the longest finite bar in the barcode of φ. Thus, for a sufficiently small $\eta > 0$,

$$\eta < \beta_{\max}\big(\Psi^\ell\big) \leq \gamma\big(\Psi^\ell\big). \tag{4.2}$$

Next, set $\delta = \eta/2$ and arguing by contradiction, assume that there exist ψ sufficiently C^∞-close to φ and a sequence $k_i \to \infty$ such that

$$\gamma\big(\psi^{k_i}\big) < \delta.$$

Since the sequence k_i is infinite and there are only finitely many residues modulo N, there exists a pair $k_i < k_j$ such that

$$k_j - k_i = \ell N$$

for some $\ell \in \mathbb{N}$.

Clearly, $\Psi = \psi^N$ is C^∞ close to φ^N when ψ is sufficiently C^∞-close to φ, and hence (4.2) holds. Then by the triangle inequality for γ, we have

$$\eta < \gamma\big(\Psi^\ell\big) \le \gamma\big(\psi^{k_j}\big) + \gamma\big(\psi^{-k_i}\big) < 2\delta = \eta.$$

This contraction concludes the proof of the theorem. ☐

Remark 4.1. It might be worth a second to examine how the invariants of φ involved in the proof depend on the isotopy φ_H^t in $Ham(M,\omega)$ generated by H and its lift to the universal covering of the group. Namely, $\gamma(\varphi)$ is *a priori* independent of the isotopy only on the universal covering. On $Ham(M,\omega)$ it is defined by passing to the infimum over often infinitely many elements. However, the boundary depth $\beta_{\max}$ is well-defined on $Ham(M,\omega)$. In the proof we bound $\beta_{\max}(\varphi)$ from below (see, e.g., [Us11]) and that bounds $\gamma(\varphi)$ from below regardless of the lift, [KS].

Remark 4.2. When $n = \dim M/2$ is odd, it is sufficient to require in Theorem 3.1 that the number of hyperbolic periodic points is greater than $b = \dim \mathrm{H}_{odd}(M)$. For instance, this is the case when M is a surface. Indeed, in the proof of the theorem by taking N even and sufficiently large, we can guarantee that the number of positive hyperbolic N-periodic points is greater than b. Such points necessarily have even Conley–Zehnder index, and hence contribute to the odd-degree homology of M under the isomorphism $\mathrm{HF}_*(\varphi^N) \cong \mathrm{H}_{*+n}(M)$. Likewise, when n is even, it suffices to require the number of hyperbolic periodic points to be greater than $\dim \mathrm{H}_{even}(M)$.

Proof of Theorem 3.3. To prove the theorem, it suffices to show that every C^∞-open set $\mathcal{U}$ in the group of Hamiltonian diffeomorphisms contains an open subset $\mathcal{W}$ such that $\underline{\gamma}(\varphi) > \delta$ for all $\varphi \in \mathcal{W}$ and some $\delta = \delta(\mathcal{W}) > 0$ independent of φ. Indeed, then fixing $\mathcal{W}$ for every $\mathcal{U}$ we can take the union of sets $\mathcal{W}$ for all $\mathcal{U}$ as the desired open and dense subset.

Let $q = \dim \mathrm{H}_*(M)$. For any $\mathcal{U}$, there are two alternatives:

(i) there exists $\varphi \in \mathcal{U}$ with more than q periodic points;
(ii) every $\varphi \in \mathcal{U}$ has at most q periodic points.

Let us first focus on Case (i). Pick $\varphi \in \mathcal{U}$ with more than q periodic points and fix $q+1$ of them. Denote these points by $x_0, \dots, x_q$, and note that arbitrarily C^∞-close to φ there exists a Hamiltonian diffeomorphism $\varphi' \in \mathcal{U}$ such that $x_0, \dots, x_q$ are non-degenerate periodic points of φ'. This is essentially a linear algebra fact and to construct φ', it suffices to perturb φ near these points, changing $D\varphi$ slightly. (Note that φ' may have many other periodic points, non-degenerate or not. We can ensure in addition that φ' is strongly non-degenerate, but we do not need this fact.) We replace φ by φ', keeping the notation φ.

If all periodic points $x_0, \dots, x_q$ are hyperbolic, we can take as $\mathcal{W}$ any C^∞-small neighborhood of φ by Theorem 3.1.

If one of the points $x_0, \dots, x_q$ is not hyperbolic, we argue by perturbing φ again. Namely, recall that by the Birkhoff–Lewis–Moser theorem (see [Mo]), whenever φ has a non-hyperbolic, non-degenerate periodic point x, there exists an arbitrarily C^∞-small perturbation $\varphi' \in \mathcal{U}$ of φ with infinitely many periodic points near x.

Moreover, φ' can be chosen so that infinitely many of these periodic points are hyperbolic; see [Ar, Prop. 8.2]. (This follows from the proof of the Birkhoff–Lewis–Moser theorem.) Thus, again by Theorem 3.1, we can take a sufficiently C^∞-small neighborhood of φ' as $\mathcal{W}$.

To deal with Case (ii), we need the following quantitative variant of the strong closing lemma.

Lemma 4.3 (Strong Closing Lemma, [ÇS]). *Let ψ be a Hamiltonian diffeomorphism of a closed symplectic manifold M. Assume that there is a closed ball $V \subset M$ containing no periodic points of ψ, i.e., $V \cap \operatorname{Per}(\psi) = \emptyset$. Let $G \geq 0$ be a Hamiltonian supported in V and such that*

$$\mathrm{c}(G) > \underline{\gamma}(\psi).$$

Then the composition $\psi\varphi_G$ has a periodic orbit passing through V.

Pick a non-degenerate Hamiltonian diffeomorphism $\varphi \in \mathcal{U}$, where $\mathcal{U}$ is as in Case (ii). Such a map exists since $\mathcal{U}$ is C^∞-open and the set of non-degenerate Hamiltonian diffeomorphisms is C^∞-dense (and open). We will show that there exists $\delta > 0$ such that $\underline{\gamma}(\psi) > \delta$ for all $\psi \in \mathcal{U}$ which are C^∞-close to φ. Hence, in this case, we can take a small C^∞-neighborhood of φ as $\mathcal{W}$.

Lemma 4.4. *Let (M, ω) be a closed symplectic manifold. Suppose that there exists a C^∞-open $\mathcal{U} \subset Ham(M, \omega)$ such that all $\varphi \in \mathcal{U}$ have at most $q = \dim H_*(M)$ periodic points. Then the function $\underline{\gamma} \colon \mathcal{U} \to [0, \infty)$ is locally uniformly bounded away from zero at every non-degenerate $\varphi \in \mathcal{U}$.*

Note that the proof of Theorem 3.3 will be completed once we prove Lemma 4.4. To prove the lemma, arguing by contradiction, fix a non-degenerate $\varphi \in \mathcal{U}$ and assume that there exists a sequence $\psi_i \to \varphi$ in $\mathcal{U}$ such that

$$\underline{\gamma}(\psi_i) \to 0.$$

Here and below convergence of maps is always understood in the C^∞-sense.

We claim that when i is large enough, all periodic points of ψ_i are close to periodic points of φ, and hence there exists a closed ball $V \subset M$ containing no periodic points of any of these maps. Indeed, since φ is non-degenerate and

$$|\operatorname{Fix}(\varphi)| \leq |\operatorname{Per}(\varphi)| \leq q = \dim \mathrm{H}_*(M),$$

by the Arnold conjecture (see [FO, LT] and also [Pa]),

$$\operatorname{Per}(\varphi) = \operatorname{Fix}(\varphi) \text{ and } |\operatorname{Per}(\varphi)| = |\operatorname{Fix}(\varphi)| = q.$$

Furthermore, when i is large enough, $\psi_i \in \mathcal{U}$ is also non-degenerate since $\psi_i \to \varphi$. Therefore, again by the Arnold conjecture,

$$\operatorname{Per}(\psi_i) = \operatorname{Fix}(\psi_i) \text{ and } |\operatorname{Per}(\psi_i)| = |\operatorname{Fix}(\psi_i)| = q.$$

It follows that $\operatorname{Per}(\psi_i)$ converges to $\operatorname{Per}(\varphi)$.

Next, take $G \geq 0$ as in Lemma 4.3, which is supported in V and small enough so that $\varphi\varphi_G \in \mathcal{U}$. Hence, $\psi_i\varphi_G \in \mathcal{U}$ when i is large; for $\psi_i \to \varphi$ and thus $\psi_i\varphi_G \to \varphi\varphi_G$.

On the other hand, due to the assumption that $\underline{\gamma}(\psi_i) \to 0$, we have

$$c(G) > \underline{\gamma}(\psi_i),$$

when again i is sufficiently large. By the strong closing lemma, the composition $\psi_i\varphi_G$ has a periodic orbit passing through V. On the other hand, the fixed points of ψ_i (or equivalently the periodic points) are among the fixed points of $\psi_i\varphi$ because $\operatorname{supp} G \subset V$. It follows that

$$|\operatorname{Per}(\psi_i\varphi_G)| \geq q + 1$$

when i is large enough, which is impossible since $\psi_i\varphi_G \in \mathcal{U}$. This contradiction completes the proof of Lemma 4.4 and hence of Theorem 3.3. □

Remark 4.5. If in Theorem 3.3 we were to find a C^1-dense (and open) set of Hamiltonian diffeomorphisms rather than C^∞-dense, the argument would be considerably simpler. Namely, in this case it would be enough to first construct a map φ with just one hyperbolic periodic point. Once this is done, we could apply the results from [Ha, Xi] to create non-trivial transverse homoclinic intersections, and hence a horseshoe (see [KH]) by a C^1-small perturbation. As a consequence, the perturbed map ψ would have infinitely many hyperbolic periodic points. For any $m \in \mathbb{N}$, having at least m such points is a C^1-open property. Now we can take any $m > \dim \mathrm{H}_*(M)$.

4.2. **Sugimoto manifolds and further remarks.** As is shown in [Su21], a strongly non-degenerate Hamiltonian diffeomorphism φ of a closed symplectic manifold M^{2n} has either a non-hyperbolic periodic point or infinitely many hyperbolic periodic points when M meets one of the following requirements:

(i) n is odd;
(ii) $\mathrm{H}_{odd}(M) \neq 0$;
(iii) the minimal Chern number of M is greater than 1.

Below we refer to a closed symplectic manifold meeting at least one of these requirements as a *Sugimoto manifold*. For this class of manifolds Theorem 3.3 has a more direct proof and can be slightly refined. We do this in two steps.

Denote by $\mathcal{V}_m$, $m \in \mathbb{N}$, the set of Hamiltonian diffeomorphisms with at least m hyperbolic points. Note that we do not require the elements of $\mathcal{V}_m$ to be strongly non-degenerate.

Proposition 4.6. *Let M be a Sugimoto manifold. Then for any $m \in \mathbb{N}$ the set $\mathcal{V}_m$ is C^1-open and C^∞-dense in the space of all Hamiltonian diffeomorphisms.*

Proof. The statement that $\mathcal{V}_m$ is C^1-open is obvious. (It is essential here that m is finite.) To show that it is C^∞-dense we argue as in [Su21] and the proof of Theorem 3.3. Let φ be a Hamiltonian diffeomorphism. To prove the proposition, we need to find $\psi \in \mathcal{V}_m$ arbitrarily C^∞-close to φ. Since the set of strongly non-degenerate Hamiltonian diffeomorphisms is C^∞-dense, we can assume that φ is in this class. As shown in [Su21], φ has infinitely many hyperbolic periodic points or a (non-degenerate) non-hyperbolic point. In the former case, $\varphi \in \mathcal{V}_m$ for all $m \in \mathbb{N}$. In the latter case, by [Ar, Prop. 8.2], for any $m \in \mathbb{N}$ there exists $\psi \in \mathcal{V}_m$ arbitrarily close to φ. □

As an immediate consequence, we obtain a slightly more precise variant of the main result from [Su21]:

Corollary 4.7. *Assume that M is a Sugimoto manifold. Then C^∞-generically a Hamiltonian diffeomorphism φ of M has infinitely many hyperbolic periodic points.*

The key difference with [Su21] is that the periodic points of φ here are specified to be hyperbolic. The residual set in this corollary is, of course,

$$\mathcal{V} := \bigcap_{m\in\mathbb{N}} \mathcal{V}_m.$$

We note that this set is not C^1- and even C^∞-open. However, one can require in addition φ to be strongly non-degenerate. Indeed, the set of such maps is residual and its intersection with $\mathcal{V}$ is still a residual set.

Closer to the immediate subject of the paper we have the following refinement of Theorem 3.3 and Corollary 3.4:

Corollary 4.8. *Assume that M is a Sugimoto manifold. Then $\underline{\gamma}$ is locally uniformly bounded away from zero on a C^1-open and C^∞-dense set of Hamiltonian diffeomorphisms of M.*

Here we can take any $\mathcal{V}_m$ with $m > \dim \mathrm{H}_*(M)$ as a C^1-open and C^∞-dense set, where $\underline{\gamma}$ is locally uniformly bounded away from zero. Note also that in this corollary we can again replace the spectral norm by the Hofer norm.

Remark 4.9. In contrast with Theorem 3.3, C^∞-generic existence of infinitely many periodic points is not known to hold without some additional assumptions on M. The class of Sugimoto manifolds is the broadest to date for which such existence has been proved, [Su21]. (See also [GG09b] for the original result and a different approach.)

Remark 4.10. Continuing the discussion from the introduction and Remark 1.2, we give here some "textbook" examples where $\gamma\big(\varphi^k\big)$ grows linearly, and hence $\underline{\gamma}(\varphi) = \infty$, and at the same time all periodic points of φ are fixed points: $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$. Namely, let $H\colon M \to \mathbb{R}$ be a non-constant autonomous Hamiltonian such that H has only finitely many critical values and all non-constant periodic orbits of the flow of H are non-contractible. Set $\varphi = \varphi_H$. Then, as is easy to see, $\gamma\big(\varphi^k\big)$ grows linearly and the only periodic points of φ are the critical points of H. For instance, we can take $H = \sin(2\pi\theta)$, where θ is the first angular coordinate θ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Alternatively, let $(\mathbb{T}^4, \omega)$ be a Zehnder's torus, i.e., a torus equipped with a sufficiently irrational translation invariant symplectic structure ω (see [Ze]), and again let $\theta\colon \mathbb{T}^4 \to \mathbb{R}/\mathbb{Z}$ be a fixed angular coordinate. Then the flow of H given by the same formula has no periodic orbits at all, contractible or not, other than the critical points of H: the 3-dimensional tori $\theta = 1/2$ and $\theta = 3/2$. In both cases, $\gamma\big(\varphi^k\big) = 2k$. More surprisingly, there exists a Hamiltonian diffeomorphism $\varphi\colon S^2 \times S^2 \to S^2 \times S^2$ such that $\gamma\big(\varphi^k\big)$ grows linearly; see [Sh22a, Rmk. 8] and [PR, Thm. 6.2.6], although the argument is quite indirect.

In all these examples, $\dim \mathrm{H}_*(\mathrm{Fix}(\varphi)) = \dim \mathrm{H}_*(M)$ over any field, in addition to the condition that $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$. Loosely following [AS], we call such a map φ a *generalized pseudo-rotation.* Generalized pseudo-rotations from the above examples have simple dynamics. However, this is not necessarily so in general. For instance, in dimension six and higher Morse-Bott non-degenerate, generalized pseudo-rotations φ with positive topological entropy have been recently constructed in [Çi23]. Such a generalized pseudo-rotation can be neither a C^0-a.i. since $\mathrm{h}_{\mathrm{top}}(\varphi) > 0$ (see [A-Z]) nor a γ-a.i. In fact, $\gamma\big(\varphi^k\big)$ also grows linearly since M is aspherical and $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$ has finitely many connected components.

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