ZEROS OF A GROWING NUMBER OF DERIVATIVES OF RANDOM POLYNOMIALS WITH INDEPENDENT ROOTS

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Abstract. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables in $\mathbb{C}$ chosen from a probability measure μ and define the random polynomial

$$P_n(z) = (z - X_1) \ldots (z - X_n).$$

We show that for any sequence $k = k(n)$ satisfying $k \leq \log n/(5 \log\log n)$, the zeros of the kth derivative of P_n are asymptotically distributed according to the same measure μ. This extends work of Kabluchko, which proved the $k = 1$ case, as well as Byun, Lee and Reddy who proved the fixed k case.

1. Introduction

Let μ be a probability measure on $\mathbb{C}$. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with values in $\mathbb{C}$ sampled from μ, and for each n define the random polynomial

$$P_n(z) = (z - X_1) \ldots (z - X_n).$$

By the law of large numbers if we consider the *empirical measure* of P_n, where we a put a point mass of $1/n$ at each root of P_n, then we see that the empirical measure converges to μ as n approaches infinity. Pemantle and Rivin [18] conjectured that the same holds for the derivative P_n. To make this precise, define $\mu_n^{(1)}$ to be the probability measure on $\mathbb{C}$ that puts a point mass at each critical point of P_n:

$$\mu_n^{(1)} := \frac{1}{n-1} \sum_{z \in \mathbb{C} : P_n'(z) = 0} \delta_z .$$

Pemantle and Rivin conjectured that $\mu_n^{(1)} \to \mu$ in distribution as $n \to \infty$ and proved their conjecture under the assumption that μ has finite 1-energy. Subramanian [19] proved the Pemantle-Rivin conjecture in the special case when μ is supported on the unit circle (see also [5] and the discussion below Conjecture 4.4 for more context and discussion on the mode of convergence). In an influential work, Kabluchko [10] confirmed Pemantle and Rivin's conjecture for *all* probability measures μ. Since then, attention has been focused on higher derivatives. To this end, for each k define the (random) probability measure $\mu_n^{(k)}$ via

$$\mu_n^{(k)} := \frac{1}{n-k} \sum_{z \in \mathbb{C} : P_n^{(k)}(z) = 0} \delta_z .$$

Byun, Lee and Reddy [3] extended Kabluchko's work and showed that for each fixed k,[1] we have $\mu_n^{(k)} \to \mu$. Looking towards very high derivatives, O'Rourke and Steinerberger [17] conjecture that for each fixed $t \in [0,1]$, one has that the random measure $\mu_n^{\lfloor tn \rfloor}$ converges to a deterministic measure μ_t. In the case when the underlying measure μ is radial, O'Rourke and Steinerberger conjecture that the logarithmic potentials of the limiting measure $(\mu_t)_t$ satisfy a certain partial differential equation. This partial differential equation has been studied by analysts [1, 13], and O'Rourke and Steinerberger's conjecture was proven in the special case when μ has real support [8] (see also [7, 11]). In the $t = o(1)$ case, the prediction of O'Rourke and Steinerberger suggests that the limiting measure should be the same as the underlying measure μ.

We confirm this in the case that k grows slightly slower than logarithmically:

Theorem 1.1. *Let μ be a probability measure on $\mathbb{C}$ and $k = k(n)$ be a sequence satisfying $k \leq \frac{\log n}{5 \log\log n}$. Then the sequence $\mu_n^{(k)} \to \mu$ in probability as $n \to \infty$.*

Further, we show that the case of $k = o(n)$ would follow from an anti-concentration conjecture for elementary symmetric polynomials evaluated at i.i.d. random variables (see Conjecture 4.1 and Remark 4.2).

Our proof of Theorem 1.1 takes inspiration from the potential-theoretic approach of Kabluchko's proof of the $k = 1$ case (as does the work of Byun, Lee and Reddy [3]); the key new step is an anti-concentration ingredient. We take a moment to sketch our proof here.

The starting point is the following classical fact from potential theory: if f is an analytic function not identically equal to zero then

$$\frac{1}{2\pi}\Delta \log |f| = \sum_{\zeta \in \mathbb{C}: f(\zeta)=0} \delta_\zeta \tag{1}$$

where the Laplacian is interpreted in the distributional sense.

In particular, if we define

$$L_n^{(k)}(z) := \frac{P_n^{(k)}(z)}{k! P_n(z)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} Y_{i_1} Y_{i_2} \ldots Y_{i_k}, \tag{2}$$

where $Y_i := \frac{1}{z - X_i}$ for $i \in [n] := \{1, \ldots, n\}$, then we have

$$\frac{1}{2\pi n}\Delta \log \left| L_n^{(k)}(z) \right| = \frac{1}{n} \sum_{z \in \mathbb{C}: P_n^{(k)}(z)=0} \delta_z - \frac{1}{n} \sum_{z \in \mathbb{C}: P_n(z)=0} \delta_z = \frac{n-k}{n}\mu_n^{(k)} - \mu_n^{(0)}. \tag{3}$$

By the law of large numbers, the measure $\mu_n^{(0)}$ tends to μ as $n \to \infty$, and so in order to show $\mu_n^{(k)}$ converges to μ it will be enough to show that $\frac{1}{n}\log |L_n^{(k)}(z)|$ goes to 0 in a sufficiently strong sense to guarantee that $\frac{1}{n}\Delta \log |L_n^{(k)}(z)|$ tends to 0.

Obtaining an upper bound on the magnitude of $\log |L_n^{(k)}(z)|$ involves controlling two different events: when $|L_n^{(k)}(z)|$ is large and when $|L_n^{(k)}(z)|$ is very small. The former event will not be too difficult to deal with, but the latter is trickier. This comes down to an anti-concentration problem for elementary symmetric polynomials evaluated at i.i.d. random variables.

[1]The work [3] does not state an explicit rate at which k can be taken to grow. An inspection of the proof shows that it depends on the Levy concentration of the non-atomic part of the random variable $(z - X)^{-1}$ as z varies in $\mathbb{C}$ (see [3, eq. (4.6)]).

Lemma 1.2. *Let $Y_1, Y_2, \dots,$ be i.i.d. copies of a complex-valued non-degenerate random variable. Then for $1 \le k \le \frac{\log n}{5 \log\log n}$ and each $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbf{P}\left[\left|\sum_{1\le i_1<i_2<\cdots<i_k\le n} Y_{i_1}Y_{i_2}\dots Y_{i_k}\right| \le e^{-\varepsilon n}\right] = 0.$$

Here we think of $Y_i = (z - X_i)^{-1}$ as above. The proof of Lemma 1.2 will be deduced from a theorem of Meka, Nguyen and Vu [16] about anti-concentration of multi-affine polynomials of Bernoulli random variables. We note that Lemma 1.2 is the only place where we need the assumption of $k \le \frac{\log n}{5 \log\log n}$; the number 5 may be replaced by any number strictly larger than 4. Lemma 1.2 will be of crucial use in the proof of Lemma 2.2.

After showing pointwise bounds on $\log|L_n^{(k)}(z)|$, we upgrade these bounds by showing that they hold in some uniform sense. To take care of this, we again follow Kabluchko and lean on a lemma from Tao and Vu [21] stating that tightness of an L^2-norm will be sufficient to deduce convergence in probability of the measures. For this, we use the Poisson-Jensen formula to relate values of $\log|L_n^{(k)}(z)|$ in a disk to its values on the boundary of a larger disk, and show uniform bounds at the origin and on the boundary of this disk.

We note that the assumption $k \le \frac{\log n}{5 \log\log n}$ is due to the use of the general anti-concentration result of Meka, Nguyen and Vu (which we reproduce as Theorem 3.1) to prove Lemma 1.2. An anti-concentration result particularly geared towards our application that holds all the way up to $k = o(n)$ would allow for Theorem 1.1 to be extend to the $k = o(n)$ case (see Conjecture 4.1).

1.1. **Notation.** Throughout, we use $B_r(z)$ to be the disk of radius r centered at z and abbreviate $B_r(0)$ as B_r. We write m for the standard (2-real-dimensional) Lebesgue measure on $\mathbb{C}$ and denote $\xrightarrow[n\to\infty]{P}$ for the convergence in probability as $n \to \infty$. For $z \ge 0$ we write $\log z = \log_+ z - \log_- z$ where

$$\log_- z = \begin{cases} |\log z|, & 0 \le z \le 1, \\ 0, & z \ge 1, \end{cases} \quad \text{and} \quad \log_+ z = \begin{cases} 0, & 0 \le z \le 1, \\ \log z, & z \ge 1, \end{cases}$$

with the convention $\log_- 0 = +\infty$.

2. Proof of the main result

We will show two main lemmas in this section. First, we recall the following result in [21].

Lemma 2.1 (Lemma 3.1 in [21]). *Suppose that $(X, \mathcal{A}, \nu)$ is a finite measure space, $(f_n)_{n\ge1} : X \to \mathbb{R}$ are random functions which are defined over a probability space $(\Omega, \mathbf{b}, \mathbf{P})$ and are jointly measurable with respect to $\mathcal{A} \otimes \mathbf{b}$. In addition,*

(1) For ν-a.e. $x \in X$ we have $f_n(x) \to 0$ in probability (resp. almost surely) , as $n \to \infty$.
(2) For some $\delta > 0$, the sequence $\int_X |f_n(x)|^{1+\delta} d\nu(x)$ is bounded in probability (resp. almost surely).

Then, $\int_X f_n(x) d\nu(x)$ converges in probability (resp. almost surely) to 0.

Now we show pointwise convergence of $\frac{1}{n}\log|L_n^{(k)}(z)|$ in probability almost everywhere:

Lemma 2.2. *There is a set $F \subset \mathbb{C}$ with $m(F) = 0$ so that for every $z \in \mathbb{C} \backslash F$ and all sequences $k \leq \frac{\log n}{5 \log \log n}$ we have*

$$\frac{1}{n} \log |L_n^{(k)}(z)| \xrightarrow[n\to\infty]{P} 0. \tag{4}$$

In order to prove Theorem 1.1, we will need to upgrade Lemma 2.2 from pointwise convergence to a more uniform mode of convergence. By Lemma 2.1, it will be sufficient to show tightness of a second moment.

Lemma 2.3. *For each $r > 0$ and $k \leq \frac{\log n}{5 \log \log n}$, the sequence of random variables*

$$\left(\frac{1}{n^2} \int_{B_r} \left(\log |L_n^{(k)}(z)| \right)^2 dm(z) \right)_{n \geq 1}$$

is tight.

We now deduce Theorem 1.1 from Lemmas 2.2 and 2.3.

Proof of Theorem 1.1. It is sufficient to show that for every smooth compactly supported function $\varphi : \mathbb{C} \to \mathbb{R}$ we have

$$\frac{1}{n} \sum_{z \in \mathbb{C}: P_n^{(k)}(z) = 0} \varphi(z) \to \int_{\mathbb{C}} \varphi(z)\, d\mu(z)$$

as $n \to \infty$ (see, e.g., [12, Theorem 14.16]). By, e.g., [9, Formula (2.4.4)], for an analytic function g and compactly supported smooth function φ we have

$$\sum_{z: g(z) = 0} \varphi(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \Delta\varphi(z) \log |g(z)|\, dm(z)$$

and so

$$\frac{1}{2\pi n} \int_{\mathbb{C}} (\log |L_n^{(k)}(z)|) \Delta\varphi(z)\, dm(z) = \frac{1}{n} \sum_{z \in \mathbb{C}: P_n^{(k)}(z) = 0} \varphi(z) - \frac{1}{n} \sum_{z \in \mathbb{C}: P_n(z) = 0} \varphi(z). \tag{5}$$

Combining Lemmas 2.3 and 2.2 with Lemma 2.1 shows

$$\frac{1}{n} \int_{\mathbb{C}} (\log |L_n^{(k)}(z)|) \Delta\varphi(z) dm(z) \xrightarrow[n\to\infty]{P} 0\,. \tag{6}$$

By the law of large numbers, the right-most term of (5) tends to $\int_{\mathbb{C}} \varphi(z) d\mu(z)$ almost-surely, completing the proof. $\square$

2.1. Proof of Lemma 2.2.

Lemma 2.4. *Let $F = \{z \in \mathbb{C} : \int_{\mathbb{C}} |y - z|^{-1} d\mu(y) = \infty\}$. Then $m(F) = 0$.*

Proof. Since μ is a probability measure we have that $\int_{y:|y-z|\geq 1} |z-y|^{-1} d\mu(y) \leq 1$ for all z. Thus F is equal to the set of z for which $\int_{B_1(z)} |y-z|^{-1}\, d\mu(y) = \infty$. Apply Fubini's theorem to compute

$$\int_{\mathbb{C}} \int_{B_1(z)} |z-y|^{-1} d\mu(y) dm(z) = \int_{\mathbb{C}} \int_{B_1(y)} |z-y|^{-1} dm(z) d\mu(y) = 2\pi.$$

Thus $m(F) = 0$. □

Proof of Lemma 2.2. If the random variable X_1 is almost surely a constant, then the lemma follows easily, and so we assume that X_1 is non-degenerate.

First, we show that for every fixed $z \in \mathbb{C} \setminus F$ and every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mathbf{P}\left[\frac{1}{n}\log|L_n^{(k)}(z)| \geq \varepsilon\right] = 0. \tag{7}$$

Indeed, defining $Y_i = (z - X_i)^{-1}$ for all $i \in [n]$, Markov's inequality bounds

$$\mathbf{P}\left[|L_n^{(k)}(z)| \geq e^{\varepsilon n}\right] \leq e^{-\varepsilon n}\mathbf{E}[|L_n^{(k)}(z)|] \leq e^{-\varepsilon n}\binom{n}{k}(\mathbf{E}|Y_1|)^k\,. \tag{8}$$

Since $z \in \mathbb{C} \setminus F$, the expectation $\mathbf{E}|Y_1|$ is finite. Thus, the right-hand-side tends to zero as long as $k = o(n)$ and so the limit (7) follows.

An application of Lemma 1.2 will provide a lower bound, i.e., we prove that for every $z \in \mathbb{C}$ which is not an atom of μ and every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left[\frac{1}{n}\log|L_n^{(k)}(z)| \leq -\varepsilon\right] = 0. \tag{9}$$

Write

$$\mathbf{P}\left[\frac{1}{n}\log|L_n^{(k)}(z)| \leq -\varepsilon\right] = \mathbf{P}\left[\left|\sum_{1\leq i_1<i_2<\cdots<i_k\leq n} Y_{i_1}Y_{i_2}\ldots Y_{i_k}\right| \leq e^{-\varepsilon n}\right]$$

and note that Lemma 1.2 shows the right-hand-side is $o(1)$, completing the proof of the lemma. □

2.2. **Proof of Lemma 2.3.** To prove tightness, we will use the Poisson-Jensen formula to write $\log|L_n^{(k)}(z)|$ in terms of a Poisson integral of $\log|L_n^{(k)}(z)|$ along the circle of radius R for some $R > r$, plus a small correction in terms of the zeros and poles of $L_n^{(k)}(z)$. From here, showing tightness of the L^2 norm will come in two steps: $\frac{1}{n}\log|L_n^{(k)}(z)|$ is tight at 0 and on the circle of radius R, and that the correction depending on the zeros and poles is not too large.

Note that by the proof of Lemma 2.4 for any $R > 0$ we have that

$$\int_{B_R}\int_{\mathbb{C}} |z-y|^{-1} d\mu(y)\, dm(z) < \infty\,.$$

By switching the outer integral into polar coordinates, we see that for Lebesgue-almost-all $R > 0$ we have that

$$\int_0^{2\pi}\int_{\mathbb{C}} \frac{1}{|Re^{i\theta} - y|}\, d\mu(y)\, d\theta < \infty\,. \tag{10}$$

Throughout, we assume that R satisfies this assumption.

We now write the Poisson-Jensen formula for $\log |L_n^{(k)}(z)|$: Let $R > r$ be chosen as above, let $x_{1,n}, \ldots, x_{j_n,n}$ denote the zeros of $P_n(z)$ in the disk B_R, and let $y_{1,n}, \ldots, y_{\ell_n,n}$ be the zeros of $P_n^{(k)}(z)$ in the disk B_R. Also note that $j_n \leq n$ and $\ell_n \leq n$. We take the following standard facts about the Poisson-Jensen formula from [15, Chapter 8]:

Lemma 2.5 (Poisson-Jensen formula).

$$\log |L_n^{(k)}(z)| = I_n^{(k)}(z;R) + \sum_{\ell=1}^{\ell_n} \log \left| \frac{R(z-y_{\ell,n})}{R^2 - \bar{y}_{\ell,n} z} \right| - \sum_{j=1}^{j_n} \log \left| \frac{R(z-x_{j,n})}{R^2 - \bar{x}_{j,n} z} \right|, \tag{11}$$

where

$$I_n^{(k)}(z;R) = \frac{1}{2\pi} \int_0^{2\pi} \log |L_n^{(k)}(Re^{i\theta})| P_R(|z|, \theta - \arg z) d\theta \tag{12}$$

where P_R denotes the Poisson kernel

$$P_R(r,\phi) = \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos\phi}, \quad r \in [0,R],\ \phi \in [0, 2\pi]. \tag{13}$$

The only property of the Poisson kernel we need is that it provides only bounded distortion when r is uniformly bounded away from R.

Fact 2.6. *For all $0 < r < R$ there exists $M \geq 1$ so that for all $z \in B_r$ and $\theta \in [0, 2\pi]$ we have*

$$\frac{1}{M} \leq P_R(|z|, \theta) \leq M\,. \tag{14}$$

We first provide an upper bound $I_n^{(k)}(z;R)$ in probability that is uniform in B_r for all k and n.

Lemma 2.7. *For all R satisfying* (10) *and $0 < r < R$ there is a constant $C > 0$ so that for all $t > 0$ and $1 \leq k \leq n$ we have*

$$\mathbf{P}\left[\frac{1}{n} \sup_{z \in B_r} I_n^{(k)}(z;R) \geq t \right] \leq \frac{C}{t}\,. \tag{15}$$

Proof. Apply Fact 2.6 and bound

$$I_n^{(k)}(z;R) \leq \frac{M}{2\pi} \int_0^{2\pi} \log_+ \left| L_n^{(k)}(Re^{i\theta}) \right| d\theta. \tag{16}$$

Bound

$$\begin{aligned} \frac{1}{n} \log_+ \left| L_n^{(k)}(Re^{i\theta}) \right| &\leq \frac{1}{n} \log\left(1 + |L_n^{(k)}(Re^{i\theta})|\right) \\ &\leq \frac{1}{n} \log\left(1 + \binom{n}{k}^{-1} |L_n^{(k)}(Re^{i\theta})|\right) + 1 \end{aligned} \tag{17}$$

where we used that $n^{-1} \log \binom{n}{k} \leq 1$.

Claim 2.8. *We have*

$$\int_0^{2\pi} \mathbf{E} \frac{1}{n} \log\left(1 + \binom{n}{k}^{-1} |L_n^{(k)}(Re^{i\theta})|\right) d\theta \leq \int_0^{2\pi} \int_{\mathbb{C}} |Re^{i\theta} - y|^{-1}\, d\mu(y)\, d\theta\,.$$

Proof of Claim 2.8. Noting that the function $x \mapsto \log(1+x)$ is concave for $x \geq 0$, Jensen's inequality bounds

$$\mathbf{E} \log \left(1 + \binom{n}{k}^{-1} |L_n^{(k)}(Re^{i\theta})|\right) \leq \log \left(1 + \mathbf{E}\binom{n}{k}^{-1} |L_n^{(k)}(Re^{i\theta})|\right).$$

Since R satisfies (10), for almost all $\theta \in [0, 2\pi]$ we may bound

$$\mathbf{E}\binom{n}{k}^{-1} |L_n^{(k)}(Re^{i\theta})| \leq \left(\int_{\mathbb{C}} \left|Re^{i\theta} - y\right|^{-1} d\mu(y)\right)^k.$$

Combining the previous two displayed equations and integrating over θ shows that

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{n}\mathbf{E} \log \left(1 + \binom{n}{k}^{-1} |L_n^{(k)}(Re^{i\theta})|\right) d\theta &\leq \int_0^{2\pi} \frac{1}{n} \log \left(1 + \left(\int_{\mathbb{C}} \left|Re^{i\theta} - y\right|^{-1} d\mu(y)\right)^k\right) d\theta \\
&\leq \int_0^{2\pi} \frac{k}{n} \int_{\mathbb{C}} |Re^{i\theta} - y|^{-1} \, d\mu(y) \, d\theta \\
&\leq \int_0^{2\pi} \int_{\mathbb{C}} |Re^{i\theta} - y|^{-1} \, d\mu(y) \, d\theta
\end{aligned}$$

where we used the elementary inequality $\log(1 + x^k) \leq kx$ for $x \geq 0$. □

Combining lines (16) and (17) with Claim 2.8 and Markov's inequality completes the proof. □

The Poisson-Jensen form will allow us to compare $I_n^{(k)}(z; R)$ to $I_n^{(k)}(0; R)$. It will be convenient to assume that $0 \notin F$. If $0 \in F$, we may find some $a \notin F$ (recall that F has measure 0) and replace each X_i with $X_i - a$; thus, we may assume without loss of generality that $0 \notin F$.

Lemma 2.9. *There is a constant C_1 depending only on $R > 0$ such that for $k \leq \frac{\log n}{5 \log\log n}$ we have*

$$\lim_{n\to\infty} \mathbf{P}\left[\frac{1}{n} I_n^{(k)}(0; R) \leq -C_1\right] = 0. \tag{18}$$

Proof. By the Poisson–Jensen formula (12) at $z = 0$, we have

$$I_n^{(k)}(0; R) = \log |L_n^{(k)}(0)| - \sum_{\ell=1}^{\ell_n} \log \left|\frac{y_{\ell,n}}{R}\right| + \sum_{j=1}^{j_n} \log \left|\frac{x_{j,n}}{R}\right|. \tag{19}$$

For the first term on the right-hand side of (19), by (9), we have

$$\lim_{n\to\infty} \mathbf{P}\left[\left(\frac{1}{n} \log |L_n^{(k)}(0)| \leq -1\right)\right] = 0. \tag{20}$$

For the second term on the right-hand side of (19), we get a trivial bound

$$\frac{1}{n}\sum_{\ell=1}^{\ell_n} \log \left|\frac{y_{\ell,n}}{R}\right| \leq 0. \tag{21}$$

For the last term on the right-hand side of (19), using the strong law of large numbers, we see

$$\frac{1}{n}\sum_{j=1}^{j_n}\log\left|\frac{x_{j,n}}{R}\right| = -\frac{1}{n}\sum_{j=1}^{n}\log_-\left|\frac{X_j}{R}\right| \xrightarrow[n\to\infty]{a.s.} -\mathbf{E}\left[\log_-\left|\frac{X_1}{R}\right|\right]. \tag{22}$$

Observe that $z\mapsto \log_-|z/R|-\log_-|z|$ is a bounded function with compact support and $\mathbf{E}\log_-|X_1|<\infty$ by the assumption $0\notin F$, so we have $\mathbf{E}\left[\log_-\left|\frac{X_1}{R}\right|\right]<\infty$. Combining (19), (20), (21) and (22) completes the proof. □

To translate Lemma 2.9 to a uniform lower bound for $z\in B_r$, we will use the fact that the Poisson kernel has bounded distortion, and so it is sufficient to consider the value at 0 and the boundary.

Lemma 2.10. *For all $0<r<R$ there is a constant C such that for all n, $t>0$ and $k\le \frac{\log n}{5\log\log n}$ we have*

$$\mathbf{P}\left[\frac{1}{n}\inf_{z\in B_r} I_n^{(k)}(z;R)\le -t\right]\le \frac{C}{t}. \tag{23}$$

Proof. By Fact 2.6 we may lower bound

$$\begin{aligned}
\frac{2\pi}{n}I_n^{(k)}(z;R) &= \frac{1}{n}\int_0^{2\pi}\log|L_n^{(k)}(Re^{i\theta})|P_R(|z|,\theta-\arg z)d\theta\\
&= \frac{1}{n}\int_0^{2\pi}\log_+|L_n^{(k)}(Re^{i\theta})|P_R(|z|,\theta-\arg z)d\theta - \frac{1}{n}\int_0^{2\pi}\log_-|L_n^{(k)}(Re^{i\theta})|P_R(|z|,\theta-\arg z)d\theta\\
&\ge \frac{1}{Mn}\int_0^{2\pi}\log_+|L_n^{(k)}(Re^{i\theta})|d\theta - \frac{M}{n}\int_0^{2\pi}\log_-|L_n^{(k)}(Re^{i\theta})|d\theta\\
&= \frac{2\pi M}{n}I_n^{(k)}(0;R) - \frac{1}{n}\left(M-\frac{1}{M}\right)\int_0^{2\pi}\log_+|L_n^{(k)}(Re^{i\theta})|d\theta.
\end{aligned} \tag{24}$$

By Lemma 2.9, we have that $\frac{2\pi M}{n}I_n^{(k)}(0;R)\ge -C$ asymptotically almost surely. By the proof of Lemma 2.7 we have

$$\mathbf{P}\left[\frac{1}{n}\int_0^{2\pi}\log_+|L_n^{(k)}(Re^{i\theta})|d\theta\ge t\right]\le\frac{C}{t}.$$

Combining these two bounds with (24) completes the lemma. □

Proof of Lemma 2.3. Applying the Cauchy-Schwarz inequality to (11) and dividing both sides by n^2, we have

$$\frac{1}{n^2}\log^2|L_n^{(k)}(z)|\le \frac{3}{n^2}I_n^{(k)}(z;R)^2+\frac{3\ell_n}{n^2}\sum_{\ell=1}^{\ell_n}\log^2\left|\frac{R(z-y_{\ell,n})}{R^2-\bar{y}_{\ell,n}z}\right|+\frac{3j_n}{n^2}\sum_{j=1}^{j_n}\log^2\left|\frac{R(z-x_{j,n})}{R^2-\bar{x}_{j,n}z}\right|. \tag{25}$$

Note that for any $y\in B_R$ and $z\in B_r$ we have $|R^2-\bar{y}z|$ is uniformly bounded below. Using integrability of $(\log|z|)^2$ near 0, this implies that

$$\sup_{y\in B_R}\int_{B_r}\log^2\left|\frac{R(z-y)}{R^2-\bar{y}z}\right|dm(z)+\sup_{x\in B_R}\int_{B_r}\log^2\left|\frac{R(z-x)}{R^2-\bar{x}z}\right|dm(z)\le C$$

for some constant C. Bounding $j_n \leq n$ and $\ell_n \leq n$ shows a deterministic bound of

$$\frac{3\ell_n}{n^2}\sum_{\ell=1}^{\ell_n}\int_{B_r}\log^2\left|\frac{R(z-y_{\ell,n})}{R^2-\bar{y}_{\ell,n}z}\right|dm(z)+\frac{3j_n}{n^2}\sum_{j=1}^{j_n}\int_{B_r}\log^2\left|\frac{R(z-x_{j,n})}{R^2-\bar{x}_{j,n}z}\right|dm(z)\leq 3C.$$

Lemma 2.7 and 2.10 show that $\frac{1}{n^2}\int_{B_r}(I_n^{(k)}(z;R))^2dm(z)$ is tight. Since a tight sequence plus a deterministically bounded sequence is tight, the proof is complete. □

3. Anticoncentration of elementary symmetric polynomials

We will deduce Lemma 1.2 from a Theorem of Meka, Nguyen and Vu concerning anti-concentration of multi-affine polynomials of Bernoulli random variables. To properly state their theorem, we will need a bit of setup.

We will consider multi-affine polynomials of the form

$$Q(z_1,\dots,z_n)=\sum_{S\subset[n]}a_S\prod_{j\in S}z_j \tag{26}$$

where the coefficients a_S are real and the *degree* of Q is the largest $|S|$ so that $a_{|S|}\neq 0$. If Q is of degree d, then the *rank* of Q is the largest integer r such that there exist disjoint sets $S_1,\dots,S_r\subset[n]$ of size d with $|a_{S_j}|\geq 1$ for all $j\in[r]$.

Theorem 3.1 (Theorem 1.7 of [16]). *There is an absolute constant B such that the following holds. Let Q be a polynomial of the form (26) whose rank $r\geq 2$. Let p be such that $\tilde{r}:=2^d\alpha^d r=2^d\alpha^d n/k\geq 3$ where $\alpha:=\min\{p,1-p\}$. Then for any interval I of length 1, if we let $\{\varepsilon_j\}_{j=1}^n$ be i.i.d. Bernoulli(p) random variables we have*

$$\mathbf{P}[Q(\varepsilon_1,\dots,\varepsilon_n)\in I]\leq\frac{Bd^{4/3}(\log\tilde{r})^{1/2}}{(\tilde{r})^{1/(4d+1)}}.$$

To apply this Theorem to the case of Lemma 1.2—where the variables are complex valued and arbitrary—we will use the non-degeneracy to first sample most of the randomness of each Y_j, leaving a Bernoulli random variable's worth of randomness behind; then we will take either the real or imaginary part of the subsequent polynomial and show that it has sufficiently high rank with high probability.

Proof of Lemma 1.2. Define

$$P(Y_1,\dots,Y_n):=\sum_{S\subset\{1,\dots,n\};|S|=k}\prod_{j\in S}Y_j. \tag{27}$$

Since Y_j is non-degenerate we have that at least one of $\operatorname{Re}Y_j$ or $\operatorname{Im}Y_j$ is non-degenerate. By replacing each Y_j with iY_j if needed, we assume without loss of generality that $\operatorname{Re}Y_j$ is non-degenerate. As such, there exist $t\in\mathbb{R},\kappa>0,q\in(0,1)$ so that

$$\mathbf{P}[\operatorname{Re}Y_j-t>\kappa]\geq q\quad\text{and}\quad\mathbf{P}[\operatorname{Re}Y_j-t<-\kappa]\geq q\,. \tag{28}$$

Note that since $k=O(\log n/\log\log n)$ we may bound $\kappa^k=o(e^{\varepsilon n/2})$, and so by replacing Y_j with Y_j/κ, we may assume without loss of generality that $\kappa=1$.

For each $j = 1, \ldots, n$, let Y_j^+ and Y_j^- be independent random variables satisfying $\mathbf{P}[Y_j^+ \in K] = \mathbf{P}[Y_j - t \in K | \operatorname{Re} Y_j - t > 0]$ and $\mathbf{P}[Y_j^- \in K] = \mathbf{P}[Y_j - t \in K | \operatorname{Re} Y_j - t \leq 0]$ for all measurable subsets $K \subset \mathbb{C}$. Let $\eta_j = \mathbb{1}_{\operatorname{Re} Y_j - t \geq 0}$, $j \in [n]$, and note that the collection $\{\eta_j\}$ are i.i.d. Bernoulli random variables with parameter $p := \mathbf{P}[\operatorname{Re} Y_j - t > 0]$. Let $Y_j' = \eta_j Y_j^+ + (1-\eta_j) Y_j^- + t$, and observe that Y_j' and Y_j have the same distribution. Therefore, it suffices show that

$$\mathbf{P}\left[|P(Y_1', \ldots, Y_n')| \leq 1/2\right] = o(1) .$$

For each fixed instance of the variables $\{Y_j^+, Y_j^-\}$, define the polynomial F in the variables $\{\eta_j\}$ via

$$\begin{aligned} F(\eta_1, \ldots, \eta_n) &:= P(\eta_1(Y_1^+ - Y_1^-) + Y_1^- + t, \ldots, \eta_n(Y_n^+ - Y_n^-) + Y_n^- + t) \\ &= \sum_{S \subset [n], |S| = k} \left(\prod_{j \in S} (Y_j^+ - Y_j^-) \right) \prod_{j \in S} \eta_j + Q \end{aligned}$$

where Q is a polynomial of degree $< k$ in terms of η_j when all the $Y_j^{\pm}$ are fixed. For $r = \lfloor n/k \rfloor$, let $S_1, \ldots, S_r$ be disjoint subsets of $[n]$ of size k. For $S \subset [n]$, define $b_S := \prod_{j \in S}(Y_j^+ - Y_j^-)$.

Claim 3.2. *With probability at least $1 - 2\exp(-cq^{2k}n/k)$, there are at least $\frac{q^{2k}n}{2k}$ coefficients b_{S_ℓ} satisfying $|b_{S_\ell}| \geq 2^k$.*

Proof of Claim 3.2. For $\ell \in [r]$, define the event $\mathcal{E}_\ell = \{|b_{S_\ell}| \geq 2^k\}$, and note that the events $\{\mathcal{E}_\ell\}_\ell$ are independent. Further, note that

$$\begin{aligned} \mathbf{P}[\mathcal{E}_\ell] &\geq \mathbf{P}[|Y_j^+ - Y_j^-| \geq 2, \forall j \in S_\ell] \\ &\geq \mathbf{P}[\operatorname{Re} Y_j^+ - \operatorname{Re} Y_j^- \geq 2\kappa, \forall j \in S_\ell] \\ &\geq \mathbf{P}[(\operatorname{Re} Y_j^+ - t) - (\operatorname{Re} Y_j^- - t) \geq 2, \forall j \in S_\ell] \\ &\geq q^{2k} \end{aligned} \tag{29}$$

by (28), since we have assumed $\kappa = 1$. Let N denote the number of ℓ for which $\mathcal{E}_\ell$ holds. Then by Bernstein's inequality there is a constant $c > 0$ so that

$$\mathbf{P}[N \leq q^{2k} r/2] \leq \mathbf{P}[N \leq \mathbf{E}N/2] \leq 2\exp(-c\mathbf{E}N) \leq 2\exp\left(-cq^{2k}n/k\right) .$$

Noting that $\mathbf{E}N \geq q^{2k}n/k$ completes the claim. ☐

Note that

$$\operatorname{Re} F(\eta_1, \ldots, \eta_n) = \sum_{S \subset [n], |S| = k} (\operatorname{Re} b_S) \prod_{j \in S} \eta_j + \operatorname{Re} Q, \quad \operatorname{Im} F(\eta_1, \ldots, \eta_n) = \sum_{S \subset [n], |S| = k} (\operatorname{Im} b_S) \prod_{j \in S} \eta_j + \operatorname{Im} Q$$

and so $\operatorname{Re} F$ and $\operatorname{Im} F$ are both multi-affine polynomials in the variables $\{\eta_j\}$ with real coefficients and degree at most k. For each coefficient $|b_{S_\ell}|$ with $|b_{S_\ell}| \geq 2^k$ we have either $|\operatorname{Re} b_{S_\ell}| \geq 2^{k-1}$ or $|\operatorname{Im} b_{S_\ell}| \geq 2^{k-1}$. Thus, if we let $\mathcal{A}$ be the event that either $\operatorname{Re} F$ or $\operatorname{Im} F$ has rank at least $q^{2k}n/(4k)$, then Claim 3.2 implies that $\mathbf{P}[\mathcal{A}^c] \leq 2\exp(-cq^{2k}n/k)$.

Thus, for n sufficiently large, we have

$$\begin{aligned}\mathbf{P}[|P(Y_1,\dots,Y_n)|\le e^{-\varepsilon n}] &\le \mathbf{P}\left[|\operatorname{Re}F(\eta_1,\dots,\eta_n)|\le\frac{1}{2},|\operatorname{Im}F(\eta_1,\dots,\eta_n)|\le\frac{1}{2}\,\Big|\,\mathcal{A}\right]+2e^{-cq^{2k}n/k}\\ &= \mathbf{P}\left[|\operatorname{Re}F(\eta_1,\dots,\eta_n)|\le\frac{1}{2},|\operatorname{Im}F(\eta_1,\dots,\eta_n)|\le\frac{1}{2}\,\Big|\,\mathcal{A}\right]+o(1) \end{aligned}\tag{30}$$

where for the second bound we used that $k=o(\log n)$.

Conditioned on the event $\mathcal{A}$, suppose that the rank of $\operatorname{Re}F$ is at least $q^{2k}n/(4k)$. Then we will apply the first bound in Theorem 3.1; if we set $\alpha=\min\{p,1-p\}$ and $\widetilde{r}=2^k\alpha^k q^{2k}n/(4k)$ then Theorem 3.1 implies

$$\mathbf{P}\left[|\operatorname{Re}F(\eta_1,\dots,\eta_n)|\le 1/2\,\Big|\,\operatorname{rank}\operatorname{Re}F(\eta_1,\dots,\eta_n)\ge q^{2k}n/(4k)\right]\le Bk^{4/3}\frac{(\log\widetilde{r})^{1/2}}{\widetilde{r}^{1/(4k+1)}}.\tag{31}$$

Write $k=\varepsilon\log n/\log\log n$, where $\varepsilon\le 1/5$ and note we first note that

$$\log\widetilde{r}=\log n\left(1+\Theta\left(\frac{\varepsilon}{\log\log n}\right)\right).$$

and so

$$\begin{aligned}\log\left(Bk^{4/3}\frac{(\log\widetilde{r})^{1/2}}{\widetilde{r}^{1/(4k+1)}}\right)&\le\frac{1}{2}\log\left(\log n\left(1+\Theta\left(\frac{\varepsilon}{\log\log n}\right)\right)\right)-\frac{\log\log n}{8\varepsilon}+\Theta\left(\log\log\left(\varepsilon\frac{\log n}{\log\log n}\right)\right)\\ &=\left(\frac{1}{2}-\frac{1}{8\varepsilon}\right)\log\log n+o(1)+o(\log\log n).\end{aligned}$$

For $\varepsilon\le 1/5$, the right-hand side tends to negative infinity, completing the proof. □

4. Comments and Open Problems

We believe that Lemma 1.2 is suboptimal, and that in fact the same statement should hold provided $k=o(n)$.

Conjecture 4.1. *Let $Y_1,Y_2,\dots,$ be i.i.d. copies of a complex-valued non-degenerate random variable. Then for any sequence $k=k(n)$ satisfying $k=o(n)$ and each $\varepsilon>0$ we have*

$$\lim_{n\to\infty}\mathbf{P}\left[\left|\sum_{1\le i_1<i_2<\cdots<i_k\le n}Y_{i_1}Y_{i_2}\dots Y_{i_k}\right|\le e^{-\varepsilon n}\right]=0.$$

The sharpest general statements for anti-concentration for multi-affine polynomials are provided by the work of Meka, Nguyen and Vu [16], for which Conjecture 4.1 lies well beyond the currently known bounds. For an example of a polynomial of very high degree for which anti-concentration is known, see Tao and Vu's work on the permanent of a random matrix [20] (see also [14]). We note that Conjecture 4.1 appears non-trivial even in the case of, say, Gaussian random variables.

Remark 4.2. We note that substituting a positive resolution to Conjecture 4.1 for Lemma 1.2 would immediately upgrade Theorem 1.1 to hold for all $k=o(n)$; indeed, throughout the proof the only assumptions on k used are that $k=o(n)$ and that the conclusion of Lemma 1.2 holds. Additionally, due to Lemma 2.4, one can further assume that the variable Y_1 in Conjecture 4.1 satisfies $\mathbf{E}|Y_1|<\infty$.

We recall that a $k = o(n)$ extensions Theorem 1.1 falls under the following more general conjecture:

Conjecture 4.3 ([17], O'Rourke-Steinerberger). *Let μ be probability measure $\mu \in \mathbb{C}$. For each $t \in [0,1]$ there is a measure μ_t so that for the random sequence of measures $\mu_n^{\lfloor tn \rfloor}$ converges to μ_t as $n \to \infty$.*

See [1, 7, 8, 11, 13] and the references therein for progress on this conjecture. Additionally, the work [6] considers the analogue of Conjecture 4.3 for the roots of high derivatives of random polynomials with independent *coefficients* rather than roots.

We also highlight a conjecture attributed both to Kabluchko[2] and Cheung, Ng, Tsai, Yam [4, Conjecture B]:

Conjecture 4.4 (Kabluchko). *For any probability measure μ on $\mathbb{C}$, the sequence of random measures $\mu_n^{(1)}$ converges to μ almost surely as $n \to \infty$.*

Subramanian's work [19] in fact proves Conjecture 4.4 in the case when μ is supported on the unit circle and not the uniform measure; this was extended to the case of the uniform measure on the unit circle by Cheung, Ng, Tsai, Yam [4] and shown to hold for the kth derivative for each fixed k provided μ is supported on the unit circle (see also [5] for a related work on random Blaschke products). We suspect that Theorem 1.1 can also be upgraded to almost sure convergence. The main difficulty appears to be in upgrading Lemma 2.2 to almost-sure convergence. After posting a draft of this work on arXiv, Conjecture 4.4 was proven by Angst, Malicet and Poly [2].

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References

[1] T. Alazard, O. Lazar, and Q. H. Nguyen. On the dynamics of the roots of polynomials under differentiation. *J. Math. Pures Appl.*, 162:1–22, 2022.

[2] J. Angst, D. Malicet, and G. Poly. Almost sure behavior of the critical points of random polynomials. *arXiv preprint arXiv:2301.06973*, 2023.

[3] S.-S. Byun, J. Lee, and T. R. Reddy. Zeros of random polynomials and their higher derivatives. *Trans. Amer. Math. Soc.*, 375(9):6311–6335, 2022.

[4] P. Cheung, T. Ng, and S. Yam. Critical points of random finite blaschke products with independent and identically distributed zeros. *Complex Analysis and Potential Theory with Applications*, 2014.

[5] P.-L. Cheung, T. W. Ng, J. Tsai, and S. Yam. Higher-order, polar and Sz.-Nagy's generalized derivatives of random polynomials with independent and identically distributed zeros on the unit circle. *Comput. Methods Funct. Theory*, 15(1):159–186, 2015.

[6] R. Feng and D. Yao. Zeros of repeated derivatives of random polynomials. *Anal. PDE*, 12(6):1489–1512, 2019.

[7] J. Hoskins and Z. Kabluchko. Dynamics of zeroes under repeated differentiation. *Exp. Math.*, pages 1–27, 2021.

[8] J. G. Hoskins and S. Steinerberger. A semicircle law for derivatives of random polynomials. *Int. Math. Res. Notices*, 2022(13):9784–9809, 2022.

[2]This conjecture was stated at the Workshop on Random Functions in April, 2021.

[9] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág. *Zeros of Gaussian analytic functions and determinantal point processes*, volume 51 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2009.
[10] Z. Kabluchko. Critical points of random polynomials with independent identically distributed roots. *Proc. Amer. Math. Soc.*, 143(2):695–702, 2015.
[11] Z. Kabluchko. Repeated differentiation and free unitary Poisson process. *arXiv preprint arXiv:2112.14729*, 2021.
[12] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, 1997.
[13] A. Kiselev and C. Tan. The flow of polynomial roots under differentiation. *Ann. PDE*, 8(2):1–69, 2022.
[14] M. Kwan and L. Sauermann. On the permanent of a random symmetric matrix. *Selecta Math. (N.S.)*, 28(1):Paper No. 15, 29, 2022.
[15] A. I. Markushevich. *Theory of functions of a complex variable. Vol. II.* Revised English edition translated and edited by Richard A. Silverman. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.
[16] R. Meka, O. Nguyen, and V. Vu. Anti-concentration for polynomials of independent random variables. *Theory Comput.*, 12:Paper No. 11, 16, 2016.
[17] S. O'Rourke and S. Steinerberger. A nonlocal transport equation modeling complex roots of polynomials under differentiation. *Proc. Amer. Math. Soc.*, 149(4):1581–1592, 2021.
[18] R. Pemantle and I. Rivin. The distribution of zeros of the derivative of a random polynomial. In *Advances in combinatorics*, pages 259–273. Springer, Heidelberg, 2013.
[19] S. D. Subramanian. On the distribution of critical points of a polynomial. *Electron. Commun. Probab.*, 17:no. 37, 9, 2012.
[20] T. Tao and V. Vu. On the permanent of random Bernoulli matrices. *Adv. Math.*, 220(3):657–669, 2009.
[21] T. Tao and V. Vu. Random matrices: universality of ESDs and the circular law. *Ann. Probab.*, 38(5):2023–2065, 2010. With an appendix by Manjunath Krishnapur.

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