

Student understanding of eigenvalue equations in quantum mechanics: Symbolic blending and sensemaking analysis

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As part of an effort to examine students' mathematical sensemaking (MSM) in a spins-first quantum mechanics course during the transition from discrete (spin) to continuous (position) systems, students were asked to construct an eigenvalue equation for a one-dimensional position operator. A subset of responses took the general form of an eigenvalue equation written in Dirac notation. Symbolic blending, a combination of symbolic forms and conceptual blending, as well as a categorical framework for MSM, were used in the analysis. The data suggest two different symbolic forms for an eigenvalue equation that share a symbol template but have distinct conceptual schemata: A transformation that reproduces the original and to operate is to act. These symbolic forms, when blended with two sets of contextual knowledge, form the basis of three different interpretations of eigenvalue equations modeled here as conceptual blends. The analysis in this study serves as a novel example of, and preliminary evidence for, student engagement in sensemaking activities in the transition from discrete to continuous systems in a spins-first quantum mechanics course.

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I. INTRODUCTION

Quantum mechanics is one of a handful of topics in which every physics major will take at least one course during their undergraduate degree due to its pervasiveness in modern research and applications in physics and beyond. Despite the ubiquity of quantum mechanics courses and the significant amount of work that has gone into improving them, the subject has still proven to be very difficult for students. The two most ubiquitous methods of teaching quantum mechanics are the traditional functions-first and the more novel spins-first. Learning quantum mechanics has been shown to be a nontrivial task across both instructional approaches [1–5].

Functions-first courses introduce quantum mechanical principles in the context of continuous systems represented as functions of position. As the name implies, a spins-first approach involves introducing students to discrete, typically two-state, spin systems, which can be modeled with relatively simple mathematics. Spins-first courses often

immediately introduce Dirac notation, adding representational sophistication. The familiarity with Dirac notation likely makes the transition to graduate study less challenging (at least in quantum mechanics). However, in order to work through other canonical quantum mechanical systems, there needs to be a point at which a spins-first course transitions from discrete systems to continuous systems that necessitate functional representations. There is an implicit expectation here that students would be able to utilize the knowledge of concepts and procedures they learn in the context of spins in these novel functional contexts. The results discussed herein are part of a project that has sought to examine student reasoning in the transition from discrete to continuous systems in a spins-first course.

Eigentheory is central to the theory of quantum mechanics; it is baked into the second and third postulates of quantum mechanics [6]. Eigenvalue equations are also one of the first mathematical entities introduced in the course and recur across a variety of different contexts in the curriculum. One could argue that an eigenvalue equation in quantum mechanics has a fundamentally different interpretation than that of an eigenvalue equation in a mathematics context, and while one could interpret them the same way, there are more productive interpretations for quantum mechanics. The majority of research on student understanding of eigenvalue equations has come from the research in undergraduate mathematics education (RUME)

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community [7–10]. The focus of this prior work has primarily been on student recognition of the geometric interpretation of an eigenvalue equation, one in which a vector is only scaled, and not rotated, by an operator. The findings on quantum mechanical eigenvalue equations in physics education research (PER) have primarily addressed difficulties or misconceptions [2,4,11,12].

The primary goals of this study are to explore two research questions.

1. What knowledge and resources are students drawing on to construct and make meaning of eigenvalue equations in quantum mechanics?
2. To what extent are students engaging in sensemaking while constructing quantum mechanical eigenvalue equations in the transition from spins to positions?

This was done by utilizing three theoretical frameworks—concept images, symbolic blending, and mathematical sensemaking—to analyze written data from student responses to a variety of prompts, which are discussed in detail in the following sections. We identified three distinct symbolic blends, composed of two distinct symbolic forms and two distinct sets of contextual knowledge. The first is consistent with the mathematical (geometric) interpretation, the second is reflective of a common difficulty in quantum mechanics, and the third is connected to the interpretation of eigentheory presented in the postulates of quantum mechanics. The identification of these symbolic forms and blends serves as some of the first evidence of students engaging in mathematical sensemaking in the transition from spins to positions. Furthermore, we were able to utilize the categorical framework for mathematical sensemaking in physics to break down one student’s reasoning in detail. Together, these data and analyses demonstrate some of the sensemaking in which students are engaging during the transition.

II. THEORETICAL FRAMEWORKS

Data analysis was an iterative process that involved working through four different theoretical frameworks. Because the main object of study was a mathematical expression, a framework from the RUME community, concept images, was utilized first to get a broad sense of student thinking. As analysis progressed and included the structure of the expressions being constructed, the symbolic forms framework was a natural progression for analysis. With that came symbolic blending to account for the different roles of context, which was especially relevant for the range of responses in the data in both templates and concepts. The categorical framework for mathematical sensemaking was used to analyze one particularly detailed student response. It has been argued that students accessing the symbolic forms and blends in their reasoning are evidence of sensemaking [13]. In addition to forms identified in this study serving as evidence of sensemaking,

one student’s response to the main task provided sufficient detail to demonstrate evidence of mathematical sensemaking as described by Gifford and Finkelstein [14], enabling that framework to be implemented in this context and providing a concrete example of the two in tandem. In this section, each of these frameworks is explained, and relevant examples of their implementations are discussed.

A. Concept image framework

Tall and Vinner developed the *concept image* framework based on mathematics education for modeling knowledge and how it is elicited [15]. A concept image is developed over time and includes the whole of a person’s understanding of a single concept, including pictures, processes, and properties. All pieces that make up a single concept image are not necessarily connected to one another and may not be elicited simultaneously. The portion of the concept image elicited in response to some stimulus (in our cases, usually a physics question) is referred to as the *evoked concept image*. In fact, it seems likely that there are a variety of factors that can influence which part of a student’s concept image is evoked in a given scenario. Given the intersectional nature of this project, the concept image framework has proven to be a useful tool in the interpretation of student responses. Since its introduction to the RUME and PER communities, the concept image framework has been utilized to analyze student thinking on integration [16,17], partial derivatives [18], eigenvalue equations [8], and vector calculus in upper-division electricity and magnetism [19,20]. Here it assisted in the identification and classification of different symbolic forms in student responses.

B. Symbolic forms

Symbolic forms [21], developed as a means of examining how students think about (physics) equations, can be seen as an extension of the knowledge-in-pieces framework [22]. Sherin intentionally modeled this framework after diSessa’s phenomenological primitives (p-prims), which were a set of intuitive chunks of knowledge or ideas [22]. These p-prims were each self-contained and relatively simple, small knowledge structures originating from nearly superficial interpretations of reality. By comparison, symbolic forms are larger structures consisting of two pieces: a pattern of symbols in an expression, the symbol template, and the rough idea expressed therein, labeled the conceptual schema. A conceptual schema is intended to have a fairly simple structure and is not inherently connected to any physical system or reasoning.

Given that the schema contains all the meaning, it is possible for a single symbol template to be a part of several different symbolic forms, depending on the interpretation and/or context. To make clear the difference between them, some examples can be quickly explored. A common expression for the total energy of a system in intermediate mechanics is

$$H = T + U,$$

where T is the total kinetic energy in the system and U is the potential energy. The sum on the right side is an example of the *parts of a whole* symbolic form, as the individual pieces are summed to compose the total energy of the system. In a senior-level quantum mechanics class, this expression would look quite different,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r}),$$

but could be interpreted in much the same way. In both cases, here the symbol template is $[\square + \square]$ and the schema is “elements that combine to make a whole quantity,” with the context of providing the meaning for the quantities represented, which is individual contributions to the total energy of a system.

Another example can be found in the kinematics equation,

$$x(t) = x_i + v_i t + \frac{1}{2} a t^2.$$

While at first glance this could be seen as *parts of a whole* due to the similarity of the symbol pattern, a more appropriate form is *base plus change*. The latter’s symbolic form consists of the template $[\square \pm \Delta]$ and the schema “a base quantity changed by some amount.” In this context, the “base” is the initial position, x_i , and the “changes” are the two time-dependent terms.

Given their intended nature as building blocks, it is possible and likely to find more than one symbolic form in a physics equation. In the examples discussed above, the focus was on the meaning of the sum, but other symbolic forms could be used to determine the meaning of the different equal signs. Due to the nature of the framework, it has been used in several contexts to analyze the intersection of physics and mathematics. An example of a more traditional implementation is seen in the work of Schermerhorn and Thompson, who used symbolic forms to investigate student construction of differential length elements, identifying the use of the *parts of a whole* form in expressions for the components of a differential length vector [20].

Dreyfus and colleagues engaged in a theory-building effort in the context of quantum mechanics meant to use data to illustrate their conjectures and explore a method for analyzing student mathematical sensemaking in quantum mechanics (QM) problem solving [23]. They also argued that the mathematical sensemaking (MSM) tools learned in introductory physics are necessary but insufficient for MSM in quantum mechanics; this argument would be consistent with the findings of Kuo *et al.* [13]. The third and final claim of Dreyfus and colleagues’ paper is that when students do not succeed in MSM in QM, it is not because they are not engaging in the process, but that they are not

TABLE I. Symbolic forms proposed by Dreyfus and colleagues [23].

Proposed form	Symbol template	Conceptual schema
Transformation	$\hat{\square} \rangle$	“Reshaping” (the idea of stuff getting molded into a different shape).
Eigenvector eigenvalue	$\hat{\square} \rangle = c \rangle$	A transformation that reproduces the original.

using particular cognitive machinery (i.e., symbolic forms) needed to engage in expert MSM in QM. They highlight episodes from a Bing and Redish study [24] to illustrate problems students could have interpreting equations with unproductive symbolic forms in the context of a flow rate problem. The forms they proposed are summarized in Table I.

The interpretation of the eigenvalue equation that results from this *eigenvector-eigenvalue* symbolic form is not necessarily productive for quantum mechanics, however. While thoroughly appropriate in mathematical contexts, the notion of transformations in quantum mechanics is not central in the same way as the additional expert-level reasoning described by Dreyfus and colleagues. We argue that although theirs is one interpretation of an eigenvalue equation, there are others that come from students in this framework. The most sophisticated of these interpretations abandons the notion of transformation altogether, and this expert-level reasoning closely resembles the associated conceptual schema. Dreyfus and colleagues conjecture that quantum mechanical symbolic forms are part of a physicist’s cognitive ecology and that students can learn them, though it may be difficult. Our data support this claim.

C. Symbolic blending

Conceptual blending is a theoretical framework for cognition that has only recently been adapted from its origins in the linguistics communities to physics and mathematics education research [25]. This framework models the generation of novel knowledge or ideas as the effective blending of existing ones. Problem solving in this framework can be modeled as consisting of two mental spaces that act as inputs, with structure provided by a generic space, that lead to a novel blended space where new understanding can develop as a result. In the blended space, ideas or relationships can be defined or recognized that would not have come from either of the individual input spaces. The construction of a blend is fluid, as are the spaces occupied by the person generating the blend. This is particularly useful when examining intersections of mathematics and physics, where the two disciplines can be considered the input spaces and the blend, or lack thereof, can be seen in student work.

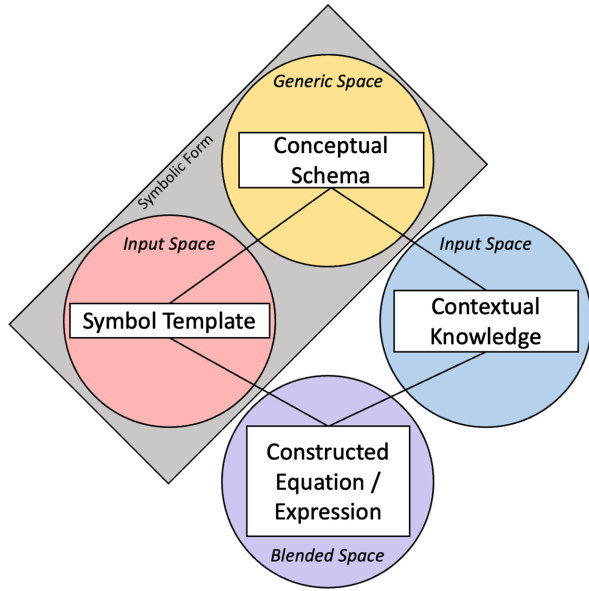


FIG. 1. Symbolic blending diagram [26] depicting the interaction between a symbolic form and conceptual blending in equation or expression construction.

In order to explicitly account for context in equation construction and interpretation, Schermerhorn and Thompson developed a model that incorporates symbolic forms into conceptual blending, which they call symbolic blending (see Fig. 1) [26]. The conceptual schema from symbolic forms fills the role of the generic space in a conceptual blend, providing structure to the reasoning and the mathematical justification for the symbol template, separate from any contextual meaning. The symbol template is one of the two input spaces and includes the structure of the representations of student-generated equations. The second input space contains the necessary contextual knowledge for the construction and interpretation of the equation, such as geometric or physical knowledge, that motivates the particular symbolic form use. The symbol template and conceptual schema spaces comprise the symbolic form, which is entirely acontextual. The two input spaces, structured by the conceptual schema, come together in the blended space, resulting in the equation generated by students and the new meaning they are able to take from it. This model extends the utility of symbolic forms without sacrificing one of its underlying principles, which is that the conceptual schemata should be entirely context independent and generalizable; it accounts for the potential of an expression to have varied meanings by blending the template to different contexts with the same conceptual schema for structure.

D. Mathematical sensemaking

Broadly, mathematical sensemaking (MSM) can be considered a part of the larger activity of coherence seeking in which students, and even experts, seek connections

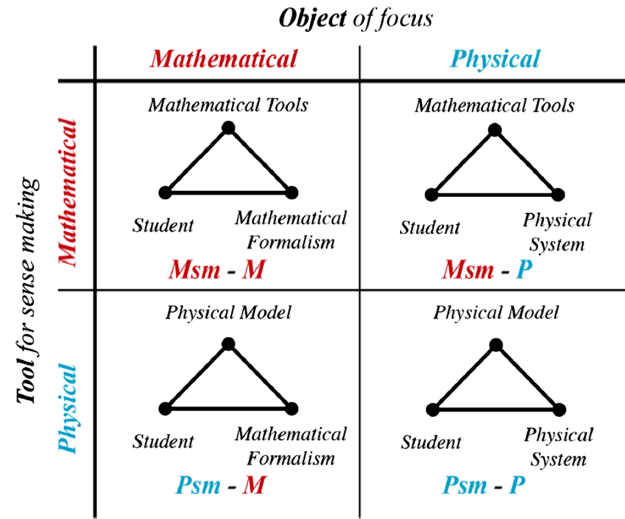


FIG. 2. The four modes of mathematical sensemaking, from Fig. 2 of Gifford and Finkelstein [14].

between conceptual and mathematical understanding in physics [27]. Part of assessing the extent to which students are engaging in MSM in the transition from discrete to continuous contexts involves assessing their ability to generalize their knowledge from spin- $\frac{1}{2}$ systems to novel systems.

Gifford and Finkelstein [14] developed a categorical framework for analyzing what they have referred to as different modes of sensemaking, using mediated cognition and activity theory. Each mode consists of three nodes (see Fig. 2). The first is the subject engaging in the reasoning (bottom left, here a student). The next is a mathematical tool or physical model (top) which is used to engage with the object of the reasoning, which is either a mathematical formalism or physical system (bottom right). The different permutations of tools, models, formalism, and system result in four different modes of sensemaking that can be used to model reasoning. The authors model transitions within and between these different modes of reasoning via three different mechanisms. Two different ways of reasoning can be related via “coordination” to provide two ways to make sense of the same idea (object). Most relevant to this study is a step in which reasoning can be “chained” (see Fig. 3). Chaining is a multistep sequence, in which the object of one mode becomes the tool in the next mode. This framework is particularly useful for analyzing responses in

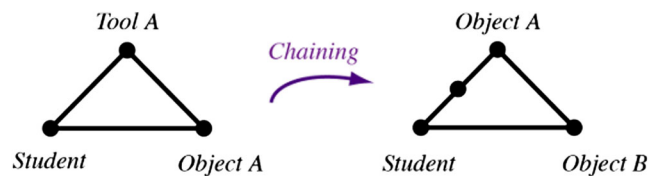


FIG. 3. Depiction of chaining, from Fig. 6 of Gifford and Finkelstein [14].

which students are explicit about the knowledge they are accessing and how they are using it to reason through their responses to our prompts.

III. BACKGROUND AND METHODS

The work discussed here is an evolution of the collaboration between the authors of Refs. [10,28,29]. In both that study and the current study, data were collected in a senior-level, spins-first quantum mechanics course that is required for the completion of physics and engineering physics B.S. degrees at the institution but is also taken by some B.A. physics majors, physics minors, and nonphysics majors. While not the students' first introduction to quantum mechanics in physics or engineering physics programs, this is the most in-depth study of the topic available to undergraduate students at this institution. In the earlier study, two different but analogous tasks were used to probe student interpretations of eigenvalue equations in both quantum mechanical and mathematical contexts and their comparisons between the two equations. Data are from the second task described in the previous article, labeled the eigenvalue equation comparison task [Fig. 1(b) in Ref. [10]; see Fig. 4], which was administered to the students in this dataset as a written homework assignment just over 2 weeks into the class, during which instruction focused on spin- $\frac{1}{2}$ systems and Dirac formalism.

There were three primary physical interpretations of quantum mechanical eigenvalue equations presented: *measurement*, *potential measurement*, and *correspondence*. Students demonstrating a measurement interpretation of the eigenvalue equation provided responses consistent with the idea that the operator acting on a state in an eigenvalue equation represented a literal measurement of the state of the quantity represented by the operator, with the right-hand side of the equation representing the outcome of that measurement. Potential measurement was a more nuanced view of the equation in which the eigenvalue

equation conveyed information about the outcomes of potential, not actual, measurements. Students who presented views consistent with the correspondence view of the eigenvalue equation gave responses that drew attention to the stated relationships between the eigenstates and their corresponding measurement values. The language used in correspondence view responses may not have mentioned measurement at all. These documented interpretations of specifically quantum mechanical eigenvalue equations serve as a starting point for the analysis of data presented in this paper. For this study, we focus on two closely related equation construction tasks.

In order to provide insight into whether students had an established symbol template for eigenvalue equations at that point in the course and to enable identification of a conceptual schema for any established symbol template, students were asked to construct a position eigenvalue equation for the position operator [see Fig. 5(a)]. The task was also intended to probe students' ability to generalize the concepts taught in the course so far to a "new" observable, position. Changing the context was important to this determination because to be a part of an eigenvalue equation symbolic form, the schema should be generalizable to different contexts. An adequate answer to this prompt would use the same symbol template as the *eigenvector-eigenvalue* symbolic form proposed by Dreyfus and colleagues [23] but specified to the position context. While still using the same symbolic form, the full symbolic blend is different due to the different context.

In previous courses you have probably seen an eigenvalue equation which looks something like this:

$$A\vec{x} = \lambda\vec{x}.$$

There are a number of ways to interpret this equation, one of which is similar to the way we interpret

$$\hat{S}_z |+\rangle = \frac{\hbar}{2} |+\rangle$$

in quantum mechanics. In general, however, the most productive ways to think about these two equations are quite different. In no more than a page, discuss the various meanings and interpretations of these two equations, their similarities and differences, and why we identify them both as eigenvalue equations.

FIG. 4. Eigenvalue equation comparison task.

(a)

Consider a quantum mechanical system constrained to be located along a straight line as shown below.



Write down an **eigenvalue equation** for an operator that represents the position of this system. Briefly explain what each term in the equation represents and why you wrote the equation the way you did.

(b)

Consider a quantum mechanical system that is physically constrained to be located along a straight line, as shown below.



- Write down an eigenvalue equation for an operator that represents the position of this system.
- Briefly explain what each term in the equation represents.
 - How do each of these relate to the physical system?
 - What, if any, connections exist between the terms in your equation?

FIG. 5. Position eigenvalue equation construction tasks (a) pre-test and (b) post-test version.

This task was given to students 6 weeks into the course after instruction on Stern-Gerlach experiments, spin- $\frac{1}{2}$ systems, general spin systems, and time evolution of spin systems, but prior to any instruction on continuous systems.

The post-test was slightly modified to address the issue where students would only provide an expression with little or no interpretation. Students were asked for an expression and an explanation of their terms in different sections of the task. The explicit additions asked students to relate these terms to the physical system and explain connections between their terms. For a condensed version of these questions, see Fig. 5(b). The task was administered after a full semester of instruction.

Since the distributions of student ideas between the pre- and postinstruction administrations were not found to be statistically significant, we will treat all responses, regardless of administration, as a combined dataset.

The authors engaged in collaborative qualitative analysis to refine the codebook until a consensus was reached [30]. This was an iterative process consisting of steps through different theoretical frameworks. As a first pass, student responses were coded holistically in accordance with evoked concept images. The goal of this analysis was to generate a broad codebook for student interpretations of the eigenvalue equations they generated. Once student interpretations were grouped broadly, we began attempting to decompose student responses into symbol templates and conceptual schemata. Separating students' overall interpretations of the equation into a symbol template and a context-independent conceptual schema proved surprisingly challenging, however. The students' overall interpretations of their equations seemed to depend on different sets of contextual knowledge in addition to a symbolic form. The students' interpretations of their constructed equations could therefore be modeled effectively utilizing the symbolic blending framework. The blends, in turn, reflected the original concept image analysis, while also providing additional information about what ideas contribute to the overall interpretations.

In coding for symbolic forms specifically, student responses were first coded for a symbol template and then an associated conceptual schema. The first pass was essentially a binary coding identifying which students provided an expression utilizing the template $\hat{O}|\cdot\rangle = c|\cdot\rangle$, where the dots inside the ket symbols indicate identical symbols inside the kets and thus identical kets. For those students who used this template, a variety of conceptual schemata were identified. This symbol template is the same as that of the *eigenvector-eigenvalue* symbolic form proposed by Dreyfus and colleagues with the exception that the “empty” kets have been given dots to denote that it must be the same ket on either side of the equal sign [23]. Conceptual schemata were derived from the portions of student responses where they interpreted the expressions they provided. Coding only for expressions that used an eigenvalue equation template proved inadequate,

however, as a variety of other expressions provided by students did not conform to this symbol template and therefore required additional categorization. Grouping similar student responses by the structure of their equation, and subsequently by ideas presented by the students, resulted in the identification of other potential symbolic forms in addition to those for eigenvalue equations.

IV. RESULTS AND ANALYSIS

Students' responses to these different tasks show evidence for a variety of interpretations of the constructed expressions. While the tasks were intended to explore specifically the prevalence of an eigenvalue equation symbolic form, student responses, especially to the eigenvalue equation construction task, have demonstrated other symbolic forms as well (see Table II). There is evidence of the utilization of these different symbolic forms in conjunction with different sets of contextual knowledge, resulting in three different symbolic blends [26] for quantum mechanical eigenvalue equations. In addition, we see evidence of mathematical sensemaking that can be categorized in Gifford and Finkelstein's MSM framework [14]. In this section, we describe the different symbolic forms and their applications by students as blends. We also provide an analysis of one student's reasoning through the MSM framework.

A. Basis expansion or parts of a whole

In response to the eigenvalue equation construction task, several students wrote expressions that took the general form of a summation rather than an operator equation. Responses in this category all had the same structure: a generic ket on the left-hand side equated to a sum of products of eigenvalues and basis kets. The response shown in Fig. 6 is representative of this group.

The symbol template of the expression seems to match that of the expansion of a state ket into its basis kets, a regularly utilized operation in quantum mechanics problem solving. A general basis expansion in quantum mechanics is $|\psi\rangle = \sum_n c_n |a_n\rangle$, where the $\{|a_n\rangle\}$ constitute some eigenvector basis, and the $\{c_n\}$ are the appropriate coefficients of each basis vector (i.e., $c_n = \langle a_n | \psi \rangle$). From a symbolic forms standpoint, this demonstrates another example of summing vector components to get a full vector but in a very different context, a Dirac notation analog to constructing a differential length vector [20].

$$|\psi\rangle = \sum a_n |a_n\rangle$$

where a_n and $|a_n\rangle$ are functions of position r
 a_n is still eigenvalue and $|a_n\rangle$ is still eigenvector

FIG. 6. Basis expansion being applied to an eigenvalue equation.

TABLE II. Summary of symbolic forms identified in the data.

Object	Symbol template	Conceptual schema	Symbolic form
Basis expansion	$\sum \square$	Amounts of a generic substance, associated with terms, that contribute to a whole.	Parts of a whole
	$c\square$	A product of factors is broken into two parts and one part is identified with an individual symbol, the coefficient.	Coefficient
	$\square \square\rangle$	A vector expression including the magnitude of a quantity and a unit vector to indicate a specific direction.	Magnitude direction
Eigenvalue equation	$\hat{\square} \square\rangle = c \square\rangle$	A transformation that reproduces the original (scaling). To operate is to act.	Reproductive transformation (cf. [23]) Operator equation as action

In that previous work, the researchers identified a *magnitude direction* symbolic form for vector components; this form nested inside the summation to constitute the parts of the whole, the whole being the full vector. As seen in Fig. 6, however, the student expressions in this group typically wrote eigenvalues not only inside the RHS kets, denoting eigenvectors, but also in front of those eigenvectors, in place of the expansion coefficients $\{c_n\}$. Thus, the resulting expressions include the summation as *parts of a whole* from a basis expansion as well as the pairing of eigenvalue and eigenvector from an eigenvalue equation, $a_n|a_n\rangle$, rather than a *magnitude direction* expression $c_n|a_n\rangle$.

This set of student responses may be indicative of a novel conception of the right-hand side of the eigenvalue equation. Dreyfus and colleagues posited that the coefficient symbolic form may show up in quantum mechanics in the

multiplication of a ket by a coefficient [23]. When blended within the context of eigentheory, this could result in the understanding of the right-hand side of an eigenvalue equation as being an eigenvalue multiplied by its eigenvectors. This blend is modeled in Fig. 7. For students who accessed *parts of a whole* in their work, this understanding seems to have informed what they thought should be inside the sum [21]. It is possible that the student brought in elements from eigentheory by labeling their $\{c_n\}$ as eigenvalues and incorporated position because the prompt asks specifically for an eigenvalue equation for an operator that represents position. This could be categorized as an unproductive activation of *parts of a whole* manifesting as a basis expansion but in the context of an eigenvalue equation (see Fig. 8).

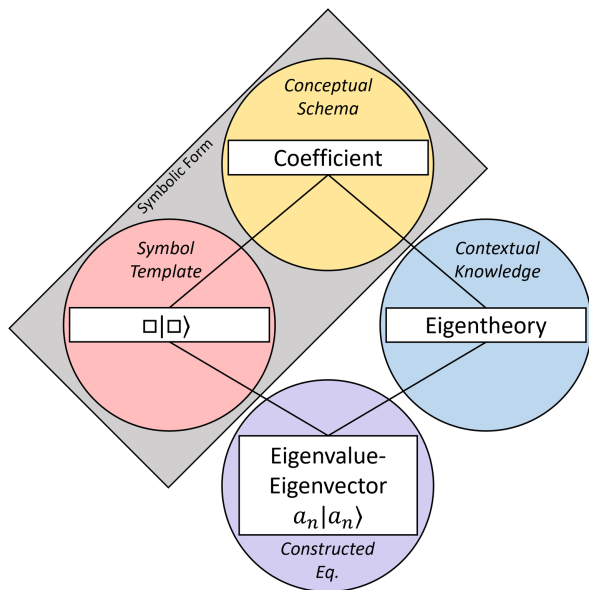


FIG. 7. A symbolic blend in which Sherin's *coefficient* symbolic form is used as a tool for understanding the right-hand side of an eigenvalue equation.

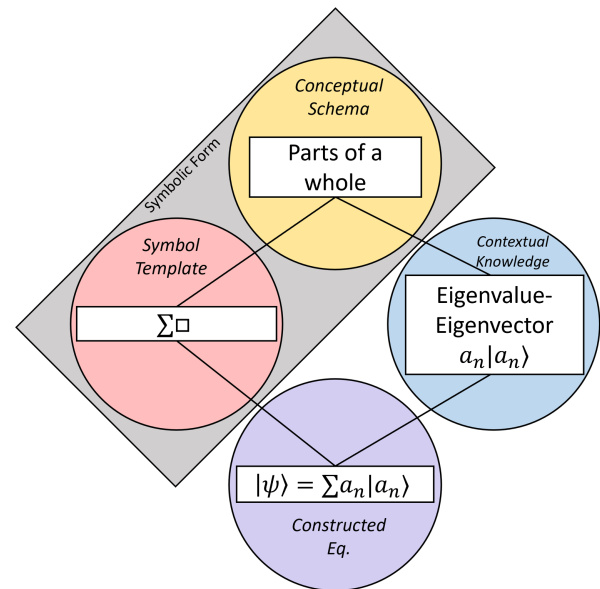


FIG. 8. A blend in which students accessed the *parts of a whole* symbolic form while generating an eigenvalue equation, resulting in an equation resembling a basis expansion but with eigenvalues instead of component amplitudes as the coefficients.

$$\hat{p} |x_n\rangle = x_n |x_n\rangle$$

FIG. 9. Response to eigenvalue equation construction task, including labels, without interpretation.

B. Eigenvalue equation symbolic forms and blends

Student responses to the two tasks have revealed two main symbolic forms, both with the same template of an eigenvalue equation (e.g., Fig. 9), that combine with two different sets of contextual knowledge, resulting in three different symbolic blends. The data supporting each of these interpretations will be presented along with a definition of the symbolic forms.

1. Eigenvector-eigenvalue/Reproductive transformation

Some students presented an interpretation of eigenvalue equations that was consistent with the *eigenvector eigenvalue* symbolic form posited by Dreyfus and colleagues [23]. The expressions all had the appropriate terms, and some even labeled the elements appropriately (e.g., see Fig. 9) but did not provide any physical reasoning or explanation for their expressions. Our proposed symbol template for this form is $\hat{\square}|\square\rangle = c|\square\rangle$, and the conceptual schema is “a transformation that reproduces the original.” These students are showing some evidence of this symbol template, and the consistency of their ket labeling can be considered evidence of the associated schema. However, we refer to this symbolic form as *reproductive*

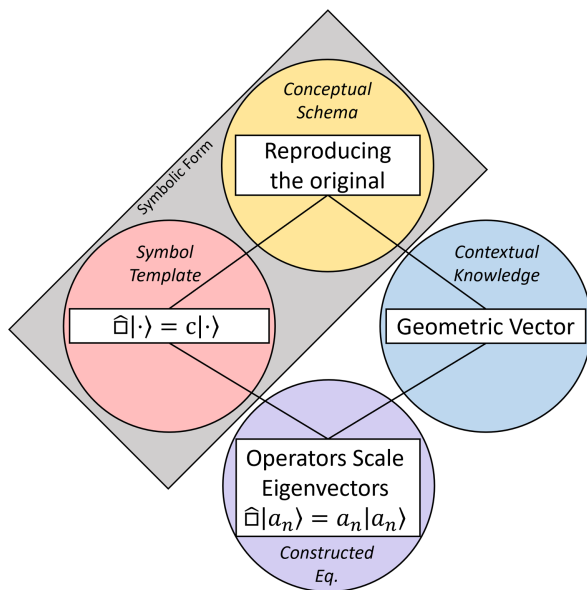


FIG. 10. Symbolic blend for the geometric interpretation of an eigenvalue equation, using the *reproductive transformation* symbolic form and a geometric vector contextual input space.

transformation as opposed to *eigenvector eigenvalue*, since we have identified more than one symbolic form for eigenvalue equations and are seeking to avoid their conflation. This symbolic form, when combined with the geometric ideas related to scaling a vector, preserving the direction of a vector, or preserving the vector altogether, results in the interpretation of the eigenvalue equation as a representation of the scaling of a ket (see Fig. 10). While they are not demonstrating any additional quantum mechanical knowledge in their responses, these students are at least demonstrating that they know that the same ket needs to be on both sides of the equation, which is consistent with the template.

2. Operating as measuring

Similar to prior work on student understanding of eigenvalue equations in quantum mechanics [10,28,29], some student responses to the eigenvalue equation construction task have the correct equation form, but the explanations are indicative of conflation of an operator acting on a state with the taking of a measurement of that state. An example of one such response and explanation is shown in Fig. 11. The student’s expression has all of the correct elements, and they are able to appropriately identify the different elements of the expression. However, in addressing how each element of the expression relates to the physical system, the student says that the operator represents “the operation of measuring position,” which is indicative of the *operating as measuring* interpretation, which seems like a distinct interpretation of an eigenvalue equation.

This type of reasoning could be an application of the idea that quantum mechanical operators typically represent measurable quantities; the overwhelming majority of operators engaged in the course do so. While this and *reproductive transformation* share a symbol template, the conceptual schema in this case is less about eigentheory and instead focuses on the generic association of operation as an action. This combination of template and schema composes another symbolic form, *operator equation as action*. When combined with the contextual knowledge of the third postulate and general ideas about the effect of quantum mechanical measurement, students arrive at an interpretation of the eigenvalue equation in which the

$$\hat{x}|x\rangle = x|x\rangle \Rightarrow \hat{x}\phi_x(x) = x\phi_x(x)$$

$\phi_x(x)$ is the prob amplitude of measuring x
 x is the value of position
 $\hat{x}$ is the operation of measuring position

FIG. 11. Response to eigenvalue equation construction task indicative of operation as a proxy for measurement.

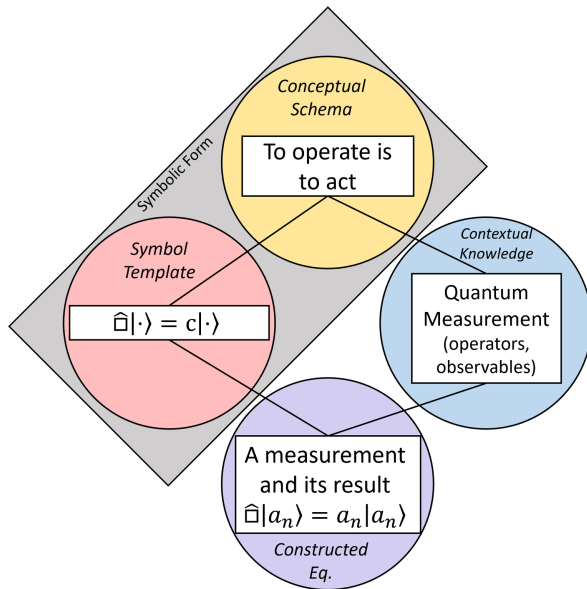


FIG. 12. Symbolic blend for the *operating as measuring* interpretation of an eigenequation.

operation of the position operator on a position eigenvector represents a measurement of position and the results of that specific measurement, which is the eigenvalue. This symbolic blend is represented in Fig. 12. These student responses are indicative of the resource “measurement as agent” identified by Gire and Manogue [31,32]. This resource is connected to the collapse of the wave function upon taking a measurement. Such a resource could be applied in combination with another resource they identified, “operator as agent,” to arrive at the conclusion that operating is measuring. “Operator as agent” is closely connected to the conceptual schema “to operate is to act,” with the primary distinction being semantic to better fulfill the role of a conceptual schema in the context of symbolic forms.

3. Potential measurement outcome

The final interpretation of eigenvalue equations with this symbol template contains all the correct elements, written as one would expect from convention [Fig. 13(a)]. Figure 13(b) shows one student’s explanation of what each term represents. When the text alone is read, this is a fairly sophisticated statement: “When you measure the position of [the eigenstate] x_i you get x_i .” The portion in brackets is an addition by the authors for coherence, which is supported by the student’s response to the question, “How does each of these relate to the physical system?”, shown in Fig. 13(c).

As with the *operating as measuring* interpretation, this student does not apply a geometric interpretation. However, neither does the student include the notion that operating is the act of taking a measurement. This is indicated by the student’s use of the phrases “When you measure ...” and “can be measured” in relation to the operator, in contrast to

(a)

$$\hat{x} |x_i\rangle = x_i |x_i\rangle$$

(b)

$\hat{x}$: when you measure the position
 $|x_i\rangle$: of x_i you get
 x_i : x_i

(c)

$\hat{x}$: operator, can be measured
 x_i : measured value of position
 $|x_i\rangle$: eigenstate

FIG. 13. Student work demonstrating *potential measurement outcome*. (a) Equation. (b) Explanation of elements. (c) Connection of terms to physical system.

the student in Fig. 11 who stated that the operator represented “the operation of measuring position.” This is a subtle but important distinction, as it separates the idea of taking a measurement from the idea that the operator represents a measurable quantity.

This student presents a more sophisticated interpretation of the eigenvalue equation than the previous ones shown: that it is a statement about the *possible* outcome of a measurement of the position, or more generally, the quantity being represented by the operator. In this case, students still recognize that the same state must be represented on either side of the equation, suggesting the same schema being accessed here as in *reproductive transformation*. As a symbolic blend, the distinction from a scaling interpretation is made by the selected contextual knowledge space, which contains information related to the third postulate and quantum measurement—the same space as for the *operating as measuring* symbolic blend. Consistent with the results from interpretation tasks [10], we label this symbolic blend *potential measurement outcome*; the blending diagram is shown in Fig. 14.

C. Interpretations of the equal sign

The student responses provided a few different ways of interpreting the equal signs. In the case of the *basis expansion*, we see an example of Sherin’s *identity* symbolic form. Sherin states, “identity allows a simple but important type of inference: It allows us to infer that whatever is true of the right hand side of the expression is also true of the left” [21] (p. 519). This is implicit in every equation students wrote that took the form of a basis expansion,

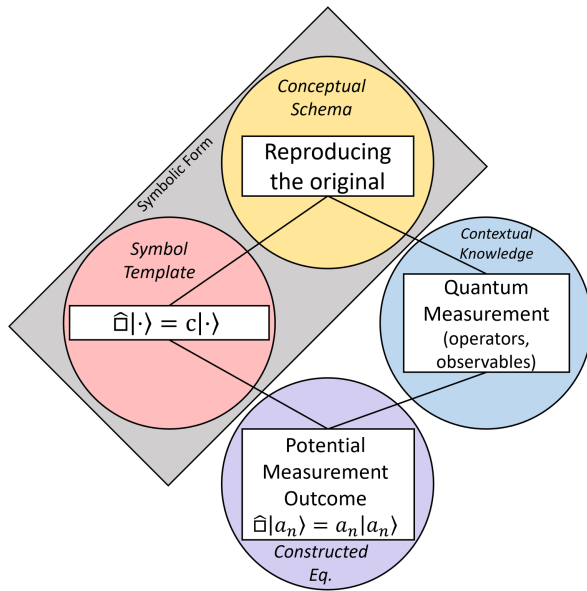


FIG. 14. Symbolic blend for the *potential measurement outcome* interpretation of an eigenequation.

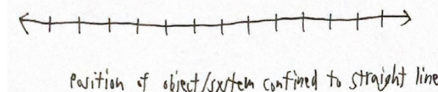
even when applying it unproductively in these cases (see Fig. 6). So while the right-hand side of each basis expansion is a novel application of *parts of a whole*, the equal sign in those expressions is part of the *identity* symbolic form. In the categories developed by Alaei and colleagues, this interpretation would fall under the *assignment* category, where the left-hand side is represented by other quantities on the right-hand side particular to a situation [33]. Framed in the context of findings from mathematics education, because the equal sign does not denote an operation from one side resulting in the other side, but more that the left-hand side is the same as the right-hand side, the equation here most closely aligns with a relational interpretation of the equal sign rather than an operational one [34–36]. However, we argue that a relational label is not a perfect fit; the equal sign is not denoting that the two sides happen to be the same as much as it is that we let the left-hand side be represented by the right-hand side in the current situation. So while there are certainly affordances to the operational/relational views, the assignment category seems more fitting.

In the case of the equal sign in *reproductive transformation*, the equal sign seems much more akin to “happens to be the same as,” an understanding that has been documented in the literature [33]. That interpretation of the equal sign is consistent with the relational view of the equal sign students have demonstrated in their interpretation of mathematical eigenvalue equations [10]. This is contrary to the proposition by Dreyfus and colleagues that the meaning of the equal sign would instead be indicative of students’ understanding of the conditional equality dependent on the eigenvectors and eigenvalues [23]. This notion of conditional equality based on the sets of eigenvectors and eigenvalues did not show up at all in our dataset.

There were also cases where the equal sign did not seem to have a distinct meaning. Neither *potential measurement outcome* nor *operating as measuring* showed evidence of distinct meanings of the equal sign, which is consistent with the conceptual schemata for these forms, both of which lack a role for the equal sign. In both cases, the interpretations of the eigenvalue equation are heavily focused on quantum mechanical concepts (independent of the correctness of those concepts), resulting in a lack of a distinct conceptualization of the equal sign. This stands in contrast to the way most physics equations are read: generally, a physics equation can be read as a direct mapping of real-world quantities [10]. For example, Newton’s second law $\sum \vec{F} = m\vec{a}$ communicates that the sum of all forces on an object is equal to the product of the object’s mass and acceleration. The relationships between the terms in a quantum mechanical eigenvalue equation are not quite so clear, which may contribute to these results.

D. Evidence of mathematical sensemaking

While the response format for the tasks discussed here does not, in general, lend themselves to analysis with the categorical framework for mathematical sensemaking developed by Gifford and Finkelstein [14], one student’s work, shown in Fig. 15, provided enough intermediary reasoning steps to allow for the framework to be implemented. The student began by defining the position operator as $\hat{x}$. They then determined the appropriate form of an eigenvalue equation by considering what appeared to be a specific and iconic eigenvalue equation, the Schrödinger equation. From this, the student extrapolated what features of the expression would look like when transferred to the context of position. In our symbolic forms analysis, this student’s



- [1] Position operator: $\hat{x}$
- [2] Format will be like the eigenvalue equation: $\hat{H}|\psi(t)\rangle = E_n|\psi(t)\rangle$
- [3] So we have a position operator multiplied/operating on $|\psi(t)\rangle$ that will give us the same $|\psi(t)\rangle$ back, but multiplied by a constant
- [4] I’ll call the constant of position x_n
- [5] The eigenvalue equation is then:
- [6]
$$\hat{x}|\psi(t)\rangle = x_n|\psi(t)\rangle$$
- [7] x_n will be an eigenvalue of the state and $\psi(t)$ is our ket or eigenvector
- [8] $\hat{x}$ is the position operator acting on $|\psi(t)\rangle$

FIG. 15. Student response to pretest version of the position eigenvalue equation construction task, analyzed with sensemaking framework. Text has been typed for legibility.

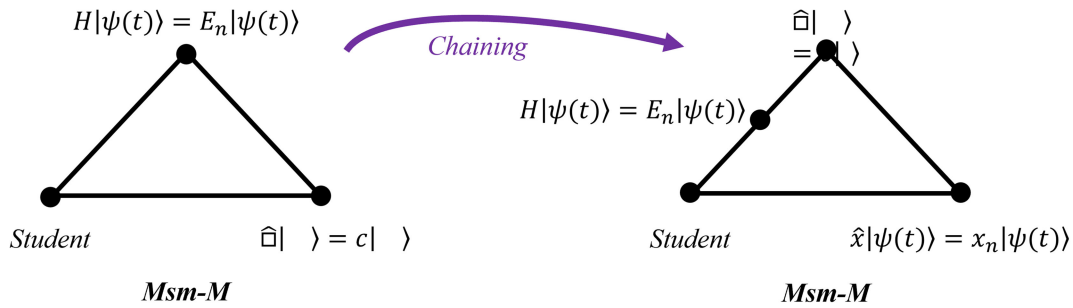


FIG. 16. Reasoning structures for student work in Fig. 15.

response would be another example of *reproductive transformation*: their reasoning in line 3 that the operation generates the same vector multiplied by a constant is consistent with the idea of scaling presented in that schema. They went on to define their eigenvalues symbolically before providing an eigenvalue equation for the position operator. This student's reasoning could be modeled very simply with a flowchart that begins with the energy eigenvalue equation, flows into their understanding of the general structure of eigenvalue equations, and terminates with the position eigenvalue equation.

The MSM framework allows for a more nuanced view of this reasoning process. Our diagram of this student's MSM is shown in Fig. 16. The first structure represents the first three lines of the student's response, in which they used the Schrödinger equation as a tool to determine the general structure—template—of an eigenvalue equation, which serves as the object of this reasoning. (Note that the student writes the template in words rather than symbolically.) This understanding of the eigenvalue equation template now becomes a tool that the student uses to write a position eigenvalue equation, the final object. This process of the student's original object of reasoning becoming a tool in their next step of reasoning is indicative of what Gifford and Finkelstein refer to as chaining. In the third and fourth lines of their response, the student is determining which pieces of their energy eigenvalue equation need to change to generate the position eigenvalue equation.

Once the tools and objects of sensemaking are identified, the mode can be determined by deciding whether the tool and/or object is mathematical or physical in nature. Despite the student starting and ending with equations that contain physically meaningful quantities—eigenvalue equations for the Hamiltonian and position operators—the student used these equations more as mathematical tools and objects. In the first structure, the student used the energy eigenvalue (Schrödinger) equation as a *mathematical* tool to make sense of the general structure of an eigenvalue equation. In the second structure, the student used the general form of an eigenvalue equation as a template to construct a position eigenvalue equation, which is the *mathematical* object here. Both context-specific eigenvalue equations are only used for their mathematical structures,

and the energy eigenvalue equation is recalled as an archetype; any physical meaning behind the expression is neither implied nor expressed. It seems similar to the position equation. Since the symbol template is a mathematical formalism, then both reasoning steps have mathematical tools and objects and thus both can be labeled as Msm-M.

There is some precedent to argue that such use of physics equations does not involve the use of physics concepts. The situation of a student recalling a well-known physics equation for a mathematical purpose (e.g., “plugging in” the relevant elements for the given system) has been described previously in the literature. Modir *et al.* described four epistemic frames for mathematical and physical reasoning in physics problem solving [37]. One frame includes students' recollection of equations, facts, and properties of physical quantities without justifications. This *algorithmic physics* frame addressed steps that physicists would often describe as “just math” in problem solving, consisting of procedural operations with physically meaningful variables. Because this work does not reflect mathematical reasoning, it is not indicative of the way mathematicians would think of mathematics, and thus *algorithmic physics* is distinguished from both *conceptual physics* and *algorithmic mathematics* for this reason. Gifford and Finkelstein acknowledged *algorithmic physics* but delineated it from any of their sensemaking modes that include physics because of the focus on conceptual physics in their definitions [14]. All three modes that include physics in Gifford and Finkelstein's categorical framework for mathematical sensemaking in physics would fall under the conceptual physics frame. Returning to the example in Figs. 15 and 16, had the student referred back to the physical system in explaining their equation, e.g., by relating the eigenvalues to the possible positions, then their work would have been classified as being in the conceptual physics frame and similarly be categorized in a sensemaking mode that includes physics in the categorical framework.

V. DISCUSSION AND CONCLUSIONS

Students' responses to the tasks discussed herein indicate that they are thinking about eigenvalue equations in a

variety of ways, three of which, highlighted here through a symbolic blending lens, are *reproductive transformation*, *operating as measuring*, and *potential measurement outcome*. These three symbolic blends all share a symbol template given by $\hat{\square}|\square\rangle = c|\square\rangle$, but have two distinct conceptual schemata, and build upon two different sets of contextual knowledge.

The findings presented here provide insight into the symbolic reasoning students may be accessing while working through eigenvalue equation construction in quantum mechanics as well as an opportunity to assess student sensemaking in the transition from discrete to continuous systems in a spins-first course. Ways in which our findings relate to prior studies will be discussed in detail in this section.

A. Symbolic blends

Dreyfus and colleagues posited that there may be symbolic forms specific to quantum mechanics that had yet to be seen in student work [23]. One such form was an eigenvalue equation symbolic form. Their eigenvalue equation symbolic form, *reproductive transformation*, which is consistent with the desired outcome of mathematics instruction on eigenvalue equations [7], is one of the two forms identified here that all use the same eigenvalue equation template, which follows the canonical structure. The other form, *operator equation as operation*, focuses on the action of the operator in the mathematical representation. These two forms blend with two different contextual knowledge spaces, resulting in three distinct interpretations of the eigenvalue equation which are summarized in Table III.

These interpretations are consistent with those identified by Wawro and colleagues in tasks in which students were provided eigenvalue equations and asked for their meaning [10]. Our current study turned the task from an interpretation task into a construction task, allowing for analysis of the prior interpretations via symbolic forms and/or symbolic blends, and providing opportunities to observe activation of additional symbol templates and/or conceptual schema that may not be productive in the context of eigenvalue equations. In fact, these data have yielded an example of a novel application of the *parts of a whole* symbolic form, as a basis expansion being applied in the context of quantum mechanical eigenvalue equations.

Operating as measuring is also consistent with earlier findings in the literature. Gire and Manogue [31,32] found

a tendency for students to associate measurements with operators when discussing sequential measurements on a superposition state; their work framed this tendency as the activation of a resource that they called “quantum measurement as an agent.” This is not an unreasonable conclusion for students to draw, especially given the resource “operator as agent” and the emphasis on measurement in quantum mechanics (e.g., the collapse of the state or wave function). When an operator acts on a state, there is often some kind of transformation made to that state, in some cases, changing it to another known state and in others, an entirely unknown state. The association of operators with both observables and transformations of a state can therefore be easily conflated as they are both in these data and those presented by Gire and Manogue. In addition, Singh and Marshman [2] showed frequent conflation by students between operating and measuring.

The third blend, *potential measurement outcome*, is significantly more relevant to and meaningful for the physical interpretation of a quantum mechanical eigenvalue equation. The fact that this interpretation shares a symbolic form with *reproductive transformation* is indicative of the idea posited by Dreyfus and colleagues [23] that there is additional expertlike reasoning that allows one to extract additional information from an eigenvalue equation in quantum mechanics. The additional information in this case is coming directly from the contextual input space in the blend. While the student who provided this response is not representative of our population in general, they serve as an example of the potential for students to reach this understanding. The fact that this is a single student can also lend some support to the claim made by Dreyfus and colleagues that while students can develop these interpretations that are part of the cognitive ecology of physicists, it is challenging [23].

All three symbolic blends are consistent with student interpretations of quantum mechanical eigenvalue equations presented by Wawro and colleagues [10]. Student ideas around potential measurement, measurement, and scaling were all present in that work and informed the analysis in this study. Interestingly, two ideas expressed by students in their interpretation task were not evident in the data collected with the eigenvalue equation construction task. Student ideas consistent with the correspondence interpretation of quantum mechanical eigenvalue equations reported by Wawro and colleagues focused heavily on the relationships between eigenvectors and eigenvalues without explicitly invoking (potential) measurement or scaling.

TABLE III. Symbolic blends for eigenvalue equations and their elements.

Symbolic blend	Symbolic form	Context
Scaling	Reproductive transformation	Geometric vector
Potential measurement outcome	Reproductive transformation	Quantum measurement
Operating as measuring	Operator equation as action	Quantum measurement

Despite the questions asking students about potential connections between the terms in their expressions in the eigenvalue equation construction task in this study, student responses were not consistent with this interpretation. Similarly, the functional interpretation of the equal sign was documented by Wawro and colleagues in student interpretations of both mathematical and quantum mechanical eigenvalue equations. In the functional interpretation, the right-hand side of the equation serves as an output of the left-hand side (e.g., $f(|a_n\rangle) = a_n|a_n\rangle$). This view was also not present in the construction task data.

Many student difficulties in quantum mechanics have been reported by others [1–3]; we see other difficulties here through the lens of symbolic forms. In addition to the conceptual, mathematical, and notational difficulties faced in quantum mechanics, Dreyfus and colleagues suggested that the potential for activation of unproductive symbolic forms can be a cause for challenges [23]. We provide evidence of this occurring with students' activation of *parts of a whole* via a basis expansion during the construction of an eigenvalue equation for a position operator. Given that both forms include the pairing of a coefficient and an eigenvector, this is not an unreasonable conflation. There are in fact situations in quantum mechanics where activation of *parts of a whole* would be appropriate and productive, such as changes of basis or perturbation theory. In those instances, the expected forms would be very similar to those identified in the construction of differential length elements reported by Schermerhorn and Thompson [20], albeit in the notation of convenience (e.g., Dirac notation).

It is also noteworthy that neither interpretation nor construction of eigenvalue equations was trivial for students. Some explicitly wrote about the difficulty of the tasks in their responses, while others showed the non-triviality of the task through their failure to provide a classifiable expression in response to the construction task.

B. Sensemaking

While generalizing has proven to be a challenge for students, the symbolic forms identified in these data suggest that students are engaging in sensemaking during this transitional period, that is, trying to apply their knowledge of eigenvalue equations and other quantum systems to generate a representative equation for a novel system. As described by Sherin [21], Kuo *et al.* [13,38], Dreyfus *et al.* [23], and Schermerhorn and Thompson [20], symbolic forms are cognitive resources students can access in order to engage in mathematical sensemaking. Differences in student reasoning during problem-solving tasks have been attributed to their use, or lack of use, of a productive symbolic form [13]. In the context of the eigenvalue equation construction task, especially before instruction on position, this is implicitly a part of what

students are being asked to do. They are given a physical system and asked to generate a representative mathematical expression. For some students, the process that ensues includes making connections between their mathematical knowledge of eigenvalue equations (form and function) and their knowledge of how quantum mechanical systems behave (primarily ideas presented in the second and third postulates). This process is facilitated, or hindered, by the activation of different symbolic forms. The fact that we can identify different symbolic forms accessed by students, whether productive in the case of some of the eigenvalue equation symbolic forms or unproductive in the case of *parts of a whole*, serves as evidence of students engaging in mathematical sensemaking.

Analysis of the student's response in Fig. 15 through the categorical framework for sensemaking [14] (Fig. 16) yields an example of what sensemaking in quantum mechanics, especially in the transition from discrete to continuous systems, can look like. It also, however, brings to light a common problem in physics education research of delineating between mathematics and physics in sensemaking and problem solving. There are many different models for modeling student reasoning and reasoning spaces that include different mathematics and physics modes/worlds/spaces and which allow, to varying degrees, the blending of the two. The application of mathematical sensemaking in this analysis highlights brilliantly the interplay of mathematics and physics. While the student's response does not include physics concepts (the criterion for being either Msm-P or Psm-P), neither is the student doing what mathematicians would typically describe as mathematics [39–41]. This explains why in the Gifford and Finkelstein [14] model, the student's work falls distinctly under Msm-M, whereas the student might be placed in an algorithmic physics frame according to Modir and colleagues [37]. While it would seem that these two interpretations of the student's work should be at odds with one another, both are self-consistent. This difference underscores the interconnections of the two disciplines. The difference in classification of this reasoning can inform future characterization of the boundary between mathematics and physics from the perspectives of problem solving and mathematical modeling.

C. Implications for instruction

The identification of new symbolic forms and blends poses some interesting possibilities with regard to quantum mechanics instruction. Kuo *et al.* reported that students who could access certain symbolic forms in problem solving were able to find conceptual shortcuts to quantitative problems [13]. The group also generated instructional materials for an introductory course that set these “calculation-concept crossover” questions as a target of instruction and found that they were indeed a feasible

target [38]. As researchers continue to determine the ways in which students interpret expressions in quantum mechanics, we should be able to develop a road map to guide them to more expertlike interpretations. The identification of both more sophisticated (*potential measurement outcome*) and less sophisticated (*operating as measuring*) symbolic blends accessed by students provides instructors with both a target and a glimpse of potential pitfalls for their students. Given the unique mathematics at play in quantum mechanics (relative to most other areas of physics) and its interplay with quantum mechanical concepts, it is unlikely that the forms discussed here span the space of those accessed by students. Therefore, it is also important to continue assessing the ways students are thinking about individual expressions in quantum mechanics.

As noted by Wawro *et al.* [10], the interpretation of a quantum mechanical eigenequation is idiosyncratic: a geometric scaling interpretation is not productive when working with normalized eigenstates, and so it is not helpful to think about the equation as a transliteration of a sentence about the relationship between the quantities in the equation, but as a representation of the third postulate of quantum mechanics. The findings in this study and in Wawro *et al.* [10] demonstrate student difficulties in recognizing this strong distinction between the mathematical and physical meanings of the equation. Thus, eliciting student explanations about both meanings is crucial to inform an instructor about the spectrum of student thinking in the classroom on this topic.

Adding to the difficulty of the transition is the interpretation of position as an observable. Quantum mechanical position is an inherently tricky quantity, particularly at the point in the course where it is introduced. Spin and energy in the context of the first few chapters of Ref. [6] are fairly abstract quantities. Position, on the other hand, is something students have been thinking about in some way since their first semester of introductory physics, if not before that. The need to consider each position as an eigenvalue may not be a challenge, but defining a state of the system at a location, a single point, as an eigenstate—an *eigenvector*—could cause confusion among students. Discussing a location as an eigenvector could be interpreted by students as a position vector rather than an eigenvector of the position Hilbert space. In a conceptual blending analysis of student’s use of electric field vector arrows, Gire and Price describe the “dual role of space” in student construction of electric field vector diagrams, that there is interference between the spatial map of the vector field and the length of a field vector as a magnitude [42]. This brings to mind the related difficulties identified by Ambrose *et al.* when investigating student understanding of electromagnetic wave representations, where students considered areas outside the envelope of the EM wave to have no field strength [43]. It is highly likely that students in

quantum mechanics conflate these dual roles of space when interpreting position as a vector, but with the added complication that both quantities under consideration are position (e.g., rather than position and electric field strength). Thus, one may expect additional student difficulties related to this dual-role issue in the transition to position as the central quantity. Although students may perform the correct computation and even use the correct vocabulary, their physical and mathematical interpretations of a position eigenvector may not be clear from a single response and may compound attempts to address difficulties related to the broader eigenequation interpretation in the context of the position.

D. Conclusion

Overall, we present evidence that students are engaging in mathematical sensemaking in the transition from spins to positions in a spins-first quantum mechanics course. This is shown by a symbolic form and symbolic blending analysis of student work constructing eigenvalue equations for position, as the first continuous variable in the course. Students implement varied symbolic forms and symbolic blends in the construction process, most with the appropriate symbol structure but with varying sophistication with respect to the quantum mechanical meaning of the equation and of operations on state vectors. Data are also analyzed through the lens of sensemaking, based on attempts to resolve inconsistencies during the process. The abstract nature of the physics concepts is tightly intertwined with the mathematical entities, leading to a challenge in distinguishing one from the other in the reasoning. This analysis serves as a novel example of and preliminary evidence for student engagement in sensemaking activities in the transition from discrete to continuous systems in a spins-first quantum mechanics course.

Spins-first quantum mechanics courses are relatively novel in the grand scheme of physics instruction. While their intent to provide students a foundation in quantum mechanics with “simple” systems and subsequently ask them to generalize to more complex systems has pedagogical validity, those generalizations are not trivial for students to make. Nevertheless, we present some of the first evidence that students are engaging in the kind of sensemaking, based on attempts to resolve inconsistencies in the process of constructing eigenvalue equations, that could generate connections between their understanding of discrete spin systems and continuous position systems, as well as the challenges of connecting the mathematical operations and entities with the physical concepts and meanings in such an abstract topic as quantum mechanics. The extent to which instruction can mediate the transition is yet to be determined, but viewing the process through the frameworks used in this study may help suggest successful strategies.

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